Evolutive Equilibrium Selection II: Quantal Response Mechanisms

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Abstract: In this paper we develop a model of Evolutive Quantal Response (EQR) mechanisms, and contrast the outcomes with the Quantal Response Equilibria (QRE) as developed by McKelvey and Palfrey(1995). A clear distinction between the two approaches can be noted; EQR is based on a dynamic formulation of individual choice in the context of evolutionary game theory in which games are played repeatedly in populations, and the aim is to determine both the micro-configuration of strategy choices across the population, and the dynamics of the population frequencies of the strategies played. Quantal Response Equilibria focuses on the more traditional aspects of non-co-operative game theory, i.e. on equilibrium in beliefs regarding strategies.We focus attention on an analytical approach which enables closed form solutions to be constructed. We consider the case of all symmetric binary choice games, which will include analysis of all well known generic games in this context, such as Prisoner's dilemma, Stag-Hunt and Pure coordination games.

Paper prepared for GAMES 2000 (Bilbao, July 24-28, 2000)
First World Congress of the Game Theory Society

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1. Introduction

The analysis of equilibrium selection based on quantal response mechanisms is currently represented by two main branches in the game theoretic literature. Firstly, there is the branch which derived its primary inspiration from models of the physical sciences for interacting particle systems. In this literature we include the analyses of Follmer(1974),Blume(1993),(1996), Herz(1993), Aoki(1995), and Brock and Durlauf(1995). Secondly, we have the branch which derives inspiration from the more traditional economic theory of probabilistic choice models applied to a game theoretic setting, in particular the analyses of McKelvey and Palfrey(1995),(1998) and Chen,Friedman and Thisse(1996). Both approaches are remarkably similar in their use of quantal response mechanisms for the specification of individual strategy choice.

In many cases the ability to contrast different equilibrium selection mechanisms is reduced because of the disparate assumptions employed by different investigators. In the case of distributional dynamics and quantal response equilibria however, there is quite an intimate relation between the literature in what appear to be currently two separate research programs. In a sequence of papers Blume established what he termed the "statistical mechanical" approach to equilibrium selection; the methodology uses techniques developed in the physical sciences to investigate the interaction of individual agents in a primarily locally interactive system; where "proximity" of agents in some context, e.g. spatial location, is important in assessing the benefits of different strategies. In the case of the analytical development of this theory the logit function plays a central role, in representing the stochastic choice behaviour of agents, and thence in determining the equilibrium strategy. The logistic choice function also plays a central role in the analytical development of McKelvey and Palfrey's "Quantal Response Equilibria" model. This model may be viewed within the context of error prone decision making, including the papers by Rosenthal(1989), Beja(1992), Ma and Manove(1993), and Fey, McKelvey and Palfrey(1996).

In this paper we develop a model of what we term Evolutive Quantal Response (EQR) mechanisms, and contrast the outcomes with the Quantal Response Equilibria (QRE) as developed by McKelvey and Palfrey. A clear distinction between the two approaches can be noted; EQR is based on a dynamic formulation of individual choice in the context of evolutionary game theory in which games are played repeatedly in populations, and the aim is to determine both the micro-configuration of strategy choices across the population, and the dynamics of the population frequencies of the strategies played. Quantal Response Equilibria focuses on the more traditional aspects of non-co-operative game theory, i.e. on equilibrium in beliefs regarding strategies.We focus attention on an analytical approach which enables closed form solutions to be constructed. We consider the case of all symmetric binary choice games, which will include analysis of all well known generic games in this context, such as Prisoner's dilemma, Stag-Hunt and Pure coordination games. In Section 2 we set out the standard model for this case. Section 3 details the equilibrium for what we term the microconfiguration of agents by strategy choice, whilst in Section 4 we derive the associated macroconfiguration of agents. Section 5 details the equilibrium selection procedure as the low noise limit of the macroconfiguration, and Section 6 notes the relationship of the implied selection and the Nash equilibria of the associated game. Section 7 provide applications of the methodology to a number of binary choice symmetric games. In the concluding section we contrast the results and methodology with existing literature in this field.

2. The Basic Model: The Microconfiguration of Agents over Strategy Choice

The basic model can be quite simply outlined. We assume a population consisting of $N \geq 2$ agents. Each agent has the same set of strategy choices; for the binary choice case, $\{s_1, s_2\}$. Time is discrete, in each time period one agent is randomly selected and this agent has the choice of sticking to the existing strategy being played, or switching to another strategy. The state of the system we define as the microconfiguration of strategy choice. At any date prior to the choice of the agent selected to make their choice it can therefore be represented by a string of length N, each component of the string being (for the binary choice case) either s_1 or s_2. The process thus proceeds, given an initial designation of the string in period 0 (the initial condition), an agent is selected to make their choice. The choice is made and a new string is therefore generated in period 1, this string is either identical to the string in period 0 (i.e. if the agent selected sticks with the original choice of strategy) or the string deviates by one component from the previous state, i.e. by one of the elements changing from s_1 to s_2 or vice versa. As the process proceeds, interest may thus focus on tracing the path of the string through time or if the interest of the researcher is on equilibrium selection, on the long run probabilities of different states of the string being observed.

The string evolves either deterministically, if the choice of agents is solely determined by the state of the string at the time they make their choice; or in a random manner if choices are made with a procedure which includes a random component. The principal focus is on the way in which agents choice of strategy is made. The switch probability has been subject to widespread research in recent years, and the relationship may be approached via a number of different model structures. These include, (i). Cognitive/Best Response; (ii) Learning/Stimulus-Response; (iii) Replicator/Evolution dynamics, and (iv). Imitation. In this paper in order to pursue our study of quantal response equilibria we utilise the Cognitive/Best Response model.

In order to relate models to experimental work particular care has to be attached to the definition of the temporal structure of the model, and the nature of the information that is passed to the agent on which choice is predicated. The above procedure we have specified may be called random sequential updating and is the form related to the randomised pair matching used by experimentalists in economics and psychology. In such experiments agents are matched in which both make a simultaneous choice of strategy which determines the payoff each makes. The information passed to the agent prior to strategy choice we assume to be a transform of the existing state of that system; i.e. the current string.

The model therefore comprises two essential elements, first there is the specification of the dynamics of the microconfigurations of agents' choices, conditional on knowledge of the methods by which agents make their choice of strategy. Secondly there is the specification of that component based on the way in which the agents' strategy choices are made.

We begin with developing a model which specifies the evolution of the microconfiguration. First some definitions:

Definition 1. A microconfiguration, K^m, of the system of agents is uniquely defined by a sequence of length N whose i^{th} component $K^m(i)$ denotes the state of agent i in microconfiguration K^m. Without restriction, we define the state of the agent to be either $+1$ or -1; indicating respectively the choice of strategy s_1 or s_2 . The configuration space R is given by the set of all possible configurations. If each agent can be in any one of two states, i.e. the choice of strategy s_1 or s_2 , the number of all possible configurations in this case is 2^N.Thus m is an index which runs over the integers from 1 to 2^N.

Definition 2. Let the system be in microconfiguration K^m, then a *neighbouring* configuration K_u^m is defined as the configuration that is obtained from K^m by changing the state of agent u from $+1$ to -1 or vice versa.

Thus we have,

$$K_u^m(j) = K^m(j) \ if \ j \neq u \tag{1}$$

$$K_u^m(j) = -K(j) \ if \ j = u \tag{2}$$

Definition 3. The neighbourhood $R_{K^m} \ \epsilon \ R$ is defined as the set of all neighbouring configurations of K^m.

We propose to specify a Markov chain defining the transition probabilities over the configuration space R. We require firstly to specify the probability that a given agent is selected to make a strategy choice in a given period; and then secondly the probability that the agent chooses a particular strategy.

We propose to define the components of the Markov Chain as follows.

Definition 4.

The probability that agent u from microconfiguration K^m is selected to make a strategy choice is,

$$G(u, K^m) \tag{3}$$

Definition 5

The probability that a change in the status of agent u from microconfiguration K^m is induced is defined by,

$$A(u, K^m) \tag{4}$$

i.e. if agent u is currently playing strategy s_1 then $A(u, K^m)$ denotes the probability of a switch to strategy s_2 and vice versa.

Definition 6.

Consider two microconfigurations K^m, K^l. The transition probability of moving from state K^m to state K^l is defined by,

$$P_{K^m,K^l} = G(u, K^m)A(u, K^m) \ if \ K^m \neq K^l \text{ and } K^l \in R_{K^m} \tag{5}$$

$$P_{K^m,K^m} = 1 - \sum_l P_{K^m,K^l} \text{ for all } l \text{ such that } K^l \in R_{K^m} \tag{6}$$

$$P_{K^m,K^l} = 0 \text{ otherwise} \tag{7}$$

Both $G(u, K^m)$ and $A(u, K^m)$ now have to be specified, and to these matters we now turn.

Assumption 1. The probability of selecting an agent for the update process is assumed to be uniform over all agents at all times, and is independent of the current state of the system, K^m; i.e.,

$$G(u, K^m) = 1/N \tag{8}$$

Now we turn to the factors affecting the switch probability $A(u, K^m)$.

Assumption 2

The functional form for $A(u, K^m)$ is assumed to be the logit function. The probability that the agent u chooses strategy $S_i = +1$ is then defined by,

$$\Pr ob(S_i = +1, u, K^m) = \frac{\exp(\beta_u Payoff_1(u, K^m))}{\exp(\beta_u Payoff_1(u, K^m)) + \exp(\beta_u Payoff_2(u, K^m))} \tag{9}$$

$$= \frac{1}{1 + \exp(-\beta_u (Payoff_1(u, K^m) - Payoff_2(u, K^m)))} \tag{10}$$

$$= \frac{1}{1+\exp(-\beta_u h_u)} \tag{11}$$

where $\beta_u \geq 0$, $Payoff_i$ $i = 1, 2$ denotes the payoffs from playing strategy 1 or 2, and

$$h_u = Payoff_1(u, K^m) - Payoff_2(u, K^m) \tag{12}$$

Whilst, we can similarly show that,

$$\Pr ob(S_i = -1, u, K^m) = \frac{1}{1+\exp(+\beta_u h_u)} \tag{13}$$

The probability of a switch from strategy $+1$ to -1 or vice-versa can thus be written as,

$$A(u, K^m) = \frac{1}{1+\exp(\beta_u K^m(u) h_u)} \tag{14}$$

The parameter β_u plays an important role in determining the probability of selection of a particular strategy. As β_u tends to infinity the logistic function approaches the step function (see Fig.1), and the probabilistic model approaches the deterministic model of best response. As β_u tends to zero the choice of $+1$ or -1 will become equally likely; i.e. at $\beta_u = 0$, $Prob(S_i = +1) = Prob(S_i = -1) = 1/2$.

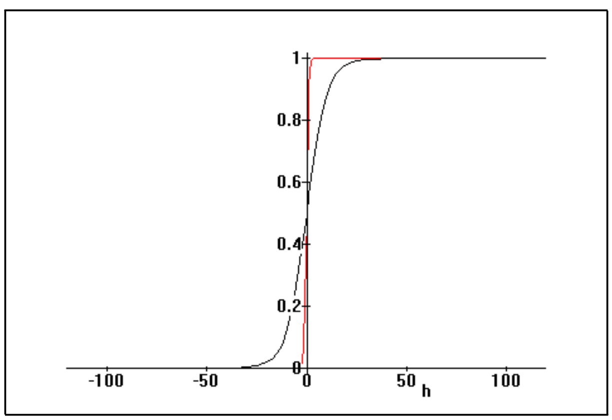

Fig.1

The logistic function is of course widely used in the econometrics literature, for a survey see e.g. Anderson,de Palma, and Thisse(1992).

3.The Micro-configuration of Strategy Choice.

Let $p_m(t)$ denote the probability mass associated with the microconfiguration K^m at time t. Let $p(t)$ denote the vector of length 2^N whose elements are $p_m(t)$. The evolution of $p(t)$ is then described by the Markov chain,

$$p(t) = Pp(t-1) \tag{15}$$

subject to the initial distribution of states, $p(0)$, and where the elements of the matrix P are defined by eqs.(5),(6),(7).

The equilibrium distribution over the micro-states follows from standard application of Markov Chain theory. We assume henceforth that agents are not distinguishable by $\beta_u, i.e.\ \beta_u = \beta$.

Theorem 1. Let the transition probabilities between states be defined by (5),(6), and (7) then there exists a unique stationary distribution, p^* , given by,

$$p^* = \lim_{t\to\infty} p(\beta, t) \tag{16}$$

where the single element $p^*_m$of the vector p^* is defined by,

$$p^*_m = (1/C)exp(\frac{1}{2}\beta\gamma(K^m)) \tag{17}$$

where C is the normalization constant defined by,

$$C = \sum_{m=1}^{m=2^N} exp(\frac{1}{2}\beta\gamma(K^m)) \tag{18}$$

and $\gamma(K^m)$ is defined by,

$$\gamma(K^m) = \sum_{i=1}^{i=N} h_i K^m(i) \tag{19}$$

and h_i defined by eq.(42).

Proof:

The existence of a unique stationary distribution p^* is guaranteed provided that the Markov chain with probabilities defined by (5),(6), and (7) is (i)finite, (ii) homogeneous, (iii) a periodic and (iv) irreducible. The proofs that (15) does indeed satisfy these properties are standard and therefore omitted. All that we require is to determine that (17) does indeed represent the limiting distribution given the transition probabilities (5),(6), and (7).

To do this we require to show that for any two states m, l of the Markov chain the balance equation,

$$p_m P_{ml} = p_l P_{lm} \tag{20}$$

holds, where p_m denotes the probability of state K^m, p_l the probability of state K^l, and P_{ml} the probability of a transference from state K^m to state K^l.

First we simply note that,

$$K^m(i)h_i = -\frac{1}{2}\sum_i h_i K_i^m(i) + \frac{1}{2}\sum_i h_i K^m(i) \tag{21}$$

since $K_i^m(i)$ differs from $K^m(i)$ by a flip of the i^{th} unit of the string.

Thus the transition probability,

$$P_{ml} = G(u, K^m)A(u, K^m) \tag{22}$$

$$= (1/N)(\frac{1}{1 + \exp(\beta K^m(i)h_i)}) \tag{23}$$

may be written as,

$$= (1/N)(\frac{1}{1 + \exp(-\frac{1}{2}\beta(\sum_i h_i K_i^m(i) + \sum_i h_i K^m(i)))} \tag{24}$$

Starting from the L.H.S. of (20) we have,

$$p_m P_{ml} = p_m G(u, K^m)A(u, K^m) \tag{25}$$

$$= (1/C)exp(\frac{1}{2}\beta\sum_i h_i K^m(i))(1/N)(\frac{1}{1 + \exp(\beta K^m(i)h_i)}) \tag{26}$$

$$= (1/C)exp(\frac{1}{2}\beta\sum_i h_i K^m(i))(1/N)(\frac{1}{1 + \exp(\frac{1}{2}\beta(-\sum_i h_i K_i^m(i) + \sum_i h_i K^m(i)))}) \tag{27}$$

$$= (1/C)exp(\frac{1}{2}\beta\sum_i h_i K_i^m(i))(1/N)(\frac{exp(\frac{1}{2}\beta(\sum_i h_i K^m(i)) - \sum_i h_i K_i^m(i)))}{1 + \exp(\frac{1}{2}\beta(-\sum_i h_i K_i^m(i) + \sum_i h_i K^m(i)))} \tag{28}$$

$$= (1/C)exp(\frac{1}{2}\beta\sum_i h_i K_i^m(i))(1/N)(\frac{1}{1 + \exp(\frac{1}{2}\beta(\sum_i h_i K_i^m(i) - \sum_i h_i K^m(i)))}) \tag{29}$$

$$= (1/C)exp(\frac{1}{2}\beta\sum_i h_i K_i^m(i))(1/N)(\frac{1}{1 + \exp(-\beta\sum_i h_i K^m(i))}) \tag{30}$$

But since,

$$-K^m(i) = K_i^m(i) \tag{31}$$

$$= (1/C)exp(\frac{1}{2}\beta\sum_i h_i K_i^m(i))(1/N)(\frac{1}{1 + \exp(\beta\sum_i h_i K_i^m(i))} \tag{32}$$

and from our definition of K^l, since

$$p_l = (1/C)exp(\frac{1}{2}\beta\sum_i h_i K_i^l(i)) \tag{33}$$

and,

$$P_{lm} = G(u, K^l)A(u, K^l) = (1/N)(\frac{1}{1 + \exp(\beta \sum_i h_i K_i^l(i))} \tag{34}$$

we thus have shown,

$$p_m P_{ml} = p_l P_{lm} \tag{35}$$

We note that in the development of the micro-configuration quite a wide generality of different model types may be allowed, in terms of the specification of the payoffs which may differ across agents or groups of agents.

4. The Macro-Configuration of Strategy Choice

Now in many instances our principal interest in the distribution p^* is to determine the numbers of agents playing the different strategies, irrespective of the named agents playing the strategy (i.e. independent of their identification number i in the string K^m). The distribution of the number of agents by strategy we term the macro-configuration.

For binary choice games each agent is assumed to face the payoff matrix.

$$\begin{array}{cccc} & & \multicolumn{2}{c}{Player\ j} \\ & & s_1 & s_2 \\ \begin{array}{c}Player\\ i\end{array} & \begin{array}{c}s_1\\ s_2\end{array} & \begin{array}{c}a(i,j)\\ c(i,j)\end{array} & \begin{array}{c}b(i,j)\\ d(i,j)\end{array} \end{array} \tag{36}$$

where $a, b, c, d \in R$.

The simplest case of information structure is where the agent knows the existing microconfiguration of strategy choice. For random pairing one statistic the agent may wish to construct is therefore the expected payoffs resulting from choice of a particular strategy. In the present analysis we restrict ourselves to the case where for each agent i the payoffs are independent of the named opponent j, and consider only symmetric games.

Letting s and r be respectively the numbers playing strategies S_1 and S_2 , including the agent making the choice. Then $s + r = N$.

Definition 7.
The normalized payoffs for agent i are defined as,

$$Payoff_i(S_1, K^m) = a(i)(s/N) + b(i)(r/N) \tag{37}$$

$$Payoff_i(S_2, K^m) = c(i)(s/N) + d(i)(r/N) \tag{38}$$

i.e. we assume that the agent has the same probability of playing against themselves as any one other opponent. This assumption leads to some notational simplicity.

The payoff difference is given by,

$$h_i(K^m) = Payoff_i(S_1, K^m) - Payoff_i(S_2, K^m) \tag{39}$$

Lemma 1.

The payoff difference h_i can be written in terms of the summation of the sequence $K^m(j)$. Letting,

$$G = (a - c) - (b - d) \tag{40}$$

$$H = (a - c) + (b - d) \tag{41}$$

then,

$$h_i = (1/2(N))(G \sum_j K^m(j) + H(N)) \tag{42}$$

Proof:

If

$$h_i = (1/2(N))(G \sum_j K^m(j) + H(N)) \tag{43}$$

then since,

$$\sum_j K(j) = s - r \tag{44}$$

$$h_i = (1/2(N))(G(s - r) + H(s + r)) \tag{45}$$

$$= (1/2(N))((G + H)(s) - (G - H)r) \tag{46}$$

$$= (1/2(N))((2(a - c)(s) + (2(b - d))r) \tag{47}$$

$$= Payoff_i(S_1) - Payoff_i(S_2) \tag{48}$$

Now consider the micro-configuration where we have,

$$p^*_m = (1/C)\exp(\frac{1}{2}\beta\gamma(K^m)) \tag{49}$$

$$= (1/C)\exp(\frac{1}{2}\beta \sum_i h_i K^m(i)) \tag{50}$$

subst. for the payoff difference we have,

$$= (1/C)\exp(\frac{1}{2}\beta\sum_i(Payoff_i(S_1,K^m) - Payoff_i(S_2,K^m))K^m(i)) \tag{51}$$

$$= (1/C)\exp(\frac{1}{2}\beta[\sum_i(1/2(N))(G\sum_j K^m(j) + HN))K^m(i))] \tag{52}$$

Considering the exponent of (52), for any population with s and r , the exponent may be written as,

$$\frac{1}{4N}\beta[G(M^2)] + \frac{1}{2}\beta HM \tag{53}$$

where,

$$M = \sum_{i=1}^{N} K(i) = s - r \tag{54}$$

Thus the probability of any configuration depends only on the numbers in total playing strategy S_1 or S_2. Since the number of the agents playing S_2 is r then the number playing S_1 is $N - r$. Thus,

$$M = \sum_{i=1}^{N} K(i) = (N - r) - r = N - 2r \tag{55}$$

and there are $\binom{N}{r}$ such arrangements of agents playing S_1 or S_2.Thus the probability of finding r agents playing S_2, is given by,

$$p(r) = \frac{N!}{r!(N-r)!}\exp(\frac{1}{4}\beta[\frac{G(M^2)}{N}] + \frac{1}{2}\beta HM] \tag{56}$$

The non-normalized frequency of the number of agents playing S_2 is then,

$$p(r) = \frac{N!}{r!(N-r)!}\exp(\frac{1}{4}\beta[\frac{G(N-2r)^2}{N}] + \frac{1}{2}\beta H(N-2r)] \tag{57}$$

which may be written in terms of the expected payoffs as

$$p(r) = \frac{N!}{r!(N-r)!}\exp(\frac{1}{2}\beta\{(Payoff_i(S_1) - Payoff_i(S_2))(N-2r)\}) \tag{58}$$

To derive the normalized distribution we divide (58) through by the normalization constant,

$$C = \sum_{r=0}^{N} p(r) = \sum_{r=0}^{N}\{\frac{N!}{r!(N-r)!}\exp(\frac{1}{2}\beta\{(Payoff_i(S_1) - Payoff_i(S_2))(N-2r)\})\} \tag{59}$$

5. The Equilibrium Selection Procedure

Equilibrium selection is predicated on the elimination of uncertainty as regards the agents choice of strategy, i.e. as agents perceive the best response to the existing configuration of strategy choice.The selection procedure thus consists of noting the limiting form of the normalized distribution of (58) as the β parameter is taken to infinity, i.e.,

$$\lim_{\beta\to\infty} p(r) = \lim_{\beta\to\infty} \frac{1}{C}\frac{N!}{r!(N-r)!}\exp(\frac{1}{2}\beta\{(Payoff_i(S_1) - Payoff_i(S_2))(N-2r)\}) \tag{60}$$

Since (60) is a discrete distribution defined over the total number of agents, N, the solution for the exact properties of the limiting distribution will rely on computational methods considered in Section 7. However, analytical approximations to the stationary values of the normalized distribution p(r) may be considered; and the resulting values compared with the exact numerical computations in Section 7.

Let us assume that the number in the population is quite large, so that,

$$r^* = r/N \tag{61}$$

may be viewed as a continuous variable. A typical plot of $p(r)$ will then appear as Fig.2; and our interest lies in determining the stationary values of $p(r)$.

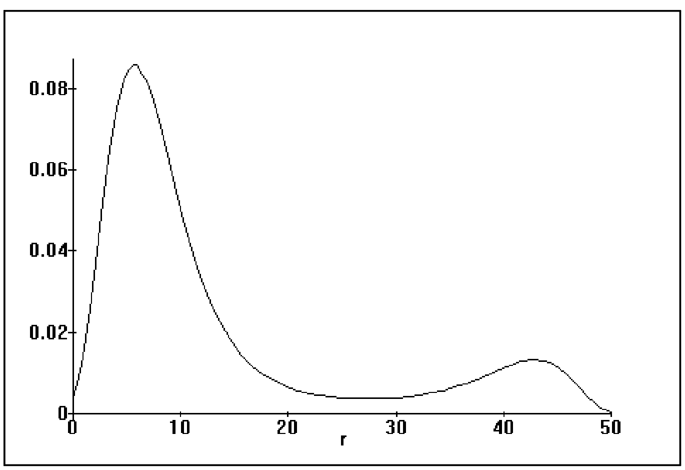

Fig.2

The assumptions under which Fig.2 was constructed may be found in Section 7 below.

An analytical approximation to the stationary values of the distribution $p(r)$ may be determined as follows.

From eq.(57) consider the ratio,

$$q(r) = \frac{p(r+1)}{p(r)} = \frac{N-r}{r+1}\exp\{-\beta G(N-2r-1)/(N) - \beta H\} \qquad (62)$$

At the maxima and minima then,

$$p(r) \sim p(r+1) \qquad (63)$$

and for large N we approximate equality, and so,

$$\exp\{-\beta G(N-2r-1)/(N) - \beta H\} = \frac{r+1}{N-r} \qquad (64)$$

let,

$$r^* = \frac{r}{N} \text{ and } q^* = \frac{N-r}{N} \qquad (65)$$

denote respectively the proportions playing strategy S_2 and strategy S_1, then (64) for large N may be written,

$$\exp\{-\beta(G(1-2r^*) + H)\} = \frac{r^*}{1-r^*} \qquad (66)$$

i.e.,

$$r^* = \frac{1}{1+\exp\{\beta(G(1-2r^*)+H)\}} \qquad (67)$$

i.e. the stationary points of the equilibrium distribution are equal to the fixed points of the logit transformation.

Note that in the present theory the fixed points of the logit transformation are not of themselves an equilibrium selection mechanism The fixed points determine the set of candidates from which the selection is made, but the primary selection mechanism remains the distribution function (60).

6. Nash Equilibria

The question that now may be asked is whether the equilibrium selection procedure always takes the population to any of the Nash equilibria of the associated game. First consider the fixed points of the logit function. Since these fixed points are the equilibrium candidates proposed by McKelvey and Palfrey(1995), then it has already been established, assuming that r^* can be treated as a continuous variable, that for finite β then the fixed points of the logit do not coincide with any of the Nash equilibria. The question therefore arises as to whether the quantal response equilibria pick out the Nash equilibria as $\beta \to \infty$.

To this end we may make use of Theorem 2 established by McKelvey and Palfrey(1995) that the limit points of the logit equilibria approach the Nash equilibria of the underlying game as $\beta \to \infty$. Given that this is the case then we have established,

(1) The values of r that determine the stationary points of the probability distribution $p(r)$ are equal to the fixed points of the logit transformation (the "logit equilibria").

(2) By McKelvey and Palfrey(1995) Theorem 2 the fixed points of the logit transform converge on the Nash equilibria as $\beta \to \infty$.

(3) Thus the stationary points of the probability distribution $p(r)$ converge on the Nash equilibria as $\beta \to \infty$.

The question of which of the Nash equilibria are selected however remains, and here there is a substantial difference between the evolutionary methodology proposed in the present paper and that proposed by McKelvey and Palfrey. McKelvey and Palfrey(1995) propose to "define a unique selection from the set of Nash equilibrium by "tracing" the graph of the logit equilibrium correspondence beginning at the centroid of the strategy simplex (the unique solution when $\beta = 0$) and continuing for larger and larger values of β." In the evolutionary selection process it is that value of r to which the normalized frequency distribution $p(r)$ converges as $\beta \to \infty$

7. Applications

We are now in a position to apply the EQR methodology to the problem of equilibrium selection in a number of games. We begin with the Stag-Hunt game.

(1).The Stag-Hunt Game

The payoff matrix for the Stag-Hunt game is given by,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & \alpha, \alpha & \alpha, 0 \\ s_2 & 0, \alpha & 1, 1 \end{array} \tag{68}$$

where $0 < \alpha < 1$. Thus s_1 is the 'safe strategy' ; irrespective of what the other player does, α is guaranteed. If s_2 is played it might yield a superior payoff of 1 , but only if the other player also plays s_2; it only yields 0 if the other player plays s_1. Two strict Nash equilibria exist $[s_1, s_1]$ and $[s_2, s_2]$.

Which equilibrium is chosen as $\beta \to \infty$. In order to pursue the question, we consider the distribution $p(r)$ generated for the case where $N = 50$, and the payoff $\alpha = 0.6$.

Thus,

$$
\begin{array}{ccc}
 & s_1 & s_2 \\
s_1 & \alpha,\alpha & \alpha,0 \\
s_2 & 0,\alpha & 1,1
\end{array}
=
\begin{array}{ccc}
 & s_1 & s_2 \\
s_1 & 0.6,0.6 & 0.6,0 \\
s_2 & 0,0.6 & 1,1
\end{array}
\tag{69}
$$

and so,

$$G = (a - c) - (b - d) = (0.6) - (0.6 - 1) = 1 \tag{70}$$

$$H = (a - c) + (b - d) = (0.6) + (0.6 - 1) = 0.2 \tag{71}$$

Thus,

$$p(r) = \frac{N!}{r!(N-r)!} \exp(\frac{1}{4}\beta[\frac{G((N-2r)^2 - N)}{(N)}] + \frac{1}{2}\beta H(N-2r)] \tag{72}$$

$$= \frac{40!}{r!(40-r)!} \exp(\frac{1}{4}\beta[\frac{((50-2r)^2 - 50)}{(50)}] + \frac{1}{2}\beta 0.2(50-2r)] \tag{73}$$

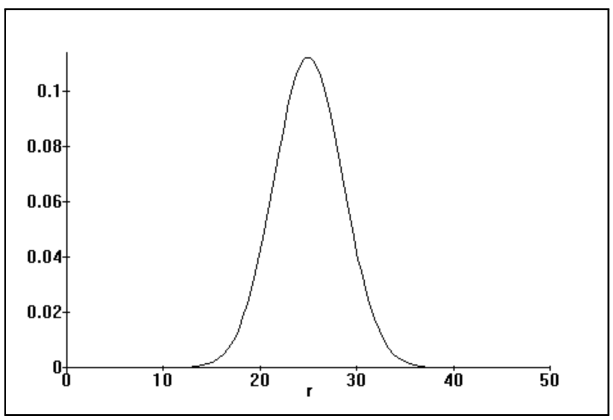

Fig.3

Fig.3. illustrates normalized $p(r)$ for the case where $\beta = 0.001$.

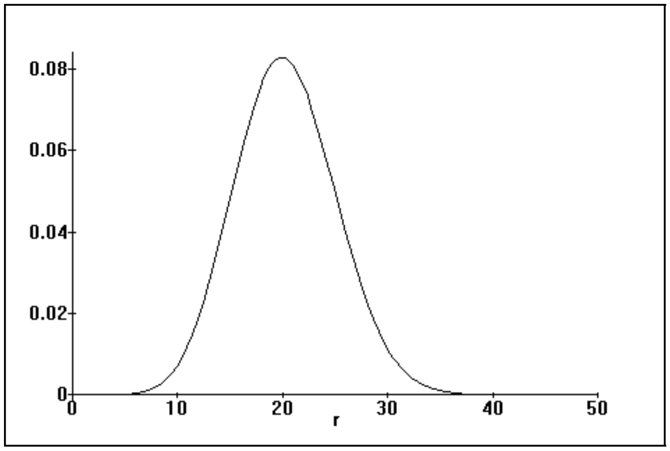

Fig.4

Fig4 illustrates $p(r)$ for the case where $\beta = 1$.

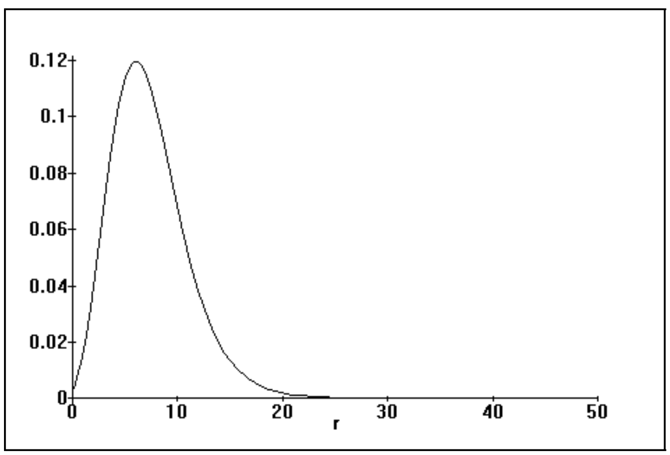

Fig.5

Fig.5 illustrates $p(r)$ for the case where $\beta = 2$.

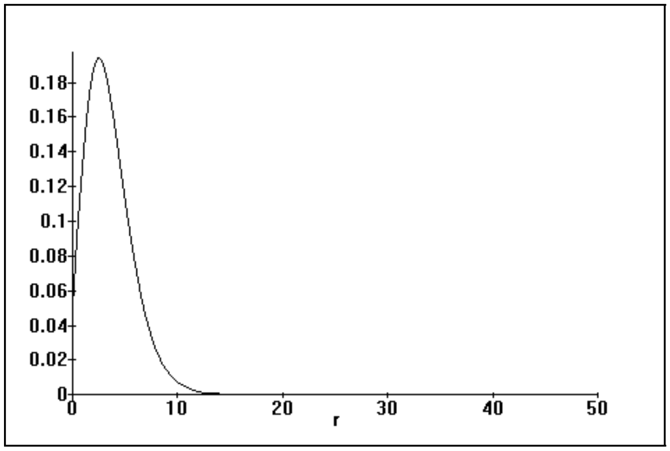

Fig.6

Fig.6 illustrates $p(r)$ for the case where $\beta = 2.5$.

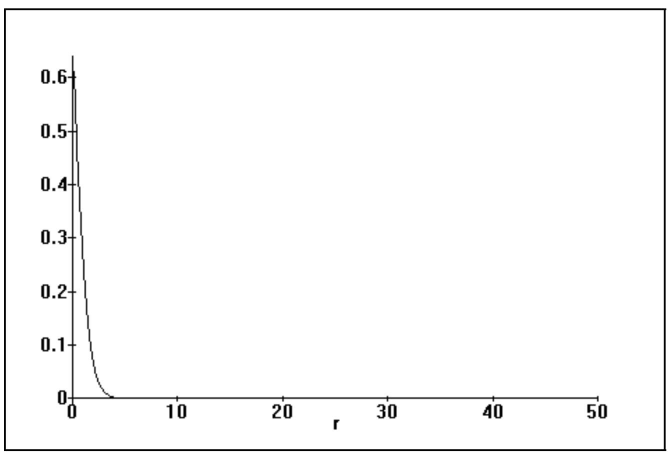

Fig.7

Fig.7 illustrates $p(r)$ for the case where $\beta = 4$. As can be seen although there are two Nash equilibria the selection mechanism results in convergence on the equilibrium in which all agents play strategy 1, the risk -neutral Pareto dominated equilibria.

For the case where $\alpha = 0.5$, we have,

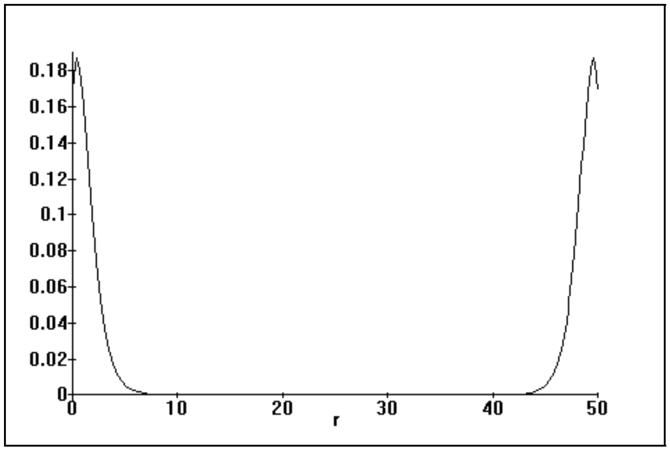

Fig.8

Fig.8 illustrates $p(r)$ for the case where $\beta = 4$.The case illustrated in Fig.2 above was for the Stag-Hunt game with $\alpha = 0.51$ and $\beta = 2.5$, $N = 50$.

(2). Pure Co-ordination

As an example of the pure co-ordination case we consider the payoff matrix,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 1,1 & 0,0 \\ s_2 & 0,0 & 2,2 \end{array} \tag{74}$$

and so,

$$G = (a - c) - (b - d) = 3 \tag{75}$$

$$H = (a - c) + (b - d) = -1 \tag{76}$$

Thus,

$$p(r) = \frac{N!}{r!(N-r)!} \exp(\frac{1}{4}\beta[\frac{G((N-2r)^2 - N)}{(N)}] + \frac{1}{2}\beta H(N-2r)] \tag{77}$$

$$= \frac{50!}{r!(50-r)!} \exp(\frac{1}{4}\beta[\frac{3((50-2r)^2 - 50)}{(50)}] + \frac{1}{2}\beta(-1)(50-2r)] \tag{78}$$

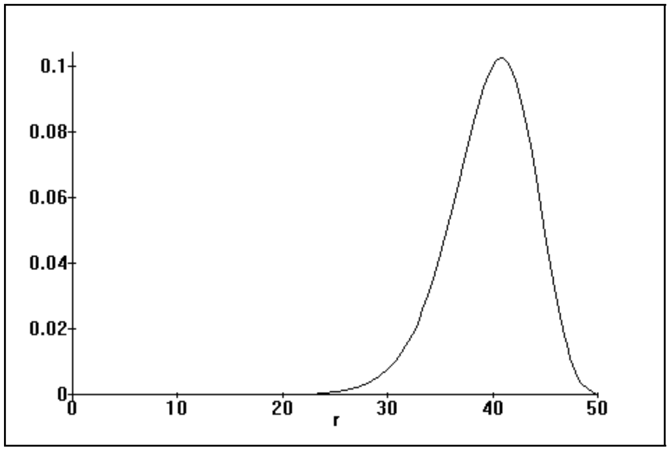

Fig.9

Fig.9 illustrates the case for $\beta = 0.5$.

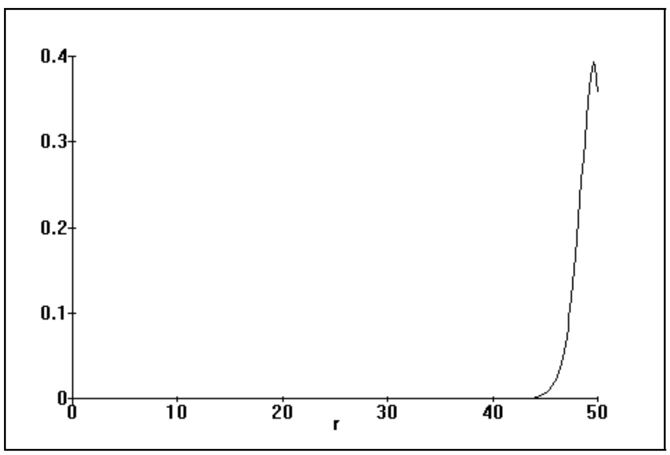

Fig.10

Fig.10 shows $p(r)$ for the case $\beta = 1$, and hence convergence on S_2 as the equilibrium selected.

(3). Hawk-Dove.

As an example of the hawk-dove game we consider the payoff matrix,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 0.5, 0.5 & 0, 1 \\ s_2 & 1, 0 & 0.5(1-c), 0.5(1-c) \end{array} \tag{79}$$

and so,

$$G = (a - c) - (b - d) = -0.5c \tag{80}$$

$$H = (a - c) + (b - d) = -1 + 0.5c \tag{81}$$

If $c > 1$ the game has a unique symmetric mixed strategy Nash equilibrium, in which case each player uses the strategy $(1 - (1/c), 1/c)$. If $c < 1$ then there is a unique mixed strategy Nash equilibrium in which each agent plays the pure strategy s_2. First consider the case $c = 3$.Then $G = -1.5, H = 0.5$, then, for $\beta = 10$

$$p(r) = \frac{N!}{r!(N-r)!}\exp(\frac{1}{4}\beta[\frac{G((N-2r)^2 - N)}{(N)}] + \frac{1}{2}\beta H(N - 2r)] \tag{82}$$

$$= \frac{50!}{r!(50-r)!}\exp(\frac{1}{4}\beta[\frac{3((50-2r)^2 - 50)}{(50)}] + \frac{1}{2}\beta(-1)(50 - 2r)] \tag{83}$$

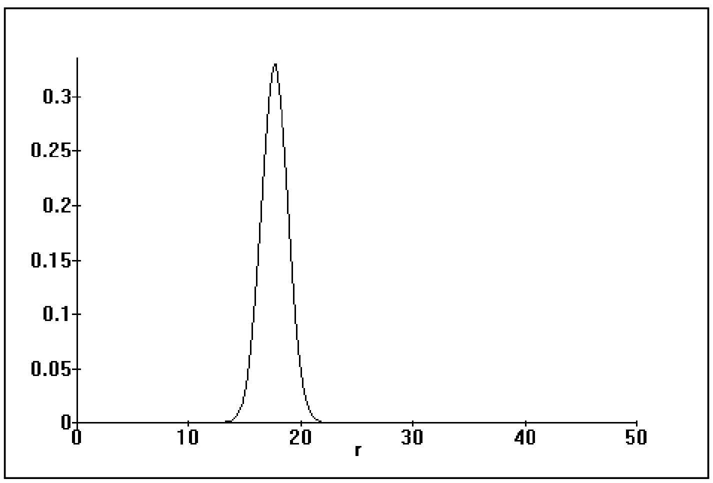

Fig.11

and we have the case illustrated in Fig 11. For the case of $c = 0.5$ we have the solution in Fig.12.

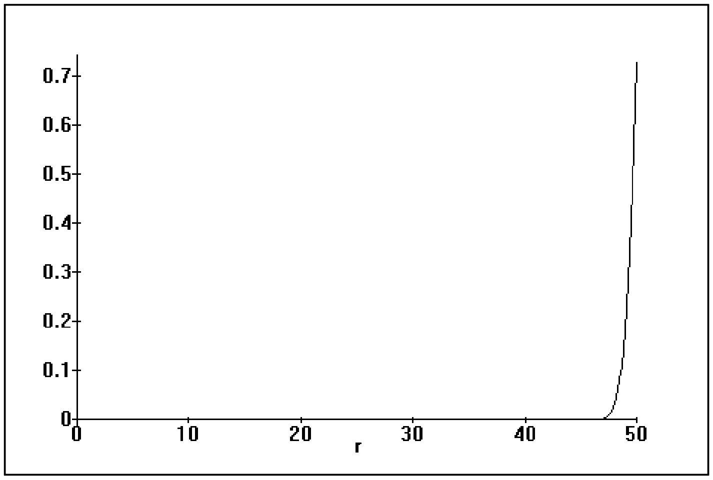

Fig.12

(4). Battle of the Sexes.

As an example of the battle of the sexes game we consider the payoff matrix,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 0,0 & 1,2 \\ s_2 & 2,1 & 0,0 \end{array} \tag{84}$$

and so,

$$G = (a - c) - (b - d) = -3 \tag{85}$$

$$H = (a - c) + (b - d) = -1 \tag{86}$$

The only nondegenerate mixed strategy Nash equilibrium of the game has $S_2 = 2/3$.

$$p(r) = \frac{N!}{r!(N-r)!} \exp(\frac{1}{4}\beta[\frac{G((N-2r)^2 - N)}{(N)}] + \frac{1}{2}\beta H(N-2r)] \tag{87}$$

$$= \frac{50!}{r!(50-r)!} \exp(\frac{1}{4}\beta[\frac{-3((50-2r)^2 - 50)}{(50)}] + \frac{1}{2}\beta(-1)(50-2r)] \tag{88}$$

Consider the case for $\beta = 10$, then Fig.13 illustrates convergence to the Nash equilibrium.

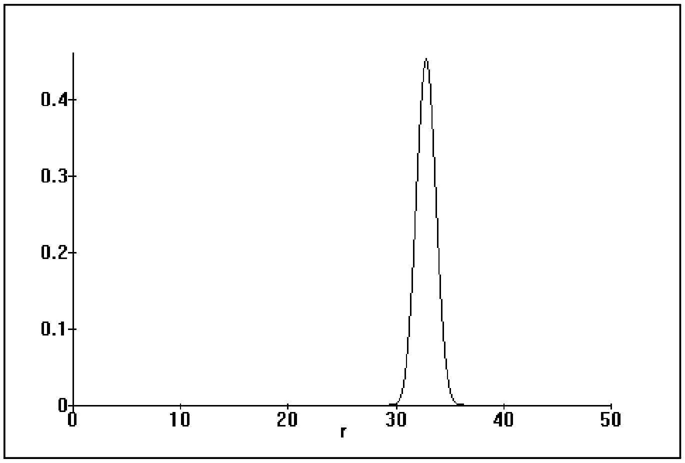

Fig.13

(5). The Prisoner's Dilemma

As an example of the prisoner's dilemma game we consider the payoff matrix,

$$\begin{array}{ccc} & s_1 & s_2 \\ s_1 & 3,3 & 0,5 \\ s_2 & 5,0 & 1,1 \end{array} \tag{89}$$

and so,

$$G = (a - c) - (b - d) = -1 \tag{90}$$

$$H = (a - c) + (b - d) = -3 \tag{91}$$

$$p(r) = \frac{N!}{r!(N-r)!} \exp(\frac{1}{4}\beta[\frac{G((N-2r)^2 - N)}{(N)}] + \frac{1}{2}\beta H(N-2r)] \tag{92}$$

$$= \frac{50!}{r!(50-r)!} \exp(\frac{1}{4}\beta[\frac{-1((50-2r)^2 - 50)}{(50)}] + \frac{1}{2}\beta(-3)(50-2r)] \tag{93}$$

Consider the case for $\beta = 10$, then Fig.14 illustrates convergence to the Nash equilibrium.

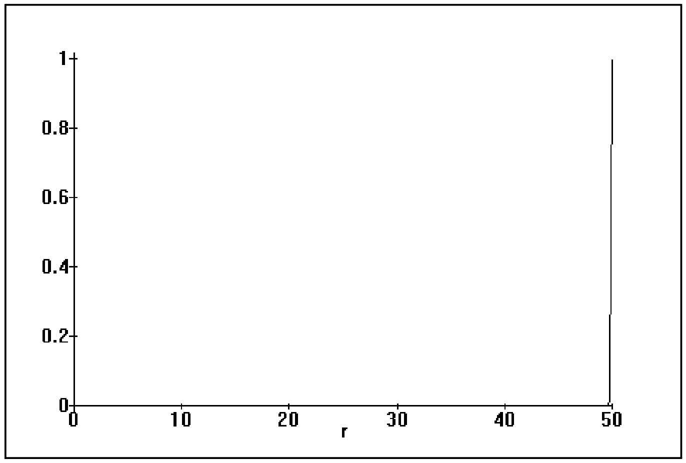

Fig.14

8.Conclusion

The proposed equilibrium selection mechanism established by the above methodology may be compared with results from the existing literature in this field. Firstly, consider the comparison with the "statistical mechanical" approach to strategic interaction studied by Blume(1993), (1996), Herz(1993), Brock and Durlauf(1995),Durlauf(1996). A major aspect of this literature was in emphasising the role of local interactive behaviour, with the consequence that the outcome of this procedure was bound to focus on the microdistributions of strategy choice, i.e.the distributions of type (17) considered above. By changing the assumption to global pairwise matching we are able as a consequence to establish the macro distributions over strategy choice, and thence establish the relationship to the more traditional econometric approach to strategy choice via the quantal response model. However the essential component of the "statistical mechanical approach" is the use of a dynamic process, in our present case the Markov chain, which allows the distribution function over choices to be determined. We regard the lack of such a formal dynamical process within the econometric quantal response approach, as being the major distinguishing feature between an evolutive and non-evolutive equilibrium selection process.

Thus when we turn to the quantal response literature, we can see that the major difference with the present approach does lie with the construction of the evolutionary process by which the change in the probability distribution over agent choices is determined, absent from the McKelvey and Palfrey(1995),(1996) and associated papers. In QRE models, the requirement for equilibrium is that the probability of each and every agents choice is equal to the current proportion in the population making that choice. The method by which the equilibrium distribution is known to agents is left unstated as is common in all deductive models of this type. An extension of the QRE to incorporate a dynamic adjustment mechanism was proposed by Chen, Friedman and Thisse(1996) in which they identify knowledge of the mixed strategies of other players with the average of observed behaviour of the players from the initial period to the present, as they note "the empirical distributions generate the beliefs that players have about one another". This mechanism provides a rationale as to why the equilibrium distribution should be known to all agents, however the proposed equilibrium selection mechanism is identical to that proposed by McKelvey and Palfrey(1995).

The clear distinction between the QRE model of McKelvey and Palfrey, and the extension of Chen, Friedman and Thisse, compared to the evolutive quantal response model can thus be clearly seen. Both QRE and EQR generate the same set of candidates from which the equilibrium selection is made, i.e.,. the fixed points of the quantal response contraction, However differ formally in terms of the selection process with regard to the equilibrium that is chosen. Selection in both is determined by the low noise limit, i.e. taking the noise parameter to zero, (i.e. equivalent to taking the β parameter to infinity).Under EQR McKelvey and Palfrey "define a unique selection from the set of Nash equilibrium by "tracing" the graph of the logit equilibrium correspondence beginning at the centroid of the strategy simplex (the unique solution when $\beta = 0$) and continuing for larger and larger values of β." In contrast the EQR equilibrium selected results from knowledge of the probability distribution over agents choices, and the limiting form of this distribution as the low noise limit is taken.

Acknowledgements. I am indebted to Dr. Joseph Swierzbinski for comments on an earlier draft of this paper. The usual disclaimer applies.

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Evolutive Equilibrium Selection II: Quantal Response Mechanisms

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