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The Isodiametric Inequality in Locally Compact Groups

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Declaration

I, Oleksandr Borisovich Metelichenko, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

In this thesis it is shown that the isodiametric inequality fails for Carnot-Carathéodory balls in the Heisenberg group $\mathbb{H}^n$ ($n \in \mathbb{N}$). Estimates for the ratio of the volume of a ball to the maximal volume of a set of the same diameter are established in this group, and the set of the maximal volume is also found among all sets of revolution about the vertical axis having the same diameter. Results of the similar nature are obtained in the additive group $\mathbb{R}^{n+1}$ ($n \in \mathbb{N}$) with non-isotropic dilations.

Using a connection between the isodiametric problem and the Besicovitch 1/2-problem it is proved that the generalized Besicovitch 1/2-conjecture fails in the Heisenberg group $\mathbb{H}^n$ ($1 \leq n \leq 8$) of the Hausdorff dimension $2n+2$ and the additive group $\mathbb{R}^{n+1}$ ($n \in \mathbb{N}$) having non-isotropic dilations and integer Hausdorff dimension greater than or equal to $n+2$. But the 1-dimensional case is shown to be exceptional – the generalized Besicovitch 1/2-conjecture is true in any locally compact group which is equipped with an invariant metric, its Haar measure and has the Hausdorff dimension 1.

A question about the relation among the Hausdorff, the spherical and the centred Hausdorff measures of codimension one restricted to a smooth surface is also investigated in the Heisenberg group $\mathbb{H}^1$. It is proved that these measures differ but coincide up to positive constant multiples, estimates for which are found.

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Notation

We list here the notation which is used throughout the thesis. Some of these concepts are explained in more details in the text.

$\mathbb{N}$	set of natural numbers: $1, 2, \ldots$
$\mathbb{Z}$	set of integer numbers
$\mathbb{R}$	set of real numbers
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, \infty\}$, extended set of real numbers
$[a, b]$, (a, b)	closed and open intervals in $\overline{\mathbb{R}}$
$[a, b)$, $(a, b]$	half-open intervals in $\overline{\mathbb{R}}$
$\mathbb{C}$	set of complex numbers
$\bar{z}$, $\operatorname{Im} z$, $\arg z$	complex conjugate, imaginary part and argument of $z \in \mathbb{C}$
$\mathbb{R}^n$	n-dimensional Euclidean space with the inner product $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ and the norm $\lvert x\rvert = \sqrt{\langle x, x\rangle}$, $x = (x_j)_{j=1}^n \in \mathbb{R}^n$ is a typical point
$\mathbb{C}^n$	n-dimensional complex vector space with the inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$, $z = (z_j)_{j=1}^n \in \mathbb{C}^n$ is a typical point
$\mathbb{H}^n$	Heisenberg group of order n
$\lvert \cdot \rvert$	Euclidean norm or absolute value on $\mathbb{C}$
$\lVert \cdot \rVert$	norm on a vector space

$\subset$, $\supset$	set inclusions, may mean equality as well
$H \lhd G$, $G \rhd H$	H is a normal subgroup of a group G, may mean equality
sign	sign function
det	determinant of a matrix
$\tau(x, y)$	some trigonometric function τ of the angle between vectors x and y in $\mathbb{R}^3$
$B(x, r)$	closed ball (with respect to a specific metric) of radius $r > 0$ centred at x
$U(x, r)$	open ball (with respect to a specific metric) of radius $r > 0$ centred at x
B_r	$B(0, r)$ in $\mathbb{R}^n$ (metric can be non-Euclidean)
U_r	$U(0, r)$ in $\mathbb{R}^n$ (metric can be non-Euclidean)
$\operatorname{diam} A$	diameter of a set A with respect to a specific metric
$\operatorname{dist}(x, A)$	distance between a point x and a set A with respect to a specific metric
$\operatorname{dist}(A, B)$	distance between two sets A and B with respect to a specific metric
proj_Π	orthogonal projection from $\mathbb{R}^n$ onto a hyperplane Π
$\overline{A}$	closure of A
∂A	boundary of A
χ_A	characteristic function of A
$\mu \llcorner A$	restriction of a measure μ to A
$A + B$	$\{x + y \mid x \in A, y \in B\}$
$A - B$	$\{x - y \mid x \in A, y \in B\}$

λA	$\{\lambda x \mid x \in A\},\ \lambda \in \mathbb{R}$
$-A$	$(-1)A = \{-x \mid x \in A\}$
C^1 surface	surface which admits a parametrization by continuously differentiable functions
$C(\Sigma)$	set of all characteristic points of a surface $\Sigma \subset \mathbb{H}^1$
$\mathcal{H}^s$	s-dimensional Hausdorff measure
$\mathcal{S}^s$	s-dimensional spherical measure
$\mathcal{C}^s$	s-dimensional centred Hausdorff measure
$\mathcal{L}^n$	n-dimensional Lebesgue outer measure
$\alpha(n)$	$\mathcal{L}^n\{x \in \mathbb{R}^n \mid \lvert x\rvert \leq 1\}$, volume of the unit ball in $\mathbb{R}^n$
$\overline{D}_s(A, x)$	upper s-density of A in x
$\underline{D}_s(A, x)$	lower s-density of A in x
$D_s(A, x)$	s-density of A in x
G/H	quotient (factor) group of a group G over its normal subgroup H
$\langle g_1, g_2, \ldots, g_n\rangle$	subgroup of a group G generated by $g_1, g_2, \ldots, g_n \in G$
$L^1[a, b]$	set of $\mathcal{L}^1$-measurable on $[a, b] \subset \mathbb{R}$ functions having finite integral over $[a, b]$

Chapter 1

Introduction

1.1 Overview of the Thesis

The main object of study in this thesis is the isodiametric inequality and its applications in metric locally compact groups. Let us briefly outline the structure of the thesis and the most interesting results we obtained. The precise definitions of notions involved here will be given in the next section and subsequent chapters.

In Chapter 3 we investigate the isodiametric inequality in the Heisenberg group $\mathbb{H}^n$ ($n \in \mathbb{N}$). This inequality states that a ball maximizes the volume for the given diameter, which is well known in Euclidean spaces. From this point of view non-Euclidean spaces are of great research interest, in particular the Heisenberg group, an important object of study in various areas of mathematics and physics, with the most interesting type of a metric on it – the Carnot-Carathéodory (geodesic) metric. Such a metric space seems to be a likely candidate to have the balls maximizing the volume for the given diameter. In fact, we show that it is not true, Carnot-Carathéodory balls don't possess this property in the Heisenberg group. The distance between the poles of a ball is strictly less than its diameter and one of the poles is the most distant point of a ball from another one. Therefore addition of a "small" set to a pole preserves the diameter of a ball but increases its volume, which is a violation of the isodiametric property.

We also give a lower bound for the ratio of the volume of a ball to the maximal volume of a set of the same diameter. This bound appears to be more than 1/2, which gives a counterexample to the generalized Besicovitch 1/2-conjecture in the Heisenberg group $\mathbb{H}^n$ $(1 \leq n \leq 8)$ of the Hausdorff dimension $2n + 2$, as we will see later on.

Next we prove that the convex hull of the Carnot-Carathéodory ball has the maximal volume among all sets of revolution about the vertical axis having the same diameter. However, an interesting question if this set has the maximal volume among all sets of the same diameter remains open.

We obtain similar results in the additive group $\mathbb{R}^{n+1}$ $(n \in \mathbb{N})$ with non-isotropic dilations. Moreover, we show that the ratio mentioned above may be arbitrary close to 1 in this group. In fact, it may converge to 1, but the rate of convergence cannot exceed $1/n^2$ as the dimension n grows.

It has recently been proved by Schechter [41] that the generalized Besicovitch 1/2-conjecture fails by constructing a purely 2-unrectifiable metric space on the real line $\mathbb{R}_\rho$ with a translation invariant metric ρ (but without dilations) that metrizes the Euclidean topology. The counterexample of Schechter extends easily to higher Hausdorff dimensions. A connection between the isodiametric problem and the Besicovitch 1/2-problem allows us to give simpler counterexamples in groups with dilations. In Chapter 3 we show that the generalized Besicovitch 1/2-conjecture (as well as the isodiametric property of balls) fails in the Heisenberg group $\mathbb{H}^n$ $(1 \leq n \leq 8)$ of the Hausdorff dimension $2n+2$ and the additive group $\mathbb{R}^{n+1}$ $(n \in \mathbb{N})$ having non-isotropic dilations and integer Hausdorff dimension greater than or equal to $n+2$.

However, in Chapter 2 we prove that the 1-dimensional case is exceptional – the generalized Besicovitch 1/2-conjecture holds in any locally compact group which is equipped with an invariant metric, its Haar measure and has the Hausdorff dimension 1.

Let us mention that the failure of the isodiametric inequality in Carnot groups (in particular the Heisenberg group) equipped with some types of homogeneous metrics and the Haar measure has recently been established by Rigot (see [36] and [37]). As a consequence it has been shown that the Hausdorff and the spherical measures of the homogeneous dimension of the group differ, but being Haar measures, they coincide up to a positive constant multiple. In recent years there have been several publications establishing connections among various measures on hypersurfaces in the Heisenberg group and more general spaces (see [15], [23], [24] and [31]). For example, Magnani [23] has found the connection between the usual Euclidean surface measure and the spherical measure (with respect to the homogeneous distance of the group) for smooth hypersurfaces. We apply results of Chapter 3 in Chapter 4 to study a similar question (see an open problem stated in the introduction of [13]). Here we investigate an interesting problem about the relation among the Hausdorff, the spherical and the centred Hausdorff measures of codimension one restricted to a smooth surface in the Heisenberg group $\mathbb{H}^1$. We prove that they are different but proportional and give estimates for proportionality constants.

As Chapters 2 and 3 deal with the Besicovitch 1/2-problem, let us explain the essence of this problem and its relation to some principal concepts of geometric measure theory in the rest of this section.

One of the fundamental results in geometric measure theory is that in Euclidean spaces $\mathbb{R}^k$ a $\mathcal{H}^n$-measurable set A of finite $\mathcal{H}^n$ measure is n-rectifiable if and only if the n-density of A exists and equals 1 in $\mathcal{H}^n$ almost all of its points ($0 \leq n \leq k$ are integers; see also [12] and [27] for different characterizations of rectifiability for subsets of Euclidean spaces).

The direct implication follows from fundamental publications of Besicovitch (see [5] and [6]). In 1928 Besicovitch [5] also proved the converse implication for $k = 2$ and $n = 1$, but the general case was accomplished only decades later: first Marstrand [25] proved this result for $k = 3$ and $n = 2$ in 1961, and then Mattila [26] generalized the proof of Marstrand to all $1 \leq n \leq k-1$ in 1975. In 1987 Preiss [34] proved a stronger result: n-rectifiability in $\mathbb{R}^k$ already follows from the existence of finite and non-zero n-density. But this cannot be true in all metric spaces, which is demonstrated by a simple example of the real line equipped with the metric $d(x, y) = |x-y|^{1/2}$.

The extension of the direct implication to arbitrary metric spaces was completely solved by Kirchheim in 1994:

Theorem 1.1 (Kirchheim [20]). *In a metric space n-rectifiability of a set A of finite $\mathcal{H}^n$ measure implies the existence of the n-density of A equal to 1 in $\mathcal{H}^n$ almost all of its points.*

However, the question if the converse implication can be extended from $\mathbb{R}^k$ to an arbitrary metric space X is still unsolved.

In relation to this matter one may ask even a deeper question: what is the smallest, "threshold density constant" $\sigma_n(X)$ such that once a subset of X of finite $\mathcal{H}^n$ measure has the lower n-density strictly greater than $\sigma_n(X)$ at $\mathcal{H}^n$ almost all of its points, it is necessary n-rectifiable (we define $\sigma_n(X)$ precisely in Definition 1.13)? In the next section we show that $\sigma_n(X) \leq 1$ (see Corollary 1.15). Let us note that the result of Mattila [26] mentioned above implies that $\sigma_n(\mathbb{R}^k) < 1$.

The following question is known as the *generalized Besicovitch* $1/2$*-problem*: is it true that $\sigma_n(X) \leq 1/2$ for an arbitrary metric space X?

The first results about these numbers are due to Besicovitch who proved the upper estimate $\sigma_1(\mathbb{R}^2) \leq 1 - 10^{-2576}$, stated[1] the lower estimate $\sigma_1(\mathbb{R}^2) \geq 1/2$ and also conjectured that $\sigma_1(\mathbb{R}^2) = 1/2$ in his famous paper [5] in 1928. Later on, in 1938, Besicovitch [6] improved the upper estimate by showing that $\sigma_1(\mathbb{R}^2) \leq 3/4$.[2]

The conjecture we have just mentioned is well known as the *Besicovitch* $1/2$*-conjecture.* It remains open since that time and it is one of the most famous and oldest open questions in classical geometric measure theory. Undoubtedly it is an extremely interesting problem and attracts a lot of attention also due to the fact that any other significant problem concerning 1-densities in Euclidean spaces has been solved many years ago.

There have been a number of publications in attempt to estimate numbers $\sigma_n(X)$ in Euclidean and non-Euclidean spaces, but the complete solution of

[1] It was proved by Dickinson [9] in 1939.

[2] Alternative approach to the proof of this estimate can be found in [11].

the problem is yet to be found. We should note that Preiss and Tišer [35] (see also Schechter [39]) managed to give an upper bound for $\sigma_1(X)$ less than 3/4 in arbitrary metric spaces. This result improves and extends estimates of Besicovitch [6] and Moore [32] in Euclidean spaces. Figure 1.1 summarizes the up-to-date progress in the generalized Besicovitch 1/2-problem since 1928. These are all positive statements except for the last one, which indicates a counterexample of Schechter [41] we have mentioned earlier.

Year	Author	Result
1928	A. S. Besicovitch [5]	$\sigma_1(\mathbb{R}^2) \leq 1 - 10^{-2576}$
1938	A. S. Besicovitch [6]	$\sigma_1(\mathbb{R}^2) \leq 3/4$
1939	D. R. Dickinson [9]	$\sigma_1(\mathbb{R}^2) \geq 1/2$
1950	E. F. Moore [32]	$\sigma_1(\mathbb{R}^k) \leq 3/4$
1961	J. M. Marstrand [25]	$\sigma_2(\mathbb{R}^3) < 1$
1975	P. Mattila [26]	$\sigma_n(\mathbb{R}^k) < 1$
1984	M. Chlebík [7]	$\sup_k \sigma_n(\mathbb{R}^k) < 1$
1992	D. Preiss and J. Tišer [35]	$\sigma_1(X) \leq (2 + \sqrt{46})/12 \approx 0.7319$
1998	A. Schechter [39]	$\sigma_1(X) < 0.7266$
2002	A. Schechter [41]	$1/2 < \sigma_2(\mathbb{R}_\rho) < 1$

Figure 1.1: The up-to-date progress in the generalized Besicovitch 1/2-problem

1.2 Basic Definitions and Standard Theorems

Let us introduce some notions extensively used in the thesis. Let X be a metric space, $A \subset X$ and $\mathcal{F}$ be a family of closed subsets of X. Let ϕ be a measure on X such that every open subset of X is ϕ-measurable and every bounded subset of X has finite ϕ measure. Throughout the thesis a measure will always mean an outer measure as in [12].

Definition 1.2. The family $\mathcal{F}$ *covers* A *finely* if for any $x \in A$ and any $\varepsilon > 0$ there is $S \in \mathcal{F}$ such that $x \in S$ and $\operatorname{diam} S < \varepsilon$.

Definition 1.3. The family $\mathcal{F}$ is said to be ϕ *adequate for* A if for any open subset V of X there is a countable subfamily $\mathcal{G} \subset \mathcal{F}$ of disjoint sets such that

$$\cup_{S \in \mathcal{G}} S \subset V \quad \text{and} \quad \phi((V \cap A) \setminus \cup_{S \in \mathcal{G}} S) = 0.$$

For any member S of the family $\mathcal{F}$ we define its τ *enlargement*

$$\hat{S} = \cup\{T \mid T \in \mathcal{F},\ T \cap S \neq \emptyset,\ \operatorname{diam} T \leq \tau \operatorname{diam} S\} \tag{1.1}$$

Theorem 1.4. *If* $1 < \tau < \infty$, *then* $\mathcal{F}$ *has a subfamily* $\mathcal{G}$ *of disjoint sets such that*

$$\bigcup_{S \in \mathcal{F}} S \subset \bigcup_{S \in \mathcal{G}} \hat{S}.$$

Proof. See Corollary 2.8.5 in [12]. □

Remark 1.5. If $S = B(x, r)$, then $\hat{S} \subset B(x, (1 + 2\tau)r)$. Therefore if $\mathcal{F}$ is a family of closed balls, we may replace the inclusion above by the following one

$$\bigcup_{S\in\mathcal{F}} S \subset \bigcup_{B(x,r)\in\mathcal{G}} B(x,(1+2\tau)r).$$

The parameter τ is often chosen to be 2.

Theorem 1.6. *If $\mathcal{F}$ covers A finely, $1<\tau<\infty$, $1<\lambda<\infty$, and*

$$\phi(\hat{S}) < \lambda\phi(S)$$

whenever $S\in\mathcal{F}$ and $\hat{S}$ is the τ enlargement of S, then $\mathcal{F}$ is ϕ adequate for A.

Proof. See Theorem 2.8.7 in [12]. □

Definition 1.7. A sequence $(A_i)_{i\in\mathbb{N}}$ of subsets of X is called a *δ-covering* of A if $A\subset\cup_i A_i$ and $\operatorname{diam} A_i<\delta$ for each $i\in\mathbb{N}$. A δ-covering of A is called *centred* if it consists only of closed balls centred in A.

Definition 1.8. For $0\le s<\infty$ and $0<\delta\le\infty$ we define

(i)

$$\mathcal{H}^s_\delta(A)=\inf\left\{\sum_i(\operatorname{diam} A_i)^s \mid (A_i)_i \text{ is a } \delta\text{-covering of } A\right\}$$

and

$$\mathcal{H}^s(A)=\sup_{\delta>0}\mathcal{H}^s_\delta(A)=\lim_{\delta\searrow 0}\mathcal{H}^s_\delta(A).$$

$\mathcal{H}^s$ is called the s-dimensional *Hausdorff measure* on X.

(ii)

$$\mathcal{S}^s_\delta(A)=\inf\left\{\sum_i(\operatorname{diam} B(x_i,r_i))^s \mid (B(x_i,r_i))_i \text{ is a } \delta\text{-covering of } A\right\}$$

and

$$\mathcal{S}^s(A)=\sup_{\delta>0}\mathcal{S}^s_\delta(A)=\lim_{\delta\searrow 0}\mathcal{S}^s_\delta(A).$$

$\mathcal{S}^s$ is called the s-dimensional *spherical measure* on X.

(iii)

$$\mathcal{C}^s_\delta(A)$$
$$= \inf\left\{\sum_i (\operatorname{diam} B(x_i, r_i))^s \mid (B(x_i, r_i))_i \text{ is a centred } \delta\text{-covering of } A\right\},$$

where $A \neq \emptyset$ and $\mathcal{C}^s_\delta(\emptyset) = 0$.

$$\mathcal{C}^s_0(A) = \sup_{\delta>0} \mathcal{C}^s_\delta(A) = \lim_{\delta\searrow 0} \mathcal{C}^s_\delta(A).$$

$\mathcal{C}^s_0$ may fail to be monotone, since a smaller set may not have centres for the "best" covering (see [38]). Therefore in order to construct an outer measure let (see [38])

$$\mathcal{C}^s(A) = \sup_{B\subset A} \mathcal{C}^s_0(B).$$

$\mathcal{C}^s$ is called the s-dimensional *centred Hausdorff measure* on X.

Definition 1.9. The *Hausdorff dimension* of a set $A \subset X$ is

$$\dim A = \inf\{s \geq 0 \mid \mathcal{H}^s(A) = 0\}.$$

The definition of measures and simple reasoning (see [38]) imply that for any $A \subset X$

$$\mathcal{H}^s(A) \leq \mathcal{S}^s(A) \leq \mathcal{C}^s(A) \leq 2^s\mathcal{H}^s(A). \tag{1.2}$$

Constructions of the Hausdorff and the spherical measures are particular cases of the more general *Carathéodory's construction*, which can be made with an arbitrary non-negative set function instead of s-th power of the diameter and an arbitrary family of covering sets.

The same measure $\mathcal{H}^s$ can be obtained by using a δ-covering by all non-empty closed (or all non-empty open) subsets of X.

It is well known that measures $\mathcal{H}^s$ and $\mathcal{S}^s$ are Borel regular (that is, all Borel sets are measurable and every set $A \subset X$ is contained in a Borel set $B \subset X$ having the same measure as A has). Borel regularity of $\mathcal{C}^s$ and more general centred measures on a separable metric space has recently been proved by Schechter in [40].

Definition 1.10. Let $0 \leq s < \infty$, $A \subset X$ and $x \in X$. The *lower* and *upper s-densities* of A at x are

$$\underline{D}_s(A, x) = \liminf_{r \searrow 0} \frac{\mathcal{H}^s(A \cap B(x, r))}{(2r)^s}$$

and

$$\overline{D}_s(A, x) = \limsup_{r \searrow 0} \frac{\mathcal{H}^s(A \cap B(x, r))}{(2r)^s}.$$

If these densities coincide at x, then we say that the *s-density* of A at x exists and the common value is denoted by $D_s(A, x)$.

Definition 1.11. Let $n \in \mathbb{N}$. A set $A \subset X$ is called *n-rectifiable* if $\mathcal{H}^n$ almost all of A can be covered by countably many Lipschitzian images of subsets of $\mathbb{R}^n$. A is called *purely n-unrectifiable* if it contains no n-rectifiable set of positive $\mathcal{H}^n$ measure.

For $n = 1$ we simply call corresponding sets as rectifiable or purely unrectifiable. Note that if a set is n-rectifiable (purely n-unrectifiable), then its every subset is also n-rectifiable (purely n-unrectifiable). The intersection of n-rectifiable and purely n-unrectifiable sets is always $\mathcal{H}^n$-null set. A countable union of n-rectifiable sets is itself n-rectifiable.

Theorem 1.12. *If $\mathcal{H}^1(A) < \infty$, A is compact and connected, then A is rectifiable.*

Proof. See Theorem 3.14 in [11] for the case $X = \mathbb{R}^n$. This result can easily be extended to an arbitrary metric space. □

Definition 1.13. Let $n \in \mathbb{N}$. By $\sigma_n(X)$ we denote the smallest number such that every subset $A \subset X$ of finite $\mathcal{H}^n$ measure having at $\mathcal{H}^n$ almost all of its points

$$\underline{D}_n(A, x) > \sigma_n(X)$$

is n-rectifiable.

Clearly, it exists once we allow $\sigma_n(X)$ to be infinite, but we even show that $\sigma_n(X) \leq 1$ in Corollary 1.15. If X is n-rectifiable, it is obvious that $\sigma_n(X) = 0$. And if X is purely n-unrectifiable and $\mathcal{H}^n$ is locally-finite on X, then it is also easy to see that $\sigma_n(X) = \operatorname{ess\,sup}_{x \in X} \underline{D}_n(X, x)$.

Now we present an extremely useful theorem, which is applied many times throughout the thesis. Let $0 \leq s < \infty$, $A \subset X$, $x \in X$ and μ be a measure on X. We use the following notation

$$\overline{D}_{\mathcal{H}^s}(\mu, A, x) = \limsup_{r \searrow 0} \left\{ \frac{\mu(A \cap S)}{(\operatorname{diam} S)^s} \mid x \in S,\ 0 < \operatorname{diam} S < r \right\},$$

$$\overline{D}_{\mathcal{S}^s}(\mu, A, x) = \limsup_{r \searrow 0} \left\{ \frac{\mu(A \cap B(y, \rho))}{(\operatorname{diam} B(y, \rho))^s} \mid x \in B(y, \rho), 0 < \operatorname{diam} B(y, \rho) < r \right\}$$

and

$$\overline{D}_{\mathcal{C}^s}(\mu, A, x)$$
$$= \limsup_{r \searrow 0} \left\{ \frac{\mu(A \cap B(y, \rho))}{(\operatorname{diam} B(y, \rho))^s} \mid x \in B(y, \rho),\ y \in A,\ 0 < \operatorname{diam} B(y, \rho) < r \right\}$$

for $x \in \overline{A}$, $\overline{D}_{\mathcal{C}^s}(\mu, A, x) = 0$ for $x \notin \overline{A}$.

It follows that

$$\overline{D}_{\mathcal{C}^s}(\mu, A, x) \leq \overline{D}_{\mathcal{S}^s}(\mu, A, x) \leq \overline{D}_{\mathcal{H}^s}(\mu, A, x). \tag{1.3}$$

Theorem 1.14. *Suppose μ is a Borel regular measure on X, $A \subset X$ and ψ is $\mathcal{H}^s$, $\mathcal{S}^s$ or $\mathcal{C}^s$ measure. Then statements (i) - (iv) hold.*

(i)

$$\mu(A) \leq \psi(A) \sup_{x \in A} \overline{D}_\psi(\mu, A, x).$$

(ii) If V is an open subset of X and $B \subset V$, then

$$\mu(V) \geq \psi(B) \inf_{x \in B} \overline{D}_\psi(\mu, X, x).$$

(iii) If $\mu(A) < \infty$ and A is μ-measurable, then

$$\overline{D}_{\mathcal{H}^s}(\mu, A, x) = 0$$

for $\mathcal{H}^s$ almost all $x \in X \setminus A$.

(iv) If $A \subset X$ and $\mathcal{H}^s(A) < \infty$, then

$$0 \leq \overline{D}_{\mathcal{H}^s}(\mathcal{H}^s, A, x) \leq 1$$

for $\mathcal{H}^s$ almost all $x \in X$.

Proof. (i) Theorem 2.10.17(2) in [12] implies (i) for $\psi = \mathcal{H}^s$ or $\psi = \mathcal{S}^s$ and also implies that

$$\mu(A) \leq \mathcal{C}_0^s(A) \sup_{x \in A} \overline{D}_{\mathcal{C}^s}(\mu, A, x),$$

which obviously holds also with $\mathcal{C}_0^s(A)$ replaced by $\mathcal{C}^s(A)$.

(ii) This follows from [12, 2.10.18(1)] for $\psi = \mathcal{H}^s$ or $\psi = \mathcal{S}^s$. It also follows that if $E \subset B \subset V$, then

$$\mu(V) \geq \mathcal{C}_0^s(E) \inf_{x \in E} \overline{D}_{\mathcal{C}^s}(\mu, X, x).$$

The infimum can be taken over all $x \in B$, then taking the supremum over all $E \subset B$ we get the required inequality for $\psi = \mathcal{C}^s$.

Statements (iii) and (iv) are particular cases of [12, 2.10.18(2, 3)]. □

The last theorem implies immediately the following statement.

Corollary 1.15. *For any $n \in \mathbb{N}$ and an arbitrary metric space X*

$$\sigma_n(X) \leq 1.$$

Proof. Let $A \subset X$ of finite $\mathcal{H}^n$ measure have

$$\underline{D}_n(A, x) > 1$$

at $\mathcal{H}^n$ almost all of its points. Therefore we get

$$\overline{D}_{\mathcal{H}^n}(\mathcal{H}^n, A, x) \geq \overline{D}_n(A, x) \geq \underline{D}_n(A, x) > 1,$$

which may hold only at $\mathcal{H}^n$-null set by Theorem 1.14(iv). Thus A is a null set and the corollary follows. □

At the end of this section let us give definitions of an invariant metric, an invariant measure and the Haar measure, which are used quite often in the thesis.

Definition 1.16. Let $(G, \cdot)$ be a group.

(i) A metric d on G is called *left invariant* if

$$d(g \cdot g_1, g \cdot g_2) = d(g_1, g_2),$$

for any $g, g_1, g_2 \in G$.

(ii) A measure μ on G is called *left invariant* if

$$\mu(g \cdot A) = \mu(A)$$

for any $g \in G$ and any $A \subset G$.

The *right invariant* metric and measure are defined similarly using multiplications by $g \in G$ from the right.

Definition 1.17. Let $(G, \cdot)$ be a locally compact topological group. A *left Haar measure* on G is a non-zero, finite on compact sets and left invariant measure μ on G, which is Borel regular and also inner regular on Borel sets B with respect to compact sets, i.e.

$$\mu(B) = \sup\{\mu(C) \mid C \subset B,\ C \text{ compact}\}.$$

As a consequence of this definition μ is positive on every non-empty open set. A right invariant *(right) Haar measure* can be defined similarly. It is well known that a left (right) Haar measure exists and is unique up to positive constant multiples in a locally compact topological group (see [19, Chapter 11]). Left and right Haar measures coincide in Abelian or compact groups. A measure which is both left and right Haar we call simply the Haar measure.

Chapter 2

Besicovitch 1/2-Conjecture in Dimension One

2.1 Preliminary Results

In this chapter we are going to prove that the generalized Besicovitch 1/2-conjecture holds in any locally compact group equipped with an invariant metric, its Haar measure and has the Hausdorff dimension 1, which makes this case exceptional from higher Hausdorff dimensions.

The aim of this section is to prove that under some conditions a compact ball in the group cannot be totally disconnected. Towards this end we carefully study our group using a connection between its metric and measure. We show that the structure of the group resembles that of the Cantor set. The result obtained here will allow us to prove our main claim in the next section.

Let $\varepsilon_0 > 0$ and $R > 0$. Let $(G, \cdot)$ be a locally compact group with an invariant metric d and a Haar measure μ (both are left and right invariant) satisfying the following properties

$$\mu(U(g,r)) > \left(\frac{1}{2} + \varepsilon_0\right) 2r \tag{2.1}$$

for every $g \in G$ and every $0 < r < R$, and

$$\mu(S) \le \operatorname{diam} S \tag{2.2}$$

for every $S \subset G$ with $\operatorname{diam} S < R$.

Let $e \in G$ be the identity element of the group. Let $0 < \Delta < R/6$ be such that the closed ball $B(e, \Delta)$ is compact. The existence of Δ is guaranteed by the fact that G is a locally compact group. We are going to prove the following theorem.

Theorem 2.1. *If G is a locally compact group with an invariant metric d and a Haar measure μ satisfying properties* (2.1) *and* (2.2)*, then the compact ball $B(e, \Delta)$ cannot be totally disconnected.*

In this section we assume that $B(e, \Delta)$ is totally disconnected compact ball. At the end of the section we prove Theorem 2.1 by deriving a contradiction. In order to do that let us establish a series of auxiliary facts.

Definition 2.2. Let $\varepsilon > 0$. We call two points $g, \tilde{g} \in G$ *ε-chain connected* if there is a chain of points in G

$$g = g_0, g_1, \ldots, g_m = \tilde{g}$$

such that

$$d(g_i, g_{i+1}) < \varepsilon, \quad i = 0, 1, \ldots, m-1.$$

Let H_ε denote a set of elements of G ε-chain connected to the identity element $e \in G$. It is clear that $e \in H_\varepsilon$.

Lemma 2.3. *For each $\varepsilon > 0$ the set H_ε is a clopen normal subgroup of G, $H_\varepsilon \lhd G$.*

Proof. By the criterion of the subgroup H_ε is a subgroup of G if and only if $h^{-1}, h\tilde{h} \in H_\varepsilon$ for any $h, \tilde{h} \in H_\varepsilon$. Let us prove that these conditions are fulfilled. As h and $\tilde{h}$ are ε-chain connected to $e \in G$, we may find two chains in H_ε

$$e = h_0, h_1, \dots, h_m = h, \quad d(h_i, h_{i+1}) < \varepsilon, \quad i = 0, 1, \dots, m-1,$$

and

$$e = \tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_n = \tilde{h}, \quad d(\tilde{h}_j, \tilde{h}_{j+1}) < \varepsilon, \quad j = 0, 1, \dots, n-1.$$

The chain

$$e = h_0^{-1}, h_1^{-1}, \dots, h_m^{-1} = h^{-1}$$

is ε-chain connecting e to h^{-1} as

$$d(h_i^{-1}, h_{i+1}^{-1}) = d(e, h_i h_{i+1}^{-1}) = d(h_{i+1}, h_i) < \varepsilon, \quad i = 0, 1, \dots, m-1.$$

Another chain

$$e = h_0, h_1, \dots, h_m = h = h\tilde{h}_0, h\tilde{h}_1, \dots, h\tilde{h}_n = h\tilde{h}$$

connects e to $h\tilde{h}$ and is also ε-chain as

$$d(h\tilde{h}_j, h\tilde{h}_{j+1}) = d(\tilde{h}_j, \tilde{h}_{j+1}) < \varepsilon, \quad j = 0, 1, \dots, n-1.$$

Therefore $h^{-1} \in H_\varepsilon$, $h\tilde{h} \in H_\varepsilon$ and it follows that H_ε is a subgroup of G.

By the criterion of the normal subgroup H_ε is a normal subgroup of G if and only if $g^{-1}hg \in H_\varepsilon$ for any $g \in G$ and any $h \in H_\varepsilon$. The chain

$$e = g^{-1}h_0g, g^{-1}h_1g, \dots, g^{-1}h_mg = g^{-1}hg$$

is ε-chain connecting e to $g^{-1}hg$, since

$$d(g^{-1}h_i g, g^{-1}h_{i+1}g) = d(h_i, h_{i+1}) < \varepsilon, \quad i = 0, 1, \dots, m-1.$$

Hence $g^{-1}hg \in H_\varepsilon$ and H_ε is the normal subgroup of G.

The definition of H_ε implies that it is a clopen set. H_ε is a closed set, since it contains all its limit points. Indeed, if h is a limit point of H_ε, then $U(h, \varepsilon)$ contains a point of H_ε distinct from h, therefore h is also in H_ε. On the other hand, H_ε is an open set, since $U(h, \varepsilon) \subset H_\varepsilon$ for every $h \in H_\varepsilon$. □

Lemma 2.4. *$H_\varepsilon \lhd H_{\tilde{\varepsilon}}$ for any $0 < \varepsilon \le \tilde{\varepsilon}$.*

Proof. Definition of H_ε implies that $H_\varepsilon \subset H_{\tilde{\varepsilon}}$ for any $0 < \varepsilon \le \tilde{\varepsilon}$. By the last theorem $H_\varepsilon \lhd G$ and $H_{\tilde{\varepsilon}} \lhd G$. H_ε is also a normal subgroup of $H_{\tilde{\varepsilon}}$ by the same reason it is a normal subgroup of G. □

Lemma 2.5. *H_ε is a non-trivial subgroup for any $\varepsilon > 0$.*

Proof. Otherwise if $H_\varepsilon = \{e\}$ for some $0 < \varepsilon < R$, then $U(e, \varepsilon) = \{e\}$. Properties (2.1) and (2.2) of μ imply

$$\varepsilon < \mu(U(e, \varepsilon)) \le \operatorname{diam} U(e, \varepsilon) = 0,$$

which cannot be true. □

Let $A^\Delta = A \cap B(e, \Delta)$ for an arbitrary set $A \subset G$. If A is a closed set, then A^Δ is compact, being the intersection of the closed set with the compact $B(e, \Delta)$. For example, H_ε^Δ is a compact set.

Theorem 2.6. *For any $\tilde{\varepsilon} > 0$ there is $0 < \varepsilon < \tilde{\varepsilon}$ such that H_ε^Δ is a proper subset of the set $H_{\tilde{\varepsilon}}^\Delta$ (hence H_ε is a proper subgroup of the group $H_{\tilde{\varepsilon}}$).*

Proof. Suppose on the contrary that $H_\varepsilon^\Delta = H_{\tilde\varepsilon}^\Delta$ for some $\tilde\varepsilon > 0$ and for any $0 < \varepsilon < \tilde\varepsilon$. In other words, for any $0 < \varepsilon < \tilde\varepsilon$ and any $g \in H_{\tilde\varepsilon}^\Delta$ there is ε-chain in G connecting g to e. By the previous lemma $U(e,\delta) \neq \{e\}$ for all $\delta > 0$. Therefore there is $h \neq e$, $h \in B(e,\delta) \subset H_{\tilde\varepsilon}^\Delta$ for such small $\delta < \min\{\Delta/3, \tilde\varepsilon\}$ that

$$\frac{\delta}{\Delta - 3\delta} < 2\varepsilon_0 \quad \text{or} \quad \delta < \frac{2\varepsilon_0\Delta}{1+6\varepsilon_0}.$$

As $B(e,\Delta)$ is totally disconnected, there are non-empty disjoint closed sets $F_1 \ni e$ and $F_2 \ni h$ which partition $B(e,\Delta)$ (see [21, §46])

$$B(e,\Delta) = F_1 \cup F_2.$$

Obviously, F_1 and F_2 are compact sets. Then letting

$$\varepsilon = \frac{1}{2}\inf\{d(g_1,g_2) \mid g_1 \in F_1, g_2 \in F_2\} \tag{2.3}$$

we have

$$0 < 2\varepsilon \leq d(e,h) \leq \delta < \tilde\varepsilon.$$

Since $H_\varepsilon^\Delta = H_{\tilde\varepsilon}^\Delta$, there is ε-chain

$$e = h_0, h_1, \ldots, h_m = h$$

in G connecting $e \in F_1$ and $h \in F_2$. By deleting intermediate points of the chain if necessary we may assume that $d(h_i,h_j) \geq \varepsilon$ if $|i-j| \geq 2$.

Now let us construct a special family of open disjoint balls centred at some points of the chain. Denote

$$i_0 = 0 \quad \text{and} \quad r_0 = d(h_0,h_1) < \varepsilon.$$

Then $i_1 = 2$ is the first index such that $d(h_{i_0},h_j) > r_0$ for any $j \geq i_1$ and

$$d(h_{i_1},h_{i_0}) \leq d(h_{i_1},h_{i_1-1}) + d(h_{i_1-1},h_{i_0}) < \varepsilon + r_0.$$

Denote

$$r_1 = d(h_{i_1}, h_{i_0}) - r_0,$$

then $0 < r_1 < \varepsilon$. It follows that open balls $U(h_{i_0}, r_0)$ and $U(h_{i_1}, r_1)$ are disjoint.

Choose the first $i_2 > i_1$ such that $d(h_{i_1}, h_j) > r_1$ for any $j \geq i_2$ (it is true for all $j \geq i_1 + 2$, since $d(h_{i_1}, h_j) \geq \varepsilon > r_1$, therefore $i_2 = i_1 + 1$ if $d(h_{i_1}, h_{i_1+1}) > r_1$, otherwise $i_2 = i_1 + 2$), then $d(h_{i_1}, h_{i_2-1}) \leq r_1$ and

$$d(h_{i_2}, h_{i_1}) \leq d(h_{i_2}, h_{i_2-1}) + d(h_{i_2-1}, h_{i_1}) < \varepsilon + r_1.$$

Denote

$$r_2 = \min\{d(h_{i_2}, h_{i_0}) - r_0, d(h_{i_2}, h_{i_1}) - r_1\},$$

then $0 < r_2 < \varepsilon$. Open balls $U(h_{i_0}, r_0)$, $U(h_{i_1}, r_1)$ and $U(h_{i_2}, r_2)$ are disjoint.

We continue until $d(h_{i_p}, h) \leq r_p$, where

$$r_p = \min\{d(h_{i_p}, h_{i_k}) - r_k \mid 0 \leq k \leq p-1\},$$

on the step p (h_{i_p} may coincide with h). The balls $U(h_{i_n}, r_n)$ are disjoint and $0 < r_n < \varepsilon$ for $n = 0, 1, \ldots, p$.

Let $0 \leq k_1 \leq p-1$ be such that $r_p + r_{k_1} = d(h_{i_p}, h_{i_{k_1}})$. Since

$$r_{k_1} = \min\{d(h_{i_{k_1}}, h_{i_k}) - r_k \mid 0 \leq k \leq k_1 - 1\},$$

there is $0 \leq k_2 \leq k_1 - 1$ such that $r_{k_1} + r_{k_2} = d(h_{i_{k_1}}, h_{i_{k_2}})$. Proceeding in this way we obtain a decreasing to zero subsequence $p > k_1 > k_2 > \cdots > 0$ of the sequence $p > p-1 > p-2 > \cdots > 0$. Without loss of generality we may assume that they coincide, i.e. in definitions of radii the last term is always minimal on all steps of the construction

$$r_n + r_{n-1} = d(h_{i_n}, h_{i_{n-1}}), \quad n = 1, 2, \ldots, p. \tag{2.4}$$

The construction implies that the chain of points

$$e = h_{i_0}, h_{i_1}, \ldots, h_{i_p}, h$$

is 2ε-chain with $e \in F_1$ and $h \in F_2$. This chain cannot be entirely in $B(e, \Delta)$ by (2.3), therefore

$$\operatorname{diam}\{h_{i_0}, h_{i_1}, \ldots, h_{i_p}\} > \Delta.$$

Let $h_{i_{q+1}}$, $0 \le q \le p-1$, be the first point in this chain outside of $B(e, \Delta - 3\delta/2)$. Then the balls

$$U(h_{i_n}, r_n) \quad \text{and} \quad U(hh_{i_n}, r_n), \quad n = 0, 1, \ldots, q, \tag{2.5}$$

are all inside of $B(e, \Delta)$ (as $d(e, h) \le \delta$ and $0 < r_n < \varepsilon \le \delta/2$). Let

$$U = \bigcup_{n=0}^{q} (U(h_{i_n}, r_n),$$

then $U \subset F_1$ and $hU \subset F_2$ by (2.3), therefore the balls (2.5) are all disjoint. Let us observe that

$$\begin{aligned} \operatorname{diam} U &\ge \operatorname{diam}\{h_{i_0}, h_{i_1}, \ldots, h_{i_q}\} \ge d(e, h_{i_q}) \ge d(e, h_{i_{q+1}}) - d(h_{i_q}, h_{i_{q+1}}) \\ &> \Delta - \frac{3}{2}\delta - (r_q + r_{q+1}) > \Delta - 3\delta. \end{aligned} \tag{2.6}$$

We notice that

$$\operatorname{diam}(U \cup hU) \le 2\Delta < \frac{R}{3},$$

since $U \cup hU \subset B(e, \Delta)$, and

$$0 < r_n < \varepsilon \le \frac{\delta}{2} < \frac{\Delta}{6} < \frac{R}{36},$$

therefore we may use properties (2.1) and (2.2) of μ to get

$$\begin{aligned} 2\left(\frac{1}{2} + \varepsilon_0\right) \sum_{n=0}^{q} 2r_n &< \sum_{n=0}^{q} \mu(U(h_{i_n}, r_n)) + \sum_{n=0}^{q} \mu(U(hh_{i_n}, r_n)) \\ &= \mu(U \cup hU) \le \operatorname{diam}(U \cup hU). \end{aligned} \tag{2.7}$$

Now let us obtain an estimate for the last term of (2.7). Taking (2.4) into account one has

$$d(g_1, g_2) \le d(g_1, h_{i_s}) + d(h_{i_s}, h_{i_{s+1}}) + \cdots + d(h_{i_{t-1}}, h_{i_t}) + d(h_{i_t}, g_2)$$
$$< r_s + (r_s + r_{s+1}) + \cdots + (r_{t-1} + r_t) + r_t = 2\sum_{n=s}^{t} r_n$$

for any pair of points of U

$$g_1 \in U(h_{i_s}, r_s) \quad \text{and} \quad g_2 \in U(h_{i_t}, r_t), \quad 0 \le s \le t \le q.$$

It follows that

$$\operatorname{diam} U \le 2\sum_{n=0}^{q} r_n \tag{2.8}$$

and

$$\operatorname{diam}(U \cup hU) \le 2\sum_{n=0}^{q} r_n + \delta. \tag{2.9}$$

Combining estimates (2.7) and (2.9) we get

$$4\left(\frac{1}{2} + \varepsilon_0\right)\sum_{n=0}^{q} r_n < 2\sum_{n=0}^{q} r_n + \delta.$$

The last inequality together with (2.8) and (2.6) implies that

$$2\varepsilon_0 < \frac{\delta}{2\sum_{n=0}^{q} r_n} \le \frac{\delta}{\operatorname{diam} U} < \frac{\delta}{\Delta - 3\delta},$$

which contradicts the choice of δ. □

Theorem 2.7.

$$\bigcap_{\varepsilon>0} H_\varepsilon = \{e\}.$$

Proof. Clearly, it is equivalent to prove that for any $h \in G$, $h \ne e$, there is $\varepsilon_h > 0$ such that $h \notin H_{\varepsilon_h}$.

We have just shown that for any small enough $\delta > 0$ and any $h \in B(e, \delta)$, $h \neq e$, there is ε_h

$$0 < \varepsilon_h \leq \frac{d(e,h)}{2} \leq \frac{\delta}{2}$$

such that there is no ε_h-chain connecting e and h (or equivalently $h \notin H_{\varepsilon_h}$). We only have to prove that the same is also true for any $h \in G \setminus B(e, \delta)$.

Consider the compact set

$$C = B(e, \delta) \setminus U(e, \delta/2)$$

and its open cover

$$\{U(h, \varepsilon_h) \mid h \in C\}.$$

Observe that $U(h, \varepsilon_h) \cap H_{\varepsilon_h} = \emptyset$. Let

$$(U(h_i, \varepsilon_{h_i}))_{i=1,\, h_i \in C}^{k}$$

be the finite subcover of the compact set C. Let

$$\varepsilon_\delta = \min\{\varepsilon_{h_1}, \varepsilon_{h_2}, \ldots, \varepsilon_{h_k}\}$$

and note that $\varepsilon_\delta \leq \delta/2$. It follows that

$$U(h_i, \varepsilon_{h_i}) \cap H_{\varepsilon_\delta} = \emptyset, \quad i = 1, 2, \ldots, k,$$

and therefore

$$C \cap H_{\varepsilon_\delta} = \emptyset.$$

There is no ε_δ-chain "crossing" C, i.e. with points in $U(e, \delta/2)$ and $G \setminus B(e, \delta)$, since

$$\operatorname{dist}(G \setminus B(e, \delta), U(e, \delta/2)) \geq \frac{\delta}{2} \geq \varepsilon_\delta,$$

and therefore

$$(G \setminus B(e, \delta)) \cap H_{\varepsilon_\delta} = \emptyset,$$

as required. □

Corollary 2.8. *For small enough $\varepsilon > 0$ the group H_ε is a subset of $B(e, \Delta)$, H_ε is a compact subgroup of G and*

$$\lim_{\varepsilon \searrow 0} \operatorname{diam} H_\varepsilon = 0.$$

Proof. The proof of Theorem 2.7 implies that for any small enough $\delta > 0$ there is $\varepsilon_\delta > 0$ such that

$$H_\varepsilon \subset U(e, \delta/2), \quad 0 < \varepsilon \leq \varepsilon_\delta,$$

and the corollary follows easily.

□

Theorem 2.9. *For $\tilde{\varepsilon} > 0$ the following statement holds*

$$\sup\{\rho \mid 0 < \rho \leq \tilde{\varepsilon},\ H_\rho^\Delta \neq H_{\tilde{\varepsilon}}^\Delta\} = \inf\{\rho \mid 0 < \rho \leq \tilde{\varepsilon},\ H_\rho^\Delta = H_{\tilde{\varepsilon}}^\Delta\} =: \varepsilon$$

and

$$H_\varepsilon^\Delta \neq H_{\tilde{\varepsilon}}^\Delta,$$

where $0 < \varepsilon < \tilde{\varepsilon}$, $\varepsilon \leq \Delta$.

Proof. The existence of the positive supremum and infimum follows from Theorem 2.6 and the fact that $H_\rho^\Delta = H_{\tilde{\varepsilon}}^\Delta$ for $\rho = \tilde{\varepsilon}$ or $\tilde{\varepsilon} \geq \rho > \Delta$. This fact is obvious for $\rho = \tilde{\varepsilon}$, but if $\tilde{\varepsilon} > \rho > \Delta$, then both H_ρ^Δ and $H_{\tilde{\varepsilon}}^\Delta$ coincide with $B(e, \Delta)$. Hence if $H_\rho^\Delta \neq H_{\tilde{\varepsilon}}^\Delta$ for some $0 < \rho \leq \tilde{\varepsilon}$, then we know that $0 < \rho < \tilde{\varepsilon}$ and $\rho \leq \Delta$.

Let us refer to the supremum and the infimum in the statement of the theorem simply as to sup and inf. Then Lemma 2.4 implies that $H_\rho^\Delta \neq H_{\tilde{\varepsilon}}^\Delta$

for any $0 < \rho < \sup$ and $H^\Delta_\rho = H^\Delta_{\tilde\varepsilon}$ for any $\inf < \rho \leq \tilde\varepsilon$, therefore $\sup \leq \inf$. If the inequality is strict, then there is ρ between sup and inf such that either $H^\Delta_\rho \neq H^\Delta_{\tilde\varepsilon}$ or $H^\Delta_\rho = H^\Delta_{\tilde\varepsilon}$. This contradicts to the definition of the supremum or the infimum, hence they are equal.

Suppose that the second part of the statement is false, $H^\Delta_\varepsilon = H^\Delta_{\tilde\varepsilon}$. But $H^\Delta_\rho \neq H^\Delta_{\tilde\varepsilon}$ for any $0 < \rho < \varepsilon$, which means that there is $g_\rho \in H^\Delta_{\tilde\varepsilon} \setminus H^\Delta_\rho$, i.e. $g_\rho \in H^\Delta_{\tilde\varepsilon}$ is not ρ-chain connected to e. As $H^\Delta_{\tilde\varepsilon}$ is compact, we may find $g_{\rho_n} \in H^\Delta_{\tilde\varepsilon} \setminus H^\Delta_{\rho_n}$ such that

$$\rho_n \nearrow \varepsilon \quad \text{and} \quad g_{\rho_n} \to g \in H^\Delta_{\tilde\varepsilon}, \quad n \to \infty.$$

Obviously, there is $N \in \mathbb{N}$ such that

$$d(g_{\rho_n}, g) < \rho_n, \quad n \geq N,$$

therefore $g \notin H^\Delta_{\rho_n}$ for $n \geq N$. But $g \in H^\Delta_{\tilde\varepsilon} = H^\Delta_\varepsilon$, hence there is ε-chain connecting e and g in G

$$e = g_0, g_1, \ldots, g_m = g, \quad d(g_i, g_{i+1}) < \varepsilon, \quad i = 0, 1, \ldots, m-1.$$

There is $n_0 \geq N$ such that

$$d(g_i, g_{i+1}) < \rho_{n_0} < \varepsilon, \quad i = 0, 1, \ldots, m-1.$$

It follows that $g \in H_{\rho_{n_0}}$, and therefore $g \in H^\Delta_{\rho_{n_0}}$, which is a contradiction. □

Let $\tilde\varepsilon > \Delta$, then $H^\Delta_{\tilde\varepsilon} = B(e, \Delta)$. The last theorem gives us $\varepsilon_1 := \varepsilon$ such that $\varepsilon_1 \leq \Delta$ and $H^\Delta_{\varepsilon_1} \neq B(e, \Delta)$. Let $\tilde\varepsilon := \varepsilon_1$, then the last theorem also gives us $\varepsilon_2 := \varepsilon$ such that $\varepsilon_2 < \varepsilon_1$ and $H^\Delta_{\varepsilon_2} \neq H^\Delta_{\varepsilon_1}$.

Repeating this procedure we obtain three infinite sequences: the decreasing sequence of numbers

$$\Delta \geq \varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n > \ldots,$$

the sequence of nested sets, each one being a proper subset of the previous one,

$$B(e,\Delta) \supset H^{\Delta}_{\varepsilon_1} \supset H^{\Delta}_{\varepsilon_2} \supset \ldots \supset H^{\Delta}_{\varepsilon_n} \supset \ldots,$$

and the sequence of nested clopen normal subgroups of G, also each one being a proper normal subgroup of the previous one,

$$G \rhd H_{\varepsilon_1} \rhd H_{\varepsilon_2} \rhd \ldots \rhd H_{\varepsilon_n} \rhd \ldots. \tag{2.10}$$

Theorem 2.10. *The series of numbers constructed above is convergent*

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

Proof. Using the fact that each set $H^{\Delta}_{\varepsilon_n}$ is a proper subset of the previous one we may find the following sequence

$$h_0 \in B(e,\Delta) \setminus H^{\Delta}_{\varepsilon_1}, \quad h_1 \in H^{\Delta}_{\varepsilon_1} \setminus H^{\Delta}_{\varepsilon_2}, \quad \ldots, \quad h_{n-1} \in H^{\Delta}_{\varepsilon_{n-1}} \setminus H^{\Delta}_{\varepsilon_n},$$

and therefore

$$U(h_0,\varepsilon_1) \subset G \setminus H_{\varepsilon_1},\ U(h_1,\varepsilon_2) \subset H_{\varepsilon_1} \setminus H_{\varepsilon_2},\ \ldots,\ U(h_{n-1},\varepsilon_n) \subset H_{\varepsilon_{n-1}} \setminus H_{\varepsilon_n}.$$

It follows that the balls

$$U(h_{i-1},\varepsilon_i), \quad i = 1,2,\ldots,n,$$

are disjoint and

$$\bigcup_{i=1}^{n} U(h_{i-1},\varepsilon_i) \subset B(e,\Delta+\varepsilon_1) \subset B(e,2\Delta).$$

Observe that $\varepsilon_n \leq \Delta < R/6$, then by properties (2.1) and (2.2) of μ one gets

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon_i < \sum_{i=1}^{n} \mu(U(h_{i-1},\varepsilon_i)) &= \mu(\cup_{i=1}^{n} U(h_{i-1},\varepsilon_i)) \leq \mu(B(e,2\Delta)) \\ &\leq \operatorname{diam} B(e,2\Delta) \leq 4\Delta < \frac{2}{3}R < \infty.\end{aligned}$$

The required statement follows immediately. □

Corollary 2.11. *There is $N \in \mathbb{N}$ such that for every $n \geq N$*

$$H_{\varepsilon_n} \subset B(e, \Delta),$$

H_{ε_n} is a compact subgroup of G and

$$\lim_{n \to \infty} \operatorname{diam} H_{\varepsilon_n} = 0.$$

Proof. Observe that the last theorem implies that $\varepsilon_n \searrow 0$ as $n \to \infty$. Then the statement follows from Corollary 2.8. □

Lemma 2.12. *If $0 < \tilde{\varepsilon} \leq \Delta$,*

$$\varepsilon := \sup\{\rho \mid 0 < \rho \leq \tilde{\varepsilon},\ H_\rho^\Delta \neq H_{\tilde{\varepsilon}}^\Delta\}, \tag{2.11}$$

H is a subgroup of G such that for any $\rho > \varepsilon$

$$H_\varepsilon \subset H \subset H_\rho \tag{2.12}$$

and

$$H \neq H_{\tilde{\varepsilon}}, \tag{2.13}$$

then

$$\operatorname{dist}(H, H_{\tilde{\varepsilon}} \setminus H) = \varepsilon.$$

Proof. Let us observe that $\tilde{\varepsilon} > \varepsilon$ by Theorem 2.9, thus $H \subset H_{\tilde{\varepsilon}}$ by (2.12), and the assumption (2.13) guarantees that $H_{\tilde{\varepsilon}} \setminus H \neq \emptyset$. The group structure implies that $h^{-1}H_{\tilde{\varepsilon}} = H_{\tilde{\varepsilon}}$ and $h^{-1}H = H$ for any $h \in H \subset H_{\tilde{\varepsilon}}$, therefore

$$\operatorname{dist}(H, H_{\tilde{\varepsilon}} \setminus H) = \operatorname{dist}(e, H_{\tilde{\varepsilon}} \setminus H) =: \rho.$$

Since $H_\varepsilon \subset H$, the sets H_ε and $H_{\tilde{\varepsilon}} \setminus H$ are disjoint, thus

$$\rho = \operatorname{dist}(e, H_{\tilde{\varepsilon}} \setminus H) \geq \varepsilon.$$

On the other hand,

$$\rho = \operatorname{dist}(H, H_{\tilde{\varepsilon}} \setminus H) < \tilde{\varepsilon},$$

otherwise there is no $\tilde{\varepsilon}$-chain with points in H and $H_{\tilde{\varepsilon}} \setminus H$, which contradicts to the definition of $H_{\tilde{\varepsilon}}$.

We have just proved

$$\varepsilon \leq \rho < \tilde{\varepsilon} \leq \Delta,$$

therefore $B(e, \Delta)$ contains points of $H_{\tilde{\varepsilon}} \setminus H$ and

$$H^\Delta \neq H^\Delta_{\tilde{\varepsilon}}. \tag{2.14}$$

If we assume that $\rho > \varepsilon$, then not only $H \subset H_\rho$ by (2.12), but also $H = H_\rho$, as there is no ρ-chain with points in H and $H_{\tilde{\varepsilon}} \setminus H$. Then according to (2.14)

$$H^\Delta_\rho \neq H^\Delta_{\tilde{\varepsilon}}, \quad \varepsilon < \rho < \tilde{\varepsilon},$$

which contradicts to the assumption (2.11). It follows that $\rho = \varepsilon$ and the lemma is proved. □

Theorem 2.13. *Let* $0 < \tilde{\varepsilon} \leq \Delta$ *and*

$$\varepsilon := \sup\{\rho \mid 0 < \rho \leq \tilde{\varepsilon},\ H^\Delta_\rho \neq H^\Delta_{\tilde{\varepsilon}}\}.$$

Then the number of cosets of the quotient group $H_{\tilde{\varepsilon}}/H_\varepsilon$ *which intersect* $B(e, \Delta)$ *is finite. Moreover, the distance between distinct cosets of* $H_{\tilde{\varepsilon}}/H_\varepsilon$ *is at least* ε *and the distance* ε *is attained from every coset to some other coset.*

Proof. As H_ε is a normal subgroup of $H_{\bar{\varepsilon}}$ by Lemma 2.4, there is a quotient (factor) group $H_{\bar{\varepsilon}}/H_\varepsilon$, which consists of disjoint cosets. Observe that $H_{\bar{\varepsilon}}^\Delta$ is covered by distinct disjoint cosets intersecting $B(e,\Delta)$. Let us take any one representative $g_\alpha \in H_{\bar{\varepsilon}}^\Delta$, $\alpha \in I$ (I is a set of indices), from each coset intersecting $B(e,\Delta)$, then

$$H_{\bar{\varepsilon}}^\Delta \subset \bigcup_{\alpha\in I} g_\alpha H_\varepsilon$$

or

$$H_{\bar{\varepsilon}}^\Delta \subset \bigcup_{\alpha\in I} (g_\alpha H_\varepsilon)^\Delta \subset \bigcup_{\alpha\in I} g_\alpha H_\varepsilon^{2\Delta},$$

since $(g_\alpha H_\varepsilon)^\Delta \subset g_\alpha H_\varepsilon^{2\Delta}$. The sets $g_\alpha H_\varepsilon^{2\Delta}$, $\alpha \in I$, are disjoint, contained in $B(e, 3\Delta)$ and have equal positive μ measure

$$\mu(g_\alpha H_\varepsilon^{2\Delta}) = \mu(H_\varepsilon^{2\Delta}), \quad \alpha \in I,$$

and

$$\mu(H_\varepsilon^{2\Delta}) \geq \mu(U(e,\varepsilon)) > 0$$

by (2.1) and the fact that $U(e,\varepsilon) \subset H_\varepsilon^\Delta$ ($\varepsilon \leq \Delta$). The number of such sets $g_\alpha H_\varepsilon^{2\Delta}$, $\alpha \in I$, is finite, since by (2.2)

$$\mu(B(e,3\Delta)) \leq \operatorname{diam} B(e,3\Delta) \leq 6\Delta < R < \infty.$$

By Theorem 2.9 $H = H_\varepsilon$ satisfies assumptions of Lemma 2.12, therefore

$$\operatorname{dist}(H_\varepsilon, H_{\bar{\varepsilon}} \setminus H_\varepsilon) = \operatorname{dist}(e, H_{\bar{\varepsilon}} \setminus H_\varepsilon) = \varepsilon. \tag{2.15}$$

It follows that for distinct cosets $H_\varepsilon \neq gH_\varepsilon$ ($g \notin H_\varepsilon$)

$$\operatorname{dist}(H_\varepsilon, gH_\varepsilon) = \operatorname{dist}(e, gH_\varepsilon) \geq \varepsilon. \tag{2.16}$$

On the other hand, as $H_{\varepsilon}^{\Delta} \neq H_{\tilde{\varepsilon}}^{\Delta}$ (Theorem 2.9) there are non-zero but finite number of distinct cosets $gH_{\varepsilon} \neq H_{\varepsilon}$ intersecting $B(e, \Delta)$. The equality (2.15) implies that for some of them

$$\mathrm{dist}(H_{\varepsilon}, gH_{\varepsilon}) = \varepsilon. \tag{2.17}$$

Then (2.16) and (2.17) imply that for any pair of cosets $gH_{\varepsilon} \neq \tilde{g}H_{\varepsilon}$ $(g^{-1}\tilde{g} \notin H_{\varepsilon})$

$$\mathrm{dist}(gH_{\varepsilon}, \tilde{g}H_{\varepsilon}) = \mathrm{dist}(H_{\varepsilon}, g^{-1}\tilde{g}H_{\varepsilon}) \geq \varepsilon,$$

and the distance ε is attained from every gH_{ε} to some $\tilde{g}H_{\varepsilon}$. □

Theorem 2.14. *Let* $0 < \tilde{\varepsilon} \leq \Delta$ *and*

$$\varepsilon := \sup\{\rho \mid 0 < \rho \leq \tilde{\varepsilon},\ H_{\rho}^{\Delta} \neq H_{\tilde{\varepsilon}}^{\Delta}\}.$$

Then the quotient group $H_{\tilde{\varepsilon}}/H_{\varepsilon}$ *is generated by all cosets* ε *distant from* H_{ε}, *i.e.*

$$H_{\tilde{\varepsilon}}/H_{\varepsilon} = \langle \tilde{g}_1 H_{\varepsilon}, \tilde{g}_2 H_{\varepsilon}, \ldots, \tilde{g}_p H_{\varepsilon} \rangle,$$

where $\tilde{g}_j \in H_{\tilde{\varepsilon}}$ *and* $\tilde{g}_j H_{\varepsilon}$ *are all cosets such that*

$$\mathrm{dist}(H_{\varepsilon}, \tilde{g}_j H_{\varepsilon}) = \varepsilon, \quad j = 1, 2, \ldots, p.$$

For any $g \in H_{\tilde{\varepsilon}}$ *there is a chain of points*

$$g = g_0, g_1, \ldots, g_m$$

in distinct cosets of $H_{\tilde{\varepsilon}}/H_{\varepsilon}$ *with* $g_m \in H_{\varepsilon}$ *such that*

$$d(g_i, g_{i+1}) = \varepsilon, \quad i = 0, 1, \ldots, m-1.$$

Proof. First of all we observe that

$$H = \langle \tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_p \rangle \cdot H_\varepsilon,$$

which is a subset of $H_{\tilde{\varepsilon}}$, is also a subgroup of $H_{\tilde{\varepsilon}}$. Indeed, for any $g, h \in H$ we have $g \in \tilde{g}H_\varepsilon$ and $h \in \tilde{h}H_\varepsilon$ ($\tilde{g}, \tilde{h} \in \langle \tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_p \rangle \subset H_{\tilde{\varepsilon}}$), and since $H_\varepsilon \lhd H_{\tilde{\varepsilon}}$ one gets

$$g^{-1} \in (\tilde{g}H_\varepsilon)^{-1} = \tilde{g}^{-1}H_\varepsilon \subset H$$

and

$$gh \in \tilde{g}H_\varepsilon \cdot \tilde{h}H_\varepsilon = \tilde{g}\tilde{h}H_\varepsilon \subset H.$$

Since H_ε is a normal subgroup of $H_{\tilde{\varepsilon}}$ and

$$H_\varepsilon \subset H \subset H_{\tilde{\varepsilon}},$$

H_ε is also a normal subgroup of H. It follows that

$$H/H_\varepsilon = \langle \tilde{g}_1 H_\varepsilon, \tilde{g}_2 H_\varepsilon, \dots, \tilde{g}_p H_\varepsilon \rangle \subset H_{\tilde{\varepsilon}}/H_\varepsilon.$$

Let us prove that for any $g \in H$ there is a chain of points

$$g = g_0, g_1, \dots, g_m$$

in distinct cosets of H/H_ε with $g_m \in H_\varepsilon$ such that

$$d(g_i, g_{i+1}) = \varepsilon, \quad i = 0, 1, \dots, m-1.$$

Since the group structure implies that

$$\operatorname{dist}(H_\varepsilon, \tilde{g}_j H_\varepsilon) = \operatorname{dist}(e, \tilde{g}_j H_\varepsilon) = \varepsilon$$

and $(\tilde{g}_j H_\varepsilon)^\Delta$ is compact, the distance ε is realized between e and some point of $\tilde{g}_j H_\varepsilon$. Therefore without loss of generality we may assume that

$$d(e, \tilde{g}_j) = \varepsilon, \quad j = 1, 2, \dots, p. \tag{2.18}$$

For any $g \in H$ we have

$$gH_\varepsilon \in H/H_\varepsilon = \langle \tilde{g}_1 H_\varepsilon, \tilde{g}_2 H_\varepsilon, \ldots, \tilde{g}_p H_\varepsilon \rangle,$$

then

$$g^{-1}H_\varepsilon = (\hat{g}_1 H_\varepsilon)^{\alpha_1}(\hat{g}_2 H_\varepsilon)^{\alpha_2}\cdots(\hat{g}_m H_\varepsilon)^{\alpha_m} = \hat{g}_1^{\alpha_1}\hat{g}_2^{\alpha_2}\cdots\hat{g}_m^{\alpha_m}H_\varepsilon,$$

where

$$\hat{g}_i \in \{\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_p\}, \quad \alpha_i \in \{-1, 1\},\ i = 1, 2, \ldots, m.$$

It follows that

$$g\hat{g}_1^{\alpha_1}\hat{g}_2^{\alpha_2}\cdots\hat{g}_m^{\alpha_m} \in H_\varepsilon.$$

We may also assume that the cosets

$$H_\varepsilon,\ (\hat{g}_1 H_\varepsilon)^{\alpha_1},\ (\hat{g}_1 H_\varepsilon)^{\alpha_1}(\hat{g}_2 H_\varepsilon)^{\alpha_2},\ \ldots,\ (\hat{g}_1 H_\varepsilon)^{\alpha_1}(\hat{g}_2 H_\varepsilon)^{\alpha_2}\cdots(\hat{g}_m H_\varepsilon)^{\alpha_m}$$

are all distinct. Then (2.18) implies that the chain

$$g,\ g\hat{g}_1^{\alpha_1},\ g\hat{g}_1^{\alpha_1}\hat{g}_2^{\alpha_2},\ \ldots,\ g\hat{g}_1^{\alpha_1}\hat{g}_2^{\alpha_2}\cdots\hat{g}_m^{\alpha_m}$$

has the required property.

The theorem follows once we show that $H = H_{\bar{\varepsilon}}$. Let us assume the opposite, $H \neq H_{\bar{\varepsilon}}$, and derive a contradiction.

Observe that $H \subset H_\rho$ for any $\rho > \varepsilon$, as we have just proved that any $g \in H$ can be ρ-chain connected to a point of H_ε, and hence to $e \in H_\varepsilon$. Then H satisfies all assumptions of Lemma 2.12, therefore

$$\operatorname{dist}(H, H_{\bar{\varepsilon}} \setminus H) = \operatorname{dist}(e, H_{\bar{\varepsilon}} \setminus H) = \varepsilon. \tag{2.19}$$

As $H^\Delta \neq H_{\bar{\varepsilon}}^\Delta$ by (2.14) there are non-zero but finite number (by the previous theorem) of distinct cosets $gH_\varepsilon \in (H_{\bar{\varepsilon}}/H_\varepsilon)\backslash(H/H_\varepsilon)$ which intersect $B(e, \Delta)$.

The equality (2.19) implies that for some of them

$$\operatorname{dist}(e, gH_\varepsilon) = \varepsilon,$$

or equivalently

$$\operatorname{dist}(H_\varepsilon, gH_\varepsilon) = \varepsilon.$$

It means that

$$gH_\varepsilon \in \{\tilde{g}_1 H_\varepsilon, \tilde{g}_2 H_\varepsilon, \dots, \tilde{g}_p H_\varepsilon\} \subset H/H_\varepsilon,$$

which is a contradiction. Therefore $H = H_{\bar{\varepsilon}}$ and the proof is finished. □

Now we are prepared to prove Theorem 2.1.

Proof of Theorem 2.1. Corollary 2.11 guarantees that nested subgroups of G in the sequence (2.10) are compact starting from some number $N \in \mathbb{N}$ and $\operatorname{diam} H_{\varepsilon_N} < R$. Since a diameter of a compact group is realized from any point, there is $g_0 \in H_{\varepsilon_N}$ ($g_0 \neq e$) such that

$$\operatorname{diam} H_{\varepsilon_N} = d(e, g_0).$$

As $\lim_{n\to\infty} \operatorname{diam} H_{\varepsilon_n} = 0$ there is $n_0 \geq N$ such that

$$g_0 \in H_{\varepsilon_{n_0}} \setminus H_{\varepsilon_{n_0+1}}.$$

By Theorem 2.14 there is a chain of points

$$g_0 = g_0^{(0)}, g_1^{(0)}, \dots, g_{m_0}^{(0)} = g_1$$

in distinct cosets of the quotient group $H_{\varepsilon_{n_0}}/H_{\varepsilon_{n_0+1}}$ with $g_1 \in H_{\varepsilon_{n_0+1}}$ such that

$$d(g_i^{(0)}, g_{i+1}^{(0)}) = \varepsilon_{n_0+1}, \quad i = 0, 1, \dots, m_0 - 1.$$

The balls

$$U(g_i^{(0)}, \varepsilon_{n_0+1}), \quad i = 0, 1, \ldots, m_0 - 1,$$

contained in $H_{\varepsilon_{n_0}} \backslash H_{\varepsilon_{n_0+1}}$ are disjoint, being in distinct cosets of $H_{\varepsilon_{n_0}}/H_{\varepsilon_{n_0+1}}$.

We continue the construction in the same way. On the step $k \geq 1$ if $g_k \neq e$, we may find $n_k > n_{k-1}$ such that

$$g_k \in H_{\varepsilon_{n_k}} \setminus H_{\varepsilon_{n_k+1}}.$$

Applying Theorem 2.14 we get a chain of points

$$g_k = g_0^{(k)}, g_1^{(k)}, \ldots, g_{m_k}^{(k)} = g_{k+1}$$

in distinct cosets of $H_{\varepsilon_{n_k}}/H_{\varepsilon_{n_k+1}}$ with $g_{k+1} \in H_{\varepsilon_{n_k+1}}$ such that

$$d(g_i^{(k)}, g_{i+1}^{(k)}) = \varepsilon_{n_k+1}, \quad i = 0, 1, \ldots, m_k - 1.$$

The balls

$$U(g_i^{(k)}, \varepsilon_{n_k+1}), \quad i = 0, 1, \ldots, m_k - 1,$$

contained in $H_{\varepsilon_{n_k}} \backslash H_{\varepsilon_{n_k+1}}$ are disjoint, being in distinct cosets of $H_{\varepsilon_{n_k}}/H_{\varepsilon_{n_k+1}}$. The construction is stopped once $g_k = e$.

We will only show how the contradiction can be derived in the case $g_k \neq e$ for all $k \geq 0$. The case $g_k = e$ for some $k \geq 0$ will lead to the contradiction in a similar way.

Suppose that $g_k \neq e$ for all $k \geq 0$. The sets

$$H_{\varepsilon_{n_k}} \setminus H_{\varepsilon_{n_k+1}}, \quad k = 0, 1, \ldots,$$

are disjoint, and so are the balls

$$U(g_i^{(k)}, \varepsilon_{n_k+1}), \quad i = 0, 1, \ldots, m_k - 1, \ k = 0, 1, \ldots.$$

Since $\varepsilon_{n_k+1} \le \Delta < R/6$ and $\varepsilon_{n_k+1} < \mu(U(g_i^{(k)}, \varepsilon_{n_k+1}))$ by (2.1), the property (2.2) and the countable additivity of μ imply

$$\sum_{j=0}^{\infty}\sum_{i=0}^{m_j-1} \varepsilon_{n_j+1} < \sum_{j=0}^{\infty}\sum_{i=0}^{m_j-1} \mu(U(g_i^{(j)}, \varepsilon_{n_j+1})) = \mu(\cup_{j=0}^{\infty} \cup_{i=0}^{m_j-1} U(g_i^{(j)}, \varepsilon_{n_j+1}))$$
$$\le \mu(H_{\varepsilon_N}) \le \operatorname{diam} H_{\varepsilon_N} < R < \infty.$$

On the other hand, we have

$$\sum_{j=0}^{k}\sum_{i=0}^{m_j-1} \varepsilon_{n_j+1} = \sum_{j=0}^{k}\sum_{i=0}^{m_j-1} d(g_i^{(j)}, g_{i+1}^{(j)}) \ge d(g_0^{(0)}, g_{m_k}^{(k)}) = d(g_0, g_{k+1})$$
$$\ge d(g_0, e) - d(g_{k+1}, e) \ge \operatorname{diam} H_{\varepsilon_N} - \operatorname{diam} H_{\varepsilon_{n_k+1}}.$$

Letting $k \to \infty$ and taking into account that $\lim_{n\to\infty} \operatorname{diam} H_{\varepsilon_n} = 0$ one gets

$$\sum_{j=0}^{\infty}\sum_{i=0}^{m_j-1} \varepsilon_{n_j+1} \ge \operatorname{diam} H_{\varepsilon_N},$$

which contradicts to the estimate obtained earlier, and the proof concludes.

□

2.2 Proof of the Main Result

Finally we are ready to prove the main result of this chapter – the generalized Besicovitch 1/2-conjecture holds in 1-dimensional locally compact group G.

Theorem 2.15. *If G is a locally compact group with an invariant metric d and the Haar measure $\mathcal{H}^1$, then*

$$\sigma_1(G) \le \frac{1}{2}.$$

Proof. In Theorem 1.14 we may put $s = 1$, $X = G$ and A to be a compact ball $B(e, \delta_0)$. Such a ball exists, as the group G is locally compact, $\mathcal{H}^1$ measure of the ball is finite by properties of the Haar measure. Then the statement (iv) implies that

$$\overline{D}_{\mathcal{H}^1}(\mathcal{H}^1, B(e, \delta_0), g) \leq 1$$

at $\mathcal{H}^1$ almost all $g \in G$, therefore

$$\overline{D}_{\mathcal{H}^1}(\mathcal{H}^1, G, g) \leq 1$$

at some $g \in U(e, \delta_0)$ as $\mathcal{H}^1(U(e, \delta_0)) > 0$. By the invariance of the measure $\mathcal{H}^1$ it is true at any $g \in G$. That is, independently of $g \in G$ for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\mathcal{H}^1(S) \leq (1 + \varepsilon) \operatorname{diam} S, \quad 0 \leq \operatorname{diam} S < \delta(\varepsilon). \tag{2.20}$$

The case $\operatorname{diam} S = 0$ follows from the definition of the measure $\mathcal{H}^1$.

Suppose a set $E \subset G$, $0 < \mathcal{H}^1(E) < \infty$, at $\mathcal{H}^1$ almost all of its points fulfills

$$\underline{D}_1(E, g) > \frac{1}{2}. \tag{2.21}$$

Once we prove that E is rectifiable, then Definition 1.13 will imply that $\sigma_1(G) \leq 1/2$.

It follows from (2.21) that

$$\underline{D}_1(G, g) \geq \underline{D}_1(E, g) > \frac{1}{2}$$

at $\mathcal{H}^1$ almost all $g \in E$, therefore by the invariance of the measure $\mathcal{H}^1$

$$\underline{D}_1(G, e) > \frac{1}{2}. \tag{2.22}$$

We may add some small positive number $5\varepsilon_0$ ($0 < \varepsilon_0 < 1/10$) to the right-hand side of (2.22). Again that inequality remains true at any $g \in G$. It follows that for some $r_0(\varepsilon_0) > 0$ and all $0 < r < r_0(\varepsilon_0)$

$$\frac{\mathcal{H}^1(B(g,(1-\varepsilon_0)r))}{(1-\varepsilon_0)2r} > \frac{1}{2} + 5\varepsilon_0,$$

therefore

$$\frac{\mathcal{H}^1(U(g,r))}{(1+\varepsilon_0)2r} \geq \frac{\mathcal{H}^1(B(g,(1-\varepsilon_0)r))}{(1+\varepsilon_0)2r} > \frac{1-\varepsilon_0}{1+\varepsilon_0}\left(\frac{1}{2}+5\varepsilon_0\right) > \frac{1}{2}+\varepsilon_0. \quad (2.23)$$

Put $R = \min\{r_0(\varepsilon_0), \delta(\varepsilon_0)\}$, then (2.20) and (2.23) show that the Haar measure

$$\mu = \frac{1}{1+\varepsilon_0}\mathcal{H}^1$$

satisfies properties (2.1) and (2.2). Let $0 < \Delta < R/8$ be such that the ball $B(e,\Delta)$ is compact, then $B(e,\Delta)$ is not totally disconnected by Theorem 2.1. Hence there is a connected subset $C \subset B(e,\Delta)$ of positive $\mathcal{H}^1$ measure, as all $\mathcal{H}^1$-null sets are totally disconnected (see Lemma 4.1 in [11]). We may also suppose that C is closed (consider its closure if it is not), and therefore compact. A connected compact C with $\mathcal{H}^1(C) < \infty$ is rectifiable by Theorem 1.12.

Let us prove that the ball $B(e,\Delta)$ is rectifiable. According to (2.20) $\mathcal{H}^1$ measure of $U(e,4\Delta)$ is finite

$$\mathcal{H}^1(U(e,4\Delta)) \leq (1+\varepsilon_0)\operatorname{diam} U(e,4\Delta) < \frac{3}{2}R.$$

Let

$$U(e,4\Delta) = G_1 \cup G_2 \quad \text{and} \quad G_1 \cap G_2 = \emptyset,$$

where G_1 is rectifiable and G_2 is purely unrectifiable parts of $U(e,4\Delta)$. Since C is rectifiable subset of $B(e,\Delta)$ of positive $\mathcal{H}^1$ measure, then up to a null set

$$C \subset G_1 \cap U(e, 2\Delta)$$

and

$$\mathcal{H}^1(G_1 \cap U(e, 2\Delta)) > 0.$$

Suppose also that

$$\mathcal{H}^1(G_2 \cap U(e, 2\Delta)) > 0.$$

Notice that for any $g \in U(e, 2\Delta)$

$$U(e, 2\Delta) \subset gU(e, 4\Delta) = gG_1 \cup gG_2,$$

where gG_1 and gG_2 are rectifiable and purely unrectifiable parts of $gU(e, 4\Delta)$. Since the intersection of rectifiable and purely unrectifiable sets is always a null set, gG_1 and gG_2 coincide on $U(e, 2\Delta)$ with G_1 and G_2 respectively up to a null set. It follows that

$$\mathcal{H}^1(G_1 \cap B(g, r)) = \mathcal{H}^1(gG_1 \cap B(g, r)) = \mathcal{H}^1(G_1 \cap B(e, r))$$

for any $g \in U(e, 2\Delta)$ and small enough $r > 0$, and if one of densities $D_1(G_1, g)$ or $D_1(G_1, e)$ exists, then so does the other one and they are equal. The same is also true for densities $D_1(G_2, g)$ and $D_1(G_2, e)$.

The statement (iii) of Theorem 1.14 with $s = 1$, $X = U(e, 2\Delta)$, $A = G_1 \cap U(e, 2\Delta)$ and $\mu = \mathcal{H}^1$ implies that

$$D_1(G_1 \cap U(e, 2\Delta), g) = 0$$

for $\mathcal{H}^1$ almost all points

$$g \in U(e, 2\Delta) \setminus (G_1 \cap U(e, 2\Delta)) = G_2 \cap U(e, 2\Delta).$$

Let g_1 be one of such points. Therefore for small enough $r > 0$ we have

$$B(g_1, r) \subset U(e, 2\Delta),$$

and thus

$$\begin{aligned} 0 = D_1(G_1 \cap U(e, 2\Delta), g_1) &= \lim_{r \searrow 0} \frac{\mathcal{H}^1(G_1 \cap U(e, 2\Delta) \cap B(g_1, r))}{2r} \\ = \lim_{r \searrow 0} \frac{\mathcal{H}^1(G_1 \cap B(g_1, r))}{2r} &= D_1(G_1, g_1) = D_1(G_1, e). \end{aligned}$$

We obtain in a similar way that

$$D_1(G_2 \cap U(e, 2\Delta), g) = 0$$

for $\mathcal{H}^1$ almost all points

$$g \in U(e, 2\Delta) \setminus (G_2 \cap U(e, 2\Delta)) = G_1 \cap U(e, 2\Delta).$$

Let g_2 be one of such points, then

$$0 = D_1(G_2, g_2) = D_1(G_2, e).$$

Hence

$$D_1(G, e) = D_1(U(e, 4\Delta), e) = D_1(G_1, e) + D_1(G_2, e) = 0,$$

which contradicts to (2.22). It follows that $G_2 \cap U(e, 2\Delta)$ is a null set, thus the ball $U(e, 2\Delta)$ and its subsets, in particular $B(e, \Delta)$, are rectifiable.

Let E_0 be a subset of E satisfying (2.21) in every point $g \in E_0$, then $E = E_0$ up to $\mathcal{H}^1$-null set and $0 < \mathcal{H}^1(E_0) = \mathcal{H}^1(E) < \infty$. According to (2.21) for every $g \in E_0$ there is some $0 < r_g \leq \Delta/5$ such that

$$\mathcal{H}^1(E_0 \cap B(g, r)) > r, \quad 0 < r \leq r_g. \tag{2.24}$$

Therefore E_0 has the following covering

$$E_0 \subset \bigcup_{g \in E_0} B(g, r_g),$$

then by Remark 1.5

$$E_0 \subset \bigcup_{g \in E_1} B(g, 5r_g),$$

where $E_1 \subset E_0$ and the balls $B(g, r_g)$, $g \in E_1$, are disjoint. Notice that the union (set E_1) is countable, since: $\mathcal{H}^1(E_0) < \infty$,

$$\bigcup_{g \in E_1} (E_0 \cap B(g, r_g)) \subset E_0,$$

the sets in the union above are disjoint and have positive $\mathcal{H}^1$ measure by (2.24).

We have already shown that $B(e, \Delta)$ is rectifiable, then so is $B(g, \Delta)$ for any $g \in G$. The balls $B(g, 5r_g) \subset B(g, \Delta)$, $g \in E_1$, are also rectifiable, and so is E_0 which they cover. Therefore E is rectifiable and the proof is finished. $\square$

Chapter 3

Isodiametric Problem in Groups with Dilations

3.1 Isodiametric Inequality and Besicovitch 1/2-Problem

As we have already mentioned in Chapter 1, the *isodiametric inequality* states that a metric ball maximizes the volume for the given diameter. This fact is well known in Euclidean spaces $\mathbb{R}^n$ (see [10, 2.2] or [12, 2.10.33])

$$\mathcal{L}^n(A) \leq \alpha(n) \left(\frac{\operatorname{diam} A}{2} \right)^n, \quad A \subset \mathbb{R}^n. \tag{3.1}$$

We call a set which maximizes the volume for the given diameter the *isodiametric set*, alternatively we say that such a set has the *isodiametric property*. If a ball is not the isodiametric set, then we say that the isodiametric inequality or the isodiametric property fails for a ball.

In this chapter we show that the isodiametric property of balls may fail in non-Euclidean spaces, in particular in the Heisenberg group $\mathbb{H}^n$ $(n \in \mathbb{N})$ and in the additive group $\mathbb{R}^{n+1}$ $(n \in \mathbb{N})$ with non-isotropic dilations.

We also give estimates for the ratio of the volume of a ball to the maximal volume of a set of the same diameter.

We have already mentioned in Chapter 1 that the first counterexample to the generalized Besicovitch 1/2-conjecture has recently been constructed by Schechter [41]. As a consequence of our results we obtain simpler counterexamples to the generalized Besicovitch 1/2-conjecture in groups with dilations.

We will work in the setting of a locally compact group $(G, \cdot)$ equipped with

(i) *dilations* $\delta_r : G \to G$, $r > 0$, which form a group of automorphisms of G such that

$$\delta_1 = \text{identity} \quad \text{and} \quad \delta_{rs} = \delta_r \circ \delta_s, \quad r, s > 0,$$

(ii) a *left invariant* and *homogeneous with respect to dilations* metric d, i.e.

$$d(g \cdot g_1, g \cdot g_2) = d(g_1, g_2), \quad g, g_1, g_2 \in G,$$

and

$$d(\delta_r g_1, \delta_r g_2) = r d(g_1, g_2), \quad g_1, g_2 \in G, \ r > 0.$$

We call such a metric *homogeneous* or *compatible with left translations and dilations.*

Let e be the identity element of G and the Hausdorff measure $\mathcal{H}^n$ $(n \in \mathbb{N})$ be a left Haar measure on G. Properties of the metric imply that $\mathcal{H}^n$ is not only left invariant

$$\mathcal{H}^n(g \cdot A) = \mathcal{H}^n(A), \quad g \in G, \ A \subset G,$$

but also homogeneous with respect to dilations, i.e.

$$\mathcal{H}^n(\delta_r A) = r^n \mathcal{H}^n(A), \quad A \subset G.$$

In the next theorem we establish a connection among the isodiametric inequality, the density $D_n(G,e)$ and the "density constant" $\sigma_n(G)$.

Theorem 3.1. *Let G be a locally compact group with a homogeneous metric d and the left Haar measure $\mathcal{H}^n$, then statements (i) - (iii) hold.*

(i) For any $g \in G$

$$D_n(G,g) = D_n(G,e) = \frac{\mathcal{H}^n(B(e,1))}{\sup\{\mathcal{H}^n(D) \mid \operatorname{diam} D \le 2\}} \le 1.$$

(ii) If a diameter of a ball is double of its radius, then the isodiametric property of balls is equivalent to $D_n(G,e) = 1$.

(iii) If $D_n(G,e) < 1$, then the group G is purely n-unrectifiable and

$$\sigma_n(G) = D_n(G,e) < 1.$$

Proof. (i) According to Definition 1.10 n-density of G at $g \in G$ is

$$D_n(G,g) = \lim_{r \searrow 0} \frac{\mathcal{H}^n(B(g,r))}{(2r)^n}.$$

The homogeneity with respect to dilations and left invariance of $\mathcal{H}^n$ imply that

$$D_n(G,g) = \frac{\mathcal{H}^n(B(g,1))}{2^n} = \frac{\mathcal{H}^n(B(e,1))}{2^n} = D_n(G,e). \tag{3.2}$$

Let $\mu = \psi = \mathcal{H}^n$, then by Theorem 1.14(i, ii) one gets

$$\overline{D}_{\mathcal{H}^n}(\mathcal{H}^n, G, e) = \lim_{r \searrow 0} \sup \left\{ \frac{\mathcal{H}^n(S)}{(\operatorname{diam} S)^n} \mid e \in S,\ 0 < \operatorname{diam} S < r \right\} = 1.$$

Again by properties of $\mathcal{H}^n$ the last equality yields

$$\sup\{\mathcal{H}^n(D) \mid \operatorname{diam} D \leq 2\} = 2^n,$$

which together with (3.2) implies (i).

(ii) This follows from (i) and the fact that a diameter of a ball is double of its radius.

(iii) If we assume otherwise, G contains a n-rectifiable set of positive $\mathcal{H}^n$ measure, then by Theorem 1.1 and Theorem 1.14(iv) there has to be $g \in G$ such that $D_n(G, g) = 1$, and therefore $D_n(G, e) = 1$, which is a contradiction. It is clear that for purely n-unrectifiable group G the equation in (iii) holds. □

We use the last theorem to estimate $D_n(G, e)$ and $\sigma_n(G)$ when G is the Heisenberg group $\mathbb{H}^n$ ($n \in \mathbb{N}$) or the additive group $\mathbb{R}^{n+1}$ ($n \in \mathbb{N}$) with non-isotropic dilations. Since a left Haar measure is unique up to positive constant multiples, we may use not only Hausdorff, but also any other left Haar measure of the group G to simplify our estimates. The Lebesgue measure, which is easy to compute, will serve us as an alternative Haar measure in groups considered in this chapter. In order to give a lower estimate for $D_n(G, e)$ and $\sigma_n(G)$ as accurate as possible we will also need a "good" upper estimate for the volume of a set of the given diameter. When we know which set maximizes the volume for the given diameter, the density $D_n(G, e)$ can be calculated precisely.

Once we show that for some group $1/2 < \sigma_n(G) < 1$, it gives immediately a counterexample to the generalized Besicovitch 1/2-conjecture. But we don't emphasize specially those counterexamples in the rest of the chapter.

3.2 The Heisenberg Group

The Heisenberg group has become a subject of increasing interest in recent years. It finds its applications in a broad range of disciplines: quantum mechanics, harmonic analysis, complex analysis, partial differential equations and geometric measure theory, to name a few. The reader may consult [4], [29] or [42] for extensive and thorough information on the subject. We only deal with those aspects of the Heisenberg group that concern the isodiametric inequality, densities and surface measures (the last question is considered in Chapter 4).

The *Heisenberg group* $(\mathbb{H}^n, \cdot)$ $(n \in \mathbb{N})$ is the set

$$\mathbb{C}^n \times \mathbb{R} = \{(z,t) \mid z \in \mathbb{C}^n,\ t \in \mathbb{R}\}$$

with the multiplication law

$$(z,t) \cdot (w,s) = (z+w, t+s+2\operatorname{Im}\langle z,w\rangle), \quad (z,t),(w,s) \in \mathbb{C}^n \times \mathbb{R}. \tag{3.3}$$

It is a non-Abelian group with the identity element being the origin $\bar{0} = (0,0) \in \mathbb{C}^n \times \mathbb{R}$ and the inverse $(z,t)^{-1} = (-z,-t)$. We write an element of the group in several ways $(x,y,t) = (x+iy,t) = (z,t)$, where $x,y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $z \in \mathbb{C}^n$. It follows from (3.3) that the group operation can also be defined as

$$(x,y,t) \cdot (x',y',t') = (x+x', y+y', t+t'+2(\langle y,x'\rangle - \langle x,y'\rangle)), \tag{3.4}$$

where $(x,y,t),(x',y',t') \in \mathbb{H}^n$. The formula (3.3) also implies that the group law is invariant under rotations of the coordinate system of the underlying space $\mathbb{R}^{2n+1}$ about the last (vertical) direction.

In the Heisenberg group there are natural *left translations*

$$\tau_h h' = h \cdot h', \quad h, h' \in \mathbb{H}^n,$$

and *dilations*

$$\delta_r(z,t) = (rz, r^2 t), \quad r > 0.$$

The differential structure of $\mathbb{H}^n$ is determined by the following so-called *horizontal vector fields*

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n, \tag{3.5}$$

which are left invariant with respect to the group law ($x_j, y_j \in \mathbb{R}$ are components of vectors $x, y \in \mathbb{R}^n$). The *Carnot-Carathéodory (CC) metric* d_c on $\mathbb{H}^n$ is defined using horizontal vector fields via admissible curves as follows. A Lipschitz curve $\gamma : [0,1] \to \mathbb{H}^n$ is called *admissible* if its tangent is spanned by X_j and Y_j

$$\dot\gamma(u) = \sum_{j=1}^{n} c_j(u) X_j(\gamma(u)) + \sum_{j=1}^{n} c_{n+j}(u) Y_j(\gamma(u))$$

for a.e. $u \in [0,1]$ with $c_j \in L^1[0,1]$, $j = 1, 2, \ldots, 2n$. Then the Carnot-Carathéodory length of γ is given by

$$l(\gamma) = \int_0^1 \left(\sum_{j=1}^{2n} c_j^2(u) \right)^{1/2} du.$$

The Carnot-Carathéodory distance $d_c : \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty]$ between points $h, h' \in \mathbb{H}^n$ is the infimum of lengths of all admissible curves joining these points and infinity if no such curves exist

$$d_c(h, h') = \inf\{ l(\gamma) \mid \gamma \text{ is admissible}, \ \gamma(0) = h, \ \gamma(1) = h' \}.$$

It appears that any pair of points can be joined by admissible curves (Chow's theorem, see [8] and also [3], [18]) and the infimum is attained on so called *geodesic curves* (see [17, Theorem 1.10]). This metric is compatible with left translations and dilations and metrizes the Euclidean topology of $\mathbb{R}^{2n+1}$. The Hausdorff dimension of $\mathbb{H}^n$ with respect to the CC metric is $2n+2$, which is strictly greater than the topological dimension of the group, $2n+1$. $\mathcal{H}^{2n+2}$ and $\mathcal{L}^{2n+1}$ are left invariant and also right invariant Haar measures on $\mathbb{H}^n$ (see [3], [29] and [42]).

The equations of geodesic curves, and thus of spheres with respect to the CC metric can be found in the literature, see for instance [3], [16], [29] and [30]. It is known that the r-sphere ∂B_r in $\mathbb{H}^n$ is a hypersurface of revolution obtained by rotating the curve given by parametric equations (3.6) about the vertical T-axis (see Fig. 3.1 and 3.2)

$$x_r(\phi) = r\frac{\sin\phi}{\phi}, \quad t_r(\phi) = r^2\left(\frac{1}{\phi} - \frac{\sin 2\phi}{2\phi^2}\right), \quad |\phi| \leq \pi. \tag{3.6}$$

From now on we will only focus on the group $\mathbb{H}^1$, but all our results can easily be extended to Heisenberg groups of higher dimensions.

The parametrization of the sphere implies that the CC distance from the origin to a point on the $\mathbb{C}$-plane is just the Euclidean distance, $d_c(\bar{0},(z,0)) = |z|$. If $|z| = r$, then both $(z,0)$ and $(-z,0)$ are in B_r and

$$d_c((-z,0),(z,0)) = d_c(\bar{0},(-z,0)^{-1}(z,0)) = d_c(\bar{0},(2z,0)) = 2|z| = 2r,$$

therefore $\operatorname{diam} B_r = 2r$.

As the poles of the r-ball are $N = (0, r^2/\pi)$ and $S = (0, -r^2/\pi)$, we have the following relation between the Euclidean $|\cdot|$ and the CC distances from

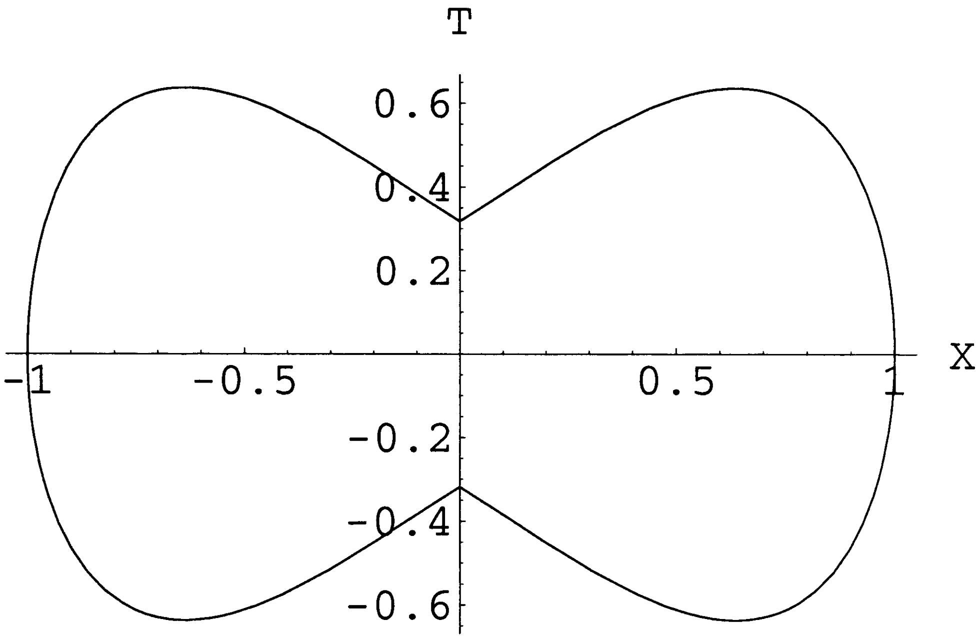

Figure 3.1: The central vertical section of the unit sphere ∂B_1

the origin to points on T-axis

$$|p| = \frac{r^2}{\pi} = \frac{d_c(\bar{0}, p)^2}{\pi}, \quad p \in \{N, S\}.$$

Then the distance between points on the same vertical line is

$$d_c((z,t),(z,s)) = d_c(\bar{0},(z,t)^{-1}(z,s)) = d_c(\bar{0},(0,s-t)) = \sqrt{\pi|s-t|}, \quad (3.7)$$

and therefore

$$d_c(N,S) = \sqrt{\pi \cdot \frac{2r^2}{\pi}} = \sqrt{2}r.$$

Let us prove the following statement.

Lemma 3.2. *One of the poles of B_r is the most distant point of the ball from another pole*

$$\sup\{d_c(S,h) \mid h \in B_r\} = d_c(N,S) = \sqrt{2}r$$

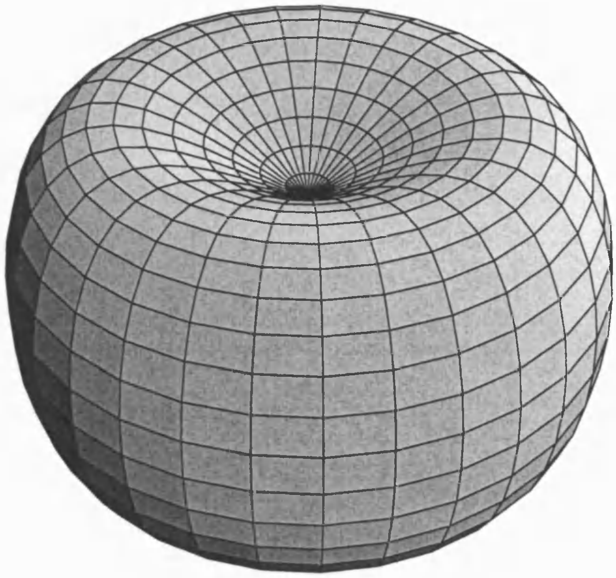

Figure 3.2: The unit ball B_1 in $\mathbb{H}^1$

and

$$\sup\{d_c(N,h) \mid h \in B_r\} = d_c(N,S) = \sqrt{2}r.$$

Proof. We will prove only the first formula, the second one is proved similarly. We have just shown the equality $d_c(N,S) = \sqrt{2}r$, hence we only need to check that $d_c(S,h) \leq \sqrt{2}r$, $h \in B_r$, or equivalently (see Fig. 3.3)

$$S^{-1}B_r \subset B_{\sqrt{2}r}. \tag{3.8}$$

Let

$$t_r(\phi) = r^2\tau\left(\frac{x_r(\phi)}{r}\right), \quad 0 \leq \phi \leq \pi. \tag{3.9}$$

Note that $\tau(0) = t_1(\pi) = 1/\pi$. Then the balls B_r and $B_{\sqrt{2}r}$ have the following representation

$$B_r = \left\{(z,t) \in \mathbb{H}^1 \mid |z| \leq r,\ |t| \leq r^2\tau\left(\frac{|z|}{r}\right)\right\},$$

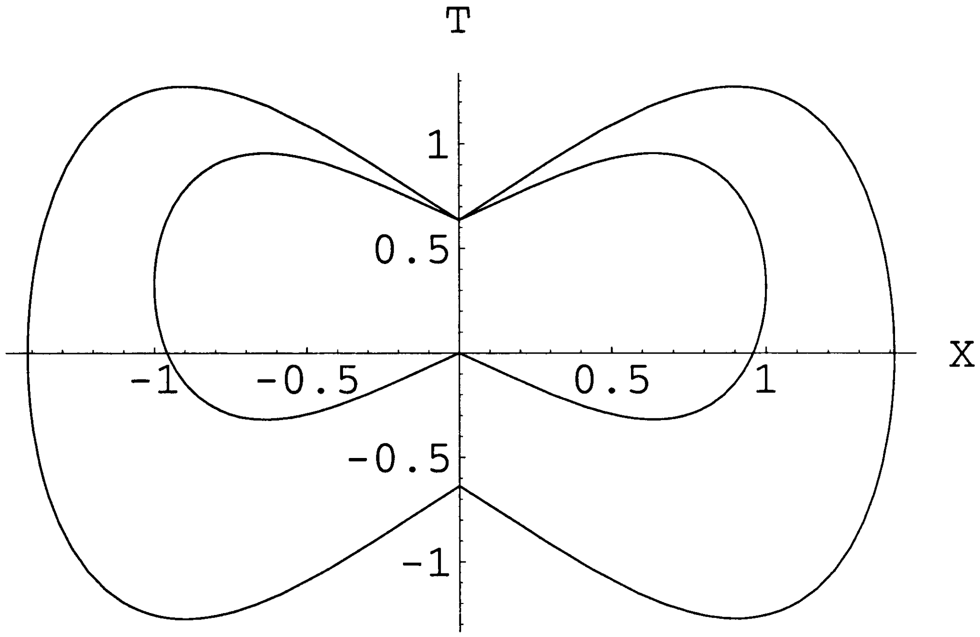

Figure 3.3: The central vertical section of spheres $S^{-1}\partial B_1$ and $\partial B_{\sqrt{2}}$

$$B_{\sqrt{2}r} = \left\{ (w,s) \in \mathbb{H}^1 \mid |w| \leq \sqrt{2}r,\ |s| \leq 2r^2\tau\left(\frac{|w|}{\sqrt{2}r}\right) \right\}.$$

As $S^{-1}B_r = (0, r^2/\pi)B_r$ the statement (3.8) is true once we show that for $(z,t) \in B_r$

$$\left| \frac{r^2}{\pi} + t \right| \leq 2r^2\tau\left(\frac{|z|}{\sqrt{2}r}\right)$$

or

$$\frac{r^2}{\pi} + r^2\tau\left(\frac{|z|}{r}\right) \leq 2r^2\tau\left(\frac{|z|}{\sqrt{2}r}\right).$$

Let $|z|/r = x \in [0,1]$, then we should check that for $0 \leq x \leq 1$

$$\frac{1}{\pi} + \tau(x) \leq 2\tau\left(\frac{x}{\sqrt{2}}\right). \tag{3.10}$$

For $x = 0$ we have the equality. If we show that

$$\frac{d}{dx}\left(-\frac{1}{\pi} - \tau(x) + 2\tau\left(\frac{x}{\sqrt{2}}\right)\right)\Bigg|_{x=0} = (\sqrt{2}-1)\frac{d\tau}{dx}(0) \geq 0$$

and for $0 \le x < 1$

$$\frac{d^2}{dx^2}\left(-\frac{1}{\pi} - \tau(x) + 2\tau\left(\frac{x}{\sqrt{2}}\right)\right) = \frac{d^2\tau}{dx^2}\left(\frac{x}{\sqrt{2}}\right) - \frac{d^2\tau}{dx^2}(x) \ge 0,$$

then the inequality (3.10) will follow immediately. It is enough to make sure that $d\tau/dx(0) \ge 0$ and $d^2\tau/dx^2(x)$ decreases on $[0,1)$.

Towards this end let us find the first three derivatives of t_r with respect to x_r

$$\frac{dt_r}{dx_r} = \frac{dt_r}{d\phi} \Big/ \frac{dx_r}{d\phi},$$

$$\frac{d^2t_r}{dx_r^2} = \frac{d}{dx_r}\left(\frac{dt_r}{dx_r}\right) = \frac{d}{dx_r}\left(\frac{dt_r}{d\phi} \Big/ \frac{dx_r}{d\phi}\right) = \frac{d}{d\phi}\left(\frac{dt_r}{d\phi} \Big/ \frac{dx_r}{d\phi}\right) \Big/ \frac{dx_r}{d\phi}$$

and

$$\frac{d^3t_r}{dx_r^3} = \frac{d}{dx_r}\left(\frac{d^2t_r}{dx_r^2}\right) = \frac{d}{d\phi}\left(\frac{d}{d\phi}\left(\frac{dt_r}{d\phi} \Big/ \frac{dx_r}{d\phi}\right) \Big/ \frac{dx_r}{d\phi}\right) \Big/ \frac{dx_r}{d\phi}.$$

Computations show that

$$\frac{dt_r}{dx_r}(\phi) = -r\frac{2\cos\phi}{\phi}, \tag{3.11}$$

$$\frac{d^2t_r}{dx_r^2}(\phi) = \frac{2(\cos\phi + \phi\sin\phi)}{\phi\cos\phi - \sin\phi} \tag{3.12}$$

and

$$\frac{d^3t_r}{dx_r^3}(\phi) = \frac{2\phi^4}{r(\phi\cos\phi - \sin\phi)^3}.$$

It is clear that

$$\frac{d\tau}{dx}(0) = \frac{dt_1}{dx_1}(\pi) = \frac{2}{\pi} \ge 0.$$

The expression

$$\phi\cos\phi - \sin\phi = \cos\phi(\phi - \tan\phi)$$

is negative for $0 < \phi \le \pi$ and vanishes at $\phi = 0$. Therefore $d^3t_r/dx_r^3(\phi)$ is also negative for $0 < \phi \le \pi$. Thus taking the definition (3.9) of τ into account we conclude that

$$\frac{d^2\tau}{dx^2}(x) = \frac{d^2t_1}{dx_1^2}(\phi), \quad x = x_1(\phi),\ 0 \le x < 1,\ 0 < \phi \le \pi,$$

and $d^2\tau/dx^2(x)$ decreases on $[0,1)$. □

The statement we have just proved allows us to add a "small" set to a pole of B_r without increasing the diameter of the ball but increasing its volume, which violates its isodiametric property.

Theorem 3.3. *The isodiametric inequality fails for the ball B_r.*

Proof. Let

$$D_r = B_r \cup B(N_1 N, (\sqrt{2}-1)r),$$

where $N = (0, r^2/\pi)$ and $N_1 = (0, (\sqrt{2}-1)^2 r^2/\pi)$ are north poles of B_r and $B_{(\sqrt{2}-1)r}$ respectively. Note that the south pole of $B(N_1N, (\sqrt{2}-1)r)$ is $N_1^{-1}N_1N = N$. Then $\operatorname{diam} D_r = \operatorname{diam} B_r = 2r$, since

$$\operatorname{diam} B(N_1N, (\sqrt{2}-1)r) = 2(\sqrt{2}-1)r < 2r$$

and by the previous lemma

$$d_c(h_1, h_2) \le d_c(h_1, N) + d_c(N, h_2) \le \sqrt{2}r + \sqrt{2}(\sqrt{2}-1)r = 2r$$

for any $h_1 \in B_r$, $h_2 \in B(N_1N, (\sqrt{2}-1)r)$.

Let us compare volumes of D_r and B_r. We will prove that there is ε such that

$$B(N_1N, \varepsilon) \subset B(N_1N, (\sqrt{2}-1)r) \setminus B_r.$$

Indeed, it is clear that

$$B(N_1N, \varepsilon) \subset B(N_1N, (\sqrt{2}-1)r) \quad \text{if} \quad \varepsilon \le (\sqrt{2}-1)r$$

and

$$B(N_1N,\varepsilon)\cap B_r=\emptyset \quad \text{if} \quad \varepsilon+r<d_c(\bar{0},N_1N).$$

According to the equation (3.7)

$$d_c(\bar{0},N_1N)=\sqrt{r^2+(\sqrt{2}-1)^2r^2},$$

then ε satisfies both conditions above if $\varepsilon < r\sqrt{1+(\sqrt{2}-1)^2}-r$. Since $\mathcal{L}^3(B(N_1N,\varepsilon))=\mathcal{L}^3(B_\varepsilon)>0$, it follows immediately that

$$\mathcal{L}^3(D_r)>\mathcal{L}^3(B_r).$$

□

Remark 3.4. The last theorem together with Theorem 3.1 implies that $\mathbb{H}^1$ is purely 4-unrectifiable and

$$\sigma_4(\mathbb{H}^1)=D_4(\mathbb{H}^1,\bar{0})<1.$$

Ambrosio and Kirchheim [1] have even proved that $\mathbb{H}^1$ is purely k-unrectifiable for $k=2,3,4$. However, authors point out that this statement doesn't hold for $k=1$. More general results of this kind can be found in Magnani's thesis [22]. The shortage of rectifiable sets in the Heisenberg group prompt researchers in the field to use an alternative, more intrinsic notion of rectifiability (see [13], [14], [28] and [33]). For example, authors of [13] and [14], Franchi, Serapioni and Serra Cassano, have successfully used a new notion of rectifiability replacing Lipschitzian images of subsets of the Euclidean space in the classical definition (see Definition 1.11) by level sets of C^1 (in the intrinsic sense involving the differential structure of $\mathbb{H}^n$) real-valued functions on the Heisenberg group.

Theorem 3.5.

$$\sigma_4(\mathbb{H}^1) = D_4(\mathbb{H}^1, \bar{0}) = \frac{\mathcal{L}^3(B_r)}{\sup\{\mathcal{L}^3(D) \mid \operatorname{diam} D \leq 2r\}} \geq 0.825.$$

Proof. Let us note that the orthogonal projection of $\mathbb{H}^1$ on the $\mathbb{C}$-plane with the Euclidean distance is Lipschitz with constant one

$$d_c((z,t),(w,s)) \geq |(z,0)-(w,0)| = |z-w|, \quad (z,t),(w,s) \in \mathbb{H}^1.$$

Indeed, we obtain

$$d_c((z,t),(w,s)) = d_c(\bar{0},(w-z, s-t-2\operatorname{Im}(z\bar{w}))) \geq d_c(\bar{0},(w-z,0)) = |w-z|,$$

since $(w-z, s-t-2\operatorname{Im}(z\bar{w})) \notin U_{|w-z|}$.

Let D be any set of the CC diameter at most δ. The closure of a set may only increase its $\mathcal{L}^3$ measure, but the diameter is unchanged by continuity of the metric d_c, therefore we may assume that D is closed, and so $\mathcal{L}^3$-measurable set. The Euclidean diameter of the orthogonal projection of D on the $\mathbb{C}$-plane is at most δ as well. According to the isodiametric inequality in Euclidean spaces (3.1) the area of this projection is at most the area of the disk of the same diameter

$$\mathcal{L}^2(\operatorname{proj}_{\mathbb{C}} D) \leq \pi \left(\frac{\delta}{2}\right)^2.$$

Let $(z,t),(z,s) \in D$, then by the equation (3.7)

$$\sqrt{\pi|s-t|} \leq \delta \quad \text{or} \quad |s-t| \leq \frac{\delta^2}{\pi}.$$

Thus the linear Lebesgue measure $\mathcal{L}^1$ of the intersection of D with a vertical line cannot be more than $\mathcal{L}^1$ of a vertical segment with the CC length δ, which is δ^2/π

$$\mathcal{L}^1(D \cap \{(z,t) \mid t \in \mathbb{R}\}) \leq \frac{\delta^2}{\pi}$$

for any $(z,0) \in \operatorname{proj}_{\mathbb{C}} D$. Fubini's theorem implies that

$$\mathcal{L}^3(D) \leq \pi \frac{\delta^2}{4} \cdot \frac{\delta^2}{\pi} = \frac{\delta^4}{4}.$$

On the other hand, the volume of the r-ball of diameter $\delta = 2r$ is

$$\mathcal{L}^3(B_r) = 2\pi \int_0^\pi x_r^2(\phi) t_r'(\phi)\, d\phi \approx 0.2064\delta^4. \tag{3.13}$$

The exact coefficient is slightly more, therefore

$$\frac{\mathcal{L}^3(B_r)}{\mathcal{L}^3(D)} \geq 4 \cdot 0.2064 \geq 0.825.$$

We apply Theorem 3.1 to conclude the proof. □

Remark 3.6. As we have already mentioned, arguments used for $\mathbb{H}^1$ can easily be extended to Heisenberg groups of higher dimensions. Thus the isodiametric inequality fails for CC balls in $\mathbb{H}^n$, by Theorem 3.1 $\mathbb{H}^n$ is purely $(2n+2)$-unrectifiable and

$$\sigma_{2n+2}(\mathbb{H}^n) = D_{2n+2}(\mathbb{H}^n, \bar{0}) < 1.$$

Let us calculate a lower bound for $\sigma_{2n+2}(\mathbb{H}^n)$ in the same way as we did for $\sigma_4(\mathbb{H}^1)$ in the last theorem. Let $\operatorname{diam} D \leq \delta = 2r$, then we have

$$\mathcal{L}^{2n+1}(D) \leq \alpha(2n) \left(\frac{\delta}{2}\right)^{2n} \cdot \frac{\delta^2}{\pi}$$

and

$$\begin{aligned}\mathcal{L}^{2n+1}(B_r) &= 2\alpha(2n) \int_0^\pi x_r^{2n}(\phi) t_r'(\phi)\, d\phi \\ &= 2\alpha(2n) \left(\frac{\delta}{2}\right)^{2n+2} \int_0^\pi \left(\frac{\sin\phi}{\phi}\right)^{2n} \left(\frac{1}{\phi} - \frac{\sin 2\phi}{2\phi^2}\right)' d\phi.\end{aligned}$$

This gives us the lower estimate for

$$\sigma_{2n+2}(\mathbb{H}^n) = D_{2n+2}(\mathbb{H}^n, \bar{0}) = \frac{\mathcal{L}^{2n+1}(B_r)}{\sup\{\mathcal{L}^{2n+1}(D) \mid \operatorname{diam} D \leq 2r\}},$$

which depends only on n and decreases to 0 as $n \to \infty$, but for n up to 8 it is still strictly greater than 1/2.

3.3 The Isodiametric Set

In this section we show that the convex hull of the ball B_r is isodiametric in the class of sets of revolution about the vertical T-axis having diameter $2r$. However, we don't know so far if this set is isodiametric in the class of all sets of diameter $2r$.

For simplicity we continue considering the Heisenberg group $\mathbb{H}^1$, but again only minor changes are needed to extend our results to $\mathbb{H}^n$. The main result of this section is preceded by several auxiliary statements.

Lemma 3.7. *The set bounded by the curve γ given by equations* (3.6), *T-axis and two tangents l_π and $l_{-\pi}$ to γ at points corresponding to $\phi = \pm\pi$ is convex (see Fig. 3.4).*

Proof. First of all we check the behaviour of d^2t_r/dx_r^2 (see the equation (3.12)). As already mentioned, $\cos\phi(\phi - \tan\phi)$, the denominator of the right-hand side of (3.12), is negative on $(0, \pi]$ and vanishes at $\phi = 0$. The numerator is $2\cos\phi(1 + \phi\tan\phi)$, it changes its sign on $[0, \pi]$ once from plus to minus at the point ϕ_0 such that $1 + \phi_0 \tan\phi_0 = 0$. Therefore $d^2t_r/dx_r^2(\phi)$ is negative on $(0, \phi_0)$, positive on $(\phi_0, \pi]$ and vanishes at $\phi = \phi_0$. It is clear that the part of the curve γ corresponding to $\phi \in (\phi_0, \pi]$ is above its tangent

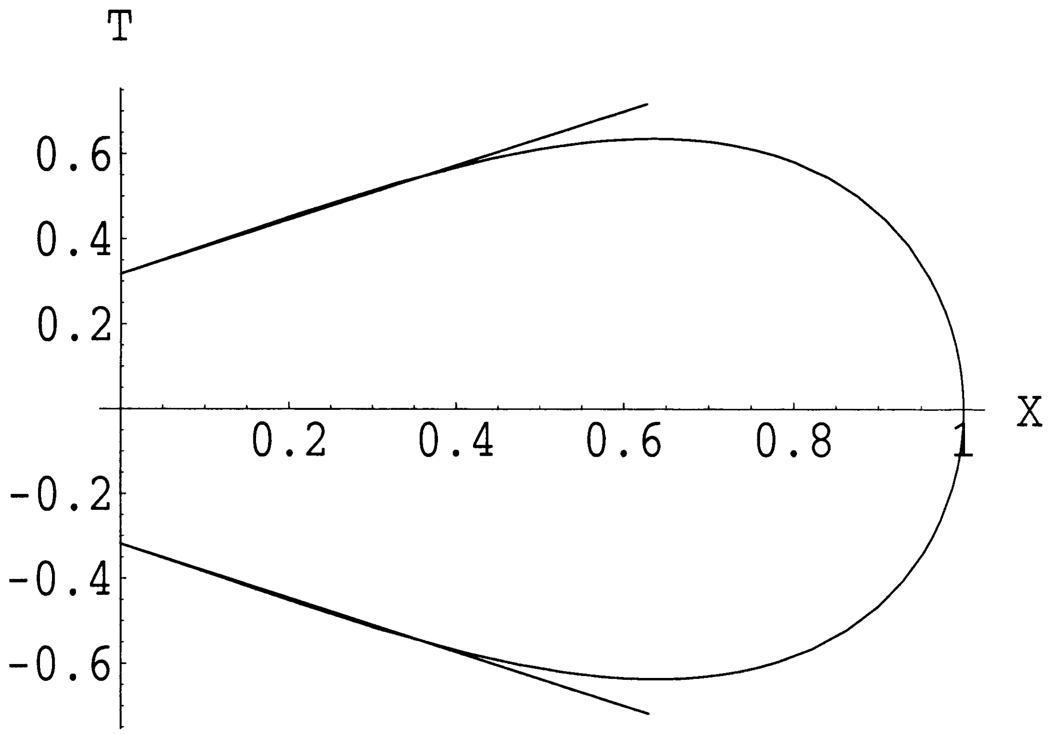

Figure 3.4: The curve γ given by equations (3.6) with tangents l_π and $l_{-\pi}$, $r = 1$

l_π at $\phi = \pi$. The parameter $\phi = \phi_0$ corresponds to the point of inflection of γ, therefore γ intersects l_π at some point with $\phi = \phi_1 < \phi_0$. Another curve composed of l_π between points $(x_r(\pi), t_r(\pi))$ and $(x_r(\phi_1), t_r(\phi_1))$ and γ between points $(x_r(\phi_1), t_r(\phi_1))$ and $(x_r(0), t_r(0))$ has non-positive curvature, and therefore concave. In order to conclude the proof we just note that γ and $l_\pi \cup l_{-\pi}$ are symmetric about the horizontal X-axis. $\square$

Lemma 3.8. *Let $z_1, z_2 \in \mathbb{C}$ and $|z_2| \leq |z_1|$, then*

$$|\operatorname{Im}(z_1\bar{z}_2)| \leq |z_2||z_1 - z_2|.$$

Proof. Let $\psi = \arg z_1 - \arg z_2$, then

$$\operatorname{Im}(z_1\bar{z}_2) = \operatorname{Im}(|z_1|\exp(\imath \arg z_1)|z_2|\exp(-\imath \arg z_2)) = |z_1||z_2|\sin\psi.$$

On the other hand, the cosine formula says

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\psi.$$

We have to check that

$$|z_1|^2|z_2|^2\sin^2\psi \leq |z_2|^2(|z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\psi)$$

or

$$|z_1|^2\sin^2\psi \leq |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\psi,$$

which is equivalent to

$$|z_1|^2\cos^2\psi + |z_2|^2 - 2|z_1||z_2|\cos\psi \geq 0.$$

Notice that the last expression is the full square of $|z_1|\cos\psi - |z_2|$, therefore the inequality holds. The equality in the original inequality is possible only if $z_2 = 0$ or $\cos\psi = |z_2|/|z_1|$ ($|z_1| \neq 0$). □

Lemma 3.9. *The diameter of any compact set $K \subset \mathbb{H}^1$ is always attained on its boundary.*

Proof. Since $d_c(h_1, h_2)$ is a continuous function on $\mathbb{H}^1 \times \mathbb{H}^1$, it attains a maximum on $K \times K$ equal to $\operatorname{diam} K$. Let this maximum be attained at some points $h_1, h_2 \in K$, $d_c(h_1, h_2) = \operatorname{diam} K$. We claim that it is necessary that $h_1, h_2 \in \partial K$. Let us assume that at least one of points, say h_2, is not on the boundary ∂K. Then h_2 is an internal point of K, hence there is some $\varepsilon > 0$ such that $B(h_2, \varepsilon) \subset K$. On the other hand, $h_2 \in \partial B(h_1, \operatorname{diam} K)$, therefore there is a point $h \in B(h_2, \varepsilon) \setminus B(h_1, \operatorname{diam} K) \subset K$ and $d_c(h_1, h) > \operatorname{diam} K$, which is a contradiction. □

Remark 3.10. A left (right) group translation is a linear transformation of $\mathbb{H}^1$, therefore a plane translates to a plane, a line to a line, and a pair of parallel planes (lines) to another pair of parallel planes (lines).

Lemma 3.11. *Let $\Gamma \subset \mathbb{H}^1$ be a plane parallel to the $\mathbb{C}$-plane, $h_2 \in \Gamma$, and $\Pi \subset \mathbb{H}^1$ be a vertical plane going through the vertical T-axis and the point $h_1^{-1}h_2$, where $h_1 = (z_1, t_1) \in \mathbb{H}^1$ and $h_2 = (z_2, t_2) \in \mathbb{H}^1$, $z_1 \neq z_2$. Then the tangent of the angle of inclination of the line $l = h_1^{-1}\Gamma \cap \Pi$ to the $\mathbb{C}$-plane is at most $2\min\{|z_1|, |z_2|\}$ and l intersects the vertical T-axis at the point $(0, t_2 - t_1)$.*

Proof. The statement of the lemma implies that

$$h_2 \in \Gamma \cap h_1\Pi \qquad \text{and} \qquad h_1 l = \Gamma \cap h_1\Pi.$$

Thus the line $h_1 l$ is parallel to the $\mathbb{C}$-plane and $h_2 \in h_1 l$. Since Π is a vertical plane going through $\bar{0}$ and $h_1^{-1}h_2$, $h_1\Pi$ is a vertical plane going through h_1 and h_2. The plane Γ and the line $h_1 l$ intersect the vertical line going through h_1 at the point $h = (z_1, t_2)$. It follows that l intersects the vertical T-axis at the point $h_1^{-1}h = (0, t_2 - t_1)$.

The vector $h_2 - h$ determines the direction of $h_1 l$ and the vector $h_1^{-1}h_2 - h_1^{-1}h$ determines the direction of l. Let us find coordinates of this vector

$$\begin{aligned} h_1^{-1}h_2 - h_1^{-1}h &= (-z_1 + z_2, -t_1 + t_2 - 2\operatorname{Im}(z_1\bar{z}_2)) - (0, -t_1 + t_2) \\ &= (-z_1 + z_2, -2\operatorname{Im}(z_1\bar{z}_2)). \end{aligned}$$

Therefore the tangent of the angle of inclination of l to the $\mathbb{C}$-plane is $2|\operatorname{Im}(z_1\bar{z}_2)|/|z_1 - z_2|$, which is bounded from above by $2\min\{|z_1|, |z_2|\}$ according to Lemma 3.8. $\square$

Let S_r be the convex hull of the ball B_r. It is clear ∂S_r is a surface of revolution generated by rotating the curve given by parametric equations (3.14) about the T-axis (see Fig. 3.5)

$$x_r(\phi) = r\frac{\sin\phi}{\phi}, \ |\phi| \leq \pi, \quad \hat{t}_r(\phi) = \begin{cases} r^2\left(\frac{1}{\phi} - \frac{\sin 2\phi}{2\phi^2}\right) & \text{if } |\phi| \leq \frac{\pi}{2}, \\ \text{sign}(\phi)\hat{t}_r\left(\frac{\pi}{2}\right) & \text{if } \frac{\pi}{2} < |\phi| \leq \pi. \end{cases} \tag{3.14}$$

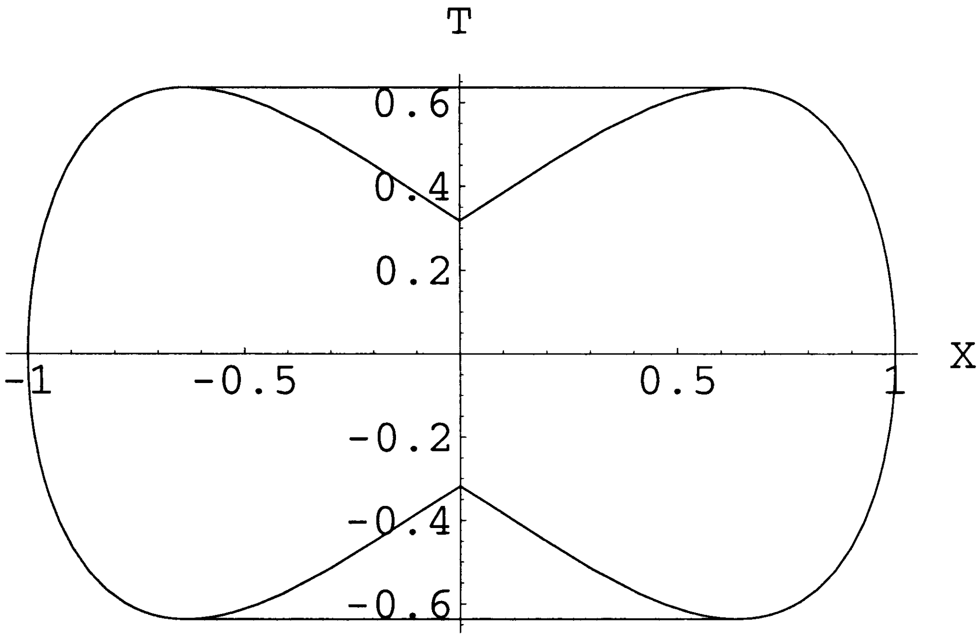

Figure 3.5: The central vertical section of ∂S_1 and the unit sphere ∂B_1

Theorem 3.12.

$$\operatorname{diam} S_r \leq 2r.$$

Proof. According to Lemma 3.9 to prove this statement we only need to show that

$$\operatorname{diam} \partial S_r \leq 2r.$$

Let us denote $A_r = \partial S_r \cap \partial B_r$ and $C_r = \partial S_r \setminus \partial B_r$, then $\partial S_r = A_r \cup C_r$ and $A_r \cap C_r = \emptyset$. We already know that $\operatorname{diam} \partial B_r \leq 2r$, therefore $\operatorname{diam} A_r \leq 2r$, and we only have to check that $d_c(h_1, h_2) \leq 2r$ or $h_1^{-1}h_2 \in B_{2r}$ for any $h_1 \in A_r \cup C_r$ and any $h_2 \in C_r$.

Consider a section of the set $h_1^{-1}\partial S_r$ by a vertical plane Π going through the vertical T-axis and the point $h_1^{-1}h_2$, then

$$h_1^{-1}h_2 \in h_1^{-1}C_r \cap \Pi.$$

Without loss of generality we may assume that $h_1^{-1}h_2$ is on the XT-plane and Π coincides with this plane. Note that C_r consists of two disks lying on parallel planes. Each disk is an open subset of the corresponding plane with the boundary in A_r. By Remark 3.10 the set $h_1^{-1}C_r$ also lies on two parallel planes. Since $h_1^{-1}C_r$ is just the linear transformation of two disks, we have

$$h_1^{-1}C_r \cap \Pi = l_1 \cup l_2,$$

where l_1 and l_2 are two parallel segments on the plane Π (they may also be points, but it only simplifies the analysis). It follows that endpoints of these segments are in $h_1^{-1}A_r$.

Let $h_1^{-1}h_2 \in l_1$, l be the line on which l_1 lies, $h_1 = (z_1, t_1) \in A_r \cup C_r$ and $h_2 = (z_2, t_2) \in C_r$, $z_1 \neq z_2$. Then by Lemma 3.11 the line l intersects the vertical T-axis at the point $h_1^{-1}h = (0, t_2 - t_1)$, $h = (z_1, t_2)$. Since $|t_2 - t_1| \leq 2\hat{t}_r(\pi/2) = t_{2r}(\pi)$, the point $h_1^{-1}h$ lies on the vertical T-axis between the poles of the ball B_{2r}. We observe that if $z_1 = z_2$, then $h_2 = h$ and $h_1^{-1}h_2 = h_1^{-1}h \in B_{2r}$, which concludes the proof.

Lemma 3.11 also says that the line l has the tangent of the angle of inclination to the $\mathbb{C}$-plane at most

$$2\min\{|z_1|,|z_2|\} \leq 2|z_2| \leq 2x_r\left(\frac{\pi}{2}\right) = \frac{4r}{\pi}.$$

As we can see from the equation (3.11)

$$\left|\frac{dt_{2r}}{dx_{2r}}(\pm\pi)\right| = \frac{4r}{\pi},$$

thus the part of the line l lying in the right half plane of Π with respect to the T-axis is always between the tangents l_π and $l_{-\pi}$ to the curve γ corresponding to B_{2r} (see Fig. 3.4). If we show that endpoints of l_1 are in $B_{2r}\cap\Pi$, then Lemma 3.7 will imply that for the whole segment, i.e. $l_1 \subset B_{2r}\cap\Pi$.

Let us consider two cases $h_1 \in A_r$ and $h_1 \in C_r$.

If $h_1 \in A_r$, then endpoints of l_1, being in $h_1^{-1}A_r$, must also be in $B_{2r}\cap\Pi$, since $\bar{0} \in h_1^{-1}A_r$ and

$$\operatorname{diam}(h_1^{-1}A_r) = \operatorname{diam} A_r \leq 2r.$$

As we have just discussed, it implies that $h_1^{-1}h_2 \in l_1 \subset B_{2r}\cap\Pi$, and therefore

$$d_c(h_1,h_2) = d_c(\bar{0}, h_1^{-1}h_2) \leq 2r \tag{3.15}$$

for any $h_1 \in A_r$ and $h_2 \in C_r$. Note that $l\cap T = h_1^{-1}h$ (or $h_1 l\cap h_1 T = h$), but $l_1\cap T = \emptyset$ or $h_1 l_1 \cap h_1 T \subset C_r \cap h_1 T = \emptyset$, since projections of A_r and C_r on the $\mathbb{C}$-plane are disjoint.

If $h_1 \in C_r$, then endpoints of l_1, being in $h_1^{-1}A_r$, must be in $B_{2r}\cap\Pi$, because according to (3.15)

$$\sup_{h'\in A_r} d_c(\bar{0}, h_1^{-1}h') = \sup_{h'\in A_r} d_c(h_1,h') \leq 2r.$$

It follows that $l_1 \subset B_{2r}\cap\Pi$, and therefore $d_c(h_1,h_2) \leq 2r$ for any $h_1,h_2 \in C_r$ or $\operatorname{diam} C_r \leq 2r$ by the same reason used to derive (3.15). Note also that

$l_1 \cap T = h_1^{-1}h$ (or $h_1 l_1 \cap h_1 T = h$). Thus we have shown that $d_c(h_1, h_2) \leq 2r$ for any $h_1 \in A_r \cup C_r$ and $h_2 \in C_r$, as required. □

Lemma 3.13.

$$d_c((z_1,t_1),(z_2,t_2)) \leq d_c((z_1,s_1),(z_2,s_2))$$

if and only if

$$|t_2 - t_1 - 2\,\mathrm{Im}(z_1\bar{z}_2)| \leq |s_2 - s_1 - 2\,\mathrm{Im}(z_1\bar{z}_2)|.$$

Proof. Let

$$d_1 = d_c((z_1,t_1),(z_2,t_2)) = d_c(\bar{0},(z_1,t_1)^{-1}(z_2,t_2))$$

and

$$d_2 = d_c((z_1,s_1),(z_2,s_2)) = d_c(\bar{0},(z_1,s_1)^{-1}(z_2,s_2)).$$

By the group multiplication law (3.3) one gets

$$(z_1,t_1)^{-1}(z_2,t_2) = (z_2 - z_1, t_2 - t_1 - 2\,\mathrm{Im}(z_1\bar{z}_2)) \in \partial B_{d_1}$$

and similarly

$$(z_1,s_1)^{-1}(z_2,s_2) = (z_2 - z_1, s_2 - s_1 - 2\,\mathrm{Im}(z_1\bar{z}_2)) \in \partial B_{d_2}.$$

It follows that

$$(z_1,t_1)^{-1}(z_2,t_2) \in B_{d_2},$$

or equivalently $d_1 \leq d_2$ if and only if

$$|t_2 - t_1 - 2\,\mathrm{Im}(z_1\bar{z}_2)| \leq |s_2 - s_1 - 2\,\mathrm{Im}(z_1\bar{z}_2)|.$$

□

Lemma 3.14. *For any* $h_1 \in \partial S_r$ *there is* $h_2 \in \partial S_r$ *such that* $d_c(h_1, h_2) = 2r$.

Proof. Let $x_r(\phi)$ and $\hat{t}_r(\phi)$ be given by equations (3.14) and $h_1 = (z, \hat{t}_r(\phi)) \in \partial S_r$, where $|z| = x_r(\phi)$, $|\phi| \leq \pi$. If $|\phi| \leq \pi/2$, we put $h_2 = (z\exp(\imath\pi + 2\imath\phi), -\hat{t}_r(\phi)) \in \partial S_r$. Let us find the distance between points h_1 and h_2. Let

$$\begin{aligned} h_2^{-1}h_1 &= (z\exp(\imath\pi + 2\imath\phi), -\hat{t}_r(\phi))^{-1}(z, \hat{t}_r(\phi)) \\ &= (z - z\exp(\imath\pi + 2\imath\phi), 2\hat{t}_r(\phi) - 2\operatorname{Im}(z\exp(\imath\pi + 2\imath\phi)\bar{z})) = (w, s), \end{aligned}$$

then one gets

$$\begin{aligned} |w| &= |z(1 + \exp(2\imath\phi))| = |z\exp(\imath\phi)(\exp(-\imath\phi) + \exp(\imath\phi))| \\ &= 2|z|\cos\phi = 2x_r(\phi)\cos\phi = 2r\frac{\sin 2\phi}{2\phi} = x_{2r}(2\phi) \end{aligned}$$

and

$$\begin{aligned} s &= 2\hat{t}_r(\phi) + 2|z|^2\operatorname{Im}(\exp(2\imath\phi)) = 2\hat{t}_r(\phi) + 2x_r^2(\phi)\sin 2\phi \\ &= 2r^2\left(\frac{1}{\phi} - \frac{\sin 2\phi}{2\phi^2}\right) + 2\left(r\frac{\sin\phi}{\phi}\right)^2 \sin 2\phi \\ &= 2r^2\left(\frac{1}{\phi} - \frac{\sin 2\phi}{2\phi^2}\right) + r^2\frac{1 - \cos 2\phi}{\phi^2}\sin 2\phi \\ &= (2r)^2\left(\frac{1}{2\phi} - \frac{\sin 4\phi}{2(2\phi)^2}\right) = t_{2r}(2\phi). \end{aligned}$$

If $\pi/2 < |\phi| \leq \pi$, we put $h_2 = (z, -\hat{t}_r(\phi)) \in \partial S_r$, then

$$\begin{aligned} h_2^{-1}h_1 &= (z, -\hat{t}_r(\phi))^{-1}(z, \hat{t}_r(\phi)) = (0, 2\hat{t}_r(\phi)) = \left(0, 2\operatorname{sign}(\phi)\hat{t}_r\left(\frac{\pi}{2}\right)\right) \\ &= (x_{2r}(\pi), \operatorname{sign}(\phi)t_{2r}(\pi)). \end{aligned}$$

In any case it follows that $h_2^{-1}h_1 \in \partial B_{2r}$ or $d_c(h_1, h_2) = 2r$. □

Theorem 3.15. *S_r has the maximal $\mathcal{L}^3$ measure among all sets of revolution about the vertical T-axis of diameter at most $2r$.*

Proof. Let D be any set of revolution about the vertical T-axis of the CC diameter at most $2r$. As explained in Theorem 3.5, we may assume that D is a closed set. Then the Euclidean diameter of the orthogonal projection of D on the $\mathbb{C}$-plane is at most $2r$ as well. This projection is a set of revolution about the origin, therefore it is contained in the closed disk of the $\mathbb{C}$-plane centred at the origin and having the Euclidean radius r. Let

$$D_z = D \cap \{(z,t) \mid t \in \mathbb{R}\}$$

and

$$u_z = \sup\{t \in \mathbb{R} \mid (z,t) \in D_z\}, \quad l_z = \inf\{t \in \mathbb{R} \mid (z,t) \in D_z\},$$

where $|z| \leq r$, and let $x_r(\phi)$ and $\hat{t}_r(\phi)$ be given by equations (3.14). We claim that for any z such that $|z| = x_r(\phi) \leq r$, $0 \leq \phi \leq \pi$,

$$u_z - l_z \leq 2\hat{t}_r(\phi). \tag{3.16}$$

Let $g_1 = (z, \hat{u}_z) \in D$ and $h_1 = (z, \hat{t}_r(\phi))$. If $0 \leq \phi \leq \pi/2$, then using the fact that D is a set of revolution, we may choose points

$$g_2 = (z\exp(\imath\pi + 2\imath\phi), \hat{l}_z) \in D,\ \hat{l}_z \leq \hat{u}_z, \quad \text{and} \quad h_2 = (z\exp(\imath\pi + 2\imath\phi), -\hat{t}_r(\phi)).$$

Since $d_c(g_2, g_1) \leq d_c(h_2, h_1) = 2r$ (Lemma 3.14), then Lemma 3.13 states that

$$|\hat{u}_z - \hat{l}_z - 2\,\mathrm{Im}(z\exp(\imath\pi + 2\imath\phi)\bar{z})| \leq |2\hat{t}_r(\phi) - 2\,\mathrm{Im}(z\exp(\imath\pi + 2\imath\phi)\bar{z})|$$

or

$$\hat{u}_z - \hat{l}_z + 2|z|^2 \sin 2\phi \leq 2\hat{t}_r(\phi) + 2|z|^2 \sin 2\phi,$$

and therefore $\hat{u}_z - \hat{l}_z \leq 2\hat{t}_r(\phi)$.

If $\pi/2 < \phi \leq \pi$, we choose

$$g_2 = (z, \hat{l}_z) \in D,\ \hat{l}_z \leq \hat{u}_z, \quad \text{and} \quad h_2 = (z, -\hat{t}_r(\phi)).$$

As $d_c(g_2, g_1) \leq d_c(h_2, h_1) = 2r$ (Lemma 3.14) again Lemma 3.13 implies that

$$\hat{u}_z - \hat{l}_z \leq 2\hat{t}_r(\phi).$$

We conclude that the equation (3.16) holds, thus for any z such that $|z| = x_r(\phi) \leq r$, $0 \leq \phi \leq \pi$,

$$\mathcal{L}^1(D_z) \leq u_z - l_z \leq 2\hat{t}_r(\phi).$$

The proof is finished by applying Fubini's theorem. □

Remark 3.16. Now we can make an upper estimate for $\sigma_4(\mathbb{H}^1)$ more accurate than just stating that $\sigma_4(\mathbb{H}^1) < 1$ (see Remark 3.4). Using parametric equations (3.14) we find

$$\mathcal{L}^3(S_r) = 2\pi \int_0^{\frac{\pi}{2}} x_r^2(\phi) t_r'(\phi)\, d\phi \approx 0.2176(2r)^4,$$

which together with (3.13) yields

$$\sigma_4(\mathbb{H}^1) = D_4(\mathbb{H}^1, \bar{0}) \leq \frac{\mathcal{L}^3(B_r)}{\mathcal{L}^3(S_r)} \leq \frac{0.2065}{0.2176} \leq 0.949.$$

Taking also the lower estimate into account (Theorem 3.5) we conclude

$$0.825 \leq \sigma_4(\mathbb{H}^1) \leq 0.949.$$

3.4 The Group $\mathbb{R}_d^{n+1}$

The same idea works also in the additive group $\mathbb{R}^{n+1}$ $(n \in \mathbb{N})$ with dilations

$$\delta_r(x,t) = (rx, r^\omega t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \ r > 0, \ \omega = 2, 3, \dots,$$

and a metric d compatible with translations and dilations of the group. We denote such a metric space as $\mathbb{R}_d^{n+1}$ and the hyperplane $t = 0$ as X. In this section we prove that the ratio of the volume of a ball to the maximal volume of a set of the same diameter may converge to 1, but the rate of such convergence cannot be faster than $1/n^2$ as the dimension n grows.

In the trivial case $\omega = 1$ the group $\mathbb{R}_d^{n+1}$ is just a normed vector space (d-distance from the origin is a norm) and the isodiametric inequality always holds for balls (follows from Theorem 1.1 and Theorem 3.1(ii)).

It is clear that $\mathcal{H}^{n+\omega}$ and $\mathcal{L}^{n+1}$ are Haar measures of the group. The identity element of the group is the origin $\bar{0} = (0,0) \in \mathbb{R}^n \times \mathbb{R}$. Observe that d-distance from the origin is some norm $\|\cdot\|$ on the hyperplane X, since the metric d is compatible with translations and dilations of $\mathbb{R}_d^{n+1}$. Let $(0,1) \in \partial B_1$, then the distance on the vertical axis is

$$d((0,t),(0,s)) = |s-t|^{1/\omega}. \tag{3.17}$$

Lemma 3.17. *The orthogonal projection of $\mathbb{R}_d^{n+1}$ on the hyperplane X is Lipschitz with constant one.*

Proof. Using the translation invariance of the metric it is enough to prove that $d(\bar{0},(x,0)) \le d(\bar{0},(x,t))$ for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. If $d(\bar{0},(x,t)) = \delta$, we have

$$2\delta \ge d((-x,-t),(x,t)) = d(\bar{0},(2x,2t)) = 2d(\bar{0},(x,2^{1-\omega}t))$$

or $d(\bar{0},(x,2^{1-\omega}t)) \leq \delta$. Continuing in this way we get

$$d(\bar{0},(x,2^{n(1-\omega)}t)) \leq \delta$$

for any $n \in \mathbb{N}$. It follows that

$$d(\bar{0},(x,0)) - d((x,0),(x,2^{n(1-\omega)}t)) \leq d(\bar{0},(x,2^{n(1-\omega)}t)) \leq \delta,$$

where by the property (3.17)

$$d((x,0),(x,2^{n(1-\omega)}t)) = d(\bar{0},(0,2^{n(1-\omega)}t)) = (2^{n(1-\omega)}t)^{1/\omega}.$$

Since $1-\omega < 0$, the last expression approaches zero as $n \to \infty$, therefore $d(\bar{0},(x,0)) \leq \delta$. □

Lemma 3.18. *The metric d induces the same topology on $\mathbb{R}^{n+1}_d$ as the Euclidean one.*

Proof. First of all we notice that for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$

$$d(\bar{0},(x,t)) \leq d(\bar{0},(x,0)) + d((x,0),(x,t)) = \|x\| + |t|^{1/\omega} \leq 2\max\{\|x\|,|t|^{1/\omega}\}. \tag{3.18}$$

On the other hand, Lemma 3.17 states that

$$d(\bar{0},(x,t)) \geq d(\bar{0},(x,0)) = \|x\|, \tag{3.19}$$

therefore

$$|t|^{1/\omega} = d((x,0),(x,t)) \leq d((x,0),\bar{0}) + d(\bar{0},(x,t)) \leq 2d(\bar{0},(x,t))$$

and

$$d(\bar{0},(x,t)) \geq \frac{1}{2}|t|^{1/\omega}.$$

Combining the last estimate with (3.19) we obtain

$$d(\bar{0},(x,t)) \geq \max\left\{\|x\|, \frac{1}{2}|t|^{1/\omega}\right\}. \tag{3.20}$$

The standard result of functional analysis says that all norms on $\mathbb{R}^{n+1}$ as a finite dimensional vector space induce the Euclidean topology. We will use the norm $\max\{\|x\|,|t|\}$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

Another standard topological result says that two metrics induce the same topology on a metric space if and only if any open ball with respect to each one of two metrics contains a concentric open ball with respect to the second metric.

We would like to show that the metric d induces the same topology as the norm $\max\{\|x\|,|t|\}$. Since d is translation invariant, we only need to check the above statement for balls centred at the origin, that is, we need to make sure that

(i) $\forall r > 0\ \exists r' > 0$ such that $\forall (x,t) \in \mathbb{R}^n \times \mathbb{R}$ if $\max\{\|x\|,|t|\} < r'$, then $d(\bar{0},(x,t)) < r$,

(ii) $\forall r > 0\ \exists r'' > 0$ such that $\forall (x,t) \in \mathbb{R}^n \times \mathbb{R}$ if $d(\bar{0},(x,t)) < r''$, then $\max\{\|x\|,|t|\} < r$.

The estimate (3.18) implies that

$$d(\bar{0},(x,t)) \leq 2\max\{\|x\|,|t|^{1/\omega}\} \leq 2(\max\{\|x\|,|t|\})^{1/\omega}$$

if $\max\{\|x\|,|t|\} < r' \leq 1$ (then $\|x\| < 1$). Hence the statement (i) holds for $r' \leq \min\{(r/2)^\omega, 1\}$. On the other hand, the estimate (3.20) leads to

$$\max\{\|x\|,|t|\} \leq 2\max\left\{\|x\|, \frac{1}{2}|t|^{1/\omega}\right\} \leq 2d(\bar{0},(x,t))$$

if $d(\bar{0},(x,t)) < r'' \leq 1/2$ (then $|t| < 1$ by (3.20)). So we may take $r'' \leq \min\{r/2, 1/2\}$ to satisfy the statement (ii). □

Remark 3.19. Lemma 3.18 implies that the closure of a set $D \subset \mathbb{R}^{n+1}$ in d-metric topology is closed in the Euclidean topology, thus $\mathcal{L}^{n+1}$-measurable. The closure of a set may only increase its $\mathcal{L}^{n+1}$ measure, but the diameter is unchanged by continuity of the metric d.

Remark 3.20. Let

$$d_\infty(\bar{0},(x,t)) = \max\left\{\|x\|, \frac{1}{2}|t|^{1/\omega}\right\}$$

and

$$d_1(\bar{0},(x,t)) = \|x\| + |t|^{1/\omega}$$

be translation invariant metrics. It follows that

$$d_\infty(\bar{0},(x,t)) \leq d(\bar{0},(x,t)) \leq d_1(\bar{0},(x,t)),$$

therefore the unit ball with respect to d contains the unit ball with respect to d_1 and itself is contained in the unit ball with respect to d_∞ (all balls being concentric).

Lemma 3.21. *The triangle inequality holds for the metric d if and only if*

$$\delta_\alpha(x,t) + \delta_{1-\alpha}(y,s) \in B_1 \tag{3.21}$$

for any $(x,t),(y,s) \in B_1$ and any $0 < \alpha < 1$.

Proof. If the triangle inequality holds for d, then (3.21) follows easily.

Now suppose that the condition (3.21) holds. The triangle inequality is true if

$$d(\bar{0}, \delta_R(x,t) + \delta_r(y,s)) \leq d(\bar{0}, \delta_R(x,t)) + d(\bar{0}, \delta_r(y,s)) = R + r$$

for any $(x,t),(y,s) \in B_1$ and any $R > 0$, $r > 0$. It is equivalent to

$$d(\bar{0}, \delta_{(R+r)^{-1}}(\delta_R(x,t) + \delta_r(y,s))) \leq 1$$

or

$$d(\bar{0}, \delta_{R(R+r)^{-1}}(x,t) + \delta_{r(R+r)^{-1}}(y,s)) \leq 1,$$

which follows from (3.21), if we put $\alpha = R(R+r)^{-1} \in (0,1)$. □

Remark 3.22.

$$\begin{aligned}\delta_\alpha(x,t) + \delta_{1-\alpha}(y,s) &= (\alpha x, \alpha^\omega t) + ((1-\alpha)y, (1-\alpha)^\omega s) \\ &= (\alpha x + (1-\alpha)y, \alpha^\omega t + (1-\alpha)^\omega s).\end{aligned}$$

Lemma 3.23.

$$\operatorname{proj}_X B_1 = \{(x,0) \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}.$$

Moreover, if $(x,t),(x,s) \in B_1$, $t \neq s$, then the whole segment connecting these points is in B_1.

Proof. Lemma 3.17 implies that if $(x,t) \in B_1$, then $(x,0) \in B_1$, therefore

$$\operatorname{proj}_X B_1 = B_1 \cap X.$$

It is clear that

$$B_1 \cap X = \{(x,0) \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\},$$

and the first part of the statement of the lemma follows.

Let $(x,t),(x,s) \in B_1$, $t \neq s$, then Lemma 3.21 and Remark 3.22 imply that for any $0 \leq \alpha \leq 1$

$$(x, \alpha^\omega t + (1-\alpha)^\omega s) \in B_1.$$

Let us define the function

$$\psi(\alpha) = \alpha^\omega t + (1-\alpha)^\omega s, \quad 0 \le \alpha \le 1.$$

The function ψ is continuous on $[0,1]$, $\psi(0) = s$ and $\psi(1) = t$. Therefore by Intermediate Value Theorem ψ takes any value between s and t, and thus the whole segment connecting (x,t) and (x,s) is in B_1. □

Lemma 3.24. *Let the unit ball B_1 be symmetric about the hyperplane X. If*

$$|t_2 - t_1| \le |s_2 - s_1|, \tag{3.22}$$

then

$$d((x_1,t_1),(x_2,t_2)) \le d((x_1,s_1),(x_2,s_2)).$$

Proof. Let

$$d_1 = d((x_1,t_1),(x_2,t_2)) = d(\bar{0},(x_2-x_1,t_2-t_1))$$

and

$$d_2 = d((x_1,s_1),(x_2,s_2)) = d(\bar{0},(x_2-x_1,s_2-s_1)).$$

The statement of the lemma is trivial if $s_2 - s_1 = 0$, therefore we may assume that $s_2 - s_1 \ne 0$. Since the unit ball is symmetric about the hyperplane X, so is $B_{d_2} = \delta_{d_2} B_1$. Thus $a = (x_2 - x_1, s_2 - s_1)$ and symmetric about X point $b = (x_2 - x_1, s_1 - s_2)$, $a \ne b$, are in B_{d_2} or $\delta_{1/d_2} a, \delta_{1/d_2} b \in B_1$. Therefore by Lemma 3.23 the segment connecting $\delta_{1/d_2} a$ and $\delta_{1/d_2} b$ is in B_1, or equivalently the segment connecting a and b is in B_{d_2}. The assumption (3.22) guarantees that the point $(x_2 - x_1, t_2 - t_1)$ lies on the latter segment, thus $d_1 \le d_2$. □

Lemma 3.25. *If*

$$\{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq 1,\ |t| \leq 1\} \subset B_1, \tag{3.23}$$

then the set

$$D_r = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq r,\ |t| \leq 2^{\omega-1} r^{\omega}\} \tag{3.24}$$

has diameter $2r$ and maximizes $\mathcal{L}^{n+1}$ measure for this diameter, it is the isodiametric set.

Proof. We use the argument similar to the one used for the Heisenberg group in the proof of Theorem 3.5. Let D be any closed set of diameter $2r$ (see Remark 3.19). By Lemma 3.17 the diameter of its projection on the hyperplane X is at most $2r$. As mentioned at the beginning of this section, the isodiametric inequality holds for balls in the normed vector space, therefore we have

$$\mathcal{L}^n(\operatorname{proj}_X D) \leq \alpha_X(n) r^n,$$

where $\alpha_X(n)$ is the volume of the unit ball $\{(x,0) \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ in the normed hyperplane X.

The intersection of D with any vertical line again has diameter at most $2r$, and hence its linear Lebesgue measure $\mathcal{L}^1$ is at most $\mathcal{L}^1$ of a vertical segment of the length $2r$ with respect to the metric d, which is $(2r)^{\omega}$

$$\mathcal{L}^1(D \cap \{(x,t) \mid t \in \mathbb{R}\}) \leq (2r)^{\omega}$$

for any $(x,0) \in \operatorname{proj}_X D$. Applying Fubini's theorem one gets

$$\mathcal{L}^{n+1}(D) \leq \alpha_X(n) r^n (2r)^{\omega} = 2^{\omega} \alpha_X(n) r^{n+\omega}$$

for any set D of diameter $2r$. $\mathcal{L}^{n+1}$ measure of the set D_r is exactly the right-hand side of the last estimate. It is easy to see that

$$D_r - D_r = 2D_r = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq 2r, \ |t| \leq (2r)^\omega\},$$

and the assumption (3.23) implies that

$$D_r - D_r \subset B_{2r},$$

therefore D_r has diameter at most $2r$. On the other hand, for any $(x,0) \in \partial D_r$ we have

$$d((-x,0),(x,0)) = d(\bar{0},(2x,0)) = 2r,$$

moreover, for any $(x,t),(-x,s) \in \partial D_r$ by Lemma 3.17

$$2r \geq d((-x,s),(x,t)) = d(\bar{0},(2x,t-s)) \geq d(\bar{0},(2x,0)) = 2r,$$

and the lemma follows. □

Consider the following example.

Example 3.26. Let $\omega = 2$ and the unit ball be

$$B_1 = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq 1, \ |t| \leq 1 + \|x\|\},$$

then by dilation rule the r-ball is

$$B_r = \left\{(y,s) \in \mathbb{R}^{n+1} \mid \frac{\|y\|}{r} \leq 1, \ \frac{|s|}{r^2} \leq 1 + \frac{\|y\|}{r}\right\}.$$

Let us prove that the balls are indeed generated by some metric d. The only non-trivial part is to verify the triangle inequality

$$d(\bar{0},(x,t)+(y,s)) \leq d(\bar{0},(x,t)) + d(\bar{0},(y,s)).$$

Without loss of generality we may assume that

$$d(\bar{0},(x,t)) = 1 \quad \text{and} \quad d(\bar{0},(y,s)) = r \leq 1.$$

It follows that

$$\|x+y\| \leq \|x\| + \|y\| \leq 1 + r$$

and

$$\begin{aligned} |t+s| &\leq 1 + \|x\| + r^2 + r\|y\| \leq 1 + (\|x+y\| + \|y\|) + r^2 + r\|y\| \\ &\leq 1 + \|x+y\| + r + r^2 + r^2 \leq (1+r)^2 + (1+r)\|x+y\|, \end{aligned}$$

therefore $d(\bar{0},(x,t)+(y,s)) \leq 1+r$ and the triangle inequality holds.

By the previous lemma D_r is the isodiametric set of diameter $2r$ and

$$\mathcal{L}^{n+1}(D_r) = 4\alpha_X(n)r^{n+2}.$$

Let us also calculate $\mathcal{L}^{n+1}$ measure of the r-ball

$$\begin{aligned} \mathcal{L}^{n+1}(B_r) &= 4\alpha_X(n)r^{n+2} - 2\alpha_X(n)\int_{r^2}^{2r^2}\left(\frac{s}{r}-r\right)^n ds \\ &= 2\alpha_X(n)\left(2-\frac{1}{n+1}\right)r^{n+2} = \left(1-\frac{1}{2(n+1)}\right)\mathcal{L}^{n+1}(D_r). \end{aligned}$$

Thus the isodiametric inequality fails for balls in this example and by Theorem 3.1

$$\sigma_{n+\omega}(\mathbb{R}_d^{n+1}) = D_{n+\omega}(\mathbb{R}_d^{n+1},\bar{0}) = 1 - \frac{1}{2(n+1)}$$

for this particular purely $(n+\omega)$-unrectifiable metric space $\mathbb{R}_d^{n+1}$.

Lemma 3.27.

$$\int_0^1 r^{n-1}(1+r)^\omega\,dr = \frac{2^\omega}{n}\left(1-\frac{\omega}{2n}+\frac{\omega(\omega+1)}{4n^2}+O\left(\frac{1}{n^3}\right)\right), \quad n\to\infty.$$

Proof. Integrating by parts we get

$$\int_0^1 r^{n-1}(1+r)^\omega \, dr = \frac{r^n}{n}(1+r)^\omega \Big|_{r=0}^{r=1} - \frac{\omega}{n}\int_0^1 r^n(1+r)^{\omega-1}\, dr.$$

If we denote the integral on the left-hand side as I_{n-1}^ω, then the last equality can be rewritten as

$$I_{n-1}^\omega = \frac{2^\omega}{n} - \frac{\omega}{n} I_n^{\omega-1}.$$

Let us apply this recursive formula several times

$$\begin{aligned}
I_{n-1}^\omega &= \frac{2^\omega}{n} - \frac{\omega}{n}\left(\frac{2^{\omega-1}}{n+1} - \frac{\omega-1}{n+1} I_{n+1}^{\omega-2}\right) \\
&= \frac{2^\omega}{n} - \frac{\omega}{n}\left(\frac{2^{\omega-1}}{n+1} - \frac{\omega-1}{n+1}\left(\frac{2^{\omega-2}}{n+2} - \frac{\omega-2}{n+2} I_{n+2}^{\omega-3}\right)\right) \\
&= \frac{2^\omega}{n} - \frac{\omega 2^{\omega-1}}{n(n+1)} + \frac{\omega(\omega-1)2^{\omega-2}}{n(n+1)(n+2)} - \frac{\omega(\omega-1)(\omega-2)}{n(n+1)(n+2)} I_{n+2}^{\omega-3}.
\end{aligned}$$

The definition of $I_{n+2}^{\omega-3}$ implies that

$$0 < I_{n+2}^{\omega-3} \leq \frac{\max\{2^{\omega-3}, 1\}}{n+3}.$$

Therefore one gets

$$I_{n-1}^\omega = \frac{2^\omega}{n} - \frac{\omega 2^{\omega-1}}{n(n+1)} + \frac{\omega(\omega-1)2^{\omega-2}}{n(n+1)(n+2)} + O\left(\frac{1}{n^4}\right), \quad n \to \infty,$$

or equivalently

$$\begin{aligned}
I_{n-1}^\omega &= \frac{2^\omega}{n} - \frac{\omega 2^{\omega-1}}{n^2} + \frac{\omega 2^{\omega-1}}{n^2(n+1)} + \frac{\omega(\omega-1)2^{\omega-2}}{n(n+1)(n+2)} + O\left(\frac{1}{n^4}\right) \\
&= \frac{2^\omega}{n} - \frac{\omega 2^{\omega-1}}{n^2} + \frac{\omega 2^{\omega-1} + \omega(\omega-1)2^{\omega-2}}{n^3} + O\left(\frac{1}{n^4}\right), \quad n \to \infty,
\end{aligned}$$

and the lemma follows. □

Theorem 3.28. *If the condition* (3.23) *holds, then*

$$\sigma_{n+\omega}(\mathbb{R}_d^{n+1}) = D_{n+\omega}(\mathbb{R}_d^{n+1}, \bar{0}) \leq 1 - \frac{\omega(\omega-1)}{4n^2} + O\left(\frac{1}{n^3}\right), \quad n \to \infty.$$

Proof. Let us define the following functions on the set $\{x \in \mathbb{R}^n \mid \|x\| \le 1\}$

$$u(x) = \sup\{t \in \mathbb{R} \mid (x,t) \in B_1\},$$

$$l(x) = \inf\{t \in \mathbb{R} \mid (x,t) \in B_1\}$$

and

$$f(x) = u(x) - l(x).$$

Points $(x, u(x))$ and $(x, l(x))$ are in B_1 for any $\|x\| \le 1$ and functions $u(x)$, $l(x)$ and $f(x)$ are $\mathcal{L}^n$-measurable, as the ball is closed. We have also assumed at the beginning of this section that $(0,1) \in \partial B_1$, therefore $u(0) = -l(0) = 1$.

Note that since the relation $(x,t) \in B_1$ is equivalent to $(-x,-t) \in B_1$, we obtain

$$\begin{aligned} u(x) &= \sup\{t \in \mathbb{R} \mid (-x,-t) \in B_1\} \\ &= -\inf\{-t \in \mathbb{R} \mid (-x,-t) \in B_1\} = -l(-x), \end{aligned}$$

hence

$$u(-x) = -l(x),$$

and thus

$$f(-x) = u(-x) - l(-x) = -l(x) + u(x) = f(x). \tag{3.25}$$

Let $y \in \mathbb{R}^n$ and $\|y\| = 1$, then

$$(-ry, u(-ry)),\ (Ry, u(Ry)) \in B_1 \quad \text{and} \quad (-ry, l(-ry)),\ (Ry, l(Ry)) \in B_1.$$

Lemma 3.21 and Remark 3.22 imply that for any $0 \le R \le 1$, $0 \le r \le 1$, $R + r \ne 0$

$$\left(0, \left(\frac{R}{R+r}\right)^{\omega} u(-ry) + \left(\frac{r}{R+r}\right)^{\omega} u(Ry)\right) \in B_1$$

and

$$\left(0, \left(\frac{R}{R+r}\right)^{\omega} l(-ry) + \left(\frac{r}{R+r}\right)^{\omega} l(Ry)\right) \in B_1.$$

It follows that

$$\left(\frac{R}{R+r}\right)^{\omega} u(-ry) + \left(\frac{r}{R+r}\right)^{\omega} u(Ry) \leq u(0)$$

and

$$\left(\frac{R}{R+r}\right)^{\omega} l(-ry) + \left(\frac{r}{R+r}\right)^{\omega} l(Ry) \geq l(0).$$

Thus the difference of the last two inequalities multiplied by $(R+r)^{\omega}$ gives

$$R^{\omega} f(-ry) + r^{\omega} f(Ry) \leq 2(R+r)^{\omega},$$

or according to the formula (3.25)

$$R^{\omega} f(ry) + r^{\omega} f(Ry) \leq 2(R+r)^{\omega} \tag{3.26}$$

for any $0 \leq R \leq 1$, $0 \leq r \leq 1$.

Let $J(ry) = j(y)r^{n-1}$, $y \in \mathbb{R}^n$, $||y|| = 1$, be the Jacobian of transformation from Cartesian to hyperspherical coordinates in the normed space X. We notice that

$$\alpha_X(n) = \int_{||y||=1} \int_0^1 j(y) r^{n-1}\, dr\, dy = \frac{1}{n} \int_{||y||=1} j(y)\, dy.$$

Then multiplying the inequality (3.26) by $j(y)R^{n-1}r^{n-1}$ and integrating with respect to R over $[0,1]$, r over $[0,1]$ and y over the set $\{y \in \mathbb{R}^n \mid ||y|| = 1\}$ we get

$$2V \int_0^1 R^{\omega+n-1}\, dR \leq 2n\alpha_X(n) \int_0^1 \int_0^1 R^{n-1} r^{n-1} (R+r)^{\omega}\, dR\, dr, \tag{3.27}$$

where

$$V = \mathcal{L}^{n+1}(B_1) = \int_{||y||=1} \int_0^1 j(y) r^{n-1} f(ry)\, dr\, dy. \tag{3.28}$$

Lemma 3.25 tells us that D_1 (see (3.24)) is the isodiametric set of diameter 2 and

$$V_{max} = \mathcal{L}^{n+1}(D_1) = 2^{\omega}\alpha_X(n). \tag{3.29}$$

The last formula and the estimate (3.27) yield

$$\frac{V}{V_{max}} \le 2^{-\omega} n(n+\omega) \int_0^1 \int_0^1 R^{n-1} r^{n-1} (R+r)^{\omega} \, dR \, dr.$$

The substitution $v = \min\{r, R\}/\max\{r, R\}$ simplifies the iterated integral on the right-hand side

$$\int_0^1 \int_0^1 R^{n-1} r^{n-1} (R+r)^{\omega} \, dR \, dr = \frac{2}{2n+\omega} \int_0^1 v^{n-1} (1+v)^{\omega} \, dv,$$

therefore

$$\frac{V}{V_{max}} \le 2^{1-\omega} \frac{n(n+\omega)}{2n+\omega} \int_0^1 v^{n-1} (1+v)^{\omega} \, dv. \tag{3.30}$$

Finally the last estimate and Lemma 3.27 imply

$$\begin{aligned}
\frac{V}{V_{max}} &\le \frac{2(n+\omega)}{2n+\omega} \left(1 - \frac{\omega}{2n} + \frac{\omega(\omega+1)}{4n^2} + O\left(\frac{1}{n^3}\right)\right) \\
&= \left(1 + \frac{\omega}{2n+\omega}\right) \left(1 - \frac{\omega}{2n} + \frac{\omega(\omega+1)}{4n^2} + O\left(\frac{1}{n^3}\right)\right) \\
&= 1 + \omega \left(\frac{1}{2n+\omega} - \frac{1}{2n}\right) - \frac{\omega^2}{2n(2n+\omega)} + \frac{\omega(\omega+1)}{4n^2} + O\left(\frac{1}{n^3}\right) \\
&= 1 - \frac{\omega(\omega-1)}{4n^2} + O\left(\frac{1}{n^3}\right), \quad n \to \infty.
\end{aligned} \tag{3.31}$$

It can be verified that the right-hand side of (3.30) is strictly less than 1 not only as $n \to \infty$, but also for any $n \in \mathbb{N}$ and $\omega = 2, 3, \dots$. Thus the isodiametric inequality fails for balls and by Theorem 3.1 the statement we are proving holds in the purely $(n+\omega)$-unrectifiable metric space $\mathbb{R}_d^{n+1}$. □

Remark 3.29. If the unit ball B_1 is the set of revolution about the vertical axis, then $f(x) = \hat{f}(\|x\|)$, $\|x\| \le 1$, and equalities (3.28) and (3.29) imply that

$$\frac{V}{V_{max}} = 2^{-\omega} n \int_0^1 r^{n-1} \hat{f}(r)\, dr.$$

Let us show on the example that the order 2 of the term $1/n$ in the estimate given by the last theorem is precise, in other words, this order can be achieved.

Example 3.30. Let the unit ball be

$$B_1 = \left\{ (x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq 1,\ |t| \leq (1 + \|x\|)^\omega - 2^{\omega-1}\|x\|^\omega \right\},$$

then by dilation rule the r-ball is

$$B_r = \left\{ (y,s) \in \mathbb{R}^{n+1} \mid \|y\| \leq r,\ |s| \leq (r + \|y\|)^\omega - 2^{\omega-1}\|y\|^\omega \right\}.$$

Let us prove that the balls are generated by some metric. Again the only non-trivial part is to make sure that the triangle inequality holds, i.e. that $(x+y, t+s) \in B_{1+r}$ for any $(x,t) \in \partial B_1$ and any $(y,s) \in \partial B_r$, $r \leq 1$. Since

$$\|x + y\| \leq \|x\| + \|y\| \leq 1 + r,$$

we only need to show that for $\|x\| \leq 1$ and $\|y\| \leq r$

$$|t + s| \leq (1 + r + \|x + y\|)^\omega - 2^{\omega-1}\|x + y\|^\omega. \tag{3.32}$$

We know that

$$|t + s| \leq (1 + \|x\|)^\omega - 2^{\omega-1}\|x\|^\omega + (r + \|y\|)^\omega - 2^{\omega-1}\|y\|^\omega. \tag{3.33}$$

Note that as long as $\omega > 1$ and $A > 0$ $(A, B \in \mathbb{R})$ the function

$$\psi(v) = (A + v)^\omega - 2^{\omega-1}v^\omega + B$$

is monotonically increasing on $[0, A]$, since for $0 \leq v \leq A$

$$\frac{d\psi}{dv}(v) = \omega(A+v)^{\omega-1} - 2^{\omega-1}\omega v^{\omega-1} = \omega((A+v)^{\omega-1} - (2v)^{\omega-1}) \geq 0.$$

As the function $\psi(v) = (1+r+v)^{\omega} - 2^{\omega-1}v^{\omega}$ increases on $[0, 1+r]$, one gets

$$(1+r+\|x+y\|)^{\omega} - 2^{\omega-1}\|x+y\|^{\omega} \geq (1+r+|\|x\|-\|y\||)^{\omega} - 2^{\omega-1}|\|x\|-\|y\||^{\omega}. \tag{3.34}$$

Thus in order to establish the inequality (3.32) it is enough to check that the right-hand side of (3.33) doesn't exceed the right-hand side of (3.34). This is equivalent to the statement

$$F(a,b) \leq 0, \quad 0 \leq a \leq 1, \ 0 \leq b \leq r,$$

where

$$F(a,b) = (1+a)^{\omega} - 2^{\omega-1}a^{\omega} + (r+b)^{\omega} - 2^{\omega-1}b^{\omega} - (1+r+|a-b|)^{\omega} + 2^{\omega-1}|a-b|^{\omega}. \tag{3.35}$$

The function $F(a,b)$ is continuous on $[0,1] \times [0,r]$ and differentiable in this region except for points where $a = b$. Let us find a maximum of $F(a,b)$ in the region above. Differentiating F with respect to its first variable we have

$$\frac{\partial F}{\partial a}(a,b) = \omega(1+a)^{\omega-1} - 2^{\omega-1}\omega a^{\omega-1} - \omega(1+r+|a-b|)^{\omega-1}\operatorname{sign}(a-b)$$
$$+ 2^{\omega-1}\omega|a-b|^{\omega-1}\operatorname{sign}(a-b), \quad a \neq b.$$

It is easy to see that

$$\frac{\partial F}{\partial a}(a,b) \leq 0 \quad \text{if} \quad 0 \leq b < a \leq 1$$

and

$$\frac{\partial F}{\partial a}(a,b) \geq 0 \quad \text{if} \quad 0 \leq a < b \leq r,$$

and therefore

$$F(a,b) \le F(b,b), \quad 0 \le a \le 1,\ 0 \le b \le r.$$

Note that the definition (3.35) of $F(a,b)$ implies

$$F(b,b) = \psi_1(b) + \psi_2(b),$$

where

$$\psi_1(b) = (1+b)^\omega - 2^{\omega-1}b^\omega$$

and

$$\psi_2(b) = (r+b)^\omega - 2^{\omega-1}b^\omega - (1+r)^\omega.$$

As both functions $\psi_1(b)$ and $\psi_2(b)$ increase on $[0,r]$,

$$F(b,b) \le \psi_1(r) + \psi_2(r) = 0, \quad 0 \le b \le r,$$

and the triangle inequality is proved.

Then Remark 3.29 gives the ratio

$$\frac{V}{V_{max}} = 2^{-\omega} n \int_0^1 2\left((1+r)^\omega - 2^{\omega-1}r^\omega\right) r^{n-1}\, dr \tag{3.36}$$

and by Lemma 3.27 we obtain

$$\begin{aligned}\frac{V}{V_{max}} &= 2\left(1 - \frac{\omega}{2n} + \frac{\omega(\omega+1)}{4n^2} + O\left(\frac{1}{n^3}\right)\right) - \frac{n}{n+\omega}\\ &= 1 + \frac{\omega}{n+\omega} - \frac{\omega}{n} + \frac{\omega(\omega+1)}{2n^2} + O\left(\frac{1}{n^3}\right)\\ &= 1 - \frac{\omega(\omega-1)}{2n^2} + O\left(\frac{1}{n^3}\right) \quad n \to \infty. \end{aligned} \tag{3.37}$$

As we can see, the order 2 of the term $1/n$ is indeed achieved in this example.

Remark 3.31. Let us denote the supremum of $\sigma_{n+\omega}(\mathbb{R}_d^{n+1})$ (or V/V_{max}) over different metrics by $\theta_{n,\omega}$. Then based on estimates (3.30) and (3.36) one gets

$$2^{-\omega} n \int_0^1 2\left((1+r)^\omega - 2^{\omega-1} r^\omega\right) r^{n-1}\, dr \leq \theta_{n,\omega}$$
$$\leq 2^{1-\omega} \frac{n(n+\omega)}{2n+\omega} \int_0^1 r^{n-1}(1+r)^\omega\, dr.$$

As we have already mentioned, it can be shown that $\theta_{n,\omega} < 1$ for any $n \in \mathbb{N}$ and $\omega = 2, 3, \ldots$. The estimates (3.31) and (3.37) imply that $\theta_{n,\omega} \to 1$ and $1 - \theta_{n,\omega} = O(1/n^2)$ as $n \to \infty$. On the other hand, the number $\theta_{n,\omega}$ tends to 0 for the fixed $n \in \mathbb{N}$ and $\omega \to \infty$.

3.5 The Group $\mathbb{R}_d^{n+1}$ (II)

We continue considering the group $\mathbb{R}_d^{n+1}$. In this section we will find the isodiametric set under some less restrictive assumptions on the shape of the ball (on the metric compatible with translations and dilations of the group) than the assumption (3.23). Convexity and symmetrization arguments will be quite helpful in this task.

In the next lemma we will make use of the Brunn-Minkowski inequality for Euclidean spaces. It states that if A and B are non-empty subsets of $\mathbb{R}^m$, then

$$\mathcal{L}^m(A+B)^{1/m} \geq \mathcal{L}^m(A)^{1/m} + \mathcal{L}^m(B)^{1/m}$$

(see Theorem 3.2.41 in [12]).

Lemma 3.32. *If $C \subset \mathbb{R}_d^{n+1}$ and the set $\tilde{C} = \frac{1}{2}(C - C)$ is a* central symmetrization *of C, then statements (i) - (iii) hold.*

(i) $\tilde{C}$ *is centrally symmetric and*

$$\mathcal{L}^{n+1}(\tilde{C}) \geq \mathcal{L}^{n+1}(C).$$

(ii) $\operatorname{diam} C \leq 2$ *if and only if* $\delta_{1/2}(C - C) \subset B_1$.

(iii) If C *is a convex set of diameter at most 2, then so is* $\tilde{C}$.

Proof. (i) The set $\tilde{C}$ is symmetric about the origin, since $-\tilde{C} = \tilde{C}$. The Brunn-Minkowski inequality implies that

$$\mathcal{L}^{n+1}(C - C)^{\frac{1}{n+1}} \geq 2\mathcal{L}^{n+1}(C)^{\frac{1}{n+1}}$$

or

$$\mathcal{L}^{n+1}(\tilde{C}) \geq \mathcal{L}^{n+1}(C).$$

(ii) It is clear that $\operatorname{diam} C \leq 2$ if and only if

$$a - b \in B_2 \quad \text{or} \quad \delta_{1/2}(a - b) \in B_1$$

for any $a, b \in C$, or equivalently

$$\delta_{1/2}(C - C) \subset B_1.$$

(iii) If C is a convex set, then so is $\tilde{C}$. More generally, if $A, B \subset \mathbb{R}^{n+1}$ are convex, then so is $A + B$. Indeed, a convex combination of any points $a_1 + b_1, a_2 + b_2 \in A + B$ ($a_1, a_2 \in A$ and $b_1, b_2 \in B$) is the sum of two convex combinations of the corresponding points of A and B, and therefore it lies in $A + B$

$$\alpha(a_1 + b_1) + (1 - \alpha)(a_2 + b_2) = (\alpha a_1 + (1 - \alpha)a_2) + (\alpha b_1 + (1 - \alpha)b_2) \in A + B,$$

where $0 \leq \alpha \leq 1$.

Notice that by central symmetry of $\tilde{C}$

$$\tilde{C} - \tilde{C} = \tilde{C} + \tilde{C}.$$

It is obvious that

$$\tilde{C} + \tilde{C} \supset 2\tilde{C},$$

and since $\tilde{C}$ is a convex set, the opposite inclusion is also true, thus

$$\tilde{C} - \tilde{C} = 2\tilde{C} = C - C.$$

Therefore the statement (ii) implies that if $\operatorname{diam} C \leq 2$, then $\operatorname{diam} \tilde{C} \leq 2$. □

Theorem 3.33.

$$D_{n+\omega}(\mathbb{R}_d^{n+1}, \bar{0}) \geq 2^{1-\omega}.$$

Proof. Let D be any subset of $\mathbb{R}_d^{n+1}$ of diameter at most 2, then by the previous lemma the set $\tilde{D} = \frac{1}{2}(D - D)$ is centrally symmetric,

$$\mathcal{L}^{n+1}(\tilde{D}) \geq \mathcal{L}^{n+1}(D)$$

and

$$\delta_{1/2}(D - D) \subset B_1.$$

Therefore

$$\mathcal{L}^{n+1}(D) \leq \mathcal{L}^{n+1}\left(\frac{1}{2}(D - D)\right) = 2^{\omega-1}\mathcal{L}^{n+1}(\delta_{1/2}(D - D)) \leq 2^{\omega-1}\mathcal{L}^{n+1}(B_1),$$

and we apply Theorem 3.1 to conclude the proof. □

Theorem 3.34. *If the unit ball B_1 contains a symmetric about the origin convex set C such that*

$$\frac{\mathcal{L}^{n+1}(C)}{\mathcal{L}^{n+1}(B_1)} > 2^{1-\omega}, \tag{3.38}$$

then statements (i) and (ii) hold.

(i) The isodiametric inequality fails for balls and

$$\sigma_{n+\omega}(\mathbb{R}_d^{n+1}) = D_{n+\omega}(\mathbb{R}_d^{n+1}, \bar{0}) < 1.$$

(ii) If the ball is a convex set itself, then

$$\sigma_{n+\omega}(\mathbb{R}_d^{n+1}) = D_{n+\omega}(\mathbb{R}_d^{n+1}, \bar{0}) = 2^{1-\omega} < 1.$$

Proof. (i) Let us denote $\tilde{C} = \frac{1}{2}\delta_2 C$, then the assumption (3.38) implies

$$\mathcal{L}^{n+1}(\tilde{C}) = 2^{\omega-1}\mathcal{L}^{n+1}(C) > \mathcal{L}^{n+1}(B_1). \tag{3.39}$$

On the other hand, since $\tilde{C}$ is also a convex symmetric about the origin set, we have

$$\tilde{C} - \tilde{C} = \tilde{C} + \tilde{C} = 2\tilde{C}$$

and

$$\delta_{1/2}(\tilde{C} - \tilde{C}) = \delta_{1/2}(2\tilde{C}) = C \subset B_1,$$

therefore by Lemma 3.32(ii) $\operatorname{diam} \tilde{C} \leq 2$. It follows that the isodiametric inequality fails for balls and by Theorem 3.1 the statement (i) is true in the purely $(n+\omega)$-unrectifiable metric space $\mathbb{R}_d^{n+1}$.

(ii) If the ball is convex itself, we may put $C = B_1$, then (3.39) implies that

$$\frac{\mathcal{L}^{n+1}(B_1)}{\mathcal{L}^{n+1}(\tilde{C})} = 2^{1-\omega}.$$

Thus $2^{1-\omega}$ is the upper bound for $D_{n+\omega}(\mathbb{R}_d^{n+1}, \bar{0})$. By the previous lemma it is also the lower bound and the statement (ii) follows. □

Definition 3.35. Let D be a bounded subset of $\mathbb{R}^{n+1}$ and H be a hyperplane in $\mathbb{R}^{n+1}$. For any line l orthogonal to H and intersecting D construct the closed segment on l symmetric about H and having the length $\mathcal{L}^1(l \cap D)$. The union of these segments is called the *Steiner symmetral* of D with respect to H.

Until the end of this section we suppose that the unit ball B_1 is symmetric about the hyperplane X. Let D^s be the Steiner symmetral of D with respect to X and l_x, $x \in \mathbb{R}^n$, be the line through $(x, 0) \in X$ orthogonal to X.

Lemma 3.36. *For a bounded set $D \subset \mathbb{R}^{n+1}_d$ statements (i) and (ii) hold.*

(i)
$$\operatorname{diam} D^s \leq \operatorname{diam} D.$$

(ii) If D is $\mathcal{L}^{n+1}$-measurable, then so is D^s and

$$\mathcal{L}^{n+1}(D^s) = \mathcal{L}^{n+1}(D).$$

Proof. (i) Let $a = (x, t)$ and $b = (y, s)$ be any points in D^s. Since D^s is the Steiner symmetral of D with respect to X, there are points $a_1 = (x, t_1)$, $a_2 = (x, t_2)$ and $b_1 = (y, s_1)$, $b_2 = (y, s_2)$ in D ($t_1 \leq t_2$ and $s_1 \leq s_2$) such that

$$t_2 - t_1 \geq \mathcal{L}^1(l_x \cap D) \quad \text{and} \quad s_2 - s_1 \geq \mathcal{L}^1(l_y \cap D).$$

Without loss of generality we may assume that $t_2 - s_1 \geq s_2 - t_1$, then

$$\begin{aligned} |t - s| &\leq |t| + |s| \leq \frac{1}{2}\mathcal{L}^1(l_x \cap D) + \frac{1}{2}\mathcal{L}^1(l_y \cap D) \\ &\leq \frac{1}{2}(t_2 - t_1) + \frac{1}{2}(s_2 - s_1) = \frac{1}{2}(t_2 - s_1) + \frac{1}{2}(s_2 - t_1) \leq t_2 - s_1. \end{aligned}$$

Lemma 3.24 implies that

$$d(a, b) \leq d(a_2, b_1) \leq \operatorname{diam} D,$$

which concludes the proof of the statement (i).

(ii) If D is $\mathcal{L}^{n+1}$-measurable, then by Fubini's theorem the function $\mathcal{L}^1(l_x \cap D)$ of $x \in \mathbb{R}^n$ is $\mathcal{L}^n$-measurable. Therefore the Steiner symmetral D^s is also $\mathcal{L}^{n+1}$-measurable and

$$\mathcal{L}^{n+1}(D^s) = \int_{\mathbb{R}^n} \mathcal{L}^1(l_x \cap D)\, dx = \mathcal{L}^{n+1}(D).$$

□

Theorem 3.37. *If* $D \subset \mathbb{R}^{n+1}_d$ *and* $\operatorname{diam} D \leq 2$, *then*

$$\mathcal{L}^{n+1}(D) \leq 2^{\omega-1} \int_{\|x\|\leq 1} \min\{2, \mathcal{L}^1(l_x \cap B_1)\}\, dx.$$

Proof. Without loss of generality we may assume that D is closed, thus $\mathcal{L}^{n+1}$-measurable (see Remark 3.19). By the previous lemma the Steiner symmetral of D with respect to X has the following properties

$$\operatorname{diam} D^s \leq \operatorname{diam} D \leq 2$$

and

$$\mathcal{L}^{n+1}(D^s) = \mathcal{L}^{n+1}(D).$$

By Lemma 3.32 the set $\tilde{D} = \frac{1}{2}(D^s - D^s)$ is centrally symmetric,

$$\mathcal{L}^{n+1}(\tilde{D}) \geq \mathcal{L}^{n+1}(D^s) = \mathcal{L}^{n+1}(D) \tag{3.40}$$

and

$$\delta_{1/2}(D^s - D^s) \subset B_1. \tag{3.41}$$

Let us observe that for any $x \in \mathbb{R}^n$

$$\mathcal{L}^1(l_x \cap D^s) \leq 2^{\omega},$$

since $\operatorname{diam} D^s \leq 2$, thus by the definition of D^s

$$|t| \leq \frac{1}{2}\mathcal{L}^1(l_x \cap D^s) \leq 2^{\omega-1}$$

for any $(x,t) \in D^s$. On the other hand, since

$$\tilde{D} = \frac{1}{2}(D^s - D^s) = \left\{ \left(\frac{x-y}{2}, \frac{t-s}{2} \right) \mid (x,t),(y,s) \in D^s \right\},$$

it follows that

$$\left| \frac{t-s}{2} \right| \leq 2^{\omega-1},$$

and therefore for any $(z,0) \in \operatorname{proj}_X \tilde{D}$

$$\mathcal{L}^1(l_z \cap \tilde{D}) \leq 2^\omega$$

and

$$\mathcal{L}^1(l_z \cap \delta_{1/2}(D^s - D^s)) \leq 2.$$

Using properties (3.40), (3.41) and Lemma 3.23 we conclude the proof as follows

$$\begin{aligned} \mathcal{L}^{n+1}(D) &\leq \mathcal{L}^{n+1}\left(\frac{1}{2}(D^s - D^s) \right) = 2^{\omega-1}\mathcal{L}^{n+1}(\delta_{1/2}(D^s - D^s)) \\ &= 2^{\omega-1} \int_{\|x\| \leq 1} \mathcal{L}^1(l_x \cap \delta_{1/2}(D^s - D^s))\, dx \\ &\leq 2^{\omega-1} \int_{\|x\| \leq 1} \min\{2, \mathcal{L}^1(l_x \cap B_1)\}\, dx. \end{aligned}$$

□

Corollary 3.38. *If the subset*

$$C = \{(x,t) \in B_1 \mid |t| \leq 1\}$$

of the unit ball is convex, then the set $\tilde{C} = \frac{1}{2}\delta_2 C$ has diameter 2 and maximizes $\mathcal{L}^{n+1}$ measure among all sets of diameter at most 2, it is the isodiametric set.

Proof. As C is convex symmetric about the origin subset of the unit ball, then $\operatorname{diam} \tilde{C} \leq 2$ as proved in Theorem 3.34(i). Moreover, the diameter of $\tilde{C}$ is 2, since

$$\tilde{C} \cap X = C \cap X = B_1 \cap X = \{(x, 0) \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}.$$

On the other hand, the fact that B_1 is symmetric about X leads to the following equality

$$\begin{aligned} \mathcal{L}^{n+1}(\tilde{C}) &= 2^{\omega-1}\mathcal{L}^{n+1}(C) = 2^{\omega-1} \int_{\|x\|\leq 1} \mathcal{L}^1(l_x \cap C)\, dx \\ &= 2^{\omega-1} \int_{\|x\|\leq 1} \min\{2, \mathcal{L}^1(l_x \cap B_1)\}\, dx. \end{aligned}$$

Taking the previous theorem into account we finish the proof. □

Remark 3.39. Lemma 3.25 follows from the last corollary under the assumption that B_1 is symmetric about X.

We finish this section and this chapter with an example which demonstrates Corollary 3.38 and appears to be quite useful in the next chapter.

Example 3.40. Let us consider a plane Π going through the vertical T-axis in the Heisenberg group $\mathbb{H}^1$ equipped with the CC metric. Π is the additive subgroup of $\mathbb{H}^1$, which can be viewed as $\mathbb{R}^2$ with $\omega = 2$ and the r-sphere (see Fig. 3.1)

$$\partial B_r \cap \Pi = \{(z, t) \in \Pi \mid |z| = x_r(\phi),\ t = t_r(\phi),\ |\phi| \leq \pi\},$$

where functions $x_r(\phi)$ and $t_r(\phi)$ are defined in (3.6).

Points $(x_1(\pm\pi), t_1(\pm\pi)) = (0, \pm 1/\pi)$ lie on the unit sphere, therefore rescaling the last coordinate by the factor π (such that $(0, \pm 1)$ are on the unit sphere) leads us to the setting considered so far.

It follows from Lemma 3.7 that the subset

$$C = \{(z,t) \in B_1 \cap \Pi \mid |t| \leq t_1(\pi)\}$$

of the unit ball is convex. Therefore by Corollary 3.38 $\tilde{C} = \frac{1}{2}\delta_2 C$ is the isodiametric set of diameter 2 on Π.

Chapter 4

Hausdorff Measures on a Surface in the Heisenberg Group

4.1 The Blow-up Formula

In this chapter we study a problem about the relation among the Hausdorff, the spherical and the centred Hausdorff measures of codimension one ($\mathcal{H}^3$, $\mathcal{S}^3$ and $\mathcal{C}^3$ measures with respect to the Carnot-Carathéodory metric d_c) restricted to a C^1 smooth surface in the Heisenberg group $\mathbb{H}^1$. The use of 3-dimensional Hausdorff measures is justified by the fact that $0 < \mathcal{H}^3(\Sigma) < \infty$ for a C^1 smooth bounded surface $\Sigma \subset \mathbb{H}^1$ (in particular the Hausdorff dimension of Σ is 3, see [18]). Saying "C^1 smooth" we always mean it in the Euclidean sense. The main result of this chapter is that these measures differ but coincide up to positive constant multiples, which we estimate as well.

Definition 4.1. Let Σ be a C^1 surface in the Heisenberg group $\mathbb{H}^1$. Let $\nu(p,\Sigma)$ be a normal of Σ at $p \in \Sigma$. Denote by $\nu_H(p,\Sigma)$ the Euclidean orthogonal projection of $\nu(p,\Sigma)$ on the plane H_p spanned by X_1 and Y_1 at $p \in \Sigma$ (see (3.5)), which is called a *horizontal plane.* We call the vector $\nu_H(p,\Sigma)$ a *horizontal normal* of Σ at p. A point $p \in \Sigma$ is called a *characteristic point* of Σ if $|\nu_H(p,\Sigma)| = 0$. The set of all characteristic points of Σ, the *characteristic set*, is denoted by $C(\Sigma)$.

The main tool of our investigation is the Blow-up formula, which is stated below. It has recently been shown by Balogh [2] that the characteristic set $C(\Sigma)$ of a C^1 surface $\Sigma \subset \mathbb{H}^1$ is $\mathcal{H}^3$-null. Therefore using the Blow-up formula for non-characteristic points of a C^1 surface Σ the connection among $\mathcal{H}^3$, $\mathcal{S}^3$ and $\mathcal{C}^3$ measures on Σ can be established quite easily.

Theorem 4.2 (Blow-up formula). *Let $\Sigma \subset \mathbb{H}^1$ be a C^1 surface, $\bar{0} \in \Sigma$ be a non-characteristic point of Σ and $q_0 \in B_1$, then*

$$\lim_{r \searrow 0} \frac{\mathcal{H}^3(\Sigma \cap \delta_r(q_0 B_1))}{r^3} = \mathcal{H}^3(\Pi_{\nu_H(\bar{0},\Sigma)} \cap q_0 B_1).$$

At first we list the additional notation used in this chapter.

Planes going through the vertical axis and only such planes we call vertical. By Π_β we denote a vertical plane with a normal β. Let $\Sigma_r = \delta_{1/r}\Sigma$ for $r > 0$ and $A^B = A \cap B$ for $A, B \subset \mathbb{H}^1$. Let V denote the convex hull of the open ball U_3 and $K = q_0 B_1$, $q_0 \in B_1$. It is clear that $\bar{0} \in K \subset B_2 \subset V$.

We use big O and little o notation in order not to specify constants and to show the rate of convergence to 0 (usually as $r \searrow 0$). By $\bar{o}$ we mean a vector of o in $\mathbb{R}^3$.

For the sake of brevity in notation $\nu(p, S)$ or $\nu_H(p, S)$ we omit S if $S = \Sigma_r$ and both arguments if $p = \bar{0}$ and $S = \Sigma_r$, thus $\nu(p) = \nu(p, \Sigma_r)$, $\nu = \nu(\bar{0}, \Sigma_r)$ and $\nu_H = \nu_H(\bar{0}, \Sigma_r)$.

Let σ measure the area of a surface in $\mathbb{R}^3$. Up to a positive constant multiple it coincides with the 2-dimensional Hausdorff measure with respect to the Euclidean distance. The measure σ restricted to a plane is also identical to $\mathcal{L}^2$ measure on this plane considered as $\mathbb{R}^2$.

In the whole chapter we suppose that $\Sigma \subset \mathbb{H}^1$ is a C^1 surface and only in this section that $\bar{0} \in \Sigma$ is a non-characteristic point of Σ.

In order to prove Theorem 4.2 let us establish a series of auxiliary lemmas. The aim of the first three lemmas is to prove the isodiametric inequality for a subset E of Σ_r^V by determining the connection of its measure and diameter with those of the projection of its translations on the plane Π_{ν_H}. Since the closure of a set may only increase its measure without changing the diameter, we may assume E to be closed, which is enough for our purpose.

Lemma 4.3. *Suppose that* $p_0 \in E \subset \Sigma_r^V$, $E' = p_0^{-1}E$ *and* $E'' = \mathrm{proj}_{\Pi_{\nu_H}} E'$, *then*

$$\sigma(E) = (1 + o(1))\sigma(E''), \quad r \searrow 0,$$

where the convergence of $o(1)$ *is uniform for* $E \subset \Sigma_r^V$ *and* $p_0 \in E$.

Proof. Let us rotate the orthonormal basis of $\mathbb{R}^3$ about the vertical direction in such a way that the first basis vector is the unit normal of the vector ν_H. Let $x = (x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbb{R}^3$ be a typical point in the new basis. Then $\nu = (\nu^{(1)}, 0, \nu^{(3)})$, $\nu_H = (\nu^{(1)}, 0, 0)$ and $|\nu^{(1)}| = |\nu_H| \neq 0$, as $\bar{0} \in \Sigma$ is a non-characteristic point of Σ, and therefore of Σ_r (see Remark 4.4).

We may assume that Σ is given locally as a level set of a C^1 function $f : V \to \mathbb{R}$ with a non-vanishing gradient. Namely Σ is given by $f(h) = 0$ in $\delta_r V$, $r \leq 1$, a small neighborhood of $\bar{0}$. Then Σ_r has the equation $f(\delta_r p) = 0$ with $p \in V$. The normal of Σ_r at $p \in \Sigma_r^V$ is the vector

$$\nu(p) = \left(\frac{\partial f(\delta_r p)}{\partial p^{(j)}}\right)_{j=1}^{3} = \delta_r \nabla f(\delta_r p) = \delta_r(\nabla f(\bar{0}) + \bar{o}(1)) = \nu + \delta_r \bar{o}(1), \quad r \searrow 0.$$

Here $\nabla f(h) = (\partial f(h)/\partial h^{(j)})_{j=1}^3$ is a gradient vector of the function f. Since $\nu = \delta_r \nabla f(\bar{0})$, it follows that

$$|\nu| = O(r), \quad |\nu^{(1)}| = |\nu_H| = O(r) \quad \text{and} \quad |\nu^{(3)}| = |\nu - \nu_H| = O(r^2), \quad r \searrow 0. \tag{4.1}$$

Let $p_0 \in \Sigma_r^V$ and consider how the normal $\nu(p)$ changes, when Σ_r is translated to $p_0^{-1}\Sigma_r$. Every point $p \in \Sigma_r$ is translated to a point $q = p_0^{-1}p \in p_0^{-1}\Sigma_r$ with p_0 translated to the origin. Therefore the equation of $p_0^{-1}\Sigma_r$ is $f(\delta_r(p_0 q)) = 0$ with $q \in p_0^{-1}V$ and the normal of $p_0^{-1}\Sigma_r$ at $q \in p_0^{-1}\Sigma_r^V$ is the vector

$$\nu'(q) = \nu(q, p_0^{-1}\Sigma_r) = \left(\frac{\partial f(\delta_r(p_0 q))}{\partial q^{(j)}}\right)_{j=1}^{3}$$
$$= \left(\sum_{i=1}^{3} \frac{\partial f(\delta_r p)}{\partial p^{(i)}} \frac{\partial p^{(i)}}{\partial q^{(j)}}\right)_{j=1}^{3}\Bigg|_{p=p_0 q} = \nu(p_0 q) J(q).$$

It is the product of the vector $\nu(p_0 q)$ and the Jacobian matrix $J(q)$ of $p = p_0 q$ as a function of q

$$J(q) = \left(\frac{\partial p^{(i)}}{\partial q^{(j)}}\right)_{i,j=1}^{3}.$$

According to the Heisenberg group multiplication law (3.4)

$$p = p_0 q = (p_0^{(1)} + q^{(1)}, p_0^{(2)} + q^{(2)}, p_0^{(3)} + q^{(3)} + 2(p_0^{(2)} q^{(1)} - p_0^{(1)} q^{(2)}), \tag{4.2}$$

therefore

$$J(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2p_0^{(2)} & -2p_0^{(1)} & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} \nu'(q) &= \nu(p_0 q)J(q) = \delta_r(\nabla f(\bar{0}) + \bar{o}(1))J(q) \\ &= \delta_r(\nabla f(\bar{0}) + \bar{o}(1)) = \nu + \delta_r \bar{o}(1), \quad r \searrow 0. \end{aligned} \tag{4.3}$$

The following well-known formula for the area of the surface piece E holds

$$\sigma(E) = \iint_{\mathrm{proj}_{\Pi_{\nu_H}} E} \frac{dp^{(2)}\, dp^{(3)}}{\cos(\nu(p), \nu_H)}, \quad p \in \Sigma_r^V.$$

Let us change variables in the double integral above from $p^{(2)}, p^{(3)}$ to $q^{(2)}, q^{(3)}$ in such a way that $p = p_0 q$, i.e.

$$\begin{cases} p^{(2)} = p_0^{(2)} + q^{(2)}, \\ p^{(3)} = p_0^{(3)} + q^{(3)} + 2(p_0^{(2)} q^{(1)} - p_0^{(1)} q^{(2)}), \end{cases}$$

where $q^{(1)} = q^{(1)}(q^{(2)}, q^{(3)})$ is such that $q \in p_0^{-1}\Sigma_r^V$. Then we obtain

$$\sigma(E) = \iint_{\mathrm{proj}_{\Pi_{\nu_H}} E'} \frac{1}{\cos(\nu(p_0 q), \nu_H)} \frac{D(p^{(2)}, p^{(3)})}{D(q^{(2)}, q^{(3)})}\, dq^{(2)}\, dq^{(3)}, \tag{4.4}$$

where

$$\frac{D(p^{(2)}, p^{(3)})}{D(q^{(2)}, q^{(3)})} = \det \left(\frac{\partial p^{(i)}}{\partial q^{(j)}} \right)_{i,j=2}^{3}$$

is the Jacobian of transformation from $p^{(2)}, p^{(3)}$ to $q^{(2)}, q^{(3)}$ coordinates on the plane Π_{ν_H}. Using the formula (4.2) we find

$$\frac{D(p^{(2)}, p^{(3)})}{D(q^{(2)}, q^{(3)})} = \det \begin{pmatrix} 1 & 0 \\ 2\left(p_0^{(2)} \frac{\partial q^{(1)}}{\partial q^{(2)}} - p_0^{(1)}\right) & 1 + 2p_0^{(2)} \frac{\partial q^{(1)}}{\partial q^{(3)}} \end{pmatrix} = 1 + 2p_0^{(2)} \frac{\partial q^{(1)}}{\partial q^{(3)}}. \tag{4.5}$$

Since the equation $f(\delta_r(p_0q)) = 0$ implicitly defines $q^{(1)}$ as a function of $q^{(2)}$ and $q^{(3)}$, differentiation of the implicit function gives

$$\frac{\partial q^{(1)}}{\partial q^{(3)}} = -\frac{\partial f(\delta_r(p_0q))/\partial q^{(3)}}{\partial f(\delta_r(p_0q))/\partial q^{(1)}} = -\frac{\nu'^{(3)}(q)}{\nu'^{(1)}(q)}. \tag{4.6}$$

Taking equations (4.5), (4.6), (4.3) and (4.1) into account we have

$$\begin{aligned}\frac{D(p^{(2)},p^{(3)})}{D(q^{(2)},q^{(3)})} &= 1 - 2p_0^{(2)}\frac{\nu'^{(3)}(q)}{\nu'^{(1)}(q)} = 1 - 2p_0^{(2)}\frac{\nu^{(3)} + o(r^2)}{\nu^{(1)} + o(r)} \\ &= 1 + \frac{O(r^2)}{O(r)} = 1 + O(r), \quad r \searrow 0,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\cos(\nu(p),\nu_H)} &= \frac{|\nu(p)||\nu_H|}{\langle\nu(p),\nu_H\rangle} = \frac{|\nu + \delta_r\bar{o}(1)||\nu_H|}{\langle\nu + \delta_r\bar{o}(1),\nu_H\rangle} = \frac{|\nu_H||\nu_H| + o(r)|\nu_H|}{\langle\nu,\nu_H\rangle + o(r^2)} \\ &= \frac{|\nu_H|^2 + o(r^2)}{|\nu_H|^2 + o(r^2)} = \frac{1 + o(1)}{1 + o(1)} = 1 + o(1), \quad r \searrow 0.\end{aligned}$$

Both expressions go to 1 as $r \searrow 0$ uniformly for $p_0, p \in \Sigma_r^V$ and together with the equation (4.4) they imply the statement of the lemma. □

Remark 4.4. Vectors $\nu_H = \nu_H(\bar{0},\Sigma_r)$ and $\nu_H(\bar{0},\Sigma)$ are both non-zero and have the same direction due to the fact that $\nu = \delta_r \nabla f(\bar{0})$ and $\nu(\bar{0},\Sigma) = \nabla f(\bar{0})$. Therefore we may use ν_H instead of $\nu_H(\bar{0},\Sigma)$ every time we are only interested in the direction of the last vector.

Lemma 4.5. *Suppose that* $p_0 \in E \subset \Sigma_r^V$, $E' = p_0^{-1}E$ *and* $E'' = \operatorname{proj}_{\Pi_{\nu_H}} E'$, *then*

$$\operatorname{diam} E'' = (1 + o(1)) \operatorname{diam} E, \quad r \searrow 0,$$

where the convergence of $o(1)$ *is uniform for* $E \subset \Sigma_r^V$ *and* $p_0 \in E$.

Proof. The statement of the lemma implies that $\bar{0} \in E'$ and $\operatorname{diam} E' = \operatorname{diam} E = \rho$, therefore $E' \subset B_\rho$. As we have already mentioned in the previous lemma, points $q \in E'$ satisfy the equation $f(\delta_r(p_0 q)) = 0$ and the normal of $p_0^{-1}\Sigma_r \supset E'$ at the origin is $\nu'(\bar{0})$. By Mean Value Theorem we have

$$0 = f(\delta_r(p_0 q)) - f(\delta_r p_0) = \langle \nu'(\theta q), q \rangle, \quad q \in E', \quad 0 < \theta < 1$$

($q \in \mathbb{H}^1$ is multiplied by a scalar θ), which combined with the equation (4.3) leads to

$$\langle \nu + \delta_r \bar{o}(1), q \rangle = 0$$

or

$$\langle \nu, q \rangle = -\langle \delta_r \bar{o}(1), q \rangle = o(r)\rho$$

as $r \searrow 0$. It follows that

$$\langle \nu_H, q \rangle + \langle \nu - \nu_H, q \rangle = o(r)\rho,$$

where $\langle \nu - \nu_H, q \rangle = O(r^2)\rho^2$ (see (4.1)), and therefore

$$\langle \nu_H, q \rangle = o(r)\rho.$$

Recalling also that $|\nu_H| = O(r)$ by (4.1) we find the Euclidean distance from a point $q \in E'$ to the plane Π_{ν_H}

$$\frac{\langle \nu_H, q \rangle}{|\nu_H|} = o(1)\rho, \quad r \searrow 0. \tag{4.7}$$

The convergence in (4.7) is uniform for $p_0 \in \Sigma_r^V$ and $q \in p_0^{-1}\Sigma_r^V$.

Let $p'' = \mathrm{proj}_{\Pi_{\nu_H}} p'$ and $q'' = \mathrm{proj}_{\Pi_{\nu_H}} q'$ for any $p', q' \in E'$. The triangle inequality implies that

$$d_c(p'', q'') \le d_c(p'', p') + d_c(p', q') + d_c(q', q'') \le d_c(p'', p') + \rho + d_c(q', q'')$$

and

$$d_c(p',q') \le d_c(p',p'')+d_c(p'',q'')+d_c(q'',q') \le d_c(p',p'')+\operatorname{diam} E''+d_c(q'',q').$$

Estimates for distances between p' and p'', q' and q'' are identical, we show one of them. Let $p' = (z_1, t)$ and $p'' = (z_2, t)$, then the formula (4.7) implies $|z_1 - z_2| = o(1)\rho$ as $r \searrow 0$. Using (3.7) and Lemma 3.8 one has

$$\begin{aligned}
d_c(p',p'') &= d_c(\bar{0}, p'^{-1}p'') = d_c(\bar{0}, (-z_1 + z_2, -2\operatorname{Im}(z_1\bar{z}_2))) \\
&\le d_c(\bar{0}, (-z_1 + z_2, 0)) + d_c(\bar{0}, (0, -2\operatorname{Im}(z_1\bar{z}_2))) \\
&= |-z_1 + z_2| + \sqrt{2\pi|\operatorname{Im}(z_1\bar{z}_2)|} \\
&\le |-z_1 + z_2| + \sqrt{2\pi|z_2||-z_1 + z_2|} \\
&\le o(1)\rho + \sqrt{2\pi\rho \cdot o(1)\rho} = o(1)\rho, \quad r \searrow 0.
\end{aligned}$$

Combining these estimates we get

$$\operatorname{diam} E'' \le (1 + o(1))\rho$$

and

$$\rho \le \operatorname{diam} E'' + o(1)\rho,$$

or equivalently

$$\operatorname{diam} E'' \ge (1 + o(1))\rho$$

as $r \searrow 0$, where the convergence of $o(1)$ has the required property, and the theorem is proved. □

As we have already mentioned in Example 3.40, any vertical plane Π is the additive subgroup of the Heisenberg group $\mathbb{H}^1$. Hence measures $\mathcal{H}^3$ and σ (which is $\mathcal{L}^2$ on Π) are identical on Π up to a positive constant multiple, being Haar measures of the subgroup. The *isodiametric constant* for subsets of the vertical plane Π is the number

$$\alpha = \sup_{S\subset\Pi} \frac{\sigma(S)}{(\operatorname{diam} S)^3}. \tag{4.8}$$

It is independent on the choice of the vertical plane Π because of the invariance of the metric, and thus the invariance of measures $\mathcal{H}^3$ and σ under rotations about the vertical axis. The definition of α implies that the isodiametric inequality for subsets of Π is

$$\sigma(S) \le \alpha(\operatorname{diam} S)^3, \quad S \subset \Pi. \tag{4.9}$$

Let us show that the similar inequality holds for subsets of the surface piece Σ_r^V for small enough $r > 0$.

Lemma 4.6. *The isodiametric inequality for subsets of* Σ_r^V *has the form*

$$\sigma(E) \le (1 + o(1))\alpha(\operatorname{diam} E)^3, \quad E \subset \Sigma_r^V, \quad r \searrow 0,$$

where the convergence of $o(1)$ *is uniform for* $E \subset \Sigma_r^V$.

Proof. Let $p_0 \in E \subset \Sigma_r^V$, $E' = p_0^{-1}E$ and $E'' = \operatorname{proj}_{\Pi_{\nu_H}} E'$. Lemmas 4.3 and 4.5 combined with the isodiametric inequality (4.9) of the vertical plane Π_{ν_H} give the required estimate

$$\begin{aligned}\sigma(E) &= (1 + o(1))\sigma(E'') \le (1 + o(1))\alpha(\operatorname{diam} E'')^3\\ &= (1 + o(1))\alpha((1 + o(1)) \operatorname{diam} E)^3 = (1 + o(1))\alpha(\operatorname{diam} E)^3, \quad r \searrow 0.\end{aligned}$$

□

The following simple statement will be useful in the proof of Lemma 4.8.

Lemma 4.7. *For any vertical plane* Π *and* $q_0 \in \mathbb{H}^1$ *the following is true*

$$\sigma(\Pi \cap \partial(q_0 B_1)) = 0.$$

Proof. Let us notice that

$$\sigma(\Pi \cap \partial(q_0 B_1)) = \sigma(q_0^{-1}\Pi \cap \partial B_1), \tag{4.10}$$

since

$$\partial(q_0 B_1) = q_0 \partial B_1$$

and multiplication of the set $\Pi \cap \partial(q_0 B_1)$ by q_0^{-1} only moves it from the vertical plane Π to another parallel plane $q_0^{-1}\Pi$ and shifts its sections by vertical lines (each section by a different value) in the vertical direction. It is clear that the right-hand side of the equation (4.10) is zero. □

It follows from the formula (4.7) that Σ_r^V approaches the plane Π_{ν_H} as $r \searrow 0$. This fact enables us to establish the continuity result for σ measure on Σ_r^V.

Lemma 4.8. *For any* $q, q_0 \in B_1$ *the following statement holds*

$$\lim_{r \searrow 0, q \to q_0} \sigma(\Sigma_r^{qB_1}) = \sigma(\Pi_{\nu_H}^{q_0 B_1}).$$

Proof. The area of the surface piece $\Sigma_r^{qB_1}$ is

$$\sigma(\Sigma_r^{qB_1}) = \iint_{\mathrm{proj}_{\Pi_{\nu_H}} \Sigma_r^{qB_1}} \frac{dp^{(2)}\, dp^{(3)}}{\cos(\nu(p), \nu_H)} = \iint_{\Pi_{\nu_H}^V} g_{r,q}(p^{(2)}, p^{(3)})\, dp^{(2)}\, dp^{(3)},$$

where

$$g_{r,q}(p^{(2)},p^{(3)}) = \frac{\chi_{\mathrm{proj}_{\Pi_{\nu_H}} \Sigma_r^{qB_1}}(p^{(2)},p^{(3)})}{\cos(\nu(p),\nu_H)}, \quad (0,p^{(2)},p^{(3)}) \in \Pi_{\nu_H}^V, \quad p \in \Sigma_r^V.$$

It follows from (4.7) that the Euclidean distance from a point $p \in \Sigma_r^V$ to the plane Π_{ν_H} is

$$\frac{\langle \nu_H, p\rangle}{|\nu_H|} = o(1)\operatorname{diam} V = o(1), \quad r \searrow 0. \tag{4.11}$$

According to Lemma 4.7 σ-a.e. point $(0,p^{(2)},p^{(3)}) \in \Pi_{\nu_H}$ is either internal or external point of q_0B_1. By (4.11) for $p \in \Sigma_r^V$ there is some $\varepsilon_p > 0$ such that for all $0 < r < \varepsilon_p$ and $|q - q_0| < \varepsilon_p$ statements (i) and (ii) hold

(i) $p \in qB_1$ if $(0,p^{(2)},p^{(3)})$ is internal point of q_0B_1,

(ii) $p \notin qB_1$ if $(0,p^{(2)},p^{(3)})$ is external point of q_0B_1.

Therefore for σ-a.e. $(0,p^{(2)},p^{(3)}) \in \Pi_{\nu_H}^V$, $0 < r < \varepsilon_p$ and $|q - q_0| < \varepsilon_p$ the following equality holds

$$\chi_{\mathrm{proj}_{\Pi_{\nu_H}} \Sigma_r^{qB_1}}(p^{(2)},p^{(3)}) = \chi_{\Pi_{\nu_H}^{q_0B_1}}(p^{(2)},p^{(3)})$$

or

$$\lim_{r\searrow 0, q\to q_0} \chi_{\mathrm{proj}_{\Pi_{\nu_H}} \Sigma_r^{qB_1}}(p^{(2)},p^{(3)}) = \chi_{\Pi_{\nu_H}^{q_0B_1}}(p^{(2)},p^{(3)}).$$

The proof of Lemma 4.3 implies

$$\lim_{r\searrow 0}\cos(\nu(p),\nu_H) = 1, \quad p \in \Sigma_r^V,$$

therefore we have

$$\lim_{r\searrow 0, q\to q_0} g_{r,q}(p^{(2)},p^{(3)}) = \chi_{\Pi_{\nu_H}^{q_0B_1}}(p^{(2)},p^{(3)}), \quad \sigma\text{-a.e. } (0,p^{(2)},p^{(3)}) \in \Pi_{\nu_H}^V.$$

Finally by Lebesgue Convergence Theorem it follows that

$$\lim_{r\searrow 0, q\to q_0} \sigma(\Sigma_r^{qB_1}) = \lim_{r\searrow 0, q\to q_0} \iint_{\Pi_{\nu_H}^V} g_{r,q}(p^{(2)}, p^{(3)})\, dp^{(2)}\, dp^{(3)}$$
$$= \iint_{\Pi_{\nu_H}^V} \chi_{\Pi_{\nu_H}^{q_0 B_1}}(p^{(2)}, p^{(3)})\, dp^{(2)}\, dp^{(3)} = \sigma(\Pi_{\nu_H}^{q_0 B_1}).$$

□

Lemma 4.9.

$$\sigma(\Pi_{\nu_H}^K) \le \alpha \liminf_{r\searrow 0} \mathcal{H}^3(\Sigma_r^K).$$

Proof. Consider an arbitrary ε-covering of Σ_r^K by at most countable family of closed sets $\{E_i \mid E_i \subset \Sigma_r^V,\ \operatorname{diam} E_i < \varepsilon\}$. The subadditivity of σ and the isodiametric inequality for subsets Σ_r^V established in Lemma 4.6 yield

$$\sigma(\Sigma_r^K) \le \sum_i \sigma(E_i) \le (1 + o(1))\alpha \sum_i (\operatorname{diam} E_i)^3, \quad r \searrow 0.$$

Taking the infimum over all such ε-coverings of Σ_r^K we have

$$\sigma(\Sigma_r^K) \le (1 + o(1))\alpha \mathcal{H}_\varepsilon^3(\Sigma_r^K) \le (1 + o(1))\alpha \mathcal{H}^3(\Sigma_r^K), \quad r \searrow 0.$$

Now applying Lemma 4.8 we obtain the required statement. □

According to Example 3.40 the set

$$S_1 = \frac{1}{2}\delta_2\{(z,t) \in B_1 \cap \Pi_{\nu_H} \mid |t| \le t_1(\pi)\}$$
$$= \left\{(z,t) \in \frac{1}{2}B_2 \cap \Pi_{\nu_H} \mid |t| \le 2t_1(\pi)\right\}$$

has diameter 2 and maximizes σ measure among all subsets of Π_{ν_H} of diameter at most 2, it is the isodiametric set on Π_{ν_H}. Therefore

$$S_\rho = \delta_\rho S_1 = \left\{(z,t) \in \frac{1}{2}B_{2\rho} \cap \Pi_{\nu_H} \mid |t| \le 2t_\rho(\pi)\right\}, \quad \rho > 0,$$

is also the isodiametric set of diameter 2ρ on Π_{ν_H}. It follows from the definition (4.8) of the isodiametric constant α that

$$\sigma(S_\rho) = \alpha(\operatorname{diam} S_\rho)^3. \tag{4.12}$$

Let us introduce the following notation

$$S_\rho^q = (\operatorname{proj}_{\Pi_{\nu_H}}^{-1} S_\rho) \cap q^{-1}\Sigma_r, \quad q \in \Sigma_r.$$

Hence

$$qS_\rho^q = (q \operatorname{proj}_{\Pi_{\nu_H}}^{-1} S_\rho) \cap \Sigma_r, \quad q \in \Sigma_r,$$

and it is clear that $\bar{0} \in S_\rho^q$ and $q \in qS_\rho^q$.

The sets qS_ρ^q are of great interest to us, as they are "almost" isodiametric for small enough $r > 0$. They satisfy the isodiametric equality similar to (4.12).

Lemma 4.10. *Let $q \in \Sigma_r^V$ and $qS_\rho^q \subset V$, then*

$$(\operatorname{diam} qS_\rho^q)^3 = (1 + o(1))\frac{\sigma(qS_\rho^q)}{\alpha}, \quad r \searrow 0,$$

where the convergence of $o(1)$ is uniform for $q \in \Sigma_r^V$ and $qS_\rho^q \subset V$.

Proof. As $q \in qS_\rho^q \subset \Sigma_r^V$ and $\operatorname{proj}_{\Pi_{\nu_H}} S_\rho^q = S_\rho$, Lemma 4.5 claims

$$\operatorname{diam} qS_\rho^q = \frac{\operatorname{diam} S_\rho}{(1 + o(1))} = (1 + o(1)) \operatorname{diam} S_\rho, \quad r \searrow 0. \tag{4.13}$$

Combining this equation with the isodiametric equality (4.12) we get

$$(\operatorname{diam} qS_\rho^q)^3 = (1 + o(1))^3 (\operatorname{diam} S_\rho)^3 = (1 + o(1))\frac{\sigma(S_\rho)}{\alpha}, \quad r \searrow 0.$$

Then according to Lemma 4.3 the required statement follows. □

Lemma 4.11. *Let*

$$\mathcal{F} = \{S \mid S = qS_\rho^q \subset V,\ q \in \Sigma_r^K,\ \rho < \varepsilon\}, \quad \varepsilon > 0,$$

be the family of closed subsets of Σ_r^V*. Then* $\mathcal{F}$ *is* $\mathcal{H}^3$ *adequate for* Σ_r^K*.*

Proof. The statement of the lemma follows from Theorem 1.6, let us only check that assumptions of Theorem 1.6 are indeed satisfied. The formula (4.13) implies that the family $\mathcal{F}$ covers Σ_r^K finely for any $\varepsilon > 0$. We need to show that for some $1 < \tau < \infty$ and $1 < \lambda < \infty$

$$\mathcal{H}^3(\hat{S}) < \lambda \mathcal{H}^3(S)$$

whenever $S \in \mathcal{F}$ and $\hat{S}$ is the τ enlargement of S given by (1.1).

Let us observe that if $S = qS_\rho^q$, then by the definition (1.1) of $\hat{S}$ one gets

$$\operatorname{diam} \hat{S} \leq (1 + 2\tau) \operatorname{diam} S, \quad r \searrow 0. \tag{4.14}$$

By Lemma 4.10 we know

$$(\operatorname{diam} S)^3 = (1 + o(1)) \frac{\sigma(S)}{\alpha}, \quad r \searrow 0, \tag{4.15}$$

and by analogy with the proof of Lemma 4.9

$$\sigma(S) \leq (1 + o(1)) \alpha \mathcal{H}^3(S), \quad r \searrow 0. \tag{4.16}$$

For any $\varepsilon > 0$ there is $R > 0$ such that $\operatorname{diam} \delta_R \hat{S} < \varepsilon$ ($R < \varepsilon / \operatorname{diam} \hat{S}$), therefore

$$\mathcal{H}_\varepsilon^3(\hat{S}) = \frac{1}{R^3} \mathcal{H}_\varepsilon^3(\delta_R \hat{S}) \leq \frac{1}{R^3} (\operatorname{diam} \delta_R \hat{S})^3 = (\operatorname{diam} \hat{S})^3$$

and

$$\mathcal{H}^3(\hat{S}) \leq (\operatorname{diam} \hat{S})^3.$$

Using this inequality together with estimates (4.14), (4.15) and (4.16) we conclude

$$\begin{aligned}\mathcal{H}^3(\hat{S}) &\leq (\operatorname{diam}\hat{S})^3 \leq (1+2\tau)^3(\operatorname{diam}S)^3\\ &\leq (1+2\tau)^3(1+o(1))\frac{(1+o(1))\alpha}{\alpha}\mathcal{H}^3(S)\\ &= (1+o(1))(1+2\tau)^3\mathcal{H}^3(S), \quad r\searrow 0.\end{aligned}$$

For small enough $r>0$ finally we get

$$\mathcal{H}^3(\hat{S}) < 2(1+2\tau)^3\mathcal{H}^3(S).$$

□

Lemma 4.12.

$$\alpha \limsup_{r\searrow 0}\mathcal{H}^3(\Sigma_r^K) \leq \sigma(\Pi_{\nu_H}^K).$$

Proof. Let us recall that $K = q_0B_1$, $q_0\in B_1$ and denote

$$V_n = q_0U_{1+1/n} \subset U_3 \subset V, \quad n\in\mathbb{N}.$$

As mentioned in the previous lemma, the family of sets

$$\mathcal{F} = \left\{S \mid S = qS_\rho^q \subset V,\ q\in\Sigma_r^K,\ \rho<\frac{\varepsilon}{4}\right\}$$

covers Σ_r^K finely for any $\varepsilon>0$. Lemma 4.11 tells us that $\mathcal{F}$ is $\mathcal{H}^3$ adequate for Σ_r^K, i.e. for $\Sigma_r^{V_n}$ open in Σ_r^V the family $\mathcal{F}$ has a countable subfamily $\mathcal{G}$ of disjoint sets such that (see Definition 1.3)

$$\cup_{S\in\mathcal{G}}S\subset\Sigma_r^{V_n} \quad\text{and}\quad \mathcal{H}^3(\Sigma_r^K\setminus\cup_{S\in\mathcal{G}}S)=0.$$

Applying Lemma 4.10 and summing over all $S \in \mathcal{G}$ we have

$$\sum_{S\in\mathcal{G}}(\operatorname{diam} S)^3 \le (1+o(1))\sum_{S\in\mathcal{G}}\frac{\sigma(S)}{\alpha}, \quad r\searrow 0.$$

The formula (4.13) guarantees that

$$\operatorname{diam} S = (1+o(1))2\rho \le 4\rho < \varepsilon$$

for small enough $r > 0$. Therefore properties of the covering $\mathcal{G}$ mentioned above imply that

$$\alpha\mathcal{H}^3_\varepsilon(\Sigma_r^K) \le (1+o(1))\sigma(\Sigma_r^{V_n}), \quad r\searrow 0. \tag{4.17}$$

Note that $V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots$ and

$$\bigcap_{n=1}^\infty V_n = \bigcap_{n=1}^\infty q_0 U_{1+1/n} = q_0 B_1 = K.$$

Therefore by continuity of σ measure on Σ_r it follows that

$$\lim_{n\to\infty}\sigma(\Sigma_r^{V_n}) = \sigma(\cap_{n=1}^\infty(\Sigma_r\cap V_n)) = \sigma(\Sigma_r\cap K) = \sigma(\Sigma_r^K).$$

Letting $\varepsilon\searrow 0$ and $n\to\infty$ the inequality (4.17) gives

$$\alpha\mathcal{H}^3(\Sigma_r^K) \le (1+o(1))\sigma(\Sigma_r^K), \quad r\searrow 0. \tag{4.18}$$

Applying Lemma 4.8 one gets the required statement. □

Now we are ready to prove the Blow-up formula.

Proof of Theorem 4.2. The Blow-up formula

$$\lim_{r\searrow 0}\frac{\mathcal{H}^3(\Sigma\cap\delta_r K)}{r^3} = \mathcal{H}^3(\Pi_{\nu_H(\bar{0},\Sigma)}\cap K)$$

is equivalent to

$$\lim_{r\searrow 0} \mathcal{H}^3(\Sigma_r^K) = \mathcal{H}^3(\Pi_{\nu_H}^K).$$

According to Lemmas 4.9 and 4.12 the above limit is

$$\lim_{r\searrow 0} \mathcal{H}^3(\Sigma_r^K) = \frac{\sigma(\Pi_{\nu_H}^K)}{\alpha}.$$

Therefore the theorem is proved once we show that for any σ-measurable subset A of a vertical plane Π the following relation between measures holds

$$\sigma(A) = \alpha\mathcal{H}^3(A). \tag{4.19}$$

Since σ is the Haar measure on Π, by Theorem 1.14(i, ii) we get

$$\sigma(A) = \mathcal{H}^3(A) \lim_{r\searrow 0} \sup \left\{ \frac{\sigma(S)}{(\operatorname{diam} S)^3} \mid \bar{0} \in S \subset \Pi,\ 0 < \operatorname{diam} S < r \right\}. \tag{4.20}$$

If we recall the definition (4.8) of the isodiametric constant α, then (4.19) follows immediately from (4.20) and the proof is finished. □

Let us prove the important consequence of Theorem 4.2, which is also of the blow-up type. We will use it to establish our main result in the next section.

Theorem 4.13. *Let* $\Sigma \subset \mathbb{H}^1$ *be a* C^1 *surface and* $\bar{0} \in \Sigma$ *be a non-characteristic point of* Σ*, then*

$$\lim_{r\searrow 0} \sup \left\{ \frac{\mathcal{H}^3(\Sigma \cap yB_\rho)}{\rho^3} \mid y \in B_\rho,\ 0 < \rho < r \right\} = \sup_{q \in B_1} \mathcal{H}^3(\Pi_{\nu_H(\bar{0},\Sigma)} \cap qB_1)$$

and

$$\lim_{r\searrow 0} \sup \left\{ \frac{\mathcal{H}^3(\Sigma \cap yB_\rho)}{\rho^3} \mid y \in \Sigma \cap B_\rho,\ 0 < \rho < r \right\} = \mathcal{H}^3(\Pi_{\nu_H(\bar{0},\Sigma)} \cap B_1).$$

Proof. Since

$$\limsup_{r\searrow 0} \left\{ \frac{\mathcal{H}^3(\Sigma \cap yB_\rho)}{\rho^3} \mid y \in B_\rho,\ 0 < \rho < r \right\}$$
$$= \lim_{r\searrow 0} \sup\{\mathcal{H}^3(\Sigma_\rho \cap \delta_{1/\rho} y B_1) \mid \delta_{1/\rho} y \in B_1,\ 0 < \rho < r\}$$
$$= \lim_{r\searrow 0} \sup\{\mathcal{H}^3(\Sigma_\rho \cap qB_1) \mid q \in B_1,\ 0 < \rho < r\}$$

and

$$\limsup_{r\searrow 0} \left\{ \frac{\mathcal{H}^3(\Sigma \cap yB_\rho)}{\rho^3} \mid y \in \Sigma \cap B_\rho,\ 0 < \rho < r \right\}$$
$$= \lim_{r\searrow 0} \sup\{\mathcal{H}^3(\Sigma_\rho \cap \delta_{1/\rho} y B_1) \mid \delta_{1/\rho} y \in \Sigma_\rho \cap B_1,\ 0 < \rho < r\}$$
$$= \lim_{r\searrow 0} \sup\{\mathcal{H}^3(\Sigma_\rho \cap qB_1) \mid q \in \Sigma_\rho \cap B_1,\ 0 < \rho < r\},$$

the formulas we need to prove are

$$\lim_{r\searrow 0} \sup\{\mathcal{H}^3(\Sigma_\rho \cap qB_1) \mid q \in B_1,\ 0 < \rho < r\} = \sup_{q \in B_1} \mathcal{H}^3(\Pi_{\nu_H} \cap qB_1) \quad (4.21)$$

and

$$\lim_{r\searrow 0} \sup\{\mathcal{H}^3(\Sigma_\rho \cap qB_1) \mid q \in \Sigma_\rho \cap B_1,\ 0 < \rho < r\} = \mathcal{H}^3(\Pi_{\nu_H} \cap B_1). \quad (4.22)$$

Let us prove (4.21). It is clear that for any $q \in B_1$ and $r > 0$

$$\sup\{\mathcal{H}^3(\Sigma_\rho \cap qB_1) \mid q \in B_1,\ 0 < \rho < r\} \geq \mathcal{H}^3(\Sigma_{r/2} \cap qB_1). \quad (4.23)$$

Letting $r \searrow 0$, applying Blow-up Theorem 4.2 and then taking the supremum over $q \in B_1$ on the right-hand side one has

$$\lim_{r\searrow 0} \sup\{\mathcal{H}^3(\Sigma_\rho \cap qB_1) \mid q \in B_1,\ 0 < \rho < r\} \geq \sup_{q \in B_1} \mathcal{H}^3(\Pi_{\nu_H} \cap qB_1). \quad (4.24)$$

Suppose the strict inequality holds. It follows that there are $\varepsilon > 0$ and $r_0 > 0$ such that for any $0 < r < r_0$[1]

$$\sup\{\mathcal{H}^3(\Sigma_\rho \cap qB_1) \mid q \in B_1,\ 0 < \rho < r\} > \sup_{q \in B_1} \mathcal{H}^3(\Pi_{\nu_H} \cap qB_1) + \varepsilon. \quad (4.25)$$

Therefore the following statement holds

$$\exists \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \exists q_n \in B_1 \quad \exists 0 < \rho_n < \frac{1}{n} \quad \forall q \in B_1$$

$$\mathcal{H}^3(\Sigma_{\rho_n} \cap q_n B_1) > \mathcal{H}^3(\Pi_{\nu_H} \cap qB_1) + \varepsilon.$$

Then there is a subsequence of $(q_n)_{n \in \mathbb{N}}$ which converges to some $q_0 \in B_1$. For the convenience of notation we may suppose that this subsequence is $(q_n)_{n \in \mathbb{N}}$ itself. Let us put $q = q_0$ in the last inequality and take the limit as $n \to \infty$

$$\lim_{n \to \infty} \mathcal{H}^3(\Sigma_{\rho_n} \cap q_n B_1) \geq \mathcal{H}^3(\Pi_{\nu_H} \cap q_0 B_1) + \varepsilon. \quad (4.26)$$

According to the inequality (4.18) we have

$$\alpha \mathcal{H}^3(\Sigma_{\rho_n} \cap q_n B_1) \leq (1 + o(1))\sigma(\Sigma_{\rho_n} \cap q_n B_1), \quad n \to \infty.$$

We may apply Lemma 4.8 and then the equation (4.19) to get

$$\lim_{n \to \infty} \mathcal{H}^3(\Sigma_{\rho_n} \cap q_n B_1) \leq \frac{\sigma(\Pi_{\nu_H} \cap q_0 B_1)}{\alpha} = \mathcal{H}^3(\Pi_{\nu_H} \cap q_0 B_1),$$

which contradicts to (4.26), and therefore (4.21) is proved.

We observe that for any $q \in \Pi_{\nu_H}$

$$\mathcal{H}^3(\Pi_{\nu_H} \cap qB_1) = \mathcal{H}^3(q^{-1}\Pi_{\nu_H} \cap B_1) = \mathcal{H}^3(\Pi_{\nu_H} \cap B_1), \quad (4.27)$$

[1] In fact, it is true for any $r > 0$, since the left-hand side of (4.25) increases together with r.

since $q^{-1}\Pi_{\nu_H} = \Pi_{\nu_H}$, being the subgroup of the group $\mathbb{H}^1$. The formula (4.22) is proved similarly with only differences listed below. According to (4.27) equations (4.23) – (4.25) are still true if on the left-hand sides we write $q \in \Sigma_\rho \cap B_1$ instead of $q \in B_1$, on the right-hand side of (4.23) $q = \bar{0}$ and on the right-hand sides of (4.24) and (4.25) $q \in \Pi_{\nu_H} \cap B_1$ instead of $q \in B_1$. Therefore in the subsequent equations we have $q_n \in \Sigma_{\rho n} \cap B_1$ and $q, q_0 \in \Pi_{\nu_H} \cap B_1$. The rest of the proof is unchanged. □

4.2 Application of the Blow-up Formula

We are going to prove the main result of this chapter about the relation among $\mathcal{H}^3$, $\mathcal{S}^3$ and $\mathcal{C}^3$ measures on a C^1 surface $\Sigma \subset \mathbb{H}^1$ of finite $\mathcal{H}^3$ measure. Formulas of the blow-up type play the crucial role in this section.

Let $X = \mathbb{H}^1$, $\mathcal{H}^3(\Sigma) < \infty$, $A \subset \Sigma$ and $\mu = \mathcal{H}^3 \llcorner \Sigma$, then Theorem 1.14(i) implies

$$\mathcal{H}^3(A) \leq \mathcal{S}^3(A) \sup_{x \in A} \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) \tag{4.28}$$

and

$$\mathcal{H}^3(A) \leq \mathcal{C}^3(A) \sup_{x \in A} \overline{D}_{\mathcal{C}^3}(\mathcal{H}^3, \Sigma, x). \tag{4.29}$$

Since μ is a Borel regular measure (we may assume Σ to be a closed set) and $\mu(\Sigma) < \infty$, then according to [12, 2.2.3 and 2.2.2]

$$\mu(A) = \inf\{\mu(V) \mid V \text{ is open}, A \subset V\}.$$

Therefore Theorem 1.14(ii) implies

$$\mathcal{H}^3(A) \geq \mathcal{S}^3(A) \inf_{x \in A} \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) \tag{4.30}$$

and

$$\mathcal{H}^3(A) \geq \mathcal{C}^3(A) \inf_{x \in A} \overline{D}_{\mathcal{C}^3}(\mathcal{H}^3, \Sigma, x). \tag{4.31}$$

Let us repeat the proof of Lemma 2.9.6 and Theorem 2.9.7 in [12] with slight modifications. Then our main result will follow easily from these facts with the aid of Theorem 4.13.

Lemma 4.14.

(i) $\overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)$ *is a* $\mathcal{S}^3$*-measurable function on* Σ.

(ii) $\overline{D}_{\mathcal{C}^3}(\mathcal{H}^3, \Sigma, x)$ *is a* $\mathcal{C}^3$*-measurable function on* Σ.

Proof. Suppose that $0 < a < b < \infty$ and A, B are bounded sets such that

$$A \subset \{x \in \Sigma \mid \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) < a\}$$

and

$$B \subset \{x \in \Sigma \mid \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) > b\}.$$

We may find Borel sets A' and B' for which

$$A \subset A', \quad \mathcal{S}^3(A) = \mathcal{S}^3(A'), \quad \mathcal{H}^3(A) = \mathcal{H}^3(A')$$

and

$$B \subset B', \quad \mathcal{S}^3(B) = \mathcal{S}^3(B'), \quad \mathcal{H}^3(B) = \mathcal{H}^3(B').$$

Therefore inequalities (4.28) and (4.30) imply

$$\mathcal{H}^3(A' \cap B') = \mathcal{H}^3(A \cap B') \leq a\mathcal{S}^3(A \cap B') = a\mathcal{S}^3(A' \cap B')$$

and

$$\mathcal{H}^3(A' \cap B') = \mathcal{H}^3(A' \cap B) \geq b\mathcal{S}^3(A' \cap B) = b\mathcal{S}^3(A' \cap B'),$$

which leads to $\mathcal{S}^3(A' \cap B') = 0$, and hence

$$\mathcal{S}^3(A \cup B) = \mathcal{S}^3((A \cup B) \cap A') + \mathcal{S}^3((A \cup B) \cap B') \geq \mathcal{S}^3(A) + \mathcal{S}^3(B).$$

Then application of the statement [12, 2.3.2(7)] concludes the proof of the first part of the lemma. The second statement is proved in the same manner using (4.29) and (4.31). □

Theorem 4.15.

(i) If $A \subset \Sigma$ is a $\mathcal{S}^3$-measurable set, then A is $\mathcal{H}^3$-measurable and

$$\mathcal{H}^3(A) = \int_A \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{S}^3.$$

(ii) If $A \subset \Sigma$ is a $\mathcal{C}^3$-measurable set, then A is $\mathcal{H}^3$-measurable and

$$\mathcal{H}^3(A) = \int_A \overline{D}_{\mathcal{C}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{C}^3.$$

Proof. There is a Borel set B containing A such that $\mathcal{S}^3(B \setminus A) = 0$, thus $\mathcal{H}^3(B \setminus A) = 0$ by (1.2) and it follows that A is $\mathcal{H}^3$-measurable.

The sets

$$Z = \{x \in \Sigma \mid \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) = 0\}$$

and

$$W = \{x \in \Sigma \mid \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) = \infty\}$$

are $\mathcal{S}^3$-measurable by the previous lemma. From (4.28) one gets

$$\mathcal{H}^3(Z) = 0 = \int_Z \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{S}^3.$$

The inequality (1.3) and the statement (iv) of Theorem 1.14 imply

$$0 \leq \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) \leq 1$$

for $\mathcal{H}^3$ almost all $x \in \Sigma$, hence $\mathcal{H}^3(W) = 0$, and thus $\mathcal{S}^3(W) = 0$ by (1.2), which leads to

$$\mathcal{H}^3(W) = 0 = \int_W \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{S}^3.$$

We note that $A \setminus (Z \cup W)$ is the union of disjoint $\mathcal{S}^3$-measurable sets (by the previous lemma)

$$A_n = \{x \in A \mid t^n \le \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) < t^{n+1}\}, \quad n \in \mathbb{Z},$$

for any $1 < t < \infty$. Therefore (4.28) and (4.30) give estimates

$$\begin{aligned}\mathcal{H}^3(A) &= \sum_{n\in\mathbb{Z}} \mathcal{H}^3(A_n) \le \sum_{n\in\mathbb{Z}} t^{n+1}\mathcal{S}^3(A_n)\\ &\le \sum_{n\in\mathbb{Z}} t \int_{A_n} \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{S}^3 = t\int_A \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{S}^3\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}^3(A) &= \sum_{n\in\mathbb{Z}} \mathcal{H}^3(A_n) \ge \sum_{n\in\mathbb{Z}} t^{n}\mathcal{S}^3(A_n)\\ &\ge \sum_{n\in\mathbb{Z}} t^{-1} \int_{A_n} \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{S}^3 = t^{-1}\int_A \overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)\, d\mathcal{S}^3.\end{aligned}$$

Letting $t \searrow 1$ finishes the proof of the first part of the theorem. The second statement is proved similarly using (4.29) and (4.31). □

Corollary 4.16. *Let Π be a vertical plane, then statements (i) and (ii) hold.*

(i) If $A \subset \Sigma$ is a $\mathcal{S}^3$-measurable set, then A is $\mathcal{H}^3$-measurable and

$$\mathcal{H}^3(A) = \frac{1}{8} \sup_{q\in B_1} \mathcal{H}^3(\Pi \cap qB_1)\mathcal{S}^3(A).$$

(ii) If $A \subset \Sigma$ is a $\mathcal{C}^3$-measurable set, then A is $\mathcal{H}^3$-measurable and

$$\mathcal{H}^3(A) = \frac{1}{8}\mathcal{H}^3(\Pi \cap B_1)\mathcal{C}^3(A).$$

Proof. Let $x \in A \backslash C(\Sigma)$. As we have already mentioned at beginning of the chapter, the characteristic set $C(\Sigma)$ is $\mathcal{H}^3$-negligible, and therefore according to (1.2) it is also $\mathcal{S}^3$- and $\mathcal{C}^3$-negligible set. Then using definitions of $\overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x)$, $\overline{D}_{\mathcal{C}^3}(\mathcal{H}^3, \Sigma, x)$ (see p. 21) and Theorem 4.13 we get

$$\begin{aligned}
\overline{D}_{\mathcal{S}^3}(\mathcal{H}^3, \Sigma, x) &= \lim_{r \searrow 0} \sup \left\{ \frac{\mathcal{H}^3(\Sigma \cap B(y, \rho))}{(2\rho)^3} \mid x \in B(y, \rho),\ 0 < \rho < r \right\} \\
&= \lim_{r \searrow 0} \sup \left\{ \frac{\mathcal{H}^3(x^{-1}\Sigma \cap x^{-1}yB_\rho)}{(2\rho)^3} \mid x^{-1}y \in B_\rho,\ 0 < \rho < r \right\} \\
&= \frac{1}{8} \sup_{q \in B_1} \mathcal{H}^3(\Pi_{\nu_H(\bar{0}, x^{-1}\Sigma)} \cap qB_1)
\end{aligned}$$

and

$$\begin{aligned}
\overline{D}_{\mathcal{C}^3}(\mathcal{H}^3, \Sigma, x) &= \lim_{r \searrow 0} \sup \left\{ \frac{\mathcal{H}^3(\Sigma \cap B(y, \rho))}{(2\rho)^3} \mid x \in B(y, \rho),\ y \in \Sigma,\ 0 < \rho < r \right\} \\
&= \lim_{r \searrow 0} \sup \left\{ \frac{\mathcal{H}^3(x^{-1}\Sigma \cap x^{-1}yB_\rho)}{(2\rho)^3} \mid x^{-1}y \in x^{-1}\Sigma \cap B_\rho,\ 0 < \rho < r \right\} \\
&= \frac{1}{8} \mathcal{H}^3(\Pi_{\nu_H(\bar{0}, x^{-1}\Sigma)} \cap B_1).
\end{aligned}$$

Therefore Theorem 4.15 implies that

$$\mathcal{H}^3(A) = \frac{1}{8} \int_A \sup_{q \in B_1} \mathcal{H}^3(\Pi_{\nu_H(\bar{0}, x^{-1}\Sigma)} \cap qB_1)\, d\mathcal{S}^3$$

and

$$\mathcal{H}^3(A) = \frac{1}{8} \int_A \mathcal{H}^3(\Pi_{\nu_H(\bar{0}, x^{-1}\Sigma)} \cap B_1)\, d\mathcal{C}^3.$$

Since $\mathcal{H}^3$ is invariant under rotations about the vertical axis and B_1 is a solid of revolution about the vertical axis, integrands are independent of x and equal to

$$\sup_{q \in B_1} \mathcal{H}^3(\Pi \cap qB_1) \quad \text{and} \quad \mathcal{H}^3(\Pi \cap B_1)$$

respectively for any vertical plane Π. Thus the corollary follows. □

Remark 4.17. From equations (4.19) and (4.12) we know that for any vertical plane Π

$$\mathcal{H}^3(\Pi \cap B_1) = \frac{\sigma(\Pi \cap B_1)}{\alpha} = (\operatorname{diam} S_\rho)^3 \frac{\sigma(\Pi \cap B_1)}{\sigma(S_\rho)},$$

where S_ρ, $\rho > 0$, is the isodiametric subset of Π. This expression is easy to compute for CC balls. Since the ρ-sphere ∂B_ρ is a surface of revolution about the vertical axis of the curve given by equations (3.6), we have

$$\sigma(\Pi \cap B_\rho) = 4 \int_0^\pi x_\rho(\phi) t'_\rho(\phi)\, d\phi \approx 2.0448\rho^3.$$

According to Example 3.40 we find

$$\sigma(S_\rho) = 8 \int_0^{\phi_0} x_\rho(\phi) t'_\rho(\phi)\, d\phi \approx 2.5125\rho^3,$$

where ϕ_0 is the unique solution of the equation $t_\rho(\phi_0) = t_\rho(\pi)$ on $(0, \pi)$ ($\phi_0 \approx 0.5022$). Also taking into account that $\operatorname{diam} S_\rho = 2\rho$ we get

$$\frac{1}{8}\mathcal{H}^3(\Pi \cap B_1) \approx \frac{2.0448}{2.5125} \approx 0.814.$$

Remark 4.18. Similarly to the previous remark we have

$$\mathcal{H}^3(\Pi \cap qB_1) = (\operatorname{diam} S_1)^3 \frac{\sigma(\Pi \cap qB_1)}{\sigma(S_1)},$$

and therefore

$$\frac{1}{8} \sup_{q \in B_1} \mathcal{H}^3(\Pi \cap qB_1) = \frac{\sup_{q \in B_1} \sigma(\Pi \cap qB_1)}{\sigma(S_1)}.$$

It is worth noticing that for any $q \in \mathbb{H}^1$

$$\sigma(\Pi \cap qB_1) = \sigma(q^{-1}\Pi \cap B_1),$$

which can be verified in the same way as the equation (4.10). Let $\Pi = \Pi_\beta$ be an arbitrary vertical plane with a unit normal β. If $\lambda \geq 0$ is the Euclidean distance from the origin to the plane $q^{-1}\Pi_\beta$, then the point $q_\lambda = \lambda\beta$ (multiplication by a scalar) lies on the plane $q^{-1}\Pi_\beta$. Planes $q^{-1}\Pi_\beta$ and $q_\lambda\Pi_\beta$ have the common point q_λ and the common normal β, therefore they coincide. Thus we may assume that $q^{-1} = q_\lambda = \lambda\beta$. Numerical computations show that

$$\sup_{q\in B_1} \sigma(q^{-1}\Pi_\beta \cap B_1) = \sup_{0\le\lambda\le 1} \sigma(q_\lambda\Pi_\beta \cap B_1)$$
$$= \sigma(q_\lambda\Pi_\beta \cap B_1)|_{\lambda\approx 0.27} \approx 2.1037,$$

and therefore

$$\frac{1}{8}\sup_{q\in B_1} \mathcal{H}^3(\Pi \cap qB_1) \approx \frac{2.1037}{2.5125} \approx 0.837.$$

The dependence of $\sigma(q_\lambda\Pi_\beta \cap B_1)$ on $\lambda \in [0,1]$ is reflected on Fig. 4.1 (the area under the graph is half of the volume of B_1).

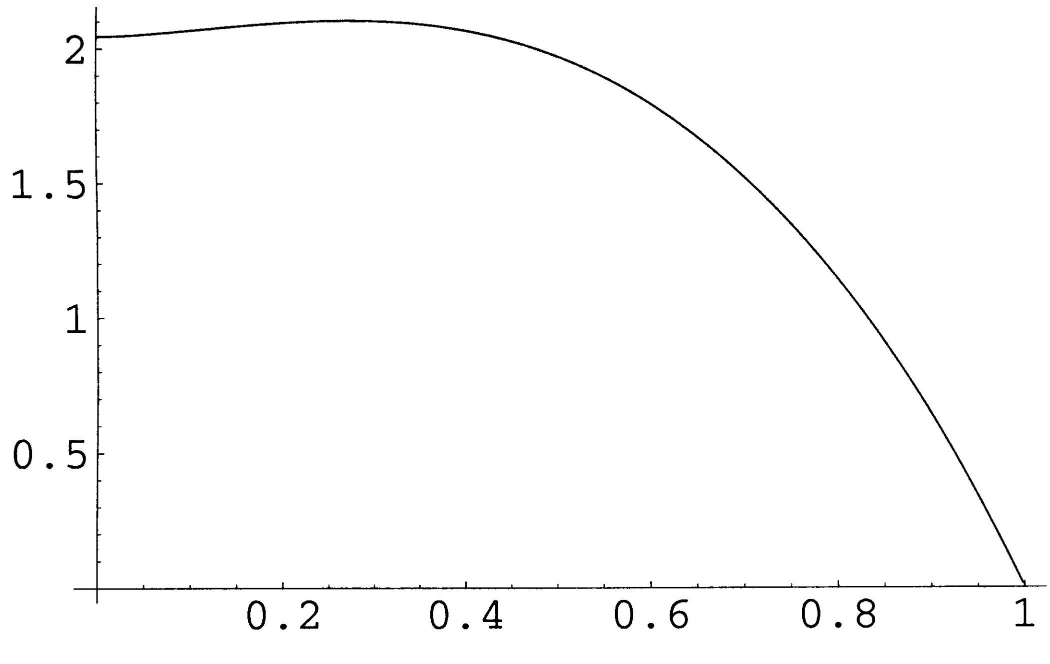

Figure 4.1: The area of the section $q_\lambda\Pi_\beta \cap B_1$ as a function of $\lambda \in [0,1]$

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