Algebraic 2-complexes over certain infinite abelian groups

Tim Edwards

Department of Mathematics
University College London
University of London

A thesis submitted for the degree of
Doctor of Philosophy.
Supervisor: Prof. F.E.A Johnson
SEPTEMBER 2006

Abstract

Whitehead's Theorem allows the study of homotopy types of two dimensional CW complexes to be phrased in terms of chain homotopy types of algebraic complexes, arising as the cellular chains on the universal cover. It is natural to ask whether the category of algebraic complexes fully represents the category of CW complexes, in particular whether every algebraic complex is realised geometrically. The case of two dimensional complexes is of special interest, partly due to the relationship between such complexes and group presentations and partly since, as was recently proved, it relates to the question as to when cohomology is a suitable indicator of dimension.

This thesis has two primary considerations. The first is the generalisation to infinite groups of F.E.A. Johnson's approach regarding problems of geometric realisation. It is shown, under certain restrictions, that the class of projective extensions containing algebraic complexes may be recognised as the unit elements of a ring, with ring elements congruence classes of extensions of the trivial module by a second homotopy module. The realisation property is shown to hold for the free abelian groups on two and three generators, and for the product of a cyclic group and a free group on a single generator.

Secondly, a reinterpretation is given of the well documented relationship between the congruence classes represented by Swan modules and the projective modules constructed via Milnor's connecting homomorphism and the relevant fibre product diagram. This relationship is shown to be typical of projective modules occurring in extensions of a two-sided ideal by a quotient ring, and we show that any two-sided ideal in a general ring results in a Mayer-Vietoris sequence which is different and complimentary to the standard excision sequence.

Acknowledgements

My foremost thanks are to my supervisor, Prof F.E.A. Johnson, whose continuing enthusiasm and interest in the subject provided my inspiration, and whose mathematical tastes and publications extend a context to much of the work in this thesis. My time at UCL has greatly benefited from his generous attitude towards research and from his unparalleled dedication to the mathematical health of his students, both post- and undergraduate. I wish to thank all the UCL students, both past and present, who have helped to create the stimulating and warm atmosphere which I have been privileged to share.

On a more personal note, I would like to express my deepest gratitude to my family; Geoff, Pip and Bron, for all the support and encouragement they have provided, and to my kind girlfriend Amy for her warmth, compassion and faith.

Lastly I wish to acknowledge the intellectual independence and financial security that I received through a Doctoral Training Account from the Engineering and Physical Sciences Research Council (EPSRC).

Contents

Chapter 1

Introduction

1.1 The D(2) and Realisation problems

In essence, the D(n) problem asks if the algebraic constructions of homology and cohomology are sufficient indicators of geometric dimension. With the exception of the case $n = 2$, this problem was solved in the affirmative by its poser, C.T.C. Wall, in the two seminal papers on finiteness conditions for CW complexes [29],[30]. In a similar spirit, the Realisation problem asks if the algebraic images of two-dimensional CW complexes are equivalent to the naturally constructed algebraic definition of a two-dimensional complex. Both questions relate to the suitability of certain algebraic constructions in fully describing the geometric properties of CW complexes. These questions have been robustly studied by F.E.A. Johnson, who succeeded in proving their equivalence on the assumption of certain highly reasonable restrictions (see Appendix B of [11]). Moreover, in his book on the subject and in several companion papers (see [12],[13],[14]), Professor Johnson achieved a resolution of several cases of the problem and highlighted some of the more computationally prohibitive obstacles to a complete solution.

Geometric 2-complexes

Suppose that Γ is a finitely presented group, with a given presentation:

$$\mathcal{G} = \langle x_1, \dots x_g | W_1, \dots, W_r \rangle.$$

Following Fox [9], we may construct a two-dimensional CW complex $X_{\mathcal{G}}$ with fundamental group naturally equal to Γ, often called the Cayley complex of $\mathcal{G}$. For further examples of this process see Johnson [11], Hogg-Angeloni, Metzler, Sieradski [16]. It can be shown that any connected two-dimensional CW complex with fundamental group Γ is homotopy equivalent to a Cayley complex constructed in this way; indeed one may read off the required presentation in a natural manner (see e.g. [16] Chapter II).

The chain complex of the universal cover $\tilde{X}_{\mathcal{G}}$ gives rise to a complex of $\mathbb{Z}[\Gamma]$-modules thus:

$$C_*(\tilde{X}_{\mathcal{G}}) = \left(0 \longrightarrow \pi_2(X_{\mathcal{G}}) \longrightarrow \mathbb{Z}[\Gamma]^r \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^g \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \to 0\right)$$

where:

(i) we have identified $\mathrm{Ker}(\partial_2) = H_2(\tilde{X}_{\mathcal{G}})$ with $\pi_2(X_{\mathcal{G}})$ via the Hurewicz isomorphism, and the isomorphism induced by the covering map $\tilde{X}_{\mathcal{G}} \to X_{\mathcal{G}}$;

(ii) since the universal cover is simply connected, and by (i) above, the complex is an exact sequence.

Note that ∂_2 and ∂_1 are completely determined by the relations and generators of $\mathcal{G}$ respectively, and may be given explicitly. The construction of the exact sequence $C_*(\tilde{X}_{\mathcal{G}})$ phrases geometric homotopy equivalence in algebraic terms, as expressed by Johnson's version of Whitehead's Theorem [11]:

Theorem 1.1.1. *Suppose that Γ is a group with two presentations:*

$$\mathcal{G} = \langle x_1, \dots x_g | W_1, \dots, W_r \rangle$$
$$\mathcal{H} = \langle y_1, \dots y_h | Z_1, \dots, Z_s \rangle.$$

Then $X_{\mathcal{G}}$ is homotopy equivalent to $X_{\mathcal{H}}$ if and only if is a commutative diagram of $\mathbb{Z}[\Gamma]$-module homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(X_{\mathcal{G}}) \arrow[r] \arrow[d, "f_4"] & \mathbb{Z}[\Gamma]^r \arrow[r] \arrow[d, "f_3"] & \mathbb{Z}[\Gamma]^g \arrow[r] \arrow[d, "f_2"] & \mathbb{Z}[\Gamma] \arrow[r] \arrow[d, "f_1"] & \mathbb{Z} \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & \pi_2(X_{\mathcal{H}}) \arrow[r] & \mathbb{Z}[\Gamma]^s \arrow[r] & \mathbb{Z}[\Gamma]^h \arrow[r] & \mathbb{Z}[\Gamma] \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

with both f_0 and f_4 isomorphisms. The horizontal homomorphisms are suppressed, but the implication is that they are those constructed from the respective presentations.

Algebraic 2-complexes

By an *algebraic 2-complex* over a group Γ we mean any exact sequence of (right) $\mathbb{Z}[\Gamma]$-modules of the form:

$$\mathbf{E} = 0 \longrightarrow M \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where each F_i is finitely generated free and $\mathbb{Z}$ denotes the trivial $\mathbb{Z}[\Gamma]$-module. It is convenient to adopt the notation $M = \pi_2\mathbf{E}$.

We emphasise the point that if $\mathcal{G}$ is a group presentation then $C_*(\tilde{X}_{\mathcal{G}})$ is an algebraic 2-complex. Two algebraic 2-complexes are said to be *chain homotopy equivalent* if there exists a collection of homomorphisms f_0, f_1, f_2, f_3, f_4 connecting the 2-complexes as above, with both f_0 and f_4 isomorphisms. As is standard, we use the symbol $\simeq$ to denote chain homotopy equivalence.

The Realisation problem

For a given group Γ, the Realisation problem asks if all algebraic 2-complexes are chain homotopy equivalent to a complex arising from a two dimensional CW Complex, i.e. the Cayley complex of some presentation for Γ. If so, we say that the *Realisation property holds for* Γ.

In what follows, we shall say that an algebraic 2-complex $\mathbf{E}$ is *realised geometrically* if there is a presentation $\mathcal{G}$ for Γ and a chain homotopy equivalence $\mathbf{E} \simeq C_*(\tilde{X}_{\mathcal{G}})$. Thus the Realisation property holds for Γ if and only if every algebraic 2-complex over Γ is geometrically realised.

The D(2) Problem

The D(2) Problem, as originally formulated by Wall in [29], asks if every three dimensional CW complex is necessarily homotopic to a complex of dimension two provided that, ranging over all possible coefficient systems, the complex has zero homology and cohomology in dimensions higher than two. The problem is parameterised by the fundamental group in the sense that, since homotopy equivalence induces an isomorphism on fundamental groups, one may prove or disprove the D(2) problem for CW complexes with a specified fundamental group. Accordingly, we say that the *D(2) property holds for* Γ if every three dimensional complex with fundamental group isomorphic to Γ, satisfying the hypothesis of the D(2) problem, is homotopy equivalent to a two dimensional complex. In [11], the D(2) problem is related to the Realisation problem by the following:

Theorem 1.1.2. *(F.E.A. Johnson) Let* Γ *be a finitely presented group such that there is an algebraic 2-complex:*

$$0 \longrightarrow \pi_2\mathbf{E} \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with $\pi_2\mathbf{E}$ *finitely generated over* $\mathbb{Z}[\Gamma]$. *Then the D(2) property holds for* Γ *if and only if the Realisation property holds for* Γ.

1.2 Statement of results

There is a natural home for algebraic 2-complexes: each algebraic 2-complex $\mathbf{E}$ represents a congruence class of 3-fold extensions of $\pi_2\mathbf{E}$ by $\mathbb{Z}$. In standard notation:

$$\mathbf{E} \in \operatorname{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E}).$$

One important calculation is to distinguish the algebraic 2-complexes from arbitrary extensions. We shall give a proof of the following:

Proposition 1.2.1. *Suppose that* Γ *satisfies the hypothesis of Theorem* 1.1.2 *and suppose also that* $\operatorname{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$. *Let* $\mathbf{E}$ *be an algebraic 2-complex. Let* H *denote the subset of extensions in* $\operatorname{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$ *which are congruent to projective extensions. Then the composition in* $\operatorname{End}_{\mathbb{Z}[\Gamma]}(\pi_2\mathbf{E})$ *induces a multiplicative group structure on* H.

A proof of the Proposition above may be constructed by combining the results of [11], where the equivalent is proved for finite groups, and the method of generalisation given in [10]. We shall give a slightly alternative proof. Our concern is the limitations placed on Johnson's approach, i.e. the condition that $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$. The first and perhaps easiest example of a group for which this condition fails is provided by C_∞^3 - the free abelian group on three generators. We shall prove:

Theorem A *The D(2) Property holds for* C_∞^3

We shall later show that if Γ is abelian and of rank one then the condition $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$ is satisfied. For such groups, the Bass-Murthy paper [1] shows that all stably free $\mathbb{Z}[\Gamma]$-modules are necessarily free; this result encouraged us to investigate the Realisation problem for the product of a cyclic group and the infinite cyclic group. We prove:

Theorem D *The D(2) Property holds for* $C_n \times C_\infty$.

This result was published in [8]. We remark that the Bass-Murthy result is essential to our proof, although its role is subtle. Indeed, in all confirmed cases of the D(2) property, the proof is dependent on a proof that stably free modules over the group ring are actually free. The class of groups for which the D(2) property has been already shown to hold is limited to finite abelian groups [18],[3], free groups [11], the dihedral groups of order $4n+2$ [12] and the dihedral group of order 8 [20].

A Mayer Vietoris Sequence

A large part of this thesis deals with a purely algebraic result regarding two-sided ideals, and provides an exact sequence of K groups which is complimentary to the standard excision sequence. Let Λ be a ring (with a unit) and suppose there is an exact sequence of Λ-modules

$$0 \longrightarrow J \longrightarrow \Lambda \longrightarrow R \longrightarrow 0$$

with J a two-sided ideal so that $(\Lambda/J) = R$ is a ring. An important condition that we shall sometimes assume is that R satisfies $\mathrm{Ext}^1(R, \Lambda) = 0$. Through the cohomological classification of Ext, one may identify $\mathrm{Ext}^1(R, J)$

with a quotient of the endomorphism ring $\mathrm{End}_\Lambda(J)$ by an additive subgroup X. Moveover X will prove to be a two-sided ideal, so that composition in $\mathrm{End}_\Lambda(J)$ defines a ring structure on $\mathrm{Ext}^1(R, J)$. We will write $\mathrm{Ext}(J)$ to denote this ring. We shall prove:

Theorem B *Let* Λ *be a ring and* J *a two-sided ideal satisfying* $\mathrm{Ext}^1(R, \Lambda) = 0$. *Then there exists a long exact sequence of abelian groups:*

$$\begin{aligned} K_1\Lambda \longrightarrow K_1S \oplus K_1R \longrightarrow K_1\mathrm{Ext}(J) \\ \longrightarrow K_0\Lambda \longrightarrow K_0S \oplus K_0R \longrightarrow K_0\mathrm{Ext}(J). \end{aligned}$$

where $S = \mathrm{End}_\Lambda(J)$

We shall construct the homomorphism $K_1\mathrm{Ext}(J) \longrightarrow K_0\Lambda$ as a map which corresponds, at the level of the units of $\mathrm{Ext}(J)$, to sending an extension with central module M to the class of $[M]$ in $K_0\Lambda$. In particular the units of $\mathrm{Ext}(J)$ represent projective extensions.

Our objective will then be to construct the homomorphisms above in as general circumstances as possible. For a two-sided ideal $J \subset \Lambda$, for each n we shall construct a series of maps:

$$\mathcal{M}_n(S) \oplus \mathcal{M}_n(R) \longrightarrow \mathcal{M}_n\mathrm{Ext}(J) \longrightarrow \mathrm{Mod}_\Lambda$$

which induce homomorphisms on K groups if defined (here $\mathcal{M}_n(R)$ means $n \times n$ matrices over R). We generalise Theorem B to the following:

Theorem C *Let* Λ *be a ring and* J *a two-sided ideal, then there is an exact sequence:*

$$K_1S \oplus K_1R \longrightarrow \tilde{K}_1\mathrm{Ext}(J) \longrightarrow K_0\Lambda$$

where:

R is the quotient ring Λ/J

$S = \mathrm{End}_\Lambda(J)$

$\tilde{K}_1\mathrm{Ext}(J)$ *is an abelian group such that*

$$\tilde{K}_1\mathrm{Ext}(J) = K_1\mathrm{Ext}(J) \qquad \textit{if} \quad \mathrm{Ext}^1(R, \Lambda) = 0$$

Chapter 2

Preliminaries regarding Ext

In this section we give a brief summary of the properties of the abelian group $\mathrm{Ext}^n(A, C)$ for modules A and C over a ring R. The content of this chapter is widely available elsewhere in most textbooks on homological algebra, our approach is most influenced by [19].

2.1 Exact sequences and congruence classes

If R is a ring, A, B, C are R modules and i, π are R-linear maps, then the sequence:

$$0 \longrightarrow C \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0$$

is said to be a short exact sequence if

1) i is injective

2) π is surjective

3) $\mathrm{Ker}(\pi) = \mathrm{Im}(i)$

Any short exact sequence as above may sometimes be called a 1-fold extension of C by A.

Lemma 2.1.1. *(The short Five Lemma) If there are two exact sequences and verti-*

cal homomorphisms between them such that the following commutes:

$$
\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & A \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & C \arrow[r, "j"] & X \arrow[r, "\sigma"] & A \arrow[r] & 0,
\end{tikzcd}
$$

then β is an isomorphism if α and γ are isomorphisms.

Two exact sequences with equal terminal modules are said to be **congruent** if there is a homomorphism β such that the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "j"] & X \arrow[r, "\sigma"] & A \arrow[r] & 0
\end{tikzcd}
$$

where the end vertical homomorphisms are the identity. By the five lemma any such β is an isomorphism and congruence defines an equivalence relation.

The set $\mathrm{Ext}^1(A, C)$ is defined as the 1-fold extensions of C by A under the equivalence relation of congruence (this may be made into a set by restricting the cardinality of the central modules).

Split extensions and the trivial extension

The trivial extension is taken to be the congruence class of the extension:

$$0 \longrightarrow C \xrightarrow{\tau} C \oplus A \xrightarrow{p} A \longrightarrow 0$$

where τ is the inclusion of C as a summand in $C \oplus A$ and p is the projection onto A. A 1-fold extension of C by A with homomorphisms i, π and central module B as above is said to be **split** if there is some homomorphism $\chi : B \to C$ such that $\chi i = \mathrm{Id}_C$.

Proposition 2.1.2. *An extension is split if and only if the extension is congruent to the trivial extension*

Proof. The trivial extension splits through the projection of $A \oplus C$ onto C. If an extension as above splits, define a function $\alpha : A \to B$ as any map such that $\pi\alpha = \mathrm{Id_A}$, and define $\hat{\alpha} = (1 - i\chi)\alpha$. Suppose that for some $r \in R$ and $a \in A$ we have $\alpha(a) = b$ and $\alpha(a.r) = x$, then $x = b.r + i(c)$ for some $c \in C$ and

$$\hat{\alpha}(a.r) = (1 - i\chi)(b.r + i(c)) = b.r - i\chi(b.r) + i(c) - i\chi i(c),$$

but $\chi i = \mathrm{Id_C}$ so

$$\hat{\alpha}(a.r) = b.r - i\chi(b.r) = \hat{\alpha}(a).r$$

and $\hat{\alpha} : A \to B$ is R-linear with $\pi\hat{\alpha} = \mathrm{Id_A}$. Define $\varphi : A \oplus C \to B$ as

$$\varphi(a, c) = \hat{\alpha}(a) + i(c).$$

Then the following commutes:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\tau"] \arrow[d, equal] & C \oplus A \arrow[r, "p"] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "i"] & B \arrow[r, "\pi"] & A \arrow[r] & 0
\end{tikzcd}$$

□

2.2 Pushouts and Pullbacks

Pushouts

Given some homomorphism $f : C \to C'$ and an 1-fold extension $\mathbf{E}$ of C by A we may construct an extension $f_*(\mathbf{E}) \in \mathrm{Ext}^1(A, C')$. Suppose that $\mathbf{E}$ is the extension:

$$\mathbf{E} = \ 0 \longrightarrow C \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0.$$

Define X to be the quotient module of $C' \oplus B$ defined by factoring out the submodule generated by elements of the form $(-f(c), i(c))$. Define

- $i' : C' \to X$ by $i'(c') = (c', 0)$,
- $\beta : B \to X$ as $\beta(b) = (0, b)$,
- $\pi' : X \to A$ as $\pi'(c, b) = \pi(b)$.

Then the following commutes:

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C' \arrow[r, "i'"] & X \arrow[r, "\pi'"] & A \arrow[r] & 0 \end{tikzcd}$$

and the bottom row is a 1-fold extension of C' by A - this provides the extension $f_*(\mathbf{E}) \in \mathrm{Ext}^1(A, C')$. We call $f_*(\mathbf{E})$ the **pushout** of **E** by f.

Proposition 2.2.1. *(see [19] 3.1.4) If there there are two exact sequences and a commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "\pi"] \arrow[d, "\beta"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C' \arrow[r, "i'"] & X \arrow[r, "\pi'"] & A \arrow[r] & 0 \end{tikzcd}$$

then the bottom row is congruent to $f_*(\mathbf{E})$

Pullbacks

Given some homomorphism $f : A' \to A$ and an 1-fold extension **E** of C by A we may construct an extension $f^*(\mathbf{E}) \in \mathrm{Ext}^1(A', C)$. Suppose that **E** is the extension:

$$\mathbf{E} = 0 \longrightarrow C \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0.$$

Define X to be the submodule of $B \oplus A'$ consisting of elements of the form (b, a') with $\pi(b) = f(a')$. Define

- $i' : C \to X$ by $i'(c) = (i(c), 0)$,
- $\beta : X \to B$ as $\beta(b, a') = b$,
- $\pi' : X \to A$ as $\pi'(b, a') = a'$.

Then the following commutes:

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "i'"] \arrow[d, equal] & X \arrow[r, "\pi'"] \arrow[d, "\beta"] & A' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & C \arrow[r, "i"] & B \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd}$$

and the top row is a 1-fold extension of C by A' - this provides the extension $f^*(\mathbf{E}) \in \mathrm{Ext}^1(A', C)$. We call $f^*(\mathbf{E})$ the **pullback** of **E** by f.

Proposition 2.2.2. *(see [19] 3.1.2) If there there are two exact sequences and a commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "i'"] \arrow[d, equal] & X \arrow[r, "\pi'"] \arrow[d, "\beta"] & A' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & C \arrow[r, "i"] & B \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd}$$

then the top row is congruent to $f^*(\mathbf{E})$

2.3 N-fold extensions and congruence

In general, if A,C and $\{E_i\}_{i=0}^{n-1}$ are R-modules and $\{\partial_i\}_{i=0}^{n}$ are R-linear maps, then the sequence

$$0 \longrightarrow C \xrightarrow{\partial_n} E_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} E_0 \xrightarrow{\partial_0} A \longrightarrow 0$$

is said to be an exact sequence if and only if

(i) ∂_n is injective.

(ii) $\mathrm{Ker}(\partial_{i-1}) = \mathrm{Im}(\partial_i)$ for each $1 \le i \le n$.

(iii) ∂_0 is surjective.

As in the statement of Theorem 1.1.1, when dealing with exact sequences we sometimes suppress the horizontal homomorphisms in order for the key point of diagrams to be clearer.

Two 1-fold extensions of the form:

$$0 \longrightarrow C \longrightarrow E_1 \xrightarrow{\pi} K \longrightarrow 0 \qquad 0 \longrightarrow K \xrightarrow{i} E_0 \longrightarrow A \longrightarrow 0$$

may be spliced together through the composite map $i\pi : E_1 \to E_0$ to produce a longer exact sequence:

$$0 \longrightarrow C \longrightarrow E_1 \xrightarrow{i\pi} E_0 \longrightarrow A \longrightarrow 0$$

and conversely any exact sequence with $n+2$ terms may be decomposed into n short exact sequences. The above exact sequence is said to be a 2-fold extension of C by A. In general an exact sequence:

$$0 \longrightarrow C \longrightarrow E_{n-1} \longrightarrow E_{n-2} \longrightarrow \cdots E_0 \longrightarrow A \longrightarrow 0$$

is said to be an **n-fold extension of** C **by** A.

A **simple congruence** $F : E \to E'$ between two n-fold extensions of C by A is a series of module homomorphisms $\{f_i\}_{i=0}^{n-1}$ and a commutative diagram:

$$\begin{tikzcd}
E: \arrow[d, "F"] & 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & E_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & E_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots E_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
E': & 0 \arrow[r] & C \arrow[r] & E'_{n-1} \arrow[r] & E'_{n-2} \arrow[r] & \cdots E'_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Two n-fold extensions E and E' of C by A are said to be **congruent** if and only if there is a finite sequence $S_1, \ldots, S_m$ of n-fold extensions of C by A such that

(i) Either E is simply congruent to S_1, or S_1 is simply congruent to E

(ii) Either E' is simply congruent to S_m, or S_m is simply congruent to E'

(iii) For all $2 \leq i \leq m$, either S_i is simply congruent to S_{i-1}, or S_{i-1} is simply congruent to S_i

Congruence is an equivalence relation and the set of congruence classes of n-fold extensions of C by A is denoted $\mathrm{Ext}^n(C, A)$

Pushouts and pullbacks of n-fold extensions

Given an n-fold extension $\mathbf{E}$ of C by A

$$0 \longrightarrow C \xrightarrow{i} E_{n-1} \xrightarrow{\delta_{n-1}} E_{n-2} \xrightarrow{\delta_{n-2}} \cdots E_0 \xrightarrow{\delta_0} A \longrightarrow 0$$

we may truncate this to form a 1-fold extension $\mathbf{E}_-$ of C by $\mathrm{Ker}(\delta_{n-2})$:

$$\mathbf{E}_- : \qquad 0 \longrightarrow C \xrightarrow{i} E_{n-1} \xrightarrow{\delta_{n-1}} \mathrm{Ker}(\delta_{n-2}) \longrightarrow 0$$

and for any $f : C \to C'$ we may form the pushout $f_*(\mathbf{E}_-)$:

$$f_*(\mathbf{E}_-) : \qquad 0 \longrightarrow C' \xrightarrow{i'} X \xrightarrow{\pi} \mathrm{Ker}(\delta_{n-2}) \longrightarrow 0.$$

This may be spliced back together with the remaining segment of $\mathbf{E}$ and defines the pushout $f_*(\mathbf{E}) \in \mathrm{Ext}^n(C', A)$:

$$f_*(\mathbf{E}) : \qquad 0 \longrightarrow C' \xrightarrow{i'} X \xrightarrow{\pi} E_{n-2} \xrightarrow{\delta_{n-2}} \cdots E_0 \xrightarrow{\delta_0} A \longrightarrow 0.$$

Similarly, let $\mathbf{E}_+$ denote the extension

$$\mathbf{E}_+ : \qquad 0 \longrightarrow \mathrm{Im}(\delta_1) \xrightarrow{j} E_0 \xrightarrow{\delta_0} A \longrightarrow 0,$$

then for any $f : A' \to A$ we may form the pullback $f^*(\mathbf{E}_+)$ and splice to define the pullback $f^*(\mathbf{E}) \in \mathrm{Ext}^n(C, A')$

2.4 The classification of Ext via cohomology

In this section we shall see that the group $\mathrm{Ext}^n(C, A)$ may be identified with the cohomology group $H^n(\mathbf{P}, C)$ for $\mathbf{P}$ a projective resolution of A. We assume that the reader is familiar with the definitions and basic properties of projective modules and cohomology groups. Let A, C be R modules and suppose that there is a projective resolution:

$$\mathbf{P} : \qquad \cdots \xrightarrow{\partial} P_n \xrightarrow{\partial} P_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_0 \xrightarrow{\partial} A \longrightarrow 0$$

and an extension $\mathbf{E} \in \mathrm{Ext}^n(C, A)$

$$\mathbf{E} : \qquad 0 \longrightarrow C \xrightarrow{\delta} E_{n-1} \xrightarrow{\delta} E_{n-2} \xrightarrow{\delta} \cdots E_0 \xrightarrow{\delta} A \longrightarrow 0.$$

Through the characteristic property of projective modules, we may pick homomorphisms $\{g_i\}_{i=0}^{n+1}$ such that there is a commutative diagram:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "\partial"] \arrow[d, "g_{n+1}"] & P_n \arrow[r, "\partial"] \arrow[d, "g_n"] & P_{n-1} \arrow[r, "\partial"] \arrow[d, "g_{n-1}"] & \cdots \arrow[r, "\partial"] & P_0 \arrow[r, "\partial"] \arrow[d, "g_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "\delta"] & E_{n-1} \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & E_0 \arrow[r, "\delta"] & A \arrow[r] & 0.
\end{tikzcd}$$

Since $g_n\partial = 0$, g_n represents an n-cocycle and is an element of the cohomology group $H^n(\mathbf{P}, C)$.

Lemma 2.4.1. *If there is also a commutative diagram:*

$$\begin{tikzcd}
P_{n+1} \arrow[r, "\partial"] \arrow[d, "f_{n+1}"] & P_n \arrow[r, "\partial"] \arrow[d, "f_n"] & P_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "\partial"] & P_0 \arrow[r, "\partial"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "\delta"] & E_{n-1} \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & E_0 \arrow[r, "\delta"] & A \arrow[r] & 0.
\end{tikzcd}$$

then there is some $h : P_{n-1} \to C$ such that $f_n - g_n = h\partial$. In particular f_n and g_n differ by a coboundary and represent the same element of $H^n(\mathbf{P}, C)$.

Proof. The first step is to note that the hypothesis imply that the following commutes:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "\partial"] & P_n \arrow[r, "\partial"] \arrow[d, "(f_n-g_n)"] & P_{n-1} \arrow[r, "\partial"] \arrow[d, "(f_{n-1}-g_{n-1})"] & \cdots \arrow[r, "\partial"] & P_0 \arrow[r, "\partial"] \arrow[d, "(f_0-g_0)"] & A \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & C \arrow[r, "\delta"] & E_{n-1} \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & E_0 \arrow[r, "\delta"] & A \arrow[r] & 0.
\end{tikzcd}$$

Define $h_0 : A \to E_0$ as $h_0 = 0$. Since $(f_0 - g_0)$ maps into $\mathrm{Ker}(\delta)$ and since P_0 is projective there is some $h_1 : P_0 \to E_1$ such that the following commutes:

$$\begin{tikzcd}
 & P_0 \arrow[dl, "h_1"'] \arrow[d, "(f_0-g_0)"] \\
E_1 \arrow[r, "\delta"] & E_0
\end{tikzcd}$$

and $\delta h_1 = (f_0 - g_0)$.

Suppose that for $i < k$ there are maps $h_{i+1} : P_i \to E_{i+1}$ such that

$$\delta h_{i+1} = (f_i - g_i) - h_i\partial.$$

Then $\delta(f_k - g_k - h_k\partial) \quad = \quad (f_{k-1} - g_{k-1})\partial - (f_{k-1} - g_{k-1} - h_{k-1}\partial)\partial = 0$ so $(f_k - g_k - h_k\partial)$ maps the projective module P_{k+1} into $\mathrm{Ker}(\delta)$ and there is a homomorphism $h_{k+1} : P_k \to E_{k+1}$ with commutative:

$$\begin{tikzcd}
 & P_k \arrow[dl, "h_{k+1}"'] \arrow[d, "(f_k-g_k-h_k\partial)"] \\
E_{k+1} \arrow[r, "\delta"] & E_k.
\end{tikzcd}$$

The diagram shows that $\delta h_{k+1} = (f_k - g_k) - h_k\partial$ and so, by induction, we may deduce that there are maps $h_n : P_{n-1} \to C$ and $h_{n+1} : P_n \to 0$ such that

$$\delta h_{n+1} = (f_n - g_n) - h_n\partial.$$

But $h_{n+1} : P_n \to 0$ is necessarily zero, so

$$f_n - g_n = h_n\partial$$

and the result is shown. □

Thus given a n-fold extension $\mathbf{E}$ of C by A and projective resolution $\mathbf{P}$, we define $\zeta(\mathbf{E}) \in H^n(\mathbf{P}, C)$ to be the corresponding cohomology class. If $\mathbf{E}$ is simply congruent to another extension $\mathbf{E}'$, the $\zeta(\mathbf{E}) = \zeta(\mathbf{E}')$ and hence ζ is a well defined map on congruence classes.

Theorem 2.4.2. *(The classification of* Ext *via cohomology - see [19] III.6.4) If A,C are R modules over a ring R and $\mathbf{P}$ is a projective resolution of A, then*

$$\zeta : \mathrm{Ext}^n(C, A) \to H^n(\mathbf{P}, C)$$

is an isomorphism of sets. When $n = 1$ the trivial extension corresponds to the zero cohomology class.

The details of the proof are technical, we shall show how to construct the inverse to ζ. The resolution $\mathbf{P}$ may be truncated to give an n-fold extension:

$$0 \longrightarrow \mathrm{Ker}(\partial) \longrightarrow P_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_0 \xrightarrow{\partial} A \longrightarrow 0$$

Given some cohomology class of $H^n(\mathbf{P}, C)$ we may represent it by some $h : P_n \to C$. Define $\hat{h} : \mathrm{Ker}(\partial) \to C$ such $\hat{h}(a) = h(b)$ for any b such that $\partial(b) = a$. Since h is a cocycle $h\partial = 0$ and hence $\hat{h}$ is well defined, indeed it is unique.

Thus we may form the pushout:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\partial) \arrow[r] \arrow[d, "\hat{h}"] & P_{n-1} \arrow[r, "\partial"] \arrow[d, "\beta"] & \mathrm{Im}(\partial) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "i"] & X \arrow[r] & \mathrm{Im}(\partial) \arrow[r] & 0
\end{tikzcd}$$

This may be spliced together with the remaining segment of $\mathbf{P}$ to give an n-fold extension $\mathbf{E}$

$$\mathbf{E}: \quad 0 \longrightarrow C \xrightarrow{i} X \longrightarrow P_{n-2} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_0 \xrightarrow{\partial} A \longrightarrow 0$$

and there is a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & P_n \arrow[r, "\partial"] \arrow[d, "h"] & P_{n-1} \arrow[r, "\partial"] \arrow[d, "\beta"] & P_{n-2} \arrow[r, "\partial"] \arrow[d, equal] & \cdots \arrow[r, "\partial"] & P_0 \arrow[r, "\partial"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "i"] & X \arrow[r] & P_{n-2} \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & P_0 \arrow[r, "\partial"] & A \arrow[r] & 0
\end{tikzcd}$$

so that $\zeta(\mathbf{E}) = \text{class(h)} \in H^n(\mathbf{P}, C)$. Note that we have not proved that the inverse to ζ is well defined on cohomology classes.

In the case where $n = 1$, define $f : P_0 \to C \oplus A$ by

$$f(x) = (0, \partial(x))$$

and, recalling the notation used in the definition of the split extension, the following commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "\partial"] \arrow[d, "0"] & P_0 \arrow[r, "\partial"] \arrow[d, "f"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r, "\tau"] & C \oplus A \arrow[r, "p"] & A \arrow[r] & 0.
\end{tikzcd}$$

The zero homology class therefore corresponds to the split extension.

2.5 The practical calculation of Ext

The classification of Ext via cohomology is often applied in the following way: for R modules A and C with truncated projective resolution

$$\mathbf{P}: \qquad 0 \longrightarrow \text{Ker}(\partial_{n-1}) \xrightarrow{i} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \longrightarrow 0,$$

the group $\text{Ext}^n(C, A)$ may be identified with the quotient group:

$$\text{Hom}(\text{Ker}(\partial_{n-1}), C) \quad / \quad X$$

where X is the subgroup consisting of homomorphisms f such that there is some $\eta : P_{n-1} \to C$ with

$$\eta i = f.$$

For each $f \in \text{Hom}(Ker(\partial_{n-1}), C)$ the corresponding class of extensions is usually denoted $[f]$ in this thesis and is given by the pushout $f_*(\mathbf{P})$.

Dimension shifting and the abelian group structure on Ext

Following the notation of [11], if R is a ring and N is any R module, we write $M \in D_n(N)$ if there is an exact sequence of R-modules:

$$\mathbf{S}: \qquad 0 \longrightarrow M \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

with every $\{P_i\}_{i=0}^{n-1}$ projective.

Corollary 2.5.1. *(Dimension shifting property of* Ext*)*
If R is a ring and A, M, N are R-modules with $M \in D_n(N)$ and $n \geq 1$, then for all $k \geq 1$:

$$\mathrm{Ext}^{k+n}(A, N) \cong \mathrm{Ext}^{k}(A, M).$$

Proof. As soon as it is noticed that the sequence S may be extended to a projective resolution of any length, the corollary is an immediate consequence of the cohomological classification of Ext. □

Through the set isomorphism $\zeta : \mathrm{Ext}^n(C, A) \to H^n(\mathbf{P}, C)$ the structure of abelian group is imposed on $\mathrm{Ext}^n(C, A)$. Moreover the trivial element of the cohomology group corresponds to the trivial or split extension in the case where $n = 1$. There is also an internal group structure on Ext groups, the *Baer Sum* - see [19] sections III.2 and III.6 for details of this group structure and a proof that ζ is a group isomorphism under these terms.

The additivity of Ext

If A, B, C, D are R-modules, then the standard isomorphism

$$\mathrm{Hom}(A \oplus B, C \oplus D) \cong \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(A, D) \oplus \mathrm{Hom}(B, C) \oplus \mathrm{Hom}(B, D)$$

extends to an isomorphism of cohomology groups and consequently Ext groups. We show how this works in practice.

If A and B are R-modules and there are projective resolutions:

$$\mathbf{P}: \quad \cdots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \longrightarrow 0$$

$$\mathbf{Q}: \quad \cdots \xrightarrow{\delta_{n+1}} Q_n \xrightarrow{\delta_n} Q_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} Q_0 \xrightarrow{\delta_0} B \longrightarrow 0$$

then there is a projective resolution of $A \oplus B$ given by:

$$\mathbf{P} \oplus \mathbf{Q}: \quad \cdots \xrightarrow{d_{n+1}} P_n \oplus Q_n \xrightarrow{d_n} P_{n-1} \oplus Q_{n-1} \xrightarrow{d_{n-1}} \cdots$$

$$\cdots \xrightarrow{d_1} P_0 \oplus Q_0 \xrightarrow{d_0} A \oplus B \longrightarrow 0$$

where

$$d_i(p, q) = (\partial_i(p), \delta_i(q))$$

and in particular

$$\mathrm{Ker}(d_n) = \mathrm{Ker}(\partial_n) \oplus \mathrm{Ker}(\delta_n).$$

If C is another R-module and $(f,h) \in \mathrm{Hom}(\mathrm{Ker}(d_n), C)$, then clearly $(f,h) = \eta d_n$ for some $\eta : P_n \oplus Q_n \to C$ if and only if $f : \mathrm{Ker}(\partial_n) \to C$ and $h : \mathrm{Ker}(\delta_n) \to C$ factor through P_n and Q_n respectively. This correspondence leads to an isomorphism:

$$\mathrm{Ext}^n(C, A \oplus B) \cong \mathrm{Ext}^n(C, A) \oplus \mathrm{Ext}^n(C, B).$$

Suppose D is another R-module, then we calculate $\mathrm{Ext}^n(A, C \oplus D)$ through homomorphisms $f : \mathrm{Ker}(\partial_n) \to C \oplus D$ modulo those which factor through P_n. But

$$\mathrm{Hom}(\mathrm{Ker}(\partial_n), C \oplus D) \cong \mathrm{Hom}(\mathrm{Ker}(\partial_n), C) \oplus \mathrm{Hom}(\mathrm{Ker}(\partial_n), D)$$

and each such f may be represented as

$$f = (h,k) \qquad h \in \mathrm{Hom}(\mathrm{Ker}(\partial_n), C), \quad k \in \mathrm{Hom}(\mathrm{Ker}(\partial_n), D)$$

and f factors through P_n if and only if both h and k do. This leads to an isomorphism

$$\mathrm{Ext}^n(C \oplus D, A) \cong \mathrm{Ext}^n(C, A) \oplus \mathrm{Ext}^n(D, A).$$

Proposition 2.5.2. *(The additivity of* Ext*) If A, B, C, D are R-modules, then for all $n \geq 1$:*

$$\mathrm{Ext}^n(A \oplus B, C \oplus D) \cong \mathrm{Ext}^n(A, C) \oplus \mathrm{Ext}^n(A, D) \oplus \mathrm{Ext}^n(B, C) \oplus \mathrm{Ext}^n(B, D)$$

This completes our brief survey of the properties of Ext.

Chapter 3

Reductions of the Realisation Problem

3.1 Schanuel's Lemma and stabilisation

Johnson outlines a general procedure for approaching the Realisation problem, which we shall now discuss. We shall use the standard results regarding extensions, available in most textbooks on homological algebra. We use (Mac Lane [19]) as our reference, the comprehensive nature of which is our justification. Immediately from (Mac Lane [19] III.5.2), we see that:

Proposition 3.1.1. *If* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *are algebraic 2-complexes which are congruent, then* $\mathbf{E}_1 \simeq \mathbf{E}_2$.

This ensures that the property of geometric realisation is well defined on congruence classes.

The first major reduction is due to Schanuel (Swan [26] Section 1) and shows that $\pi_2\mathbf{E}$ is determined up to a form of algebraic stability. We shall use the following version of Schanuel's Lemma (W. Mannan [20]), which is proved for general rings with a unit.

Theorem 3.1.2. *(Schanuel's Lemma) Suppose that* P_i, Q_i *are projective modules occurring in exact sequences:*

$$0 \longrightarrow M \xrightarrow{i} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} N \longrightarrow 0$$

$$0 \longrightarrow K \xrightarrow{j} Q_2 \xrightarrow{\partial_2'} Q_1 \xrightarrow{\partial_1'} Q_0 \xrightarrow{\partial_0'} N \longrightarrow 0.$$

Set

$$R = Q_2 \oplus P_1 \oplus Q_0$$
$$S = P_2 \oplus Q_1 \oplus P_0.$$

Then the 2-complexes:

$$0 \longrightarrow M \oplus R \xrightarrow{i \oplus Id} P_2 \oplus R \xrightarrow{\partial_2 \oplus 0} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} N \longrightarrow 0$$
$$0 \longrightarrow K \oplus S \xrightarrow{j \oplus Id} Q_2 \oplus S \xrightarrow{\partial'_2 \oplus 0} Q_1 \xrightarrow{\partial'_1} Q_0 \xrightarrow{\partial'_0} N \longrightarrow 0.$$

are chain homotopy equivalent.

Geometric and algebraic stabilisation

Schanuel's Lemma has two important corollaries. Suppose that a presentation $\mathcal{G}$ for Γ realises the extension:

$$C_*(\tilde{X}_{\mathcal{G}}) = 0 \to \pi_2(\mathcal{G}) \xrightarrow{i} \mathbb{Z}[\Gamma]^{a_1} \xrightarrow{\delta_2} \mathbb{Z}[\Gamma]^{a_2} \xrightarrow{\delta_1} \mathbb{Z}[\Gamma]^{a_3} \to \mathbb{Z} \to 0,$$

then the addition of n relations of the form $e = e$ to $\mathcal{G}$ realises the extension:

$$\Sigma^n(C_*(\tilde{X}_{\mathcal{G}})) = 0 \to \pi_2(\mathcal{G}) \oplus \mathbb{Z}[\Gamma]^n \xrightarrow{i \oplus Id} \mathbb{Z}[\Gamma]^{a_1+n} \xrightarrow{\delta_2 \oplus 0} \mathbb{Z}[\Gamma]^{a_2} \xrightarrow{\delta_1} \mathbb{Z}[\Gamma]^{a_3} \to \mathbb{Z} \to 0.$$

If $\mathbf{E}$ is an arbitrary extension in $\mathrm{Ext}^3(\mathbb{Z}, \pi_2(\mathcal{G})$ then one may define $\Sigma^n(\mathbf{E})$ to be the extension in $\mathrm{Ext}^3(\mathbb{Z}, \pi_2(\mathcal{G} \oplus \mathbb{Z}[\Gamma])$ obtained similarly. Schanuel's Lemma shows:

Corollary 3.1.3. *Given algebraic 2-complexes* $\mathbf{E}_1$ *and* $\mathbf{E}_2$*, there are natural numbers* a *and* b *such that there is a chain homotopy equivalence*

$$\Sigma^a(\mathbf{E}_1) \simeq \Sigma^b(\mathbf{E}_2).$$

We remark that the geometric equivalent of Corollary 3.1.3 is well known, and may be derived from Tietze's Theorem regarding the moves required to transform one group presentation into another [11].

Corollary 3.1.4. *Given algebraic 2-complexes* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *over a group* Γ*, there are natural numbers* a *and* b *and an isomorphism*

$$\pi_2\mathbf{E}_1 \oplus \mathbb{Z}[\Gamma]^a \quad \cong \quad \pi_2\mathbf{E}_2 \oplus \mathbb{Z}[\Gamma]^b.$$

We describe such a relationship by saying that $\pi_2\mathbf{E}_1$ is *stably equivalent* (or stably isomorphic) to $\pi_2\mathbf{E}_2$.

3.2 Johnson's approach

For a specified fundamental group, Johnson's approach to the Realisation problem is to identify all chain homotopy equivalence classes of algebraic 2-complexes, and then determine which are realised by a presentation. This breaks up into several steps:

(i) An initial presentation is used to construct a first algebraic 2-complex with corresponding $\pi_2\mathbf{E}$.

(ii) The resulting 2-complex may be used for the purposes of calculating the cohomology group $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$.

(iii) Assuming the hypothesis of Theorem 1.1.2 are satisfied, and in view of Corollary 3.1.4, the first non-trivial task is to describe the modules which are stably isomorphic to a given $\pi_2\mathbf{E}$; this has historically presented the more difficult objective.

(iv) The final step determines each class of algebraic 2-complexes with a given terminal module M stably equivalent to $\pi_2\mathbf{E}$, and asks which of these are chain homotopy equivalent to a complex arising from a group presentation.

In this section we give an overview of the existing simplifications for the final step above. There is an immediate consequence of the definition of the function Σ^n, of which we shall sketch the proof. This result is not explicitly found elsewhere, although it constitutes a fairly intuitive result.

Proposition 3.2.1. *Let $\pi_2\mathbf{E}$ be as in Theorem 1.1.2. If $\mathbf{X}$ is an arbitrary extension in $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$, then $\mathbf{X}$ is congruent to an algebraic 2-complex if and only if $\Sigma(\mathbf{X})$ is.*

Proof. If $\mathbf{X}$ is congruent to some algebraic complex $\mathbf{E}$, then clearly $\Sigma(\mathbf{X})$ is congruent to the algebraic complex $\Sigma(\mathbf{E})$.

Suppose that $\Sigma(\mathbf{X})$ is congruent to an algebraic complex $\mathbf{E}'$. By hypothesis there exists an exact sequence:

$$0 \longrightarrow \pi_2\mathbf{E} \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

with each F_i a finitely generated free $\mathbb{Z}[\Gamma]$-module. By the standard classification of Ext through cohomology (Mac Lane [19] III.6.4) we may assume each congruence class $\mathbf{X}$ to be represented by an extension of the form:

$$\mathbf{X} = 0 \longrightarrow \pi_2\mathbf{E} \longrightarrow E_X \xrightarrow{f} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,$$

so that the congruence class of $\Sigma(\mathbf{X})$ is represented by:

$$\Sigma(\mathbf{X}) = 0 \longrightarrow \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma] \longrightarrow E_X \oplus \mathbb{Z}[\Gamma] \xrightarrow{f\oplus 0} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0.$$

It may be shown (Mannan [20] 1.5.1) that the algebraic 2-complex $\mathbf{E}'$ is necessarily congruent to an extension:

$$\mathbf{E}' = 0 \longrightarrow \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma] \longrightarrow S \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

with S stably free. By the dimension shifting property of Ext and since $\mathbf{E}'$ is congruent to $\Sigma(\mathbf{X})$, the two extensions:

$$0 \longrightarrow \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma] \longrightarrow E_X \oplus \mathbb{Z}[\Gamma] \longrightarrow \mathrm{Ker}(\partial_1) \longrightarrow 0$$

$$0 \longrightarrow \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma] \longrightarrow S \longrightarrow \mathrm{Ker}(\partial_1) \longrightarrow 0$$

are congruent. Then by the five lemma E_X is stably free and

$$E_X \oplus \mathbb{Z}[\Gamma]^b \cong \mathbb{Z}[\Gamma]^a$$

for some suitable chosen a and b. We complete the proof by noticing that there is a simple congruence between $\mathbf{X}$ and the algebraic 2-complex given by:

$$0 \longrightarrow \pi_2\mathbf{E} \longrightarrow E_X \oplus \mathbb{Z}[\Gamma]^b \xrightarrow{f\oplus Id} F_1 \oplus \mathbb{Z}[\Gamma]^b \xrightarrow{\partial_1\oplus 0} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

where the vertical maps on each of the component modules of $\mathbf{X}$ are the identity maps into the corresponding summand in the sequence above. □

We have shown that $\pi_2\mathbf{E}$ is determined up to stability, and in some cases we may pick a *minimal* module $\pi_2\mathbf{E}$, in the sense that for any module N :

$$\pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]^a \cong N \oplus \mathbb{Z}[\Gamma]^b \quad \Rightarrow \quad a \geq b.$$

Johnson has shown that for any group satisfying the hypothesis of Theorem 1.1.2, a minimal module $\pi_2\mathbf{E}$ must exist [15]. For finite groups, it is shown in [11] that it is sufficient to realise extensions at a minimal level, and we elicit the key condition required in order to prove this:

Proposition 3.2.2. *(F.E.A. Johnson) Suppose that we are given Γ satisfying the hypothesis of Theorem* 1.1.2 *and such that* $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$. *Suppose also that we are given some $\mathbb{Z}[\Gamma]$ module M such that each algebraic complex $\mathbf{E}$ with $\pi_2\mathbf{E} = M$ is geometrically realised. Then each algebraic complex with $\pi_2\mathbf{E} = M \oplus \mathbb{Z}[\Gamma]$ is geometrically realised.*

Proof. Note that the hypotheses of the Proposition ensure that each congruence class containing an algebraic 2-complex in $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$ is geometrically realised. The stablisation operation

$$\Sigma^1 : \mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E}) \to \mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma])$$

corresponds to the inclusion in the standard isomorphism

$$\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E}) \oplus \mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) \cong \mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]).$$

See e.g. [19] for a proof that the functor Ext is additive in both variables. Furthermore since $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$ we see that Σ^1 is an isomorphism, and by Proposition 3.2.1, Σ^1 is a bijection on sequences congruent to algebraic 2-complexes. The result follows. □

Supposing the modules stably equivalent to a given $\pi_2\mathbf{E}$ have been determined, one wishes to distinguish the congruence classes of algebraic 2-complexes. In general, this is non-trivial even for finite groups, but under certain restrictions the *projective* extensions may be identified as follows:

Proposition 3.2.3. *(F.E.A. Johnson) Suppose that we are given Γ satisfying the hypothesis of Theorem* 1.1.2 *and such that* $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$. *Then there is a natural ring structure on* $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$, *under which congruence classes of projective extensions are precisely the units of* $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$.

We shall sketch a proof of Johnson's result in a later section. Clearly the condition $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$ is highly desirable, but will not hold for all groups; it fails for example if Γ is a group of cohomological dimension three.

3.3 An introductory example

We begin our investigations with a quick and easy proof that the D(2) property holds for the fundamental group of the torus.

Theorem E *The Realisation property holds for $C_\infty \times C_\infty$.*

We remark before the proof that it follows from the Quillen-Suslin proof of Serre's conjecture that if R is any principal ideal domain and Γ is a free abelian group, then every projective $R[\Gamma]$-module is free, see for example Proposition 4.12 in Chapter 5 of T.Y. Lam's exposition [17]. In particular all stably free $\mathbb{Z}[\Gamma]$-modules are free.

Proof. We may take the presentation $\mathcal{G} = \langle x, t | xt = tx \rangle$ for Γ, leading to the algebraic complex:

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z}[\Gamma] \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} 1-t \\ x-1 \end{pmatrix} \quad , \quad \partial_1 = (x-1 \quad t-1).$$

Immediately we deduce that $\mathrm{Ext}^3(\mathbb{Z}, M) = 0$ for any $\mathbb{Z}[\Gamma]$-module M, and if there is an algebraic 2-complex

$$\mathbf{E} = 0 \longrightarrow \pi_2\mathbf{E} \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

then by Schanuel's Lemma and the remark above $\pi_2\mathbf{E}$ is free of rank (say) n. Thus the addition of n trivial relations to the presentation $\mathcal{G}$ results in an algebraic 2-complex which is geometrically realised and congruent to $\mathbf{E}$. Since $\mathbf{E}$ was arbitrary, this implies that each algebraic 2-complex over $\mathbb{Z}[\Gamma]$ is realised up to homotopy equivalence by a presentation for Γ. □

Indeed, since the second homotopy module of any 2-manifold is necessarily zero, other than S^2 and $\mathbb{R}P^2$ which may be dealt with separately, if Γ is the fundamental group of *any* surface and if all stably free $\mathbb{Z}[\Gamma]$-modules are free, a similar proof shows that the D(2) property holds for Γ.

Chapter 4

K_1 of projective extensions

4.1 Introduction

We shall return to geometric considerations in chapter 5, the next two chapters are algebraic in subject matter and pertain to general two-sided ideals. The proofs will then adapt to geometric applications.

Suppose that Λ is a ring with a unit and J is a two-sided ideal in Λ, giving rise to an exact sequence:

$$\mathbf{X} = \ 0 \longrightarrow J \xrightarrow{j} \Lambda \xrightarrow{\sigma} R \longrightarrow 0,$$

with j inclusion and σ the natural map onto the quotient. Through j and σ, J and R have natural (right) Λ-module structures and $\mathbf{X}$ may be considered to represent a congruence class of the abelian group $\mathrm{Ext}^1(R, J)$.

Notation: $\mathrm{Ext}^1(R, J)$ is completely determined by the ideal $J \leq \Lambda$: define
$\mathrm{Ext}^1(R, J) = \mathrm{Ext}(J)$. Also set $S = \mathrm{End}_\Lambda(J)$.

Theorem 4.1.1. *If Λ is a ring and J a two-sided ideal in Λ, then there is a fibre product of additive groups:*

$$\begin{tikzcd} \Lambda \arrow[r, "i_1"] \arrow[d, "i_2"] & R \arrow[d, "j_1"] \\ S \arrow[r, "j_2"] & \mathrm{Ext}(J) \end{tikzcd}$$

with i_1, j_2 surjective.

Construction of i_1,i_2

Letting Λ be a right module over itself, there is a natural ring isomorphism $\mathrm{End}_\Lambda(\Lambda) \cong \Lambda$; through left multiplication any $f \in \Lambda$ represents an endomorphism of Λ. Since J is a two-sided ideal, there are $f_- \in S$, $f_+ \in \mathrm{End}(R)$ for each $f \in \Lambda$ such that the following commutes:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "j"] \arrow[d, "f_-"] & \Lambda \arrow[r, "\sigma"] \arrow[d, "f"] & R \arrow[r] \arrow[d, "f_+"] & 0 \\
0 \arrow[r] & J \arrow[r, "j"] & \Lambda \arrow[r, "\sigma"] & R \arrow[r] & 0.
\end{tikzcd}$$

Since there are ring isomorphisms $\mathrm{End}_\Lambda(R) = \mathrm{End}_R(R) \cong R$, we may consider f_+ as an element of R. Indeed $f_+ = \sigma(f)$. Define

$$i_1(f) = f_+ \qquad i_2(f) = f_-.$$

Construction of j_2

Explicitly, for $f \in S$ we take $j_2(f)$ to be the congruence class of the bottom row in the pushout:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "j"] \arrow[d, "f"] & \Lambda \arrow[r, "\sigma"] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J \arrow[r] & M \arrow[r] & R \arrow[r] & 0.
\end{tikzcd}$$

Proposition 4.1.2. *j_2 is a well defined surjective homomorphism.*

Proof. By standard homological algebra ([19] III.9.1), there is an exact sequence of group homomorphisms:

$$\mathrm{Hom}(R, J) \longrightarrow \mathrm{Hom}(\Lambda, J) \longrightarrow S \xrightarrow{j_2} \mathrm{Ext}^1(R, J) \longrightarrow \mathrm{Ext}^1(\Lambda, J)$$

where in Mac Lane's notation $j_2 = \mathbf{X}^*$ (recall $\mathbf{X}$ is the initial sequence). Note that since Λ is projective, $\mathrm{Ext}^1(\Lambda, J) = 0$ and consequently j_2 is surjective. $\square$

Construction of j_1

There is also an exact sequence:

$$\mathrm{Hom}(R, J) \longrightarrow \mathrm{Hom}(R, \Lambda) \longrightarrow R \xrightarrow{j_1} \mathrm{Ext}^1(R, J) \longrightarrow \mathrm{Ext}^1(R, \Lambda)$$

Where in Mac Lane's notation $j_1 = \mathbf{X}_*$. Representing endomorphisms as left multiplication, for $f \in R$ we may take $j_1(f)$ to be the congruence class of the bottom row in the pullback:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "j"] & \Lambda \arrow[r, "\sigma"] & R \arrow[r] & 0 \\ 0 \arrow[r] & J \arrow[r] \arrow[u, equal] & M \arrow[r] \arrow[u] & R \arrow[r] \arrow[u, "f"] & 0. \end{tikzcd}$$

Clearly if $\mathrm{Ext}^1(R, \Lambda) = 0$ then j_1 is surjective, but this may not necessarily be the case.

An important condition

We have already seen that the condition

$$\mathrm{Ext}^1(R, \Lambda) = 0 \qquad\qquad (\star)$$

may be used to imply certain properties of $\mathrm{Ext}(J)$, in particular the surjectivity of j_1. Indeed, the condition will prove to be remarkably useful, and we shall say that condition $(\star)$ holds in this case.

Proof of Theorem 4.1.1. By construction i_1, i_2, j_1, j_2 are group homomorphisms. We have already shown that j_2 is surjective. By the projectivity of Λ, any Λ-module endomorphism of R lifts to an endomorphism of Λ (see e.g. [19] III.6.1), and i_1 is surjective. Suppose that $f \in \Lambda$, then as in ([19] III.1.5), the morphism $(f_-, f, f_+) : \mathbf{X} \to \mathbf{X}$ factors and there is a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "j"] \arrow[d, "f_-"] & \Lambda \arrow[r, "\sigma"] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "f_+"] & 0 \\ 0 \arrow[r] & J \arrow[r, "j"] & \Lambda \arrow[r, "\sigma"] & R \arrow[r] & 0. \end{tikzcd}$$

which shows that

$$j_1 i_1(f) = j_2 i_2(f).$$

Given f_-, f_+ with $j_2(f_-) = j_1(f_+)$, a similar diagram shows that there exists an f with $i_1(f) = f_+, i_2(f) = f_-$ and hence we have shown that the square in question is indeed a fibre product of abelian groups.

The product structure on Ext

We would like to show that in fact we have a fibre product of rings, but certainly this requires a multiplicative structure on $\mathrm{Ext}(J)$. Under the restriction $(\star)$ we may impose a product on $\mathrm{Ext}(J)$ through the surjective $j_2 : S \to \mathrm{Ext}(J)$.

Proposition 4.1.3. *Suppose that*

$$\mathrm{Ext}^1(R, \Lambda) = 0$$

then $\mathrm{Ker}(j_2)$ *is a two-sided ideal in* S

Proof. For $h \in S$, by definition $h \in \mathrm{Ker}(j_2)$ if and only if the corresponding extension $j_2(h)$ splits and there is a homomorphism $\varphi : \Lambda \to J$ such that $h = \varphi j$, where $j : J \to \Lambda$ as before. Under this characterisation if $h \in \mathrm{Ker}(j_2)$ then for any $f \in S$ it is clear that $fh \in \mathrm{Ker}(j_2)$. Thus, with or without condition $(\star)$, $\mathrm{Ker}(j_2)$ is a left ideal in S.

Given $f \in S$, consider the homomorphism

$$jf : J \to \Lambda.$$

By the cohomological classification of $\mathrm{Ext}^1(R, \Lambda)$ this homomorphism represents an extension. But since $\mathrm{Ext}^1(R, \Lambda) = 0$, the corresponding extension splits and there is some homomorphism $\eta : \Lambda \to \Lambda$ such that $jf = \eta j$. Then for any $h \in \mathrm{Ker}(j_2)$ with factorisation φ as before: $hf = \varphi\eta j$ and $hf \in \mathrm{Ker}(j_2)$. □

Thus under such restrictions the ring structure on $\mathrm{Ext}(J)$ is well defined and j_2 is a ring homomorphism.

Remark: If $M \to P \to N$ is a short exact sequence of Λ-modules with P projective and such that $\mathrm{Ext}^1(N, \Lambda) = 0$, then the above easily generalises to show that composition in $\mathrm{End}(M)$ induces a ring structure on $\mathrm{Ext}^1(N, M)$.

4.2 The fibre square associated to a two-sided ideal

Theorem 4.2.1. *If Λ is a ring and J a two-sided ideal satisfying $(\star)$, then there is a fibre square of rings and ring homomorphisms:*

$$\begin{tikzcd} \Lambda \arrow[r, "i_1"] \arrow[d, "i_2"] & R \arrow[d, "j_1"] \\ S \arrow[r, "j_2"] & \mathrm{Ext}(J) \end{tikzcd}$$

with all maps surjective.

Proof. It remains to show that all maps are surjective and multiplicative.

We have already shown that j_2 and i_1 are surjective.

(i) To see that i_2 is surjective, if $f \in S$ then, again by $(\star)$ and as in the proof of 4.1.3, $jf = \eta j$ for some $\eta : \Lambda \to \Lambda$ and hence $i_2(\eta) = f$.

(ii) As we remarked earlier, by the exact sequence used in the construction of j_1 and by $(\star)$, j_1 is surjective.

(iii) Clearly i_1, j_2 and i_2 are ring homomorphisms. This implies that j_1 is multiplicative, since we already have shown that the square commutes and all maps are surjective.

□

Proof of Theorem B

We may apply the standard Mayer-Vietoris sequence to the fibre product (Milnor [21], Theorem 3.3) to obtain:

Theorem B *Let Λ be a ring and J a two-sided ideal satisfying $(\star)$, then there exists a long exact sequence of abelian groups:*

$$K_1\Lambda \longrightarrow K_1S \oplus K_1R \longrightarrow K_1\mathrm{Ext}(J) \longrightarrow K_0\Lambda \longrightarrow K_0S \oplus K_0R \longrightarrow K_0\mathrm{Ext}(J).$$

We remark that the standard homomorphism $K_1\mathrm{Ext}(J) \to K_0\Lambda$, as described by Milnor, will not feature in our generalisation, although the two maps are simply related to each other (c.f. Proposition 4.5.2).

4.3 Stabilisation of Ext and projective extensions

The process of passing from rings to matrices over that ring is mimicked over extensions by the process of stabilisation, and the 'invertible matrices' will correspond to projective extensions. In this section we explain what we mean by this and give some results towards a deeper understanding of Milnor's long exact sequence, as applied in Theorem B.

Let $J^n = J \oplus \ldots \oplus J$, so that for each n there is an extension

$$0 \longrightarrow J^n \xrightarrow{j} \Lambda^n \xrightarrow{\sigma} R^n \longrightarrow 0,$$

with $j(a_1, \ldots, a_n) = (j(a_1), \ldots, j(a_n))$ and $\sigma(b_1, \ldots, b_n) = (\sigma(b_1), \ldots \sigma(b_n))$. As before, the group $\mathrm{Ext}^1(R^n, J^n)$ is completely determined by the ideal J and we may define:

$$\mathrm{Ext}(J^n) = \mathrm{Ext}^1(R^n, J^n).$$

Proposition 4.3.1. *If Λ is a ring and J is a two-sided ideal in Λ then for each n there is an additive group isomorphism:*

$$\mathcal{M}_n(\mathrm{Ext}(J)) \cong \mathrm{Ext}(J^n).$$

($\star$) *If $\mathrm{Ext}^1(R, \Lambda) = 0$ then the above is a ring isomorphism.*

Proof. Recall that $\mathrm{Ext}(J^n)$ may be classified, through cohomology, as a quotient group of $\mathrm{End}_\Lambda(J^n)$. There is a natural identification $\mathcal{M}_n(S) \cong \mathrm{End}_\Lambda(J^n)$, which takes any $n \times n$ matrix $F = \{f_{hk}\}$ to the homomorphism F' defined by

$$F'(j_1, \ldots, j_n) = \left(\sum_k f_{1,k}(j_k) \quad , \quad \ldots \quad , \quad \sum_k f_{n,k}(j_k) \right).$$

This is a ring isomorphism under any circumstances, and we shall write F to denote both the homomorphism and corresponding matrix. Thus, as an abelian group:

$$\mathrm{Ext}(J^n) \cong (\mathcal{M}_n(S)/\sim)$$

, where $\sim$ denotes the cohomology relation. The additive property of Ext ensures that $[F]$ represents the zero extension of $\mathrm{Ext}(J^n)$ if and only if each $[f_{hk}]$ represents a zero extension of $\mathrm{Ext}(J)$ and, abusing the notation $\sim$, we have:

$$\mathrm{Ext}(J^n) \cong \mathcal{M}_n(S/\sim).$$

Here $\sim$ now used to denote the cohomology relation such that $\mathrm{Ext}(J) \cong S/\sim$.. This is a ring isomorphism when required and the result follows.

□

Notation: For each $F \in \mathcal{M}_n(S)$ we write $[F]$ to specify the congruence class represented by the bottom row in the pushout:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r, "j"] \arrow[d, "F"] & \Lambda^n \arrow[r, "\sigma"] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r, "i_{[F]}"] & M[F] \arrow[r, "\pi_{[F]}"] & R^n \arrow[r] & 0.
\end{tikzcd}$$

If F is given, we shall always use the notation $M[F]$ to denote the central module determined in this way.

Definition 4.3.2. *We say that a congruence class $\mathbf{x} \in \mathcal{M}_n\mathrm{Ext}(J)$ is invertible if there exist F, H such that $[F] = \mathbf{x}$ and*

$$[HF] = [Id] = [FH].$$

Lemma 4.3.3. *(Whitehead's Lemma) Suppose that $[F] \in \mathcal{M}_n(\mathrm{Ext}(J))$ is invertible with $H \in \mathcal{M}_n(S)$ such that $[HF] = [Id] = [FH]$. Then there is an isomorphism $\eta : J^{2n} \to J^{2n}$ such that*

$$[\eta] = \left[\begin{pmatrix} F & 0 \\ 0 & H \end{pmatrix}\right]$$

Proof. Elementary calculations show that

$$\left[\begin{pmatrix} F & 0 \\ 0 & H \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -H & 1 \end{pmatrix}\begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$$

and since each of the factors is an isomorphism, their product provides the required η. □

Proposition 4.3.4. *If $(\star)$ holds then:*

(1) $M[F]$ is projective if and only if $[F]$ is invertible.

Without the condition $(\star)$, we have instead:

(2.1) If $M[F]$ is projective then $[F]$ has a left inverse.

(2.2) If $[F]$ is invertible then $M[F]$ is projective.

We remark that a proof of the statement (1) may be inferred from ([11], Proposition 23.3) and the proof of the result identifying 'Johnson cohomology' and standard cohomology in [10]. We shall give an alternative proof that the invertible extensions are projective.

Proof. *(2.1)* If $M[F]$ is projective then, by the universal property of projective modules there is some H and a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r, "j"] \arrow[d, "F"] & \Lambda^n \arrow[r, "\sigma"] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r, "i_{[F]}"] \arrow[d, "H"] & M[F] \arrow[r, "\pi_{[F]}"] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r, "j"] & \Lambda^n \arrow[r, "\sigma"] & R^n \arrow[r] & 0
\end{tikzcd}$$

i.e. $[HF] = [Id]$ and hence $[H]$ is a left inverse for $[F]$. We highlight the diagram above for subsequent reference.

(1) If $(\star)$ holds then for any $F \in \mathcal{M}_n(S)$ representing a projective extension, there is a homomorphism F^* given by

$$F^*[A] = [AF].$$

Note that F^* is surjective since there is some H such that $[HF] = [Id]$. We shall show that F^* is injective. Since $M[F]$ is projective and $(\star)$ holds, by additivity $\mathrm{Ext}^1(R^n, M[F]) = 0$. Thus the homomorphism $i_{[F]} : J^n \to M[F]$ has a factorisation $i_{[F]} = \eta j$ for some $\eta : \Lambda^n \to M[F]$. Suppose that there is some $[A]$ such that $[AF] = [0]$, since $M[F]$ is projective and the cohomological classification of Ext is independent of the resolution taken (see [19] III.6.3), there is some $\varphi : M[F] \to J^n$ such that $A = \varphi i_{[F]}$. Then $A = \varphi\eta j$ and hence $[A] = [0]$. Therefore F^* is injective and it follows that $[F]$ is necessarily invertible with $[FH] = [Id]$.

(2.2) Suppose that $[F]$ is invertible, with inverse $[H] = [F]^{-1}$. Then by

Whitehead's Lemma there is an isomorphism η with

$$[\eta] = \left[\begin{pmatrix} F & 0 \\ 0 & H \end{pmatrix}\right]$$

and so there exists a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J^{2n} \arrow[r, "j"] \arrow[d, "\eta"] & \Lambda^{2n} \arrow[r, "\sigma"] \arrow[d, "\eta'"] & (R)^{2n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^{2n} \arrow[r] & M[F] \oplus M[H] \arrow[r] & (R)^{2n} \arrow[r] & 0.
\end{tikzcd}$$

By the five lemma η' is an isomorphism and $M[F]$ is projective, with projective inverse $M[H]$. □

4.4 The action of elementary matrices

The key point may be expressed as follows:

Proposition 4.4.1. *If $[X] \in \mathcal{M}_n(\mathrm{Ext}(J))$ is elementary then:*

(i) There exists an elementary matrix $E \in \mathcal{M}_n(S)$ such that $[E] = [X]$.

(ii) For all $F \in \mathcal{M}_n(S)$,

$$M[F] \cong M[XF].$$

(iii) For all $F \in \mathcal{M}_n(S)$, such that $[F]$ is invertible,

$$M[FX] \oplus \Lambda^n \cong M[F] \oplus \Lambda^n.$$

We remark before our proof that clearly (iii) is unsatisfactory, and conjecture that the stronger result $M[F] \cong M[FX]$ holds. In the case of $(\star)$, this may be seen to hold by arguments derived from Johnson's work, since the class of endomorphisms which represent split (i.e. zero) extensions is then independent of the projective resolution chosen to compute the cohomology group (see [11] Chapter 4). However this will suffice for our purposes.

Proof. (i) This is trivial.
(ii) We may suppose that $[X] = [E]$ with E elementary, in particular E is an

isomorphism. Then there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r, "j"] \arrow[d, "F"] & \Lambda^n \arrow[r, "\sigma"] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] \arrow[d, "E"] & M[F] \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] & M[EF] \arrow[r] & R^n \arrow[r] & 0
\end{tikzcd}$$

and by the five lemma $M[F] \cong M[EF]$.

(iii) Suppose that $[F]$ has an inverse $[H]$. We have

$$\left[\begin{pmatrix} FE & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} F & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}\right]$$

and by Whitehead's lemma and (ii) above we see that:

$$M[FE] \oplus \Lambda^n \cong M[F] \oplus M[E].$$

Since E is an isomorphism, $M[E] \cong \Lambda^n$ by the five lemma and the result is shown. □

Thus, assuming the familiar condition $(\star)$, the elements of $K_1\mathrm{Ext}(J)$ may be represented as equivalence classes $[F]$ with $F \in \mathcal{M}_n(S)$ such that $[F]$ is invertible, and hence $M[F]$ is projective. Furthermore, Proposition 4.4.1 shows that $M[F]$ is determined up to stability in the sense that

$$cls[F] = cls[H] \Rightarrow M[F] \oplus \Lambda^b \cong M[H] \oplus \Lambda^a$$

for some suitably chosen $a, b \in \mathbb{N}$.

Corollary 4.4.2. *If $(\star)$ holds then there is a well defined homomorphism $\partial_1 : K_1\mathrm{Ext}(J) \to K_0\Lambda$ such that for any invertible extension $[F]$:*

$$\partial_1([F]) = M[F].$$

Proof. ∂_1 is well defined by Proposition 4.4.1 and statement (1) of Proposition 4.3.4. It remains to show that ∂_1 is a homomorphism. Given $[A], [B] \in K_1\mathrm{Ext}(J)$, let $[A']$ be the inverse of $[A]$. Then

$$\begin{pmatrix} [AB] & 0 \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \cdot \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}\right]$$

and so by Whitehead's lemma and Proposition 4.4.1,

$$M[AB] \oplus \Lambda^n \cong M[B] \oplus M[A]$$

so that $\partial_1(AB) = \partial_1(A) + \partial_1(B)$ in $K_0\Lambda$. □

4.5 Milnor's construction

The reader may wish to see the connection between our construction of a map $K_1\mathrm{Ext}(J) \to K_0\Lambda$ and the standard map occurring in the Mayer-Vietoris sequence, as given by Milnor. In order to compare our map with Milnor's, we need to assume the condition $\mathrm{Ext}^1(\Lambda/J, \Lambda) = 0$ and work with the fibre square:

$$\begin{tikzcd} \Lambda \arrow[r, "i_1"] \arrow[d, "i_2"] & R \arrow[d, "j_1"] \\ S \arrow[r, "j_2"] & \mathrm{Ext}^1(J). \end{tikzcd}$$

Recall that for any projective module P over any ring $\mathcal{R}_1$ and ring homomorphism $f : \mathcal{R}_1 \to \mathcal{R}_2$, there is an induced projective $\mathcal{R}_2$-module $f_{\#}P$ given by

$$f_{\#}P = P \otimes_{\mathcal{R}_1} \mathcal{R}_2.$$

There is also a canonical map $f_* : P \to f_{\#}P$ given by

$$f_*(p) = p \otimes_{\mathcal{R}_1} 1.$$

See Chapter 3 of Milnor's book [21] for a more detailed description of the construction. It suffices to say that $f_{\#}P$ is equivalent to the intuitive description of the tensor product.

Milnor defines a map δ as follows: given projective modules P_1 over R, P_2 over S and an $\mathrm{Ext}(J)$ isomorphism

$$h\colon\ j_{1\#}(P_1) \cong j_{2\#}(P_2),$$

let $M(P_1, P_2, h)$ denote the subgroup of $P_1 \times P_2$ consisting of all pairs (p_1, p_2) such that $hj_{1*}(p_1) = j_{2*}(p_2)$. Then $M(P_1, P_2, h)$ has a natural Λ-module structure.

Theorem 4.5.1. *(Milnor) The module $M(P_1, P_2, h)$ is projective over Λ, and every projective Λ-module is isomorphic to some $M(P_1, P_2, h)$. Moreover, the modules P_1 and P_2 are naturally isomorphic to $i_{1\#}M(P_1, P_2, h)$ and $i_{2\#}M(P_1, P_2, h)$ respectively.*

For each invertible matrix h over $\mathrm{Ext}(J)$ of rank n, one may form the projective Λ module

$$\delta(h) = M(R^n, S^n, h).$$

This construction leads to Milnor's homomorphism:

$$\delta : K_1\mathrm{Ext}(J) \to K_0\Lambda.$$

Proposition 4.5.2. *(Equivalence to Milnor's construction.) Let δ be Milnor's map $\mathrm{GL_n(Ext(J))} \to \mathrm{Proj}_\Lambda$ and ∂_1 be as in section 5.1. Then for each $h \in \mathrm{GL_n(Ext(J))}$,*

$$\partial_1(h) = \delta(h).$$

Suppose that the Λ-module P occurs in a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r] \arrow[d, "h"] & \Lambda^n \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] & P \arrow[r] & R^n \arrow[r] & 0
\end{tikzcd}$$

with $[h] \in \mathrm{GL_n(Ext(J))}$ (so P is projective) and in particular

$$\partial_1(h) = [P].$$

By Theorem 2.2 of (Milnor [21]) it is sufficient to construct a fibre product diagram of homomorphisms:

$$\begin{tikzcd}
P \arrow[r, "i_1^P"] \arrow[d, "i_2^P"] & R^n \arrow[d, "hj_1"] \\
S^n \arrow[r, "j_2"] & \mathrm{Ext}(J)^n
\end{tikzcd}$$

since any such P is unique and is necessarily isomorphic to $M(S^n, R^n, h)$.

Lemma 4.5.3. *For $p \in P$, set $\varphi : \Lambda \to P$ to be the homomorphism such that $\varphi(1) = p$. Then there are homomorphisms $i_2^P(p) \in S^n$ and $i_1^P(p) \in R^n$ such that* *the following commutes:*

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "i_2^P(p)"] & \Lambda \arrow[r] \arrow[d, "\varphi"] & R \arrow[r] \arrow[d, "i_1^P(p)"] & 0 \\ 0 \arrow[r] & J^n \arrow[r] & P \arrow[r] & R^n \arrow[r] & 0. \end{tikzcd}$$

Proof. Recall that for any module M

$$\mathrm{Hom}(M, M^n) \cong \mathrm{End}(M)^n$$

and that we have identified R with its endomorphism ring. Since condition $(\star)$ holds the map $j_1 : \mathrm{End}(R^n) \to \mathcal{M}_n(\mathrm{Ext}(J))$ defined in earlier sections is surjective. Thus there is a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & J^n \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & J^n \arrow[r] & \Lambda^n \arrow[r] & R^n \arrow[r] & 0. \end{tikzcd}$$

Any homomorphism $\eta : \Lambda \to \Lambda^n$ satisfies $\eta(J) \subseteq J^n$, so from the two diagrams above, and since we have the identity on the right hand side, $i_1^P(p)$ is well defined and we may deduce the result. □

Proof of Theorem 4.5.2: It is required to show the commutativity of the diagram:

$$\begin{tikzcd} P \arrow[r, "i_1^P"] \arrow[d, "i_2^P"] & R^n \arrow[d, "hj_1"] \\ S^n \arrow[r, "j_2"] & \mathrm{Ext}(J)^n \end{tikzcd}$$

For each $p \in P$ the element $[j_2 i_2^P(p)]$ is represented by the bottom row in the diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "i_2^P(p)"] & \Lambda \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J^n \arrow[r] & P_2 \arrow[r] & R \arrow[r] & 0. \end{tikzcd}$$

$[j_1 i_1^P(p)]$ is represented by the top row of

$$\begin{tikzcd} 0 \arrow[r] & J^n \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "i_1^P(p)"] & 0 \\ 0 \arrow[r] & J^n \arrow[r] & \Lambda^n \arrow[r] & R^n \arrow[r] & 0. \end{tikzcd}$$

Note that $h \circ [j_1 i_1^P(p)]$ is represented by any exact sequence occurring in the bottom row of a diagram:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r] \arrow[d, "h"] & P_1 \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] & M \arrow[r] & R^n \arrow[r] & 0.
\end{tikzcd}$$

Combining four of the given diagrams leads to the composite:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "i_1^P(p)"] & 0 \\
0 \arrow[r] & J^n \arrow[r] \arrow[d, "h"] & \Lambda^n \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] & P \arrow[r] & R^n \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[u, "i_2^P(p)"] \arrow[d, "i_2^P(p)"] & \Lambda \arrow[r] \arrow[u, "\varphi"] \arrow[d] & R \arrow[r] \arrow[u, "i_1^P(p)"] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] & P_2 \arrow[r] & R \arrow[r] & 0.
\end{tikzcd}$$

This may be compressed to form the diagram:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r] \arrow[d, "h"] & P_1 \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] & P_2 \arrow[r] & R^n \arrow[r] & 0
\end{tikzcd}$$

and hence:

$$h \circ [j_1 i_1^P(p)] = [j_2 i_2^P(p)]$$

and the square commutes. The proof that the square satisfies the fibre product condition follows as in the proof of Theorem 4.1.1. □

Remark. Our construction is somewhat more general than Milnor's: we may construct a module $M[A]$ for any $A \in \mathcal{M}_n(\mathrm{Ext}(J))$ without assuming that A be invertible. One is tempted to suggest that, under reasonable restrictions, Theorem 4.5.1 above will generalise to arbitrary (rather than projective) modules.

4.6 Example: Swan modules

The canonical example for our fibre product and resulting Mayer-Vietoris sequence is provided by taking $\Lambda = \mathbb{Z}[G]$ to be an integral group ring of a finite group and J to be the two-sided ideal of $\mathbb{Z}$[G] on which the action of G is trivial. All of the results in this section have been previously shown elsewhere, and may be found for example in [5], [28] and [11].

Suppose that $G = \{g_i\}_{i=1}^n$ is a finite group of order n. We follow Swan in using the notation

$$N = \sum g_i,$$

so that N is fixed and denotes the sum of all the group elements. Then the trivial $\mathbb{Z}$[G]-module $\mathbb{Z}$ embeds into $\mathbb{Z}$[G] through identification with the ideal generated by the element N and there is a resulting exact sequence:

$$\mathbf{E} = \ 0 \longrightarrow \mathbb{Z} \xrightarrow{\varepsilon^*} \mathbb{Z}[\mathrm{G}] \xrightarrow{s} \mathbb{Z}[\mathrm{G}]/(N) \longrightarrow 0.$$

One may show computationally that $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}[\mathrm{G}]) = 0$, i.e. condition $(\star)$ holds, or use the fact that this result is true in general for all finitely generated torsion free $\mathbb{Z}$[G]-modules ([11] Chapter 5). Since $\mathrm{Hom}_{\mathbb{Z}[\mathrm{G}]}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ this sequence leads to the fibre product:

$$\begin{tikzcd}
\mathbb{Z}[\mathrm{G}] \arrow[r, "s"] \arrow[d, "\varepsilon"] & \mathbb{Z}[\mathrm{G}]/(N) \arrow[d, "j_1"] \\
\mathbb{Z} \arrow[r, "j_2"] & \mathrm{Ext}^1(J)
\end{tikzcd}$$

We remark in passing that $\mathbb{Z}[\mathrm{G}]/(N) \cong \mathrm{Hom}_{\mathbb{Z}[\mathrm{G}]}(I, \mathbb{Z}[\mathrm{G}])$ is the dual of the augmentation ideal, where the augmentation ideal is the kernel of the ring homomorphism $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$. Indeed, one may take as a starting point the extension:

$$0 \longrightarrow I \xrightarrow{s^*} \mathbb{Z}[\mathrm{G}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

and work with this sequence, which arises naturally in consideration of algebraic 2-complexes. The two sequences are naturally dual to each other. Either sequence results in the same fibre product of rings, and it is preferable to work with the former simply because the relevant $\mathbb{Z}$[G]-modules may then be identified with the corners of the fibre square.

Any homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ is representable as multiplication by some element $f \in \mathbb{Z}$, and it is easy to see that f factors though ε^* if and only if $f \equiv 0 \mod n$ (recall that $\varepsilon(N) = n$). Thus

$$\mathrm{Ext}(\mathbb{Z}) = \mathrm{Ext}^1(\mathbb{Z}[\mathrm{G}]/(N), \mathbb{Z}) \cong \mathbb{Z}_n$$

where $\mathbb{Z}_n$ denotes the quotient ring $\mathbb{Z}/n\mathbb{Z}$. One sees directly that the natural multiplicative structure on $\mathbb{Z}_n$ represents the multiplication in $\mathrm{Ext}(\mathbb{Z})$.

For each $r \in \mathbb{N}^+$ define (N, r) to be the submodule of $\mathbb{Z}[\mathrm{G}]$ generated by the elements N and r. Then $\mathbb{Z}$ embeds in (N, r) by identification with the submodule generated by N and there is a resulting exact sequence:

$$\mathbf{E}(r) = \ 0 \longrightarrow \mathbb{Z} \xrightarrow{i_r} (N, r) \xrightarrow{\pi_r} \mathbb{Z}[\mathrm{G}]/(N) \longrightarrow 0.$$

with $\pi_r(r) = 1 + (N)$. Moreover:

$$\mathbf{E}(r) \to cls(r) \in \mathbb{Z}_n$$

is an isomorphism on restriction to congruence classes. Originally defined in [26], Swan proved that the projective extensions in $\mathrm{Ext}(\mathbb{Z})$ are represented by the modules (N, r) for r coprime to n. The modules (N, r) form a well studied subgroup of the projective class group of finite groups and are often called *Swan modules*. If r is coprime to n with modular inverse s, then the proof of Theorem 4.3.4 recovers Lemma 6.1 of [26], i.e.

$$(N, r) \oplus (N, s) \cong \mathbb{Z}[\mathrm{G}]^2.$$

It has often been remarked (see e.g. [5], [28], [22]), that there is a fibre product of rings:

$$\begin{tikzcd} \mathbb{Z}[\mathrm{G}] \arrow[r] \arrow[d] & \mathbb{Z}[\mathrm{G}]/(N) \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}_n \end{tikzcd}$$

and that the Swan module (N, r) corresponds to the projective $\mathbb{Z}[\mathrm{G}]$-module constructed from the unit $r \in \mathbb{Z}_n$ via Milnor's construction. The homological classification of such modules is achieved by noting that for any $r \in \mathbb{N}^+$

there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "i_1"] \arrow[d, "r"] & \mathbb{Z}[G] \arrow[r, "\pi_1"] \arrow[d, "r"] & \mathbb{Z}[G]/(N) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "i_r"] & (N, r) \arrow[r, "\pi_r"] & \mathbb{Z}[G]/(N) \arrow[r] & 0.
\end{tikzcd}$$

There is a pre-established connection between Swan modules and cancellation problems arising from considerations of algebraic 2-complexes. For certain finite groups of period four, it is a celebrated result of Swan's [27] that there are stably free $\mathbb{Z}[\Gamma]$ modules which are not free. Letting G denote the quaternion group of order $4n$ with $n \geq 6$, or in general any finite group of period four such that there are stably free modules $\mathbb{Z}[G]$ which are not free, we say that *weak cancellation* holds for G if all stably free Swan modules are free. Johnson has shown in [13] that if weak cancellation holds for G, then one may construct modules which are stably equivalent to $\pi_2(\mathcal{G})$ for some presentation $\mathcal{G}$ for G, but which are *not* isomorphic to $\pi_2(\mathcal{G}) \oplus \mathbb{Z}[G]^m$. We refer the interested reader to the recently published [2] for an explicit construction of such a module. We remark that for G a 2-group, or for G of order $4p$ with p an odd prime, Swan has shown that weak cancellation holds ([27] - Theorem VI).

We shall use the following result later:

Corollary 4.6.1. *$K_1(\mathbb{Z}[G]/(N))$ is finitely generated.*

Proof. Using the Mayer-Vietoris sequence resulting from the fibre square, we consider the portion:

$$K_1\mathbb{Z}[G] \longrightarrow K_1(\mathbb{Z}[G]/(N)) \longrightarrow K_1\mathrm{Ext}(\mathbb{Z}).$$

It is well known that $K_1\mathbb{Z}[G]$ is always finitely generated for finite groups (see e.g. Oliver [24] Chapter 2), and $K_1\mathrm{Ext}(\mathbb{Z}) \cong K_1\mathbb{Z}_n$ is finitely generated. The result then follows from the classification of finitely generated abelian groups. $\square$

Chapter 5

The general case of a two-sided ideal

Our attempts to remove the condition $\mathrm{Ext}^1(R, \Lambda) = 0$ are partially successful, and we shall show that in the more general case we may define an abelian group $\tilde{K}_1\mathrm{Ext}(J)$, which is equivalent to $K_1\mathrm{Ext}(J)$ if the latter defined. We also construct a homomorphism of abelian groups $\partial_1 : \tilde{K}_1\mathrm{Ext}(J) \to K_0\Lambda$ occurring in an exact sequence:

$$K_1S \oplus K_1R \xrightarrow{\partial_2} \tilde{K}_1\mathrm{Ext}(J) \xrightarrow{\partial_1} K_0\Lambda.$$

5.1 Definition of $\tilde{K}_1\mathrm{Ext}(J)$

In the case where there is no natural product structure on $\mathrm{Ext}(J)$, we shall show that one may be induced on the invertible extensions. Recall that congruence class $\mathbf{x}$ is *invertible* if there exists an $F, H \in \mathcal{M}_n(S)$ such that $\mathbf{x} = [F]$ and

$$[HF] = [Id] = [FH].$$

Let $Proj_n$ denote the set of classes $[F]$ in $\mathrm{Ext}(J^n)$ such that $[F]$ is invertible. Define a relation on $Proj_n$ by setting $[F] \simeq [H]$ if there exists an elementary matrix E such that

$$[F] = [EH].$$

Proposition 5.1.1. *$\simeq$ is a well defined additive equivalence relation.*

Proof. As in the proof that Ker(j$_2$) is a left ideal (c.f. Proposition 4.1.3), it is clear that for all $H, K \in \mathcal{M}_n(S)$, if $[H] = [K]$, then $[FH] = [FK]$ and thus the relation is well defined on equivalence classes. The rest is trivial. □

For each n there is an inclusion $Proj_n \to Proj_{n+1}$ given by:

$$[F] \to \left[\begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Let $Proj$ denote the limit of these inclusions.

Theorem 5.1.2. *Composition defines a product structure on $Proj/\simeq$*

Proof. Given invertible congruence classes $[F], [H], [K], [L]$ such that $[F] = [K]$ and $[H] = [L]$, let $[F']$ be an inverse for $[F]$. Then

$$\begin{aligned}
\left[\begin{pmatrix} FH & 0 \\ 0 & 1 \end{pmatrix}\right] &= \left[\begin{pmatrix} F & 0 \\ 0 & F' \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & F \end{pmatrix}\right] \\
&\simeq \left[\begin{pmatrix} H & 0 \\ 0 & F \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} L & 0 \\ 0 & K \end{pmatrix}\right] \\
&\simeq \left[\begin{pmatrix} KL & 0 \\ 0 & 1 \end{pmatrix}\right]
\end{aligned}$$

and hence $[FH] = [KL]$ in $Proj/\simeq$. □

Since the product structure is well defined, we may now factor out on both the right and left by elementary matrices over S and define

$$\tilde{K}_1\mathrm{Ext}(J) = (Proj/\simeq)/E(S).$$

Then $\tilde{K}_1\mathrm{Ext}(J)$ is an abelian group and clearly:

Corollary 5.1.3. *If $(\star)$ holds there is a natural identification:*

$$\tilde{K}_1\mathrm{Ext}(J) \equiv K_1\mathrm{Ext}(J).$$

Construction of ∂_1

Recall that if $[F]$ is an invertible congruence class then $M[F]$ is projective. Define a map $\partial_1 : \tilde{K}_1\mathrm{Ext}(J) \to K_0\Lambda$ by

$$\partial_1([F]) = M[F].$$

Proposition 5.1.4. $\partial_1 : \tilde{K}_1\mathrm{Ext}(J) \to K_0\Lambda$ *is a well defined homomorphism.*

Proof. As before, Propositions 4.3.4 and 4.4.1 show that ∂_1 is well defined. The proof that ∂_1 is additive follows as in 4.4.2. □

Construction of ∂_2

We define a map $\partial_2 : K_1S \oplus K_1R \to \tilde{K}_1\mathrm{Ext}(J)$ by sending any $(\alpha, \beta) \in \mathrm{GL_n(S)} \oplus \mathrm{GL_n(R)}$ to the congruence class of the *bottom* row of the pushout / pullback:

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r, "j"] & \Lambda^n \arrow[r, "\sigma"] & R^n \arrow[r] & 0 \\
0 \arrow[r] & J^n \arrow[r] \arrow[u, equal] \arrow[d, "\alpha"] & X \arrow[r] \arrow[u] \arrow[d] & R^n \arrow[r] \arrow[u, "\beta"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] & M \arrow[r] & R^n \arrow[r] & 0.
\end{tikzcd}$$

We wish to describe the resulting exact sequence $\partial_2(\alpha, \beta)$ in terms of endomorphisms of J. We may find a $f(\beta)$ such that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r, "j"] \arrow[d, "f(\beta)"] & \Lambda^n \arrow[r, "\sigma"] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J^n \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & J^n \arrow[r] & \Lambda^n \arrow[r] & R^n \arrow[r] & 0.
\end{tikzcd}$$

From the two diagrams above we deduce that the following commutes:

$$\begin{tikzcd} 0 \arrow[r] & J^n \arrow[r, "j"] \arrow[d, "f(\beta)"] & \Lambda^n \arrow[r, "\sigma"] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J^n \arrow[r] \arrow[d, "\alpha"] & X \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J^n \arrow[r] & M \arrow[r] & R^n \arrow[r] & 0. \end{tikzcd}$$

So that $\partial_2(\alpha,\beta) = [\alpha f(\beta)]$. Note that $[f(\beta)]$ is an invertible element, with inverse provided by $[f(\beta^{-1})]$ and f is injective on congruence classes. Indeed, if $(\star)$ holds then it may be shown that f is a ring isomorphism [11]. A large diagram which we omit should convince the reader that $[f(\alpha\beta)] = [f(\alpha)f(\beta)]$.

Proposition 5.1.5. *If β is an elementary matrix, then so is $[f(\beta)]$.*

Proof. Suppose that β is elementary with one non-zero off diagonal entry b, so that $\beta = \epsilon_{i,j}(b)$. Then we may pick an $f(b) \in S$ so that the following commutes:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "j"] \arrow[d, "f(b)"] & \Lambda \arrow[r, "\sigma"] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "b"] & 0 \\ 0 \arrow[r] & J \arrow[r, "j"] & \Lambda \arrow[r, "\sigma"] & R \arrow[r] & 0. \end{tikzcd}$$

Set $F = \epsilon_{i,j}(f(b))$, then there exists a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & J^n \arrow[r, "j"] \arrow[d, "F"] & \Lambda^n \arrow[r, "\sigma"] \arrow[d] & R^n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & J^n \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & J^n \arrow[r, "j"] & \Lambda^n \arrow[r, "\sigma"] & R^n \arrow[r] & 0. \end{tikzcd}$$

and hence $[F] = [f(\beta)]$ is elementary. $\square$

Corollary 5.1.6. *∂_2 is a well defined homomorphism.*

Proof. By Proposition 5.1.5, if (α, β) is such that both α and β are elementary then $\partial_2(\alpha, \beta)$ is a product of elementary matrices, so ∂_2 is well defined. We have already shown that $[f(\alpha\beta)] = [f(\alpha)f(\beta)]$, and hence

$$\begin{aligned}
\partial_2[(\alpha, \beta) \cdot (\gamma, \delta)] &= [\alpha\gamma f(\beta\delta)] \\
&= [\alpha][\gamma][f(\beta)][f(\delta)] \\
(\tilde{K}_1 Ext(J) \text{ is commutative}) \qquad &= [\alpha][f(\beta)][\gamma][f(\delta)] \\
&= \partial_2(\alpha, \beta)\partial_2(\gamma, \delta)
\end{aligned}$$

and ∂_2 is multiplicative. □

5.2 Exactness at $\tilde{K}_1\mathrm{Ext}(J)$

Suppose that $[F] \in \mathrm{Ker}(\partial_1)$. Then by definition $M[F]$ is stably free, and we may assume there is a $b \in \mathbb{N}$ such that $M[F \oplus I_b]$ is free, where for matrices A and B we take $A \oplus B$ to be the matrix:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Set $F' = F \oplus I_b$. We shall assume for convenience that the free rank of $M[F']$ is $rk(F) + b$, that is

$$M[F'] \cong \Lambda^{rk(F)+b}$$

and return later to the case where this is not a justifiable assumption. For the moment set $n = (rk(F) + b)$ and suppose that $\varphi : M[F'] \cong \Lambda^n$ is an isomorphism. Then φ induces a congruence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J^n & \xrightarrow{i_{[F']}} & M[F'] & \xrightarrow{\pi_{[F']}} & R^n & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle\varphi} & & \Vert & & \\
0 & \longrightarrow & J^n & \xrightarrow{\varphi i_{[F']}} & \Lambda^n & \xrightarrow{\pi_{[F']}\varphi^{-1}} & R^n & \longrightarrow & 0
\end{array}$$

so that we may take $M[F'] = \Lambda^n$ and the congruence class of $[F']$ to be represented by the bottom row in the above diagram.

Proposition 5.2.1. *If $\eta : \Lambda^n \to R^n$ is a surjective Λ-module homomorphism then*

$$\mathrm{Ker}(\eta) = J^n.$$

We remark before the proof that we shall need to make the assumption that if $f : R^n \to R^n$ is a surjective R-module homomorphism then f is an isomorphism. This holds in many cases - indeed it always holds for the rings considered in this thesis - but fails for example if $R^n \cong R^m$ with $n \neq m$.

Proof. Recall that the Λ-module structure on R is determined by the surjective $\sigma : \Lambda \to R$ and $J = \text{Ker}(\sigma)$.

$J^n \subseteq \text{Ker}(\eta)$: In the case where $n = 1$, if $a \in J$ then

$$\eta(a) = \eta(1) \cdot a = \eta(1)\sigma(a) = 0.$$

The case where $n \geq 2$ follows by writing each column $a \in J^n$ as the sum of the components and applying a similar reasoning.

$\text{Ker}(\eta) \subseteq J^n$: Again we start with the case $n = 1$. Suppose $\eta(1) = s \in R$, then since η is surjective, s is a unit. If $\eta(a) = 0$ then $s\sigma(a) = 0$ and hence $a \in J$. In the general case, notice that if $\{e_1, \ldots e_n\}$ the standard basis for Λ^n, then $\{\eta(e_1), \ldots \eta(e_n)\}$ forms a basis for R^n (see remark before proof) and the matrix $\eta(Id)$ formed by adjoining the columns $\eta(e_i)$ is invertible. Moreover if $\eta(a) = 0$ then $\eta(a) = \eta(Id).\sigma(a) = 0$ and hence $a \in J^n$. □

Corollary 5.2.2. *There are induced isomomorphisms α and γ so that the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & J^n \arrow[r, "\varphi i_{[F']}"] \arrow[d, "\alpha"] & \Lambda^n \arrow[r, "\pi_{[F']}\varphi^{-1}"] \arrow[d, equal] & R^n \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & J^n \arrow[r, "i"] & \Lambda^n \arrow[r, "\sigma"] & R^n \arrow[r] & 0
\end{tikzcd}$$

Proof. It follows from Proposition 5.2.1 that α and γ are well defined.

(i) α is injective by the injectivity of $\varphi i_{[F']}$ and by commutativity.

(ii) α is surjective by Proposition 5.2.1.

(iii) γ is surjective since σ is surjective and the diagram commutes.

(iv) γ is injective since if $\gamma(r) = 0$ and $\pi_{[F']}\varphi^{-1}(\lambda) = r$ then $\sigma(\lambda) = 0$ by commutativity and again by Proposition 5.2.1, $\lambda \in \text{Ker}(\pi_{[F']}\varphi^{-1})$ so that $r = 0$.

□

Note that in the corollary above the congruence class of the top row is equal to $[F \oplus Id_b]$ This is all that is required to prove:

Theorem 5.2.3. *The sequence*

$$K_1S \oplus K_1R \xrightarrow{\partial_2} \tilde{K}_1\mathrm{Ext}(J) \xrightarrow{\partial_1} K_0\Lambda$$

is exact at $\tilde{K}_1\mathrm{Ext}(J)$.

Proof. Immediately from the five lemma we see that $\partial_2\partial_1 = 0$. If $[F]$ is in the kernel of ∂_1, then by Corollary 5.2.2 there is an isomorphism of extensions $(\alpha, \beta, \gamma) : [Id_n] \to [F \oplus Id_b]$ and

$$\partial_2(\alpha, \gamma^{-1}) = [F \oplus Id_b].$$

Since we have identified $[F]$ and $[F \oplus Id_b]$ in $\tilde{K}_1\mathrm{Ext}(J)$,

$$[F] \in \mathrm{Im}(\partial_1)$$

and the result follows. □

Some restrictions on the ring and on the ideal

We deal now with the case where we may not assume that the free rank of $M[F']$ is $rk(F) + b$. Suppose that there is some $n \neq a$ and extensions

$$0 \longrightarrow J^n \longrightarrow \Lambda^n \longrightarrow R^n \longrightarrow 0$$
$$0 \longrightarrow J^n \longrightarrow \Lambda^a \longrightarrow R^n \longrightarrow 0$$

then by Schanuel's Lemma there is an isomorphism

$$J^n \oplus \Lambda^a \cong J^n \oplus \Lambda^n \qquad n \neq a.$$

A priori, this is possible. P.M. Cohn has shown, for example, that there are many examples of rings Λ such that there are isomorphisms $\Lambda^a \cong \Lambda^n$ with $n \neq a$. The following is essentially due to Cohn:

Definition. A ring Λ is said to satisfy the *strong basis number property* (SBN property) if $\alpha : \Lambda^n \to \Lambda^a$ is a surjective homomorphism implies $a \leq n$.

Proposition 5.2.4. *(P.M. Cohn) Suppose that Λ has SBN and M is a finitely generated module, then*

$$M \oplus \Lambda^n \cong M \oplus \Lambda^a \quad \Rightarrow \quad n = a.$$

Proof. Suppose that we are given some finitely generated M, i.e. there is a surjective homomorphism

$$\Lambda^b \to M$$

and suppose that there is some $a > n$ and isomorphism

$$M \oplus \Lambda^n \cong M \oplus \Lambda^a.$$

Then there is a surjective homomorphism

$$\Lambda^{b+2nb} \to M \oplus \Lambda^{2ab}$$

which induces surjective

$$\Lambda^{b(2n+1)} \to \Lambda^{2ab}$$

and since $b(2n+1)$ is strictly less than $2ab$ this is impossible if Λ has SBN. $\square$

Thus we impose the conditions that Λ has the SBN property, that J is finitely generated and that any surjective R-linear map $f : R^n \to R^n$ is an isomorphism. Cohn has also shown in [7] that the group ring of any finitely presented group satisfies the SBN property.

5.3 Further generalisations

The results of all the previous sections may be considered as an easier and more complete version of a slightly more general phenomenon. Suppose now that Λ is a ring and there is an exact sequence of Λ-modules:

$$0 \longrightarrow M \xrightarrow{j} V \xrightarrow{\sigma} N \longrightarrow 0$$

with V a finitely generated free module. Then the pushout and pullback constructions generate abelian group homomorphisms:

$$j_2 : \mathrm{End}_\Lambda(M) \to \mathrm{Ext}^1(N, M)$$
$$j_1 : \mathrm{End}_\Lambda(N) \to \mathrm{Ext}^1(N, M)$$

with j_2 surjective. We may consider the module M to be identified with its image in V, so that $\mathrm{Ext}^1(N, M)$ is completely determined by M and we may write

$$\mathrm{Ext}^1(N, M) = \mathrm{Ext}(M)$$
$$\mathrm{Ext}^1(N^n, M^n) = \mathrm{Ext}(M^n).$$

As before, we may represent each congruence class of $\mathrm{Ext}(M^n)$ by $[F]$ for some $F \in \mathcal{M}_n(\mathrm{End}_\Lambda(M))$, and we retain the notation $M[F]$ for the central module. We shall now assume that the condition $(\star)$ refers to the statement:

$$(\star) \qquad \mathrm{Ext}^1(N, \Lambda) = 0$$

A substitution of variables in the previous section (c.f. 4.3.4, 4.1.3) proves the following:

Proposition 5.3.1. *If $(\star)$ holds then* $\mathrm{Ker}(j_2)$ *is a two-sided ideal in* $\mathrm{End}_\Lambda(V)$*, so that in particular there is a well defined ring structure on* $\mathrm{Ext}(M)$*.*

Proposition 5.3.2. *If $(\star)$ holds then:*

(1) $M[F]$ is projective if and only if $[F]$ is invertible.

Without the condition $(\star)$, we have instead:

(2.1) If $M[F]$ is projective then $[F]$ has a left inverse.

(2.2) If $[F]$ is invertible then $M[F]$ is projective.

One may similarly define the group $\tilde{K}_1\mathrm{Ext}(M)$, which represents the projective extensions if $(\star)$ holds, and moreover there is a well defined map:

$$\partial_1 : \tilde{K}_1\mathrm{Ext}(M) \to K_0\Lambda$$

which corresponds to sending an extension with central module P to the class of P in $K_0\Lambda$.

Considerable profit was made in previous sections from the fact that $\varphi(J) \subseteq J$ for every $\varphi \in \mathrm{End}_\Lambda(\Lambda)$. This motivates the following definition:

Definition. Given a submodule M of V with quotient module N, M is said to be a *characteristic submodule* of V if $\varphi(M) \subseteq M$ for each $\varphi \in \mathrm{End}_\Lambda(V)$. If $\mathrm{Hom}_\Lambda(M, N) = 0$ holds then we say that $\mathrm{Ext}(M)$ is characteristic.

Examples of characteristic submodules arise naturally when considering torsion free modules over the group ring of a finite group, see for example section 34A of [5]. Of special interest is the case where the stronger condition $\mathrm{Hom}_\Lambda(M, N) = 0$ holds, such submodules M may be seen to be characteristic by a simple diagram chase. In anticipation of later sections we remark here that the isomorphism classes of the central modules are more easily determined in this case:

Theorem 5.3.3. *(Curtis & Reiner [4] 34.5) Suppose that* $\mathrm{Hom}_\Lambda(M, N) = 0$*, then the isomorphism classes of central modules* $M[F]$ *are classified by the orbits of* $\mathrm{Ext}(M^n)$ *under the action of* $\mathrm{GL_n(End(M))} \times \mathrm{GL_n(End(N))}$*; the action is determined by the homomorphisms* j_1 *and* j_2*.*

However, our main point is that if M is a characteristic submodule of V then given any $F \in \mathrm{End}_\Lambda(V)$ there is a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "F_-"] & V \arrow[r, "\sigma"] \arrow[d, "F"] & N \arrow[r] \arrow[d, "F_+"] & 0 \\ 0 \arrow[r] & M \arrow[r, "j"] & V \arrow[r, "\sigma"] & N \arrow[r] & 0 \end{tikzcd}$$

so that we may define

$$i_2(F) = F_- \qquad i_1(F) = F_+.$$

Theorem 5.3.4. *Suppose that* Λ *is a ring and there is an exact sequence of* Λ*-modules:*

$$0 \longrightarrow M \xrightarrow{j} V \xrightarrow{\sigma} N \longrightarrow 0$$

with V *a finitely generated free module and* M *a characteristic submodule of* V*. Suppose also that* $\mathrm{Ext}^1(N, \Lambda) = 0$*. Then is a fibre product of rings:*

$$\begin{tikzcd} \mathrm{End}_\Lambda(V) \arrow[r, "i_1"] \arrow[d, "i_2"] & \mathrm{End}_\Lambda(N) \arrow[d, "j_1"] \\ \mathrm{End}_\Lambda(M) \arrow[r, "j_2"] & \mathrm{Ext}^1(N, M) \end{tikzcd}$$

with all maps surjective.

The proof follows simply from the proof of Theorem A. Note also that there is a ring isomorphism $\mathrm{End}_\Lambda(V) \cong \mathcal{M}_k(\Lambda)$, where k is the free rank of V, so that by Morita invariance the Mayer-Vietoris sequence for the above fibre square becomes

$$K_1\Lambda \longrightarrow K_1\mathrm{End}_\Lambda(M) \oplus K_1\mathrm{End}_\Lambda(N) \longrightarrow K_1\mathrm{Ext}(M)$$
$$\longrightarrow K_0\Lambda \longrightarrow K_0\mathrm{End}_\Lambda(M) \oplus K_0\mathrm{End}_\Lambda(N) \longrightarrow K_0\mathrm{Ext}(M).$$

Indeed, with the exception of the isomorphism $\mathrm{End}_\Lambda(N) \cong N$, it is easily checked that the results of the previous sections may be adapted to this case, with the exact sequence arising from a two-sided ideal replaced by the exact sequence arising from a characteristic submodule of a finitely generated free module.

Chapter 6

Applications to 2-complexes

Some of the results of the previous section, in particular the methods to detect projective extensions, may be generalised and are now applied to 2-complexes. Recall that the Realisation problem asks, for a particular fundamental group, if each chain homotopy class of algebraic 2-complexes is realised geometrically by some presentation. The work of Johnson and others has provided a useful strategy for approaching problems of this sort, aimed at providing a positive result, which we now summarise.

Program of realisation

We assume that we are given a group Γ with an accompanying presentation $\mathcal{G}$ and that the hypothesis of Theorem 1.1.2 are satisfied.

1) The presentation $\mathcal{G}$ may be used to construct a (realised) algebraic 2-complex $C_*(\tilde{X}_{\mathcal{G}})$ with some terminal module $\pi_2(\mathcal{G})$

   $$C_*(\tilde{X}_{\mathcal{G}}) = \; 0 \longrightarrow \pi_2(\mathcal{G}) \xrightarrow{i} \mathbb{Z}[\Gamma]^3 \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^3 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

   This gives a reference point for the ensuing investigation.

2) The class of $\mathbb{Z}[\Gamma]$-modules stably equivalent to $\pi_2(\mathcal{G})$ is calculated.

3) The complex $C_*(\tilde{X}_{\mathcal{G}})$ is considered as a truncated projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[\Gamma]$-modules and may be used to calculate $\mathrm{Ext}^3(\mathbb{Z}, M)$ for all possible modules M stably equivalent to $\pi_2(\mathcal{G})$.

4) The congruence classes of algebraic 2-complexes in each $\mathrm{Ext}^3(\mathbb{Z}, M)$ are identified and each is either realised or shown not to be realisable.

In both step (3) and step (4) and if $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$, there are important simplifications of the program. These were given in Propositions 3.2.2 and 3.2.3. Together these propositions ensure that steps (3) and (4) need only be completed at the minimal level, where 'minimal' corresponds to the size of the terminal module M (see remarks before proposition 3.2.2). Furthermore, Proposition 3.2.3 is often a crucial simplification of step (4), since the projective extensions are then known.

Without the condition $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$, we do not have any existing method of detecting projective extensions. In Chapter 5 of this thesis, in our discussion of projective versus invertible extensions, we were presented with a similar problem and surmounted it by passing to K_1 from general matrices. The analogue of this process is not delicate enough to deal with chain homotopy types of algebraic 2-complexes, but some of the ideas will transfer successfully.

6.2 Relaxing the condition $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$

We first set up the notation that we shall be using. We suppose that Γ is a finitely presented group and that we have an exact sequence of $\mathbb{Z}[\Gamma]$-modules:

$$\mathbf{E} = 0 \longrightarrow \pi_2\mathbf{E} \xrightarrow{i} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,$$

with F_i finitely generated free over $\mathbb{Z}[\Gamma]$ and $\pi_2\mathbf{E}$ finitely generated. The extension $\mathbf{E}$ remains fixed throughout this section, under the assumption that we wish to determine all possible algebraic 2-complexes with terminal module isomorphic to $\pi_2\mathbf{E}$.

Using the classification of extensions through cohomology and by analogy to Proposition 4.3.1, the congruence classes of $\mathrm{Ext}^3(\mathbb{Z}^n, \pi_2\mathbf{E}^n)$ for $n \geq 1$ are determined by equivalence classes of homomorphisms $[F]$ for some

$$F \in \mathrm{End}_{\mathbb{Z}[\Gamma]}(\pi_2\mathbf{E}^n) \cong \mathcal{M}_n(S) \qquad \text{where} \qquad S = \mathrm{End}_{\mathbb{Z}[\Gamma]}(\pi_2\mathbf{E}).$$

Note that we have appropriated the notation $[F]$ to represent the congruence class of the extension constructed as a pushout through the homomorphism F, as applied to a suitable number of direct sums of the resolution **E**. Compare with the remark on notation following Proposition 4.3.1.

6.3 Detection of algebraic 2-complexes

As before we say that a congruence class $\mathbf{x} \in \mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$ is invertible if there is some F, H such that $\mathbf{x} = [F]$ and

$$[FH] = [HF] = [Id].$$

We say that $\mathbf{x}$ has a left inverse if there is some F, H such that $\mathbf{x} = [F]$ and $[HF] = [Id]$.

Theorem 6.3.1. *Any projective extension of* $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$ *has a left inverse; if* $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$ *then projective extensions are precisely the invertible extensions.*

Proof. A proof may easily be constructed along the lines of 4.3.4, the only subtlety is in showing that invertible extensions are projective.

As a constituent part of the algebraic 2-complex **E** fixed earlier, there is an exact sequence:

$$0 \longrightarrow \pi_2\mathbf{E} \xrightarrow{i} F_2 \xrightarrow{\partial_2} \mathrm{Im}(\partial_2) \longrightarrow 0$$

and by dimension shifting:

$$\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E}) \cong \mathrm{Ext}^1(\mathrm{Im}(\partial_2), \pi_2\mathbf{E}).$$

Thus we may assume that any $\mathbf{x} \in \mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$ is congruent to some extension:

$$\begin{tikzcd} 0 \arrow[r] & \pi_2\mathbf{E} \arrow[r, "i_F"] & M[F] \arrow[rr] \arrow[dr] & & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[r, "\partial_0"] & \mathbb{Z} \arrow[r] & 0. \\ & & & \mathrm{Im}(\partial_2) \arrow[ur] & & & & \end{tikzcd}$$

If x is invertible then, as in the proof of Whitehead's Lemma (c.f. Proposition 4.3.3), we may find an isomorphism η of $\pi_2\mathbf{E}^2$ such that

$$[\eta] = \left[\begin{pmatrix} F & 0 \\ 0 & H \end{pmatrix}\right].$$

Thus we may construct a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (\pi_2\mathbf{E})^2 \arrow[r, "i"] \arrow[d, "\eta"] & F_2^2 \arrow[r, "\partial_2"] \arrow[d] & \mathrm{Im}(\partial_2)^2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\pi_2\mathbf{E})^2 \arrow[r, "i"] & M[F] \oplus M[H] \arrow[r] & \mathrm{Im}(\partial_2)^2 \arrow[r] & 0
\end{tikzcd}$$

and by the five lemma $M[F]$ is projective. □

This distinguishes the projective extension as those with a left inverse. The chain homotopy types may be distinguished by the following:

Proposition 6.3.2. *Suppose that $[F], [H]$ represent projective extensions of $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E})$. Then $[F]$ is chain homotopy equivalent to $[H]$ if and only if there exists an isomorphism η of $\pi_2\mathbf{E}$ and*

$$[\eta F] = [H].$$

Proof. Suppose that there is a homotopy equivalence:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2\mathbf{E} \arrow[r] \arrow[d, "f_4"] & P_2 \arrow[r] \arrow[d, "f_3"] & P_1 \arrow[r] \arrow[d, "f_2"] & P_0 \arrow[r] \arrow[d, "f_1"] & \mathbb{Z} \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & \pi_2\mathbf{E} \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

with the top row $[F]$ and bottom row $[H]$. Since f_0 is an isomorphism it may be represented as multiplication by ± 1 and hence, replacing f_i with $-f_i$ if necessary, there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2\mathbf{E} \arrow[r] \arrow[d, "f_4"] & P_2 \arrow[r] \arrow[d, "f_3"] & P_1 \arrow[r] \arrow[d, "f_2"] & P_0 \arrow[r] \arrow[d, "f_1"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_2\mathbf{E} \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

Then $[H] = [f_4 F]$ and f_4 is an isomorphism. Since congruent extensions are chain homotopy equivalent, the converse is trivial. □

For ease of reading we adopt the notation

$$\mathrm{Ext}(\pi_2\mathbf{E}) = \mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E}).$$

We now show how the algebraic 2-complexes may be distinguished from the general projective extensions. Recall that we have supposed there is some exact sequence of $\mathbb{Z}[\Gamma]$-modules:

$$\mathbf{E} = \ 0 \longrightarrow \pi_2\mathbf{E} \xrightarrow{i} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,$$

with each F_i a finitely generated free $\mathbb{Z}[\Gamma]$-module, and $\pi_2\mathbf{E}$ finitely generated over $\mathbb{Z}[\Gamma]$. Elements of $\mathrm{End}(\pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]^n)$ may be represented by the left action of $(n+1) \times (n+1)$ matrices

$$\begin{pmatrix} \alpha & v \\ w & A \end{pmatrix}$$

with

$$\begin{aligned} \alpha \in \mathrm{End}(\pi_2\mathbf{E}) \quad &: \quad w = (w_1, \ldots, w_n), \quad w_i \in \mathrm{Hom}(\pi_2\mathbf{E}, \mathbb{Z}[\Gamma]) \\ v = (v_1, \ldots, v_n), \quad v_i \in \mathrm{Hom}(\mathbb{Z}[\Gamma], \pi_2\mathbf{E}) \quad &: \quad A \in \mathcal{M}_n(\mathbb{Z}[\Gamma]). \end{aligned}$$

Let $[\alpha] \in \mathrm{Ext}(\pi_2\mathbf{E})$ represent the class of an algebraic 2-complex, say:

$$[\alpha] = \ 0 \longrightarrow \pi_2\mathbf{E} \xrightarrow{j} X_2 \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0 \xrightarrow{\delta_0} \mathbb{Z} \longrightarrow 0.$$

Then by Corollary 3.1.3 and Proposition 6.3.2 there is natural number n and an isomorphism φ such that the following commutes:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]^n \arrow[r, "i\oplus 1"] \arrow[d, "\varphi"] & F_2 \oplus \mathbb{Z}[\Gamma]^n \arrow[r, "\partial_2\oplus 0"] \arrow[d] & F_1 \arrow[r, "\partial_1"] \arrow[d] & F_0 \arrow[r, "\partial_0"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]^n \arrow[r, "j\oplus 1"] & X_2 \oplus \mathbb{Z}[\Gamma]^n \arrow[r, "\delta_2\oplus 0"] & X_1 \arrow[r, "\delta_1"] & X_0 \arrow[r, "\delta_0"] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

Thus:

Corollary 6.3.3. $[\alpha] \in \mathrm{Ext}(\pi_2\mathbf{E})$ *represents the class of an algebraic 2-complex only if there is natural number n and an isomorphism φ such that:*

$$\left[\begin{pmatrix} \alpha & 0 \\ 0 & Id_n \end{pmatrix}\right] = [\varphi]\,.$$

Note that using Proposition 3.2.1 it is easy to reinforce the above statement to an 'if and only if', and note also that $[\varphi]$ is an invertible extension. This gives a computational method to detect algebraic 2-complexes, assuming only the hypothesis of Theorem 1.1.2.

Remark 6.3.4. *Recall that we have classified extensions* $\mathbf{x}$ *in* $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]^n)$ *with matrices*

$$\begin{pmatrix} \alpha & v \\ w & A \end{pmatrix}$$

with

$$\begin{aligned} \alpha \in \mathrm{End}(\pi_2\mathbf{E}) \quad &: \quad w = (w_1, \dots, w_n), \quad w_i \in \mathrm{Hom}(\pi_2\mathbf{E}, \mathbb{Z}[\Gamma]) \\ v = (v_1, \dots, v_n), \quad v_i \in \mathrm{Hom}(\mathbb{Z}[\Gamma], \pi_2\mathbf{E}) \quad &: \quad A \in \mathcal{M}_n(\mathbb{Z}[\Gamma]). \end{aligned}$$

By inspection of the resolution used to compute this group:

$$0 \longrightarrow \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]^n \xrightarrow{i\oplus 1} F_2 \oplus \mathbb{Z}[\Gamma]^n \xrightarrow{\partial_2\oplus 0} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0$$

we see that in $\mathrm{Ext}^3(\mathbb{Z}, \pi_2\mathbf{E} \oplus \mathbb{Z}[\Gamma]^n)$*:*

$$\left[\begin{pmatrix} \alpha & v \\ w & A \end{pmatrix}\right] = \left[\begin{pmatrix} \alpha & 0 \\ w & 0 \end{pmatrix}\right].$$

In the appendix we give a description of stably free modules which would seem highly suggestive in relation to the above.

6.4 Realisation for C_∞^3

Given a finitely generated group Γ, we have indicated how the condition $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$ may be used to reduce the Realisation problem to realising algebraic 2-complexes at the minimal level, and to simplify the detection of algebraic 2-complexes. In this section we take an example of a group for which this condition does not hold, and solve the Realisation problem in the affirmative.

Throughout this section let Γ be the group determined by the presentation

$$\mathcal{G} = \langle a, b, c \mid ab = ba, \quad ca = ac, \quad bc = cb \rangle,$$

so that $\Gamma = C_\infty^3$ is a free abelian group on the generators a, b, c. The corresponding chain complex is given by:

$$C_*(\tilde{X}_{\mathcal{G}}) = \ 0 \longrightarrow \pi_2(\mathcal{G}) \xrightarrow{i} \mathbb{Z}[\Gamma]^3 \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^3 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} 1-b & c-1 & 0 \\ a-1 & 0 & 1-c \\ 0 & 1-a & b-1 \end{pmatrix} \quad , \quad \partial_1 = \begin{pmatrix} a-1 & b-1 & c-1 \end{pmatrix}$$

and $\pi_2(\mathcal{G})$ may be identified with the submodule of $\mathbb{Z}[\Gamma]^3$ generated by the element:

$$\begin{pmatrix} c-1 \\ b-1 \\ a-1 \end{pmatrix}.$$

Since each co-ordinate is not a zero divisor of $\mathbb{Z}[\Gamma]$ we see that

$$\pi_2(\mathcal{G}) \cong \mathbb{Z}[\Gamma].$$

This allows us to prove a cancellation result:

Corollary 6.4.1. *If* $\mathbf{E}$ *is an algebraic 2-complex over* $\mathbb{Z}[\Gamma]$, *then* $\pi_2\mathbf{E} \cong \mathbb{Z}[\Gamma]^a$ *for some non zero* a.

Proof. Any $\pi_2\mathbf{E}$ is a stably free module by Schanuel's Lemma. Then by Proposition 4.12 in (T.Y. Lam [17]), as mentioned in section 3.3, $\pi_2\mathbf{E}$ is free. Note that $\pi_2(\mathcal{G})$ is minimal since if $\pi_2\mathbf{E} = 0$ was possible then $\mathrm{Ext}^3(\mathbb{Z}, N) = 0$ would hold for all modules N, and one may deduce that this is false from Lemma 6.4.2. □

At this point we have completed steps (1) and (2) of the realisation program. Steps (3) and (4) are mostly completed - at the minimal level - by the following:

Lemma 6.4.2. *As an abelian group,* $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]) = \mathbb{Z}$. *The projective extensions are precisely the elements* $\pm 1 \in \mathbb{Z}$.

Recall there is an augmentation homomorphism: $\varepsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ with $\varepsilon(g) = 1$ for all $g \in \Gamma$.

Proof. Using the resolution $C_*(\tilde{X}_{\mathcal{G}})$ and the characterisation of Ext given in Section 2.5, we classify the elements of $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma])$ as equivalence classes of homomorphisms in $\mathrm{End}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma])$. In particular we wish to calculate homomophisms $f : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ modulo homomorphisms h such that there is some $\eta : \mathbb{Z}[\Gamma]^3 \to \mathbb{Z}[\Gamma]$ and commutative diagram:

$$\begin{tikzcd} \mathbb{Z}[\Gamma] \arrow[r, "i"] \arrow[d, "h"] & \mathbb{Z}[\Gamma]^3 \arrow[dl, "\eta"] \\ \mathbb{Z}[\Gamma] & \end{tikzcd}$$

where

$$i(1) = \begin{pmatrix} c-1 \\ b-1 \\ a-1 \end{pmatrix}$$

To detect the projective extensions, we wish to determine the cohomology classes of elements with a left inverse in $\mathrm{End}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma])$. We may consider each map in $\mathrm{End}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}[\Gamma]) \cong \mathbb{Z}[\Gamma]$ to be left multiplication by some element of $\mathbb{Z}[\Gamma]$. Since Γ is commutative so is $\mathbb{Z}[\Gamma]$. Thus any extension that has a left inverse also has a right inverse and the projective extensions are represented by the invertible elements.

We claim that each of multiplication by $(a-1), (b-1)$ and $(c-1)$ represent the zero element of $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma])$. For example, if $f : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ is multiplication by $(a-1)$ then $f = \eta i$ where $\eta : \mathbb{Z}[\Gamma]^3 \to \mathbb{Z}[\Gamma]$ is projection onto the last coordinate. Thus for each $\lambda \in \mathbb{Z}[\Gamma]$, the class of extensions represented by (left multiplication by) λ is only dependent on $\varepsilon(\lambda)$. So we may assume that each extension is represented by multiplication by some $n \in \mathbb{Z}$. Moreover, multiplication by n never represents a split (i.e. zero) extension unless $n = 0$, since for any homomorphism $\mathbb{Z}[\Gamma]^3 \to \mathbb{Z}[\Gamma]$, the image of $\pi_2(\mathcal{G})$ is in the kernel of the augmentation ε. Composition of homomorphisms corresponds to the natural multiplicative structure on $\mathbb{Z}$ and hence the only invertible extensions are those represented by 1 and -1. $\square$

Proof of Theorem A: Consider the group $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]^{n+1})$. This may be calculated from the resolution:

$$0 \longrightarrow \mathbb{Z}[\Gamma] \oplus \mathbb{Z}[\Gamma]^n \xrightarrow{i'} \mathbb{Z}[\Gamma]^3 \oplus \mathbb{Z}[\Gamma]^n \xrightarrow{\partial_2'} \mathbb{Z}[\Gamma]^3 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where

$$i' = \begin{pmatrix} i & 0 \\ 0 & Id_n \end{pmatrix} \quad , \quad \partial_2' = \begin{pmatrix} \partial_2 & 0 \\ 0 & 0 \end{pmatrix}$$

and $i, \partial_2, \partial_1$ are as before (see definition of $C_*(\tilde{X}_{\mathcal{G}})$ given at the start of this section).

If $F = (f_{i,j})$ is an endomorphism of $\mathbb{Z}[\Gamma]^{n+1}$, that is $F \in \mathcal{M}_{n+1}(\mathbb{Z}[\Gamma])$, then by inspection of the resolution above:

$$\left[\begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n+1} \\ \vdots & & & \vdots \\ f_{n+1,1} & f_{n+1,2} & \cdots & f_{n+1,n+1} \end{pmatrix}\right] = \left[\begin{pmatrix} f_{1,1} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ f_{n+1,1} & 0 & \ldots & 0 \end{pmatrix}\right].$$

Thus:

(I) The equivalence class $[F] \in \mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]^n)$ is not affected by any of the entries not in the first column of F.

Moreover there is an isomorphism of abelian groups $\mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma]^{n+1}) \to \mathrm{Ext}^3(\mathbb{Z}, \mathbb{Z}[\Gamma])^n$ given by

$$[F] \to ([f_{1,1}], [f_{1,2}], \ldots, [f_{1,n+1})]).$$

This isomorphism is a direct consequence of the additivity of Ext, but may also be justified by inspection of the resolution. Moreover by Lemma 6.4.2 and the isomorphism above:

(II) We may assume without loss of generality that each $f_{1,i}$ is an integer.

By Corollary 6.3.3, $[F]$ represents an algebraic 2-complex only if there is an invertible matrix φ such that

$$\left[\begin{pmatrix} F & 0 \\ 0 & Id \end{pmatrix}\right] = [\varphi].$$

Note that since φ is invertible, $\varepsilon(\varphi)$ is an invertible matrix over $\mathbb{Z}$ of the same rank as φ. In order to represent the same congruence class, the elements of the first column of $\varepsilon(\varphi)$ must be equal to the first column of $\varepsilon(F) \oplus Id$. Consequently and by statement (II):

(III) $[F]$ represents an algebraic 2-complex only if the $\mathbb{Z}$-linear span of the elements in the first row of F contains 1.

By statement (I) we may alter any entry of F apart from the first column without altering the congruence class $[F]$, so if $[F]$ represents an algebraic 2-complex then by statement (III) we may take F to be an invertible matrix over $\mathbb{Z}$. Then there exists products of elementary matrices E_1, E_2 such that

$$\begin{array}{lcc} \text{either} & E_1F = I_n \quad \text{and} & FE_2 = I_n \\ \text{or} & E_1F = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \quad \text{and} & FE_2 = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}. \end{array}$$

Since

$$[Id] \simeq \left[\begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \right]$$

and elementary matrices are representable by isomorphisms, we conclude that for any endomorphism F of $\mathbb{Z}[\Gamma]^{n+1}$: $[F]$ is congruent to an algebraic 2-complex if and only if $[F] \simeq [Id]$.

This completes the proof, since if $\mathbf{E}$ is an algebraic 2-complex over $\mathbb{Z}[\Gamma]$ then by Corollary 6.4.1, $\pi_2\mathbf{E}$ is free of rank (say) $n+1$, and then $\mathbf{E}$ is realised geometrically by the addition of n trivial relations to $\mathcal{G}$.

Chapter 7

The D(2) Property for $C_n \times C_\infty$

7.1 One dimensional groups

In this chapter we shall prove Theorem D, i.e. the Realisation property and the D(2) property hold for groups of the form $C_n \times C_\infty$. Our proof is largely independent of the work in chapters 3-5, coming earlier in discovery. However the proof involves a cancellation result which is demonstrated through a method which inspired the more general concerns of the last few chapters. We shall see that the relevant second homotopy module occurs as the central module in an extension, and that extensions of this form are a simple generalisation of those pertinent to Swan modules.

We shall assume throughout that G is a finite group and take Ψ to be the product $\Psi = G \times C_\infty$. We shall later specify that G be a cyclic group, but some progress may be made in the more general case. Here the most important simplification of the problem is given by viewing $\mathbb{Z}[\Psi]$ as

$$\mathbb{Z}[\Psi] = R[G]$$

where R is the commutative integral domain $\mathbb{Z}[C_\infty]$. Modules over $\mathbb{Z}[\Psi]$ which are free as R-modules are analogous to torsion free $\mathbb{Z}$[G]-modules and many of the results regarding lattices over finite groups (such as Maschke's Theorem) generalise easily to this case. The results contained in Chapters 4 and 5 of Johnson's book [11] are particularly relevant and we shall borrow from them freely.

Let

$$\mathcal{G} = \langle x_1, \dots x_g | W_1, \dots W_r \rangle$$

be a presentation for the finite group G, with corresponding complex of $\mathbb{Z}[G]$-modules:

$$C_*(\tilde{X}_{\mathcal{G}}) = \; 0 \longrightarrow \mathrm{Ker}(\partial_2) \longrightarrow \mathbb{Z}[G]^r \xrightarrow{\partial_2} \mathbb{Z}[G]^g \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where we assume ∂_n is a matrix over $\mathbb{Z}[G]$. Note that $\mathrm{Ker}(\partial_2)$ is finitely generated and hence we may pick a homomorphism (matrix):

$$\partial_3 : \mathbb{Z}[G]^n \to \mathbb{Z}[G]^r$$

such that $\mathrm{Im}(\partial_3) = \mathrm{Ker}(\partial_2)$.

There is a canonical choice of presentation $\mathcal{H}$ for Ψ given by

$$\mathcal{H} = \langle x_1, \dots x_g, t \quad | \quad W_1, \dots, W_r, C_1, \dots, C_g \rangle,$$

where $C_i = t^{-1} x_i^{-1} t x_i$ is the commutator relation for t and x_i. Then $\mathcal{H}$ has corresponding algebraic 2-complex:

$$C_*(\tilde{X}_{\mathcal{H}}) = \; 0 \longrightarrow \mathrm{Ker}(\delta_2) \longrightarrow \mathbb{Z}[\Psi]^{r+g} \xrightarrow{\delta_2} \mathbb{Z}[\Psi]^{g+1} \xrightarrow{\delta_1} \mathbb{Z}[\Psi] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where the δ_n are represented by matrices

$$\delta_2 = \begin{pmatrix} \partial_2 & (t-1) \cdot Id \\ 0 & -\partial_1 \end{pmatrix} \quad , \quad \delta_1 = \begin{pmatrix} \partial_1 & (t-1) \end{pmatrix},$$

letting ∂_n represent a matrix over $\mathbb{Z}[\Psi]$ through the ring inclusion of $\mathbb{Z}[G]$ in $\mathbb{Z}[\Psi]$. Note that $C_*(\tilde{X}_{\mathcal{H}})$ is naturally the complex arising from the universal cover of the 2-skeleton of the product $X_{\mathcal{G}} \times S^1$.

We can find a representation for the second homotopy module $\pi_2(\mathcal{H}) = \mathrm{Ker}(\delta_2)$ as the submodule of $\mathbb{Z}[\Psi]^{r+g}$ generated by the columns of the matrix:

$$\delta_3 = \begin{pmatrix} \partial_3 & (t-1) \cdot Id \\ 0 & -\partial_2 \end{pmatrix}.$$

Computations of Ext groups

We show that Ψ satisfies the hypothesis of the more specific reductions of the Realisation problem, as determined by Johnson and detailed in the introduction, and calculate the relevant Ext groups.

Lemma 7.1.1. *For G a finite group and $\Psi = G \times C_\infty$:*

$$\mathrm{Ext}^n(\mathbb{Z}, \mathbb{Z}[\Psi]^r) = 0 \quad \text{foral} \quad \mathrm{n} \geq 2 \quad \text{andall} \quad \mathrm{r} \geq 1$$

Proof. Consider the kernel of the standard augmentation:

$$I = \mathrm{Ker}(\varepsilon : \mathbb{Z}[\Psi] \to \mathbb{Z}).$$

This may be seen to be free as an R-module, generated by the elements $(1-t)$ and $\{(1-g) \quad ; \quad g \in G\}$. It may be shown that for any $\mathbb{Z}[\Psi]$-module M such that M is free over R:

$$\mathrm{Ext}^n(M, \mathbb{Z}[\Psi]) = 0 \qquad \text{for all } n \geq 1.$$

See e.g. (Johnson [11] Chapter 4) for a proof of this statement.

Thus

$$\mathrm{Ext}^n(I, \mathbb{Z}[\Psi]^r) = 0 \qquad \forall n \geq 1$$

and since there is an exact sequence:

$$0 \longrightarrow \mathrm{Ker}(\delta_2) \longrightarrow \mathbb{Z}[\Psi]^{r+g} \xrightarrow{\delta_2} \mathbb{Z}[\Psi]^{g+1} \xrightarrow{\delta_1} I \longrightarrow 0,$$

by dimension shifting

$$\mathrm{Ext}^n(\mathbb{Z}, \mathbb{Z}[\Psi]^r) = \mathrm{Ext}^{n-1}(I, \mathbb{Z}[\Psi]^r)$$

and the result is shown. □

We also need to calculate the Ext group containing algebraic 2-complexes, we shall do so using the following:

Proposition 7.1.2. *There exists an exact sequence of $\mathbb{Z}[\Gamma]$-modules:*

$$E = \qquad 0 \longrightarrow \mathbb{Z}[\Psi]^r \xrightarrow{j} \pi_2(\mathcal{H}) \xrightarrow{\pi} \pi_2(\mathcal{G}) \longrightarrow 0$$

where $\pi_2(\mathcal{G}) = \mathrm{Ker}(\partial_2 : \mathbb{Z}[\mathrm{G}]^r \to \mathbb{Z}[\mathrm{G}]^g)$ is given a $\mathbb{Z}[\Psi]$-module structure through the augmentation ring homomorphism $\varepsilon_G : \mathbb{Z}[\Psi] \to \mathbb{Z}[\mathrm{G}]$.

Proof. $\pi_2(\mathcal{H})$ consists of elements of the form $(\partial_3(v_1) + (t-1)v_2, -\partial_2(v_2))$, where $v_1 \in \mathbb{Z}[\Psi]^b, v_2 \in \mathbb{Z}[\Psi]^r$. Define $j : \mathbb{Z}[\Psi]^r \to \pi_2(\mathcal{H})$ as

$$j(v_2) = ((t-1)v_2, -\partial_2(v_2)).$$

Then since $(t-1)$ is not a divisor of zero, j is a well defined injection. Note that j identifies the i^{th} generator of $\mathbb{Z}[\Psi]^r$ with the submodule of $\pi_2(\mathcal{H})$ generated by the i^{th} column of:

$$\begin{pmatrix} (t-1)I_r \\ -\partial_2 \end{pmatrix}.$$

Define $\pi : \pi_2(\mathcal{H}) \to \pi_2(\mathcal{G})$ as

$$\pi((\partial_3(v_1) + (t-1)v_2, -\partial_2(v_2)) = \varepsilon_G \partial_3(v_1)$$

then π is well defined and $\mathrm{Im}(\pi) = \pi_2(\mathcal{G})$.

The reader may verify that $\mathrm{Ker}(\pi) = \mathrm{Im}(j)$ and we have constructed the required exact sequence. □

Thus, by (Mac Lane [19] III.9.1), we may form the long exact sequence:

$$\cdots \to \mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \mathbb{Z}[\Psi]^r) \xrightarrow{j_*} \mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{H}))$$
$$\xrightarrow{\pi_*} \mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{G})) \xrightarrow{E_*} \mathrm{Ext}^4_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \mathbb{Z}[\Psi]^r) \to \cdots$$

where we indicate the ambient ring over which the Ext groups are constructed because they will soon be manipulated.

Corollary 7.1.3. $\pi_* : \mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{H})) \to \mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{G}))$ *is an isomorphism.*

Proof. This follows directly from Lemma 7.1.1 and the long exact sequence above. □

Proposition 7.1.4. *There is an isomorphism of abelian groups*

$$f : \mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{G})) \to \mathrm{Ext}^3_{\mathbb{Z}[G]}(\mathbb{Z}, \pi_2(\mathcal{G})).$$

Note the change of rings.

Proof. If E is an extension in $\mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{G}))$, then through the ring inclusion $\mathbb{Z}[\mathrm{G}] \hookrightarrow \mathbb{Z}[\Psi]$ and restriction of scalars, E may be considered as an extension in $\mathrm{Ext}^3_{\mathbb{Z}[\mathrm{G}]}(\mathbb{Z}, \pi_2(\mathcal{G}))$. Write $f(E)$ for E considered as an extension over $\mathbb{Z}[\mathrm{G}]$. Conversely if F is an extension in $\mathrm{Ext}^3_{\mathbb{Z}[\mathrm{G}]}(\mathbb{Z}, \pi_2(\mathcal{G}))$, then F may be considered as an extension in $\mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{G}))$ through the augmentation homomorphism $\varepsilon_G : \mathbb{Z}[\Psi] \to \mathbb{Z}[\mathrm{G}]$. This correspondence commutes with the Baer Sum.

It is sufficient to show that $f(E)$ splits as an extension if and only if E splits. Trivially, if E splits then $f(E)$ splits, since all $\mathbb{Z}[\Psi]$ maps are $\mathbb{Z}[\mathrm{G}]$ linear.

Suppose that $f(E)$ represents a split extension in $\mathrm{Ext}^3_{\mathbb{Z}[\mathrm{G}]}(\mathbb{Z}, \pi_2(\mathcal{G}))$, then there exists a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & \pi_2(\mathcal{G}) \arrow[r, "i"] \arrow[d, "\varphi"] & \mathbb{Z}[\mathrm{G}]^r \arrow[r, "\partial_2"] \arrow[d] \arrow[dl, "\eta"] & \mathbb{Z}[\mathrm{G}]^g \arrow[r, "\partial_1"] \arrow[d] & \mathbb{Z}[\mathrm{G}] \arrow[r, "\varepsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
f(E) = & 0 \arrow[r] & \pi_2(\mathcal{G}) \arrow[r] & E^2 \arrow[r] & E^1 \arrow[r] & E^0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where our notation is taken to imply that the extension $f(E)$ is represented by the bottom row. Define $\pi_2 : \mathbb{Z}[\Psi]^{r+g} \to \mathbb{Z}[\mathrm{G}]^r$ as projection on to the first r factors composed with ε_G. Define $\pi_1 : \mathbb{Z}[\Psi]^{g+1} \to \mathbb{Z}[\mathrm{G}]^r$ as projection on to the first g factors composed with ε_G. Then the following commutes:

$$\begin{tikzcd}
 & 0 \arrow[r] & \pi_2(\mathcal{H}) \arrow[r] \arrow[d, "\pi"] & \mathbb{Z}[\Psi]^{r+g} \arrow[r, "\delta_2"] \arrow[d, "\pi_2"] & \mathbb{Z}[\Psi]^{g+1} \arrow[r, "\delta_1"] \arrow[d, "\pi_1"] & \mathbb{Z}[\Psi] \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon_G"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0\,. \\
 & 0 \arrow[r] & \pi_2(\mathcal{G}) \arrow[r] \arrow[d, "\varphi"] & \mathbb{Z}[\mathrm{G}]^r \arrow[r, "\partial_2"] \arrow[d] \arrow[dl, "\eta"] & \mathbb{Z}[\mathrm{G}]^g \arrow[r, "\partial_1"] \arrow[d] & \mathbb{Z}[\mathrm{G}] \arrow[r, "\varepsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
f(E) = & 0 \arrow[r] & \pi_2(\mathcal{G}) \arrow[r] & E^2 \arrow[r] & E^1 \arrow[r] & E^0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

so that $\varphi\pi$ splits as $\eta\pi_2 i$ and hence E is a split extension. This completes the proof. $\square$

It is well known that $\mathrm{Ext}^3_{\mathbb{Z}[\mathrm{G}]}(\mathbb{Z}, \pi_2(\mathcal{G})) \cong \mathbb{Z}_n$ for any finite group G (see e.g. [11] Chapter 6) and hence:

Corollary 7.1.5. $\mathrm{Ext}^3_{\mathbb{Z}[\Psi]}(\mathbb{Z}, \pi_2(\mathcal{G})) \cong \mathbb{Z}_n$

Another exact sequence

We show that $\pi_2(\mathcal{H})$ is related to $\pi_2(\mathcal{G})$ by another short exact sequence. This sequence exists in general for $\Psi = G \times C_\infty$ with G finite and so we shall construct it here, although little use is suggested beyond the fundamental role it will play in our later investigations with G a cyclic group. Recall that we have already constructed $\mathbb{Z}$[G]-modules $\mathrm{Ker}(\partial_2) = \pi_2(\mathcal{G})$ and $\mathrm{Ker}(\partial_1)$. These may be transformed into $\mathbb{Z}[\Psi]$-modules through tensoring with $\mathbb{Z}[C_\infty]$ over $\mathbb{Z}$. As already mentioned, through the inclusion of $\mathbb{Z}$[G] in $\mathbb{Z}[\Psi]$, ∂_2 and ∂_1 may be considered as matrices over $\mathbb{Z}[\Psi]$. Then there is a natural identification

$$\mathrm{Ker}(\partial_i) \otimes_{\mathbb{Z}} \mathbb{Z}[C_\infty] = \mathrm{Ker}(\partial_i)$$

where ∂_i is considered as a $\mathbb{Z}$[G] matrix on the left and as a $\mathbb{Z}[\Psi]$ matrix on the right.

Proposition 7.1.6. *There is a short exact sequence of $\mathbb{Z}[\Psi]$-modules:*

$$0 \longrightarrow \pi_2(\mathcal{G}) \otimes \mathbb{Z}[C_\infty] \longrightarrow \pi_2(\mathcal{H}) \longrightarrow \mathrm{Ker}(\partial_1) \longrightarrow 0$$

where ∂_1 is considered as a matrix over $\mathbb{Z}[\Psi]$.

Proof. Recall that we have identified $\pi_2(\mathcal{H})$ with the submodule of $\mathbb{Z}[\Psi]^{r+g}$ generated by the columns of the matrix:

$$\delta_3 = \begin{pmatrix} \partial_3 & (t-1) \\ 0 & -\partial_2 \end{pmatrix}.$$

Then since

$$\pi_2(\mathcal{G}) \otimes \mathbb{Z}[C_\infty] = \mathrm{Im}(\partial_3),$$

$\pi_2(\mathcal{G}) \otimes \mathbb{Z}[C_\infty]$ embeds into $\pi_2(\mathcal{H})$ through identification with the submodule generated by the columns of:

$$\begin{pmatrix} \partial_3 \\ 0 \end{pmatrix}.$$

The quotient under this embedding is easily seen to be equivalent to $\mathrm{Ker}(\partial_1)$, which is represented as the columns of the matrix:

$$\begin{pmatrix} 0 \\ -\partial_2 \end{pmatrix}.$$

□

We wish to compare the above sequence with a projective extension. Through tensoring the sequence $C_*(\tilde{X}_{\mathcal{G}})$ with $\mathbb{Z}[C_\infty]$ there is an exact sequence:

$$0 \longrightarrow \pi_2(\mathcal{G}) \otimes \mathbb{Z}[C_\infty] \longrightarrow \mathbb{Z}[\Psi]^r \longrightarrow \mathrm{Ker}(\partial_1) \longrightarrow 0$$

and the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(\mathcal{G}) \otimes \mathbb{Z}[C_\infty] \arrow[r] \arrow[d, "\times(t-1)"'] & \mathbb{Z}[\Psi]^r \arrow[r] \arrow[d, "j"] & \mathrm{Ker}(\partial_1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_2(\mathcal{G}) \otimes \mathbb{Z}[C_\infty] \arrow[r] & \pi_2(\mathcal{H}) \arrow[r] & \mathrm{Ker}(\partial_1) \arrow[r] & 0
\end{tikzcd}$$

where j is as in Proposition 7.1.2.

This reaches the extent of the general case. We now specify that $G = C_n$ is cyclic.

7.2 The product of a cyclic group and the integers

Now consider the case where $G = C_n$ is the cyclic group with n elements and Γ is the product $G \times C_\infty$. We take the presentation

$$\mathcal{H} = \langle x, t \mid x^n = 1, tx = xt \rangle.$$

Writing N for $\sum_{g \in C_n} g$, the corresponding chain complex is:

$$C_*(\tilde{X}_{\mathcal{H}}) = \left(0 \longrightarrow \pi_2(\mathcal{H}) \longrightarrow \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_2} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \to 0\right)$$

$$\partial_2 = \begin{pmatrix} 1-t & N \\ x-1 & 0 \end{pmatrix} \qquad \partial_1 = (x-1 \quad t-1),$$

and $\pi_2(\mathcal{H})$ may be identified with the submodule of $\mathbb{Z}[\Gamma]^2$ generated by the elements

$$\begin{pmatrix} 0 \\ x-1 \end{pmatrix} \text{ and } \begin{pmatrix} N \\ t-1 \end{pmatrix}.$$

We wish to determine the class of modules which are stably equivalent to $\pi_2(\mathcal{H})$, and we shall do so in Theorem 7.2.8.

A resolution for the integers over $\mathbb{Z}[G]$ is given by the period 2 resolution:

$$\cdots \longrightarrow \mathbb{Z}[G] \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\partial_2} \mathbb{Z}[G] \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where ∂_1 is multiplication by $(1-x)$ and ∂_2 multiplication by the sum of the elements in G. The image of ∂_1 is the augmentation ideal and its kernel is isomorphic to the trivial module $\mathbb{Z}$.

Since cyclic groups have cohomological period two, the exact sequence constructed in Proposition 7.1.6 becomes:

$$0 \longrightarrow \hat{I} \longrightarrow \pi_2(\mathcal{H}) \longrightarrow \mathbb{Z}[C_\infty] \longrightarrow 0$$

where $\hat{I}$ is the tensor product over $\mathbb{Z}$ of $\mathbb{Z}[C_\infty]$ and the $\mathbb{Z}[C_n]$-augmentation ideal. Through tensoring the standard augmentation sequence with $\mathbb{Z}[C_\infty]$ there is also an exact sequence:

$$0 \longrightarrow \hat{I} \longrightarrow \mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z}[C_\infty] \longrightarrow 0$$

with $\hat{I}$ a two-sided ideal.

This means that $\pi_2(\mathcal{H})$ occurs as the central module in an extension with terminal modules occurring as such in another extension, as a quotient ring and two sided ideal. Thus, informally, $\pi_2(\mathcal{H})$ takes the form of a non-projective generalised Swan module. As was the case with Swan modules, discussed in section 4.6, the notation and proofs are clearer when working with the dual sequence, and it shall prove easier to calculate the stable class of the dual module $\pi_2(\mathcal{H})^* = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\pi_2(\mathcal{H}), \mathbb{Z}[\Gamma])$. We justify doing so by [11] Prop 28.1, which shows that cancellation holds for $\pi_2(\mathcal{H})^*$ if and only if it holds for $\pi_2(\mathcal{H})$.

Directly, one may verify that $\pi_2(\mathcal{H})^*$ may be identified with the submodule of $\mathbb{Z}[\Gamma]^2$ generated by the elements:

$$\begin{pmatrix} t-1 \\ 1-x \end{pmatrix}, \begin{pmatrix} N \\ 0 \end{pmatrix}.$$

$\mathbb{Z}[C_\infty]$ imbeds in $\pi_2(\mathcal{H})^*$ by identification with the module generated by the second element above, resulting in a short exact sequence of $\mathbb{Z}[\Gamma]$-modules:

$$0 \longrightarrow \mathbb{Z}[C_\infty] \longrightarrow \pi_2(\mathcal{H})^* \longrightarrow \mathbb{Z}[\Gamma]/\langle N \rangle \longrightarrow 0\,.$$

This is the dual of the exact sequence above. We fix the notation:

$$S = \mathbb{Z}[\Gamma]/\langle N \rangle \qquad R = \mathbb{Z}[C_\infty].$$

This notation clashed with our earlier use of R for the quotient ring and S for the endomorphism group of a two-sided ideal, but is entirely consistent with the concept that we are using the dual of the original sequence. Embed R into $\mathbb{Z}[\Gamma]$ by identifying R with the submodule generated by N, so that there is an exact sequence:

$$0 \longrightarrow R \longrightarrow \mathbb{Z}[\Gamma] \longrightarrow S \longrightarrow 0$$

and R is a two-sided ideal in $\mathbb{Z}[\Gamma]$.

Defining $f : \mathbb{Z}[\Gamma] \to \pi_2(\mathcal{H})^*$ by setting $f(1) = (t-1, 1-x)^T$, there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "(t-1)"] & \mathbb{Z}[\Gamma] \arrow[r] \arrow[d, "f"] & S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \arrow[r] & \pi_2(\mathcal{H})^* \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

which gives a cohomological classification for the bottom extension.

For each $k \in \mathbb{N}$ there is an exact sequence:

$$0 \longrightarrow R^k \longrightarrow \pi_2(\mathcal{H})^* \oplus \mathbb{Z}[\Gamma]^{k-1} \longrightarrow S^k \longrightarrow 0\,.$$

This gives a naïve model for modules potentially stably isomorphic to $\pi_2(\mathcal{H})^*$ as a subclass of the central modules occurring in extensions of R^k by S^k. Through a detailed study of such extensions we shall show that this model is appropriate and prove the required cancellation result.

Proposition 7.2.1. *Let $\mathbb{Z}_n$ denote the integers modulo n, and $\mathcal{M}_k(R_n)$ the ring of k by k matrices over the ring $R_n = \mathbb{Z}_n[C_\infty]$. There is a ring isomorphism*

$$\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(S^k, R^k) \cong \mathcal{M}_k(R_n).$$

Proof. Note that since S is free as an R-module, $\mathrm{Ext}^1(S, \mathbb{Z}[\Gamma]) = 0$ (see [11] Chapter 5). The corresponding fibre product diagram is given by:

$$\begin{tikzcd}
\mathbb{Z}[\Gamma] \arrow[r, "i_1"] \arrow[d, "i_2"] & R \arrow[d, "j_1"] \\
S \arrow[r, "j_2"] & \mathrm{Ext}^1(S, R) \cong R_n
\end{tikzcd}$$

i_1 is the augmentation map $\mathbb{Z}[\Gamma] \to R$.
i_2 is the natural quotient map $\mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]/\langle N \rangle$ with $S = \mathbb{Z}[\Gamma]/\langle N \rangle$.
j_1 is the natural quotient map $R \to R_n$.
j_2 is defined for each $\alpha + \langle N \rangle \in S$ by $j_2(\alpha + \langle N \rangle) = j_1 i_1(\alpha)$.
The result then follows by Proposition 4.3.1. □

To give concrete examples of the congruence classes of extensions and central modules, given any $A \in \mathcal{M}_k(R_n)$, we may pick a lift of A in $\mathcal{M}_k(\mathbb{Z}[\Gamma])$ and define $M(A)$ to be the submodule of $\mathbb{Z}[\Gamma]^{2k}$ generated by the columns of the matrix

$$A' = \begin{pmatrix} A & (N)I_k \\ (x-1)I_k & 0 \end{pmatrix}.$$

Then R^k embeds in $M(A)$ by identifying the i^{th} basis element of R^k with the $i + k^{th}$ column of A', and the image of R^k in $M(A)$ is the kernel of the surjective map $\pi_A \colon M(A) \to S^k$ given by sending the i^{th} column of A' (for $1 \leq i \leq k$) to the i^{th} generator of S^k. We represent the resulting exact sequence as:

$$E(A) = \qquad 0 \longrightarrow R^k \xrightarrow{i_A} M(A) \xrightarrow{\pi_A} S^k \longrightarrow 0$$

and each extension of R^k by S^k is congruent to $E(A)$ for some unique $A \in \mathcal{M}_k(R_n)$.

We wish to characterise the isomorphism classes of the modules $M(A)$. Since the action of N is zero on S and multiplication by n on R, and since R is torsion free over $\mathbb{Z}$:

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(R^k, S^k) = 0,$$

so that any isomorphism $\phi \colon M(A) \cong M(B)$ of $\mathbb{Z}[\Gamma]$-modules induces an isomorphism of extensions $E(A) \cong E(B)$. Note the isomorphisms on each end may be represented as matrices $C \in \mathcal{M}_n(R)$ and $D \in \mathcal{M}_n(S)$. We distinguish the matrices in $\mathcal{M}_k(R_n)$ which are images of such isomorphisms:

- $GR(k)$ denotes the image of $\mathrm{GL_k(R)}$ in $\mathrm{GL_k(R_n)}$ under j_1.

- $GS(k)$ denotes the image of $\mathrm{GL_k(S)}$ in $\mathrm{GL_k(R_n)}$ under j_2.

The following Theorem may be recognised as an application of Proposition 34.4 in (Curtis & Reiner [4]), as discussed in section 5.3.

Theorem 7.2.2. *For arbitrary matrices $A, B \in \mathcal{M}_k(R_n)$, $M(A) \cong M(B)$ if and only if there exists $C \in GR(k), D \in GS(k)$ such that $CAD = B$.*

Having characterised the isomorphism classes of the modules $M(A)$, we wish to characterise their stable isomorphism classes. Clearly

$$M(A) \oplus \mathbb{Z}[\Gamma]^m \cong M(A \oplus I_m),$$

where as before the matrix $A \oplus B$ is taken to be the matrix:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Corollary 7.2.3. *$M(A)$ is stably equivalent to $M(B)$ if and only if there exists $C \in GR(k+m), D \in GS(k+m)$ and $m \in \mathbb{N}$ such that $C(A \oplus I_m)D = (B \oplus I_m)$.*

As is standard, we write $E_k(R_n)$ for the group of $k \times k$ matrices over R_n which are products of elementary matrices, and R^+ for the units of R. Note that $E_k(R_n) \subseteq GS(k)$ and $E_k(R_n) \subseteq GR(k)$.

Proposition 7.2.4. *The determinant homomorphism* $\det\colon \mathrm{GL_k(R_n)/E_k(R_n)} \to \mathrm{R_n^+}$ *is an isomorphism for all $k \geq 1$. Equivalently, every invertible matrix in $\mathcal{M}_k(R_n)$ with determinant one is a product of elementary matrices.*

Proof. Let the prime decomposition of n be $\Pi_{i=1}^{s} p_i^{e_i}$. Then R_n is isomorphic to the product $\Pi_{i=1}^{s} R_{p_i^{e_i}}$. The multiplicative groups $\mathcal{M}_k(R_n)$ and R_n^+ decompose similarly as the products of the matrix and unit ring respectively of the rings $R_{p_i^{e_i}}$, as does the determinant homomorphism. It is enough to prove the Proposition in the case where $n = p^e$ is a power of a prime.

Suppose that $\det(E) = 1$ where $E \in \mathcal{M}_k(R_n)$, and let E_p denote the equivalence class of E in $\mathcal{M}_k(R_p)$. We shall show that we may reduce E by elementary row and column operations to the identity. Since $R_p = \mathbb{F}_p[t, t^{-1}]$ is a Euclidean domain there are elementary matrices $E_p^1, \dots E_p^j$ such that

$E_p \cdot E_p^1 \cdot \ldots E_p^j = Id_p$. Thus we may pick elementary matrices $E^1, \ldots E^j \in E_k(R_n)$ such that:

$$E \cdot E^1 \ldots E^j = \begin{pmatrix} 1 + c_{1,1}p & c_{1,2}p & \ldots & c_{1,k}p \\ c_{1,2}p & 1 + c_{2,2}p & \ldots & c_{2,k}p \\ \vdots & \vdots & & \vdots \\ c_{k,1}p & c_{k,2}p & \ldots & 1 + c_{k,k}p \end{pmatrix} \qquad c_{i,j} \in R_n.$$

By the Binomial Theorem each $(1+c_{i,i}p)$ is a unit (take the $(p^e)^{th}$ power!) and thus the above matrix has diagonal entries which are all units. Thus the above matrix, and hence E, may be reduced by the action of column operations to a matrix of the form

$$\begin{pmatrix} 1 + d_{1,1}p & 0 & \ldots & 0 \\ 0 & 1 + d_{2,2}p & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 + d_{k,k}p \end{pmatrix} \qquad d_{i,i} \in R_n.$$

By Whitehead's Lemma, $Diag(1, \ldots, 1, u, u^{-1}, 1, \ldots, 1) \in E_k(R_n)$ and E may be further reduced to

$$E' = Diag(u, 1, \ldots, 1)$$

for some $u \in R_n^+$. Since all elementary matrices have determinant one, $\det(E') = \det(E) = 1$, i.e. $u = 1$ and $E' = I_k$. Hence E is a product of elementary matrices. □

Recall that for any Euclidean domain $\mathcal{R}$ and any $k \times k$ matrix A over $\mathcal{R}$, there are products of elementary matrices E_1 and E_2 over $\mathcal{R}$ such that

- $E_1 \cdot A \cdot E_2 = Diag(a_1, a_2, \ldots, a_k)$ is a diagonal matrix with $|a_{i+1}|$ a divisor of $|a_i|$.
- $Diag(a_1, a_2, \ldots, a_k)$ is unique up to multiplication by diagonal elementary matrices and is sometimes called the *Smith Normal Form* of A, denoted $SNF(A)$.

E_1 and E_2 may be constructed through a process derived from the Gauss algorithm for Euclidean domains. Recall from the section immediately after Proposition 7.2.1 the definition of the modules $M(A)$ for each $A \in \mathcal{M}_k(R_n)$.

Theorem 7.2.5. *For each non-zero $\alpha \in R_n = \mathcal{M}_1(R_n)$ and all $B \in \mathcal{M}_k(R_n)$, $M(\alpha)$ is stably equivalent to $M(B)$ and only if $M(B) \cong M(\alpha) \oplus \mathbb{Z}[\Gamma]^{k-1}$.*

Proof. Suppose that $M(\alpha)$ is stably equivalent to $M(B)$ for some non-zero $\alpha \in R_n$ and some $B \in \mathcal{M}_k(R_n)$. Then by $\mathbb{Z}[C_\infty]$ rank considerations we may assume that there exists an $m \in \mathbb{N}$ and an isomorphism $M(B) \oplus \mathbb{Z}[\Gamma]^m \cong M(\alpha) \oplus \mathbb{Z}[\Gamma]^{k+m-1}$, so that by Theorem 7.2.2 there exist matrices $C \in GR(k+m)$, $D \in GS(k+m)$ such that

$$C(B \oplus I_m)D = (\alpha \oplus I_{m+k-1}).$$

Define $B_{new} = (\det(C) \oplus I_{k-1}) \cdot B \cdot (\det(D) \oplus I_{k-1})$ and note that by hypothesis $\det(B_{new}) = \alpha$. Since R and S are commutative rings, the determinant homomorphisms are well defined and

$$(\det(C) \oplus I_{k-1}) \in GR(k)$$
$$(\det(D) \oplus I_{k-1}) \in GS(k).$$

By Theorem 7.2.2 $M(B) \cong M(B_{new})$, so that we may assume that $B = B_{new}$ and $\det(B) = \alpha$. We shall show that B must be reducible by the action of elementary matrices to $(\alpha \oplus I_{k-1})$, which would imply the result. Again, this is essentially a statement about matrices over R_n and we may assume that $n = p^e$ is a power of a prime. As in the proof of the lemma, let B_p denote the congruence class of B mod p. Since R_p is a Euclidean domain, we may reduce B_p by row and column operations to a matrix of the form $SNF(B_p) = Diag(b_1, \ldots b_k)$ and moreover we may assume that $SNF(B_p) = Diag(\alpha_p, 1, \ldots 1)$. Thus, over R_n with $n = p^e$, we may reduce B by the action of elementary matrices to a matrix of the form:

$$\begin{pmatrix} \alpha + c_{1,1}p & c_{1,2}p & \cdots & c_{1,k}p \\ c_{1,2}p & 1 + c_{2,2}p & \cdots & c_{2,k}p \\ \vdots & & & \vdots \\ c_{k,1}p & c_{k,2}p & \cdots & 1 + c_{k,k}p \end{pmatrix} \qquad c_{i,j} \in R_n.$$

Again, each $1 + c_{i,i}p$ is a unit and we may reduce B to a matrix of the form $Diag(\alpha + dp, 1, \ldots, 1)$ for some $d \in R_n$. This completes the proof, since we have insisted that $\det(B) = \alpha$ and hence B may be reduced by the action of elementary matrices to $(\alpha \oplus I_{k-1})$. □

Theorem 7.2.5 shows that a limited form of cancellation holds within the class of modules $M(A)$, of which $\pi_2(\mathcal{H})^* = M(t-1)$ is a member.

Theorem 7.2.6. *If M is any module such that $M \oplus \mathbb{Z}[\Gamma]^m \cong M(B)$ for some $B \in \mathcal{M}_{k+m}(R_n)$, then $M \cong M(A)$ for some $A \in \mathcal{M}_k(R_n)$.*

We remark that it is here that we use the work of Bass-Murthy. In particular we shall use the details in section 9 of [1].

Proof. It is sufficient to show that there exists an exact sequence

$$0 \to R^k \to M \to S^k \to 0$$

where as before $S = \mathbb{Z}[\Gamma]/ < N >$. It is clear that, in the extension $E(B)$, R^{k+m} is identified with the submodule of $M(B)$ on which the action of x is trivial. Let M_G denote the submodule of M on which x acts trivially, so that there is an exact sequence:

$$0 \to M_G \to M \to M/M_G \to 0.$$

Any isomorphism $M \oplus \mathbb{Z}[\Gamma]^m \cong M(B)$ induces isomorphisms $M_G \oplus R^m \cong R^{k+m}$, and $M/M_G \oplus S^m \cong S^{k+m}$. Thus M_G is stably free, and hence free, as an R-module. Since the action of x on M_G is trivial, we deduce that $M_G \cong R^k$ as a $\mathbb{Z}[\Gamma]$-module. It remains to show that $M/M_G \cong S^k$. Note that M/M_G has an S-module structure which is then stably free, and if we can show that M/M_G is free over S we may deduce the result.

We claim that all stably free S-modules are free. Let $S_\mathbb{Z}$ denote $\mathbb{Z}[G]/\langle N\rangle$, where as before $G = C_n$, so that there is a natural identification of S with $S_\mathbb{Z}[C_\infty]$. By [9.1 of Bass-Murthy] it is enough to prove that $S_\mathbb{Z}$ has finitely many non-projective maximal ideals. We shall deduce it from the fact that $\mathbb{Z}[G]$ has finitely many non-projective maximal ideals, which [Bass-Murthy] claims is "not difficult to verify", and we prove for the sake of completeness after this proof.

Let $i_1\colon \mathbb{Z}[G] \to S_\mathbb{Z}$ be the natural map onto the quotient, $i_2\colon \mathbb{Z}[G] \to \mathbb{Z}$ be augmentation, $j_1\colon S_\mathbb{Z} \to \mathbb{Z}_n$ be defined as $j_1(\alpha + (N)) = i_2(\alpha) \mod n$ and $j_2\colon \mathbb{Z} \to \mathbb{Z}_n$ be the natural map. Then the following is a commutative diagram of rings and surjective ring homomorphisms:

$$\begin{tikzcd} \mathbb{Z}[G] \arrow[r, "i_1"] \arrow[d, "i_2"] & S_{\mathbb{Z}} \arrow[d, "j_1"] \\ \mathbb{Z} \arrow[r, "j_2"] & \mathbb{Z}_n \end{tikzcd}$$

and $\mathbb{Z}[G]$ may be identified with the fibre product of $S_{\mathbb{Z}}$ and $\mathbb{Z}$ over $\mathbb{Z}_n$. Let I be a maximal ideal of $S_{\mathbb{Z}}$, and let I' denote the (necessarily projective) maximal ideal of $\mathbb{Z}$ given by $I' = j_2^{-1}(j_1(I))$. Then (I, I') is a maximal ideal of $\mathbb{Z}[G]$ which is isomorphic to the module $M(I, I', Id)$ constructed as in 4.5.1. Furthermore, by 4.5.1, (I,I') is projective if and only if I is projective. Lastly, if J is another maximal ideal of $S_{\mathbb{Z}}$, then $(J, J') = (I, I')$ if and only if $J = I$ and hence there is an injective map from the maximal ideals of $S_{\mathbb{Z}}$ to the maximal ideals of $\mathbb{Z}[G]$ which preserves projectivity. □

Proposition 7.2.7. *For $G = C_n$, the integral group ring $\mathbb{Z}[G]$ has finitely many non-projective maximal ideals.*

Proof. Let J be a maximal ideal of $\mathbb{Z}[G]$. Then $\mathbb{Z}[G]/J = \mathbb{F}$ is a field, which is necessarily finite with ground ring $\mathbb{Z}_p$ for some prime p. Define $J\colon \mathbb{Z}[G]$ to be the set of elements $r \in \mathbb{Z}$ such that $r\mathbb{Z}[G] \subseteq J$. Clearly, $J\colon \mathbb{Z}[G] = (p)$ and so by Proposition 7.1 of [25] J is projective unless p divides the order of G.

Thus, if J is not projective, we may assume that $J \cap \mathbb{Z} = (p)$ for some p dividing n, where we consider $\mathbb{Z}$ to be a subring of $\mathbb{Z}[G]$. Again, since $\mathbb{Z}[G]/J = \mathbb{F}$ is a field, the generator x for C_n has some minimal polynomial $\omega(x)$ over $\mathbb{F}$. Then J is necessarily the ideal generated by p and $\omega(x)$. Since the degree of $\omega(x)$ is less than or equal to n, there are finitely many $\omega(x)$ such that the ideals $(p, \omega(x))$ are distinct. This completes the proof. □

Corollary 7.2.8. *Any module stably isomorphic to $\pi_2(\mathcal{H})$ is necessarily isomorphic to $\pi_2(\mathcal{H}) \oplus \mathbb{Z}[\Gamma]^m$ for some m, where $\pi_2(\mathcal{H})$ is the module defined in the beginning of this section.*

Proof. Observe, from our description of the generators of $\pi_2(\mathcal{H})^*$, that there is an isomorphism $\pi_2(\mathcal{H})^* \cong M(t-1)$. The required result is then a clear consequence of 7.2.5 and 7.2.6. □

7.3 The D(2) Problem for $C_n \times C_\infty$

We bring together the work of the previous sections:

Proposition 7.3.1. *In order to prove the D(2) Problem for $C_n \times C_\infty$ it is sufficient to realise geometrically all algebraic 2-complexes of the form:*

$$0 \longrightarrow \pi_2(\mathcal{H}) \longrightarrow \mathbb{Z}[\Gamma]^a \longrightarrow \mathbb{Z}[\Gamma]^b \longrightarrow \mathbb{Z}[\Gamma]^c \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Proof. By Theorem 1.1.2 we may consider the D(2) problem to be equivalent to the Realisation problem for Γ. The result then follows from Proposition 3.2.2 and from Corollary 7.2.8. □

Theorem 7.3.2. *The D(2) Property holds for $C_n \times C_\infty$*

Proof. We have already shown that it is sufficient to realise all extensions of the form

$$0 \longrightarrow \pi_2(\mathcal{H}) \longrightarrow \mathbb{Z}[\Gamma]^a \longrightarrow \mathbb{Z}[\Gamma]^b \longrightarrow \mathbb{Z}[\Gamma]^c \longrightarrow \mathbb{Z} \longrightarrow 0$$

and by Proposition 3.2.2, $\mathrm{Ext}^3(\mathbb{Z}, \pi_2(\mathcal{H}))$ has the structure of a ring under which algebraic 2-complexes are necessarily units. By Proposition 7.1.5

$$\mathrm{Ext}^3(\mathbb{Z}, \pi_2(\mathcal{H})) \cong \mathbb{Z}_n$$

For each unit $w \in \mathbb{Z}_n^+$ we shall realise the class of w by a geometric complex.

An obvious change one may make to the standard presentation

$$\mathcal{H} = \langle x, t | x^n = 1, tx = xt \rangle$$

is to replace the generator x for C_n by the generator $y(v) = x^v$ where $1 \leq v \leq n-1$ is a natural number coprime to n. Denote each such presentation by

$$\mathcal{H}(v) = \langle\ y(v),\ t \mid y(v)^n = 1\ ,\ y(v)t = ty(v)\ \rangle$$

where $y(v) = x^v$.

We remark that the Cayley complex of $\mathcal{H}(v)$ is homotopy equivalent to the standard one, since as presentations they are identical. Indeed, one may view this complex as that arising from changing the isomorphism between the fundamental group of the original complex determined by $\mathcal{H}$ and the abstract group determined by the presentation. The aforementioned homotopy equivalence does not induce the identity on the fundamental groups of the spaces, and so may not imply that the resulting algebraic complexes are chain homotopic, although the distinction is meaningless from a geometric viewpoint.

The corresponding chain complex of the universal cover of $X_{\mathcal{H}(v)}$ is then

$$0 \longrightarrow \mathrm{Ker}(\partial_2^v) \longrightarrow \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_2^v} \mathbb{Z}[\Gamma]^2 \xrightarrow{\partial_1^v} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where $\partial_2^v = \begin{pmatrix} 1-t & N \\ x^v - 1 & 0 \end{pmatrix}$ and $\partial_1^v = (x^v - 1 \quad t-1)$

so that $\mathrm{Ker}(\partial_2^v) = \mathrm{Ker}(\partial_2) = \pi_2(\mathcal{H})$. Let w denote the inverse of v mod n. Set τ to be the element $1 + x^v + \ldots x^{v(w-1)}$, then $(1-x^v)\tau = (1-x)$ and the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(\mathcal{H}) \arrow[r, "j"] \arrow[d, "f_3"] & \mathbb{Z}[\Gamma]^2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathbb{Z}[\Gamma]^2 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \mathbb{Z}[\Gamma] \arrow[r, "\varepsilon"] \arrow[d, "Id"] & \mathbb{Z} \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & \pi_2(\mathcal{H}) \arrow[r, "j"] & \mathbb{Z}[\Gamma]^2 \arrow[r, "\partial_2^v"] & \mathbb{Z}[\Gamma]^2 \arrow[r, "\partial_1^v"] & \mathbb{Z}[\Gamma] \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where each f_i may be represented as multiplication on the left by:

$$f_3 = \tau, f_2 = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}, f_1 = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, since $\varepsilon(\tau) = w$, $C_*(\tilde{X}_{\mathcal{H}(v)})$ realises geometrically the congruence class of $w \in \mathrm{Ext}^3(\mathbb{Z}, \pi_2(\mathcal{H}))$. This completes the proof. □

Chapter 8

Generalised Swan modules

Suppose that $\Gamma = G \times H$ is a product of groups, with G a finite group of order n. Then the previous example of Swan modules generalises to an exact sequence:

$$0 \longrightarrow \mathbb{Z}[H] \xrightarrow{j} \mathbb{Z}[\Gamma] \xrightarrow{\sigma} \mathbb{Z}[\Gamma]/(N) \longrightarrow 0$$

where as before $N = \sum_{g \in G} g$ and $j(1) = N$. Through the natural ring homomorphism $\mathbb{Z}[\Gamma] \to \mathbb{Z}[\mathrm{H}]$ every $\mathbb{Z}[\Gamma]$-module has a well defined $\mathbb{Z}[\mathrm{H}]$-module structure, and if H is commutative then Theorem 28.5 of [11] shows that:

Proposition 8.1.3. *For each $\mathbb{Z}[\Gamma]$-module M such that the underlying $\mathbb{Z}[\mathrm{H}]$-module structure of M is finitely generated and projective:*

$$\mathrm{Ext}^1(M, \mathbb{Z}[\Gamma]) = 0$$

It is easy to see that if G is a finite group then $\mathbb{Z}[\Gamma]/(N)$ is finitely generated and free as a $\mathbb{Z}[\mathrm{H}]$-module. Thus, if G is finite and H is commutative then condition $(\star)$ holds.

By the change of rings formula given in [6] we see that

$$\mathrm{Ext}(\mathbb{Z}[\mathrm{H}]) = \mathrm{Ext}^1(\mathbb{Z}[\Gamma]/(N), \mathbb{Z}[\mathrm{H}]) \cong \mathbb{Z}_n[H]$$

and one may show directly that the composition operation represents the natural multiplication on $\mathbb{Z}_n[H]$. To give examples of the congruence classes, we state without proof that one may take as central module the submodule of $\mathbb{Z}[\Gamma]$ generated by the elements N and r, with $r \in \mathbb{Z}[H]$ uniquely defining a congruence class modulo n.

We consider in detail the case where $H = C_\infty$ is a free abelian group of rank one, and reprove a result of Bass-Murthy [1], which states that for finite abelian G: $K_0\mathbb{Z}[G \times C_\infty]$ is finitely generated if and only if the order of G is square free. The proof given in [1] is somewhat unwieldy, and our proof shows in a reasonably clean manner that the image of generalised Swan modules provides an infinitely generated subgroup of $K_0\mathbb{Z}[\Gamma]$ when the order of G is not square free.

Lemma 8.1.4. *$K_1\mathrm{Ext}(\mathbb{Z}[C_\infty])$ is finitely generated if and only if n is square free.*

Proof. We have already shown that $K_1\mathrm{Ext}(\mathbb{Z}[C_\infty]) \cong K_1\mathbb{Z}_n[C_\infty]$, and we shall use this representation. We use the notation R^+ to denote the units of any ring R. As in the proof of 7.2.4 it is enough to prove the lemma in the case where $n = p^e$ is a power of a prime.

Suppose that $n = p^e$ with $e > 1$ and let t be the generator of C_∞. As before, by the Binomial Theorem each $1 + pt^i$ is a unit of $\mathbb{Z}_n[C_\infty]$ of order less than or equal to p^e. Letting i range over $\mathbb{Z}$, such units form an infinitely generated subgroup of $\mathbb{Z}_n[C_\infty]^+$. Since no finitely generated abelian group contains an infinitely generated subgroup, we conclude that $\mathbb{Z}_n[C_\infty]^+$ is infinitely generated. Finally, since $\mathbb{Z}_n[C_\infty]$ is commutative there is a surjective determinant homomorphism

$$\det : K_1\mathbb{Z}_n[C_\infty] \to \mathbb{Z}_n[C_\infty]^+$$

and if $\mathbb{Z}_n[C_\infty]^+$ is not finitely generated then neither is $K_1\mathbb{Z}_n[C_\infty]$.

Alternatively, suppose that n is square free, so that we may assume n is prime. Then $\mathbb{Z}_n$ is a field and $\mathbb{Z}_n[C_\infty]$ is a Euclidean domain. In particular

$$K_1 Z_n[C_\infty] \cong \mathbb{Z}_n[C_\infty]^+.$$

Since $\mathbb{Z}_n$ is an integral domain, we see that

$$\mathbb{Z}_n[C_\infty]^+ \cong \mathbb{Z}_n^* \times \mathbb{Z}$$

where if t is the generator for C_∞, the image of $\mathbb{Z}$ in $\mathbb{Z}_n[C_\infty]^+$ is taken to be the elements $\{t^i : i \in \mathbb{Z}\}$. Thus $K_1\mathbb{Z}_n[C_\infty]$ is finitely generated. □

We remark that the above proof generalises easily to show that if H is a finitely generated free abelian group, then, in the notation used above, Ext($\mathbb{Z}$[H]) is infinitely generated if $|G|$ is not square free.

Theorem 8.1.5. *Let $\Gamma = G \times C_\infty$, where G is a finite abelian group. Then $K_0\mathbb{Z}[\Gamma]$ is finitely generated if and only if the order of G is square free.*

Proof. Set S to be the quotient ring and $\mathbb{Z}[\Gamma]$-module

$$S = \mathbb{Z}[\Gamma]/N$$

and as before we write

$$R = \mathbb{Z}[C_\infty].$$

By Theorem B there is a long exact sequence:

$$K_1R \oplus K_1S \longrightarrow K_1\mathrm{Ext}(R) \longrightarrow K_0\mathbb{Z}[\Gamma] \longrightarrow K_0R \oplus K_0S$$

Suppose that $|G|$ is square free. We shall use the portion of the Mayer-Vietoris sequence:

$$K_1\mathrm{Ext}(R) \longrightarrow K_0\mathbb{Z}[\Gamma] \longrightarrow K_0R \oplus K_0S.$$

Since G is abelian with square free order, G is a cyclic group; say $G = C_n$. Then

$$\mathbb{Z}[\mathrm{G}]/(N) \cong \mathbb{Z}[\xi]$$

where $\xi \in \mathbb{C}$ is a primitive n^{th} root of unity. Thus

$$S \cong \mathbb{Z}[\xi][C_\infty]$$

and it is a classical result that both K_0S and K_0R are finitely generated, see e.g. Corollary 4.12 in Chapter 5 of [17].

By the lemma, $K_1\mathrm{Ext}(R)$ is finitely generated. By the given portion of the Mayer-Vietoris sequence and by the classification of finitely generated abelian groups, it follows $K_0\mathbb{Z}[\Gamma]$ is finitely generated.

Suppose now that n is not square free. There are determinant homomorphisms and a commutative diagram

$$\begin{tikzcd}
K_1R \oplus K_1S \arrow[r] \arrow[d, "det"] & K_1\mathrm{Ext}(R) \arrow[r] \arrow[d, "det"] & K_0\mathbb{Z}[\Gamma] \arrow[d] \\
R^+ \oplus S^+ \arrow[r] & \mathrm{Ext}(R)^+ \arrow[r, two heads] & Coker
\end{tikzcd}$$

with all vertical homomorphisms surjective. If *Coker* can be shown to be infinitely generated, then we may deduce the result. By the Lemma, $\mathrm{Ext}(R)^+$ is infinitely generated, so *Coker* is necessarily infinitely generated if both R^+ and S^+ are finitely generated.

Since $\mathbb{Z}$ is an integral domain, we see that

$$R^+ \cong \mathbb{Z}^+ \times \mathbb{Z}$$

which is clearly finitely generated.

By Wedderburn's Theorem, since G is abelian we have $\mathbb{C}[G] \cong \mathbb{C}^n$ and we have an isomorphism

$$\varphi : \mathbb{C} \otimes \mathbb{Z}[\mathrm{G}]/(N) \cong \mathbb{C}^{n-1}$$

which clearly extends to an isomorphism

$$\varphi : \mathbb{C} \otimes S \cong \mathbb{C}[C_\infty]^{n-1}.$$

Let $u \in S^+$ be a unit, and consider the image of u in $\mathbb{C}[C_\infty]^{n-1}$, this will be of the form

$$\varphi(u) = (u_1 t^{i_1}, \ldots, u_{n-1} t^{i_{n-1}})$$

and since there is a natural (split) ring homomorphism $S \to \mathbb{Z}[\mathrm{G}]/(N)$ which sends t to 1, we have

$$\varphi^{-1}(u_1, \ldots, u_{n-1}) \in \mathbb{Z}[\mathrm{G}]/(N)^+.$$

Thus there is an embedding

$$S^+/(\mathbb{Z}[\mathrm{G}]/(N)^+) \hookrightarrow \mathbb{Z}^{n-1}.$$

We have already seen that $K_1(\mathbb{Z}[\mathrm{G}]/(N))$ is finitely generated, hence $\mathbb{Z}[\mathrm{G}]/(N)^+$ is finitely generated and we may deduce the result. $\square$

Appendix A

Stably free modules and matrices

The ensuing description of stably free modules is similar to the description of projective modules through idempotent matrices, given for example in Rosenberg [23] Chapter 1. We shall assume that every finitely generated free module has a well defined rank.

Suppose that Λ is a ring and SF is a stably free Λ-module. Then there are some $a, b \in \mathbb{N}$ and an exact sequence:

$$0 \longrightarrow \Lambda^b \xrightarrow{\alpha} \Lambda^a \xrightarrow{\chi} SF \longrightarrow 0.$$

We may represent α as a $(b \times a)$ matrix and identify Λ^b with the submodule of Λ^a generated by the columns of α. We say that the stably free module is *represented* by α.

Proposition A.1.1. *SF is free if and only if $(a - b)$ columns may be adjoined to the matrix α in order to produce an invertible matrix.*

Proof. If $(a-b)$ columns may be adjoined to the matrix α in order to produce an invertible matrix, then the resulting columns are linearly independent. In this case, SF will be isomorphic to the submodule of Λ^a generated by the $(a - b)$ columns, and hence free.

If SF is free then we may assume that the free rank of SF is $(a - b)$ and there is a splitting homomorphism $SF \to \Lambda^a$ and the image of the free generators of SF provide the required $(a - b)$ columns. □

Since there is an isomorphism:

$$\varphi : SF \oplus \Lambda^a \cong \Lambda^b,$$

there is an exact sequence:

$$0 \longrightarrow \Lambda^b \xrightarrow{A\oplus 0} \Lambda^a \oplus \Lambda^b \xrightarrow{\sigma} \Lambda^a \longrightarrow 0,$$

where $\sigma = \varphi \circ (\chi \oplus Id)$. Then by the Proposition we see:

Corollary A.1.2. *A $(b \times a)$ matrix α represents a stably free module if and only if there is a matrix of the form:*

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

which may be completed to an invertible matrix by adjoining columns.

Bibliography

[1] **Hyman Bass and M. Pavaman Murthy** ; *Grothendieck Groups and Picard groups of Abelian Group Gings* : Ann. of Math. 86 (1967) 16-73

[2] **F. Rudolf Beyl and Nancy Waller** ; *A stably free nonfree module and its relevance for homotopy classification, case Q_{28}* : Algebr. Geom. Topol. 5 (2005), 899-910 (electronic)

[3] **W. Browning** ; *Homotopy types of certain finite C.W. complexes with finite fundamental group* : PhD Thesis, Cornell University, (1978)

[4] **C.W. Curtis and I. Reiner** ; *Methods of Representation Theory : Volume I* : Wiley-Interscience, (1981)

[5] **C.W. Curtis and I. Reiner** ; *Methods of Representation Theory : Volume II* : Wiley-Interscience, (1987)

[6] **H. Cartan and S. Eilenberg** ; *Homological Algebra* : Princeton Mathematical Series vol. 19 (1956)

[7] **P.M. Cohn** ; *Some remarks on the invariant basis property* : Topology, 5 (1966) 215-218

[8] **Tim Edwards** ; *Generalised Swan Modules and the D(2) Problem* : Algebr. Geom. Topol. 6 (2006), 71-89 (electronic)

[9] **R.H. Fox** ; *Free differential calculus. I. Derivation in the free group ring.* : Ann. of Math. (2) 57, (1953). 547-560

[10] **J.J.A.M. Humphreys** ; *Algebraic properties of semi-simple lattices and related groups* : PhD Thesis, University of London, (2005)

[11] **F.E.A Johnson** ; *Stable modules and the D(2) Problem* : LMS Lecture Notes in Mathematics, vol. 301. CUP (2003)

[12] **F.E.A Johnson** ; *Explicit Homotopy Equivalences in Dimension Two* : Math. Proc. Camb. Phil. Soc. 133 (2002), 411-430

[13] **F.E.A Johnson** ; *Minimal 2-complexes and the D(2)-Problem* Proc. Amer. Math. Soc. 132 (2004), 579-586.

[14] **F.E.A Johnson** ; *Stable modules and the structure of Poincaré 3-complexes* : Contemp. Math. 258 (2000), 227-247.

[15] **F.E.A Johnson** ; *Personal correspondence* : (2006)

[16] **Cynthia Hogg-Angeloni, Wolfgang Metzler & Allan J. Sieradski** ; *Two-dimensional Homotopy and Combinatorial Group Theory* : LMS Lecture Notes in Matheamtics vol. 197 (1993)

[17] **T.Y. Lam** ; *Serre's Conjecture* : Springer-Verlag, (1978)

[18] **M. P. Latiolais** ; *The simple homotopy type of finite two-complexes with finite abelian fundamental group* : Trans. Amer. Math. Soc. 293 (1986). 655-662

[19] **S. Mac Lane** ; *Homology* : Springer-Verlag (1963)

[20] **Wajid Mannan** ; *Algebraic 2-complexes and five dimensional manifolds over finite groups* : In preparation.

[21] **J. Milnor** ; *Introduction to Algebraic K - Theory* : Ann. of Math. Studies, vol. 72 : Princeton Univ. Press (1971)

[22] **I. Reiner, S. Ullum** ; *A Mayer-Vietoris sequence for classgroups* : J. of Algebra, 31 (1974) 305-342

[23] **J. Rosenberg** : *Algebraic K-Theory and Its Applications* : Springer-Verlag (1994)

[24] **R. Oliver** ; *Whitehead Groups of Finite Groups* : LMS Lecture Notes in Mathematics vol. 132 (1988)

[25] **R.G. Swan** ; *Induced Representations and Projective Modules* : Ann. of Math. 71 (1960), 552-578.

[26] **R.G. Swan** ; *Periodic resolutions for finite groups* : Ann. of Math. 72 (1960), 267-291.

[27] **R.G. Swan** ; *Projective modules over binary polyhedral groups* : J. reine und angew. Math. 342 (1983) 66-172

[28] **M. Taylor** ; *Classgroups of Group Rings* : LMS Lecture Notes in Mathematics, vol. 91. CUP (1984)

[29] **C.T.C. Wall** ; *Finiteness Conditions for CW complexes* : Ann. of Math. 81 (1965), 56-69

[30] **C.T.C. Wall** ; *Finiteness conditions for CW complexes. II.* Proc. Roy. Soc. Ser. A 295 (1966) 129-139.

[31] **C.T.C. Wall** ; *Poincaré complexes I* : Ann. of Math. 86 (1967), 213-245

[32] **J.H.C. Whitehead** ; *Elements of Homotopy Theory* : Springer-Verlag (1978)