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Analysis of first order systems of partial differential equations

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Abstract. The paper deals with a formally self-adjoint first order linear differential operator acting on m-columns of complex-valued half-densities over an n-manifold without boundary. We study the distribution of eigenvalues in the elliptic setting and the propagator in the hyperbolic setting, deriving two-term asymptotic formulae for both. We then turn our attention to the special case of a two by two operator in dimension four. We show that the geometric concepts of Lorentzian metric, Pauli matrices, spinor field, connection coefficients for spinor fields, electromagnetic covector potential, Dirac equation and Dirac action arise naturally in the process of our analysis.

1. The playing field

Let L be a formally self-adjoint first order linear differential operator acting on m-columns $v = \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix}^T$ of complex-valued half-densities over a connected n-dimensional manifold M without boundary. Throughout this paper we assume that $m, n \geq 2$. The coefficients of the operator L are assumed to be infinitely smooth.

In local coordinates $x = (x^1, \dots, x^n)$ our operator reads

$$L = P^\alpha(x) \frac{\partial}{\partial x^\alpha} + Q(x), \tag{1.1}$$

where $P^\alpha(x)$ and $Q(x)$ are some $m \times m$ matrix-functions and summation is carried out over $\alpha = 1, \dots, n$. The full symbol of the operator L is the matrix-function

$$L(x, p) := iP^\alpha(x)\, p_\alpha + Q(x), \tag{1.2}$$

where $p = (p_1, \dots, p_n)$ is the dual variable (momentum). Note that the tradition in microlocal analysis is to denote momentum by ξ. We choose to denote it by p instead because in Sections 8 and 9 we will need the letter ξ for the spinor.

The problem with the full symbol is that it is not invariant under changes of local coordinates. In order to overcome this problem we decompose the full symbol into components homogeneous in momentum, $L(x, p) = L_1(x, p) + L_0(x)$,

$$L_1(x, p) := iP^\alpha(x)\, p_\alpha, \qquad L_0(x) := Q(x), \tag{1.3}$$

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and define the principal and subprincipal symbols as

$$L_{\mathrm{prin}}(x,p) := L_1(x,p), \tag{1.4}$$

$$L_{\mathrm{sub}}(x) := L_0(x) + \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x), \tag{1.5}$$

where the subscripts indicate partial derivatives. It is known that L_{prin} and L_{sub} are invariantly defined matrix-functions on T^*M and M respectively, see subsection 2.1.3 in [**11**] for details. As we assumed our operator L to be formally self-adjoint, the matrix-functions L_{prin} and L_{sub} are Hermitian.

Examination of formulae (1.1)–(1.5) shows that $L_{\mathrm{prin}}(x,p)$ and $L_{\mathrm{sub}}(x)$ uniquely determine the first order differential operator L. Thus, the notions of principal symbol and subprincipal symbol provide an invariant analytic way of describing a first order differential operator.

We say that a formally self-adjoint first order differential operator L is *elliptic* if

$$\det L_{\mathrm{prin}}(x,p) \ne 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}, \tag{1.6}$$

and *non-degenerate* if

$$L_{\mathrm{prin}}(x,p) \ne 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}. \tag{1.7}$$

The ellipticity condition (1.6) is a standard condition in the spectral theory of differential operators, see, for example, [**2**]. Our non-degeneracy condition (1.7) is less restrictive, and we will see later, in Section 4, that in the special case $m = 2$ and $n = 4$ this condition describes a hyperbolic operator.

2. Distribution of eigenvalues and the propagator

In this section we assume that the manifold M is compact (and without boundary) and that the operator L is elliptic, see (1.6).

Remark 2.1. Ellipticity and the fact that dimension n is greater than or equal to two imply that m, the number of equations, is even. Indeed, let us fix an arbitrary point $x \in M$ and consider $L_{\mathrm{prin}}(x,p)$ as a function of momentum $p \in T^*_xM$. As throughout this paper the operator L is assumed to be formally self-adjoint, the matrix-function $L_{\mathrm{prin}}(x,p)$ is Hermitian, and, hence, $\det L_{\mathrm{prin}}(x,p)$ is real. For $n \ge 2$ the set $T^*_xM \setminus \{0\}$ is connected, so the ellipticity condition (1.6) implies that the polynomial $\det L_{\mathrm{prin}}(x,p)$ preserves sign on $T^*_xM \setminus \{0\}$. But our $m \times m$ matrix-function $L_{\mathrm{prin}}(x,p)$ is linear in p, so $\det L_{\mathrm{prin}}(x,-p) = (-1)^m \det L_{\mathrm{prin}}(x,p)$, therefore the sign of $\det L_{\mathrm{prin}}(x,p)$ can only be preserved if m is even.

Let $h^{(j)}(x,p)$ be the eigenvalues of the principal symbol. We assume that these are simple for all $(x,p) \in T^*M \setminus \{0\}$. We enumerate the eigenvalues of the principal symbol $h^{(j)}(x,p)$ in increasing order, using a negative index $j = -m/2, \dots, -1$ for negative $h^{(j)}(x,p)$ and a positive index $j = 1, \dots, m/2$ for positive $h^{(j)}(x,p)$.

It is known that our differential operator L has a discreet spectrum accumulating to $+\infty$ and to $-\infty$. Let λ_k and $v_k = \begin{pmatrix} v_{k1}(x) & \dots & v_{km}(x) \end{pmatrix}^T$ be the eigenvalues and eigenfunctions of the operator L; the particular enumeration of these eigenvalues (accounting for multiplicities) is irrelevant for our purposes.

We will be studying the following two objects.

Object 1. Our first object of study is the *propagator*, which is the one-parameter family of operators defined as

$$\begin{aligned} U(x^{n+1}) &:= e^{-ix^{n+1}L} \\ &= \sum_k e^{-ix^{n+1}\lambda_k}\, v_k(x^1,\dots,x^n) \int_M [v_k(y^1,\dots,y^n)]^*(\,\cdot\,)\, dy^1 \dots dy^n\,, \end{aligned} \tag{2.1}$$

where $x^{n+1} \in \mathbb{R}$ is an additional 'time' coordinate. The propagator provides a solution to the Cauchy problem

$$w|_{x^{n+1}=0} = v \tag{2.2}$$

for the hyperbolic system

$$(-i\partial/\partial x^{n+1} + L)w = 0\,. \tag{2.3}$$

Namely, it is easy to see that if the column of half-densities $v = v(x^1,\dots,x^n)$ is infinitely smooth, then, setting $w := U(x^{n+1})\,v$, we get a 'time-dependent' column of half-densities $w(x^1,\dots,x^n,x^{n+1})$ which is also infinitely smooth and which satisfies the equation (2.3) and the initial condition (2.2). The use of the letter "U" for the propagator is motivated by the fact that for each x^{n+1} the operator $U(x^{n+1})$ is unitary.

Note that the operator $-i\partial/\partial x^{n+1} + L$ appearing in the LHS of formula (2.3) is a formally self-adjoint $m \times m$ first order differential operator on the $(n+1)$-dimensional manifold $M \times \mathbb{R}$. Moreover, it is easy to see that this 'extended' operator $-i\partial/\partial x^{n+1} + L$ automatically satisfies the non-degeneracy condition from Section 1.

Object 2. Our second object of study is the *counting function*

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1\,. \tag{2.4}$$

In other words, $N(\lambda)$ is the number of eigenvalues λ_k between zero and a positive λ.

Here it is natural to ask the question: why, in defining the counting function (2.4), did we choose to count all *positive* eigenvalues up to a given positive λ rather than all *negative* eigenvalues up to a given negative λ? There is no particular reason. One case reduces to the other by the change of operator $L \mapsto -L$. This issue is known as *spectral asymmetry* and is discussed in [**1**], as well as in Section 10 of [**2**] and in [**4**].

Our objectives are as follows.

Objective 1. We aim to construct the propagator (2.1) explicitly in terms of oscillatory integrals, modulo an integral operator with an infinitely smooth, in the variables $x^1,\dots,x^n,x^{n+1},y^1,\dots,y^n$, integral kernel.

Objective 2. We aim to derive a two-term asymptotic expansion for the counting function (2.4)

$$N(\lambda) = a\lambda^n + b\lambda^{n-1} + o(\lambda^{n-1}) \tag{2.5}$$

as $\lambda \to +\infty$, where a and b are some real constants. More specifically, our objective is to write down explicit formulae for the asymptotic coefficients a and b.

Here one has to have in mind that the two-term asymptotic expansion (2.5) holds only under appropriate assumptions on periodic trajectories, see Theorem 8.4 from [**2**] for details. In order to avoid dealing with the issue of periodic trajectories, in this paper we understand the asymptotic expansion (2.5) in a regularised fashion. One way of regularising the asymptotic formula (2.5) is to take a convolution with a function from Schwartz space $\mathcal{S}(\mathbb{R})$; see Theorem 7.2 in [**2**] for details. Alternatively, one can look at the eta function $\eta(s) := \sum |\lambda_k|^{-s} \operatorname{sign} \lambda_k$, where summation is carried out over all nonzero eigenvalues λ_k and $s \in \mathbb{C}$ is the independent variable. The results of Section 10 of [**2**] imply that the series converges absolutely for $\operatorname{Re} s > n-1$ and defines a holomorphic function in this half-plane. Moreover, it is known [**1**] that the eta function extends meromorphically to the whole s-plane with simple poles. It is easy to see that the residue of the eta function at the point $s = n-1$ is $2(n-1)b$, where b is the coefficient from (2.5).

It is well known that the above two objectives are closely related: if one achieves Objective 1, then Objective 2 follows via a Fourier transform in the variable x^{n+1}, see Sections 6 and 7 in [**2**].

We are now in a position to state our results.

Result 1. We construct the propagator as a sum of m oscillatory integrals (Fourier integral operators)

$$U(x^{n+1}) \overset{\operatorname{mod} C^\infty}{=} \sum_j U^{(j)}(x^{n+1}) ,$$

where the phase function of each oscillatory integral $U^{(j)}(x^{n+1})$ is associated with the corresponding Hamiltonian $h^{(j)}(x^1, \dots, x^n, p_1, \dots, p_n)$ and summation is performed over nonzero integers j from $-m/2$ to $+m/2$. The notion of a phase function associated with a Hamiltonian is defined in Section 2 of [**2**] and Section 2.4 of [**11**].

We will now write down explicitly the principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$. The notion of a principal symbol of an oscillatory integral is defined in accordance with Definition 2.7.12 from [**11**]. The principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ is a complex-valued $m \times m$ matrix-function on

$$M \times \mathbb{R} \times (T^*M \setminus \{0\}).$$

We denote the arguments of this principal symbol by $x^1, \dots, x^n$ (local coordinates on M), x^{n+1} ('time' coordinate on $\mathbb{R}$), $y^1, \dots, y^n$ (local coordinates on M) and $q_1, \dots, q_n$ (variable dual to $y^1, \dots, y^n$).

Further on in this section we use x, y, p and q as shorthand for $x^1, \dots, x^n$, $y^1, \dots, y^n$, $p_1, \dots, p_n$ and $q_1, \dots, q_n$ respectively. The additional 'time' coordinate x^{n+1} will always be written separately.

In order to write down the principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ we need to introduce some auxiliary objects first.

Curly brackets will denote the Poisson bracket on matrix-functions $\{P, R\} := P_{x^\alpha} R_{p_\alpha} - P_{p_\alpha} R_{x^\alpha}$ and its further generalisation

$$\{F, G, H\} := F_{x^\alpha} G H_{p_\alpha} - F_{p_\alpha} G H_{x^\alpha} , \tag{2.6}$$

where the subscripts x^α and p_α indicate partial derivatives and the repeated index α indicates summation over $\alpha = 1, \dots, n$.

Let $v^{(j)}(x,p)$ be the normalised eigenvector of the principal symbol $L_{\mathrm{prin}}(x,p)$ corresponding to the eigenvalue $h^{(j)}(x,p)$. We define the scalar function $f^{(j)} : T^*M \setminus \{0\} \to \mathbb{R}$ in accordance with the formula

$$f^{(j)} := [v^{(j)}]^* L_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, L_{\mathrm{prin}} - h^{(j)}, v^{(j)}\} - i[v^{(j)}]^*\{v^{(j)}, h^{(j)}\}.$$

By $(x^{(j)}(x^{n+1};y,q), p^{(j)}(x^{n+1};y,q))$ we denote the Hamiltonian trajectory originating from the point (y,q), i.e. solution of the system of ordinary differential equations (the dot denotes differentiation in x^{n+1})

$$\dot{x}^{(j)} = h_p^{(j)}(x^{(j)}, p^{(j)}), \qquad \dot{p}^{(j)} = -h_x^{(j)}(x^{(j)}, p^{(j)})$$

subject to the initial condition $(x^{(j)}, p^{(j)})\big|_{x^{n+1}=0} = (y,q)$.

The formula for the principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ is known [**10, 9, 2**] and reads as follows:

$$[v^{(j)}(x^{(j)}(x^{n+1};y,q), p^{(j)}(x^{n+1};y,q))]\,[v^{(j)}(y,q)]^* \\ \times \exp\left(-i\int_0^{x^{n+1}} f^{(j)}(x^{(j)}(\tau;y,q), p^{(j)}(\tau;y,q))\,d\tau\right). \tag{2.7}$$

This principal symbol is positively homogeneous in momentum q of degree zero.

Let us now examine the lower order terms of the symbol of the oscillatory integral $U^{(j)}(x^{n+1})$. The algorithm described in Section 2 of [**2**] provides a recursive procedure for the calculation of all lower order terms, of any degree of homogeneity in momentum q. However, there are two issues here. Firstly, calculations become very complicated. Secondly, describing these lower order terms in an invariant way is problematic because, as far as the authors are aware, the concept of subprincipal symbol has never been defined for time-dependent oscillatory integrals (Fourier integral operators). We thank Yuri Safarov for drawing our attention to the latter issue.

We overcome the problem of invariant description of lower order terms of the symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ by restricting our analysis to $U^{(j)}(0)$. It turns out that knowing the properties of the lower order terms of the symbol of $U^{(j)}(0)$ is sufficient for the derivation of the two-term asymptotic expansion (2.5). And $U^{(j)}(0)$ is a pseudodifferential operator, so one can use here the standard notion of subprincipal symbol of a pseudodifferential operator, see subsection 2.1.3 in [**11**] for definition.

The following result was established recently in [**2**].

Theorem 2.2.

$$\operatorname{tr}[U^{(j)}(0)]_{\mathrm{sub}} = -i\{[v^{(j)}]^*, v^{(j)}\}. \tag{2.8}$$

It is interesting that the RHS of formula (2.8) admits a geometric interpretation: it can be interpreted as the scalar curvature of a U(1) connection on $T^*M \setminus \{0\}$, see Section 5 of [**2**] for details. This connection is to do with gauge transformations of the normalised eigenvector $v^{(j)}(x,p)$ of the principal symbol $L_{\mathrm{prin}}(x,p)$ corresponding to the eigenvalue $h^{(j)}(x,p)$. Namely, observe that if $v^{(j)}(x,p)$ is an eigenvector and $\phi^{(j)}(x,p)$ is an arbitrary real-valued function, then $e^{i\phi^{(j)}(x,p)}v^{(j)}(x,p)$ is also an eigenvector, and careful analysis of this gauge transformation leads to the appearance of a curvature term.

Result 2. The formula for the first coefficient of the asymptotic expansion (2.5) reads [**7, 8**]

$$a = (2\pi)^{-n} \sum_{j=1}^{m/2} \int\limits_{h^{(j)}(x,p)<1} dx\, dp\,, \tag{2.9}$$

where $dx = dx^1 \dots dx^n$ and $dp = dp_1 \dots dp_n$.

The formula for the second coefficient of the asymptotic expansion (2.5) was established recently in [**2**].

Theorem 2.3.

$$b = -n(2\pi)^{-n} \sum_{j=1}^{m/2} \int\limits_{h^{(j)}(x,p)<1} \Big([v^{(j)}]^* L_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, L_{\mathrm{prin}} - h^{(j)}, v^{(j)}\} + \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big)(x,p)\, dx\, dp\,. \tag{2.10}$$

Note that Theorem 2.2 plays an important role in the proof of Theorem 2.3. The information contained in formula (2.7) is, on its own, insufficient for the derivation of the formula for the coefficient b.

Recall also that according to Remark 2.1 the number m is even, so the upper limit of summation in formulae (2.9) and (2.10) is a natural number.

A bibliographic review of the subject is provided in Section 11 of [**2**].

3. Two by two operators are special

Suppose that we are dealing with a 2×2 operator, i.e. suppose that

$$m = 2. \tag{3.1}$$

Observe that in this case the determinant of the principal symbol is a quadratic form in the dual variable (momentum) p:

$$\det L_{\mathrm{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta\,. \tag{3.2}$$

We interpret the real coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, \dots, n$, appearing in formula (3.2) as components of a (contravariant) metric tensor. Thus, 2×2 formally self-adjoint first order linear partial differential operators are special in that the concept of a metric is encoded within such operators. This opens the way to the geometric interpretation of analytic results. So further on in this paper we work under the assumption (3.1).

4. Dimension four is special

It is easy to see that if $n \ge 5$, then our metric defined in accordance with formula (3.2) has the property $\det g^{\alpha\beta}(x) = 0$, $\forall x \in M$. Hence,

$$n = 4 \tag{4.1}$$

is the highest dimension in which it makes sense to define the metric as we do. So further on in this section and the next two we work under the assumption (4.1).

It is natural to ask the question: what is the signature of our metric? The answer is given by the following lemma.

LEMMA 4.1. *Suppose that we have (3.1) and (4.1) and suppose that our operator L satisfies the non-degeneracy condition (1.7). Then our metric tensor defined in accordance with formula (3.2) is Lorentzian, i.e. it has three positive eigenvalues and one negative eigenvalue.*

Lemma 4.1 is proved by a straightforward calculation, see Section 2 in [**6**].

Lemma 4.1 tells us that under the conditions (3.1), (4.1) and (1.7) our operator L is hyperbolic. This indicates that one could, in principle, perform a comprehensive microlocal analysis of the corresponding propagator and, moreover, do this in a relativistically invariant fashion. Here the relativistic propagator is, loosely speaking, the Fourier integral operator mapping a 2-column of half-densities v to a 2-column of half-densities w which is a solution of the hyperbolic system $Lw = v$. One would expect the construction of this relativistic propagator to proceed along the lines of the construction sketched out in Section 2 and, in more detail, in [**2**], only without reference to a particular choice of time coordinate.

We are currently a long way from developing relativistic microlocal techniques. However, we are able to perform a gauge-theoretic analysis of 2×2 operators in dimension four. The results of this gauge-theoretic analysis are presented in the next two sections and these results reveal additional geometric structures encoded within the operator L. We hope that the identification of these geometric structures will eventually help us develop relativistic microlocal techniques.

5. Gauge-theoretic analysis in 4D

Take an arbitrary matrix-function

$$Q : M \to \mathrm{GL}(2, \mathbb{C}) \tag{5.1}$$

and consider the transformation of our 2×2 differential operator

$$L \mapsto Q^* L Q. \tag{5.2}$$

We interpret (5.2) as a gauge transformation. Note that in spectral theory it is customary to apply unitary transformations rather than general linear transformations. However, in view of Lemma 4.1 we are working in a relativistic (hyperbolic) setting without a specified time coordinate and in this setting restricting our analysis to unitary transformations would be unnatural.

The transformation (5.2) of the differential operator L induces the following transformations of its principal and subprincipal symbols:

$$L_{\mathrm{prin}} \mapsto Q^* L_{\mathrm{prin}} Q, \tag{5.3}$$

$$L_{\mathrm{sub}} \mapsto Q^* L_{\mathrm{sub}} Q + \frac{i}{2}\left(Q^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} Q - Q^*(L_{\mathrm{prin}})_{p_\alpha} Q_{x^\alpha}\right). \tag{5.4}$$

Comparing formulae (5.3) and (5.4) we see that, unlike the principal symbol, the subprincipal symbol does not transform in a covariant fashion due to the appearance of terms with the gradient of the matrix-function $Q(x)$. In order to identify the sources of this non-covariance we observe that any matrix-function (5.1) can be written as a product of three terms: a complex matrix-function of determinant one, a positive scalar function and a complex scalar function of modulus one. Hence, we examine the three gauge-theoretic actions separately.

Take an arbitrary scalar function $\psi : M \to \mathbb{R}$ and consider the transformation of our differential operator

$$L \mapsto e^{\psi} L e^{\psi}. \tag{5.5}$$

The transformation (5.5) is a special case of the transformation (5.2) with $Q = e^{\psi} I$, where I is the 2×2 identity matrix. Substituting this Q into formula (5.4), we get

$$L_{\mathrm{sub}} \mapsto e^{2\psi} L_{\mathrm{sub}}, \tag{5.6}$$

so the subprincipal symbol transforms in a covariant fashion.

Now take an arbitrary scalar function $\phi : M \to \mathbb{R}$ and consider the transformation of our differential operator

$$L \mapsto e^{-i\phi} L e^{i\phi}. \tag{5.7}$$

The transformation (5.7) is a special case of the transformation (5.2) with $Q = e^{i\phi} I$. Substituting this Q into formula (5.4), we get

$$L_{\mathrm{sub}}(x) \mapsto L_{\mathrm{sub}}(x) + L_{\mathrm{prin}}(x, (\operatorname{grad} \phi)(x)), \tag{5.8}$$

so the subprincipal symbol does not transform in a covariant fashion. We do not (and can not) take any action with regards to the non-covariance of (5.8).

Finally, take an arbitrary matrix-function $R : M \to \mathrm{SL}(2, \mathbb{C})$ and consider the transformation of our differential operator

$$L \mapsto R^* L R. \tag{5.9}$$

Of course, the transformation (5.9) is a special case of the transformation (5.2): we are looking at the case when $\det Q(x) = 1$. It turns out that it is possible to overcome the resulting non-covariance in (5.4) by introducing the *covariant subprincipal symbol* $L_{\mathrm{csub}}(x)$ in accordance with formula

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} + \frac{i}{16}\, g_{\alpha\beta} \{L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}, \tag{5.10}$$

where subscripts p_α, p_β indicate partial derivatives, curly brackets denote the generalised Poisson bracket on matrix-functions (2.6) and adj stands for the operator of matrix adjugation

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} P \tag{5.11}$$

from elementary linear algebra.

Lemma 5.1. *The transformation (5.9) of the differential operator induces the transformation*

$$L_{\mathrm{csub}} \mapsto R^* L_{\mathrm{csub}} R \tag{5.12}$$

of its covariant subprincipal symbol.

The proof of Lemma 5.1 is given in [**6**].

Formula (5.12) tells us that when working with 2×2 operators in dimension four it makes sense to use the covariant subprincipal symbol rather than the standard subprincipal symbol because the covariant subprincipal is 'more invariant'.

It is easy to check that, like the standard subprincipal symbol, the covariant subprincipal symbol of a formally self-adjoint non-degenerate first order differential operator is an Hermitian matrix-function on the base manifold M. Also, it is easy to see that $L_{\mathrm{prin}}(x, p)$ and $L_{\mathrm{csub}}(x)$ uniquely determine our operator L.

Substituting (1.5) into (5.10) we get

$$L_{\mathrm{csub}} = L_0 + \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha} + \frac{i}{16}\,g_{\alpha\beta}\{L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}. \tag{5.13}$$

Comparing formulae (1.5) and (5.13) we see that the standard subprincipal symbol and covariant subprincipal symbol have the same structure, only the covariant subprincipal symbol has a second correction term designed to 'take care' of special linear transformations.

Examination of formulae (5.10), (5.6) and (5.8) shows that the transformations (5.5) and (5.7) of the differential operator induce the transformations

$$L_{\mathrm{csub}} \mapsto e^{2\psi} L_{\mathrm{csub}}\,, \tag{5.14}$$

$$L_{\mathrm{csub}}(x) \mapsto L_{\mathrm{csub}}(x) + L_{\mathrm{prin}}(x, (\operatorname{grad}\phi)(x)) \tag{5.15}$$

of its covariant subprincipal symbol. Thus, the switch from standard subprincipal symbol to covariant subprincipal symbol 'does not spoil' the behaviour under the transformations (5.5) and (5.7).

The non-degeneracy condition (1.7) implies that for each $x \in M$ the matrices

$$\sigma^\alpha(x) := (L_{\mathrm{prin}})_{p_\alpha}(x), \qquad \alpha = 1,2,3,4, \tag{5.16}$$

form a basis in the real vector space of 2×2 Hermitian matrices. Decomposing the covariant subprincipal symbol $L_{\mathrm{csub}}(x)$ with respect to this basis, we get $L_{\mathrm{csub}}(x) = \sigma^\alpha(x)\,A_\alpha(x)$ with some real coefficients $A_\alpha(x)$, $\alpha = 1,2,3,4$. The latter formula can be rewritten in more compact form as

$$L_{\mathrm{csub}}(x) = L_{\mathrm{prin}}(x, A(x)), \tag{5.17}$$

where A is a covector field with components $A_\alpha(x)$, $\alpha = 1,2,3,4$. Formula (5.17) tells us that the covariant subprincipal symbol L_{csub} is equivalent to a real-valued covector field A, the electromagnetic covector potential.

Examination of formulae (5.12) and (5.14)–(5.17) shows that our electromagnetic covector potential A is invariant under the transformations (5.9) and (5.5) of our differential operator L, whereas the transformation (5.7) induces the transformation $A \mapsto A + \operatorname{grad}\phi$.

The geometric and theoretical physics interpretations of the transformations (5.5), (5.7) and (5.9) are discussed in detail in [**6**].

6. A non-geometric representation of the massive Dirac equation in 4D

As explained in the previous section, in dimension four our formally self-adjoint non-degenerate 2×2 first order differential operator L is completely determined by its principal symbol $L_{\mathrm{prin}}(x,p)$ and covariant subprincipal symbol $L_{\mathrm{csub}}(x)$. Namely, in local coordinates the formula for the differential operator L reads

$$\begin{aligned} L = -i[(L_{\mathrm{prin}})_{p_\alpha}(x)]\frac{\partial}{\partial x^\alpha} \\ - \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x) - \frac{i}{16}(g_{\alpha\beta}\{L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta})(x) + L_{\mathrm{csub}}(x)\,. \end{aligned} \tag{6.1}$$

Further on we use the notation

$$L = \operatorname{Op}(L_{\mathrm{prin}}, L_{\mathrm{csub}}) \tag{6.2}$$

as shorthand for (6.1). We call (6.2) the *covariant representation* of the differential operator L.

Using the covariant representation (6.2) and matrix adjugation (5.11) we define the adjugate of the differential operator L as

$$\operatorname{Adj} L := \operatorname{Op}(\operatorname{adj} L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{csub}}).$$

We define the Dirac operator as the differential operator

$$D := \begin{pmatrix} L & mI \\ mI & \operatorname{Adj} L \end{pmatrix} \tag{6.3}$$

acting on 4-columns v of complex-valued scalar fields. Here m is the electron mass and I is the 2×2 identity matrix.

We claim that the system of four scalar equations $Dv = 0$ is equivalent to the Dirac equation in its traditional geometric formulation. In order to justify this claim we need to compare our Dirac operator (6.3) with the traditional Dirac operator D_{trad}, see Appendix A in [**6**] for definition.

THEOREM 6.1. *The two operators are related by the formula*

$$D = |\det g_{\kappa\lambda}|^{1/4}\, D_{\mathrm{trad}}\, |\det g_{\mu\nu}|^{-1/4},$$

where the Lorentzian metric is defined in accordance with formula (3.2).

The proof of Theorem 6.1 is given in [**6**].

Our representation (6.3) of the massive hyperbolic Dirac operator in dimension four is given in an analytic language different from the traditional geometric language. Hence, we feel the need to reassure the reader that all the standard ingredients are implicitly contained in (6.3).

The matrices (5.16) are our Pauli matrices. Moreover, it is easy to see that our definition of the metric (3.2) ensures that our Pauli matrices (5.16) automatically satisfy the standard defining relation $\sigma^\alpha(\operatorname{adj}\sigma^\beta) + \sigma^\beta(\operatorname{adj}\sigma^\alpha) = -2Ig^{\alpha\beta}$.

The traditional representation of the Dirac operator involves covariant derivatives of spinor fields with respect to the Levi-Civita connection. Technical calculations given in [**6**] show that these connection coefficients are contained within the Poisson bracket term in our definition of the covariant subprincipal symbol (5.10). More precisely, the Poisson bracket term in formula (5.10) does not give each spinor connection coefficient separately, it rather gives their sum, the way they appear in the Dirac operator.

7. Dimension three is special

In the remainder of the paper we retain the assumption (3.1), and we also assume that

$$n = 3 \tag{7.1}$$

and that the principal symbol of our formally self-adjoint first order differential operator L is trace-free,

$$\operatorname{tr} L_{\mathrm{prin}}(x,p) = 0. \tag{7.2}$$

In addition, in the remainder of the paper we assume ellipticity (1.6). Note that under the assumptions (3.1), (7.1) and (7.2) the ellipticity condition (1.6) is equivalent to the non-degeneracy condition (1.7).

We define the metric tensor in accordance with formula (3.2). It is easy to see that under the assumptions (3.1), (7.1), (7.2) and (1.6) our metric is Riemannian, i.e. the metric tensor is positive definite.

8. Gauge-theoretic analysis in 3D

The metric tensor defined in accordance with formula (3.2) does not determine the Hermitian matrix-function $L_{\mathrm{prin}}(x,p)$ uniquely. Hence, in this section we identify a further geometric object encoded within the principal symbol of our differential operator L. To this end, we will now start varying this principal symbol, assuming the metric g, defined by formula (3.2), to be fixed (prescribed).

Let us fix a reference principal symbol $\mathring{L}_{\mathrm{prin}}(x,p)$ corresponding to the prescribed metric g and look at all principal symbols $L_{\mathrm{prin}}(x,p)$ which correspond to the same prescribed metric g.

Lemma 8.1. *If our principal symbol $L_{\mathrm{prin}}(x,p)$ is sufficiently close to the reference principal symbol $\mathring{L}_{\mathrm{prin}}(x,p)$, then there exists a unique special unitary matrix-function $R : M \to \mathrm{SU}(2)$ close to the identity matrix such that*

$$L_{\mathrm{prin}}(x,p) = R^*(x)\, \mathring{L}_{\mathrm{prin}}(x,p)\, R(x)\,.$$

The proof of Lemma 8.1 is given in Section 2 of [**5**].

The choice of reference principal symbol $\mathring{L}_{\mathrm{prin}}(x,p)$ in our construction is arbitrary, as long as this principal symbol corresponds to the prescribed metric g, i.e. as long as we have $\det \mathring{L}_{\mathrm{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta$. It is natural to ask the question: what happens if we choose a different reference principal symbol $\mathring{L}_{\mathrm{prin}}(x,p)$? The freedom in choosing the reference principal symbol $\mathring{L}_{\mathrm{prin}}(x,p)$ is a gauge degree of freedom in our construction and our results are invariant under changes of the reference principal symbol, see Section 6 of [**5**] for details.

In order to work effectively with special unitary matrices we need to choose coordinates on the 3-dimensional Lie group SU(2). It is convenient to describe a 2×2 special unitary matrix by means of a spinor ξ, i.e. a pair of complex numbers ξ^a, $a = 1, 2$. The relationship between a matrix $R \in \mathrm{SU}(2)$ and a nonzero spinor ξ is given by the formula

$$R = \frac{1}{\|\xi\|} \begin{pmatrix} \xi^1 & -\overline{\xi^2} \\ \xi^2 & \overline{\xi^1} \end{pmatrix}, \tag{8.1}$$

where the overline stands for complex conjugation and $\|\xi\| := \sqrt{|\xi^1|^2 + |\xi^2|^2}\,$.

Formula (8.1) establishes a one-to-one correspondence between SU(2) matrices and nonzero spinors, modulo a rescaling of the spinor by an arbitrary positive real factor. The matrices

$$\mathring{\sigma}^\alpha(x) := (\mathring{L}_{\mathrm{prin}})_{p_\alpha}(x), \qquad \alpha = 1, 2, 3, \tag{8.2}$$

are the Pauli matrices in our construction.

Remark 8.2. The gauge-theoretic analysis performed in the current section is somewhat different from that of Section 5. Namely, the differences are as follows.

- In the current section we applied gauge transformations to the principal symbol whereas in Section 5 we applied gauge transformations to the operator itself.
- In the current section we chose a particular principal symbol as a reference, which led to a somewhat different definition of Pauli matrices: compare formulae (8.2) and (5.16).
- In the current section we did not discuss the subprincipal symbol.

9. A non-geometric representation of the massless Dirac action in 3D

In this section we retain the assumptions (3.1), (7.1), (7.2) and (1.6). In addition, we assume that our manifold M is compact (and without boundary).

We study the eigenvalue problem

$$Lv = \lambda s v, \tag{9.1}$$

where $s(x)$ is a given positive scalar weight function. Obviously, the problem (9.1) has the same spectrum as the problem

$$s^{-1/2} L s^{-1/2} v = \lambda v, \tag{9.2}$$

so it may appear that the weight function $s(x)$ is redundant. We will, however, work with the eigenvalue problem (9.1) rather than with (9.2) because we want our problem to possess a gauge degree of freedom associated with conformal scalings of the metric, see Section 5 of [**5**] for details. Note also that the subprincipal symbol of the operator $s^{-1/2} L s^{-1/2}$ is $s^{-1} L_{\mathrm{sub}}$: here we are looking at formulae (5.5) and (5.6) with $\psi = -\frac{1}{2} \ln s$.

The eigenvalue problem (9.1) can be thought of as the result of separation of variables $w(x^1, x^2, x^3, x^4) = e^{-i\lambda x^4} v(x^1, x^2, x^3)$ in the hyperbolic system

$$(-is\, \partial/\partial x^4 + L) w = 0\,, \tag{9.3}$$

compare with formula (2.3). The operator $-is\, \partial/\partial x^4 + L$ appearing in the LHS of formula (9.3) is a special case of the 'relativistic' hyperbolic operator introduced in Section 4, its special feature being that it has a naturally defined 'time' coordinate x^4 which does not 'mix up' with the 'spatial' coordinates x^1, x^2, x^3 (local coordinates on the manifold M).

We define the counting function $N(\lambda)$ in the usual way (2.4) as the number of eigenvalues λ_k of the problem (9.1) between zero and a positive λ. The results from Section 2 give us explicit formulae for the coefficients a and b of the asymptotic expansion (2.5), see formulae (2.9) and (2.10), but these formulae do not have a clear geometric meaning. Our aim in the current section is to rewrite these formulae in a geometrically meaningful form.

The coefficients a and b are expressed via the principal and subprincipal symbols of the operator L as well as the scalar weight function $s(x)$. But in Section 8 we established that the principal symbol is described by a metric and a nonvanishing spinor field $\xi(x)$, with the latter defined modulo rescaling by an arbitrary positive real function. We choose to specify the scaling of our spinor field $\xi(x)$ in accordance with the formula

$$\|\xi(x)\| = s(x). \tag{9.4}$$

The coefficients a and b can now be expressed via the metric, spinor field and subprincipal symbol of the operator L.

THEOREM 9.1. *The coefficients in the two-term asymptotics (2.5) of the counting function (2.4) of the eigenvalue problem (9.1) are given by the formulae*

$$a = \frac{1}{6\pi^2} \int_M \|\xi\|^3 \sqrt{\det g_{\alpha\beta}}\; dx\,, \tag{9.5}$$

$$b = \frac{S(\xi)}{2\pi^2} - \frac{1}{4\pi^2} \int_M \|\xi\|^2\, (\operatorname{tr} L_{\mathrm{sub}}) \sqrt{\det g_{\alpha\beta}}\; dx\,, \tag{9.6}$$

where $S(\xi)$ is the massless Dirac action with Pauli matrices (8.2) and $dx = dx^1 dx^2 dx^3$.

Theorem 9.1 follows from Theorem 1.1 of [**3**] and Theorem 1.1 of [**5**]. The massless Dirac action is defined in Appendix A of [**5**].

Theorem 9.1 allows us to define the concept of massless Dirac action in dimension three in a non-geometric way. Namely, if the subprincipal symbol of the operator L is zero, then the second asymptotic coefficient of the counting function is, up to the factor $\frac{1}{2\pi^2}$, the massless Dirac action.

10. The covariant subprincipal symbol in 3D

The assumptions in this section are the same as in Section 9.

As pointed out in Remark 8.2, the gauge-theoretic analysis performed in Sections 8 and 5 is somewhat different. In this section we rewrite Theorem 9.1 in the gauge-theoretic language of Section 5.

The central element of the gauge-theoretic analysis of Section 5 was the notion of covariant subprincipal symbol, see formula (5.10) for definition. This definition of covariant subprincipal symbol works equally well in dimension three, only it becomes slightly simpler. Namely, observe that if P is a 2×2 trace-free matrix, then $\operatorname{adj} P = -P$. Hence, the definition of the covariant subprincipal symbol can now be rewritten as

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} - \frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}. \tag{10.1}$$

Let us define the massless Dirac operator on half-densities in accordance with formula (A.19) from [**3**].

LEMMA 10.1. *Our operator L is a massless Dirac operator on half-densities if and only if $L_{\mathrm{csub}}(x) = 0$.*

PROOF. The subprincipal symbol of a massless Dirac operator on half-densities was calculated explicitly in Section 6 of [**3**]. Straightforward calculations show that this explicit formula can be rewritten as $\frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}$. □

We can now reformulate Theorem 9.1 in the following equivalent form

THEOREM 10.2. *The coefficients in the two-term asymptotics (2.5) of the counting function (2.4) of the eigenvalue problem (9.1) are given by the formulae*

$$a = \frac{1}{6\pi^2}\int_M s^3 \sqrt{\det g_{\alpha\beta}}\; dx\,, \tag{10.2}$$

$$b = -\frac{1}{4\pi^2}\int_M s^2\,(\operatorname{tr} L_{\mathrm{csub}})\sqrt{\det g_{\alpha\beta}}\; dx\,. \tag{10.3}$$

PROOF. Formula (10.2) is a consequence of formulae (9.5) and (9.4). Formula (10.3) is a consequence of formulae (9.6), (9.4), (10.1) and Lemma 10.1 from the current paper and Theorem 1.2 from [**3**]. □

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