$G-$CONVERGENCE, DIRICHLET TO NEUMANN MAPS AND INVISIBILITY

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Abstract. We establish optimal conditions under which the G-convergence of linear elliptic operators implies the convergence of the corresponding Dirichlet to Neumann maps. As an application we show that the approximate cloaking isotropic materials from [19] are independent of the source.

1. Introduction

We start with the definition of the Dirichlet to Neumann map (Voltage to current) map. Given an elliptic matrix $\sigma \in L^\infty(\Omega)$, for a given boundary data $\varphi \in H^{1/2}(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ to the Dirichlet problem;

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = \varphi. \end{cases} \tag{1.1}$$

When the boundary is sufficiently smooth, the measurements on the boundary consist of the classical Dirichlet–to–Neumann map

$$\Lambda_\sigma(\varphi) = \langle \sigma \nabla u, \nu \rangle\big|_{\partial\Omega}, \tag{1.2}$$

where ν denotes the exterior unit normal to the boundary. In this way $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. It follows by integration by parts that Λ_σ can also be described in the weak form as

$$\langle \Lambda_\sigma(\varphi), \psi \rangle = \int_\Omega \langle \sigma \nabla u, \nabla \tilde{\psi} \rangle, \tag{1.3}$$

where $\psi \in H^{1/2}(\partial\Omega)$ and $\tilde{\psi} \in H^1(\Omega)$ is an extension of ψ into Ω. In case $\partial\Omega$ lacks of a proper normal, the weak formulation is still valid.

The Calderón inverse problem consists of the stable determination of σ from Λ_σ, see [30, 20, 28, 7] for the uniqueness in the isotropic case, [3, 5, 9, 10, 13, 16, 12] for stability and [27, 28] for the reconstruction. Much less is known in the anisotropic case except in dimension d=2 [8]. Notice that when the Dirichlet to Neumann map is known for all energies, uniqueness and stability are studied also for the anisotropic case, see e.g. [22, 6].

The results from [3, 9, 10, 13, 16, 12] require some uniform control of the oscillations of σ (conditional stability). Unfortunately, wild oscillations of a sequence of conductivities σ_n creates an instability of the Calderón problem.

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This is well expressed in terms of the G-topology [15, 21]. It is not hard to see that if σ_n G-converges to σ, the corresponding Dirichlet to Neumann maps converge weakly. Namely, for each $\varphi, \psi \in H^{1/2}(\partial\Omega)$,

$$\langle \Lambda_{\sigma_h}(\varphi), \psi\rangle \to \langle \Lambda_\sigma(\varphi), \psi\rangle. \tag{1.4}$$

Now, if σ_n G-converges to σ but does not convergence pointwise, we deduce that the convergence (1.4) does not imply any sort of L^p convergence. (Notice σ_n, σ could be choosen to be C^∞!).

However, the stability estimates are normally stated in terms of the operator norm and (1.4) by itself does not imply the convergence in the operator norm $\|\|_{\mathcal{L}(H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega))}$. In [1], it is proved that if, in addition to the $G-$convergence, we have that $\sigma_n = \sigma = I$ on $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$, with Ω being the unit disc and $\sigma = I$, then in fact the G-convergence implies the convergence in the operator norm. On the other hand, the stability at the boundary of the inverse problem implies that, in order to obtain operator norm convergence, some control on the behaviour of the conductivities at the boundary is needed. For example, it was proved, see [31], [11], [4] and [14], that, for isotropic conductivities, if $\lim_{n\to\infty} \|\Lambda_{\rho_n} \to \Lambda_\rho\|_{H^{1/2}\to H^{-1/2}(\partial\Omega)} = 0$ then

$$\lim_{n\to\infty} \|\rho_n - \rho\|_{L^\infty(\partial\Omega)} = 0 \tag{1.5}$$

Thus, the G-convergence by itself can not guarantee the operator norm convergence. Let $\Omega \subset \mathbb{R}^n$ and define, for $K \geq 1, \delta > 0$,

$$\begin{aligned} M_K(\Omega) = \{\sigma \in L^\infty(\Omega, \mathcal{M}^{n\times n}) : \frac{1}{K}|\xi|^2 \leq \sigma\xi\cdot\xi \leq K|\xi|^2 \\ \text{for almost every } x \in \Omega \text{ and } \xi \in \mathbb{R}^n\}; \\ \Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}. \end{aligned} \tag{1.6}$$

The following theorem seems to be essentially sharp (see comments below).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a domain. Assume that*

$$\lim_{\delta\to 0} \delta^{-1}\left(\limsup_{n\to\infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega_\delta)}\right) = 0 \tag{1.7}$$

and that $\sigma_n \in M_K$ converges to σ in the sense of the $G-$convergence. Then

$$\lim_{n\to\infty} \|\Lambda_{\sigma_n} - \Lambda_\sigma\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} = 0.$$

Let us emphasize that no regularity assumption is made on the domain or on the conductivities. The condition (1.7) can be read as a weak version of

$$\lim_{n\to\infty} (\|\nabla_\nu(\sigma_n - \sigma)\|_{L^\infty(\partial\Omega)} + \|\sigma_n - \sigma\|_{L^\infty(\partial\Omega)}) = 0. \tag{1.8}$$

Note that the above conditions are natural, since the convergence of the D-N maps is known to imply convergence of the conductivities and their normal derivatives at the boundary under mild regularity assumptions ([31, 5, 11, 4, 14]). Moreover, in Theorem 4.9 we provide an explicit example which shows that, just the convergence σ_n to σ in $L^\infty(\partial\Omega)$ together with their convergence in $L^p(\Omega)$, for any $p < \infty$, are not sufficient for the norm-convergence of the DN maps.

The proof of Theorem 1.1 is very different in spirit to that from [1] and we believe it to be of an independent interest. The proof in [1] uses the decay properties of the spherical harmonics away from the boundary. Under some regularity assumptions on σ, which in turn imply certain properties of the corresponding Poisson kernel, a related strategy works (estimating decay properties of solutions with oscillating boundary data away from the boundary). To prove theorem 1.1 we argue in a different manner. Namely, we sudy the behaviour of the solutions near the boundary. If, for example, $\sigma_n = \sigma$ on $\Omega \setminus \Omega'$, where $\Omega' \subset\subset \Omega$, then the difference of two solutions of the Dirichlet problem associated with σ_n and σ solves the same elliptic equation in $\Omega \setminus \Omega'$. It turns out that the resulting operators from the boundary into $\Omega \setminus \Omega'$ are compact in a proper space. Our way to codify this is to factorize $\Lambda_{\sigma_n} - \Lambda_\sigma = T \circ A_n$, where T is compact. The arguments behind this procedure are quite robust and allow to relax the condition $\sigma_n = \sigma$ on $\Omega \setminus \Omega'$ to (1.7).

Next we turn to applications of our techniques to what is called an approximate cloaking. In the last decade it has been shown that the failure of uniqueness in the Calderón problem is related to the modeling of invisible materials and what is called acoustic and electromagnetic cloaking, see [17, 29, 24, 18]. It is shown there that the available conductivities yielding perfect cloaking are singular and anisotropic. Recently it has been shown that they can be approximated by elliptic isotropic materials in the G-convergence sense [19], see also [23, 26, 25] for different approachs. Leaving precise formulations of the involved operators and a general case to section 3, assume that $\Omega = B_3$, i.e. the ball of radius 3 in $\mathbb{R}^3$ and $q \in L^\infty(B_1)$ is an arbitrary potential. Consider the D-N maps associated with the Dirichlet problems, with spectral parameter λ, for the free space,

$$\begin{cases} -\nabla \cdot \nabla u = \lambda u \\ u\big|_{\partial\Omega} = \varphi, \end{cases} \tag{1.9}$$

cf. (1.1), and for cloaked space,

$$\begin{cases} -g_n^{-1/2} \nabla \cdot \sigma_n \nabla u + qu = \lambda u \\ u\big|_{\partial\Omega} = \varphi. \end{cases} \tag{1.10}$$

Here the weight factors g_n and the isotropic conductivities $\sigma_n = \gamma_n I$, which are supported in $\{x : 1 \le |x| \le 2\}$, are chosen independent of q. Denoting by Λ^λ_{out} and Λ^λ_n the corresponding D-N operators, we have

Theorem 1.2. *The exists a sequence g_n, γ_n such that, for all except a countable number of λ,*

$$\|\Lambda^\lambda_n - \Lambda^\lambda_{out}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \to 0, \quad \text{as } n \to \infty.$$

This theorem means that, by a proper choice of g_n, γ_n, one can better and better hide from an external observer any potential in B_1. The novelty here is that we have an operator norm convergence instead of the strong convergence in [19]. We refer the reader to section 3 for the more general case as well as the explanation of the nature of the exceptional points λ.

Let us note that, motivated by the applications to acoustic, quantum and electromagnetic cloaking, we extend our results of the type of Theorem 1.1, to the operators

$$\text{Ł}_n u = -\nabla_{A_n} \cdot \sigma_n \nabla_{A_n} u + q_n u, \quad u|_{\partial\Omega} = 0, \tag{1.11}$$

where $\nabla_A = \nabla + iA$ with A being a real one-form. However, for the sake of brevity, we do so only for the case when $\sigma_n = \sigma$, $A_n = A$, $q_n = q$ in $\Omega \setminus \Omega'$.

Finally, we point out that, since the G-convergence rules out general stability results with respect to L^p classes, one is tempted to conjecture that the convergence of the D-N maps implies the G-convergence. Recall that if F is a diffeomorphism of Ω, which is the identity at the boundary, then $\Lambda_{F^*(\sigma)} = \Lambda_\sigma$. As discussed for example in [2], the isotropic conductivities are G-dense in the set of anisotropic conductivities, so that the only hope is to recover from the D-N maps the G-limit up to a gauge transformation. In contrast to the previous results on the conditional stability, the compactness of the sets M_K in the G-topology indeed provides a stability result which is unconditional respect to regularity (we still require ellipticity).

Theorem 1.3. *Let $d = 2, \sigma_n \in M_K$. Then*

$$\lim_{n\to\infty} \Lambda_{\sigma_n} = \Lambda_\sigma, \tag{1.12}$$

weakly in $H^{-1/2}(\partial\Omega)$ if and only if there exists a sequence of quasiconformal maps $F_n : \Omega \to \Omega$, $F_n|_{\partial\Omega} = id|_{\partial\Omega}$, such that, in the sense of the $G-$convergence,

$$F_n^*(\sigma_n) \to \sigma. \tag{1.13}$$

Let us emphasize that since there is no requirement at the boundary here we speak only of weak convergence of the D-N maps.

The paper is structured as follows. In section 2 we start by proving the convergence of the D-N maps for the operators of form (1.11), assuming that $\sigma_n = \sigma$, $A_n = A$, $q_n = q$ in a neighborhood of the boundary, see Theorem 2.1. Note that this is the case which will be needed for applications to aprroximate cloaking considered in section 3. In section 4 we prove Theorem 1.1 and in section 5 we prove Theorem 1.3.

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2. Operators which coincide near the boundary

Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. We consider the conductivity equations with magnetic potential A_n and electrical potential $q_n \in L^\infty(\Omega)$ and the spectral parameter $\lambda \in \mathbb{C}$. Namely, for $u \in H^1(\Omega, \mathbb{C})$ we define the Dirichlet problem for the corresponding differential operator $Ł_n$:

$$Ł_n u = -\nabla_{A_n} \cdot \sigma_n \nabla_{A_n} u + q_n u, \quad u|_{\partial\Omega} = 0. \tag{2.1}$$

Here

$$\sigma_n \in M_K,\ K > 1, \quad \text{i.e. } \frac{1}{K} I \le \sigma_n(x) \le KI,\ x \in \Omega, \tag{2.2}$$

and

$$A_n \in L^\infty(\Omega, \mathbb{R}^d),\ \|A_n\|_\infty \le K, \quad q_n \in L^\infty(\Omega, \mathbb{R}),\ \|q_n\|_\infty \le K, \tag{2.3}$$

where for simplicity we assume all $K's$ to be the same. The magnetic gradient is given by $\nabla_{A_n} u = \nabla u + iA_n u$. Note that conditions (2.2), (2.3) imply the existence of $\lambda(K)$, such that

$$(-\infty, \lambda(K)) \cap \operatorname{spec}(Ł_n) = \emptyset.$$

If $\lambda \notin \operatorname{spec}(L_n)$, then for a given boundary data $\psi \in H^{1/2}(\partial\Omega)$ there is a unique solution $u_n = u_n^\psi(\lambda) \in H^1(\Omega)$ to the Dirichlet problem;

$$\begin{cases} Ł_n^\lambda u_n := (Ł_n - \lambda) u_n = 0 \\ u_n\big|_{\partial\Omega} = \psi. \end{cases} \tag{2.4}$$

For the regular domains we define the Dirichlet to Neumann map, $\Lambda_n^\lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_n^\lambda(\psi) = \nu \cdot \sigma_n \nabla_{A_n} u_n.$$

It follows by integration by parts that Λ_n^λ can also be described in the weak form as

$$\langle \Lambda_n^\lambda(\psi), \varphi \rangle = \int_\Omega (\sigma_n \nabla_{A_n} u_n \cdot \overline{\nabla_{A_n} \tilde\varphi} + (q_n - \lambda) u_n \overline{\tilde\varphi}), \tag{2.5}$$

where $\varphi \in H^{1/2}(\partial\Omega)$ and $\tilde\varphi \in H^1(\Omega)$ is an extension of φ to Ω. We will denote the solution of (2.4) by u_n^ψ or even u_n (we omit the dependence on λ). In this section we prove that

Theorem 2.1. *Let L_n, $Ł$ be operators of form (2) which satisfy (2.2), (2.3). Assume that there is Ω' with $\overline{\Omega}' \subset\subset \Omega$ such that $\sigma_n = \sigma$, $A_n = A$ and $q_n = q$ on $\Omega \setminus \Omega'$. Then, if the operators L_n G- converge to L, we have*

$$\|\Lambda_n^\lambda - \Lambda_0^\lambda\|_{H^{1/2} \to H^{-1/2}} \to 0. \tag{2.6}$$

Here $\lambda \notin spec(Ł)$ and, for any $\mathbb{K} \subset \mathbb{C}$ being a compact set such that $\mathbb{K} \cap spec(Ł) = \emptyset$, the convergence is uniform for $\lambda \in \mathbb{K}$.

The proof of the theorem is rather long and will consist of several steps.

Let us first note that there are several equivalent definitions of the $G-$convergence of operators, see e.g. Th. 13.6 and Example 13.13, [15] which essentially amount to the convergence of the solutions. For our purpose we use

Definition 2.2. The operator $Ł_n$ G-converges to Ł if, for any $f \in H^{-1}(\Omega)$ and $\lambda < \lambda(K)$, it holds that

$$(Ł_n - \lambda I)^{-1} f \to (Ł - \lambda I)^{-1} f, \quad \text{as } n \to \infty, \tag{2.7}$$

weakly in $H^1_0(\Omega)$.

The following two lemmata follow more or less straightforward from the definition of the G-convergence. To this end we first introduce the quadratic form, ℓ_n, associated with $Ł_n$,

$$\ell_n[u] = \int_\Omega (\sigma_n \nabla_{A_n} u, \nabla_{A_n} u) + q_n |u|^2, \;\; u \in H^1_0(\Omega). \tag{2.8}$$

Lemma 2.3. *Let now $\lambda \in \mathbb{K}$. Then, for any $f \in H^{-1}(\Omega)$,*

$$(Ł_n - \lambda I)^{-1} f \to (Ł - \lambda I)^{-1} f, \quad \text{as } n \to \infty, \tag{2.9}$$

weakly in $H^1_0(\Omega)$ and uniformly in $\mathbb{K}$.

Proof. Using the coercivity of ℓ_n the proof follows the lines of [19, Lemma 2.7]. Note that this fact does not require the coincidence of σ_n, A_n and q_n with σ, A and q near $\partial\Omega$ since it follows from the uniform ellipticity of forms ℓ_n together with the strong resolvent convergence of (2.7). □

Lemma 2.4. *Let $Ł_n$ G-converge to Ł and $\lambda \notin spec(Ł)$. Then, for each $\psi \in \mathcal{H}^{1/2}(\partial\Omega)$,*

$$(\Lambda^\lambda_n - \Lambda^\lambda)(\psi) \rightharpoonup 0 \text{ in } \mathcal{H}^{-1/2}.$$

Proof. Denoting by u_n, u the solutions to (2.4), we have, by Theorem 22.9 [15], that

$$\ell_n[u_n] - \lambda \|u_n\|^2 \to \ell[u] - \lambda \|u\|^2, \quad \text{as } n \to \infty.$$

Polarising this equality, we arrive at

$$\langle \Lambda^\lambda_n(\psi), \varphi \rangle \to \langle \Lambda^\lambda(\psi), \varphi \rangle.$$

□

We denote by $\mathring{H}^1(\Omega \setminus \Omega')$ the functions in $H^1(\Omega \setminus \Omega')$ with trace 0 on $\partial\Omega$. A key fact in our arguments is the following Caccioppoli type inequality.

Lemma 2.5. *Let $w \in \mathring{H}^1(\Omega \setminus \Omega')$ be a weak solution of*

$$Ł^\lambda w = f + \operatorname{div} F \text{ on } \Omega \setminus \Omega', \tag{2.10}$$

for $f \in L^2(\Omega)$ and F being a vector field in $L^2(\Omega \setminus \Omega')$. Then, for any Ω'', $\overline{\Omega}' \Subset \Omega''$, $\overline{\Omega}'' \Subset \Omega$, there exists a $C = C(\Omega, \Omega', \Omega'', K, \lambda)$ such that

$$\int_{\Omega\setminus\Omega''} |\nabla w|^2 \le C \left(\int_{\Omega\setminus\Omega'} |w|^2 + \int_{\Omega\setminus\Omega'} |F|^2 + \int_{\Omega\setminus\Omega'} |f|^2 \right). \tag{2.11}$$

Moreover if we choose $\Omega' = \Omega_{2\delta}$ and $\Omega'' = \Omega_\delta$ the estimate is

$$\int_{\Omega_\delta} |\nabla w|^2 \le C \left(\delta^{-2} \int_{\Omega_{2\delta}} |w|^2 + \int_{\Omega_{2\delta}} |F|^2 + \int_{\Omega_{2\delta}} |f|^2 \right). \tag{2.12}$$

Proof. Choose $\eta \in C^\infty(\mathbb{R}^n \setminus \Omega')$ such that $\eta = 1$ on $\Omega \setminus \Omega''$ and $\eta = 0$ near $\partial\Omega'$. Since w has zero trace on $\partial\Omega$, $\eta^2 w \in H_0^1(\Omega \setminus \Omega')$. Thus it is a proper test function for the weak formulation of (2.10). Thus,

$$\int_{\Omega\setminus\Omega'} \langle \sigma\nabla_A w, \nabla_A(\eta^2 w)\rangle = \int_{\Omega\setminus\Omega'} \langle F, \nabla(\eta^2 w)\rangle - \int_{\Omega\setminus\Omega'} \eta^2(q-\lambda)|w|^2 + \int_{\Omega\setminus\Omega'} f\overline{\eta^2 w}$$

Hence

$$|\int_{\Omega\setminus\Omega'} \eta^2\langle\sigma\nabla_A w, \nabla_A w\rangle|$$

$$\leq \underbrace{\Big|2\int_{\Omega\setminus\Omega'} \eta\langle\sigma\nabla_A w, \nabla\eta\rangle w\Big|}_{=I_1} + \underbrace{\Big|\int_{\Omega\setminus\Omega'}\langle F, \nabla(\eta^2 w)\rangle\Big|}_{=I_2} + C\int_{\Omega\setminus\Omega'}|w|^2 + \int_{\Omega\setminus\Omega'}|f\eta^2 w|$$

We can bound the first term on the right hand side by

$$I_1 \leq \Big|2\int_{\Omega\setminus\Omega'} \eta\langle\sigma\nabla_A w, \nabla_A w\rangle^{1/2}\langle\sigma\nabla\eta, \nabla\eta\rangle^{1/2}|w|\Big|$$

$$\leq 2K\|\nabla\eta\|_{L^\infty}\left(\int_{\Omega\setminus\Omega'}\eta^2\langle\sigma\nabla_A w, \nabla_A w\rangle\right)^{1/2}\left(\int_{\Omega\setminus\Omega'}|w|^2\right)^{1/2}$$

$$\leq 1/2\left(\int_{\Omega\setminus\Omega'}\eta^2\langle\sigma\nabla_A w, \nabla_A w\rangle\right) + 16K^2\|\nabla\eta\|^2_{L^\infty}\int_{\Omega\setminus\Omega'}|w|^2,$$

where we have used that $\sigma \leq KI$. Hence we absorb the term $1/2\left(\int \eta^2\langle\sigma\nabla_A w, \nabla_A w\rangle\right)$ by the left hand side to obtain

$$(2.13)\quad \begin{aligned} |\int_{\Omega\setminus\Omega'}\eta^2\langle\sigma\nabla_A w, \nabla_A w\rangle| &\leq C\|\nabla\eta\|^2_{L^\infty}\int_{\Omega\setminus\Omega'}|w|^2 \\ &\quad + 2|\int_{\Omega\setminus\Omega'}\langle F, \nabla(\eta^2 w)\rangle| + C\int_{\Omega\setminus\Omega'}|fw|. \end{aligned}$$

Now we deal with the term $I_2 = |\int_{\Omega\setminus\Omega'}\langle F, \nabla(\eta^2 w)\rangle|$. By integrating by parts and the definition of ∇_A we have that,

$$I_2 \leq |\int_{\Omega\setminus\Omega'}\langle F, 2\eta\nabla\eta\rangle w| + |\int_{\Omega\setminus\Omega'}\eta^2\langle F, \nabla_A w\rangle| + \int_{\Omega\setminus\Omega'}|\langle F, \eta^2 iAw\rangle|.$$

Next, we use the Cauchy-Schwartz inequality for the first and the third terms of the right and the Hölder inequality for the second, in order to bound I_2 by

$$\leq C(\int_{\Omega\setminus\Omega'}|F\eta|^2 + \int_{\Omega\setminus\Omega'}|\nabla\eta|^2|w|^2) + \int_{\Omega\setminus\Omega'}\eta^2|F||\nabla_A w| + \int_{\Omega\setminus\Omega'}|F|^2\eta^2 + \int_{\Omega\setminus\Omega'}|Aw|^2$$

$$\leq C\int_{\Omega\setminus\Omega'}\eta^2|F|^2 + C\|\nabla\eta\|^2_{L^\infty}\int_{\Omega\setminus\Omega'}|w|^2 + (\int_{\Omega\setminus\Omega'}\eta^2|F|^2)^{1/2}(\int_{\Omega\setminus\Omega'}\eta^2|\nabla_A w|^2)^{1/2}.$$

Since,

$$(\int_{\Omega\setminus\Omega'}\eta^2|F|^2)^{1/2}(\int_{\Omega\setminus\Omega'}\eta^2|\nabla_A w|^2)^{1/2} \leq C\int_{\Omega\setminus\Omega'}\eta^2|F|^2 + \frac{1}{2K}\int_{\Omega\setminus\Omega'}\eta^2|\nabla_A w|^2,$$

we have obtained the bound

$$(2.14)\quad I_2 \leq C\int_{\Omega\setminus\Omega'}\eta^2|F|^2 + \frac{1}{2K}\int_{\Omega\setminus\Omega'}\eta^2|\nabla_A w|^2 + C\|\nabla\eta\|^2_{L^\infty}\int_{\Omega\setminus\Omega'}|w|^2.$$

Now we incorporate estimate (2.14) into (2.13) estimating the term $\int_{\Omega\setminus\Omega'}|f\eta^2 w|$ by Cauchy-Schwarz. We obtain that

$$|\int_{\Omega\setminus\Omega'}\eta^2\langle\sigma\nabla_A w,\nabla_A w\rangle|$$

$$\leq C\int_{\Omega\setminus\Omega'}\eta^2|F|^2+\frac{1}{2K}\int_{\Omega\setminus\Omega'}\eta^2|\nabla_A w|^2+C\|\nabla\eta\|^2_{L^\infty}\int|w|^2+\int|f|^2.$$

Since $\frac{1}{K}I\leq\sigma$,

$$\frac{1}{K}\int_{\Omega\setminus\Omega'}|\eta\nabla_A w|^2\leq|\int_{\Omega\setminus\Omega'}\eta^2\langle\sigma\nabla_A w,\nabla w\rangle|$$

and thus we can absorb the term $\frac{1}{2K}\int_{\Omega\setminus\Omega'}\eta^2|\nabla_A w|^2$ to the left hand side to obtain the bound

(2.15)
$$\int_{\Omega\setminus\Omega'}|\eta\nabla_A w|^2\leq C\left(\int_{\Omega\setminus\Omega'}|F|^2+\|\nabla\eta\|^2_{L^\infty}\int_{\Omega\setminus\Omega'}|w|^2+\int_{\Omega\setminus\Omega'}|f|^2\right),$$

where C depends on (K,A,q) but not of Ω',Ω''.

In order to obtain (2.11) we simply expand $\nabla_A w$ and observe that $\eta=1$ on $\Omega\setminus\Omega''$.

In order to obtain (2.12) we define the cut-off more carefully. Let $\eta_\delta\in C^{0,1}(\Omega)$ be defined by

$$\eta_\delta(x)=\eta(d(x,\partial\Omega)/\delta), \tag{2.16}$$

where $\eta(s)\in C_0^\infty(\mathbb{R}_+)$, $\eta(s)=1$ for $s<1$, $\eta(s)=0$ for $s>2$. Observe that $\mathrm{supp}(\eta_\delta)\subset\Omega_{2\delta}$ and

$$\|\eta_\delta\|_{C^{0,1}(\Omega)}\leq C\delta^{-1}. \tag{2.17}$$

Plugging η_δ into (2.15) yields (2.12). □

Let us fix a boundary value $\psi\in H^{1/2}(\partial\Omega)$ and, as in the beginning of this section, denote the corresponding solutions to (2.4) by u_n^ψ, with u^ψ being the solution to (2.4) for Ł^λ. It will be convenient for us to work with the difference

$$d_n^\psi(\lambda)=d_n^\psi=u_n^\psi-u^\psi\in H_0^1(\Omega).$$

Due to (2.2) and (2.3), it follows that

$$\|d_n^\psi\|_{H^1(\Omega)}\leq\|u_n^\psi\|_{H^1(\Omega)}+\|u^\psi\|_{H^1(\Omega)}\leq C\|\psi\|_{H^{1/2}(\partial\Omega)}$$

It is convenient to state the above inequality as a separate lemma.

Lemma 2.6. *Let*

$$\mathcal{A}_n^\lambda(\psi)=d^\psi_{n\,|_{\Omega\setminus\Omega'}},\quad \mathcal{A}_n^\lambda: H^{1/2}(\partial\Omega)\to\mathring{H}^1(\Omega\setminus\Omega'). \tag{2.18}$$

Then these operators are uniformly bounded wrt n and $\lambda\in\mathbb{K}$.

We prove now the strong convergence of the Dirichlet to Neumann maps.

Proposition 2.7. *Let $Ł_n$, $Ł$ and $\mathbb{K}$ satisfy conditions of Theorem 2.1. Then, for any $\psi \in H^{1/2}(\partial\Omega)$,*

$$\lim_{n\to\infty} \|(\Lambda_n^\lambda - \Lambda^\lambda)(\psi)\|_{H^{-1/2}} = 0,$$

the convergence being uniform for $\lambda \in \mathbb{K}$.

Proof. Let us fix a boundary value $\psi \in H^{1/2}(\partial\Omega)$ and define $u_n^\psi, u^\psi, d_n^\psi = \mathcal{A}_n^\lambda \psi$ as above. Observe that, with $\tilde\psi \in H^1(\Omega)$, $\tilde\psi = 0$ in Ω', $\tilde\psi|_{\partial\Omega} = \psi$,

$$u_n^\psi = \tilde\psi + (Ł_n - \lambda I)^{-1} F, \quad u^\psi = \tilde\psi + (Ł - \lambda I)^{-1} F,$$

where

$$F = \nabla_A \cdot \sigma \nabla_A \tilde\psi - (q-\lambda)\tilde\psi \in H^{-1}(\Omega), \quad \mathrm{supp}(F) \subset \Omega \setminus \Omega'.$$

Then it follows from G-convergence that that

$$d_n^\psi \to 0,$$

where convergence is weak in $\mathring{H}^1(\Omega\setminus\Omega')$ and strong in $L^2(\Omega\setminus\Omega')$. (A direct proof under condition 1.7 is given in Lemma 4.2). We continue by applying Caccioppoli inequality (2.5) and taking into the account that

$$\nabla_A \cdot \sigma \nabla_A d_n^\psi - (q-\lambda) d_n^\psi = 0 \quad \text{in } \Omega \setminus \Omega', \tag{2.19}$$

we see that

$$d_n^\psi|_{\Omega\setminus\Omega''} \to 0 \quad \text{in } \mathring{H}^1(\Omega \setminus \Omega'').$$

This implies the desired result taking into the account the weak definition of the Dirichlet-to-Neumann map (2.5) and the ability to take $\tilde\phi$ there so that $\tilde\phi = 0$ in Ω'' and

$$\|\tilde\phi\|_{H^1(\Omega\setminus\Omega'')} \le C\|\phi\|_{H^{1/2}(\partial\Omega)}.$$

□

In order to utilize that the functions d_n^ψ satisfy (2.19), we introduce the following subspace:

Definition 2.8. We denote $L_s^2(\Omega \setminus \Omega')$ to be the $L^2(\Omega\setminus\Omega')$-closure of the set $\{u : u \in \mathring{H}^1_{loc}(\Omega \setminus \Omega') \text{ and } Ł^\lambda u = 0 \text{ in } \Omega \setminus \Omega'\}$.

Lemma 2.9. *Let $v \in L_s^2(\Omega \setminus \Omega')$. Then $v \in \mathring{H}^1_{loc}(\Omega \setminus \Omega')$ and is a solution in $\Omega \setminus \Omega'$ of equation (2.19).*

Proof. Let $v_k \subset \mathring{H}^1_{\mathrm{loc}}(\Omega \setminus \Omega') \cap L_s^2(\Omega\setminus\Omega')$ satisfy (2.19) and

$$\lim_{k\to\infty} \|v_k - v\|_{L^2(\Omega\setminus\Omega')} = 0.$$

Then, for any Ω'' such that $\Omega' \subset \Omega'' \subset \Omega$, v_k is a Cauchy sequence in $\mathring{H}^1(\Omega\setminus\Omega'')$. Indeed, since v_k vanish on $\partial\Omega$, this follows from Lemma 2.5. Thus, $v_k \to v$ strongly in $\mathring{H}^1_{loc}(\Omega\setminus\Omega')$. As a strong limit of solutions is a weak solution to (2.19), the claim follows. □

Lemma 2.10. *Fix $\lambda \in \mathbb{K}$. Let $\mathcal{A}_n^\lambda$ be defined by (2.18) and $\mathcal{T}^\lambda : L^2_s(\Omega\backslash\Omega') \to H^{-1/2}(\partial\Omega)$ be defined for $\varphi \in H^{1/2}(\partial\Omega)$ as*

$$(\mathcal{T}^\lambda(v), \varphi) = \int (\langle \sigma\nabla_A v, \nabla_A \tilde{\varphi}\rangle + (q-\lambda) v\overline{\tilde{\varphi}}), \tag{2.20}$$

where $\tilde{\varphi}$ is a $H^1(\Omega)$ extension of φ such that $\tilde{\varphi} = 0$ on Ω''. Then

$$\Lambda_n^\lambda - \Lambda^\lambda = \mathcal{T}^\lambda \circ \mathcal{A}_n^\lambda.$$

Moreover, the operators $\mathcal{A}_n^\lambda$ are uniformly bounded and the operator $\mathcal{T}^\lambda$ is compact.

Proof. Notice that the composition makes sense since

$$\text{Range}(\mathcal{A}_n^\lambda) \subset L^2_s(\Omega \setminus \Omega') \subset \mathring{H}^1_{loc}(\Omega \setminus \Omega').$$

The factorization is obvious and the uniform boundedness of $\mathcal{A}_n^\lambda$ is proven in Lemma 2.6. Let us show that $\mathcal{T}^\lambda$ is compact. To this end it is sufficient to show that, if $v_k \in L^2_s(\Omega \setminus \Omega')$ is a bounded sequence, then $\mathcal{T}^\lambda(v_k)$ is precompact in $H^{-1/2}(\partial\Omega)$.

Let us take a sequence of nested compact sets $\Omega' \subset \Omega'' \subset \Omega''' \subset \Omega$.

It follows from Definition 2.8 together with Cacciopoli inequality (2.5) that

$$\|v_k\|_{H^1(\Omega\backslash\Omega'')} \le C\|v_k\|_{L^2(\Omega\backslash\Omega')}.$$

By Banach-Alaoglu theorem there is a (not relabeled) subsequence v_k such that

$$v_k \to v_\infty \in H^1(\Omega \setminus \Omega'') \to 0,$$

where the convergence is weak in $H^1(\Omega\backslash\Omega'')$ and strong in $L^2(\Omega \setminus \Omega'')$. Then v_∞ is also a solution to the equation

$$Ł^\lambda v_\infty = 0 \text{ on } \Omega \setminus \Omega''.$$

Thus, it follows by Caccioppoli inequality (2.5) that

$$\|v_k - v_\infty\|_{H^1(\Omega\backslash\Omega''')} \to 0. \tag{2.21}$$

Now, by the definition of $\mathcal{T}^\lambda$,

$$\|\mathcal{T}^\lambda v_k - \mathcal{T}^\lambda v_\infty\|_{H^{-1/2}} = \sup_{\{\|\varphi\|_{H^{1/2}(\partial\Omega)}=1\}} \int (\langle \sigma\nabla_A(v_k - v_\infty), \nabla_A\tilde{\varphi}\rangle + (q-\lambda)(v_k - v_\infty)\overline{\tilde{\phi}}),$$

where we take the extension function $\tilde{\varphi}$ so that $\text{supp}(\tilde{\varphi}) \subset \Omega \setminus \Omega'''$. Choosing $\tilde{\varphi}$ so that

$$\|\tilde{\varphi}\|_{H^1(\Omega)} \le C\|\varphi\|_{H^{1/2}(\partial\Omega)},$$

we see that

$$\|\mathcal{T}^\lambda v_k - \mathcal{T}^\lambda v_\infty\|_{H^{-1/2}} \le CK\|v_k - v_\infty\|_{H^1(\Omega\backslash\Omega''')}\|\varphi\|_{H^{1/2}},$$

which tends to zero by (2.21). Thus, the desired compactness of $\mathcal{T}^\lambda(v_k)$ is proved.

□

We are now in position to complete the proof of Theorem 2.1.

Proof. Notice that $(H^{1/2})^*(\partial\Omega) = H^{-1/2}(\partial\Omega)$. Taking $\tilde\phi$ in (2.5) to be the solution of (2.4) with $\overline{\lambda}$ instead of λ and ϕ instead of ψ, we see that $(\Lambda_n^\lambda - \Lambda^\lambda) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfies

$$(\Lambda_n^\lambda - \Lambda^\lambda)^* = (\Lambda_n^{\bar\lambda} - \Lambda^{\bar\lambda}).$$

Thus,

$$(\Lambda_n^\lambda - \Lambda^\lambda) = \left(\mathcal{T}^{\bar\lambda} \circ \mathcal{A}_n^{\bar\lambda}\right)^* = (\mathcal{A}_n^{\bar\lambda})^* \circ (\mathcal{T}^{\bar\lambda})^*,$$

where $(\mathcal{T}^{\bar\lambda})^* : H^{1/2}(\partial\Omega) \to L^2_s(\Omega\backslash\Omega')$ is a compact operator. Thus, for every $\epsilon > 0$, there exists a finite dimensional projection operator $P_\epsilon : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$, such that

$$\|(\mathcal{T}^{\bar\lambda})^*(I - P_\epsilon)\|_{H^{1/2}(\partial\Omega)\to L^2_s(\Omega\backslash\Omega')} \le \epsilon. \tag{2.22}$$

Since P_ϵ is finite dimensional, it follows from the strong convergence of $\Lambda_n^\lambda - \Lambda^\lambda$, Proposition 2.7, that

$$\lim_{n\to\infty} \|\Lambda_n^\lambda - \Lambda^\lambda)P_\epsilon\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} = 0.$$

Moreover, the above limit is uniform wrt $\lambda \in \mathbb{K}$. On the other hand, since $\|(\mathcal{A}_n^{\bar\lambda})^*\|_{L^2_s(\Omega\backslash\Omega')\to H^{-1/2}(\partial\Omega)} \le C(\mathbb{K})$, $\lambda \in \mathbb{K}$, we obtain from (2.22) that

$$\|(\Lambda_n^\lambda - \Lambda^\lambda)(I - P_\epsilon)\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \le C\epsilon.$$

Since ϵ is arbitrary, these estimates prove the theorem. □

Remark 2.11. *Assuming $\sigma, A \in C^{0,1}(\Omega \setminus \Omega')$, equation (2.6) remains valid for the operator norm in $H^{1/2}(\partial\Omega)$. Moreover, assuming further smoothness of σ, A and q, we obtain equation (2.6) with the operator norm from $H^{1/2}$ to H^s with larger s.*

3. Application to cloaking

In this section we apply the previous construction to study an approximate invisibility as introduced in [19]. To start, we recall the main result in [19].

Let us consider $\Omega = B_r$, $r = 3$, where $B_r \subset \mathbb{R}^3$ is a ball of radius r centered at 0. Denote by $Ł_{out}$ the operator

$$\begin{aligned} Ł_{out} u &= -\nabla_{A_{out}} \cdot \nabla_{A_{out}} u + q_{out} u, \\ \mathcal{D}(A_{out}) &= \{u \in H^1_0(\Omega) : \nabla_{A_{out}} \cdot \nabla_{A_{out}} u \in L^2(\Omega)\}. \end{aligned} \tag{3.1}$$

Here the magnetic potential A_{out} and electric potential q_{out} satisfy,

$$|A_{out}| \cdot |x| \in L^\infty(\Omega), \quad q_{out} \in L^\infty(\Omega),$$

see (15) and preceeding discussion in [19], where β_1 stands for A_{out} and κ_1 stands for q_{out}. Denote by Λ^λ_{out} the Dirichlet-to-Neumann map corresponding to $Ł_{out} - \lambda$.

Next, consider the Dirichlet-to-Neumann map $\Lambda^\lambda_{R,m,\epsilon}$ associated to the approximate cloaking. To this end, consider the Dirichlet problem of the type (2.4),

$$\begin{cases} Ł^\lambda_{R,m,\epsilon} u = -g_m^{-1/2} \nabla_A \cdot \sigma_{R,\epsilon} \nabla_A u + qu - \lambda u = 0 \\ u\big|_{\partial\Omega} = \psi, \end{cases} \tag{3.2}$$

cf. (126), (127) in [19]. Here $\sigma_{R,\epsilon}$ is a regular, isotropic $G-$approximation to the singular cloaking conductivity σ_s and g_m is a truncation of g_s, namely, $g_m(x) = \max\{m^{-1}, g_s(x)\}$, where $g_s = (\det \sigma_s)^{2/(n-2)}$. To define σ_R, $R \geq 1$, we start with the diffeomorphism $F_R = (F_{1,R}, F_{2,R}) : (B_3 \setminus B_\rho) \sqcup B_R \to \Omega$, where $F_{2,R}$ is the identity on B_R, while

$$F_{1,R}(x) = \left(\frac{|x|}{2} + 1\right) \frac{x}{|x|}, \ \rho < |x| < 2, \ \rho = 2(R-1); \quad F_{1,R}(x) = x, \ |x| > 2.$$

Then $\sigma_R = (F_R)_*(\gamma_0, \gamma_0)$, where $\gamma_0^{ij} = \delta^{ij}$ so that σ_R degenerates on ∂B_1 when $R = 1$ but is bounded for $R > 1$, with however lower bound going to 0 if $R \to 1$. We note that in [19], for technical reasons, γ_0 is substituted by $2\gamma_0$ in B_R, however, the constructions in [19] can be readily modified for the considered case.

With $A \in L^\infty(\Omega)$, $q \in L^\infty(\Omega)$, the operator $Ł_R$ is defined as in (2) with σ_R instead of σ_n and an extra factor $g_s^{-1/2}$ in front of the main term in the right-hand side of (2). The operator $Ł_1$ represents perfect cloaking but it is singular. To avoid further confusing in terminology we will denote this operator by $Ł_{sing}$. The operators $Ł_R$ are self-adjoint in $L^2(\Omega, g_s^{1/2}dx)$. Note that then

$$A_{out} = (F_{1,1})^*(A|_{B_3 \setminus B_1}), \quad q_{out} = (F_{1,1})^*(q|_{B_3 \setminus B_1}),$$

which, in particular, produces the $1/|x|$ singularity of A_{out}.

At last, the isotropic $\sigma_{R,\epsilon}$ are obtained from σ_R by de-homogenization, see S.3, [19], so that, if $\lambda \notin \mathrm{spec}(Ł_{sing})$, then, for $f \in L^2(\Omega)$,

$$(Ł_{R,\epsilon,m} - \lambda I)^{-1} f \to (Ł_{R,m} - \lambda I)^{-1} f,$$

see Lemma 3.3, [19]. Note that the condition $\lambda \notin \mathrm{spec}(Ł_{sing})$ implies that, for R close to 1, large m and small ϵ, λ is outside the spectra of all the operators considered above so all the objects are well-defined. Then it is shown in [19], see Corollary 4.4, that there exists a sequence $R(n) \to 1$, $m(n) \to \infty$, $\epsilon(n) \to 0$ such that, for any $h \in H^{3/2}(\partial\Omega)$,

$$\|\Lambda^\lambda_{R(n),m(n),\epsilon(n)} h - \Lambda^\lambda_{out} h\|_{H^{1/2}(\partial\Omega)} \to 0.$$

Here $\Lambda^\lambda_{R(n),m(n),\epsilon(n)}$, Λ^λ_{out} are Dirichlet-to-Neumann maps associated with the operators $Ł_{R(n),\epsilon(n),m(n)}$, $Ł_{out}$ and $\lambda \notin \mathrm{spec}(Ł_{sing})$.

Using the methods of section 2, we have

Theorem 3.1. *The exists a sequence $R(n) \to 1$, $m(n) \to \infty$, $\epsilon(n) \to 0$ such that, for any $\lambda \notin spec(Ł_{sing})$,*

$$\|\Lambda^\lambda_{R(n),m(n),\epsilon(n)} - \Lambda^\lambda_{out}\|_{H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)} \to 0, \quad \textit{as } n \to \infty.$$

Observe that Λ_{out} does not depend upon the behaviour of A and q inside B_1. Thus, Theorem 3.1 means that $A|_{B_1}$, $q|_{B_1}$ are almost cloaked from an external observer by a proper choice of $\sigma_{R,\epsilon}$.

Proof. By Theorem 4.3, [19], for $f \in L^2(\Omega)$ and $\lambda \notin \mathrm{spec}(Ł_{sing})$,

(3.3)
$$\lim_{n\to\infty} \left(Ł_{R(n),\epsilon(n),m(n)} - \lambda I\right)^{-1} f = (Ł_{sing} - \lambda I)^{-1} f \quad \text{in } L^2(\Omega, g_s^{1/2}dx),$$

where the convergence is uniform for $\lambda \in \mathbb{K}$, $\mathbb{K}$ being a compact in $\mathbb{C} \setminus \text{spec}(\L_{sing})$.

Let, for $f \in L^2(\Omega \setminus B_2)$

$$\mathcal{R}_{R,m,\epsilon}(\lambda)f = \left((\L_{R,m,\epsilon} - \lambda I)^{-1} f\right)|_{\Omega\setminus B_2} \in \mathring{H}^1(\Omega \setminus B_2), \tag{3.4}$$

where in the right-hand side we continue f by 0 to B_2 and we use similar notation for $\mathcal{R}_{R,m}(\lambda)$, etc. Our next goal is to show that, for $f \in L^2(\Omega \setminus B_2)$ and $\lambda \in \mathbb{K}$,

$$\mathcal{R}_n(\lambda)f \to \mathcal{R}_{sing}(\lambda)f, \quad \|\mathcal{R}_n(\lambda)\|_{L^2\to\mathring{H}^1} < C(\mathbb{K}). \tag{3.5}$$

Here $\mathcal{R}_n(\lambda) = \mathcal{R}_{R(n),m(n),\epsilon(n)}(\lambda)$ and the convergence in (3.5) is the weak convergence in $\mathring{H}^1(\Omega \setminus B_2)$.

Indeed, using Lemmata 2.7, 2.8, [19], for $f \in L^2(\Omega, g_s^{1/2}dx)$ and $\lambda \in \mathbb{K}$,

$$\lim_{n\to\infty} \left(\L_{R(n)} - \lambda I\right)^{-1} f = (\L_{sing} - \lambda I)^{-1} f \quad \text{in } H_0^1(\Omega, g_s^{1/2}dx),$$

and there are $C(\mathbb{K}), R(\mathbb{K}) > 1$ such that, for $R < R(\mathbb{K})$,

$$\| (\L_R - \lambda I)^{-1} \|_{L^2(g_s^{1/2}dx)\to H_0^1(g_s^{1/2}dx)} < C(\mathbb{K}).$$

Since $\sigma_R(x) = \gamma_0$, $g(x) = 1$ for $|x| > 2$, these two equations imply that

$$\mathcal{R}_R(\lambda)f \to \mathcal{R}_{sing}(\lambda)f \text{ in } \mathring{H}^1(\Omega \setminus B_2), \quad \|\mathcal{R}_R(\lambda)\|_{L^2\to\mathring{H}^1} < C(\mathbb{K}). \tag{3.6}$$

Next, using Lemma 2.11, [19], we see that equation (3.6) remains valid if we put in $\mathcal{R}_{R,m}(\lambda)$ instead of $\mathcal{R}_R(\lambda)$ and first take the limit as $m \to \infty$ and then as $R \to 1$. Here $\mathcal{R}_{R,m}(\lambda)$ are defined by (3.4) with $\L_{R,m}$.

At last, by means of Lemma 3.3, [19], we see that (3.6) remains valid, in the sense of the weak-convergence, if we put $\mathcal{R}_{R,m,\epsilon}(\lambda)$ instead of $\mathcal{R}_{R,m}(\lambda)$ and (3.5) follows.

Since $\mathcal{R}^*_{R,\epsilon,m}(\lambda)h = \mathcal{R}_{R,\epsilon,m}(\overline{\lambda})h$, if $h \in L^2(\Omega \setminus B_2)$, it follows from (3.5) that, for $h \in \left(\mathring{H}^1(\Omega \setminus B_2)\right)^*$ and $\lambda \in K$,

$$\lim_{R\to 1} \lim_{m\to\infty} \lim_{\epsilon\to 0} \mathcal{R}_{R,\epsilon,m}(\lambda)h \to \mathcal{R}_1(\lambda)h \text{ in } L^2(\Omega \setminus B_2),$$

in the sense of the weak convergence in $L^2(\Omega\setminus B_2)$ and, when R is sufficiently close to 1, m is sufficiently large and ϵ is sufficiently close to 0,

$$\|\mathcal{R}_n(\lambda)\|_{(\mathring{H}^1(\Omega\setminus B_2))^*\to L^2(\Omega\setminus B_2)} < C(\mathbb{K}), \tag{3.7}$$

Then, similar to the proof of Theorem 4.3, [19], there is a sequence $R(n) \to 1$, $m(n) \to \infty$, $\epsilon(n) \to 0$, such that

$$\langle \mathcal{R}_n(\lambda)h,\, f\rangle \to \langle \mathcal{R}_1(\lambda)h,\, f\rangle, \quad h \in \left(\mathring{H}^1(\Omega \setminus B_2)\right)^*,\ f \in L^2(\Omega \setminus B_2). \tag{3.8}$$

Moreover, operators $\mathcal{R}_n(\lambda)$ satisfy (3.7).

To continue, consider the solutions $u_n^\psi(\lambda)$ and $u_1^\psi(\lambda)$ to the Dirichlet problems (3.2) for $\L_n^\lambda$ and $\L_{sing}^\lambda$ with $\psi \in H^{1/2}(\partial\Omega)$. Using a bounded extension, $\tilde{\psi} \in H^1(\Omega)$, $\text{supp}(\tilde{\psi}) \subset \Omega \setminus B_{5/2}$, we see that $d_n^\psi(\lambda) = \left(u_n^\psi(\lambda) - u_{sing}^\psi(\lambda)\right)|_{\Omega\setminus B_2}$ satisfies

$$d_n^\psi(\lambda) = \left(\mathcal{R}_n(\lambda) - \mathcal{R}_{sing}(\lambda)\right) h,$$

where h is given in terms of the extension as

$$h = \nabla_{A_{out}} \cdot \nabla_{A_{out}} \tilde{\psi} - (q_{out} - \lambda)\tilde{\psi}.$$

Note that since $\mathrm{supp}(h) \subset \Omega \setminus B_{5/2}$, we have $h \in (\mathring{H}^1(\Omega \setminus B_2)^*$. Moreover,

$$\|d_n^\psi(\lambda)\|_{L^2(\Omega\setminus B_2)} < C(\mathbb{K})\|\psi\|_{H^{1/2}(\partial\Omega)}, \quad w - \lim d_n^\psi(\lambda) = 0. \tag{3.9}$$

where $w - \lim$ is the weak limit in $L^2(\Omega \setminus B_2)$.

Now, notice that $Ł^\lambda_{sing}(d_n^\psi(\lambda)) = 0$ on $\Omega \setminus B_2$. Thus, we can use the Cacciopoli inequality (2.11) to obtain that

$$\|d_n^\psi(\lambda)\|_{H^1(\Omega\setminus B_{5/2})} \le \|d_n^\psi(\lambda)\|_{L^2(\Omega\setminus B_2)} < C(\mathbb{K})\|\psi\|_{H^{1/2}(\partial\Omega)} \tag{3.10}$$

and thus, by compactness of the Sobolev embeddding and (3.9) (and Kuratowski-Zorn Lemma), it follows that

$$\limsup_{n\to\infty} \|d_n^\psi(\lambda)\|_{H^1(\Omega\setminus B_{11/4})} \le c \lim_{n\to\infty} \|d_n^\psi(\lambda)\|_{L^2(\Omega\setminus B_{5/2})} = 0 \tag{3.11}$$

We can now mimic the arguments in section 2. Namely, recall that, for $\psi, \varphi \in H^{1/2}(\partial\Omega)$, we have that,

$$\langle (\Lambda^\lambda_{R(n),m(n),\epsilon(n)} - \Lambda^\lambda_{out})\psi, \varphi\rangle$$
$$= \int_{\Omega\setminus B_{11/4}} \left(\nabla_{A_{out}} d_n^\psi \cdot \overline{\nabla_{A_{out}}\tilde{\varphi}} + (q_{out} - \lambda) d_n^\psi \bar{\tilde{\varphi}}\right).$$

Here $\tilde{\varphi}$, $\mathrm{supp}(\tilde{\varphi}) \subset \Omega \setminus B_{11/4}$ is the extension of φ. Thus, cf. the proof of Proposition 2.7 we have that

$$\|(\Lambda^\lambda_{R(n),m(n),\epsilon(n)} - \Lambda^\lambda_{out})\psi\|_{H^{-1/2}(\partial\Omega)} \le C\|d_n^\psi(\lambda)\|_{H^1(\Omega\setminus B_{11/4})} \tag{3.12}$$

and hence (3.11) yields the strong convergence

$$\lim_{n\to\infty} \|(\Lambda^\lambda_{R(n),m(n),\epsilon(n)} - \Lambda^\lambda_{out})\psi\|_{H^{-1/2}(\partial\Omega)} = 0 \tag{3.13}$$

Next, we introduce the intermediate space

Definition 3.2. We denote $L^2_s(\Omega \setminus B_{11/4})$ to be the $L^2(\Omega \setminus B_{11/4})$-closure of the set $\{u \in \mathring{H}^1_{loc}(\Omega \setminus B_{11/4}) : Ł^\lambda_{sing} u = 0\}$.

We factorize the difference of Dirichlet to Neumann maps by

$$\Lambda^\lambda_{R(n),m(n),\epsilon(n)} - \Lambda^\lambda_{out} = \mathcal{T}^\lambda \circ \mathcal{A}^\lambda_n.$$

Exactly as in the end of proof of Theorem 2.1 in section 2, $\mathcal{A}^\lambda_n : H^{1/2}(\partial\Omega) \to L^2_s(\Omega \setminus B_{11/4})$, defined by $A_n(\psi) = d_n^\psi(\lambda)$, is uniformly bounded in n and, due to (3.10), (3.12), $\mathcal{T}^\lambda : L^2_s(\Omega \setminus B_{11/4}) \to H^{-1/2}(\partial\Omega)$ are compact. Since

$$\left(\Lambda^\lambda_{R(n),m(n),\epsilon(n)} - \Lambda^\lambda_{out}\right)^* = \Lambda^{\bar\lambda}_{R(n),m(n),\epsilon(n)} - \Lambda^{\bar\lambda}_{out},$$

this gives rise to the factorization $(\mathcal{A}^{\bar\lambda}_n)^* \circ (\mathcal{T}^{\bar\lambda})^*$ with compact $(\mathcal{T}^{\bar\lambda})^*$. Thus, we can find P_ϵ so that $(\mathcal{T}^{\bar\lambda})^*(I - P_\epsilon)$ is small in norm and prove the theorem. □

Remark 3.3. *By a slight modification of the arguments we can show that Theorem 3.1 remains valid if B_R is changed into an arbitrary smooth Riemannian manifold (M, g) with ∂M diffeomorphic to ∂B_1, cf.* [18].

Remark 3.4. *Similar to Remark 2.11, Theorem 3.1 remains valid for the operator norm in* $H^{1/2}(\partial\Omega)$ *if* σ *and* A *are* $C^{0,1}-$*smooth near* $\partial\Omega$.

4. General Condition

In this section we relax the conditions on the behaviour of σ_n and σ near the boundary under which the $G-$convergence implies the convergences of the Dirichlet-to-Neumann maps in the operator norm. It will be desirable to be able to deal with the situation when, for every n,

$$|\sigma_n(x) - \sigma(x)| \le Cd(x, \partial\Omega)^{1+\epsilon}, \epsilon > 0$$

(this is the condition suggested by G.Alessandrini as mentioned in Introduction), or when, for some $\Omega' \Subset \Omega$, we have that

$$\lim_{n\to\infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega\setminus\Omega')} = 0.$$

As discussed in the introduction, we prove that actually a condition resembling the convergence of the conductivities and their normal derivatives at the boundary and weaker than both conditions above suffices.

Theorem 4.1. *Let for* $\delta > 0$, *let* $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \le \delta\}$. *Assume that*

$$\lim_{\delta\to 0} \delta^{-1} \left(\limsup_{n\to\infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega_\delta)} \right) = 0 \tag{4.1}$$

and that $\sigma_n \in M_K$ *converges to* σ *in the sense of the* $G-$*convergence. Then*

$$\lim_{n\to\infty} \|\Lambda_{\sigma_n} - \Lambda_\sigma\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} = 0.$$

For the sake of simplicity we will consider only the case of the conductivity equation at $\lambda = 0$, however, with the obvious modifications the proof will work as well for the more general operators treated in section 2. We also recall the smooth extensions and restrictions of Sobolev functions to Ω_δ.

Let $\eta_\delta \in C^{0,1}(\Omega)$ be supported in Ω_δ with $\|\nabla\eta_\delta\|_{L^\infty(\Omega_{2\delta})} \le C/\delta$, see (2.16) from the proof of (2.11).

Then set $\psi_\delta = \eta_\delta\tilde{\psi} \in H^1(\Omega)$ and observe that $\text{supp}(\psi_\delta) \subset \Omega_{2\delta}$ and

$$\|\psi_\delta\|_{H^1(\Omega)} \le \|\eta_\delta\|_{C^{0,1}(\Omega)}\|\tilde{\psi}\|_{H^1(\Omega)} \le C\delta^{-1}\|\psi\|_{H^{1/2}(\partial\Omega)}. \tag{4.2}$$

Now recall Caccioppoli estimate (2.12),

$$\int_{\Omega_\delta} |\nabla w|^2 \le C\left(\delta^{-2}\int_{\Omega_{2\delta}} |w|^2 + \int_{\Omega_{2\delta}} |F|^2 + \int_{\Omega_{2\delta}} |f|^2\right), \tag{4.3}$$

where $w \in \mathring{H}^1(\Omega_{2\delta})$ satisfies (2.10) in $\Omega_{2\delta}$.

In order to prove the convergence of the Dirichlet-to-Neumann maps in the operator norm, we treat the current case as a perturbation of the one considered in section 2, where $\sigma_n = \sigma$ near the boundary. As in section 2, we introduce the function $d_n^\psi = u_n^\psi - u^\psi$.

Let us recall that the $G-$convergence implies the convergence of the solutions to the corresponding Dirichlet problems (see [15, Thm22.9]). In the next lemma we give a quick proof under the condition (1.7) valid also for the situation in section 3 where we do not have uniform ellipticity.

Lemma 4.2. *Let σ_n G-converge to σ. Then, for any $\psi \in H^{1/2}(\partial\Omega)$,*

$$\|d_n^\psi\|_{L^2(\Omega)} \to 0, \quad as\ n \to \infty. \tag{4.4}$$

Proof. With ψ_δ as above, we have

$$Ł_n(u_n^\psi - \psi_\delta) = \nabla \cdot \Psi_\delta + \nabla \cdot F_{n,\delta}, \quad Ł(u^\psi - \psi_\delta) = \nabla \cdot \Psi_\delta,$$
$$\Psi_\delta = -\sigma \nabla \psi_\delta, \quad F_{n,\delta} = (\sigma - \sigma_n)\nabla \psi_\delta.$$

Thus,

$$d_n^\psi = (Ł_n^{-1} - Ł^{-1})(\nabla \cdot \Psi_\delta) + Ł_n^{-1}(\nabla \cdot F_{n,\delta}) = I^1_{n,\delta} + I^2_{n,\delta}.$$

Notice that in the case $\sigma = \sigma_n$ on Ω_δ, $I^2_{n,\delta} = 0$. Due to (2.2) and (4.2),

$$\|I^2_{n,\delta}\|_{H^1_0(\Omega)} \le C(K)\|\sigma_n - \sigma\|_{L^\infty(\Omega_{2\delta})}\|\psi_\delta\|_{H^1(\Omega)}$$
$$\le C(K)\delta^{-1}\|\sigma_n - \sigma\|_{L^\infty(\Omega_{2\delta})}\|\psi\|_{H^{1/2}(\partial\Omega)}.$$

Thus, due to condition (1.7), for any $\epsilon > 0$ there are $\delta(\epsilon)$, $n(\delta, \epsilon)$ such that if $\delta < \delta(\epsilon)$, $n > n(\delta, \epsilon)$,

$$\|I^2_{n,\delta}\|_{H^1_0(\Omega)} < \epsilon.$$

Fixing $\delta < \delta(\epsilon)$ and taking into the account that σ_n $G-$converges to σ, we see that

$$\|I^1_{n,\delta}\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

These two equations imply (4.4). □

Next we represent d_n^ψ in $\Omega_{2\delta}$ as

$$d_n^\psi = v_{n,\delta} + m_{n,\delta},$$

where $v^\psi_{n,\delta}$, $m^\psi_{n,\delta}$ are the solutions to the following equations

$$\begin{cases} \nabla \cdot \sigma \nabla v^\psi_{n,\delta} = \nabla \cdot ((\sigma - \sigma_n)\nabla u_n^\psi) \\ v^\psi_{n,\delta} = 0 \text{ on } \partial(\Omega_{2\delta}) \end{cases} \tag{4.5}$$

and

$$\begin{cases} \nabla \cdot \sigma \nabla m^\psi_{n,\delta} = 0 \text{ in } \Omega_{2\delta} \\ m^\psi_{n,\delta} = d_n^\psi \text{ on } \partial(\Omega_{2\delta}). \end{cases} \tag{4.6}$$

Notice that $v^\psi_{n,\delta} \in H^1_0(\Omega_{2\delta})$, $m^\psi_{n,\delta} \in \mathring{H}^1(\Omega_{2\delta})$.

Therefore, using the weak definition of the Dirichlet-to-Neumann map, we have

(4.7)

$$\langle(\Lambda_{\sigma_n} - \Lambda_\sigma)\psi, \varphi\rangle = \int_{\Omega_{2\delta}} (\sigma_n \nabla u_n^\psi - \sigma \nabla u^\psi) \cdot \nabla \varphi_\delta$$
$$= \int_{\Omega_{2\delta}} \sigma \nabla d_n^\psi \cdot \nabla \varphi_\delta + \int_{\Omega_{2\delta}} (\sigma_n - \sigma)\nabla u_h^\psi \cdot \nabla \varphi_\delta$$
$$= \int_{\Omega_{2\delta}} (\sigma_n - \sigma)\nabla u_h^\psi \cdot \nabla \varphi_\delta + \int_{\Omega_{2\delta}} \sigma \nabla v^\psi_{n,\delta} \cdot \nabla \varphi_\delta + \int_{\Omega_{2\delta}} \sigma \nabla m^\psi_{n,\delta} \cdot \nabla \varphi_\delta$$
$$= \langle D_{n,\delta}\psi,\ \varphi\rangle + \langle V_{n,\delta}\psi,\ \varphi\rangle + \langle M_{n,\delta}\psi,\ \varphi\rangle.$$

We summarize the above in the following lemma.

Lemma 4.3. *Let $D_{n,\delta}, V_{n,\delta}, M_{n,\delta} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be defined by the last equation in (4.7). Then, for each $\delta > 0$,*

$$\Lambda_{\sigma_n} - \Lambda_\sigma = D_{n,\delta} + V_{n,\delta} + M_{n,\delta}. \tag{4.8}$$

We start bounding the first and second terms on the above decomposition.

Lemma 4.4. *Let $D_{n,\delta}$ and $V_{n,\delta}$ be defined as above. Then*

$$\begin{aligned} &\|D_{n,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} + \|V_{n,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \\ &\le C\delta^{-1}\|\sigma_n - \sigma\|_{L^\infty(\Omega_{2\delta})}. \end{aligned} \tag{4.9}$$

Proof. The case of $D_{n,\delta}$ follows from Cauchy-Schwartz inequality and (4.2). The definition of $V_{n,\delta}$ implies that

$$\|V_{n,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \le C\delta^{-1} \sup_{\|\psi\|_{H^{1/2}(\partial\Omega)}=1} \|\nabla v^\psi_{n,\delta}\|_{L^2(\Omega_{2\delta})}$$

Using $v^\psi_{n,\delta} \in H^1_0(\Omega_{2\delta})$ as a test function for the equation (4.5), we see that

$$\begin{aligned} &\int_{\Omega_{2\delta}} |\nabla v^\psi_{n,\delta}|^2 \le K \int_{\Omega_{2\delta}} |\sigma\nabla v^\psi_{n,\delta}\cdot\nabla v^\psi_{n,\delta}| \\ &\le K\int_{\Omega_{2\delta}} |(\sigma-\sigma_n)\nabla u^\psi_n \cdot \nabla v^\psi_{n,\delta}| \\ &\le K\|(\sigma-\sigma_n)\nabla u^\psi_n\|_{L^2(\Omega_{2\delta})}\,\|\nabla v^\psi_n\|_{L^2(\Omega_{2\delta})} \\ &\le K\|\sigma-\sigma_n\|_{L^\infty(\Omega_{2\delta})}\,\|\nabla u^\psi_n\|_{L^2(\Omega_{2\delta})}\,\|\nabla v^\psi_n\|_{L^2(\Omega_{2\delta})} \end{aligned} \tag{4.10}$$

Dividing both terms by $\|\nabla v^\psi_{n,\delta}\|_{L^2(\Omega_{2\delta})}$ and recalling that $\|\nabla u^\psi_n\|_{L^2(\Omega)} \le C\|\psi\|_{H^{1/2}(\partial\Omega)}$ the claim follows. □

Note that, by continuing $v^\psi_{n,\delta}$ by 0 into $\Omega\setminus\Omega_{2\delta}$, (4.10) implies that

$$\|v^\psi_{n,\delta}\|_{\mathring{H}^1(\Omega)} \le C\|\sigma-\sigma_n\|_{L^\infty(\Omega_{2\delta})}\,\|\psi\|_{H^{1/2}(\partial\Omega)}. \tag{4.11}$$

Now we deal with the term $M_{n,\delta}$. Namely, we prove the strong uniform boundedness of both $M_{n,\delta}(\psi)$ and $M^*_{n,\delta}(\psi)$.

Lemma 4.5. *Let $\psi \in H^{1/2}(\partial\Omega)$ and $\delta > 0$. Then there exists a constant $C(K)$ such that , for any $\delta > 0$,*

$$\limsup_{n\to\infty}\|M_{n,\delta}(\psi)\|_{H^{-1/2}(\partial\Omega)} \le C\delta^{-1}\limsup_{n\to\infty}\|\sigma_n-\sigma\|_{L^\infty(\Omega_{2\delta})}\|\psi\|_{H^{1/2}(\partial\Omega)}. \tag{4.12}$$

Also,

$$\limsup_{n\to\infty}\|M^*_{n,\delta}(\psi)\|_{H^{-1/2}(\partial\Omega)} \le C\delta^{-1}\limsup_{n\to\infty}\|\sigma_n-\sigma\|_{L^\infty(\Omega_{2\delta})}\|\psi\|_{H^{1/2}(\partial\Omega)}. \tag{4.13}$$

Proof. Notice that, as follows from (4.2) and (4.7), for each $\delta > 0$ and $\varphi \in H^{1/2}(\partial\Omega)$,

$$|\langle M_{n,\delta}(\psi),\varphi\rangle| \le C\delta^{-1}\|\nabla m^\psi_{n,\delta}\|_{L^2(\Omega_\delta)}\|\varphi\|_{H^{1/2}(\partial\Omega)} \tag{4.14}$$

Now we use $m^\psi_{n,\delta} - d^\psi_n = -v^\psi_{n,\delta} \in H^1_0(\Omega_{2\delta})$ as a test function in equation (4.6). By the strong ellipticity we get that

$$\|\nabla m^\psi_{n,\delta}\|_{L^2(\Omega_{2\delta})} \le K^2\|\nabla d^\psi_n\|_{L^2(\Omega_{2\delta})} \le C\|\psi\|_{H^{1/2}(\partial\Omega)}.$$

Now observe that

$$\begin{cases} \nabla\cdot\sigma_n\nabla d_n^\psi = \nabla\cdot(\sigma_n-\sigma)\nabla u^\psi \text{ in } \Omega_{2\delta} \\ d_n^\psi = 0 \text{ on } \partial\Omega. \end{cases}$$

Thus, by (4.3) it follows that

$$\begin{aligned} \|\nabla d_n^\psi\|_{L^2(\Omega_\delta)} &\le \frac{C}{\delta}\|d_n^\psi\|_{L^2(\Omega)} + C\|(\sigma-\sigma_n)\nabla u^\psi\|_{L^2(\Omega_{2\delta})} \\ &\le \frac{C}{\delta}\|d_n^\psi\|_{L^2(\Omega)} + C\|(\sigma-\sigma_n)\|_{L^2(\Omega_{2\delta})}\|\nabla u^\psi\|_{L^2(\Omega)}. \end{aligned}$$

Fix $\delta > 0$ and let n go to ∞. Then, using Lemma 4.2 and (4.14), we arrive at the desired estimate (4.12).

At last, (4.13) follows from (4.12) and (4.9) since, due to the fact that the DN maps are self-adjoint, we have from (4.8) that

$$M_{n,\delta}^* = M_{n,\delta} + V_{n,\delta} + D_{n,\delta} - {V^*}_{n,\delta} - D_{n,\delta}^*.$$

□

Next, following section 2, we reintroduce the following definitions:

Definition 4.6. We denote by $L^2_s(\Omega_{2\delta})$ the $L^2(\Omega_{2\delta})$-closure of the set $\{u \in \mathring{H}^1_{loc}(\Omega_{2\delta}) :\ \nabla\cdot\sigma\nabla u = 0\}$.

We introduce also the modified operators,

$$\mathcal{A}_{n,\delta} : H^{1/2}(\partial\Omega) \to L^2_s(\Omega_{2\delta}), \quad \mathcal{A}_{n,\delta}(\psi) = m^\psi_{n,\delta}, \tag{4.15}$$

and

$$\mathcal{T}^\delta : L^2_s(\Omega_{2\delta}) \to H^{-1/2}(\partial\Omega), \quad \langle\mathcal{T}^\delta(v),\varphi\rangle = \int_{\Omega_{2\delta}} \sigma\nabla v\cdot\nabla\varphi_\delta, \tag{4.16}$$

so that

$$M_{n,\delta} = \mathcal{T}^\delta \circ \mathcal{A}_{n,\delta}. \tag{4.17}$$

Here ψ_δ is defined as in the beginning of this section.

Lemma 4.7. *The operators $\mathcal{T}^\delta$ are compact and, for any fixed $\delta > 0$, the operators $\mathcal{A}_{n,\delta}$ are uniformly bounded wrt n.*

Proof. As in section 2, we notice that the Caccioppoli inequality, Lemma 2.5 and the compactness of the Sobolev embedding imply that the restriction operator,

$$R_\delta : L^2_s(\Omega_{2\delta}) \to \mathring{H}^1(\Omega_{3\delta/2}) \to L^2_s(\Omega_{3\delta/2}) \to \mathring{H}^1(\Omega_\delta),$$

is a bounded and, due to the second embedding above, compact operator. Now the operator $\widetilde{\mathcal{T}}^\delta : H^1(\Omega_\delta) \to H^{-1/2}(\partial\Omega)$ defined by

$$\langle\tilde{T}_\delta(v),\varphi\rangle = \int_{\Omega_\delta} \sigma\nabla v\cdot\nabla\varphi_{\delta/2}, \quad \varphi\in H^{1/2}(\partial\Omega),$$

is continuous. Hence $\mathcal{T}^\delta = \widetilde{\mathcal{T}}^\delta \circ R_\delta$ is also compact.

As for the uniform boundedness, for a fixed δ, of the operators $\mathcal{A}_{n,\delta}$, it follows from the decomposition of $A_{n,\delta}$ in form:

$$\psi \in H^{1/2} \mapsto d^\psi_{n,\delta} \in H^1(\Omega) \mapsto \mathrm{trace}(d_n^\psi) \in H^{1/2}(\partial\Omega_{2\delta}) \mapsto m^\psi_{n,\delta} \in L^2(\Omega_{2\delta}).$$

□

We can now return to the uniform estimate for the operators $M_{n,\delta}$.

Lemma 4.8.

$$\limsup_{n\to\infty} \|M_{n,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \le C\delta^{-1} \limsup_{n\to\infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega_{2\delta})}. \tag{4.18}$$

Proof. By (4.17), it follows from Lemma 4.7 that, for any $\epsilon > 0$, $\delta > 0$, there is a finite dimensional projector $P_{\epsilon,\delta} : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ such that

$$\|M^*_{n,\delta} \circ (I - P_{\epsilon,\delta})\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \le \epsilon. \tag{4.19}$$

Then, by using estimate (4.13) and the fact that $Range(P_{\epsilon,\delta})$ has finite dimension, it follows that, for a fixed δ,

$$\limsup_{n\to\infty} \|M^*_{n,\delta} \circ P_{\epsilon,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \le C\delta^{-1} \limsup_{n\to\infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega_{2\delta})}. \tag{4.20}$$

Note that the constant C in the above estimate is chosen independent of ϵ, δ.

Thus, from (4.19) and (4.20), we have

$$\limsup_{n\to\infty} \|M^*_{n,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \le C\delta^{-1} \limsup_{n\to\infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega_{2\delta})} + \epsilon,$$

for arbitrary $\epsilon > 0$. Since

$$\|M^*_{n,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} = \|M_{n,\delta}\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)},$$

the claim follows. □

To complete the proof of Theorem 4.1 , by Lemmata 4.3, 4.4 and 4.8 we get that, for every $\delta > 0$,

$$\limsup_{n\to\infty} \|\Lambda_{\sigma_n} - \Lambda_\sigma\|_{H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)} \le C\delta^{-1} \limsup_{n\to\infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega_{2\delta})}.$$

We complete this section by an example which shows that, in order to achieve the uniform convergence of the DN maps, one should indeed control the behavior of σ_n in some vicinity of $\partial\Omega$ and to see, in fact, that the control of σ_n and even all its derivatives on $\partial\Omega$ is not sufficient.

Theorem 4.9. *For any $\alpha > 0$, there exists a sequence $\sigma_n \in M_{1+\alpha}(B(0,1))$, $\sigma_n = I$ on Ω_{δ_n}, $\delta_n \to 0$, such that $\sigma_n \to I$ in the sense of the $G-$convergence, but*

$$\limsup \|(\Lambda_{\sigma_n} - \Lambda_I)\|_{H^{1/2}\to H^{-1/2}} > \frac{\alpha}{16(2+\alpha)}.$$

Proof. Take $\alpha > 0$, $\Omega = B(0,1) \subset \mathbb{R}^2$, and consider the family of isotropic conductivities, σ_R, $R < 1$,

$$\sigma_R = \gamma_R\, I = (\chi_{B(0,1)}(x) + \alpha\chi_{B(0,R)-B(0,R^2)}(x))\, I$$

For these conductivities we have, on one hand, that when $R \to 1$, then σ_R G-converge to I. On the other hand, we have the expression,

$$\langle(\Lambda_R - \Lambda_1)e^{ik\theta}, e^{il\theta}\rangle = \delta_k^l |k| m_k,$$

where

$$m_k = \frac{2\alpha(2+\alpha)(R^{2|k|} - R^{4|k|})}{(2+\alpha)^2 - \alpha^2 R^{2|k|} - \alpha(2+\alpha)(R^{2|k|} - R^{4|k|})}.$$

Taking the $H^\alpha-$norm on $\partial B(0,1)$ of the form

$$\|u\|_{H^\alpha}^2 = |u_0|^2 + \sum_{k\neq 0} |k|^{2\alpha}|u_k|^2, \quad \text{if } u = \sum u_k e^{ik\theta},$$

we thus have

$$\|(\Lambda_R - \Lambda_1)e^{ik\theta}\|_{H^{-1/2}} = |k|^{1/2} m_k = |m_k| \|e^{ik\theta}\|_{H^{1/2}}.$$

Then, assuming $R > (3/4)^{1/4}$ and choosing $k = [\frac{-1}{2\log_2 R}]$, we see that

$$\|(\Lambda_R - \Lambda_1)\|_{H^{1/2}\to H^{-1/2}} > \frac{\alpha}{16(2+\alpha)}.$$

Hence, choosing $\sigma_n = \sigma_{R(n)}$, $R(n) \to 1$ as $n \to \infty$, we see that there is no convergence $\Lambda_n \to \Lambda_I$ in the operator norm. □

5. Stability with respect to the G-convergence

Stability with respect to the G-convergence has been proved in 2D by Alessandrini and Cabib in [2] assuming further that $\nabla \cdot \sigma = 0$. As discussed also in that paper, the lack of uniqueness in the Calderón problem in the anisotropic case prevents stability in the general case. Compactness arguments show that this is the only obstruction. Moreover in dimension 2, it is known [8, Theorem 1] that the lack of uniqueness is due to a quasiconformal change of variables. Since changes of variables preserve the G-convergence, the unconditional stability, in the introduction, follows. We will first recall the basic definitions, then prove that indeed the G-convergence is preserved by the change of variables and finally will combine it all to prove Theorem 1.3.

For a constant $K \geq 1$, a K-quasiconformal mappings is a homeomophism $F : \mathbb{C} \to \mathbb{C}$ which belongs to $W^{1,2}_{\text{loc}}(\mathbb{C})$ and such that

$$\|DF(x)\|^2 \leq K J_F, \quad J_F = \det(DF). \tag{5.1}$$

Given $\sigma \in M_K(\Omega)$, its associated quadratic form $l_\sigma : H^1_0(\Omega) \to \mathbb{R}$ is defined by

$$l_\sigma[u] = \int_\Omega \langle \sigma \nabla u, \nabla u\rangle,$$

cf. (2.8). Let $F = I$ at $\partial\Omega$. Then $F_*(\sigma)$ is formally given by

$$l_{F_*(\sigma)}(u,v) = l_\sigma(F^*(u), F^*(v)) = l_\sigma(u \circ F, v \circ F). \tag{5.2}$$

Expressing the push forward F_* in coordinates, we have

$$F_*(\sigma)(y) = J_F^{-1}(x) DF(x) \sigma DF(x)^t_{|F^{-1}(y)=x}.$$

It is straightforward to see that that, if F is $K-$quasiconformal, then

$$F_* : M_K(\Omega) \to M_{K^2}(\Omega). \tag{5.3}$$

Together with the fact that $F = I$ on $\partial\Omega$, this implies that F_* is bijective in $H^1_0(\Omega)$ and, by duality, in $H^{-1}(\Omega)$. Explicitly, if $f \in H^{-1}(\Omega)$ we solve

$$\triangle v = f, \quad f \in H^{-1}(\Omega) \quad \text{with } v \in H^1_0(\Omega), \tag{5.4}$$

and notice that

$$F^*(f) = \nabla \cdot (F_*(I)\nabla(v \circ F^{-1})) \in H^{-1}(\Omega). \tag{5.5}$$

Now, since $\langle f, F^*(\varphi)\rangle = \langle F^*(f), \varphi\rangle$ it follows that

$$L^{-1}_{F_*(\sigma)}(F^*(f)) = F^*(L^{-1}_\sigma(f)) \tag{5.6}$$

Lemma 5.1. *Let F be a quasiconformal homeomorphism fixing the boundary of a planar domain Ω. Let $\sigma_n, \sigma \in M_K(\Omega)$. Then $\sigma_n \to \sigma$ in the sense of G convergence if and only if $F_*(\sigma_n) \to F_*(\sigma)$ in the sense of G convergence.*

Proof. By the definition of the G-convergence and (5.6) it is enough to show that if $u_n \in H^1_0(\Omega)$ converges weakly to u then $F^*(u_n)$ converges weakly to $F^*(u)$. Since $F_* : H^1_0(\Omega) \to H^1_0(\Omega)$, there is $C > 0$ such that

$$\|u_n \circ F^{-1}\|_{H^1_0} \le C.$$

By the weak compactness of $H^1_0(\Omega)$, it follows that the sequence $u_n \circ F^{-1}$ has a subsequence converging weakly in H^1_0 and, therefore, strongly in L^2, to some $v \in H^1_0$. On the other hand, the subsequence of u_n has a further subsequence which converges almost everywhere to u and thus $F^*(u_n) \to F^*(u)$ almost everywhere. By the uniqueness of the weak limits $v = F^*(u)$. □

Proof of Theorem 1.3 Due to Lemma 2.4, it is sufficient to prove that (1.12) implies (1.13).

Recall that M_K is compact ([15, Theorem 22.3]) and metrizable ([15, Corollary 10.23]) with respect to the G-topology. Hence, if (1.12) is valid, the sequence σ_n (and any its subsequence) has a subsequence $\sigma_{n(k)}$, $k = 1, 2, \ldots,$ which converges in the $G-$sense to a limit conductivity denoted by $\tilde\sigma$, $\tilde\sigma \in M_K$. It then follows from the weak definition of the DN map, (2.5), that

$$\Lambda_\sigma = \Lambda_{\tilde\sigma}.$$

Thus, since the G-topology is metrizable, if we define, for $\tilde K > 1$,

$$M_{\tilde K}(\sigma) = \{\tilde\sigma \in M_{\tilde K} : \Lambda_\sigma = \Lambda_{\tilde\sigma}\}, \tag{5.7}$$

it follows that

$$\lim_{n\to\infty} d_G(\sigma_n, M_K(\sigma)) = 0,$$

where d_G is a distance in M_K inducing the $G-$topology.

The above arguments work in any dimension but in 2D the sets $M_{\tilde K}(\sigma)$ are described in [8, Theorem 1]. Namely,

$$M_{\tilde K}(\sigma) = \{\tilde\sigma \in M_{\tilde K} : \tilde\sigma = F^*(\sigma) \text{ for some quasiconformal map } F \text{ with } F_{|\partial\Omega} = I\}, \tag{5.8}$$

Thus, it is enough to prove that

$$\lim_{n\to\infty} d_G(\sigma, M_{K^2}(\sigma_n)) = 0,$$

see (5.3). We prove this by contradiction. Assume that there is $\epsilon > 0$ and a subsequence $n(k)$, such that

$$d_G(\sigma, M_k) > \epsilon, \text{ where } \quad M_k = M_{K^2}(\sigma_{n(k)}).$$

Since $M_{K^2}(\Omega)$ is G-compact there is a non-relabelled subsequence of $\sigma_{n(k)}$ which is $G-$convergent to $\tilde{\sigma} \in M_{K^2}$. Since, for any $\tilde{\sigma}_n \in M_{K^2}(\sigma_n)$, we have $\Lambda_{\tilde{\sigma}_n} = \Lambda_{\sigma_n}$, and $\Lambda_{\sigma_n} \to \Lambda_\sigma$ in the weak sense, we see that $\Lambda_{\tilde{\sigma}} = \Lambda_\sigma$. By [8], there is a quasiconformal F, with $F = \mathrm{id}$ on $\partial\Omega$, such that

$$F_*(\tilde{\sigma}) = \sigma.$$

Now Lemma 5.1 implies that $F_*(\sigma_{n_k}) \in M_k$ converges to σ. This is a contradiction.

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