CONVEX BODIES, ECONOMIC CAP COVERINGS, RANDOM POLYTOPES

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§1. *Introduction.* Let K be a convex compact body with nonempty interior in the d-dimensional Euclidean space R^d and let $x_1, \ldots, x_n$ be random points in K, independently and uniformly distributed. Define $K_n = \text{conv}\{x_1, \ldots, x_n\}$. Our main concern in this paper will be the behaviour of the deviation of vol K_n from vol K as a function of n, more precisely, the expectation of the random variable vol $(K \backslash K_n)$. We denote this expectation by $E(K, n)$.

There are few results known about $E(K, n)$, mainly when $d = 2$. (The case $d = 1$ is trivial.) Rényi and Sulanke [**18, 19**] proved that for smooth enough convex bodies $K \subset R^2$

$$E(K, n) \approx \text{const}\,(K) n^{-2/3}, \tag{1.1}$$

where const (K) denotes a constant depending on K only and the notation $f(n) \approx g(n)$ means that f and g are asymptotically equal, *i.e.*, $\lim f(n)/g(n) = 1$ when $n \to \infty$. This has been extended to smooth enough convex bodies $K \subset R^3$ by Wieacker [**24**], who obtained

$$E(K, n) \approx \text{const}\,(K) n^{-1/2}. \tag{1.2}$$

Buchta [**4**], see also Renyi and Sulanke [**18**], proved that for a convex polygon $P \subset R^2$

$$E(P, n) \approx \text{const}\,(P) n^{-1} \log n. \tag{1.3}$$

Little is known in R^d. Wieacker [**24**] determined $E(B^d, n)$ where B^d denotes the unit ball of R^d, obtaining

$$E(B^d, n) \approx \text{const}\,(d) n^{-2/(d+1)}. \tag{1.4}$$

Groemer [**11**] proved that for a convex compact body $K \subset R^d$ with vol $K = \text{vol}\, B^d$

$$E(K, n) \leq E(B^d, n), \tag{1.5}$$

with equality, if, and only if, K is an ellipsoid.

Until quite recently, nothing has been known about the case of general d-polytopes when $d > 2$. Buchta [**5**] proved $E(T, n) \approx \frac{3}{4} n^{-1} (\log n)^2$ where T denotes the three dimensional simplex. Dwyer, Kannan and Lovász [**14**] proved that

$$E(P, n) \leq \text{const}\,(P) n^{-1} (\log n)^d$$

for a polytope $P \subset R^d$. This was improved later by Dwyer [**8**] to

$$E(P, n) \leq \text{const}\,(P) n^{-1} (\log n)^{d-1}. \tag{1.6}$$

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They also proved that for a polytope P having a simple vertex (*i.e.*, a vertex where exactly d facets meet)

$$E(P, n) \geqslant \text{const}\,(P) n^{-1} (\log n)^{d-1}. \tag{1.7}$$

For further information on the expectation of the number of vertices, surface area, mean width, *etc.* of K_n we refer the reader to Buchta [**5**], Dwyer [**8**], Gruber [**12**], Schneider [**22**].

We are going to relate $E(K, n)$ to another quantity which we now describe. First, define a map $v: K \to R$ as

$$v(x) = \min \{\text{vol}\,(K \cap H) \colon x \in H,\ H \text{ a halfspace}\}.$$

Next, for $\varepsilon > 0$ define

$$K(v \leqslant \varepsilon) = \{x \in K \colon v(x) \leqslant \varepsilon\}.$$

Sometimes we will write $K(\varepsilon)$ as a shorthand for $K(v \leqslant \varepsilon)$. Here our main result is

THEOREM 1. *Assume K is a convex compact body in R^d with* vol $K = 1$. *Then, for $n \geqslant n_0(d)$ we have*

$$\text{const vol}\, K(1/n) \leqslant E(K, n) \leqslant \text{const}\,(d) \text{ vol}\, K(1/n). \tag{1.8}$$

Theorem 1 means that $E(K, n)$ is of the same order of magnitude as vol $K(1/n)$. We will write this as vol $K(1/n) \sim E(K, n)$ so the notation $f(n) \sim g(n)$ means that $\liminf f(n)/g(n) > 0$ and $\liminf g(n)/f(n) > 0$. This notation implies two constants that are independent of n. We mention that in Theorem 1 the constants are independent of K as well. Actually, one of them is universal and the other depends on d only.

Theorem 1 can be used to determine the order of magnitude of $E(K, n)$ for different classes of convex bodies in R^d. First we prove a general upper bound for vol $K(1/n) \sim E(K, n)$.

THEOREM 2. *Let $K \subset R^d$ be a convex compact body with* vol $K = 1$ *and let $\varepsilon > 0$. Then*

$$\text{const}\,(d) \varepsilon (\log\,(1/\varepsilon))^{d-1} \leqslant \text{vol}\, K(\varepsilon). \tag{1.9}$$

This theorem is best possible (apart from the constant) as shown by the polytopes.

THEOREM 3. *Let $P \subset R^d$ be a polytope with* vol $P = 1$ *and let $\varepsilon \geqslant 0$. Then*

$$\text{vol}\, P(\varepsilon) \leqslant \text{const}\,(P) \varepsilon (\log\,(1/\varepsilon))^{d-1}. \tag{1.10}$$

Theorem 2 and 3 show that vol $P(\varepsilon) \sim \varepsilon(\log\,(1/\varepsilon))^{d-1}$ with the implied constant depending on P. This, together with Theorem 1 proves that for the class of polytopes $E(P, n) \sim n^{-1}(\log n)^{d-1}$. This result has been obtained independently by Dwyer [**8**]. The other extreme class of convex bodies is that of the smooth ones. We state an asymptotic result for this class without proof (see Leichweiss [**26**]).

THEOREM. *For a $\mathscr{C}^3$ convex body $K \subset R^d$ with* vol $K > 0$ *and positive Gaussian curvature κ and for $\varepsilon > 0$ we have*

$$\text{vol } K(\varepsilon) \approx \text{const } (d)\left(\int_{\delta K} \kappa^{1/(d+1)} dS \right) \varepsilon^{2/(d+1)}, \tag{1.11}$$

where the integration is taken on the boundary, δK, of K.

This theorem was also proved by Buchta, Gruber, Müller [6]. They noticed that the right-hand side here is a constant multiple of the affine surface area of K (*cf.* Blaschke [3]) and so Blaschke's affine isoperimetric inequality implies that among all $\mathscr{C}^3$ convex bodies of unit volume vol $K(\varepsilon)$ is the largest for the ellipsoids.

Theorem 1, the Theorem above and Groemer's result (1.5) show that for a $\mathscr{C}^3$ convex body $K \subset R^d$, $E(K, n) \sim n^{-2/(d+1)}$ with the implied constants depending on K. We are going to prove a theorem that will also yield this. Some preparations are needed. We write $B(\rho, x)$ for the ball of radius ρ and with centre $x \in R^d$. Let p be a point on the boundary δK of the convex compact set $K \subset R^d$. Assume there is a unique outer normal a (with $|a| = 1$) to K at p. Then we call the point p ρ-circular if $\rho > 0$ and

$$K \subset B(\rho, p - \rho a).$$

The set of points that are ρ-circular for some $\rho > 0$ are called circular.

THEOREM 4. *If the set of circular points of the boundary of K has positive $(d-1)$-dimensional measure in δK, then*

$$\text{vol } K(\varepsilon) \geqslant \text{const } (K) \varepsilon^{2/(d+1)}.$$

It is clear that for smooth enough ($\mathscr{C}^3$, say) convex bodies the conditions of Theorem 4 are satisfied. Thus for smooth convex bodies K we have from Theorem 1, 4 and (1.5) that $E(K, n) \sim n^{-2/(d+1)}$.

What happens between these two extreme classes is not a mystery: it is the usual unpredictable behaviour. Using (1.5), (1.10) and a general theorem of Gruber [13] (see Schneider [22] for a similar application) one can show this.

THEOREM 5. *Assume $\omega(n) \to 0$ and $\Omega(n) \to \infty$ as $n \to \infty$. Then for most (in the Baire-category sense) convex bodies $K \subset R^d$ with* vol $K = 1$ *one has, for infinitely many n*

$$E(K, n) < n^{-1} (\log n)^{d-1} \Omega(n),$$

and also, for infinitely many n

$$E(K, n) > n^{-2/(d+1)} \omega(n).$$

§2. *Economic cap coverings.* One of the main tools in proving Theorem 1 will be the construction of an economic cap covering of $K(v \leqslant \varepsilon)$. For a closed halfpsace H the set $K \cap H$ is called a cap of K whenever it is nonempty.

THEOREM 6. *Assume a convex body* $K \subset R^d$ *is given with* vol $K > 0$. *Take* ε *with* $0 < \varepsilon < \varepsilon_0(d) = (2d)^{-2d}$. *Then there are caps* $K_1, \ldots, K_m$ *of* K *and pairwise disjoint subsets* $K'_1, \ldots, K'_m$, $K'_i \subset K_i$ $(i = 1, \ldots, m)$ *such that*

(i) $\bigcup_1^m K'_i \subset K(v \leqslant \varepsilon) \subset \bigcup_1^m K_i$,

(ii) vol $K_i \leqslant 6^d \varepsilon \qquad (i = 1, \ldots, m)$,

(iii) vol $K'_i \geqslant (6d)^{-d} \varepsilon \qquad (i = 1, \ldots, m)$.

This is what is called economic cap covering in the title. In Ewald, Larman, Rogers [9] there is another economic cap covering theorem for the inner parallel body of K (instead of $K(v \leqslant \varepsilon)$). Our proof of Theorem 6 is an adaptation of the one in Ewald, Larman, Rogers [9].

Actually, $K \setminus K(v \leqslant \varepsilon)$ is a certain kind of "inner parallel" body to K. One may wonder then if its volume is a convex function of ε or not, or, what is the same, if vol $K(v \leqslant \varepsilon)$ is a concave function of ε or not. Maybe the d-th root of vol $K(v \leqslant \varepsilon)$ is concave. We do not know the answer to these questions. However we can prove some concavity type property of vol $K(v \leqslant \varepsilon)$ that will be useful.

THEOREM 7. *Under the assumptions of Theorem* 6 *one has, for* $\lambda \geqslant 1$,

$$\text{vol } K(v \leqslant \varepsilon) \geqslant \text{const } (d) \lambda^{-d} \text{ vol } K(v \leqslant \lambda \varepsilon). \tag{2.1}$$

We mention a Heilbronn type consequence of Theorem 6.

THEOREM 8. *Assume* $P \subset R^d$ *is a convex polytope with* n *vertices and* vol $P > 0$. *Then* P *has* $(d+1)$ *vertices* $x_0, x_1, \ldots, x_d$ *such that*

$$\text{vol } (\text{conv } \{x_0, \ldots, x_d\}) / \text{vol } P \leqslant \text{const } (d) n^{-(d+1)/(d-1)}. \tag{2.2}$$

This is a Heilbronn type result (*cf.* [**15, 20**]) because it says that among n points in convex position in R^d there is a simplex with small relative volume. This result is known in the plane in a sharper form, see Rényi, Sulanke [**19**].

Theorem 8 is related to a theorem of Arnold [**1**] (when $d = 2$) and Konyagin, Sevastyanov [**16**] (when $d > 2$) which states that for a lattice polytope $P \subset R^d$ with n vertices and positive volume one has

$$\text{const } (d) n^{(d+1)/(d-1)} \leqslant \text{vol } P. \tag{2.3}$$

Theorem 8 can be regarded as an extension of (2.3) to the case of general (non-lattice) polytopes. Actually, (2.2) implies (2.3) if the lattice polytope P has no $d+1$ vertices on a hyperplane because then in the left-hand side of (2.2) the volume of the simplex is at least $1/(d!)$. In fact, the results of Arnold and Konyagin, Sevastyanov are contained in the results of G. E. Andrews [**25**].

§3. *Notation, definitions, basic properties.* A cap C of K is a set $C = K \cap H$ where H is a closed halfspace and $K \cap H$ is nonempty. Then $H = \{x \in R^d : a \,.\, x \leqslant \alpha\}$ for some $a \in R^d$ with $|a| = 1$ and $\alpha \in R^1$. Here $a \,.\, x$ denotes the scalar product of a and $x \in R^d$. It will be convenient to write $H = H(a, t)$ with $t = h(a) - \alpha$ where $h(a) = \max \{a \,.\, x : x \in K\}$ is the support function of K. With this notation t is the width of the cap C in direction a. In the same spirit we write $H(a, t_1, t_2)$ for the strip between the hyperplanes $H(a, t_1)$ and $H(a, t_2)$.

For a cap $C = K \cap H(a, t)$ a point $z \in C$ is called the centre of C if $a \,.\, z = h(a)$. A cap may have several centres but we think of a cap as having a fixed centre, say, the centre of gravity of all centres.

For a cap $C = K \cap H(a, t)$ with centre z we define (when $\lambda \geq 0$)

$$C^\lambda = z + \lambda(C - z). \tag{3.1}$$

Obviously $C^1 = C$. It is easy to see that for $\lambda \geq 1$ one has

$$C^\lambda \supset K \cap H(a, \lambda t). \tag{3.2}$$

When $x \in K$, a minimal cap is defined as a cap $C(x)$ with $x \in C(x)$ and

$$\text{vol } C(x) = v(x) = \min \{\text{vol } H \cap K : x \in H \text{ a halfspace}\}.$$

Let us write $H(a = t)$ for the bounding hyperplane of the halfspace $H(a, t)$. A standard variational argument shows that for a minimal cap

$$C(x) = K \cap H(a, t)$$

the point x is the centre of gravity of the section $K \cap H(a = t)$.

For $x \in K$ and $\lambda > 0$ we call the set

$$M(x, \lambda) = M_K(x, \lambda) = x + \lambda\{(K - x) \cap (x - K)\} \tag{3.3}$$

a Macbeath region. Such regions were studied by A. M. Macbeath [**17**] and Ewald, Larman, Rogers [**9**]. A Macbeath region is obviously convex and centrally symmetric with centre x. We will write $M(x) = M_K(x) = M(x, 1)$ when convenient. Define a map $u : K \to R$ as

$$u(x) = \text{vol } M(x).$$

Macbeath [**17**] has shown that the set $K(u \geq \varepsilon) = \{x \in K : u(x) \geq \varepsilon\}$ is convex. The convexity of the set $K(v \geq \varepsilon)$ is trivial because it is the intersection of closed halfspaces. It turns out that $K(v \geq \varepsilon)$ is "close" to $K(u \geq \varepsilon)$.

THEOREM 9. *Assume* $0 < \varepsilon < \varepsilon_1(d)$. *Then*

$$K(v \leq \varepsilon) \subset K(u \leq 2\varepsilon) \subset K(v \leq 2(3d)^d \varepsilon)$$

and

$$\text{vol } K(v \leq \varepsilon) \leq \text{vol } K(u \leq 2\varepsilon) \leq c_1(d) \text{ vol } K(v \leq \varepsilon)$$

where $\varepsilon_1(d)$ *and* $c_1(d)$ *are constants depending on d only.*

Here one can take

$$\varepsilon_1(d) = \tfrac{1}{2}(12d^3)^{-d}. \tag{3.4}$$

We will postpone the proof of this theorem till the last section because we will not use it in the paper.

Denote by $B(r)$ or $B^d(r)$ the ball of radius r and centre 0 in R^d. Throughout the paper we will assume that the given compact convex body $K \subset R^d$ (with vol $K>0$) is in "standard form", *i.e.*,

$$B(r)\subset K\subset B(R) \quad \text{and} \quad dr\geqslant R. \tag{3.5}$$

It is well-known (see [7] for instance) that any convex compact body can be transformed by a volume preserving affine transformation into a body K in standard form. Further, it is clear that such a transformation does not change the quantities vol $K(v\leqslant\varepsilon)$, vol $K(u\leqslant\varepsilon)$ or $E(K, n)$ when vol $K=1$.

The assumption vol $K=1$ in the theorems is made for convenience. What we really need is vol $K>0$. At some points we will have to consider sets K with vol $K\neq 1$. Then vol $K(v\leqslant\varepsilon)$ is not affine invariant and it is better to consider instead

$$\text{vol } K(v\leqslant\varepsilon \text{ vol } K)/\text{vol } K \tag{3.6}$$

which is affine invariant.

§4. *Proof of Theorem* 6. We start with two lemmas.

LEMMA 1. $u(x)\leqslant 2v(x)$.

Proof. Take a halfspace H with $x\in H$. Then

$$u(x)=\text{vol } M(x)\leqslant 2 \text{ vol } (M(x)\cap H)\leqslant 2 \text{ vol } (K\cap H),$$

so

$$u(x)\leqslant 2 \min\{\text{vol } (K\cap H)\colon x\in H\}=2v(x).$$

LEMMA 2. $v(x)\leqslant(3d)^d u(x)$ *if* $v(x)\leqslant(2d)^{-2}d$ *or if* $u(x)\leqslant(12d^3)^{-d}$.

Proof. We prove first that $v(x)\leqslant(2d)^{-2d}$ implies $v(x)\leqslant(3d)^d u(x)$.

Take a minimal cap $C(x)=K\cap H(a, t)$. As we mentioned earlier x is the centre of gravity of the section $K\cap H(a=t)$. Then, by Lemma 2 of Ewald, Larman, Rogers [**9**],

$$C(x)\subset M(x, 3d) \tag{4.1}$$

provided $B(r/2)\cap H(a, t)$ is empty and $t\leqslant r/4$.

Assume now that (4.1) fails. Then either $B(r/2)\cap H(a, t)$ is nonempty or $t\geqslant r/4$. We show now that both cases contradict the condition $v(x)\leqslant(2d)^{-2d}$.

In the first case, *i.e.*, when $B(r/2)\cap H(a, t)\neq\varnothing$, the set $B(r)\cap H(a, t)$ contains a cap C_r of $B(r)$ whose width is $r/2$. Moreover, by (3.5),

$$C=K\cap H(a, t)\supset B(r)\cap H(a, t)\supset C_r$$

so vol $C(x)\geqslant$ vol C_r. A simple computation shows now that vol $C_r\geqslant(2d)^{-d}$.

In the second case when $t\geqslant r/4$, *i.e.*, the width of $C(x)$ in direction a is at least $r/4$, let z be the centre of $C(x)$. Consider the cone L with apex z whose base is the intersection of $B(r)$ with the hyperplane through 0 and orthogonal to a. The height of this cone is $h(a)\leqslant R$ and its volume is

$$\text{vol } L=(1/d)r^{d-1}\omega_{d-1}h(a)$$

where ω_{d-1} is the volume of B^{d-1}, the unit ball of R^{d-1}. The cap $C(x)$ contains the part of this cone L lying in the strip $H(a, 0, t)$. The volume of this part is

$$\left(\frac{t}{h(a)}\right)^d \operatorname{vol} L = \left(\frac{t}{h(a)}\right)^{d-1} \frac{t}{d} r^{d-1}\omega_{d-1} \geqslant \left(\frac{r}{4R}\right)^{d-1} \frac{r}{4d} r^{d-1}\omega_{d-1}$$
$$\geqslant \left(\frac{1}{4d}\right)^{d-1} \frac{1}{4d} d^d r^d \omega_d(\omega_{d-1}/\omega_d) d^{-d}$$
$$\geqslant (4d)^{-d} R^d \omega_d(\omega_{d-1}/\omega_d) d^{-d} \geqslant (2d)^{-2d}(\omega_{d-1}/\omega_d)$$
$$\geqslant (2d)^{-2d}.$$

So $\operatorname{vol} C(x) \geqslant (2d)^{-2d}$. This contradiction shows that (4.1) holds. Then obviously, $v(x) \leqslant (3d)^d u(x)$.

To finish the proof of the lemma we prove now that $u(x) \leqslant (12d^3)^{-d}$ implies $v(x) \leqslant (2d)^{-2d}$. To see this we claim that

$$K(v \geqslant (2d)^{-2d}) \subset K(u \geqslant (12d^3)^{-d}).$$

Both sets here are convex (the second by Macbeath's result [**17**]) and both of them contain the origin. When x is a point on the boundary of $K(v \geqslant (2d)^{-2d})$, *i.e.*, when $v(x) = (2d)^{-2d}$, then by the first part of this proof,

$$u(x) \geqslant (3d)^{-d} v(x) = (12d^3)^{-d},$$

i.e., $x \in K(u \geqslant (12d^3)^{-d})$.

Now we turn to the proof of Theorem 6. Consider the set $K(v \geqslant \varepsilon)$ and choose a maximal system of points $x_1, x_2, \ldots, x_m$ from the boundary $\partial K(v \geqslant \varepsilon)$ of the set $K(v \geqslant \varepsilon)$ subject to the condition that

$$M(x_i, \tfrac{1}{2}) \cap M(x_j, \tfrac{1}{2}) = \varnothing \qquad \text{when} \qquad x_i \neq x_j. \tag{4.2}$$

This maximal system is indeed finite because the sets $M(x_i, \frac{1}{2})$ are pairwise disjoint, all of them lie in K and

$$\operatorname{vol} M(x_i, \tfrac{1}{2}) = 2^{-d} \operatorname{vol} M(x_i, 1) = 2^{-d} u(x_i) \geqslant (6d)^{-d} v(x_i) = (6d)^{-d}\varepsilon \tag{4.3}$$

according to Lemma 2.

CLAIM 1. $K(v \leqslant \varepsilon) \subset \bigcup\{M(x_i, 5): i = 1, \ldots, m\}$.

Proof. Consider any point $y'' \in K(v \leqslant \varepsilon)$. As $0 \in \operatorname{int} K(v \geqslant \varepsilon)$, the halfline stemming from 0 in direction y'' intersects the boundary of the convex set $K(v \geqslant \varepsilon)$ and K at the points y and y', respectively. Now $x_1, \ldots, x_m$ form a maximal system in $\delta K(v \geqslant \varepsilon)$ with respect to (4.2) and $y \in \delta K(v \geqslant \varepsilon)$. So there is an i such that

$$M(x_i, \tfrac{1}{2}) \cap M(y, \tfrac{1}{2}) \neq \varnothing.$$

Then, by Lemma 1 of Ewald, Larman, Rogers [**9**],

$$M(y, 1) \subset M(x_i, 5). \tag{4.4}$$

We will prove now that $y' \in M(y, 1)$. This will show that the line segment $[y, y']$ and, consequently, the point $y'' \in [y, y']$ lie in $M(y, 1)$ and this will prove the Claim.

Assume $y' \notin M(y, 1)$. On the line through 0 and y let z be the point at distance r from 0 and such that 0 lies between z and y. Then $z \in B(r) \subset K$ and so $y' \notin M(y, 1)$ implies

$$|z-y| < |y-y'|.$$

Consider now the minimal cap $C = C(y) = K \cap H$. Clearly, H cannot contain 0 for otherwise C would contain "half" of the ball $B(r)$ which has volume $\frac{1}{2}r^d\omega_d \geq \frac{1}{2}d^{-d} > \varepsilon = \text{vol } C$. Then H must contain y'. Then H must contain "half" of the cone L whose apex is y' and whose base is the intersection of the set conv $(\{y\} \cup B(r))$ with the halfspace orthogonal to, and passing through, the vector y. Computing volumes again

$$\text{vol } C \geq \tfrac{1}{2} \text{vol } L \geq \tfrac{1}{2}\frac{1}{d}|y'|r^{d-1}\omega_{d-1}\left(\frac{|y-y'|}{|y'|}\right)^d$$

$$\geq \frac{1}{2d} r^d \omega_{d-1}\left(\frac{|z-y|}{|y'|}\right)^d \geq \frac{1}{2d} r^d \omega_d (r/R)^d (\omega_{d-1}/\omega_d) > (2d)^{-2d} \geq \varepsilon.$$

Now we have an economic cap covering of $K(v \leq \varepsilon)$ by Macbeath regions. We are going to turn it into a covering by caps.

For this end consider the minimal cap $C_i = C(x_i) = K \cap H(a_i, t_i)$, for $i = 1, \ldots, m$. Define

$$K_i = K \cap H(a_i, 6t_i),$$

$$K_i' = M(x_i, \tfrac{1}{2}) \cap H(a_i, t_i).$$

We claim that the sets K_i, K_i' satisfy the requirements of the theorem. First, as the sets $M(x_i, \frac{1}{2})$ are pairwise disjoint, so are the sets K_i'. According to (4.1), $C_i \subset M(x_i, 3d)$ so

$$\text{vol } K_i' = \tfrac{1}{2} \text{vol } M(x_i, \tfrac{1}{2}) = \tfrac{1}{2}(6d)^{-d} \text{ vol } M(x_i, 3d)$$

$$\geq \tfrac{1}{2}(6d)^{-d} \text{ vol } C_i = \tfrac{1}{2}(6d)^{-d}\varepsilon.$$

One can get vol $K_i' \geq (6d)^{-d}\varepsilon$ from here by observing that the central symmetry of $M(x_i, 3d)$ and (4.1) imply $2 \text{ vol } C_i \leq \text{vol } M(x_i, 3d)$.

Notice that $M(x_i, 1)$ lies in the strip $H(a_i, 0, 2t_i)$. Then $M(x_i, 5)$ lies in the strip $H(a_i, -4t_i, 6t_i)$ as the centre of $M(x_i, \lambda)$ is on the hyperplane $H(a_i = t_i)$. Thus

$$K \cap M(x_i, 5) \subset K \cap H(a_i, -4t_i, 6t_i) = K \cap H(a_i, 6t_i) = K_i$$

and indeed

$$K(v \leq \varepsilon) \subset \bigcup_1^m K_i.$$

According to (3.1) and (3.2)

$$\text{vol } K_i \leq 6^d \text{ vol } C_i = 6^d \varepsilon.$$

Finally, $K_i' \subset K_i$ is evident.

§5. *Proof of Theorem* 7. Let $K_1, \ldots, K_m$ be the economic cap covering of $K(v \leq \varepsilon)$ from Theorem 6. We will prove that the union of $K_i^{d\lambda}$ $(i=1,\ldots,m)$ covers $K(v \leq \lambda\varepsilon)$.

So we take a point $x \in K(v \leq \lambda\varepsilon)$. We have to show that

$$x \in K_1^{d\lambda} \cup \ldots \cup K_m^{d\lambda},$$

thus we may assume that $x \notin K_1 \cup \ldots \cup K_m$.

The minimal cap $C(x) = K \cap H(a, t)$ has centre z (say), and the line segment $[x, z]$ intersects the boundary of $K(v \leq \varepsilon)$ at the point y. Clearly $v(y) = \varepsilon$. Let t' be the distance of y from the hyperplane $H(a, 0)$ (which supports K at z). Then $y \in H(a = t')$ and

$$\varepsilon = v(y) \leq \text{vol}\,(K \cap H(a, t')) = \int_0^{t'} \text{vol}_{d-1}(K \cap H(a = \tau))\,d\tau$$

$$\leq t' \max\{\text{vol}_{d-1}(K \cap H(a = \tau)) \colon 0 \leq \tau \leq t'\}$$

$$\leq t' \max\{\text{vol}_{d-1}(K \cap H(a = \tau)) \colon 0 \leq \tau \leq t\}.$$

On the other hand

$$\lambda\varepsilon \geq v(x) = \text{vol}\, K \cap H(a, t) \geq \frac{1}{d} t \max\{\text{vol}_{d-1}(K \cap H(a = \tau)) \colon 0 \leq \tau \leq t\},$$

where the last inequality follows from the fact that the double cone whose base is the maximal section $K \cap H(a = \tau)$ is contained in $C(x)$. Now $t/t' = |z - x|/|z - y|$ and so we get

$$|z - x| \leq \lambda d |z - y|.$$

Consider now the cap $K_i = K \cap H(a_i, t_i)$ from the cap covering that contains y. Let z_i be the centre of K_i and write y_i for the intersection $[z_i, x] \cap H(a_i = t_i)$. The line L through z and x intersects the hyperplanes $H(a_i = 0)$ and $H(a_i = t_i)$ at the points z' and y', respectively. It is easy to check that the points z', z, y, y', x come on L in this order. Then

$$\frac{|x - z_i|}{|y_i - z_i|} = \frac{|x - z'|}{|y' - z'|} \leq \frac{|x - z'|}{|y - z'|} = \frac{|x - z| + |z - z'|}{|y - z| + |z - z'|} \leq \frac{|x - z|}{|y - z|} \leq \lambda d.$$

So indeed $x \in K_1^{\lambda d} \cup \ldots \cup K_i^{\lambda d}$. Now

$$\text{vol}\, K(v \leq \lambda\varepsilon) \leq \sum_{i=1}^m \text{vol}\, K_i^{\lambda d} \leq (\lambda d)^d \sum_{i=1}^m \text{vol}\, K_i \leq (\lambda d)^d 6^d m\varepsilon$$

$$\leq (6\lambda d)^d (6d)^d \sum_{i=1}^m \text{vol}\, K_i' \leq (36\lambda d^2)^d \,\text{vol}\, K(v \leq \varepsilon).$$

§6. *Proof of Theorem* 1. To establish the lower bound let $x \in K$ and let $C(x)$ be the corresponding minimal cap. Then

$$\text{Prob}\,(x \notin K_n) \geq \text{Prob}\,(C(x) \cap K_n = \varnothing) = (1 - v(x))^n.$$

Consequently, for $\varepsilon > 0$, we get

$$E(K, n) = \int_K \text{Prob}\,(x \notin K_n) \geq \int_K (1 - v(x))^n \geq \int_{K(v \leq \varepsilon)} (1 - v(x))^n$$
$$\geq \int_{K(v \leq \varepsilon)} (1 - \varepsilon)^n = (1 - \varepsilon)^n \operatorname{vol} K(v \leq \varepsilon).$$

Choosing now $\varepsilon = 1/n$ (and assuming $n \geq 3$) we have

$$\tfrac{1}{4} \operatorname{vol} K\left(v \leq \frac{1}{n}\right) \leq E(K, n).$$

Proving the upper bound is more involved. First we use an idea from Bárány and Füredi [**2**]. Let $x_1, \dots, x_n$ be randomly chosen points and write $N(x) = \{x_1, \dots, x_n\} \cap M(x)$ when $x \in K$. Further, denote by $n(x)$ the cardinality of $N(x)$. Now

$$\text{Prob}\,(x \notin K_n) = \sum_{m=0}^{n} \text{Prob}\,(x \notin K_n \mid n(x) = m)\, \text{Prob}\,(n(x) = m)$$
$$\leq \sum_{m=0}^{n} \text{Prob}\,(x \notin \operatorname{conv} N(x) \mid n(x) = m)\, \text{Prob}\,(n(x) = m). \tag{6.1}$$

According to a theorem of Wendel [**24**] (*cf.* Füredi [**10**] as well)

$$\text{Prob}\,(x \notin \operatorname{conv} N(x) \mid n(x) = m) = 2^{-(m-1)} \sum_{i=1}^{d-1} \binom{m}{i}.$$

Using this

$$\text{Prob}\,(x \notin K_n) \leq 2 \sum_{m=0}^{n} 2^{-m} \sum_{i=0}^{d-1} \binom{m}{i} [u(x)]^m [1 - u(x)]^{n-m} \binom{n}{m}$$
$$= 2 \sum_{i=0}^{d-1} \sum_{m=i}^{n} \binom{m}{i} \binom{n}{m} [u(x)/2]^m [1 - u(x)]^{n-m}$$
$$= 2 \sum_{i=0}^{d-1} \sum_{m=i}^{n} \binom{n}{i} \binom{n-i}{m-i} [u(x)/2]^m [1 - u(x)]^{n-m}$$
$$= 2 \sum_{i=0}^{d-1} \binom{n}{i} \sum_{m=i}^{m} \binom{n-i}{m-i} [u(x)/2]^m [1 - u(x)]^{n-m}$$
$$= 2 \sum_{i=0}^{d-1} \binom{n}{i} \sum_{k=0}^{n-i} \binom{n-i}{k} [u(x)/2]^{k+i} [1 - u(x)]^{n-i-k}$$
$$= 2 \sum_{i=0}^{d-1} \binom{n}{i} [u(x)/2]^i [1 - \tfrac{1}{2} u(x)]^{n-i}. \tag{6.2}$$

Now we integrate

$$E(K,n)=\int \operatorname{Prob}(x\notin K_n)\,dx$$

$$\leq 2\sum_{i=0}^{d-1}\binom{n}{i}\int_K [\tfrac{1}{2}u(x)]^i[1-\tfrac{1}{2}u(x)]^{n-i}\,dx$$

$$=2\sum_{i=0}^{d-1}\binom{n}{i}\sum_{\lambda=1}^{n}\int_{[(\lambda-1)/n]\leq u(x)\leq(\lambda/n)}[\tfrac{1}{2}u(x)]^i[1-\tfrac{1}{2}u(x)]^{n-i}dx$$

$$<2\sum_{i=0}^{d-1}\binom{n}{i}\sum_{\lambda=1}^{n}\int_{[(\lambda-1)/n]\leq u(x)\leq(\lambda/n)}\left(\frac{\lambda}{2n}\right)^i\left(1-\frac{\lambda-1}{2n}\right)^{n-i}$$

$$\leq 2\sum_{i=0}^{d-1}\binom{n}{i}\sum_{\lambda=1}^{n}\left(\frac{\lambda}{2n}\right)^i\left(1-\frac{\lambda-1}{2n}\right)^{n/2}\operatorname{vol}K\left(u\leq\frac{\lambda}{n}\right). \tag{6.3}$$

Here

$$2\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\leq\lambda^i/(2^i i!)\leq\lambda^{d-1}$$

because $i\geq d-1$. Moreover, when $n\geq 2d$,

$$\left(1-\frac{\lambda-1}{2n}\right)^{n-i}\leq\left(1-\frac{\lambda-1}{2n}\right)^{n-i}\leq e^{-(\lambda-1)/4}.$$

According to Lemma 2

$$K(u\leq\lambda/n)\subset K(v\leq(3d)^d\lambda/n)$$

provided $\lambda/n<(12d^3)^{-d}$. We set $\Lambda=\lfloor(12d^3)^{-d}n\rfloor$ and continue (6.3).

$$E(K,n)\leq\sum_{\lambda=1}^{\lambda=\Lambda}d\lambda^{d-1}e^{-(\lambda-1)/4}\operatorname{vol}K(v\leq(3d)^d\lambda/n)$$

$$+\sum_{\lambda=\Lambda+1}^{n}d\lambda^{d-1}e^{-(\lambda-1)/4}, \tag{6.4}$$

as $\operatorname{vol}K(v\leq\varepsilon)\leq\operatorname{vol}K=1$ for every $\varepsilon\geq 0$. Here in the first sum

$$\operatorname{vol}K(v\leq(3d)^d\lambda/n)\leq[36\lambda(3d)^d d^2]^d\operatorname{vol}K(v\leq 1/n),$$

by Theorem 7, so the first sum is at most

$$\operatorname{vol}K\left(v\leq\frac{1}{n}\right)\sum_{\lambda=1}^{\infty}d\lambda^{d-1}e^{-(\lambda-1)/4}\lambda^d[36d^2(3d)^d]^d<\operatorname{const}(d)\operatorname{vol}K\left(v\leq\frac{1}{n}\right). \tag{6.5}$$

To estimate the second sum in (6.4) observe that it is less than

$$\sum_{\lambda=1}^{\infty}d\lambda^{d-1}e^{-(\lambda-1)/8}e^{-(12d^3)^{-d}(n/8)}<c_1(d)e^{-(12d^3)^{-d}(n/8)},$$

where $c_1(d)$ is a constant depending on d only. We need a lower bound on vol $K(v\leq 1/n)$. We could use Theorem 2, but we prefer the very simple vol $K(v\leq \varepsilon)\geq \varepsilon$ inequality, which follows from the fact that $C(x)\subset K(v\leq \varepsilon)$ for any x with $v(x)=\varepsilon$. Using this the second sum in (6.4) is less than

$$c_1(d)e^{-(12d^3)^{-d}(n/8)} < c_2(d)\frac{1}{n}\leq c_2(d)\text{ vol } K(v\leq 1/n). \tag{6.6}$$

With (6.5) and (6.6) we get from (6.4) that indeed

$$E(K, n)\leq \text{const}(d)\text{ vol } K(v\leq 1/n).$$

We mention here that a byproduct of (6.2) is that:

Prob $(x\notin K_n)$

≤ 2 Prob (less that d points from $\{x_1, \dots, x_n\}$ lie in $M(x)\cap C(x)$).

§7. *Proof of Theorem* 2. We start with some notation. Fix $a\in R^d$, $|a|=1$ somehow and let $H(a=t_0)$ be the hyperplane whose intersection with K has the largest $(d-1)$-dimensional volume among all hyperplanes $H(a=t)$. Assume the width of K in direction a is at most $2t_0$. If this were not the case we would take $-a$ instead of a. As a will be fixed throughout this proof we will write $H(t)=H(a=t)$. Define, further,

$$Q(t)=H(t)\cap K \quad \text{and} \quad q(t)=\text{vol}_{d-1}\,Q(t).$$

Our choice for t_0 insures that

$$q(t)\geq (t/t_0)^{d-1}q(t_0) \quad \text{for } 0\leq t\leq t_0, \tag{7.1}$$

$$2t_0q(t_0)\geq \text{vol } K=1. \tag{7.2}$$

LEMMA 3. *For* $\varepsilon>0$ *and* $0<t<t_0$

$$K(u_K\leq \varepsilon)\cap H(t)\supset Q(t)(u_{Q(t)}\leq \varepsilon/2t).$$

Proof. We are going to show that $x\in H(t)\cap K$ implies $u_K(x)\leq 2tu_{Q(t)}(x)$. This will prove the lemma.

Notice, first that $M(x)$ lies in the strip $H(a, 0, 2t)$. Then

$$u(x)=\int_0^{2t}\text{vol}_{d-1}(M(x)\cap H(\tau))d\tau\leq 2t\text{ vol}_{d-1}(M(x)\cap H(t))$$

because $M(x)$ is centrally symmetric so its largest section is the middle one, $M(x)\cap H(t)$. Next

$$\begin{aligned} M(x)\cap H(t)&=\{x+[(K-x)\cap(x-K)]\}\cap H(t)\\ &=x+\{[(K\cap H(t))-x]\cap[x-(K\cap H(t))]\}\\ &=x+[(Q(t)-x)\cap(x-Q(t))]\\ &=M_{Q(t)}(x). \end{aligned}$$

Then

$$u(x) \leqslant 2t \operatorname{vol}_{d-1} M_{Q(t)}(x) = 2tu_{Q(t)}(x).$$

We will now show that for $0 < \varepsilon \leqslant 1$

$$\operatorname{vol} K(u \leqslant \varepsilon) \geqslant \operatorname{const}(d)\varepsilon(\log(1/\varepsilon))^{d-1}. \tag{7.3}$$

Recalling Lemma 2 this proves that for $\varepsilon \leqslant (2d)^{-2d}$

$$\operatorname{vol} K(v \leqslant \varepsilon) \geqslant \operatorname{vol} K(u \leqslant (3d)^{-e}\varepsilon) \geqslant \operatorname{const}(d)\varepsilon(\log(1/\varepsilon))^{d-1}.$$

When $\varepsilon > (2d)^{-2d}$, the statement of the theorem follows from the fact that $\operatorname{vol} K(v \leqslant \varepsilon)$ is an increasing function of ε.

We prove (7.3) by induction on d. The case $d = 1$ is trivial. We will need the induction hypothesis in the invariant form (3.6): for $Q \subset R^{d-1}$ compact, convex with $\operatorname{vol}_{d-1} Q > 0$ and for $0 < \eta \leqslant 1$

$$\operatorname{vol} Q(u_Q \leqslant \eta \operatorname{vol} Q)/\operatorname{vol} Q \geqslant c_{d-1}\eta(\log(1/\eta))^{d-2}.$$

Assuming this holds we prove (7.3). Write

$$\operatorname{vol} K(u \leqslant \varepsilon) = \operatorname{vol}[K(u \leqslant \varepsilon) \cap H(a, 0, t_0)] = \int_0^{t_0} \operatorname{vol}_{d-1}[K(u \leqslant \varepsilon) \cap H(t)]dt$$

$$\geqslant \int_0^{t_0} \operatorname{vol}_{d-1} Q(t)(u_{Q(t)} \leqslant \varepsilon/2t)dt, \tag{7.4}$$

according to Lemma 3. Define $\eta = \eta(t) = \varepsilon/(2tq(t))$ and let t_1 be the unique solution to $\eta(t) = 1$ between 0 and t_0. Then, by the induction hypothesis, for $t_1 \leqslant t \leqslant t_0$

$$\begin{aligned} \operatorname{vol}_{d-1} Q(t)(u_{Q(t)} \leqslant \eta q(t)) &\geqslant c_{d-1}q(t)\eta(\log(1/\eta))^{d-2} \\ &= c_{d-1}\frac{\varepsilon}{2t}\left[\log\left(\frac{2tq(t)}{\varepsilon}\right)\right]^{d-2} \\ &\geqslant c_{d-1}\frac{\varepsilon}{2t}\left[\log\left(\frac{2t}{\varepsilon}\left(\frac{t}{t_0}\right)^{d-1} q(t_0)\right)\right]^{d-2}, \end{aligned}$$

where the last inequality follows from (7.1). We continue (7.4).

$$\operatorname{vol} K(u \leqslant \varepsilon) \geqslant \int_{t_1}^{t_0} c_{d-1}\frac{\varepsilon}{2t}\left[\log\left(\frac{2t^d q(t_0)}{\varepsilon t_0^{d-1}}\right)\right]^{d-2}. \tag{7.5}$$

Define α by $\alpha^d = 2q(t_0)/(\varepsilon t_0^{d-1})$ and let $t_2 = 1/\alpha$. Then, by (7.1) again, $t_1 < t_2 < t_0$. Substitute now $\tau = \alpha t$ with $\tau_i = \alpha t_i$, $i = 0, 2$. Continue (7.5).

$$\operatorname{vol} K(u \leqslant \varepsilon) \geqslant \int_{\tau_2}^{\tau_0} c_{d-1}\frac{\varepsilon}{2}\frac{1}{\tau}(\log \tau)^{d-2}d\tau = \frac{\varepsilon c_{d-1}}{2(d-1)}(\log \tau)^{d-1}\Big|_{\tau=\tau_2}^{\tau=\tau_0}$$

$$= \frac{\varepsilon c_{d-1}}{2(d-1)}\left[\log\left(\frac{t_0(2q(t_0))^{1/d}}{(\varepsilon t_0^{d-1})^{1/d}}\right)\right]^{d-1} \geqslant \frac{\varepsilon c_{d-1}}{2(d-1)}\left(\frac{1}{d}\log\frac{1}{\varepsilon}\right)^{d-1}$$

where the last inequality follows from (7.2).

§8. *Proof of Theorem* 3. We prove this theorem for simplices first and then for general polytopes. We may take any simplex $S \subset R^d$ because for a nonsingular linear transformation A one clearly has

$$\text{vol } S(v_S \leq \varepsilon)/\text{vol } S = \text{vol } AS(v_{AS} \leq |\det A|\varepsilon)/\text{vol } AS. \tag{8.1}$$

We take a regular simplex $S = \text{conv}\{y_0, \dots, y_d\}$ with vol $S = 1$.

LEMMA 4. *Assume $z \in \text{int } S$ and the nearest vertex to z is y_0. Then*

$$C(z) \supset M(\tfrac{1}{2}(z + y_0)).$$

Proof. Let $C(z) = S \cap H(a, t)$ be the minimal cap for z. Recall the definition:

$$H(a, t) = \{x \in R^d : a \,.\, x = h(a) - t\} \text{ with } h(a) = \max\{a \,.\, x : x \in S\}.$$

We know that $z \in H(a = t)$ and that z is the centre of gravity of the section $S \cap H(a = t)$.

Obviously, $h(a) = a \,.\, y_i$ for some vertex y_i. Consider

$$x \in M(\tfrac{1}{2}(z + y_i)) = S \cap (z + y_i - S).$$

Then $x = z + y_i - y$ with $y \in S$, so

$$a \,.\, x = a \,.\, z + a \,.\, y_i - a \,.\, y \geq h(a) - t.$$

This shows that

$$M(\tfrac{1}{2}(z + y_i)) \subset S \cap H(a, t) = C(z)$$

for some vertex, y_i, of S.

Assume now that z is closer to y_j than to y_k. We will prove then that $M(\frac{1}{2}(z + y_k)) \subset C(z)$ does not hold. This will prove the lemma.

Consider the reflection, z', of z to the hyperplane bisecting the line segment $[y_j, y_k]$. We show that $z' \notin C(z)$ and $z' \in M(\frac{1}{2}(z + y_k))$. By the symmetry of the regular simplex we have $v(z) = v(z')$. Now $z' \in \text{int } C(z)$ would imply $v(z') < v(z)$, a contradiction. And if z' were on the bounding hyperplane of $C(z)$, then $C(z) = C(z')$ must hold. But this cannot be the case because both z and z' cannot be the centre of gravity of the section $S \cap H(a = t)$. So $z' \notin C(z)$. On the other hand $z' = z + \alpha(y_k - y_j) \in S$ for some $\alpha \in (0, 1)$. Then

$$z' = z + \alpha(y_k - y_j) = z + y_k - [(1 - \alpha)y_k + \alpha y_j] \in z + y_k - S.$$

Thus $z' \in M(\frac{1}{2}(z + y_k))$ and then $M(\frac{1}{2}(z + y_k)) \subset C(z)$ is indeed impossible.

Define $T_i = \{x \in S : |x - y_i| = \min\{|x - y_j| : j = 0, \dots, d\}$. Then

$$S(v \leq \varepsilon) = \bigcup_{i=0}^{d} (T_i \cap S(v \leq \varepsilon)) \subset \bigcup_{i=0}^{d} \{x \in T_i : u(\tfrac{1}{2}(x + y_i)) \leq \varepsilon\},$$

by Lemma 4. Thus

$$\text{vol } S(v \leq \varepsilon) \leq (d+1) \text{ vol}\{x \in T_0 : u(\tfrac{1}{2}(x + y_0)) \leq \varepsilon\}.$$

Define now an affine transformation $A : R^d \to R^d$ with $Ay_0 = 0$ and $Ay_i = e_i$ $(i = 1, \dots, d)$ where $e_1, \dots, e_d$ form an orthonormal basis of R^d. Write

$Ax = (\xi_1, \ldots, \xi_d) = \xi_1 e_1 + \ldots + \xi_d e_d$. Then $x \in T_0$, $u_S(\frac{1}{2}(x+y_0)) \le \varepsilon$ imply

$$\xi_1 \ldots \xi_d \le |\det A|\varepsilon \qquad \text{and} \qquad \max\{\xi_i : i = 1, \ldots, d\} \le 1.$$

Similarly as in (8.1) and (3.6)

$$\frac{\operatorname{vol}\{x \in T_0 : u(\frac{1}{2}(x+y_0)) \le \varepsilon\}}{\operatorname{vol} T_0} \le \frac{\operatorname{vol}\{\xi \in R^d : \xi_1 \ldots \xi_d \le |\det A|\varepsilon, 0 \le \xi_i \le 1\}}{\operatorname{vol} AT_0}.$$

A simple induction argument shows that for $0 < \eta \le 1$

$$\operatorname{vol}\{\xi \in R^d : \xi_1 \ldots \xi_d \le \eta, 0 < \xi_i < 1\} = \eta \sum_{j=0}^{d-1} \frac{1}{j!}\left(\log \frac{1}{\eta}\right)^j.$$

But $\det A$ is a constant depending on d only so for $\varepsilon < \varepsilon_0(d)$ we get

$$\operatorname{vol} S(v \le \varepsilon) \le (d+1) \operatorname{vol}(T_0 \cap S(v \le \varepsilon)) \le \operatorname{const}(d)\varepsilon(\log(1/\varepsilon))^{d-1}.$$

Now we prove the theorem for general polytopes $P \subset R^d$. Take a triangulation of P into simplices $S_1, \ldots, S_m$ using vertices of P only. Then

$$P(v_P \le \varepsilon) \subset \bigcup_{i=1}^m S_i(v_{S_i} \le \varepsilon).$$

With suitable (nonsingular) affine transformations $A_i : R^d \to R^d$ such that $A_i S_i = S$ we have

$$\operatorname{vol} P(v_P \le \varepsilon) \le \sum_{i=1}^m \operatorname{vol} S_i(v_{S_i} \le \varepsilon) = \sum_{i=1}^m \frac{\operatorname{vol} S_i}{\operatorname{vol} AS_i} \operatorname{vol} AS_i(v_{AS_i} \le |\det A_i|\varepsilon)$$

$$\le \varepsilon \sum_{i=1}^m (\log(1/(|\det A_i|\varepsilon)))^{d-1} \le \operatorname{const}(P)\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{d-1}.$$

We mention here that there is an alternative proof for this theorem using the arguments of the proof of Theorem 2.

§9. *Proof of Theorem* 4. It is clear that for some $\rho > 0$, $\delta > 0$ the set of (ρ, δ)-circular points Ω form a set of positive measure in ∂K. Take $p \in \Omega$ and consider $z = p - \alpha q$ where q is the outer unit normal to K at P. Assume

$$\alpha \le \min\left(\frac{\delta}{d}, \varepsilon^{2/(d+1)} \rho^{-(d-1)/(d+1)} \omega_{d-1}^{-(d-1)/(d+1)}\right).$$

CLAIM 2. $z \in K(v \le \varepsilon)$.

Proof. Assume this is false. Then for a minimal cap $C(z) = K \cap H(a, t)$ one has $\operatorname{vol} C(z) > \varepsilon$. Take a chord $[x, y]$ through z of K lying in the bounding hyperplane $H(a = t)$ of $C(z)$. Consider a minimal cap $C'(z)$ of the ball $B^\rho = B(\rho, p - \rho a)$. As p is ρ-circular, one of the endpoints of the chord $[x, y]$, x (say), lies in the cap $C'(z)$. But z is the centre of gravity of the section $K \cap H(a = t)$ and so, according to a well-known result (see [7], *e.g.*)

$$(d-1)|z - x| \ge |y - z|.$$

This shows that y lies in the minimal cap $C'(z')$ of B^ρ where $z'=p-d\alpha a$. As y is an arbitrary point of the section $K\cap H(a=t)$, we have that $C(z)\subset C'(z')$. Then

$$\text{vol } C(z)\leqslant \text{vol } C'(z')\leqslant (\alpha d)^{(d+1)/2}(2\rho)^{(d-1)/2}\omega_{d-1}.$$

By the choice of α this is less than ε. A contradiction.

We claim now that

$$\text{vol } K(v\leqslant\varepsilon)\geqslant \text{const }(K)\alpha \text{ vol}_{d-1}\,\Omega. \tag{9.1}$$

This will prove the theorem for $\alpha=\text{const }\varepsilon^{2/(d+1)}$ if ε is small enough. Define first

$$L_s=\{p\in\delta K\colon K\supset B(s,p-sa)\}.$$

It is well-known [27] that $\text{vol}_{d-1}(\delta K\backslash L_s)\to 0$ as $s\to 0$. Choose $s>0$ so that $\text{vol}_{d-1}(\Omega\cap L_s)\geqslant\frac{1}{2}\text{vol}_{d-1}\,\Omega$. Now to see (9.1) we use the proof of the cap covering theorem (Theorem 6). So choose a maximal system of points $x_1,\ldots,x_m$ from $\delta K(v\geqslant\varepsilon)$ subject to the conditions:

$$M(x_i,\tfrac{1}{2})\cap M(x_j,\tfrac{1}{2})=\varnothing; \tag{9.2}$$

$$\text{the centre of the minimal cap } C(x_i) \text{ lies in } \Omega\cap L_s. \tag{9.3}$$

So let $C(x_i)=K\cap H(a_i,t_i)$ with centre $p_i\in\Omega\cap L_s$. We know from the proof of the cap covering theorem that

$$\Omega\cap L_s\subset\bigcup_{i=1}^m M(x_i,5)\subset\bigcup_{i=1}^m K_i$$

where $K_i=K\cap H(a_i,6t_i)$. According to Claim 2, the width of the cap $C(x_i)$ is at least α, so the width of K_i is at least 6α. Then

$$\begin{aligned}\text{vol } K(v\leqslant\varepsilon)&\geqslant\sum_{i=1}^m\tfrac{1}{2}\text{vol } M(x_i,\tfrac{1}{2})\\&\geqslant\text{const }(d)\sum_{i=1}^m\text{vol } K_i\\&\geqslant\text{const }(d)\frac{6\alpha}{d}\sum_{i=1}^m\text{vol}_{d-1}(K\cap H(a_i=6t_i)).\end{aligned}$$

Now $\text{vol}_{d-1}(K\cap H(a_i=6t_i))\geqslant\text{const }(d,\rho,s)\text{ vol}_{d-1}(\partial K\cap K_i)$. This follows from the fact that the outer normals to K at the points of $\partial K\cap K_i$ cannot differ much from a_i (if ε is small enough) because p_i is in L_s. Using this we get

$$\begin{aligned}\text{vol } K(v\leqslant\varepsilon)&\geqslant\text{const }(d,\rho,s)\alpha\sum_{i=1}^m\text{vol}_{d-1}(\partial K\cap K_i)\\&\geqslant\text{const }(K)\alpha\text{ vol}_{d-1}(\Omega\cap L_s)\geqslant\text{const }(K)\tfrac{1}{2}\alpha\text{ vol}_{d-1}(\Omega).\end{aligned}$$

§10. *Proof of Theorem* 8. Let $P\subset R^d$ be a convex polytope having n vertices and assume vol $P=1$. Set

$$\varepsilon=c_0(d)n^{-(d+1)/(d-1)}$$

where the constant $c_0(d)$ is to be determined later.

We assume that n is large enough to ensure that $\varepsilon < (2d)^{-2d}$. Then Theorem 6 applies: there are caps $K_1, \ldots, K_m$ and subsets $K'_1, \ldots, K'_m$ satisfying (i), (ii) and (iii) of Theorem 6. Then

$$m(6d)^{-d}\varepsilon \leqslant \sum_{i=1}^{m} \text{vol } K'_i \leqslant \text{vol } P(v \leqslant \varepsilon)$$
$$\leqslant 4E(P, \lceil 1/\varepsilon \rceil) \leqslant 4c(d)(\lceil 1/\varepsilon \rceil)^{-2/(d+1)} \leqslant 8c(d)\varepsilon^{2/(d+1)}$$

where the third inequality follows from Theorem 1 and the fourth from (1.5) and (1.4) with a suitable constant $c(d)$. This shows that

$$m \leqslant 8(6d)^d c(d)\varepsilon^{-(d-1)/(d+1)} \leqslant 8(6d)^d c(d) c_0(d)^{-(d-1)/(d+1)} n \leqslant \frac{1}{d+1} n$$

if we choose $c_0(d)$ large enough.

Now the caps $K_1, \ldots, K_m$ cover $P(v \leqslant \varepsilon)$ so they cover the boundary of P as well. Then there is a cap, K_i say, containing at least $n/m \geqslant d+1$ vertices, $y_0, \ldots, y_d$ of P. Consequently $\text{conv}\{y_0, \ldots, y_d\} \subset K$ and

$$\text{vol conv}\{y_0, \ldots, y_d\} \leqslant \text{vol } K_i \leqslant 6^d \varepsilon \leqslant \text{const}(d) n^{-(d+1)/(d-1)}.$$

§11. *Proof of Theorem* 9. Lemma 1 implies $K(v \leqslant \varepsilon) \subset K(u \leqslant 2\varepsilon)$. By Lemma 2, if $\varepsilon \leqslant \varepsilon_1(d) = \frac{1}{2}(12d^3)^{-d}$, then $K(u \leqslant 2\varepsilon) \leqslant K(v \leqslant 2(3d)^d \varepsilon)$ so indeed

$$K(v \leqslant \varepsilon) \subset K(u \leqslant 2\varepsilon) \subset K(v \leqslant 2(3d)^d \varepsilon).$$

Computing volumes here and applying Theorem 7 gives

$$\text{vol } K(v \leqslant \varepsilon) \leqslant \text{vol } K(u \leqslant 2\varepsilon) \leqslant c_1(d) \text{ vol } K(v \leqslant \varepsilon).$$

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