Cosserat elasticity, spectral theory of first order systems, and the massless Dirac operator

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by

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Declaration

I, Robert James Downes, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

This thesis is concerned with the study of the massless Dirac operator in dimension three and is, in part, based upon [12, 22, 21, 26, 25].

An introduction is given in Chapter 1.

In Chapter 2 we study a special version of Cosserat elasticity, with deformations induced by rotations only, and no displacements. We prove that for a particular choice of elastic moduli and in the stationary setting (harmonic dependence on time) our mathematical model reduces to the massless Dirac equation.

Chapter 3 contains a description of the progress recently made in the spectral theory of first order systems, with a particular focus on dimension three presented in Chapter 4.

We prove in Chapter 5 that the second asymptotic coefficient of the counting function of a first order system has the geometric meaning of the massless Dirac action.

Finally, in Chapter 6 we examine the spectral asymmetry of the massless Dirac operator. We work on a 3-torus equipped, initially, with a Euclidean metric and consider the behaviour of the spectrum under a perturbation of the metric. We derive an explicit asymptotic formula for the eigenvalue closest to zero.

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Contents

Chapter 1

Introduction

1.1 The massless Dirac operator: an overview

This thesis is concerned with the massless Dirac operator in dimension three. This is directly related to the accepted mathematical model for a massless neutrino or antineutrino.

The massless Dirac operator is a particular self-adjoint complex two-by-two matrix partial differential operator acting on two-columns of complex-valued scalar fields. In theoretical physics and differential geometry this pair of complex-valued scalars is referred to as a *spinor*. When we say that the components of a spinor are scalars we mean that they are anholonomic objects, i.e. they do not depend on the choice of local coordinates.

Although reasonably familiar when defined on $\mathbb{R}^3$, one can work with a more general operator defined on a Riemannian 3-manifold (that is, a 3-manifold equipped with a Riemannian metric). For simplicity, in this thesis we will always work on either $\mathbb{R}^3$ or an oriented compact Riemannian 3-manifold, M, unless otherwise stated.

We give two explicit formulae for the massless Dirac operator, W, both of which are utilised in the thesis. In matrix form we have

$$W := -i\sigma^{\alpha}\left[\frac{\partial}{\partial x^{\alpha}} + \frac{1}{4}\sigma_{\beta}\left(\frac{\partial \sigma^{\beta}}{\partial x^{\alpha}} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\}\sigma^{\gamma}\right)\right]. \tag{1.1.1}$$

Here summation is carried out over repeated tensor indices $\alpha = 1, 2, 3$ and $\sigma^{\beta} = \sigma^{\beta}(x)$, $\beta = 1, 2, 3$, are a set of Pauli matrices. These matrices span the 3-dimensional real vector space of trace-free two-by-two Hermitian matrices and satisfy the defining relation

$$\sigma^{\alpha}\sigma^{\beta} + \sigma^{\beta}\sigma^{\alpha} = 2g^{\alpha\beta}I_{2\times 2}, \tag{1.1.2}$$

where $g^{\alpha\beta}$ is the metric tensor and $I_{2\times 2}$ the two-by-two identity matrix. In general, the Pauli matrices depend upon local coordinates $x^{\beta} \in M$, $\beta = 1, 2, 3$. In addition, in (1.1.1) $\left\{ \begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix} \right\}$ are the Christoffel symbols, the Levi-Civita connection coefficients, given by

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} = \frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\gamma\delta}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right). \tag{1.1.3}$$

An alternative formulation of the massless Dirac operator, utilising so-called spinor notation, is

$$W := -i\sigma^{\alpha}{}_{\dot{a}b}\nabla_{\alpha}. \tag{1.1.4}$$

Aside from the appearance of the covariant derivative ∇ with respect to the Levi-Civita connection, see Section I.A in [81], this formula differs from (1.1.1) in its use of spinor indices $\dot{a} = \dot{1}, \dot{2}$, $b = 1, 2$.

The appearance of a dotted index in formula (1.1.4) is a matter of tradition. We could have written all formulae containing spinor indices without using dots. The necessity to distinguish dotted and undotted indices arises when working in the more general 4-dimensional Lorentzian setting. In dimension four the spin group, which determines transformation properties of spinors under changes of the corresponding anholonomic basis, is reducible, having two distinct irreducible

subgroups. This necessitates the introduction of two types of index (dotted/undotted) indicating the so-called left- or right-handedness of spinors, i.e. distinguishing two possible types of spinor determined by behaviour under changes of anholonomic basis.

In three dimensions this is not a critical issue: the spin group is irreducible and, as such, all spinors transform in the same manner under changes of the anholonomic basis. We need not, therefore, use dotted indices at all. However, to preserve accord between the formulae appearing in this thesis and those in the literature, we maintain the dotted indices.

The use or spinor indices, or lack thereof, in formulae (1.1.1) and (1.1.4) is made clear by the formulation of the massless Dirac equation, the accepted mathematical model for a massless neutrino or antineutrino. Forming the massless Dirac equation is equivalent to the introduction of time into our problem, that is, considering the dynamic equation

$$(D_t + W)\, w = 0. \tag{1.1.5}$$

Here, $D_t := -i\partial/\partial t$ and $w(t, x)$ is a column-function on $\mathbb{R} \times M$. Rewriting this equation in spinor form, for example, we arrive at the standard formulation of the massless Dirac equation

$$-i\left(-\sigma^0{}_{\dot{a}b}\partial_t + \sigma^\alpha{}_{\dot{a}b}\nabla_\alpha\right)\zeta^b = 0. \tag{1.1.6}$$

Here ζ denotes a two-component complex-valued spinor field on $\mathbb{R} \times M$, summation is carried out over repeated spinor indices $b = 1, 2$, and $\sigma^0{}_{\dot{a}b}$ is the negative identity matrix (this is again a relic of the 4-dimensional setting), see Appendix 2.A. In addition, the subscript attached to the partial derivative ∂ indicates the variable with respect to which partial differentiation occurs (e.g. $\partial_1 := \partial/\partial x^1$).

We see that the massless Dirac equation is a system of two ($\dot{a} = 1, 2$) complex linear partial differential equations on the 4-manifold $\mathbb{R} \times M$ for the unknown

spinor field w. Note that the mapping ${\sigma^0}_{\dot{a}b} \mapsto -{\sigma^0}_{\dot{a}b}$ is equivalent to switching our object of study from a neutrino to an antineutrino.

Of course, (1.1.5) and (1.1.6) are two equivalent ways of writing the same hyperbolic system of equations. In particular, the elements of the 2-column w are the components of the spinor ζ. We use different notation in (1.1.5) and (1.1.6) as a matter of tradition.

For clarity, we state the explicit form of the massless Dirac equation when the massless Dirac operator is defined on $\mathbb{R}^3$. In this instance the formulae are particularly simple and will allow us to demonstrate various properties of the massless Dirac operator with ease throughout this thesis. On $\mathbb{R}^3$ the Pauli matrices σ^α are constant and so, taking the canonical choice, see (2.A.4) in Appendix 2.A, we are able to write the massless Dirac equation as

$$-i\begin{pmatrix} \partial_t + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_t - \partial_3 \end{pmatrix}\begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = 0. \tag{1.1.7}$$

In this thesis we eventually consider a spectral problem arising from the separation of variables in equation (1.1.5). In particular, we seek a separable solution $w(t,x) = e^{-it\lambda}v(x)$ and consider the corresponding spectral problem

$$Wv = \lambda v. \tag{1.1.8}$$

In fact, we study this problem for a more general operator than W in arbitrary dimension (greater than, or equal to, 2), focusing on the massless Dirac operator at a later stage, see Chapter 3 and Chapter 4.

We study the *spectral* and *counting functions*, defined respectively as

$$e(\lambda, x, x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2, \tag{1.1.9}$$

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1 = \int_M e(\lambda, x, x)\sqrt{\det g_{\alpha\beta}}\, dx, \tag{1.1.10}$$

where λ_k is an eigenvalue of the operator W determined from (1.1.8) and $v_k(x)$ a corresponding normalised eigenfunction evaluated at the point $x \in M$, $\lambda \in \mathbb{R}$ is a positive parameter, $\|\cdot\|^2$ is the square of the Euclidean norm, and $dx = dx^1 dx^2 dx^3$. In other words, the counting function $N(\lambda)$ is the number of eigenvalues λ_k between zero and λ, and $e(\lambda, x, x)$ its local (pointwise) version. Again, we begin by studying (1.1.9) and (1.1.10) for more general operators before eventually considering the special case of the massless Dirac operator.

The purpose of this thesis is to consider the massless Dirac operator from a new geometric point of view. The formulae (1.1.1)-(1.1.10) are our basic objects of study in this endeavour.

We start in Chapter 2 by developing a novel theory of elasticity centred around rotations of material points. This is based somewhat on the work of the Cosserat brothers and is linked to the study of Teleparallel gravity, an alternative field theory developed by A. Einstein and É. Cartan. Although seemingly disconnected from the study of the massless Dirac operator, we are able to give a theorem connecting our theory with plane wave solutions of the massless Dirac equation, an unusual geometric link.

We go on in Chapter 3 to present recent developments in the theory of general first order matrix (pseudo)differential operators. Reducing this to the case of the massless Dirac operator in three dimensions we then compute two-term spectral asymptotics for the spectral and counting functions: this work is found in Chapter 4.

For a specific class of operators (those featured in Chapter 4 under the additional assumption of vanishing subprincipal symbol), the second asymptotic term is shown to be related to the massless Dirac action. A comprehensive discussion concerning this relationship is given in Chapter 5.

Finally, in Chapter 6 we consider the issue of spectral asymmetry of the massless Dirac operator. We show that, by perturbing the metric on a 3-torus, we are able to achieve spectral asymmetry. We develop a perturbation theory taking

into account some of the unique properties of the massless Dirac operator in dimension three and present several families of Riemannian metrics for which we can compute explicitly the shift of the zero eigenvalue.

Note that, as this thesis draws on a number of different areas of mathematics, it has not been possible to achieve complete consistency in the use of notation. For example, in Chapter 2 we denote by ξ a nonvanishing spinor field, whereas ξ in Chapters 3–6 stands for the dual variable (momentum). Within each particular chapter the notation is consistent.

In addition, a number of lengthy calculations that would add significantly to the overall length of this thesis have been omitted, although, of course, details are always provided to assist the interested reader in the derivation. The author welcomes requests for further details regarding such calculations, and the omission of a detailed calculation is always noted in the text of this thesis.

For greater clarity, a more detailed introduction to each chapter of this thesis is given below.

1.2 Cosserat elasticity and the massless Dirac equation

In Chapter 2 we develop a new theory of elastic media which we name *Rotational elasticity*. Working in 3-dimensional Euclidean space, which we view as an elastic continuum, we develop a theory of elasticity in which deformations of our medium correspond only to rotations of material points.

While this may seem exotic, it is a natural limit case of a more general theory of elastic media developed by the Cosserat brothers in the early twentieth century. A general deformation in this case corresponds to both a displacement and rotation of material points. It is interesting to note that classical linear elasticity, wherein

deformations are induced only by displacements of material points, is a well-studied limit case of Cosserat elasticity.

Rotational elasticity is an interesting topic of study for several reasons. Unlike its more famous cousin, linear elasticity, Rotational elasticity has received very little attention from the academic community. It is therefore a beneficial exercise to contrast these two theories. In addition, as we show, there are unexpected links both with gravitation and the massless Dirac operator.

The essential geometric object in this construction is the coframe, the field of orthonormal covector fields e^j, $j = 1, 2, 3$, defined over $\mathbb{R}^3$. The orthogonality condition is expressed as a kinematic constraint on the coframe:

$$g = \delta = \delta_{jk} e^j \otimes e^k \tag{1.2.1}$$

or, equivalently, component-wise

$$g_{\alpha\beta} = \delta_{\alpha\beta} = \delta_{jk} e^j{}_\alpha e^k{}_\beta, \quad \alpha, \beta = 1, 2, 3. \tag{1.2.2}$$

Here, for simplicity, we assume the metric to be Euclidean (i.e. $g_{\alpha\beta} = \delta_{\alpha\beta}$ where δ is the Kronecker delta). Note that we select coframes satisfying (1.2.1) which leaves us with three real degrees of freedom at each point in $\mathbb{R}^3$. This justifies our description of (1.2.1) as a kinematic constraint.

We take as the dynamical variables of our theory the coframe e^j and a real positive density ρ. Following the same logic as in classical elasticity, we define an appropriate measure of deformation of our medium which, in our case, is the torsion tensor of the connection generated by our coframe. More specifically, the connection generated by the coframe is the *Teleparallel connection*, a metric compatible affine connection with vanishing curvature but non-vanishing torsion. The connection coefficients can be expressed explicitly in terms of the coframe and its partial derivatives, and are given by

$$\Gamma^\alpha{}_{\beta\gamma} = e_k{}^\alpha \frac{\partial e^k{}_\gamma}{\partial x^\beta}, \quad \alpha, \beta, \gamma = 1, 2, 3, \tag{1.2.3}$$

where

$$e_k{}^{\alpha} := \delta_{kj} g^{\alpha\beta} e^j{}_{\beta} \tag{1.2.4}$$

is the "inverse" of the coframe: see Section 4.3 for further discussion. The components of the torsion tensor are defined as the difference

$$T^{\alpha}{}_{\beta\gamma} := \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta}. \tag{1.2.5}$$

This is an important observation that links Chapter 2 with Chapter 3 and Chapter 4.

We are then able to define both a potential and kinetic energy in terms of the torsion tensor, coframe, and density which allows us to construct a Lagrangian (energy functional) from which we can determine appropriate equations of motion for waves propagating in our elastic continuum.

However, achieving this aim while maintaining the kinematic constraint (1.2.1) is extremely difficult. To ameliorate this problem, we reformulate our model in terms of spinors. It is known that, in three dimensions, a coframe and positive density e^j, ρ are equivalent to a spinor field ζ^b (modulo the sign of the field). Taking advantage of this fact, we are able to rewrite our Lagrangian in a more user-friendly manner, avoiding the constraint (1.2.1), and immediately seek plane wave solutions.

The main results presented in Chapter 2 are the following theorems.

Firstly, we present a theorem concerning plane wave solutions, i.e. spinors satisfying the field equations of Rotational elasticity of the form

$$\zeta(t, x^1, x^2, x^3) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(p_0 t + p_1 x^1 + p_2 x^2 + p_3 x^3)}, \tag{1.2.6}$$

where $p_i \in \mathbb{R}$, $i = 0, 1, 2, 3$, and $p_0 \neq 0$. Note that we can always map one of the components of our plane wave spinor field to zero by performing a rotation of the coordinate system. This justifies our choice of spinor field in (1.2.6).

We can then state the following theorem in which v_1 and v_2 are real constants expressed via the elastic moduli of the underlying elastic continuum, see formula (2.8.22).

Theorem 1.2.1. *Plane wave solutions of Rotational elasticity can, up to rescaling and rotation, be explicitly written down in the form (1.2.6) with arbitrary non-zero p_0 and $p = (p_1, p_2, p_3)$ determined as follows.*

- *If $v_1 > 0$ and $v_2 > 0$ and $v_1 \neq v_2$ then we have two possibilities:*
  - $p = \left(0, 0, \pm\dfrac{p_0}{v_1}\right)$ *(type 1 wave), or*
  - $p = \left(\dfrac{|p_0|}{v_2}\cos\varphi, \dfrac{|p_0|}{v_2}\sin\varphi, 0\right)$ *where $\varphi \in \mathbb{R}$ is arbitrary (type 2 wave).*
- *If $v_1 > 0$ and $v_2 > 0$ and $v_1 = v_2$ then p is an arbitrary 3-vector satisfying $\|p\| = \dfrac{|p_0|}{v_1}$.*
- *If $v_1 > 0$ and $v_2 = 0$ then $p = \left(0, 0, \pm\dfrac{p_0}{v_1}\right)$.*
- *If $v_1 = 0$ and $v_2 > 0$ then $p = \left(\dfrac{|p_0|}{v_2}\cos\varphi, \dfrac{|p_0|}{v_2}\sin\varphi, 0\right)$ where $\varphi \in \mathbb{R}$ is arbitrary.*

This theorem classifies all plane wave solutions of Rotational elasticity. This allows us to compare our model directly with linear elasticity, in particular the existence of longitudinal and transverse waves and their modes of propagation.

The existence of plane wave solutions is highly nontrivial due to the nonlinear nature of the system of partial differential equations governing our model of elasticity. This nonlinearity persists even when we simplify matters by writing our equations in terms of spinors. As such, this theorem is a surprising result. While we are not sure exactly why this holds, we suspect it is related to underlying group-theoretic properties of our model.

Our next result forges a link between Rotational elasticity and the massless Dirac equation for the case when we choose to deal with an axial material, that is, a

material for which the two constants are specified as follows:

$$v_1 = 1, \quad v_2 = 0. \tag{1.2.7}$$

Under this condition we have the following theorem.

Theorem 1.2.2. *In the case of a purely axial material a plane wave spinor field is a solution of Rotational elasticity if and only if it is a solution of one of the two (±) massless Dirac equations (1.1.6)*

$$-i \begin{pmatrix} \pm\partial_t + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \pm\partial_t - \partial_3 \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = 0. \tag{1.2.8}$$

This somewhat surprising result motivated us to investigate the structure of the massless Dirac operator in more detail with a particular focus on the role of geometry. To achieve this aim we first had to develop an understanding of more general first order systems of partial differential equations. An overview of this work is provided in the following section.

1.3 Microlocal analysis of first order systems I

In order to develop a deeper understanding of the massless Dirac operator we first consider the more general case of an arbitrary first order system of partial differential equations under certain assumptions. The material appearing in Chapter 3 contains an abridged version of [22]. Certain aspects of this material are also investigated in [20].

Initially, we consider a classical first order self-adjoint pseudodifferential operator A acting on m-columns of complex-valued half-densities over a connected compact n-dimensional manifold without boundary, M, where both $m, n \geq 2$.

Note that the dynamical variable in this case is not a spinor field as in Section 1.1 and Section 1.2. We work instead with half-densities. A half-density $v \in C^\infty(M)$

changes in a specific manner under changes of local coordinates. We say that, given two choices of local coordinates $x, \tilde{x} \in M$, $v(x)$ is a half-density if $v(x) = J^{1/2}(x)\tilde{v}(\tilde{x}(x))$ where $\tilde{v}$ is the representation of v in the coordinates $\tilde{x}$ and $J(x) = |\det \partial\tilde{x}/\partial x|$ is the Jacobian determinant.

Half-densities are useful to work with as one is able to invariantly define an inner product without any additional assumptions or constructions. In particular, define the inner product for half-densities $v, w \in C^\infty(M)$

$$\langle v, w \rangle := \int_M w^* v \, dx. \tag{1.3.1}$$

Here the $*$ indicates complex conjugation. Note that (1.3.1) as defined here extends to the case when v, w are columns of half-densities: in this case the $*$ must be understood as indicating both complex conjugation *and* transposition, i.e. as Hermitian conjugation.

As we work with matrix operators throughout this thesis this more general notion of an inner product will be useful. Under changes of local coordinates the inner product (1.3.1) is invariant. This gives an invariantly defined notion of formal self-adjointness for our operator A:

$$\langle Av, w \rangle = \int_M w^*(Av)\, dx = \int_M (Aw)^* v \, dx = \langle v, Aw \rangle. \tag{1.3.2}$$

Again, (1.3.2) extends to the case when v, w are columns of half-densities and A is an appropriate matrix operator.

Our aim is to construct the propagator, i.e. the one-parameter family of operators defined as

$$U(t) := e^{-itA} = \sum_k e^{-it\lambda_k} v_k(x) \int_M [v_k(y)]^*(\,\cdot\,) dy, \tag{1.3.3}$$

$t \in \mathbb{R}$. The propagator provides a solution of the Cauchy problem

$$(D_t + A)w = 0, \quad w|_{t=0} = v, \tag{1.3.4}$$

cf. (1.1.5). Of course, the solution of the Cauchy problem (1.3.4) can be written in terms of the propagator (1.3.3) as $w = U(t)v$. We construct the propagator (1.3.3) explicitly in terms of oscillatory integrals, modulo an integral operator with an infinitely smooth kernel, see Chapter 3 for technical details.

This allows us to derive two-term asymptotics for the spectral function (1.1.9) of the operator A in the form

$$e(\lambda, x, x) = a(x)\lambda^n + b(x)\lambda^{n-1} + o(\lambda^{n-1}) \tag{1.3.5}$$

as $\lambda \to +\infty$. Formal integration of (1.3.5) over the manifold M gives an asymptotic formula for the counting function (1.1.10) of the operator A. Section 8 of [22] gives the details of the assumptions under which this argument holds, in particular Theorems 8.3 and 8.4 are required to justify this statement.

Our major result is being able to state explicit formulae for the coefficients $a(x)$ and $b(x)$ appearing in (1.3.5). This is significant as previous publications on the subject have either stopped short of providing such formulae or have given incorrect expressions, see Section 3.6 in Chapter 3. We show that

$$a(x) = \sum_{j=1}^{m^+} \int\!\!\!\!\!\!\diagup_{h^{(j)}<1} đ\xi, \tag{1.3.6}$$

$$\begin{aligned} b(x) = -n \sum_{j=1}^{m^+} \int\!\!\!\!\!\!\diagup_{h^{(j)}<1} \Big[& [v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} \\ & - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} + \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big] đ\xi, \end{aligned} \tag{1.3.7}$$

where m^+ is the number of positive eigenvalues $h^{(j)} = h^{(j)}(x, \xi)$ of the principal symbol A_1 of the operator A, $v^{(j)}(x)$ are the corresponding eigenvectors of the principal symbol, A_{sub} is the subprincipal symbol of A, $\{\cdot,\cdot\}$ and $\{\cdot,\cdot,\cdot\}$ are the Poisson bracket on matrix-functions and its further generalisation, and $đ\xi =$

$(2\pi)^{-n}d\xi = (2\pi)^{-n}d\xi_1 \dots d\xi_n$, see Chapter 3 for further details and complete definitions.

1.4 Microlocal analysis of first order systems II

In Chapter 4 we reduce the general setting described in Section 1.3 to the particular case wherein:

Assumption 1: The dimension of our manifold is three, $n = 3$.

Assumption 2: We are dealing with a 2×2 matrix operator, $m = 2$.

Assumption 3: Our operator is differential as opposed to pseudodifferential.

Assumption 4: The principal symbol of our operator is trace-free.

This reduction is motivated by our interest in the massless Dirac operator. However, the massless Dirac operator defined in Section 1.1 is not an operator of the type discussed above: it acts on spinors, not half-densities. Fortunately, on a parallelizable manifold one can identify the components of a spinor with half-densities (and, as we show in Chapter 4, the existence of the principal symbol of the operator A implies the 3-manifold M is parallelizable). When we perform this identification for the massless Dirac operator (1.1.1) we refer to the resulting operator as the *massless Dirac operator on half-densities*, see (4.A.3) in Appendix 4.A. Using this fact we will, in due course, be able to apply the tools developed in Chapter 3 to the massless Dirac operator on half-densities. The results can then be mapped to the case of the massless Dirac operator (1.1.1) itself.

Under Assumptions 1–4, the principal symbol of our operator A admits a complete geometric description. In particular, a number of geometric objects are encoded in the principal symbol:

- *The metric.* We observe that the determinant of the principal symbol is a negative definite quadratic form

$$\det A_1(x,\xi) = -g^{\alpha\beta}(x)\xi_\alpha\xi_\beta \tag{1.4.1}$$

and the coefficients $g^{\alpha\beta}(x)$, $\alpha,\beta = 1,2,3$, in (1.4.1) can be interpreted as the components of a symmetric contravariant Riemannian metric. In particular, this implies that the two eigenvalues of the (trace-free) principal symbol are of the form

$$\pm\sqrt{g^{\alpha\beta}(x)\xi_\alpha\xi_\beta}. \tag{1.4.2}$$

- *The Teleparallel connection.* This is the affine connection defined in an operator-theoretic manner as follows. Given a point $x \in M$ and a covector $\xi \in T'_xM$ ($T'_xM = T^*_xM\backslash\{\xi = 0\}$, i.e. the cotangent fibre with zero removed) we can construct a parallel (co)vector $\tilde{\xi} \in T'_{\tilde{x}}M$ at the point $\tilde{x} \in M$ by solving the system of linear equations

$$A_1(x,\xi) = A_1(\tilde{x},\tilde{\xi}). \tag{1.4.3}$$

Equation (1.4.3) is equivalent to a real algebraic system of three linear equations for three unknowns, the components of $\tilde{\xi}$. This system has a unique solution and, furthermore, preserves the Riemannian norm of (co)vectors, i.e. the connection is metric compatible. As the parallel transport defined by (1.4.3) is independent of the curve along which we transport the (co)vector, the connection has zero curvature. In other words, (1.4.3) defines a Teleparallel connection.

- *The topological charge.* As is mentioned above, the existence of a principal symbol implies our 3-manifold M is parallelizable and, hence, orientable. Specifying a given orientation, we only allow changes of local coordinates $x \in M$ which preserve orientation. We then define the topological charge as

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\,\mathrm{tr}\left[(A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\right]. \tag{1.4.4}$$

As we demonstrate in Chapter 4, the quantity **c** takes one of only two values, ± 1, and expresses the orientation of the principal symbol relative to the specified orientation of local coordinates.

Note that the Teleparallel connection appears in Chapter 2 as (1.2.3), although (1.4.3) is expressed in an operator-theoretic manner. In addition, the topological charge (1.4.4) is implicitly present in Chapter 2 and takes the value $+1$ by definition.

We are able to express the asymptotic coefficients of the spectral function (1.3.6), (1.3.7) for operators obeying Assumptions 1–4 in terms of the three geometric objects given above. Indeed, the following theorem is one of the main results of Chapter 4.

Theorem 1.4.1. *The coefficients of the two-term asymptotics of the spectral function (1.3.6), (1.3.7) for operators under Assumptions 1–4 are given by the formulae*

$$a(x) = \frac{1}{6\pi^2}\sqrt{\det g_{\alpha\beta}(x)}\,, \tag{1.4.5}$$

$$b(x) = \frac{1}{8\pi^2}\big([\,3\,\mathbf{c}\,{*}T^{\mathrm{ax}} - 2\,\mathrm{tr}\,A_{\mathrm{sub}}\,]\,\sqrt{\det g_{\alpha\beta}}\,\big)(x)\,, \tag{1.4.6}$$

where

$$T^{\mathrm{ax}}_{\alpha\beta\gamma} := \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha}) \tag{1.4.7}$$

is axial torsion *(totally antisymmetric piece of the torsion tensor) and $*$ is the Hodge star, see formula (4.3.4) in Chapter 4.*

Turning our attention to the massless Dirac operator on half-densities, using Theorem 1.4.1 we can state a second theorem addressing the following question: is a given operator A, under Assumptions 1–4, a massless Dirac operator on half-densities? The answer is given by the following:

Theorem 1.4.2. *The operator A is a massless Dirac operator on half-densities if and only if the following conditions are satisfied at every point of the manifold M:*

- *The subprincipal symbol of the operator A is proportional to the identity matrix.*
- *The second asymptotic coefficient of the spectral function, $b(x)$, is zero.*

This theorem completely characterizes the massless Dirac operator in terms of spectral theoretic quantities.

1.5 The massless Dirac action as the second coefficient in the asymptotics of the counting function

In Chapter 5 we study the *massless Dirac action* rather than the massless Dirac operator, i.e. the variational functional corresponding to the operator (1.1.1). This is motivated by the following observations.

Observation 1. The formula for the second asymptotic coefficient of the spectral function for an operator A under Assumptions 1–4, (1.4.6), contains a term proportional to the subprincipal symbol, A_{sub}. If we consider operators under Assumptions 1–4 with the additional assumption that the subprincipal principal is identically zero, then:

- The operator A is completely determined by its principal symbol.
- The second asymptotic coefficient of the spectral function of the operator A, (1.4.6), is proportional to the Hodge star of the axial torsion of the Teleparallel connection (1.4.7).

Observation 2. It is known that the massless Dirac action is directly related to axial torsion, see for example [23].

These two facts motivate our interest in the massless Dirac action. Our aim in Chapter 5 is to investigate the geometric meaning of the second asymptotic

coefficient of the spectral and counting functions and, as we will show, link this to the massless Dirac action. Therefore, in this chapter, we work with a first order differential 2×2 matrix operator acting on two-columns of complex-valued half-densities over a compact oriented 3-manifold under the following assumptions.

Assumption 5: The principal symbol of our operator is trace-free.

Assumption 6: The subprincipal symbol of our operator is identically zero.

As in Chapter 4, there are a number of objects encoded in our operator A.

- *The metric.* As we again work with a 2×2 matrix operator we are free to use the metric defined by (1.4.1).
- *The nonvanishing spinor field.* The determinant of the principal symbol (metric) does not uniquely determine the principal symbol itself. Supposing that the metric (1.4.1) is fixed, we are able to define a new geometric object by considering operators corresponding to the same metric. To make this more precise, fixing a reference principal symbol B_1 corresponding to the prescribed metric and considering all principal symbols A_1 corresponding to the same metric that are sufficiently close (a more detailed discussion is presented in Chapter 5) we are able to relate the two through

$$A_1(x,\xi) = R(x)B_1(x,\xi)R^*(x) \tag{1.5.1}$$

  where R is an infinitely smooth SU(2) matrix-function close to the identity. In other words, particular SU(2) matrix-functions parametrise principal symbols corresponding to a given metric.

  It is known that SU(2) matrices can be conveniently described by a (Weyl) spinor, i.e. a pair of complex numbers ζ^a, $a = 1, 2$. The relationship between the spinor field and SU(2) matrices is given by the formula

$$R = \frac{1}{\|\zeta\|}\begin{pmatrix} \overline{\zeta^1} & \overline{\zeta^2} \\ -\zeta^2 & \zeta^1 \end{pmatrix}, \tag{1.5.2}$$

where complex conjugation is denoted by the overline and $\|\cdot\|$ is the appropriate Euclidean norm. Equation (1.5.2) establishes a one-to-one correspondence between SU(2) matrices and non-zero spinors, modulo a rescaling of the spinor by an arbitrary positive real factor.

Note that in this introduction we assume, for simplicity, that the spinor field is normalised, i.e. $\|\zeta\|^2 = |\zeta^1|^2 + |\zeta^2|^2 = 1$. In Chapter 5 we actually deal with a more general weighted spectral problem which leads to a natural scaling of the spinor field, but we want to minimise the technical details at this point.

- *The topological charge.* As indicated in Chapter 4, the existence of a principal symbol implies our manifold is parallelizable and, hence, orientable. In accordance with the argument given in Chapter 4, we use the topological charge as defined by (1.4.4).

Having identified these three geometric objects encoded within our operator A, it is reasonable to assume that the corresponding coefficients of the spectral/counting functions would be expressed through these same objects. The key result of Chapter 5 is the following theorem which confirms this assertion.

Theorem 1.5.1. *The coefficients in the two-term asymptotics of the counting function (1.1.10) for the operator A as described in this section and under Assumptions 5 and 6 above are given by the formulae*

$$a = \frac{1}{6\pi^2} \int_M \|\zeta\|^3 \sqrt{\det g_{\alpha\beta}}\, dx\,, \tag{1.5.3}$$

$$b = \frac{S(\zeta)}{2\pi}, \tag{1.5.4}$$

where $S(\zeta)$ is the massless Dirac action (5.A.1) with Pauli matrices

$$\sigma^\alpha := (B_1)_{\xi_\alpha} \tag{1.5.5}$$

Note that the topological charge **c** does not appear in (1.5.1) explicitly. However, it is implicitly present in our choice of Pauli matrices.

In addition formula (1.5.3) contains the term $\|\zeta\|^3$. As indicated, in this introduction we assume the spinor field to be normalised. We wrote formula (1.5.3) in a way that would make it applicable in the more general case dealt with in Chapter 5. The theorem applies equally well to a normalised spinor field nonetheless.

Theorem 1.5.1 provides a new perspective on two-term spectral asymptotics for first order systems. The operator we have described in this section (first order differential 2×2 matrix operator in dimension three with trace-free principal symbol and vanishing subprincipal symbol) is, compared to a general system, a very special mathematical object. However, even for this relatively simple case, the two-term spectral asymptotics have a highly nontrivial geometric meaning.

1.6 Spectral asymmetry and the massless Dirac operator

In the final chapter of this thesis we consider the issue of spectral asymmetry of the massless Dirac operator on a 3-torus. It is known that, taking the standard Euclidean metric on the 3-torus, the spectrum of the massless Dirac operator can be calculated explicitly. Indeed, the spectrum is symmetric about zero and zero is itself an eigenvalue. We will show that, by perturbing the metric, one can shift the zero eigenvalue and thus achieve spectral asymmetry.

The main reason for studying this problem is that producing explicit examples of spectral asymmetry for the massless Dirac operator is difficult. While it is known that there is no particular reason that the spectrum of the massless Dirac operator should be symmetric on a general oriented Riemannian 3-manifold, to our knowledge the only explicit example is given in [65]. The example is based on selecting a 3-manifold with Euclidean metric but highly nontrivial topology.

In Chapter 6 we take a different approach. We maintain a simple topology and perturb the metric instead. We present a theorem concerning a general perturbation of this kind and then explicitly construct several families of Riemannian metrics which produce various shifts of the zero eigenvalue.

From this point we work on the unit torus $\mathbb{T}^3$ parametrised by periodic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π. Taking the (unperturbed) metric to be Euclidean ($g_{\alpha\beta} = \delta_{\alpha\beta}$) the massless Dirac operator (corresponding to the standard spin structure, see Section 5.4) reads

$$W = -i \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}, \tag{1.6.1}$$

cf. (1.1.7) and (1.1.8). The operator (1.6.1) admits separation of variables, i.e. one can seek eigenfunctions of the form $v(x) = ue^{im_\alpha x^\alpha}$ with $u \in \mathbb{C}^2 \setminus \{0\}$ and $m \in \mathbb{Z}^3$: eigenvalues and eigenfunctions can be calculated explicitly in this case.

The spectrum of (1.6.1) is as follows.

- Zero is an eigenvalue of multiplicity two.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$, $\|m\|$ is an eigenvalue with corresponding eigenfunction of the form $ue^{im_\alpha x^\alpha}$, where u is unique up to multiplication by a constant factor.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$, $-\|m\|$ is an eigenvalue with corresponding eigenfunction of the form $ue^{im_\alpha x^\alpha}$, where u is unique up to multiplication by a constant factor.

The spectrum of (1.6.1) is clearly symmetric about zero.

We now perturb the metric, i.e. consider a new metric $g_{\alpha\beta}(x; \epsilon)$, the components of which are smooth functions of the cyclic coordinates x^α, $\alpha = 1, 2, 3$, *and* a small real parameter ϵ such that

$$g_{\alpha\beta}(x; 0) = \delta_{\alpha\beta}. \tag{1.6.2}$$

We aim to demonstrate that, by choosing an appropriate perturbation, we can break the symmetry of the spectrum of the massless Dirac operator corresponding to the perturbed metric.

Demonstrating spectral asymmetry for the massless Dirac operator is complicated by the following issues.

Issue 1. One way of establishing spectral asymmetry is to compare the asymptotic distribution of large positive eigenvalues and large negative eigenvalues, see Section 10 of [22] for details of the application of this technique to generic first order differential operators. However, the massless Dirac operator is special in the sense that the second asymptotic coefficient of its counting function is zero, see Theorem 1.4.2. Therefore, in the first two approximations, its large positive eigenvalues are distributed in the same way as its large negative eigenvalues. To circumvent this issue, we will deal instead with small eigenvalues to demonstrate spectral asymmetry.

Issue 2. The massless Dirac operator commutes with the antilinear operator of *charge conjugation*, C, defined by

$$v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \to \begin{pmatrix} -\overline{v^2} \\ \overline{v^1} \end{pmatrix} =: \mathrm{C}(v). \tag{1.6.3}$$

Therefore, if v is an eigenfunction of the massless Dirac operator, so is $\mathrm{C}(v)$. This means that all eigenvalues have even multiplicity. As a result of this property we are unable to apply standard techniques to calculate the shift of the zero eigenvalue under a perturbation of the metric as described above. To counter this difficulty, we develop a perturbation theory that takes into account the charge conjugation symmetry of the massless Dirac operator. The double eigenvalue at zero does not split under perturbation, and we show that it can, effectively, be treated as a simple eigenvalue.

The perturbation theory we develop culminates in the theorem presented below. Note that we raise and lower tensor indices using the Euclidean metric throughout

the rest of this section. In addition, given a function $f : \mathbb{T}^3 \to \mathbb{C}$, we denote by

$$\hat{f}(m) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} e^{-im_\alpha x^\alpha} f(x) dx, \quad m \in \mathbb{Z}^3, \tag{1.6.4}$$

its Fourier coefficients. We can now state the following theorem, the main result of Chapter 6.

Theorem 1.6.1. *Let $W(\epsilon)$ be the massless Dirac operator corresponding to the perturbed metric $g_{\alpha\beta}(x;\epsilon)$ so that*

$$W(0) = -i \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}. \tag{1.6.5}$$

Let $\lambda_0(\epsilon)$ be the eigenvalue of $W(\epsilon)$ with smallest modulus and

$$h_{\alpha\beta}(x) := \left. \frac{\partial g_{\alpha\beta}}{\partial \epsilon} \right|_{\epsilon=0}. \tag{1.6.6}$$

Then, we have

$$\lambda_0(\epsilon) = c\epsilon^2 + O(\epsilon^3) \quad as \quad \varepsilon \to 0, \tag{1.6.7}$$

where the constant c is given by the formula

$$c = \frac{i}{16} \varepsilon_{\alpha\beta\gamma} \sum_{m \in \mathbb{Z}^3 \setminus \{0\}} \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2} \right) m_\alpha \hat{h}_{\beta\mu}(m) \overline{\hat{h}_{\gamma\nu}(m)}, \tag{1.6.8}$$

with the overline standing for complex conjugation and where ε is the totally antisymmetric quantity, $\varepsilon_{123} := +1$.

At the end of Chapter 6 we present two families of Riemannian metrics on $\mathbb{T}^3$ for which the eigenvalue with smallest modulus can be calculated explicitly. One family gives a quadratic shift of the zero eigenvalue, and the other produces a quartic shift. We show that, for both families, the explicit formulae for the eigenvalue with smallest modulus are in agreement with Theorem 1.6.1. Finally, we consider the relationship between Theorem 1.6.1 and the eta invariant, the traditional measure of spectral asymmetry.

Chapter 2

Rotational elasticity

2.1 Introduction

We work in 3-dimensional Euclidean space and view it as an elastic continuum whose material points can experience no displacements, only rotations, with rotations of different material points being kinematically independent. Rotations of material points of the 3-dimensional elastic continuum are described mathematically by attaching to each geometric point an orthonormal basis. This gives a field of orthonormal bases called the *coframe*.

The purpose of Chapter 2 is to develop a theory of elasticity on rotations, i.e. a theory of elasticity in which the coframe plays the role of the dynamical variable (unknown quantity). Recall that in classical elasticity the vector field of displacements is the dynamical variable.

Our motivation for studying such a seemingly exotic problem comes from four main sources.

The first source is *Cosserat elasticity*. In 1909 the Cosserat brothers proposed a theory of elasticity [24] which generalised classical elasticity by giving each material point rotational degrees of freedom. Cosserat elasticity has since become an accepted part of solid mechanics, though for most real life materials effects

resulting from rotations of material points are small compared to effects resulting from displacements. From a purely mathematical point of view classical elasticity and rotational elasticity are two limit cases of Cosserat elasticity. One of these limit cases, classical elasticity, has been extensively studied so it seems natural to now examine the other limit case.

The second source is *Teleparallelism* (= absolute parallelism = fernparallelismus), a subject promoted by A. Einstein and É. Cartan [18, 73, 79] in the late 1920s. The idea of rotating material points lies at the heart of Teleparallelism and can easily be traced back to Cosserat elasticity: when in 1922 Cartan started developing what eventually became modern differential geometry he acknowledged [17] that he drew inspiration from the 'beautiful' work of the Cosserat brothers. The relationship between Cosserat elasticity and Teleparallelism is examined in detail in the review paper [39]. The basic difference between Teleparallelism and our model is that in Teleparallelism the metric is viewed as a dynamical variable rather than as a prescribed tensor field: formula (2.2.1) ceases being a kinematic constraint and becomes the definition of the metric. Consequently, in Teleparallelism the coframe is varied without any constraints whatsoever which, on the one hand, makes calculations easier and, on the other, allows one to model gravitational phenomena.

The third source is the theory of liquid crystals and, in particular, the concept of an *Ericksen fluid*. According to [29], in a liquid crystal one can observe 'orientation waves which propagate, inducing little or no motion of the fluid' and Ericksen's mathematical model is the natural way of describing this phenomenon. The only difference between Ericksen's model and ours is that in Ericksen's model one attaches to each geometric point a single unit vector rather than an orthonormal basis as we do. More precisely, in Ericksen's model there is also no distinction between the head and tail of the unit vector, which leads to mathematical analysis based on projective geometry.

Finally, the fourth source is the concept of *gyrocontinuum* proposed by M. Brocato and G. Capriz [13]. Quoting from [13], Brocato and Capriz "consider a body

endowed with the fine structure of very many, very small gyroscopes and describe it as a sort of composite (perhaps constrained) Cosserat continuum (gyrocontinuum)". Note that even though the basic idea of a rotating material point in [13] is similar to ours, there are substantial differences between our model and that of [13]. In particular, the equations in [13] are nonlinear both in terms of geometry and material behaviour (while the material behaves linearly in our model) and the displacements of material points in [13] are nonzero.

From a physical point of view we are studying a classical field theory which gives a novel description of fermions, i.e. elementary particles such as the neutrino and the electron. Our theory produces a wave description for particles, however, a formal quantisation scheme is yet to be developed. This means that we are attempting to model "world aether" rather than some realistic macroscopic material. A more modern way of rephrasing this is that we are modelling possible excitations of the vacuum and giving them a particle interpretation.

Chapter 2 has the following structure. In Section 2.2 we define our dynamical variables (unknowns of our theory), in Section 2.3 we write down the kinetic energy and in Section 2.4 we write down the potential energy. The Lagrangian of rotational elasticity is written down in Section 2.5. In Section 2.6 we reformulate our model in the language of spinors and in Section 2.7 we discuss the corresponding Euler–Lagrange equation. In Section 2.8 we construct an explicit class of solutions which we call plane wave solutions; this construction is summarised in Theorem 2.8.1 which is the main result of Chapter 2. Finally, in Section 2.9 we compare our model with the massless Dirac equation and in Section 2.10 we briefly outline a linearised version of rotational elasticity.

2.2 Setting the playing field

We work in Euclidean space $\mathbb{R}^3$ equipped with Cartesian coordinates x^α, $\alpha = 1, 2, 3$, and standard Euclidean metric. We denote time by t. Partial differentiation in t and x^α, $\alpha = 1, 2, 3$, is denoted by ∂_t and ∂_α respectively.

The coframe e is a triple of orthonormal covector fields e^j, $j = 1, 2, 3$, in $\mathbb{R}^3$. Each covector field e^j can be written more explicitly as ${e^j}_\alpha$ where the tensor index $\alpha = 1, 2, 3$ enumerates the components. The orthonormality condition for the coframe can be represented as a single tensor identity

$$g = \delta_{jk} e^j \otimes e^k \tag{2.2.1}$$

where δ is the Kronecker delta and $g_{\alpha\beta} = \delta_{\alpha\beta}$ is the Euclidean metric. For the sake of clarity we repeat formula (2.2.1) giving tensor indices explicitly and performing summation over frame indices explicitly:

$$\delta_{\alpha\beta} = {e^1}_\alpha {e^1}_\beta + {e^2}_\alpha {e^2}_\beta + {e^3}_\alpha {e^3}_\beta \tag{2.2.2}$$

where α and β run through the values $1, 2, 3$. We view the identity (2.2.1) as a kinematic constraint: the covector fields e^j are chosen so that they satisfy (2.2.1), which leaves us with three real degrees of freedom at every point of $\mathbb{R}^3$.

We work only with coframes which have positive orientation, i.e. which satisfy the condition

$$\det {e^j}_\alpha = +1 > 0. \tag{2.2.3}$$

If one views ${e^j}_\alpha$ as a 3×3 real matrix function, then conditions (2.2.1) and (2.2.3) mean that this matrix function is special orthogonal. Thus, the coframe can be thought of as a field of special orthogonal matrices. This matrix approach is explained in greater detail in Appendix 2.E.

As dynamical variables in our model we choose the coframe e and a positive density ρ. Our coframe and density are functions of Cartesian coordinates x^α, $\alpha = 1, 2, 3$, as well as of time t. At a physical level, making the density ρ a dynamical variable means that we view our continuum more like a fluid rather than a solid. In other words, we allow the material to redistribute itself so that it finds its equilibrium density distribution. Observe that the total number of real

dynamical degrees of freedom contained in the coframe e and positive density ρ is four, exactly as in a two-component complex-valued spinor field.

The underlying motivation for taking the density to be a dynamical variable is our aim of modelling quantum mechanical phenomena. This will become clear when we eventually switch to spinors: we will arrive at formula (2.6.3) which is the standard formula for the density of a massless neutrino field (zeroth component of the neutrino 4-current).

Note that there is nothing fundamentally wrong in taking a prescribed density (as opposed to a density which is a dynamical variable): the theory one gets is very similar to the one described in the current chapter and most formulae carry through with minimal changes. We will briefly examine a version of our theory with prescribed ρ in the very end of Chapter 2, in Section 2.10.

Below is the list of the main assumptions on which our model will be based.

Assumption 1: our model is geometrically nonlinear. This means that we do not linearise rotations and we do not linearise the density. In other words, we allow our material points to experience full turns and we allow our density to experience changes comparable to the density itself.

Assumption 2: our material is physically linear. This means that our potential energy is chosen to be quadratic in torsion (the latter serves as the measure of rotational deformations, see Section 2.4.1). Note that physical linearity does not contradict geometric nonlinearity: locally (in space and time) material points "do not know" that they may eventually experience full rotations and the density "does not know" that it may eventually experience a change comparable to its current value. This assumption is similar to the hyperelasticity assumption sometimes made in classical elasticity. It describes a highly idealised situation which is not likely to occur in realistic materials.

Assumption 3: our material is homogeneous and isotropic. Homogeneity means that physical properties of the material are the same at all points of our continuum and isotropy means that there are no preferred directions.

Assumption 4: our model is invariant under rigid rotations of the coframe. By a rigid rotation of the coframe we understand the transformation

$$e^j \mapsto O^j{}_k e^k \tag{2.2.4}$$

where $O^j{}_k$ is a *constant* special orthogonal matrix. The thinking here is that when we attach an orthonormal basis to each geometric point of our continuum there is no reason to associate one particular direction with e^1, another with e^2 and a third with e^3. What matters is how these directions change when we move from one point to another, i.e. how orthonormal bases at different points differ relative to each other. A rigid rotation of the coframe means that we simultaneously rotate all our orthonormal bases by the same angle around the same axis. We view rigid rotations of the coframe as gauge transformations and assume that our model does not feel them. See also [51] for a detailed exposition of gauge theory for problems similar to the ones considered in Chapter 2.

Assumption 4 effectively means that the actual dynamical variable in our model is not the coframe itself, but the coframe modulo a rigid rotation. A loose way of rephrasing this would be to say that the actual dynamical variable is not the coframe but rather its rotation.

Assumption 4 is similar to the assumption of objectivity or frame-indifference in classical (without displacements) elasticity theory. However, it is not the same. When in classical elasticity one talks of objectivity or frame-indifference this normally refers to rotations of the coordinate system. Assumption 4, on the other hand, refers to "internal" degrees of freedom present in the rotating material point and the coframe here is unrelated to the choice of coordinate system.

2.3 Kinetic energy

Kinetic energy is given by the formula

$$K(t) = c^{\text{kin}} \int \|\omega\|^2 \rho \, dx^1 dx^2 dx^3 \tag{2.3.1}$$

where c^{kin} is some positive constant and ω is the (pseudo)vector of angular velocity

$$\omega = \frac{1}{2} * (\delta_{jk} e^j \wedge \partial_t e^k). \tag{2.3.2}$$

Here $\wedge$ is the exterior product and $*$ is the Hodge star (2.A.1).

In writing the formula for kinetic energy (2.3.1) we think of each material point as a uniform ball possessing a moment of inertia and without a preferred axis of rotation.

If we were to consider a more complete model, with displacements, the formula for angular velocity (2.3.2) would change as displacements generate macrorotations.

We give for reference a more explicit version of the formula for angular velocity (2.3.2):

$$\omega_\alpha = \frac{1}{2} \sum_{j=1}^{3} \begin{pmatrix} e^j{}_2 \partial_t e^j{}_3 - e^j{}_3 \partial_t e^j{}_2 \\ e^j{}_3 \partial_t e^j{}_1 - e^j{}_1 \partial_t e^j{}_3 \\ e^j{}_1 \partial_t e^j{}_2 - e^j{}_2 \partial_t e^j{}_1 \end{pmatrix}. \tag{2.3.3}$$

2.4 Potential energy

2.4.1 Measuring rotational deformations

In order to write down the formula for the potential energy we need to measure deformations caused by rotations of the coframe. More specifically, we need to measure deformations caused by the fact that at different points the coframe is oriented differently. Obvious candidates for a measure of deformations are the

rank two tensors

$$K^j := \partial e^j, \qquad j = 1, 2, 3, \tag{2.4.1}$$

or, in more explicit form, $K^j{}_{\alpha\beta} := \partial_\alpha e^j{}_\beta$. The problem is that taken separately the three rank two tensors (2.4.1) are not invariant under rigid rotations of the coframe (2.2.4). The natural way of forming a truly invariant object is to make one rank three tensor out of the three rank two tensors K^j according to the formula

$$K := \delta_{jk} e^j \otimes K^k = \delta_{jk} e^j \otimes \partial e^k. \tag{2.4.2}$$

The rank three tensor K is invariant under rigid rotations of the coframe (2.2.4) and, moreover, the individual rank two tensors K^j can be recovered from K as $K^j{}_{\gamma\delta} = e^{j\alpha} K_{\alpha\gamma\delta}$, where $e^{j\alpha} = e^j{}_\beta g^{\alpha\beta}$, so there is no loss of information.

Let us examine the symmetries of the tensor K. Observe that formula (2.2.1) implies

$$0 = \partial_\alpha g_{\beta\gamma} = \partial_\alpha(\delta_{jk} e^j{}_\beta e^k{}_\gamma) = \delta_{jk}(\partial_\alpha e^j{}_\beta) e^k{}_\gamma + \delta_{jk} e^j{}_\beta (\partial_\alpha e^k{}_\gamma) = K_{\gamma\alpha\beta} + K_{\beta\alpha\gamma}$$

which means that the rank three tensor K is antisymmetric in the first and third indices,

$$K_{\gamma\alpha\beta} = -K_{\beta\alpha\gamma}. \tag{2.4.3}$$

Now, let us introduce another rank three tensor

$$T := \delta_{jk} e^j \otimes de^k \tag{2.4.4}$$

where d stands for the exterior derivative, see Definition 5.5, Section 5.4.2, in [53]. The tensor (2.4.4) is obviously antisymmetric in the second and third indices

$$T_{\alpha\beta\gamma} = -T_{\alpha\gamma\beta} \tag{2.4.5}$$

and is expressed via our original deformation tensor (2.4.2) as

$$T_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} - K_{\alpha\gamma\beta}. \tag{2.4.6}$$

Formulae (2.4.6) and (2.4.3) imply

$$\begin{aligned} T_{\alpha\beta\gamma} &= K_{\alpha\beta\gamma} + K_{\beta\gamma\alpha}, \\ T_{\gamma\alpha\beta} &= K_{\gamma\alpha\beta} + K_{\alpha\beta\gamma}, \\ T_{\beta\gamma\alpha} &= K_{\beta\gamma\alpha} + K_{\gamma\alpha\beta} \end{aligned}$$

where the last two identities were obtained from the first one by a cyclic relabelling of tensor indices. Adding up the first and second identities and subtracting the third one we get

$$K_{\alpha\beta\gamma} = \frac{1}{2}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} - T_{\beta\gamma\alpha}) = \frac{1}{2}(T_{\alpha\beta\gamma} + T_{\beta\alpha\gamma} + T_{\gamma\alpha\beta}) \tag{2.4.7}$$

(here we also used (2.4.5)). Note that the argument carried out above is a rephrasing of the standard argument that for a metric compatible affine connection contortion can be expressed via torsion, see Section 7.2.6 in [53].

Formulae (2.4.6) and (2.4.7) show that the tensors K and T are expressed via each other so either of them can be used as a measure of rotational deformations. We choose to use the tensor T because it has a clear geometric meaning: it is the torsion of the Teleparallel connection generated by the coframe e, see Appendix A of [15] for a concise exposition. An additional advantage of using the tensor T is that the definition (2.4.4) of this tensor does not require the use of covariant derivatives so it works when the metric g appearing in formula (2.2.1) is not assumed to be Euclidean. The latter was important for Einstein and Cartan who arrived at the mathematical model similar to the one described in in Chapter 2 coming from general relativity. Recall that in general relativity the metric plays the role of dynamical variable so for someone with a relativistic background assuming the metric to be Euclidean (i.e. space to be flat) is unnatural.

Starting from Einstein's works [79] torsion is traditionally used as a measure of deformations when modelling elastic continua with rotations. We shall follow this tradition and construct our potential energy as a function(al) of T. However, before writing down the formula for potential energy we will simplify matters by using the fact that we are working in three dimensions (our previous arguments were dimension-independent).

Applying the Hodge star (2.A.1) in the second and third indices we switch from the original torsion tensor T to the tensor

$$\overset{*}{T}_{\alpha\beta} := \frac{1}{2} T_\alpha{}^{\gamma\delta} \varepsilon_{\gamma\delta\beta}. \tag{2.4.8}$$

Of course, the tensor T can be recovered from $\overset{*}{T}$ as

$$T_{\alpha\beta\gamma} = \overset{*}{T}_\alpha{}^\delta \varepsilon_{\delta\beta\gamma}. \tag{2.4.9}$$

Formulae (2.4.8) and (2.4.9) show that the tensors T and $\overset{*}{T}$ are expressed via each other so either of them can be used as a measure of rotational deformations. We choose to use the tensor $\overset{*}{T}$ because it has lower rank, two instead of three.

Formulae (2.4.4) and (2.4.8) imply

$$\overset{*}{T} = \delta_{jk} e^j \otimes *de^k = \delta_{jk} e^j \otimes \operatorname{curl} e^k. \tag{2.4.10}$$

We see that $\overset{*}{T}$ is a rank two tensor without any symmetries and with arbitrary trace. This is the tensor we will be using as a measure of rotational deformations when writing down the formula for potential energy. The tensor $\overset{*}{T}$ is sometimes called the *dislocation density tensor* [49] .

We give for reference a more explicit version of formula (2.4.10):

$$\overset{*}{T}_{\alpha\beta} = \sum_{j=1}^{3} \begin{pmatrix} e^j{}_1\partial_2 e^j{}_3 - e^j{}_1\partial_3 e^j{}_2 & e^j{}_1\partial_3 e^j{}_1 - e^j{}_1\partial_1 e^j{}_3 & e^j{}_1\partial_1 e^j{}_2 - e^j{}_1\partial_2 e^j{}_1 \\ e^j{}_2\partial_2 e^j{}_3 - e^j{}_2\partial_3 e^j{}_2 & e^j{}_2\partial_3 e^j{}_1 - e^j{}_2\partial_1 e^j{}_3 & e^j{}_2\partial_1 e^j{}_2 - e^j{}_2\partial_2 e^j{}_1 \\ e^j{}_3\partial_2 e^j{}_3 - e^j{}_3\partial_3 e^j{}_2 & e^j{}_3\partial_3 e^j{}_1 - e^j{}_3\partial_1 e^j{}_3 & e^j{}_3\partial_1 e^j{}_2 - e^j{}_3\partial_2 e^j{}_1 \end{pmatrix} . \tag{2.4.11}$$

2.4.2 Irreducible decomposition of rotational deformations

Recall the logic of classical linear elasticity [50]: after identifying the measure of deformation one decomposes it into irreducible pieces. We follow this logic by decomposing the tensor $\overset{*}{T}$ into irreducible pieces. The construction presented below is similar to [50], the only difference being that instead of a symmetric rank two tensor, strain, we deal with a rank two tensor, $\overset{*}{T}$, without any symmetries.

Decomposing the rank two tensor $\overset{*}{T}$ into irreducible pieces means the following. We fix a point in $\mathbb{R}^3$ and at this point look at all real rank two tensors P. Such tensors can be viewed as elements of a real 9-dimensional vector space V equipped with inner product

$$(P, Q)_V := P_{\alpha\beta}Q^{\alpha\beta} \tag{2.4.12}$$

and corresponding norm

$$\|P\|_V = \sqrt{(P, P)_V} = \sqrt{P_{\alpha\beta}P^{\alpha\beta}}\,. \tag{2.4.13}$$

Let us now examine what happens when we rotate our Cartesian coordinate system x^α, i.e. when we perform a linear change of coordinates preserving the metric $g_{\alpha\beta}$ and orientation. The components of our tensors $P_{\alpha\beta}$ change in a particular way under rotations of the coordinate system, so we get an action of the group SO(3) on the vector space V. Looking for irreducible pieces of torsion means identifying subspaces of V which are invariant under the action of the group SO(3), i.e. which map into themselves, and which do not contain smaller nontrivial invariant subspaces.

In our case the invariant subspaces can be shown to be

- the 1-dimensional subspace of real rank two tensors proportional to the metric,
- the 3-dimensional subspace of real antisymmetric rank two tensors and
- the 5-dimensional subspace of real symmetric trace-free rank two tensors.

These three subspaces are irreducible and mutually orthogonal in the inner product (2.4.12); explicit calculations are omitted for brevity.

Our rank two tensor $\overset{*}{T}$ can now be written as a sum of three irreducible pieces

$$\overset{*}{T} = \overset{*}{T}^{\mathrm{ax}} + \overset{*}{T}^{\mathrm{vec}} + \overset{*}{T}^{\mathrm{ten}} \tag{2.4.14}$$

where

$$\overset{*}{T}^{\mathrm{ax}}_{\alpha\beta} := \frac{\overset{*}{T}^{\gamma}{}_{\gamma}}{3}\, g_{\alpha\beta}\,, \tag{2.4.15}$$

$$\overset{*}{T}^{\mathrm{vec}}_{\alpha\beta} := \frac{\overset{*}{T}_{\alpha\beta} - \overset{*}{T}_{\beta\alpha}}{2}\,, \tag{2.4.16}$$

$$\overset{*}{T}^{\mathrm{ten}}_{\alpha\beta} := \overset{*}{T}_{\alpha\beta} - \overset{*}{T}^{\mathrm{ax}}_{\alpha\beta} - \overset{*}{T}^{\mathrm{vec}}_{\alpha\beta} = \frac{\overset{*}{T}_{\alpha\beta} + \overset{*}{T}_{\beta\alpha}}{2} - \frac{\overset{*}{T}^{\gamma}{}_{\gamma}}{3}\, g_{\alpha\beta}\,. \tag{2.4.17}$$

We label the irreducible pieces (2.4.15), (2.4.16) and (2.4.17) by the adjectives *axial*, *vector* and *tensor* respectively, which is terminology traditional in alternative theories of gravity [39]. Note that authors working in "mainstream" Cosserat elasticity use different terminology. Say, Neff's [46, 57, 58, 59, 60, 61, 62] use of the adjective "axial" corresponds to our vector piece (2.4.16).

2.4.3 Formula for potential energy

Following the logic of classical linear elasticity [50] we now write down the explicit formula for potential energy:

$$P(t) = \int \bigl(c^{\mathrm{ax}} \|\overset{*}{T}{}^{\mathrm{ax}}\|_V^2 + c^{\mathrm{vec}} \|\overset{*}{T}{}^{\mathrm{vec}}\|_V^2 + c^{\mathrm{ten}} \|\overset{*}{T}{}^{\mathrm{ten}}\|_V^2\bigr) \rho \, dx^1 dx^2 dx^3 \tag{2.4.18}$$

where c^{ax}, c^{vec} and c^{ten} are some nonnegative constants (elastic moduli), not all zero, and $\| \cdot \|_V$ is the norm (2.4.13). Comparing our formula (2.4.18) with formula (4.3) from [50] we see a difference between classical and rotational elasticity: classical elasticity involves two elastic moduli whereas rotational elasticity involves three. The extra elastic modulus c^{vec} is needed because the tensor $\overset{*}{T}$ which we use as measure of rotational deformations is not necessarily symmetric.

Formula (2.4.18) is traditionally used in Teleparallelism. This formula already appears in the original papers of Einstein [79], though for some reason[1] Einstein did not include the axial term $c^{\mathrm{ax}} \|\overset{*}{T}{}^{\mathrm{ax}}\|^2$. Subsequent authors always used three terms, see, for example, formula (26) in [39].

2.4.4 Simplifying the formula for potential energy

Let us introduce the (pseudo)scalar

$$f := \overset{*}{T}{}^{\alpha}{}_{\alpha} \tag{2.4.19}$$

and the vector

$$v_\alpha := \overset{*}{T}{}^{\beta\gamma} \varepsilon_{\beta\gamma\alpha} \, . \tag{2.4.20}$$

Formulae (2.4.13)–(2.4.17), (2.4.19) and (2.4.20) imply

$$\|\overset{*}{T}{}^{\mathrm{ax}}\|_V^2 = \frac{1}{3} f^2, \tag{2.4.21}$$

[1] The reason could be that Einstein was primarily interested in providing a geometric interpretation of electromagnetism and might have felt that the axial term would not contribute to the electromagnetic field.

$$\|\overset{*}{T}{}^{\mathrm{vec}}\|_V^2 = \frac{1}{2}\|v\|^2, \tag{2.4.22}$$

$$\|\overset{*}{T}{}^{\mathrm{ten}}\|_V^2 = \|\overset{*}{T}\|_V^2 - \frac{1}{3}f^2 - \frac{1}{2}\|v\|^2. \tag{2.4.23}$$

Substituting (2.4.11) into (2.4.19) and (2.4.20) we get more explicit formulae for f and v:

$$f = \sum_{j=1}^{3}(e^j{}_1\partial_2 e^j{}_3 - e^j{}_1\partial_3 e^j{}_2 + e^j{}_2\partial_3 e^j{}_1 - e^j{}_2\partial_1 e^j{}_3 + e^j{}_3\partial_1 e^j{}_2 - e^j{}_3\partial_2 e^j{}_1), \tag{2.4.24}$$

$$v_\alpha = \sum_{j=1}^{3}\begin{pmatrix} e^j{}_2\partial_1 e^j{}_2 - e^j{}_2\partial_2 e^j{}_1 - e^j{}_3\partial_3 e^j{}_1 + e^j{}_3\partial_1 e^j{}_3 \\ e^j{}_3\partial_2 e^j{}_3 - e^j{}_3\partial_3 e^j{}_2 - e^j{}_1\partial_1 e^j{}_2 + e^j{}_1\partial_2 e^j{}_1 \\ e^j{}_1\partial_3 e^j{}_1 - e^j{}_1\partial_1 e^j{}_3 - e^j{}_2\partial_2 e^j{}_3 + e^j{}_2\partial_3 e^j{}_2 \end{pmatrix}. \tag{2.4.25}$$

Substituting formulae (2.4.19)–(2.4.21) into formula (2.4.18) we get

$$P(t) = \int\left(\frac{c^{\mathrm{ax}} - c^{\mathrm{ten}}}{3}f^2 + \frac{c^{\mathrm{vec}} - c^{\mathrm{ten}}}{2}\|v\|^2 + c^{\mathrm{ten}}\overset{*}{T}_{\alpha\beta}\overset{*}{T}{}^{\alpha\beta}\right)\rho\, dx^1dx^2dx^3\,. \tag{2.4.26}$$

The advantage of writing potential energy in the form (2.4.26) is that the geometric quantities f, v and $\overset{*}{T}$ appearing in this formula have relatively compact explicit representations (2.4.24), (2.4.25) and (2.4.11).

2.5 Lagrangian of Rotational elasticity

We combine our potential energy (2.4.26) and kinetic energy (2.3.1) to form the action (variational function) of dynamic rotational elasticity

$$S_{\mathrm{Re}}(e,\rho) = \int(P(t) - K(t))dt = \int L_{\mathrm{Re}}(e,\rho)\, dt\, dx^1dx^2dx^3 \tag{2.5.1}$$

where

$$L_{\mathrm{Re}}(e,\rho) = \left(\frac{c^{\mathrm{ax}} - c^{\mathrm{ten}}}{3}f^2 + \frac{c^{\mathrm{vec}} - c^{\mathrm{ten}}}{2}\|v\|^2 + c^{\mathrm{ten}}\overset{*}{T}_{\alpha\beta}\overset{*}{T}{}^{\alpha\beta} - c^{\mathrm{kin}}\|\omega\|^2\right)\rho \tag{2.5.2}$$

is the Lagrangian density. Recall that the geometric quantities f, v, $\overset{*}{T}$ and ω appearing in formula (2.5.2) are defined by formulae (2.4.24), (2.4.25), (2.4.11) and (2.3.3) respectively.

An equivalent matrix representation of the Lagrangian density (2.5.2) is provided in Appendix 2.E.

It is worth noting that the choice of parameters in the Lagrangian of Cosserat elasticity is highly controversial. The Cosserat model as such is not accepted in the mainstream elasticity theory community and it has never been shown to be superior to classical models. On the positive side, the Cosserat model allows for the rigorous derivation [59, 60] of certain plate theories.

A comprehensive analysis of the equations of Cosserat elasticity in the complete (with both displacements and microrotations) setting was performed in [46, 57, 58, 59, 60, 61, 62]. In particular, the authors of [62] performed a scale analysis which led them to the conclusion that the constants c^{ax} and c^{vec} appearing in formula (2.5.2) should be zero. We do not believe that the arguments presented in [62] are directly relevant to our case because the authors of [62] assumed the strain part of the Lagrangian to be nonzero. In other words, the authors of [62] had in mind a realistic macroscopic material whereas we do not aim at modelling such a material.

2.6 Reformulating the problem in the language of spinors

Our field equations (Euler–Lagrange equations) are obtained by varying the action (2.5.1) with respect to the coframe e and density ρ. Varying with respect to the density ρ is easy: this gives the field equation

$$\frac{c^{\mathrm{ax}} - c^{\mathrm{ten}}}{3} f^2 + \frac{c^{\mathrm{vec}} - c^{\mathrm{ten}}}{2} \|v\|^2 + c^{\mathrm{ten}} \overset{*}{T}_{\alpha\beta} \overset{*}{T}^{\alpha\beta} - c^{\mathrm{kin}} \|\omega\|^2 = 0 \tag{2.6.1}$$

which is equivalent to $L_{\mathrm{Re}}(e,\rho)=0$. Varying with respect to the coframe e is more difficult because we have to maintain the kinematic constraint (2.2.1).

This technical difficulty can be overcome by switching to a different dynamical variable. Namely, it is known [23] that in dimension 3 a coframe e and a (positive) density ρ are equivalent to a 2-component complex-valued spinor field

$$\xi=\xi^a=\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} \tag{2.6.2}$$

modulo the sign of ξ (i.e. modulo a factor ±1). The explicit formulae establishing this equivalence are

$$\rho=\bar\xi^{\dot a}\sigma_{0\dot a b}\xi^b, \tag{2.6.3}$$

$$(e^1+ie^2)_\alpha=\rho^{-1}\epsilon^{\dot c\dot b}\sigma_{0\dot b a}\xi^a\sigma_{\alpha\dot c d}\xi^d, \tag{2.6.4}$$

$$e^3{}_\alpha=\rho^{-1}\bar\xi^{\dot a}\sigma_{\alpha\dot a b}\xi^b. \tag{2.6.5}$$

Here σ are Pauli matrices and ϵ is "metric spinor" (see (2.A.2)–(2.A.4)), the free tensor index α runs through the values $1,2,3$, and the spinor summation indices run through the values $1,2$ or $\dot1,\dot2$. The advantage of switching to a spinor field ξ is that there are no kinematic constraints on its components, so the derivation of field equations becomes straightforward.

We give for reference more explicit versions of formulae (2.6.3)–(2.6.5):

$$\rho=\bar\xi^{\dot1}\xi^1+\bar\xi^{\dot2}\xi^2, \tag{2.6.6}$$

$$(e^1+ie^2)=(\bar\xi^{\dot1}\xi^1+\bar\xi^{\dot2}\xi^2)^{-1}\begin{pmatrix}(\xi^1)^2-(\xi^2)^2\\ i(\xi^1)^2+i(\xi^2)^2\\ -2\xi^1\xi^2\end{pmatrix}, \tag{2.6.7}$$

$$e^3=(\bar\xi^{\dot1}\xi^1+\bar\xi^{\dot2}\xi^2)^{-1}\begin{pmatrix}\bar\xi^{\dot2}\xi^1+\bar\xi^{\dot1}\xi^2\\ i\bar\xi^{\dot2}\xi^1-i\bar\xi^{\dot1}\xi^2\\ \bar\xi^{\dot1}\xi^1-\bar\xi^{\dot2}\xi^2\end{pmatrix}. \tag{2.6.8}$$

Let us rewrite the geometric quantities f, v, $\overset{*}{T}$ and ω appearing in formula (2.5.2) in terms of the spinor field ξ. The spinor representation of angular velocity ω was derived in [23]:

$$\omega_\alpha = i\frac{\bar{\xi}^{\dot{a}}\sigma_{\alpha\dot{a}b}\partial_t\xi^b - \xi^b\sigma_{\alpha\dot{a}b}\partial_t\bar{\xi}^{\dot{a}}}{\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d} \tag{2.6.9}$$

or, more explicitly,

$$\omega = \frac{1}{\bar{\xi}^{\dot{1}}\xi^1 + \bar{\xi}^{\dot{2}}\xi^2}\begin{pmatrix} i\bar{\xi}^{\dot{2}}\partial_t\xi^1 + i\bar{\xi}^{\dot{1}}\partial_t\xi^2 \\ -\bar{\xi}^{\dot{2}}\partial_t\xi^1 + \bar{\xi}^{\dot{1}}\partial_t\xi^2 \\ i\bar{\xi}^{\dot{1}}\partial_t\xi^1 - i\bar{\xi}^{\dot{2}}\partial_t\xi^2 \end{pmatrix} + \text{c.c.} \tag{2.6.10}$$

where the "c.c." stands for "complex conjugate term". The spinor representation of the tensor $\overset{*}{T}$ is derived in Appendix 2.B, see formula (2.B.1) or its more explicit version (2.B.7). Substituting (2.B.1) or (2.B.7) into (2.4.19) and (2.4.20) we arrive at spinor representations for the (pseudo)scalar f and vector v:

$$f = -2i\frac{\bar{\xi}^{\dot{a}}\sigma^\alpha{}_{\dot{a}b}\partial_\alpha\xi^b - \xi^b\sigma^\alpha{}_{\dot{a}b}\partial_\alpha\bar{\xi}^{\dot{a}}}{\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d}\,, \tag{2.6.11}$$

$$v_\alpha = -i\varepsilon_{\beta\gamma\alpha}\frac{\bar{\xi}^{\dot{a}}\sigma^\beta{}_{\dot{a}b}\partial^\gamma\xi^b - \xi^b\sigma^\beta{}_{\dot{a}b}\partial^\gamma\bar{\xi}^{\dot{a}}}{\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d} \tag{2.6.12}$$

where $\partial^\alpha := g^{\alpha\beta}\partial_\beta = \partial_\alpha$, or, more explicitly,

$$f = \frac{2}{\bar{\xi}^{\dot{1}}\xi^1 + \bar{\xi}^{\dot{2}}\xi^2}(-i\bar{\xi}^{\dot{1}}\partial_1\xi^2 - i\bar{\xi}^{\dot{2}}\partial_1\xi^1 - \bar{\xi}^{\dot{1}}\partial_2\xi^2 + \bar{\xi}^{\dot{2}}\partial_2\xi^1 - i\bar{\xi}^{\dot{1}}\partial_3\xi^1 + i\bar{\xi}^{\dot{2}}\partial_3\xi^2) + \text{c.c.}, \tag{2.6.13}$$

$$v = \frac{1}{\bar{\xi}^{\dot{1}}\xi^1 + \bar{\xi}^{\dot{2}}\xi^2}\begin{pmatrix} i\bar{\xi}^{\dot{1}}\partial^2\xi^1 - i\bar{\xi}^{\dot{2}}\partial^2\xi^2 - \bar{\xi}^{\dot{1}}\partial^3\xi^2 + \bar{\xi}^{\dot{2}}\partial^3\xi^1 \\ i\bar{\xi}^{\dot{1}}\partial^3\xi^2 + i\bar{\xi}^{\dot{2}}\partial^3\xi^1 - i\bar{\xi}^{\dot{1}}\partial^1\xi^1 + i\bar{\xi}^{\dot{2}}\partial^1\xi^2 \\ \bar{\xi}^{\dot{1}}\partial^1\xi^2 - \bar{\xi}^{\dot{2}}\partial^1\xi^1 - i\bar{\xi}^{\dot{1}}\partial^2\xi^2 - i\bar{\xi}^{\dot{2}}\partial^2\xi^1 \end{pmatrix} + \text{c.c.} \tag{2.6.14}$$

Note that formula (2.6.11) is a rephrasing of formula (B.5) from [23].

From now on we write our action (2.5.1) and Lagrangian density (2.5.2) as $S_{\text{Re}}(\xi)$ and $L_{\text{Re}}(\xi)$ rather than $S_{\text{Re}}(e,\rho)$ and $L_{\text{Re}}(e,\rho)$, thus indicating that we have switched to spinors. The explicit formula for $L_{\text{Re}}(\xi)$ is obtained by substituting

formulae (2.6.3), (2.6.11), (2.6.12), (2.B.1) and (2.6.9) into (2.5.2). The nonvanishing spinor field ξ is the new dynamical variable and it will be varied without any constraints.

2.7 Euler–Lagrange equation

Let us perform a formal variation of our spinor field $\xi \mapsto \xi + \delta\xi$, where $\delta\xi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^2$ is an arbitrary (infinitely) smooth function with compact support. Then, after integration by parts, the variation of our action can be written as

$$\delta S_{\mathrm{Re}} = \int (F_{\dot{a}} \delta\bar{\xi}^{\dot{a}} + \bar{F}_a \delta\xi^a)\, dt\, dx^1 dx^2 dx^3 \tag{2.7.1}$$

where F is a dotted spinor field uniquely determined by the undotted spinor field ξ. The Euler–Lagrange for our unknown spinor field ξ is, therefore,

$$F = 0. \tag{2.7.2}$$

The map

$$\xi \mapsto F \tag{2.7.3}$$

defines a nonlinear second order partial differential operator in the variables t (time) and x^α, $\alpha = 1, 2, 3$ (Cartesian coordinates).

We shall refrain from writing down the Euler–Lagrange equation (2.7.2) explicitly. The reason for this is that in the current chapter we are interested in finding a particular class of solutions for which the procedure is much simpler.

Note that for the special case of a purely axial material, i.e. material with

$$c^{\mathrm{vec}} = 0, \qquad c^{\mathrm{ten}} = 0, \tag{2.7.4}$$

the Euler–Lagrange equation (2.7.2) was written down explicitly in [23]. The calculations in [23] were carried out under the additional assumption

$$c^{\mathrm{kin}} = \frac{4}{3} c^{\mathrm{ax}} \tag{2.7.5}$$

which can always be achieved by rescaling time t.

2.8 Plane wave solutions

In this section, as well as in Section 2.9 and Appendix 2.C, we denote time by x^0 and the time derivative by ∂_0. This will allow us to simplify notation.

We seek solutions of the form

$$\xi(\mathbf{x}) = e^{-i\mathbf{p}\cdot\mathbf{x}}\zeta \tag{2.8.1}$$

where $\zeta \neq 0$ is a constant (complex) spinor and $\mathbf{p}$ is a constant real covector. Here we use relativistic notation, incorporating time x^0 into our coordinates. This means that $\mathbf{x} = (x^0, x^1, x^2, x^3)$ and $\mathbf{p} = (p_0, p_1, p_2, p_3)$; bold type indicates that we are working in (1+3)-dimensional spacetime. The number $|p_0|$ is the wave frequency and the covector (p_1, p_2, p_3) is the wave vector in original 3-dimensional Euclidean space. The 4-component covector $\mathbf{p} = (p_0, p_1, p_2, p_3)$ has the meaning of relativistic 4-momentum.

Throughout this section, as well as the next one, we assume that

$$p_0 \neq 0 \tag{2.8.2}$$

which means that we are not interested in static (time-independent) solutions. The sign of p_0 can be arbitrary.

Our Euler–Lagrange equation (2.7.2) is highly nonlinear so it is by no means obvious that one can seek solutions in the form of plane waves (2.8.1). Fortunately

(and miraculously) this is the case. In order to see this, we rewrite our Euler–Lagrange equation (2.7.2) in equivalent form

$$e^{i\mathbf{p}\cdot\mathbf{x}}F = 0. \tag{2.8.3}$$

Note that the sign in the exponent in (2.8.3) is opposite to that in (2.8.1). We have

Lemma 2.8.1. *If the spinor field ξ is a plane wave (2.8.1) then the left-hand side of equation (2.8.3) is constant, i.e. it does not depend on* $\mathbf{x}$.

Of course, Lemma 2.8.1 can be equivalently reformulated as follows: the nonlinear partial differential operator (2.7.3) maps a plane wave (2.8.1) into a plane wave with the same relativistic 4-momentum $\mathbf{p}$.

The proof of Lemma 2.8.1 is quite technical and is given in Appendix 2.C.

Lemma 2.8.1 justifies separation of variables, i.e. it reduces the study of the nonlinear partial differential equation (2.7.2) for the unknown spinor field ξ to the study of the rational algebraic equation (2.8.3) for the unknown constant spinor ζ. We suspect that the underlying group-theoretic reason for our nonlinear partial differential equation (2.7.2) admitting separation of variables is the fact that our model is U(1)-invariant, i.e. it is invariant under the multiplication of the spinor field ξ by a complex constant of modulus 1. Hence, it is feasible that one could prove Lemma 2.8.1, as well as Lemma 2.8.2 stated further down in this section, without performing the laborious calculations presented in Appendix 2.C.

We are now faced with the task of writing down the LHS of equation (2.8.3) explicitly and with minimal calculations. To this end we address a seemingly different issue: we examine what happens when we substitute our plane wave (2.8.1) into our Lagrangian density $L_{\mathrm{Re}}(\xi)$ (i.e. (2.5.2) rephrased in terms of spinors, see Section 2.6 for details), rather than the Euler–Lagrange equation (2.7.2).

Substituting formula (2.8.1) into formulae (2.6.3), (2.6.11), (2.6.12), (2.B.1) and (2.6.9) we get

$$\rho = \bar{\zeta}^{\dot{a}}\sigma_{0\dot{a}b}\zeta^b, \tag{2.8.4}$$

$$f = -\frac{4\bar{\zeta}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}p_\alpha\zeta^b}{\bar{\zeta}^{\dot{c}}\sigma_{0\dot{c}d}\zeta^d}\,, \tag{2.8.5}$$

$$v_\alpha = -\frac{2\varepsilon_{\beta\gamma\alpha}\bar{\zeta}^{\dot{a}}\sigma^{\beta}{}_{\dot{a}b}p^\gamma\zeta^b}{\bar{\zeta}^{\dot{c}}\sigma_{0\dot{c}d}\zeta^d}\,, \tag{2.8.6}$$

$$\overset{*}{T}_{\alpha\beta} = 2\frac{\bar{\zeta}^{\dot{a}}\sigma_{\beta\dot{a}b}p_\alpha\zeta^b - \bar{\zeta}^{\dot{a}}\sigma^{\gamma}{}_{\dot{a}b}p_\gamma\zeta^b g_{\alpha\beta}}{\bar{\zeta}^{\dot{c}}\sigma_{0\dot{c}d}\zeta^d}\,, \tag{2.8.7}$$

$$\omega_\alpha = \frac{2\bar{\zeta}^{\dot{a}}\sigma_{\alpha\dot{a}b}p_0\zeta^b}{\bar{\zeta}^{\dot{c}}\sigma_{0\dot{c}d}\zeta^d}\,, \tag{2.8.8}$$

or, more explicitly,

$$\rho = \bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2, \tag{2.8.9}$$

$$f = \frac{4}{\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2}\left(p_1(-\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1) + ip_2(\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1) + p_3(-\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2)\right), \tag{2.8.10}$$

$$v = \frac{2}{\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2}\begin{pmatrix} p_2\left(\bar{\zeta}^{\dot{1}}\zeta^1 - \bar{\zeta}^{\dot{2}}\zeta^2\right) + ip_3\left(\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1\right) \\ p_3\left(\bar{\zeta}^{\dot{1}}\zeta^2 + \bar{\zeta}^{\dot{2}}\zeta^1\right) + p_1\left(-\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2\right) \\ ip_1\left(-\bar{\zeta}^{\dot{1}}\zeta^2 + \bar{\zeta}^{\dot{2}}\zeta^1\right) + p_2\left(-\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1\right) \end{pmatrix}, \tag{2.8.11}$$

$$\begin{pmatrix} \overset{*}{T}_{11} \\ \overset{*}{T}_{12} \\ \overset{*}{T}_{13} \\ \overset{*}{T}_{21} \\ \overset{*}{T}_{22} \\ \overset{*}{T}_{23} \\ \overset{*}{T}_{31} \\ \overset{*}{T}_{32} \\ \overset{*}{T}_{33} \end{pmatrix} = \frac{2}{\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2}\begin{pmatrix} ip_2\left(\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1\right) + p_3\left(-\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2\right) \\ ip_1\left(-\bar{\zeta}^{\dot{1}}\zeta^2 + \bar{\zeta}^{\dot{2}}\zeta^1\right) \\ p_1\left(\bar{\zeta}^{\dot{1}}\zeta^1 - \bar{\zeta}^{\dot{2}}\zeta^2\right) \\ p_2\left(\bar{\zeta}^{\dot{1}}\zeta^2 + \bar{\zeta}^{\dot{2}}\zeta^1\right) \\ p_3\left(-\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2\right) + p_1\left(-\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1\right) \\ p_2\left(\bar{\zeta}^{\dot{1}}\zeta^1 - \bar{\zeta}^{\dot{2}}\zeta^2\right) \\ p_3\left(\bar{\zeta}^{\dot{1}}\zeta^2 + \bar{\zeta}^{\dot{2}}\zeta^1\right) \\ ip_3\left(-\bar{\zeta}^{\dot{1}}\zeta^2 + \bar{\zeta}^{\dot{2}}\zeta^1\right) \\ p_1\left(-\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1\right) + ip_2\left(\bar{\zeta}^{\dot{1}}\zeta^2 - \bar{\zeta}^{\dot{2}}\zeta^1\right) \end{pmatrix}, \tag{2.8.12}$$

$$\omega = \frac{2p_0}{\bar{\zeta}^{\dot{1}}\zeta^1 + \bar{\zeta}^{\dot{2}}\zeta^2}\begin{pmatrix} -i\bar{\zeta}^{\dot{1}}\zeta^2 - i\bar{\zeta}^{\dot{2}}\zeta^1 \\ -\bar{\zeta}^{\dot{1}}\zeta^2 + \bar{\zeta}^{\dot{2}}\zeta^1 \\ -i\bar{\zeta}^{\dot{1}}\zeta^1 + i\bar{\zeta}^{\dot{2}}\zeta^2 \end{pmatrix}. \tag{2.8.13}$$

Substituting formulae (2.8.4)–(2.8.8) or their more explicit versions (2.8.9)–(2.8.13) into formula (2.5.2) we arrive at a Lagrangian density $L_{\mathrm{Re}}(\zeta;\mathbf{p})$ which does not depend on $\mathbf{x}$. The self-contained formula for this Lagrangian density, written in terms of 4-momentum $\mathbf{p}$ and 4-current

$$\mathbf{j}_{\boldsymbol{\alpha}} := \bar{\zeta}^{\dot{a}}\sigma_{\boldsymbol{\alpha}\dot{a}b}\zeta^{b}, \tag{2.8.14}$$

is

$$L_{\mathrm{Re}}(\zeta;\mathbf{p}) = \frac{2}{j_0}\left(c^{\mathrm{vec}} + c^{\mathrm{ten}}\right)\|p\|^2\|j\|^2 + \frac{4}{j_0}\left(\frac{4}{3}c^{\mathrm{ax}} - \frac{1}{2}c^{\mathrm{vec}} + \frac{1}{6}c^{\mathrm{ten}}\right)(j\cdot p)^2 - \frac{4}{j_0}c^{\mathrm{kin}}p_0^2\|j\|^2. \tag{2.8.15}$$

Here we write our 4-covectors as $\mathbf{p} = (p_0, p)$ and $\mathbf{j} = (j_0, j)$, where p and j are 3-covectors.

We view the 4-momentum $\mathbf{p}$ as a parameter and the constant spinor $\zeta \neq 0$ as the dynamical variable. The Lagrangian density $L_{\mathrm{Re}}(\zeta;\mathbf{p})$ is a smooth function of $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$, so varying ζ we get

$$\delta L_{\mathrm{Re}} = G_{\dot{a}}\delta\bar{\zeta}^{\dot{a}} + \bar{G}_a\delta\zeta^a \tag{2.8.16}$$

where G is a dotted constant spinor expressed via the partial derivatives of $L_{\mathrm{Re}}(\zeta;\mathbf{p})$ with respect to $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$. It is natural to ask the question: what is the relationship between the spinor field F appearing in formula (2.7.1) and the constant spinor G appearing in formula (2.8.16)? The answer is given by

Lemma 2.8.2. *If the spinor field ξ is a plane wave (2.8.1) then $G = e^{i\mathbf{p}\cdot\mathbf{x}}F$.*

The proof of Lemma 2.8.2 is presented in Appendix 2.C.

Lemma 2.8.2 reduces the construction of plane wave solutions of rotational elasticity to finding the critical, with respect to ζ, points of the function $L_{\mathrm{Re}}(\zeta;\mathbf{p})$.

Varying (2.8.15), we arrive at the following equation for critical points:

$$\frac{4}{j_0}\left(c^{\mathrm{vec}}+c^{\mathrm{ten}}\right)\|p\|^2 j^\alpha \sigma_{\alpha\dot{a}b}\zeta^b - \frac{2}{j_0^2}\left(c^{\mathrm{vec}}+c^{\mathrm{ten}}\right)\|p\|^2\|j\|^2\sigma_{0\dot{a}b}\zeta^b$$
$$+\frac{8}{j_0}\left(\frac{4}{3}c^{\mathrm{ax}}-\frac{1}{2}c^{\mathrm{vec}}+\frac{1}{6}c^{\mathrm{ten}}\right)(j\cdot p)p^\alpha\sigma_{\alpha\dot{a}b}\zeta^b - \frac{4}{j_0^2}\left(\frac{4}{3}c^{\mathrm{ax}}-\frac{1}{2}c^{\mathrm{vec}}+\frac{1}{6}c^{\mathrm{ten}}\right)(j\cdot p)^2\sigma_{0\dot{a}b}\zeta^b$$
$$-\frac{8}{j_0}c^{\mathrm{kin}}p_0^2 j^\alpha\sigma_{\alpha\dot{a}b}\zeta^b+\frac{4}{j_0^2}c^{\mathrm{kin}}{p_0}^2\|j\|^2\sigma_{0\dot{a}b}\zeta^b=0. \quad (2.8.17)$$

Recall that the 4-current $\mathbf{j}=(j_0,j)$ appearing in the above equation is defined in accordance with formula (2.8.14).

It now remains to find the 4-momenta $\mathbf{p}$ and spinors $\zeta\neq 0$ which satisfy equation (2.8.17). We carry out the analysis of equation (2.8.17) assuming that

$$\rho=j_0=\bar{\zeta}^{\dot{1}}\zeta^1+\bar{\zeta}^{\dot{2}}\zeta^2=1. \quad (2.8.18)$$

Condition (2.8.18) is a normalisation of the density: general plane wave solutions differ from those satisfying condition (2.8.18) by a real scaling factor. Furthermore, we assume that

$$\zeta^b=\begin{pmatrix}1\\0\end{pmatrix}. \quad (2.8.19)$$

Indeed, any spinor ζ satisfying condition (2.8.18) can be transformed into the spinor (2.8.19) by the linear transformation

$$\zeta\mapsto U\zeta \quad (2.8.20)$$

where U is a special ($\det U=1$) unitary matrix. The transformation (2.8.20) leads to a rotation of the spatial part of the 4-current (2.8.14), so plane wave solutions with general ζ differ from those with ζ of the form (2.8.19) by a rotation.

Substituting (2.8.17), (2.8.19), (2.A.3), (2.A.4) and (2.8.13) into (2.8.16) we get

$$\begin{aligned}
&4\left(c^{\mathrm{vec}}+c^{\mathrm{ten}}\right)\|p\|^2\begin{pmatrix}1\\0\end{pmatrix}-2\left(c^{\mathrm{vec}}+c^{\mathrm{ten}}\right)\|p\|^2\begin{pmatrix}1\\0\end{pmatrix}\\
&+8\left(\frac{4}{3}c^{\mathrm{ax}}-\frac{1}{2}c^{\mathrm{vec}}+\frac{1}{6}c^{\mathrm{ten}}\right)p_3\begin{pmatrix}p_3\\p_1+ip_2\end{pmatrix}-4\left(\frac{4}{3}c^{\mathrm{ax}}-\frac{1}{2}c^{\mathrm{vec}}+\frac{1}{6}c^{\mathrm{ten}}\right)p_3^2\begin{pmatrix}1\\0\end{pmatrix}\\
&\qquad\qquad-4c^{\mathrm{kin}}\begin{pmatrix}p_0^2\\0\end{pmatrix}=0,
\end{aligned}$$

or, equivalently,

$$2\left(c^{\mathrm{vec}}+c^{\mathrm{ten}}\right)\begin{pmatrix}\|p\|^2\\0\end{pmatrix}+4\left(\frac{4}{3}c^{\mathrm{ax}}-\frac{1}{2}c^{\mathrm{vec}}+\frac{1}{6}c^{\mathrm{ten}}\right)\begin{pmatrix}p_3^2\\2p_3(p_1+ip_2)\end{pmatrix}-4c^{\mathrm{kin}}\begin{pmatrix}p_0^2\\0\end{pmatrix}=0. \tag{2.8.21}$$

Put

$$v_1:=\sqrt{\frac{4c^{\mathrm{ax}}+2c^{\mathrm{ten}}}{3c^{\mathrm{kin}}}}\,,\qquad v_2:=\sqrt{\frac{c^{\mathrm{vec}}+c^{\mathrm{ten}}}{2c^{\mathrm{kin}}}}\,. \tag{2.8.22}$$

Note that because we assumed our three elastic moduli to be nonnegative and not all zero, our v_1 and v_2 are nonnegative and not both zero. Using (2.8.22) we can now rewrite equation (2.8.21) in more compact form

$$v_2^2\begin{pmatrix}\|p\|^2\\0\end{pmatrix}+(v_1^2-v_2^2)\begin{pmatrix}p_3^2\\2p_3(p_1+ip_2)\end{pmatrix}-\begin{pmatrix}p_0^2\\0\end{pmatrix}=0. \tag{2.8.23}$$

The analysis of equation (2.8.23) is elementary and the outcome is summarised in the following theorem, which is the main result of Chapter 2.

Theorem 2.8.1. *Plane wave solutions of rotational elasticity can, up to rescaling and rotation, be explicitly written down in the form (2.8.1), (2.8.19) with arbitrary nonzero p_0 and $p=(p_1,p_2,p_3)$ determined as follows.*

- *If $v_1>0$ and $v_2>0$ and $v_1\neq v_2$ then we have two possibilities:*
  - $p=\left(0,0,\pm\dfrac{p_0}{v_1}\right)$ *(type 1 wave), or*

– $p = \left(\dfrac{|p_0|}{v_2} \cos\varphi, \dfrac{|p_0|}{v_2} \sin\varphi, 0 \right)$ *where* $\varphi \in \mathbb{R}$ *is arbitrary (type 2 wave).*

- *If* $v_1 > 0$ *and* $v_2 > 0$ *and* $v_1 = v_2$ *then* p *is an arbitrary 3-vector satisfying* $\|p\| = \dfrac{|p_0|}{v_1}$.
- *If* $v_1 > 0$ *and* $v_2 = 0$ *then* $p = \left(0, 0, \pm\dfrac{p_0}{v_1} \right)$.
- *If* $v_1 = 0$ *and* $v_2 > 0$ *then* $p = \left(\dfrac{|p_0|}{v_2} \cos\varphi, \dfrac{|p_0|}{v_2} \sin\varphi, 0 \right)$ *where* $\varphi \in \mathbb{R}$ *is arbitrary.*

Theorem 2.8.1 shows that rotational elasticity, like classical linear elasticity, produces two distinct types of plane wave solutions. We call these solutions type 1 and type 2 and they propagate with velocities v_1 and v_2 respectively, with v_1 and v_2 given by formulae (2.8.22).

However, unlike with classical linear elasticity, in rotational elasticity the two wave velocities, v_1 and v_2, are not ordered, i.e. we do not know a priori which one, v_1 or v_2, is bigger. The reason the two wave velocities are not ordered is because rotational elasticity has three elastic moduli compared to the two elastic moduli of classical linear elasticity. The "extra" elastic modulus is c^{vec}, the one associated with the antisymmetric part of the rank two tensor $\overset{*}{T}$ which we use as a measure of rotational deformations. If we set $c^{\text{vec}} = 0$, we end up with the inequality

$$v_1 \geq \sqrt{\frac{4}{3}}\, v_2 \tag{2.8.24}$$

similar to the well known inequality from classical linear elasticity, see formula (22.5) in [50].

Throughout this chapter we considered our equations of rotational elasticity in the whole of $\mathbb{R}^3$. It would be natural to examine, in due course, boundary value problems in domains of $\mathbb{R}^3$, starting with the static case. We do not yet have existence or uniqueness theorems for static boundary value problems generated by the Lagrangian of rotational elasticity. Note that for the complete (with both displacements and microrotations) Cosserat model theorems of this type have

been proved in [46, 57, 58, 59, 60, 61, 62]. Note also that the result of [63] may be particularly useful in the proof of possible existence theorems for rotational elasticity: as shown in [63], partial derivatives of the coframe are bounded from above by the norm of torsion.

2.9 The massless Dirac equation

In this section we consider a purely axial material (2.7.4), (2.7.5). Substituting (2.7.4) and (2.7.5) into (2.8.22) we get $v_1 = 1$ and $v_2 = 0$, so a purely axial material supports only type 1 waves. Throughout this section we also retain the assumption (2.8.2).

Note that a purely axial material has the remarkable property that its potential energy is invariant under conformal rescalings of the spatial metric by an arbitrary positive scalar function. We do not elaborate on this issue in the current chapter because we chose to work with a specific (standard Euclidean) metric. The appropriate arguments are presented in Section 2 of [23].

Our aim is to compare our model with the linear partial differential equation (or, more precisely, system of two linear partial differential equations)

$$i(\pm\sigma^0{}_{\dot{a}b}\partial_0 + \sigma^\alpha{}_{\dot{a}b}\partial_\alpha)\xi^b = 0. \tag{2.9.1}$$

Here ∂_0 is the time derivative, ∂_α are spatial derivatives, σ are Pauli matrices (2.A.3), (2.A.4), the free spinor index $\dot{a}$ runs through the values $\dot{1}, \dot{2}$, summation is carried out over the tensor index $\alpha = 1, 2, 3$ as well as the spinor index $b = 1, 2$, and ξ is the unknown spinor field. We give for reference a more explicit version of equation (2.9.1):

$$i\begin{pmatrix} \mp\partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \mp\partial_0 - \partial_3 \end{pmatrix}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = 0. \tag{2.9.2}$$

Equation (2.9.1) is called *the massless Dirac equation* or *the Weyl equation*. This equation is the accepted mathematical model for a massless neutrino field. The two choices of sign in (2.9.1) give two versions of the massless Dirac equation which differ by time reversal. Thus, we have a pair of massless Dirac equations.

We want to compare plane wave solutions (see (2.8.1)) of our model with those of the massless Dirac equation. As both models are invariant under the rescaling of the spinor field by a positive real constant as well as the rotations of Euclidean 3-space, it is sufficient to compare plane wave solutions for ζ of the form (2.8.19). Substituting (2.8.1) and (2.8.19) into (2.9.1) or its more explicit version (2.9.2) we get $p = (0, 0, \pm p_0)$ which is exactly what Theorem 2.8.1 gives us. Thus, we have established

Theorem 2.9.1. *In the case of a purely axial material a plane wave spinor field is a solution of rotational elasticity if and only if it is a solution of one of the two massless Dirac equations (2.9.1).*

It turns out that, in fact, a much stronger result holds. Consider a spinor field of the form

$$\xi(x^0, x^1, x^2, x^3) = e^{-ip_0x^0}\eta(x^1, x^2, x^3). \tag{2.9.3}$$

We will call spinor fields of the form (2.9.3) *stationary*. Here, stationary refers to "time-harmonic" as opposed to "time-independent" as is shown by (2.9.3): while this is slightly non-standard language, it preserves accord between this thesis and [23]. In considering stationary spinor fields what we are doing is separating out time only as opposed to separating out all the variables.

The following result generalises Theorem 2.9.1.

Theorem 2.9.2. *In the case of a purely axial material a nonvanishing stationary spinor field is a solution of rotational elasticity if and only if it is a solution of one of the two massless Dirac equations (2.9.1).*

Theorem 2.9.2 was proved in [23] and the proof is quite delicate. It involves an argument which reduces a nonlinear second order partial differential equation of

a particular type to a pair of linear first order partial differential equations, which is, effectively, a form of integrability. An abstract self-contained version of this argument is given in Appendix B of [16].

It is interesting that Brocato and Capriz wrote in Section 6 of [13] "we are not aiming at a model for ether". Theorem 2.9.2 shows that one can, in fact, model world aether (ether) based on their idea of gyrocontinuum.

2.10 The linearised problem

Let R be the matrix function (2.E.1). Suppose that $R \approx I$, i.e. the coframe is almost aligned with the coordinate axes. Then

$$R = e^{*w} = I + *w + O(w^2) \tag{2.10.1}$$

where w is a vector function. In Cosserat elasticity literature the vector function w is called the *vector of microrotations*. It should not be confused with the *vector of macrorotations* which is one half of the curl of the vector function of displacements.

Formula (2.10.1) can be equivalently rewritten as

$$e^j \mapsto e^j + w \times e^j + O(w^2), \qquad j = 1, 2, 3, \tag{2.10.2}$$

where $e^j = e^j{}_\alpha$ is the jth element of the coframe and $\times$ denotes the cross (vector) product.

Substituting formula (2.10.2) into formulae (2.4.24), (2.4.25), (2.4.11) and (2.3.3) and linearising in w, we get

$$f = -2 \operatorname{div} w, \tag{2.10.3}$$

$$v = \operatorname{curl} w, \tag{2.10.4}$$

$$\overset{*}{T}_{\alpha\beta} = \partial_\alpha w_\beta - \delta_{\alpha\beta} \operatorname{div} w, \tag{2.10.5}$$

$$\omega = \dot{w} \tag{2.10.6}$$

where the dot indicates the time derivative. Finally, substituting formulae (2.10.3)–(2.10.6) into the Lagrangian density (2.5.2) of our rotational theory of elasticity, we get

$$L_{\mathrm{Re}}(w,\rho) = \left(\frac{4c^{\mathrm{ax}}}{3}(\operatorname{div} w)^2 + \frac{c^{\mathrm{vec}}}{2}\|\operatorname{curl} w\|^2 + c^{\mathrm{ten}}\left\|\frac{\partial_\alpha w_\beta + \partial_\beta w_\alpha}{2} - \frac{\delta_{\alpha\beta}}{3}\operatorname{div} w\right\|_V^2 - c^{\mathrm{kin}}\|\dot{w}\|^2\right)\rho \tag{2.10.7}$$

where the norm $\|\cdot\|_V$ of a rank two tensor is defined in accordance with (2.4.13). Formula (2.10.7) is similar to the formula for the Lagrangian density of classical linear elasticity, the only difference being the appearance of the term with $\operatorname{curl} w$.

Up till now in this chapter we viewed the density ρ as a *dynamical variable*. The aim of this section is to formulate a linear analogue of the nonlinear problem considered in Sections 2.2–2.9 and in order to achieve this we have to make our density ρ *prescribed*. Moreover, as we assume our material to be homogeneous, the prescribed value of ρ will be constant. This leaves us with only one dynamical variable, the vector function w.

The linear field equation corresponding to the quadratic Lagrangian density (2.10.7) is

$$\ddot{w} = v_1^2 \operatorname{grad}\operatorname{div} w - v_2^2 \operatorname{curl}\operatorname{curl} w \tag{2.10.8}$$

where v_1 and v_2 are defined by (2.8.22). Equation (2.10.8) is the equation of classical linear elasticity, the only difference being that we do not have the a priori inequality (2.8.24).

The equations of linearised Cosserat elasticity were previously examined by a number of authors. Publications [56, 55, 54] give a good overview of the subject. The difference between [56, 55, 54] and what is done in the current section is that in [56, 55, 54] the authors consider a model with both displacements and

microrotations whereas in our model the vector of microrotations is the only dynamical variable. Our Lagrangian density (2.10.7) is a special case of the Lagrangian density from [56, 55, 54] and our equation (2.10.8) is a special case of the equations from [56, 55, 54].

Equation (2.10.8) gives two types of waves, longitudinal waves propagating with velocity v_1 and transversal waves propagating with velocity v_2. Of course, care is required in interpreting the terms "longitudinal" and "transversal" as w has the geometric meaning of the vector function of microrotations rather than the vector function of displacements (as it would be in classical linear elasticity).

Comparing the results of this section with Theorem 2.8.1 we see that the wave velocities are the same. This indicates that the mathematical model presented in the main part of Chapter 2 (Sections 2.2–2.5) is a natural geometrically nonlinear extension of linear rotational elasticity. Type 1 waves from Theorem 2.8.1 are a nonlinear version of longitudinal waves from equation (2.10.8) whereas type 2 waves are a nonlinear version of transversal ones.

2.A Notation

In Chapter 2 our notation follows [15, 23, 16, 81]. The only difference with [15, 81] is that in the latter the spacetime metric has opposite signature. In [23, 16] the signature is the same as in the current chapter, i.e. the 3-dimensional spatial metric has signature $+++$.

We use Greek letters for tensor (holonomic) indices and Latin letters for frame (anholonomic) indices. We identify differential forms with covariant antisymmetric tensors.

We define the action of the Hodge star on a rank r antisymmetric tensor X as

$$(*X)_{\alpha_{r+1}\ldots\alpha_3} := (r!)^{-1}\, X^{\alpha_1\ldots\alpha_r}\varepsilon_{\alpha_1\ldots\alpha_3} \tag{2.A.1}$$

where ε is the totally antisymmetric quantity, $\varepsilon_{123} := +1$.

We use two-component complex-valued spinors (Weyl spinors) whose indices run through the values $1, 2$ or $\dot{1}, \dot{2}$. Complex conjugation makes the undotted indices dotted and vice versa.

We define the "metric spinor"

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.A.2}$$

and choose Pauli matrices

$$\sigma_{0\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma^0{}_{\dot{a}b}, \tag{2.A.3}$$

$$\sigma_{1\dot{a}b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1{}_{\dot{a}b}, \qquad \sigma_{2\dot{a}b} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^2{}_{\dot{a}b}, \qquad \sigma_{3\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3{}_{\dot{a}b}\,. \tag{2.A.4}$$

Here the first index enumerates rows and the second enumerates columns.

2.B Spinor representation of torsion

We show in this appendix that the tensor $\overset{*}{T}_{\alpha\beta}$ defined by formula (2.4.10) (it is the Hodge dual, in the last pair indices, of the torsion tensor) is expressed via the spinor field ξ as

$$\overset{*}{T}_{\alpha\beta} = i\frac{\bar{\xi}^{\dot{a}}\sigma_{\beta\dot{a}b}\partial_\alpha\xi^b - \xi^b\sigma_{\beta\dot{a}b}\partial_\alpha\bar{\xi}^{\dot{a}} - (\bar{\xi}^{\dot{a}}\sigma^\gamma{}_{\dot{a}b}\partial_\gamma\xi^b - \xi^b\sigma^\gamma{}_{\dot{a}b}\partial_\gamma\bar{\xi}^{\dot{a}})g_{\alpha\beta}}{\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d}\,. \tag{2.B.1}$$

Note that formula (2.B.1) is invariant under the rescaling of our spinor field by an arbitrary positive scalar function.

Formula (2.B.1) is proved by direct substitution of formulae (2.6.4) and (2.6.5) into (2.4.10). In order to simplify calculations we observe that the expressions in

the left- and right-hand sides of formula (2.B.1) have an invariant nature, hence it is sufficient to prove formula (2.B.1) at a point at which the spinor field takes the value $\xi^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then at this point we have

$$e^j{}_\beta = \delta^j{}_\beta \,, \tag{2.B.2}$$

$$\partial_\alpha(e^1 + ie^2) = \begin{pmatrix} \partial_\alpha\xi^1 - \partial_\alpha\bar{\xi}^{\dot{1}} \\ i\partial_\alpha\xi^1 - i\partial_\alpha\bar{\xi}^{\dot{1}} \\ -2\partial_\alpha\xi^2 \end{pmatrix}, \qquad \partial_\alpha e^3 = \begin{pmatrix} \partial_\alpha\xi^2 + \partial_\alpha\bar{\xi}^{\dot{2}} \\ -i\partial_\alpha\xi^2 + i\partial_\alpha\bar{\xi}^{\dot{2}} \\ 0 \end{pmatrix}, \tag{2.B.3}$$

where $\alpha = 1, 2, 3$. Note that formulae (2.B.3) imply

$$\operatorname{curl}(e^1 + ie^2) = \begin{pmatrix} -2\partial_2\xi^2 - \partial_3(i\xi^1 - i\bar{\xi}^{\dot{1}}) \\ 2\partial_1\xi^2 + \partial_3(\xi^1 - \bar{\xi}^{\dot{1}}) \\ \partial_1(i\xi^1 - i\bar{\xi}^{\dot{1}}) - \partial_2(\xi^1 - \bar{\xi}^{\dot{1}}) \end{pmatrix}, \tag{2.B.4}$$

$$\operatorname{curl} e^3 = \begin{pmatrix} -\partial_3(-i\xi^2 + i\bar{\xi}^{\dot{2}}) \\ \partial_3(\xi^2 + \bar{\xi}^{\dot{2}}) \\ \partial_1(-i\xi^2 + i\bar{\xi}^{\dot{2}}) - \partial_2(\xi^2 + \bar{\xi}^{\dot{2}}) \end{pmatrix}. \tag{2.B.5}$$

We now rewrite formula (2.4.10) in the form

$$\overset{*}{T} = \frac{1}{2}(e^1 - ie^2) \otimes \operatorname{curl}(e^1 + ie^2) + \frac{1}{2}(e^1 + ie^2) \otimes \operatorname{curl}(e^1 - ie^2) + e^3 \otimes \operatorname{curl} e^3. \tag{2.B.6}$$

Substituting formulae (2.B.2), (2.B.4) and (2.B.5) into formula (2.B.6) we get

$$\overset{*}{T} = \begin{pmatrix} -\partial_2(\xi^2 + \bar{\xi}^{\dot{2}}) - i\partial_3(\xi^1 - \bar{\xi}^{\dot{1}}) & \partial_1(\xi^2 + \bar{\xi}^{\dot{2}}) & i\partial_1(\xi^1 - \bar{\xi}^{\dot{1}}) \\ i\partial_2(\xi^2 - \bar{\xi}^{\dot{2}}) & -i\partial_1(\xi^2 - \bar{\xi}^{\dot{2}}) - i\partial_3(\xi^1 - \bar{\xi}^{\dot{1}}) & i\partial_2(\xi^1 - \bar{\xi}^{\dot{1}}) \\ i\partial_3(\xi^2 - \bar{\xi}^{\dot{2}}) & \partial_3(\xi^2 + \bar{\xi}^{\dot{2}}) & -i\partial_1(\xi^2 - \bar{\xi}^{\dot{2}}) - \partial_2(\xi^2 + \bar{\xi}^{\dot{2}}) \end{pmatrix}$$

which coincides with the RHS of formula (2.B.1). This completes the proof.

We give for reference a more explicit version of formula (2.B.1):

$$\begin{pmatrix} \overset{*}{T}_{11} \\ \overset{*}{T}_{12} \\ \overset{*}{T}_{13} \\ \overset{*}{T}_{21} \\ \overset{*}{T}_{22} \\ \overset{*}{T}_{23} \\ \overset{*}{T}_{31} \\ \overset{*}{T}_{32} \\ \overset{*}{T}_{33} \end{pmatrix} = \frac{1}{\bar{\xi}^{\dot{1}}\xi^1 + \bar{\xi}^{\dot{2}}\xi^2} \begin{pmatrix} \bar{\xi}^{\dot{2}}\partial_2\xi^1 - \bar{\xi}^{\dot{1}}\partial_2\xi^2 + i\bar{\xi}^{\dot{2}}\partial_3\xi^2 - i\bar{\xi}^{\dot{1}}\partial_3\xi^1 \\ \bar{\xi}^{\dot{1}}\partial_1\xi^2 - \bar{\xi}^{\dot{2}}\partial_1\xi^1 \\ i\bar{\xi}^{\dot{1}}\partial_1\xi^1 - i\bar{\xi}^{\dot{2}}\partial_1\xi^2 \\ i\bar{\xi}^{\dot{2}}\partial_2\xi^1 + i\bar{\xi}^{\dot{1}}\partial_2\xi^2 \\ -i\bar{\xi}^{\dot{1}}\partial_1\xi^2 - i\bar{\xi}^{\dot{2}}\partial_1\xi^1 - i\bar{\xi}^{\dot{1}}\partial_3\xi^1 + i\bar{\xi}^{\dot{2}}\partial_3\xi^2 \\ i\bar{\xi}^{\dot{1}}\partial_2\xi^1 - i\bar{\xi}^{\dot{2}}\partial_2\xi^2 \\ i\bar{\xi}^{\dot{1}}\partial_3\xi^2 + i\bar{\xi}^{\dot{2}}\partial_3\xi^1 \\ \bar{\xi}^{\dot{1}}\partial_3\xi^2 - \bar{\xi}^{\dot{2}}\partial_3\xi^1 \\ -i\bar{\xi}^{\dot{1}}\partial_1\xi^2 - i\bar{\xi}^{\dot{2}}\partial_1\xi^1 + \bar{\xi}^{\dot{2}}\partial_2\xi^1 - \bar{\xi}^{\dot{1}}\partial_2\xi^2 \end{pmatrix} + \text{c.c.} \quad (2.B.7)$$

2.C Separation of variables

In this appendix we prove Lemmata 2.8.1 and 2.8.2. Note that it would suffice to prove Lemma 2.8.2 only, because Lemma 2.8.1 follows from Lemma 2.8.2. However, we prove Lemma 2.8.1 first for the sake of clarity of exposition.

Throughout this appendix we denote time by x^0 and the time derivative by ∂_0.

Let us arrange the (pseudo)scalar f, the three components of the vector v_α, the nine components of the tensor $\overset{*}{T}_{\alpha\beta}$ and the three components of the (pseudo)vector ω_α into one 16-component "vector" V_J, $J = 1, \ldots, 16$. Then our Lagrangian density (2.5.2) can be written as

$$L_{\text{Re}}(\xi) = \rho \sum_{J=1}^{16} A_J V_J^2 \quad (2.C.1)$$

where the A_J are some real constants. Put $W_J := \sqrt{\rho}\, V_J$. Then formula (2.C.1) takes the form

$$L_{\text{Re}}(\xi) = \sum_{J=1}^{16} A_J W_J^2. \quad (2.C.2)$$

According to formulae (2.6.11), (2.6.12), (2.B.1), (2.6.9) and (2.6.3) the components of the "vector" W are expressed via the spinor field ξ as

$$W_J = i\frac{\bar{\xi}^{\dot{a}} B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\partial_{\boldsymbol{\alpha}}\xi^b - \xi^b B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\partial_{\boldsymbol{\alpha}}\bar{\xi}^{\dot{a}}}{(\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d)^{1/2}} \tag{2.C.3}$$

where $B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}$ are some constants and summation is carried out over the spinor indices $\dot{a} = \dot{1}, \dot{2}$, $b = 1, 2$, and over $\boldsymbol{\alpha} = 0, 1, 2, 3$. Here we use bold type to indicate relativistic notation, when time x^0 is viewed as one of the coordinates in (1+3)-dimensional spacetime.

Note that for given J and $\boldsymbol{\alpha}$ the 2×2 matrices $B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}$ are Hermitian. This is because each of these matrices is a linear combination with real coefficients of the Pauli matrices $\sigma^{\beta}{}_{\dot{a}b}$, $\beta = 1, 2, 3$.

Our action is $S_{\mathrm{Re}}(\xi) = \int L_{\mathrm{Re}}(\xi)\, dx^0 dx^1 dx^2 dx^3$. Substituting (2.C.3) into (2.C.2) and varying the spinor field ξ we get

$$\delta S_{\mathrm{Re}}(\xi) = 2\int \sum_{J=1}^{16} A_J \left(i\frac{(\delta\bar{\xi}^{\dot{a}}) B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\partial_{\boldsymbol{\alpha}}\xi^b - \xi^b B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\partial_{\boldsymbol{\alpha}}\delta\bar{\xi}^{\dot{a}}}{(\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d)^{1/2}} - \frac{(\delta\bar{\xi}^{\dot{a}})\sigma_{0\dot{a}b}\xi^b}{2\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d} W_J \right) W_J + \text{c.c.}$$

where we wrote down explicitly the terms with $\delta\bar{\xi}$ and incorporated the terms with $\delta\xi$ into the "c.c." (complex conjugate term). Here, when writing the integral, we dropped, for the sake of brevity, $dx^0 dx^1 dx^2 dx^3$. Integrating the term with $\partial_{\boldsymbol{\alpha}}\delta\bar{\xi}^{\dot{a}}$ by parts and taking out the common factor $\delta\bar{\xi}^{\dot{a}}$ we rewrite the above formula as

$$\delta S_{\mathrm{Re}}(\xi) = 2\int (\delta\bar{\xi}^{\dot{a}}) \sum_{J=1}^{16} A_J \left[\frac{iW_J B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\partial_{\boldsymbol{\alpha}}\xi^b}{(\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d)^{1/2}} - \frac{W_J^2\sigma_{0\dot{a}b}\xi^b}{2\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d} + i\partial_{\boldsymbol{\alpha}}\left(\frac{W_J B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\xi^b}{(\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d)^{1/2}} \right) \right] + \text{c.c.}$$

Comparing this formula with formula (2.7.1) we conclude that the spinor field F appearing in the latter is given by formula

$$F_{\dot{a}} = 2\sum_{J=1}^{16} A_J \left[\frac{iW_J B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\partial_{\boldsymbol{\alpha}}\xi^b}{(\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d)^{1/2}} - \frac{W_J^2\sigma_{0\dot{a}b}\xi^b}{2\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d} + i\partial_{\boldsymbol{\alpha}}\left(\frac{W_J B^{\boldsymbol{\alpha}}_{J\,\dot{a}b}\xi^b}{(\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d)^{1/2}} \right) \right] \tag{2.C.4}$$

where the W_J are, in turn, given by formula (2.C.3).

If we now substitute the plane wave (2.8.1) into formula (2.C.3) we get

$$W_J = \frac{2\bar{\zeta}^{\dot{a}} B^{\boldsymbol{\alpha}}_{J\,\dot{a}b} \mathbf{p}_{\boldsymbol{\alpha}} \zeta^b}{(\bar{\zeta}^{\dot{c}} \sigma_{0\dot{c}d} \zeta^d)^{1/2}} \,. \tag{2.C.5}$$

Note that the above W_J are constant (do not depend on $\mathbf{x}$), which simplifies the next step: substituting (2.8.1) into (2.C.4) and dividing through by the common factor $e^{-i\mathbf{p}\cdot\mathbf{x}}$ we get

$$e^{i\mathbf{p}\cdot\mathbf{x}} F_{\dot{a}} = 2 \sum_{J=1}^{16} A_J \left[\frac{2 W_J B^{\boldsymbol{\alpha}}_{J\,\dot{a}b} \mathbf{p}_{\boldsymbol{\alpha}} \zeta^b}{(\bar{\zeta}^{\dot{c}} \sigma_{0\dot{c}d} \zeta^d)^{1/2}} - \frac{W_J^2 \sigma_{0\dot{a}b} \zeta^b}{2 \bar{\zeta}^{\dot{c}} \sigma_{0\dot{c}d} \zeta^d} \right] . \tag{2.C.6}$$

The remarkable feature of formula (2.C.6) is that its RHS is constant, i.e. it does not depend on $\mathbf{x}$. This completes the proof of Lemma 2.8.1.

Let us now substitute the plane wave (2.8.1) directly into our Lagrangian density (2.C.2). Our Lagrangian density takes the form

$$L_{\mathrm{Re}}(\zeta; \mathbf{p}) = \sum_{J=1}^{16} A_J W_J^2 \tag{2.C.7}$$

where the W_J are given by formula (2.C.5). The Lagrangian density (2.C.7) does not depend on $\mathbf{x}$. The dynamical variable in this Lagrangian density is the constant 2-component complex spinor ζ, whereas the relativistic 4-momentum $\mathbf{p}$ plays the role of a parameter. Varying the spinor ζ we get

$$\delta L_{\mathrm{Re}}(\zeta; \mathbf{p}) = 2 \sum_{J=1}^{16} A_J \left(\frac{2(\delta\bar{\zeta}^{\dot{a}}) B^{\boldsymbol{\alpha}}_{J\,\dot{a}b} \mathbf{p}_{\boldsymbol{\alpha}} \zeta^b}{(\bar{\zeta}^{\dot{c}} \sigma_{0\dot{c}d} \zeta^d)^{1/2}} - \frac{(\delta\bar{\zeta}^{\dot{a}}) \sigma_{0\dot{a}b} \zeta^b}{2 \bar{\zeta}^{\dot{c}} \sigma_{0\dot{c}d} \zeta^d} W_J \right) W_J + \text{c.c.}$$

Comparing this formula with formula (2.8.16) we conclude that the constant spinor G appearing in the latter is given by formula

$$G_{\dot{a}} = 2 \sum_{J=1}^{16} A_J \left[\frac{2 W_J B^{\boldsymbol{\alpha}}_{J\,\dot{a}b} \mathbf{p}_{\boldsymbol{\alpha}} \zeta^b}{(\bar{\zeta}^{\dot{c}} \sigma_{0\dot{c}d} \zeta^d)^{1/2}} - \frac{W_J^2 \sigma_{0\dot{a}b} \zeta^b}{2 \bar{\zeta}^{\dot{c}} \sigma_{0\dot{c}d} \zeta^d} \right] . \tag{2.C.8}$$

It remains to observe that the right-hand sides of formulae (2.C.6) and (2.C.8) are the same. This completes the proof of Lemma 2.8.2.

2.D Conservation of mass

In this appendix we address the issue of conservation of total mass of our continuum. We feel the need to clarify this matter because in the main part of this chapter (Sections 2.2–2.9) we took the density ρ to be a dynamical variable, making it an arbitrary function of time t and spatial coordinates x^α, $\alpha = 1, 2, 3$.

A variable density ρ generates a material flow with velocity V. These are related by the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho V) = 0. \tag{2.D.1}$$

If we take the flow to be potential, $V = \operatorname{grad} \phi$, then equation (2.D.1) becomes

$$\partial_t \rho + \operatorname{div}(\rho \operatorname{grad} \phi) = 0. \tag{2.D.2}$$

Equation (2.D.2) is Poisson's equation for the unknown velocity potential ϕ.

As equation (2.D.2) is written in the whole of $\mathbb{R}^3$, it has to be supplemented by an appropriate condition at infinity. Assume, for definiteness, that $\partial_t \rho \to 0$ and $\rho \to \text{const} > 0$ as $\|x\| \to \infty$, and that convergence is sufficiently fast. Then the natural condition at infinity is

$$\phi \to 0 \quad \text{as} \quad \|x\| \to \infty. \tag{2.D.3}$$

The problem (2.D.2), (2.D.3) has a unique solution ϕ which admits the asymptotic expansion

$$\phi = c\|x\|^{-1} + O(\|x\|^{-2}) \quad \text{as} \quad \|x\| \to \infty. \tag{2.D.4}$$

This means that the total flow at infinity is $-4\pi c$. More specifically, if the constant c in equation (2.D.4) is positive then we have material flowing from infinity, and if it is negative then we have material flowing to infinity. In other words, we implicitly assume in our model that infinity can act as a source or a sink.

It remains to examine what happens when we seek plane wave solutions (2.8.1) or, more generally, stationary solutions (2.9.3). Substituting (2.8.1) or (2.9.3) into (2.6.3) we see that in this case the density ρ is static, hence the velocity potential ϕ is identically zero and there is no flow to or from infinity.

2.E Matrix representation of the Lagrangian of Rotational elasticity

In Chapter 2 we predominantly use tensor notation which comes from alternative theories of gravity [39]. This requires care in distinguishing upper indices from lower indices and tensor (holonomic) indices from frame (anholonomic) indices. In this appendix we establish correspondence between our notation and the more user-friendly matrix notation of [46, 57, 58, 59, 60, 61, 62, 63] the advantage of which is that it does not require such a high degree of sophistication: all indices become lower indices, there is no distinction between tensor and frame indices, and only the order of indices is important.

Define the 3×3 special orthogonal matrix function R which is expressed via our coframe e as

$$R_{j\alpha} := e^{j}{}_{\alpha} \tag{2.E.1}$$

with the index j enumerating the rows of the matrix R and the index α enumerating the columns. This matrix function R is the matrix function $\overline{R}$ from [59], we simply dropped the bar because in our setting we do not have to distinguish macro- and microrotations (we have only microrotations).

Formulae (2.3.2) and (2.E.1) imply that the vector of angular velocity ω can be rewritten in terms of the matrix function R as

$$\omega = *(R^T \dot{R}) \tag{2.E.2}$$

where the dot indicates the time derivative and the application of the Hodge star is justified because the matrix $R^T \dot{R}$ is antisymmetric (we can identify it with a rank two antisymmetric tensor).

Define the 3×3 matrix function $\operatorname{Curl} R$ in accordance with formula (2.1) from [63]:

$$(\operatorname{Curl} R)_{j\gamma} := (\operatorname{curl} e^j)^\gamma = (*de^j)^\gamma = \varepsilon^{\alpha\beta\gamma} \partial_\alpha e^j{}_\beta \tag{2.E.3}$$

with the index j enumerating the rows of the matrix $\operatorname{Curl} R$ and the index γ enumerating the columns.

Let us now examine our rank two tensor $\overset{*}{T}$ defined by formula (2.4.10). We can view the tensor $\overset{*}{T}$ as a matrix function with the first tensor index enumerating the rows and the second index enumerating the columns. Formulae (2.4.10) and (2.E.3) imply the matrix identity

$$\overset{*}{T} = R^T \operatorname{Curl} R. \tag{2.E.4}$$

Formulae (2.E.2), (2.E.4), (2.4.19) and (2.4.20) imply that the Lagrangian density of rotational elasticity (2.5.2) can be rewritten as

$$L_{\mathrm{Re}}(R, \rho) = \Big(\frac{c^{\mathrm{ax}}}{3} [\operatorname{tr}(R^T \operatorname{Curl} R)]^2 + c^{\mathrm{vec}} \| \operatorname{skew}(R^T \operatorname{Curl} R) \|_V^2 + c^{\mathrm{ten}} \| \operatorname{dev} \operatorname{sym}(R^T \operatorname{Curl} R) \|_V^2 - \frac{c^{\mathrm{kin}}}{2} \| R^T \dot{R} \|_V^2 \Big) \rho \tag{2.E.5}$$

where $\operatorname{dev} X = X - \frac{1}{3}(\operatorname{tr} X) I$, $\operatorname{sym} X = \frac{1}{2}(X + X^T)$, $\operatorname{skew} X = \frac{1}{2}(X - X^T)$ and $\| \cdot \|_V$ is the Frobenius norm (2.4.13).

Chapter 3

Microlocal analysis of first order systems I

The purpose of this chapter is to study an elliptic self-adjoint $m \times m$ first order pseudodifferential operator A acting on m-columns of half-densities over a connected compact n-dimensional manifold M without boundary, where $m, n \geq 2$. We construct the propagator which provides a solution to the Cauchy problem for the dynamic equation, evolving the solution to the Cauchy problem in time. This object is related to the spectral function and counting function of the operator A. We go on to present the two-term asymptotic expansions of the spectral and counting functions, correcting errors in the literature.

The structure of this chapter is as follows. We give the absolute minimum microlocal analysis theory, providing definitions and references. We state the key results of [22] and give an overview of the iterative process developed to construct the propagator. The appearance of a U(1) connection in our construction is discussed, and we consider briefly the issue of spectral asymmetry (which, in part, led us towards the work presented in Chapter 6). Finally, we provide a short review discussing the literature and motivating this study.

3.1 Key ideas in microlocal analysis

In this chapter we work, as stated, on an n-dimensional connected compact manifold M without boundary. By T^*M we denote the cotangent bundle of the manifold M, and by

$$T'M = T^*M \setminus \{\xi = 0\} \tag{3.1.1}$$

the cotangent bundle with the zero section removed. Local coordinates on our manifold M are denoted by $x = (x^1, \ldots, x^n)$ and their dual coordinates (referred to as momentum in the physics literature) $\xi = (\xi_1, \ldots, \xi_n)$ on the fibre $T^*_x M$. Then (x, ξ) are local coordinates on the cotangent bundle.

A highly detailed treatment of the differential geometry required in this thesis (including (co)tangent bundles) can be found in [53], Chapter 9.

3.1.1 Hamiltonian trajectories

A *Hamiltonian* $h^{(j)}(x, \xi)$ is a nonvanishing smooth function on $T'M$, positively homogeneous of degree 1 in ξ, that is, for $\lambda \in \mathbb{R}$ s.t. $\lambda > 0$:

$$h^{(j)}(x, \lambda\xi) = \lambda h^{(j)}(x, \xi). \tag{3.1.2}$$

Although seemingly redundant at this stage, the use of the additional superscript (j) will become clear in due course: the Hamiltonians $h^{(j)}$, $j = 1, \ldots, m$, will be the eigenvalues of the principal symbol of our pseudodifferential operator A.

Given a Hamiltonian $h^{(j)}(x, \xi)$ and a point $(y, \eta) \in T'M$ we denote by

$$\left(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta)\right), \quad t \in \mathbb{R}, \tag{3.1.3}$$

the integral curve of the Hamiltonian vector field generated by $h^{(j)}(x,\xi)$, i.e. the parametric curve in $T'M$ providing the solution of the Hamiltonian system

$$\dot{x}^{(j)}(t;y,\eta) = h^{(j)}_{\xi}(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta)), \tag{3.1.4}$$

$$\dot{\xi}^{(j)}(t;y,\eta) = -h^{(j)}_{x}(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta)), \tag{3.1.5}$$

where the dot denotes differentiation in the parameter $t \in \mathbb{R}$, with initial condition

$$\big(x^{(j)}(0;y,\eta),\xi^{(j)}(0;y,\eta)\big) = (y,\eta). \tag{3.1.6}$$

The curve $(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))$ is referred to as the Hamiltonian trajectory generated by the Hamiltonian $h^{(j)}$ starting at the point (y,η). The Hamiltonian is preserved along Hamiltonian trajectories.

The abstract situation described above becomes relevant to our study when the Hamiltonians are taken to be the eigenvalues of the principal symbol (see Section 3.2). Roughly speaking, the Hamiltonian trajectories trace out the path of waves governed by the hyperbolic equation (3.2.3) and, therefore, are a basic building block in the construction of the propagator.

3.1.2 Oscillatory integrals

Basic to our construction is the notion of an oscillatory integral. Using this object we will be able to define pseudodifferential operators. Oscillatory integrals are also important for the construction of the propagator: in this case we must extend the most basic definition of an oscillatory integral given in this section to a more general time-dependent object.

Let Ω be a sufficiently small open set in M. (By "sufficiently small" we mean that by introducing local coordinates x we can identify Ω with an open set in $\mathbb{R}^n$.)

We denote by S^l the class of complex-valued infinitely smooth $m \times m$ matrix-functions $a(x, y, \xi)$ which admit the asymptotic expansion

$$a(x, y, \xi) \sim \sum_{k=0}^{\infty} a_{l-k}(x, y, \xi), \quad |\xi| \to \infty \tag{3.1.7}$$

with each $a_{l-k}(x, y, \xi)$ positively homogeneous in ξ of degree $l - k$.

We assume that for $a \in S^l$

$$\operatorname{supp} a \subset K \times K \times (\mathbb{R}^n \setminus \{\xi = 0\}) \tag{3.1.8}$$

where $K \subset \Omega$ is some compact set. This can be achieved by introducing a suitable cutoff function, see Example 3.10 and Definition 3.11 from [71].

Note that $S^p \subset S^q$ for $q > p$, and define $S^{-\infty} := \cap_l S^l$. Then $S^{-\infty}$ consists of all infinitely smooth matrix-functions decreasing with all derivatives faster than any power of $|\xi|$ as $|\xi| \to \infty$.

The formal integral

$$\mathcal{I}_a(x, y) = \int e^{i(x-y)^\alpha \xi_\alpha} a(x, y, \xi) đ\xi \tag{3.1.9}$$

with $a \in S^l$ is called an *oscillatory integral of order* l with (matrix) amplitude a. While this integral may not converge in the usual sense, it can be treated as a distribution in the variable x or the variable y. This means that one can operate with (3.1.9) as though it were a convergent integral, that is, one can formally integrate by parts, differentiate under the integral sign, etc.

Note that it is possible to replace the function $(x - y)^\alpha \xi_\alpha$ appearing in (3.1.9) with a more general phase function which must satisfy certain nondegeneracy conditions. See [72], Section 2.1, for further details.

3.1.3 Pseudodifferential operators

Consider the oscillatory integral (3.1.9) to be the distribution kernel of an operator A, which is then expressed explicitly as

$$Au(x) = \int e^{i(x-y)^\alpha \xi_\alpha} a(x,y,\xi) u(y) dy đ\xi \tag{3.1.10}$$

where x, y are coordinates on $\Omega \subset M$ and $u(x) = (u_1(x) \ \dots \ u_m(x))^T$ is a column of complex-valued half-densities, cf. Subsection 1.3. An operator A of the form (3.1.10) is called an *$m \times m$ matrix pseudodifferential operator of order l* on Ω acting in the space of half-densities. Throughout Chapter 3 we will always consider operators acting on m-columns of half-densities.

We can extend this definition to the whole manifold M as follows. We say the operator A is an $m \times m$ matrix pseudodifferential operator of order l on M if its distribution kernel (3.1.9) is infinitely smooth outside the diagonal

$$\{(x,y) \in M \times M \ : \ x = y\}, \tag{3.1.11}$$

and if, for any point on the diagonal, there exists a neighbourhood $\Sigma \subset M \times M$ such that the distribution kernel of A can be written in the form (3.1.9) with $a \in S^l$ and $\Sigma \subset \Omega \times \Omega$.

We can remove the dependence on the variable y (or x) in the amplitude of the pseudodifferential operator (3.1.10). Indeed, if A is an $m \times m$ pseudodifferential operator of order l with amplitude $a \in S^l$, then A differs by an integral operator with a C^∞ kernel from the pseudodifferential operator with amplitude $A(x,\xi)$, $(x,\xi) \in T'M$, given by

$$A(x,\xi) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\right)\big|_{y=x}. \tag{3.1.12}$$

This is achieved by expanding the amplitude a at the point $y = x$ using Taylor's formula and then integrating by parts in the variable ξ. Equivalently, we can

express (3.1.10) as follows:

$$Au(x) = \int e^{i(x-y)^\alpha \xi_\alpha} A(x,\xi) u(y)\, dy đ\xi \quad (\text{mod } C^\infty) \tag{3.1.13}$$

where $A(x,\xi)$ is given by (3.1.12). The reduced amplitude $A(x,\xi)$ is called the (full) symbol of the operator A. The symbol $A(x,\xi)$ determines the operator A modulo an integral operator with infinitely smooth kernel. Conversely, the operator A determines the symbol $A(x,\xi)$ modulo an amplitude from $S^{-\infty}$.

As $A(x,\xi) \in S^l$ it can be expanded asymptotically as the sum

$$A(x,\xi) \sim \sum_{k=0}^{\infty} A_{l-k}(x,\xi), \quad |\xi| \to +\infty \tag{3.1.14}$$

where each term $A_{l-k}(x,\xi)$ is positively homogeneous in ξ of degree $l-k$. The leading term $A_l(x,\xi)$ in the expansion (3.1.14) is called the *principal symbol* of the pseudodifferential operator A.

The principal symbol $A_l(x,\xi)$ is invariant under changes of local coordinates. However, the term $A_{l-1}(x,\xi)$ is not invariant. To combat this issue, we introduce the *subprincipal symbol* of the operator A, defined as

$$A_{\text{sub}}(x,\xi) = A_{l-1}(x,\xi) + \frac{i}{2} \left(A_l\right)_{x^\alpha \xi_\alpha} (x,\xi). \tag{3.1.15}$$

The subprincipal symbol (3.1.15) is positively homogeneous of degree $l-1$ in ξ and invariant under changes of local coordinates. The (full) symbol behaves in a more complicated manner under changes of local coordinates.

As stated in the introduction of this chapter, we work with elliptic pseudodifferential operators. An $m \times m$ pseudodifferential operator A of order l is said to be elliptic if $\det A_l(x,\xi) \neq 0$, $\forall (x,\xi) \in T'M$.

Note that, as stated in Subsection 3.1.1, the nonzero Hamiltonians $h^{(j)}$, $j = 1,\dots,m$, will be the eigenvalues of the principal symbol of our intended object of

study, a first order elliptic pseudodifferential operator. The fact that our Hamiltonians are nonvanishing is encoded in the ellipticity condition, i.e. the nonvanishing of the determinant of the principal symbol, see Section 3.2 for further details.

3.1.4 Time-dependent oscillatory integrals

The final component of established theory we need to proceed to the key results from [22] pertains to so-called time-dependent oscillatory integrals. At a formal level, one can view the extension from the local case, see Subsection 3.1.2, to the time-dependent case as the introduction of time dependence into the definition of the oscillatory integral (specifically into the phase function and amplitude). At a more fundamental level, we relate this time dependence to an underlying Hamiltonian system as described in Subsection 3.1.1.

The purpose of this construction is reasonably clear. One of the key applications of the microlocal techniques presented here is to allow us to study the propagation of the singularities of solutions of certain hyperbolic systems. We construct the propagator for our system as a sum of global oscillatory integrals. The propagator is the object that shows how a solution evolves in time, and the singularities of our initial data under its action will propagate along Hamiltonian trajectories.

The limitations of the definition of oscillatory integrals in this regard are obvious: oscillatory integrals of the form (3.1.9) are defined only on (sufficiently small) open sets $\Omega \subset M$. The highly local nature of this definition means we are, in effect, unable to propagate information between, say, any two points of our compact connected manifold M.

The definition of time-dependent oscillatory integrals is a delicate issue and must be handled with care. Our construction of time-dependent oscillatory integrals follows [72]. The remarkable feature of the technique proposed in [72] is that the time-dependent oscillatory integrals developed are global, in the sense that they are global in time, $t \in \mathbb{R}$, and do not, in general, depend on the choice of local coordinates on M.

The superlative reference for scalar operators is [72]: we follow this treatment, using notation adapted for systems. A matrix version of this construction was previously considered in [64].

From this point until the end of this section we associate a seemingly redundant index (j) with a number of different quantities. This will become essential in Section 3.2 when the index j will enumerate the eigenvalues of the principal symbol of our operator of study.

The (formal) integral

$$\mathcal{I}_{\varphi^{(j)},a^{(j)}}(t,x;y) = \int e^{i\varphi^{(j)}(t,x;y,\eta)} a^{(j)}(t,x;y,\eta)\varsigma^{(j)}(t,x;y,\eta) d_{\varphi^{(j)}}(t,x;y,\eta) đ\eta, \tag{3.1.16}$$

is called a *time-dependent oscillatory integral of order* l, where the expressions appearing in (3.1.16) have the following meaning.

- The function $\varphi^{(j)}(t,x;y,\eta)$ is a *phase function*, i.e. a function $\mathbb{R}\times M\times T'M\to\mathbb{C}$ positively homogeneous in η of degree 1 and satisfying the conditions

$$\varphi^{(j)}(t,x;y,\eta) = (x-x^{(j)}(t;y,\eta))^{\alpha}\,\xi^{(j)}_{\alpha}(t;y,\eta) + O(|x-x^{(j)}(t;y,\eta)|^2), \tag{3.1.17}$$

$$\operatorname{Im}\varphi^{(j)}(t,x;y,\eta)\geq 0, \tag{3.1.18}$$

$$\det\varphi^{(j)}_{x^{\alpha}\eta_{\beta}}(t,x^{(j)}(t;y,\eta);y,\eta)\neq 0. \tag{3.1.19}$$

  Recall that according to Corollary 2.4.5 from [72] we are guaranteed to have (3.1.19) if we choose a phase function

$$\begin{aligned}\varphi^{(j)}(t,x;y,\eta) = {}& (x-x^{(j)}(t;y,\eta))^{\alpha}\,\xi^{(j)}_{\alpha}(t;y,\eta)\\ &+\frac{1}{2}C^{(j)}_{\alpha\beta}(t;y,\eta)\,(x-x^{(j)}(t;y,\eta))^{\alpha}\,(x-x^{(j)}(t;y,\eta))^{\beta}\\ &+O(|x-x^{(j)}(t;y,\eta)|^3)\end{aligned} \tag{3.1.20}$$

with complex-valued symmetric matrix-function $C^{(j)}_{\alpha\beta}$ satisfying the strict inequality $\operatorname{Im} C^{(j)} > 0$ (our original requirement (3.1.18) implies only the non-strict inequality $\operatorname{Im} C^{(j)} \geq 0$). Note that even though the matrix-function $C^{(j)}_{\alpha\beta}$ is not a tensor, the inequalities $\operatorname{Im} C^{(j)} \geq 0$ and $\operatorname{Im} C^{(j)} > 0$ are invariant under transformations of local coordinates x; see Remark 2.4.9 in [72] for details.

- The quantity $a^{(j)}(t, x; y, \eta)$ is the amplitude of our oscillatory integral, i.e. a complex-valued $m \times m$ matrix-function $\mathbb{R} \times M \times T'M \to \mathbb{C}^{m^2}$ which admits the asymptotic expansion

$$a^{(j)}(t, x; y, \eta) \sim \sum_{k=0}^{\infty} a^{(j)}_{l-k}(t, x; y, \eta) \tag{3.1.21}$$

as $|\eta| \to +\infty$, where each $a^{(j)}_{l-k}$ is positively homogeneous of degree $l - k$ in η.

Remark 3.1.1. *A great deal of this chapter deals with the construction of the propagator, a formal definition of which is given later in (3.2.1). We construct the propagator as a sum of time-dependent oscillatory integrals, i.e. integrals of the type we are defining in this section. An important property of these particular time-dependent oscillatory integrals is that their amplitudes are independent of the variable x, that is, they are actually time-dependent* symbols *$u^{(j)}(t; y, \eta)$ rather than amplitudes (3.1.21).*

- The quantity $d_{\varphi^{(j)}}(t, x; y, \eta)$ is defined in accordance with formula (2.2.4) from [72] as

$$d_{\varphi^{(j)}}(t, x; y, \eta) := (\det{}^2 \varphi^{(j)}_{x^\alpha \eta_\beta})^{1/4} = |\det \varphi^{(j)}_{x^\alpha \eta_\beta}|^{1/2} \, e^{\, i \arg(\det{}^2 \varphi^{(j)}_{x^\alpha \eta_\beta})/4}. \tag{3.1.22}$$

Note that in view of (3.1.19) our $d_{\varphi^{(j)}}$ is well-defined and smooth for x close to $x^{(j)}(t; y, \eta)$. It is known [72] that under coordinate transformations $d_{\varphi^{(j)}}$ behaves as a half-density in x and as a half-density to the power -1 in y.

In formula (3.1.22) we wrote $(\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})^{1/4}$ rather than $(\det\varphi^{(j)}_{x^\alpha\eta_\beta})^{1/2}$ in order to make this expression truly invariant under coordinate transformations. Recall that local coordinates x and y are chosen independently and that η is a covector based at the point y. Consequently, $\det\varphi^{(j)}_{x^\alpha\eta_\beta}$ changes sign under inversion of one of the local coordinates x^α, $\alpha = 1,\dots,n$, or y^β, $\beta = 1,\dots,n$, whereas $\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta}$ retains sign under inversion.

The choice of (smooth) branch of $\arg(\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})$ is assumed to be fixed. Thus, for a given phase function $\varphi^{(j)}$ formula (3.1.22) defines the quantity $d_{\varphi^{(j)}}$ uniquely up to a factor $e^{ik\pi/2}$, $k = 0,1,2,3$. Observe now that if we set $t = 0$ and choose the same local coordinates for x and y, we get $\varphi^{(j)}_{x^\alpha\eta_\beta}(0,y;y,\eta) = I$. This implies that we can fully specify the choice of branch of $\arg(\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})$ by requiring that $d_{\varphi^{(j)}}(0,y;y,\eta) = 1$.

The introduction of the factor $d_{\varphi^{(j)}}$ in (3.1.16) is important in the construction of the propagator, discussed in Remark 3.1.1. The purpose of the factor is twofold.

(a) It ensures that the symbol of the propagator, $u^{(j)}$, is a function on $\mathbb{R}\times T'M$ in the full differential geometric sense of the word, i.e. that it is invariant under transformations of local coordinates x and y.

(b) It ensures that the principal symbol $u^{(j)}_0$ does not depend on the choice of phase function $\varphi^{(j)}$. See Remark 2.2.8 in [72] for more details.

- The quantity $\varsigma^{(j)}(t,x;y,\eta)$ is a smooth cut-off function $\mathbb{R}\times M\times T'M\to\mathbb{R}$ satisfying the following conditions.

(a) $\varsigma^{(j)}(t,x;y,\eta) = 0$ on the set $\{(t,x;y,\eta):\ |h^{(j)}(y,\eta)|\le 1/2\}$.

(b) $\varsigma^{(j)}(t,x;y,\eta) = 1$ on the intersection of a small conic neighbourhood of the set

$$\{(t,x;y,\eta):\ x = x^{(j)}(t;y,\eta)\} \tag{3.1.23}$$

with the set $\{(t,x;y,\eta):\ |h^{(j)}(y,\eta)|\ge 1\}$.

(c) $\varsigma^{(j)}(t,x;y,\lambda\eta) = \varsigma^{(j)}(t,x;y,\eta)$ for $|h^{(j)}(y,\eta)|\ge 1,\ \lambda\ge 1$.

- It is known (see Section 2.3 in [72] for details) that Hamiltonian trajectories generated by a Hamiltonian $h^{(j)}(x,\xi)$ positively homogeneous in ξ of degree 1 satisfy the identity

$$(x^{(j)}_\eta)^{\alpha\beta}\xi^{(j)}_\alpha = 0, \tag{3.1.24}$$

  where $(x^{(j)}_\eta)^{\alpha\beta} := \partial(x^{(j)})^\alpha/\partial\eta_\beta$. Formulae (3.1.17) and (3.1.24) imply

$$\varphi^{(j)}_\eta(t, x^{(j)}(t;y,\eta);y,\eta) = 0. \tag{3.1.25}$$

  This allows us to apply the stationary phase method in the neighbourhood of the set (3.1.23) and disregard what happens away from it.

The definition of a time-dependent oscillatory integral (3.1.16) is the last piece of established theory needed to proceed with the statement of the key results from [22].

3.2 Main results

Consider a first order classical pseudodifferential operator A acting on columns $v = \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix}^T$ of complex-valued half-densities over a connected compact n-dimensional manifold M without boundary. As stated above we assume that $m, n \geq 2$.

We assume the symbol of the operator A to be infinitely smooth. We also assume that the operator A is formally self-adjoint (symmetric):

$$\int_M w^*(Av)\,dx = \int_M (Aw)^* v\,dx$$

for all infinitely smooth $v, w : M \to \mathbb{C}^m$. Here and further on the superscript * in matrices, rows and columns indicates Hermitian conjugation in $\mathbb{C}^m$.

Let $A_1(x,\xi)$ be the principal symbol of the operator A and $h^{(j)}(x,\xi)$ its (Hamiltonian) eigenvalues. We assume these eigenvalues to be nonzero (this is a version of the ellipticity condition) but do not make any assumptions on their sign. We also assume that the eigenvalues $h^{(j)}(x,\xi)$ are simple for all $(x,\xi) \in T'M$. The techniques developed in [22] do not work in the case when eigenvalues of the principal symbol have variable multiplicity, though they could probably be adapted to the case of constant multiplicity different from multiplicity 1.

We enumerate the eigenvalues of the principal symbol $h^{(j)}(x,\xi)$ in increasing order, using a positive index $j = 1, \dots, m^+$ for positive $h^{(j)}(x,\xi)$ and a negative index $j = -1, \dots, -m^-$ for negative $h^{(j)}(x,\xi)$. Here m^+ is the number of positive eigenvalues of the principal symbol and m^- is the number of negative ones. Of course, $m^+ + m^- = m$.

Under the above assumptions A is a self-adjoint operator, in the full functional analytic sense, in the Hilbert space $L^2(M;\mathbb{C}^m)$ (Hilbert space of square integrable complex-valued column "functions") with domain $H^1(M;\mathbb{C}^m)$ (Sobolev space of complex-valued column "functions" which are square integrable together with their first partial derivatives) and the spectrum of A is discrete.

Let λ_k and $v_k = \begin{pmatrix} v_{k1}(x) & \dots & v_{km}(x) \end{pmatrix}^T$ be the eigenvalues and normalised eigenfunctions of the operator A. The eigenvalues λ_k are enumerated in increasing order with account of multiplicity, using a positive index $k = 1, 2, \dots$ for positive λ_k and a nonpositive index $k = 0, -1, -2, \dots$ for nonpositive λ_k. If the operator A is bounded from below (i.e. if $m^- = 0$) then the index k runs from some integer value to $+\infty$; if the operator A is bounded from above (i.e. if $m^+ = 0$) then the index k runs from $-\infty$ to some integer value; and if the operator A is unbounded from above and from below (i.e. if $m^+ \neq 0$ and $m^- \neq 0$) then the index k runs from $-\infty$ to $+\infty$.

We will be studying the following three objects.

Object 1. Our first object of study is the *propagator*, which is the one-parameter family of operators defined as

$$U(t) := e^{-itA} = \sum_k e^{-it\lambda_k} v_k(x) \int_M [v_k(y)]^*(\,\cdot\,)\, dy\,, \tag{3.2.1}$$

$t \in \mathbb{R}$. The propagator provides a solution to the Cauchy problem

$$w|_{t=0} = v \tag{3.2.2}$$

for the dynamic equation

$$D_t w + Aw = 0\,, \tag{3.2.3}$$

where $D_t := -i\partial/\partial t$. Namely, it is easy to see that if the column of half-densities $v = v(x)$ is infinitely smooth, then, setting $w := U(t)\, v$, we get a time-dependent column of half-densities $w(t, x)$ which is also infinitely smooth and which satisfies the equation (3.2.3) and the initial condition (3.2.2). The use of the letter "U" for the propagator is motivated by the fact that for each t the operator $U(t)$ is unitary.

Object 2. Our second object of study is the *spectral function*, which is the real density defined as

$$e(\lambda, x, x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2, \tag{3.2.4}$$

where $\|v_k(x)\|^2 := [v_k(x)]^* v_k(x)$ is the square of the Euclidean norm of the eigenfunction v_k evaluated at the point $x \in M$ and λ is a positive parameter (spectral parameter).

Object 3. Our third and final object of study is the *counting function*

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1 \;= \int_M e(\lambda, x, x)\, dx\,. \tag{3.2.5}$$

In other words, $N(\lambda)$ is the number of eigenvalues λ_k between zero and λ.

It is natural to ask the question: why, in defining the spectral function (3.2.4)

and the counting function (3.2.5), did we choose to perform summation over all *positive* eigenvalues up to a given positive λ rather than over all *negative* eigenvalues up to a given negative λ? There is no particular reason. One case reduces to the other by the change of operator $A \mapsto -A$. This issue will be revisited in Section 3.5.

Further on we assume that $m^+ > 0$, i.e. that the operator A is unbounded from above.

Our objectives are as follows.

Objective 1. We aim to construct the propagator (3.2.1) explicitly in terms of oscillatory integrals, modulo an integral operator with an infinitely smooth, in the variables t, x and y, integral kernel.

Objectives 2 and 3. We aim to derive, under appropriate assumptions on Hamiltonian trajectories, two-term asymptotics for the spectral function (3.2.4) and the counting function (3.2.5), i.e. formulae of the type

$$e(\lambda, x, x) = a(x)\,\lambda^n + b(x)\,\lambda^{n-1} + o(\lambda^{n-1}), \tag{3.2.6}$$

$$N(\lambda) = a\lambda^n + b\lambda^{n-1} + o(\lambda^{n-1}) \tag{3.2.7}$$

as $\lambda \to +\infty$. Obviously, here we expect the real constants a, b and real densities $a(x)$, $b(x)$ to be related in accordance with

$$a = \int_M a(x)\,dx, \tag{3.2.8}$$

$$b = \int_M b(x)\,dx. \tag{3.2.9}$$

It is well known that the above three objectives are closely related: if one achieves Objective 1, then Objectives 2 and 3 follow via Fourier Tauberian theorems [27, 72, 45, 70].

We are now in a position to state the main results from [22].

Result 1. We construct the propagator as a sum of m oscillatory integrals

$$U(t) \overset{\mathrm{mod}\, C^\infty}{=} \sum_j U^{(j)}(t)\,, \tag{3.2.10}$$

where the phase function of each oscillatory integral $U^{(j)}(t)$ is associated with the corresponding Hamiltonian $h^{(j)}(x,\xi)$. The symbol of the oscillatory integral $U^{(j)}(t)$ is a complex-valued $m \times m$ matrix-function $u^{(j)}(t;y,\eta)$, where $y = (y^1,\dots,y^n)$ is the position of the source of the wave (i.e. this is the same y that appears in formula (3.2.1)) and $\eta = (\eta_1,\dots,\eta_n)$ is the corresponding dual variable. When $|\eta| \to +\infty$, the symbol admits an asymptotic expansion

$$u^{(j)}(t;y,\eta) = u_0^{(j)}(t;y,\eta) + u_{-1}^{(j)}(t;y,\eta) + \dots \tag{3.2.11}$$

into components positively homogeneous in η, with the subscript indicating degree of homogeneity.

The formula for the principal symbol of the oscillatory integral $U^{(j)}(t)$ is known [69, 64] and reads as follows:

$$u_0^{(j)}(t;y,\eta) = [v^{(j)}(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))]\,[v^{(j)}(y,\eta)]^* \\ \times \exp\left(-i\int_0^t q^{(j)}(x^{(j)}(\tau;y,\eta),\xi^{(j)}(\tau;y,\eta))\,d\tau\right), \tag{3.2.12}$$

where $v^{(j)}(z,\zeta)$ is the normalised eigenvector of the principal symbol $A_1(z,\zeta)$ corresponding to the eigenvalue (Hamiltonian) $h^{(j)}(z,\zeta)$, $(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))$ is the Hamiltonian trajectory originating from the point (y,η), and $q^{(j)} : T'M \to \mathbb{R}$ is the function

$$q^{(j)} := [v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} - i[v^{(j)}]^*\{v^{(j)}, h^{(j)}\}. \tag{3.2.13}$$

Curly brackets in formula (3.2.13) denote the Poisson bracket on matrix-functions

$$\{P,R\} := P_{z^\alpha} R_{\zeta_\alpha} - P_{\zeta_\alpha} R_{z^\alpha} \tag{3.2.14}$$

and its further generalisation

$$\{P, Q, R\} := P_{z^\alpha} Q R_{\zeta_\alpha} - P_{\zeta_\alpha} Q R_{z^\alpha} \,. \tag{3.2.15}$$

Formula (3.2.12) is invariant under changes of local coordinates on the manifold M, i.e. elements of the $m \times m$ matrix-function $u_0^{(j)}(t; y, \eta)$ are scalars on $\mathbb{R} \times T'M$. Moreover, formula (3.2.12) is invariant under the transformation of the eigenvector of the principal symbol

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)}, \tag{3.2.16}$$

where

$$\phi^{(j)} : T'M \to \mathbb{R} \tag{3.2.17}$$

is an arbitrary smooth function. When some quantity is defined up to the action of a certain transformation, theoretical physicists refer to such a transformation as a *gauge transformation*. We follow this tradition. Note that our particular gauge transformation (3.2.16), (3.2.17) is quite common in quantum mechanics: when $\phi^{(j)}$ is a function of the position variable x only (i.e. when $\phi^{(j)} : M \to \mathbb{R}$) this gauge transformation is associated with electromagnetism.

Both Yu. Safarov [69] and W.J. Nicoll [64] assumed that the operator A is semi-bounded from below but this assumption is not essential and their formula (3.2.12) remains true in the more general case that we are dealing with.

However, knowing the principal symbol (3.2.12) of the oscillatory integral $U^{(j)}(t)$ is not enough if one wants to derive two-term asymptotics (3.2.6) and (3.2.7). One needs information about $u_{-1}^{(j)}(t; y, \eta)$, the component of the symbol of the oscillatory integral $U^{(j)}(t)$ which is positively homogeneous in η of degree -1, see formula (3.2.11), but here the problem is that $u_{-1}^{(j)}(t; y, \eta)$ is not a true invariant in the sense that it depends on the choice of phase function in the oscillatory integral. We overcome this difficulty by observing that $U^{(j)}(0)$ is a pseudodifferential operator, hence, it has a well-defined subprincipal symbol $[U^{(j)}(0)]_{\text{sub}}$. We

prove that

$$\mathrm{tr}[U^{(j)}(0)]_{\mathrm{sub}} = -i\{[v^{(j)}]^*, v^{(j)}\} \tag{3.2.18}$$

and subsequently show that information contained in formulae (3.2.12) and (3.2.18) is sufficient for the derivation of two-term asymptotics (3.2.6) and (3.2.7).

Note that the RHS of formula (3.2.18) is invariant under the gauge transformation (3.2.16), (3.2.17).

Formula (3.2.18) plays a central role in [22].

Let us elaborate briefly on the geometric meaning of the RHS of (3.2.18) (a more detailed exposition is presented in Section 3.4). The eigenvector of the principal symbol is defined up to a gauge transformation (3.2.16), (3.2.17) so it is natural to introduce a U(1) connection on $T'M$ as follows: when parallel transporting an eigenvector of the principal symbol along a curve in $T'M$ we require that the derivative of the eigenvector along the curve be orthogonal to the eigenvector itself. This is equivalent to the introduction of an (intrinsic) electromagnetic field on $T'M$, with the $2n$-component real quantity

$$i\,(\,[v^{(j)}]^* v^{(j)}_{x^\alpha}\,,\,[v^{(j)}]^* v^{(j)}_{\xi_\gamma}\,) \tag{3.2.19}$$

playing the role of the electromagnetic covector potential. Our quantity (3.2.19) is a 1-form on $T'M$, rather than on M itself as is the case in "traditional" electromagnetism. The above U(1) connection generates curvature which is a 2-form on $T'M$, an analogue of the electromagnetic tensor. Out of this curvature 2-form one can construct, by contraction of indices, a real scalar. This scalar curvature is the expression appearing in the RHS of formula (3.2.18).

Observe now that $\sum_j U^{(j)}(0)$ is the identity operator on half-densities. The subprincipal symbol of the identity operator is zero, so formula (3.2.18) implies

$$\sum_j \{[v^{(j)}]^*, v^{(j)}\} = 0. \tag{3.2.20}$$

One can check the identity (3.2.20) directly, without constructing the oscillatory integrals $U^{(j)}(t)$: it follows from the fact that the $v^{(j)}(x,\xi)$ form an orthonormal basis, see end of Section 3.4 for details. We mentioned the identity (3.2.20) in order to highlight, once again, the fact that the curvature effects we have identified are specific to systems and do not have an analogue in the scalar case.

Results 2 and 3. We prove, under appropriate assumptions on Hamiltonian trajectories (see Theorems 8.3 and 8.4 from [22], asymptotic formulae (3.2.6) and (3.2.7) with

$$a(x) = \sum_{j=1}^{m^+} \int\!\!\!\!\!\!- \limits_{h^{(j)}(x,\xi)<1} d\!\!\!^{-}\xi\,, \tag{3.2.21}$$

$$\begin{aligned} b(x) = -n \sum_{j=1}^{m^+} \int\!\!\!\!\!\!- \limits_{h^{(j)}(x,\xi)<1} \Big(& [v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} \\ & - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} + \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big)(x,\xi)\, d\!\!\!^{-}\xi\,, \end{aligned} \tag{3.2.22}$$

and a and b expressed via the above densities (3.2.21) and (3.2.22) as (3.2.8) and (3.2.9). In (3.2.21) and (3.2.22) the Poisson bracket on matrix-functions $\{\cdot,\cdot\}$ and its further generalisation $\{\cdot,\cdot,\cdot\}$ are defined by formulae (3.2.14) and (3.2.15) respectively.

To our knowledge, formula (3.2.22) is a new result. Note that in [69] this formula (more precisely, its integrated over M version (3.2.9)) was written incorrectly, without the curvature terms

$$-\frac{ni}{n-1} \int h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} d\!\!\!^{-}\xi. \tag{3.2.23}$$

See also Section 3.6 where we give a more detailed bibliographic review.

It is easy to see that the right-hand sides of (3.2.21) and (3.2.22) behave as densities under changes of local coordinates on the manifold M and that these expressions are invariant under gauge transformations (3.2.16), (3.2.17) of the eigenvectors of the principal symbol. Moreover, the right-hand sides of (3.2.21)

and (3.2.22) are unitarily invariant, i.e. invariant under the transformation of the operator

$$A \mapsto RAR^*, \tag{3.2.24}$$

where

$$R : M \to \mathrm{U}(m) \tag{3.2.25}$$

is an arbitrary smooth unitary matrix-function. The fact that the RHS of (3.2.22) is unitarily invariant is non-trivial: the appropriate calculations are presented in Section 9 of [22]. The observation that without the curvature terms (3.2.23) (as in [69]) the RHS of (3.2.22) is not unitarily invariant was a major motivating factor in the writing of [22].

Formula (3.2.22) is the main result of this chapter. Note that even though the two-term asymptotic expansion (3.2.6) holds only under certain assumptions on Hamiltonian trajectories (loops), the second asymptotic coefficient (3.2.22) is, in itself, well-defined irrespective of how many loops we have. If one wishes to reformulate the asymptotic expansion (3.2.6) in such a way that it remains valid without assumptions on the number of loops, this can easily be achieved, say, by taking a convolution with a function from Schwartz space $\mathcal{S}(\mathbb{R})$: see Theorem 7.1 from [22].

3.3 Overview of the algorithm for the construction of the propagator

We construct the propagator as a sum of m oscillatory integrals (3.2.10) where each integral is of the form

$$U^{(j)}(t) = \int e^{i\varphi^{(j)}(t,x;y,\eta)}\, u^{(j)}(t;y,\eta)\, \varsigma^{(j)}(t,x;y,\eta)\, d_{\varphi^{(j)}}(t,x;y,\eta)\,(\,\cdot\,)\, dy\, đ\eta\,, \tag{3.3.1}$$

defined in accordance with Subsection 3.1.4. Our task now is to construct the symbols $u_0^{(j)}(t;y,\eta)$, $j = 1,\dots,m$, so that our oscillatory integrals $U^{(j)}(t)$, $j =$

$1, \dots, m$, satisfy the dynamic equations

$$(D_t + A(x, D_x))\, U^{(j)}(t) \overset{\text{mod}\, C^\infty}{=} 0 \tag{3.3.2}$$

and initial condition

$$\sum_j U^{(j)}(0) \overset{\text{mod}\, C^\infty}{=} I\,, \tag{3.3.3}$$

where I is the identity operator on half-densities; compare with formulae (3.2.3), (3.2.2) and (3.2.10). Note that the pseudodifferential operator A in formula (3.3.2) acts on the oscillatory integral $U(t)$ in the variable x; say, if A is a differential operator this means that in order to evaluate $A\,U^{(j)}(t)$ one has to perform the appropriate differentiations of the oscillatory integral (3.3.1) in the variable x. Following the conventions of Section 3.3 of [72], we emphasise the fact that the pseudodifferential operator A in formula (3.3.2) acts on the oscillatory integral $U(t)$ in the variable x by writing this pseudodifferential operator as $A(x, D_x)$, where $D_{x^\alpha} := -i\partial/\partial x^\alpha$.

We examine first the dynamic equation (3.3.2). We have

$$(D_t + A(x, D_x))\, U^{(j)}(t) = F^{(j)}(t)\,,$$

where $F^{(j)}(t)$ is the oscillatory integral

$$F^{(j)}(t) = \int e^{i\varphi^{(j)}(t,x;y,\eta)}\, f^{(j)}(t, x; y, \eta)\, \varsigma^{(j)}(t, x; y, \eta)\, d_{\varphi^{(j)}}(t, x; y, \eta)\, (\,\cdot\,)\, dy\, đ\eta$$

whose matrix-valued amplitude $f^{(j)}$ is given by the formula

$$f^{(j)} = D_t u^{(j)} + \big(\varphi_t^{(j)} + (d_{\varphi^{(j)}})^{-1}(D_t d_{\varphi^{(j)}}) + s^{(j)}\big)\, u^{(j)}, \tag{3.3.4}$$

where the matrix-function $s^{(j)}(t, x; y, \eta)$ is defined as

$$s^{(j)} = e^{-i\varphi^{(j)}} (d_{\varphi^{(j)}})^{-1}\, A(x, D_x)\, (e^{i\varphi^{(j)}} d_{\varphi^{(j)}})\,. \tag{3.3.5}$$

When $|\eta| \to +\infty$ the matrix-valued amplitude $f^{(j)}(t,x;y,\eta)$ defined by formula (3.3.4) admits an asymptotic expansion

$$f^{(j)}(t,x;y,\eta) = f_1^{(j)}(t,x;y,\eta) + f_0^{(j)}(t,x;y,\eta) + f_{-1}^{(j)}(t,x;y,\eta) + \dots \qquad (3.3.6)$$

into components positively homogeneous in η, with the subscript indicating degree of homogeneity.

Note that in (3.3.1) there are no terms containing derivatives of $\varsigma^{(j)}(t,x;y,\eta)$: as $\varsigma^{(j)}(t,x;y,\eta)$ is infinitely smooth in all independent variables, its' derivatives give an infinitely smooth contribution which is dealt with by the "mod C^∞" in (3.3.2).

Note the following differences between formulae (3.2.11) and (3.3.6).

- The leading term in (3.3.6) has degree of homogeneity 1, rather than 0 as in (3.2.11). In fact, the leading term in (3.3.6) can be easily written out explicitly

$$f_1^{(j)}(t,x;y,\eta) = (\varphi_t^{(j)}(t,x;y,\eta) + A_1(x,\varphi_x^{(j)}(t,x;y,\eta)))\, u_0^{(j)}(t;y,\eta)\,, \quad (3.3.7)$$

  where $A_1(x,\xi)$ is the (matrix-valued) principal symbol of the pseudodifferential operator A.

- Unlike the symbol $u^{(j)}(t;y,\eta)$, the amplitude $f^{(j)}(t,x;y,\eta)$ depends on x.

We now need to exclude the dependence on x from the amplitude $f^{(j)}(t,x;y,\eta)$. This can be done by means of the algorithm described in Subsection 2.7.3 of [72]. We outline this algorithm below.

Working in local coordinates, define the matrix-function $\varphi_{x\eta}^{(j)}$ in accordance with $(\varphi_{x\eta}^{(j)})_\alpha{}^\beta := \varphi_{x^\alpha\eta_\beta}^{(j)}$ and then define its inverse $(\varphi_{x\eta}^{(j)})^{-1}$ from the identity

$$(\varphi_{x\eta}^{(j)})_\alpha{}^\beta [(\varphi_{x\eta}^{(j)})^{-1}]_\beta{}^\gamma := \delta_\alpha{}^\gamma.$$

Define the "scalar" first order linear differential operators

$$L_\alpha^{(j)} := [(\varphi_{x\eta}^{(j)})^{-1}]_\alpha{}^\beta \, (\partial/\partial x^\beta), \qquad \alpha = 1, \dots, n. \tag{3.3.8}$$

Note that the coefficients of these differential operators are functions of the position variable x and the dual variable ξ (as well as t and y). It is known, see part 2 of Appendix E in [72], that the operators (3.3.8) commute: $L_\alpha^{(j)} L_\beta^{(j)} = L_\beta^{(j)} L_\alpha^{(j)}$, $\alpha, \beta = 1, \dots, n$.

Denote $L_{\boldsymbol{\alpha}}^{(j)} := (L_1^{(j)})^{\alpha_1} \cdots (L_n^{(j)})^{\alpha_n}$, $(-\varphi_\eta^{(j)})^{\boldsymbol{\alpha}} := (-\varphi_{\eta_1}^{(j)})^{\alpha_1} \cdots (-\varphi_{\eta_n}^{(j)})^{\alpha_n}$, and, given an $r \in \mathbb{N}$, define the "scalar" linear differential operator

$$\mathfrak{P}_{-1,r}^{(j)} := i(d_{\varphi^{(j)}})^{-1} \frac{\partial}{\partial \eta_\beta} \, d_{\varphi^{(j)}} \left(1 + \sum_{1 \le |\boldsymbol{\alpha}| \le 2r-1} \frac{(-\varphi_\eta^{(j)})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}! \, (|\boldsymbol{\alpha}| + 1)} \, L_{\boldsymbol{\alpha}}^{(j)} \right) L_\beta^{(j)} , \tag{3.3.9}$$

where $|\boldsymbol{\alpha}| := \alpha_1 + \ldots + \alpha_n$, $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_n!$, and the repeated index β indicates summation over $\beta = 1, \dots, n$.

Recall Definition 2.7.8 from [72]: the linear operator L is said to be positively homogeneous in η of degree $p \in \mathbb{R}$ if for any $q \in \mathbb{R}$ and any function f positively homogeneous in η of degree q the function Lf is positively homogeneous in η of degree $p + q$. It is easy to see that the operator (3.3.9) is positively homogeneous in η of degree -1 and the first subscript in $\mathfrak{P}_{-1,r}^{(j)}$ emphasises this fact.

Let $\mathfrak{S}_0^{(j)}$ be the (linear) operator of restriction to $x = x^{(j)}(t; y, \eta)$,

$$\mathfrak{S}_0^{(j)} := (\,\cdot\,)|_{x = x^{(j)}(t;y,\eta)} , \tag{3.3.10}$$

and let

$$\mathfrak{S}_{-r}^{(j)} := \mathfrak{S}_0^{(j)} (\mathfrak{P}_{-1,r}^{(j)})^r \tag{3.3.11}$$

for $r = 1, 2, \ldots$. Observe that our linear operators $\mathfrak{S}_{-r}^{(j)}$, $r = 0, 1, 2, \ldots$, are positively homogeneous in η of degree $-r$. This observation allows us to define

the linear operator

$$\mathfrak{S}^{(j)} := \sum_{r=0}^{+\infty} \mathfrak{S}^{(j)}_{-r} \,, \tag{3.3.12}$$

where the series is understood as an asymptotic series in inverse powers of η.

According to Subsection 2.7.3 of [72], the dynamic equation (3.3.2) can now be rewritten in the equivalent form

$$\mathfrak{S}^{(j)} f^{(j)} = 0 \,, \tag{3.3.13}$$

where the equality is understood in the asymptotic sense, as an asymptotic expansion in inverse powers of η. Recall that the matrix-valued amplitude $f^{(j)}(t, x; y, \eta)$ appearing in (3.3.13) is defined by formulae (3.3.4) and (3.3.5).

Substituting (3.3.12) and (3.3.6) into (3.3.13) we obtain a hierarchy of equations

$$\mathfrak{S}^{(j)}_0 f^{(j)}_1 = 0, \tag{3.3.14}$$

$$\mathfrak{S}^{(j)}_{-1} f^{(j)}_1 + \mathfrak{S}^{(j)}_0 f^{(j)}_0 = 0, \tag{3.3.15}$$

$$\mathfrak{S}^{(j)}_{-2} f^{(j)}_1 + \mathfrak{S}^{(j)}_{-1} f^{(j)}_0 + \mathfrak{S}^{(j)}_0 f^{(j)}_{-1} = 0,$$

$$\dots$$

positively homogeneous in η of degree 1, 0, -1, ... These are the *transport* equations for the determination of the unknown homogeneous components $u^{(j)}_0(t; y, \eta)$, $u^{(j)}_{-1}(t; y, \eta)$, $u^{(j)}_{-2}(t; y, \eta)$, ..., of the symbol of the oscillatory integral (3.3.1).

Let us now examine the initial condition (3.3.3). Each operator $U^{(j)}(0)$ is a pseudodifferential operator, only written in a slightly nonstandard form. The issues here are as follows.

- We use the invariantly defined phase function $\varphi^{(j)}(0, x; y, \eta) = (x-y)^\alpha \, \eta_\alpha + O(|x-y|^2)$ rather than the linear phase function $(x-y)^\alpha \, \eta_\alpha$ written in local coordinates.

- When defining the (full) symbol of the operator $U^{(j)}(t)$ we excluded the variable x from the amplitude rather than the variable y. Note that when dealing with pseudodifferential operators it is customary to exclude the variable y from the amplitude; exclusion of the variable x gives the dual symbol of a pseudodifferential operator, see Subsection 2.1.3 in [72]. Thus, at $t = 0$, our symbol $u^{(j)}(0; y, \eta)$ resembles the dual symbol of a pseudodifferential operator rather than the "normal" symbol.

- We have the extra factor $d_{\varphi^{(j)}}(0, x; y, \eta)$ in our representation of the operator $U^{(j)}(0)$ as an oscillatory integral.

The (full) dual symbol of the pseudodifferential operator $U^{(j)}(0)$ can be calculated in local coordinates in accordance with the following formula which addresses the issues highlighted above:

$$\sum_{\boldsymbol{\alpha}} \frac{(-1)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \left(D_x^{\boldsymbol{\alpha}} \, \partial_\eta^{\boldsymbol{\alpha}} \left[u^{(j)}(0; y, \eta) \, e^{i\omega^{(j)}(x;y,\eta)} \, d_{\varphi^{(j)}}(0, x; y, \eta) \right] \right) \big|_{x=y} , \qquad (3.3.16)$$

where $\omega^{(j)}(x; y, \eta) = \varphi^{(j)}(0, x; y, \eta) - (x - y)^\beta \, \eta_\beta$. Formula (3.3.16) is a version of the formula from Subsection 2.1.3 of [72], only with the extra factor $(-1)^{|\boldsymbol{\alpha}|}$. The latter is needed because we are writing down the dual symbol of the pseudodifferential operator $U^{(j)}(0)$ (no dependence on x) rather than its "normal" symbol (no dependence on y).

The initial condition (3.3.3) can now be rewritten in explicit form as

$$\sum_j \sum_{\boldsymbol{\alpha}} \frac{(-1)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \left(D_x^{\boldsymbol{\alpha}} \, \partial_\eta^{\boldsymbol{\alpha}} \left[u^{(j)}(0; y, \eta) \, e^{i\omega^{(j)}(x;y,\eta)} \, d_{\varphi^{(j)}}(0, x; y, \eta) \right] \right) \big|_{x=y} = I \, , \qquad (3.3.17)$$

where I is the $m \times m$ identity matrix. Condition (3.3.17) can be decomposed into components positively homogeneous in η of degree $0, -1, -2, \ldots$, giving us a hierarchy of initial conditions. The leading (of degree of homogeneity 0) initial condition reads

$$\sum_j u_0^{(j)}(0; y, \eta) = I \, , \qquad (3.3.18)$$

whereas lower order initial conditions are more complicated and depend on the choice of our phase functions $\varphi^{(j)}$.

Sections 3 and 4 of [22] go on to apply the algorithm presented above. In particular, (3.2.12) and (3.2.18) are proved. This provides us with sufficient information concerning the propagator $U^{(j)}(t)$ to apply Fourier Tauberian theorems which, in turn, allow us to determine the asymptotics of the spectral and counting functions (3.2.6) and (3.2.7). The appropriate calculations, under suitable assumptions on Hamiltonian trajectories, are given in Sections 6-8 of [22]. In particular, Theorems 8.3 and 8.4 from [22] provide the required conditions on Hamiltonian trajectories.

3.4 U(1) connection

In the Sections 2-4 of [22] we presented technical details of the construction of the propagator; an overview of this scheme is presented in Section 3.3. We saw that the eigenvectors of the principal symbol, $v^{(j)}(x,\xi)$, play a major role in this construction. As pointed out in Section 3.2, each of these eigenvectors is defined up to a U(1) gauge transformation (3.2.16), (3.2.17). In the end, the full symbols (3.2.11) of our oscillatory integrals $U^{(j)}(t)$ do not depend on the choice of gauge for the eigenvectors $v^{(j)}(x,\xi)$. However, the effect of the gauge transformation (3.2.16), (3.2.17) is not as trivial as it may appear at first sight. We will demonstrate in this section that the gauge transformation (3.2.16), (3.2.17) shows up, in the form of invariantly defined curvature, in the lower order terms $u^{(j)}_{-1}(t;y,\eta)$ of the symbols of our oscillatory integrals $U^{(j)}(t)$. More precisely, we will show that the RHS of formula (3.2.18) is the scalar curvature of a connection associated with the gauge transformation (3.2.16), (3.2.17). Further on in this section, until the very last paragraph, the index j enumerating eigenvalues and eigenvectors of the principal symbol is assumed to be fixed.

Consider a smooth curve $\Gamma \subset T'M$ connecting points (y,η) and (x,ξ). We write this curve in parametric form as $(z(t),\zeta(t))$, $t \in [0,1]$, so that $(z(0),\zeta(0)) = (y,\eta)$

and $(z(1), \zeta(1)) = (x, \xi)$. Put

$$w(t) := e^{i\phi(t)} v^{(j)}(z(t), \zeta(t)) \,, \tag{3.4.1}$$

where $\phi : [0,1] \to \mathbb{R}$ is an unknown function which is to be determined from the condition

$$iw^* \dot{w} = 0 \tag{3.4.2}$$

with the dot indicating the derivative with respect to the parameter t. Substituting (3.4.1) into (3.4.2) we get an ordinary differential equation for ϕ which is easily solved, giving

$$\begin{aligned} \phi(1) = \phi(0) + \int_0^1 (\dot{z}^\alpha(t)\, P_\alpha(z(t), \zeta(t)) + \dot{\zeta}_\gamma(t)\, Q^\gamma(z(t), \zeta(t)))\, dt \\ = \phi(0) + \int_\Gamma (P_\alpha dz^\alpha + Q^\gamma d\zeta_\gamma) \,, \end{aligned} \tag{3.4.3}$$

where

$$P_\alpha := i[v^{(j)}]^* v^{(j)}_{z^\alpha}, \qquad Q^\gamma := i[v^{(j)}]^* v^{(j)}_{\zeta_\gamma}. \tag{3.4.4}$$

Note that the $2n$-component real quantity (P_α, Q^γ) is a covector field (1-form) on $T'M$. This quantity already appeared in Section 3.2 as formula (3.2.19).

Put $f(y, \eta) := e^{i\phi(0)}$, $f(x, \xi) := e^{i\phi(1)}$ and rewrite formula (3.4.3) as

$$f(x, \xi) = f(y, \eta)\, e^{i \int_\Gamma (P_\alpha dz^\alpha + Q^\gamma d\zeta_\gamma)}. \tag{3.4.5}$$

Let us identify the group U(1) with the unit circle in the complex plane, i.e. with $f \in \mathbb{C}$, $|f| = 1$. We see that formulae (3.4.5) and (3.4.4) give us a rule for the parallel transport of elements of the group U(1) along curves in $T'M$. This is the natural U(1) connection generated by the normalised field of columns of complex-valued scalars

$$v^{(j)}(z, \zeta) = \big(v_1^{(j)}(z, \zeta) \quad \dots \quad v_m^{(j)}(z, \zeta)\big)^T. \tag{3.4.6}$$

Recall that the Γ appearing in formula (3.4.5) is a curve connecting points (y, η)

and (x, ξ), whereas the $v^{(j)}(z, \zeta)$ appearing in formulae (3.4.4) and (3.4.6) enters our construction as an eigenvector of the principal symbol of our $m \times m$ matrix pseudodifferential operator A.

In practice, dealing with a connection is not as convenient as dealing with the covariant derivative ∇. The covariant derivative corresponding to the connection (3.4.5) is determined as follows. Let us view the (x, ξ) appearing in formula (3.4.5) as a variable which takes values close to (y, η), and suppose that the curve Γ is a short straight (in local coordinates) line segment connecting the point (y, η) with the point (x, ξ). We want the covariant derivative of our function $f(x, \xi)$, evaluated at (y, η), to be zero. Examination of formula (3.4.5) shows that the unique covariant derivative satisfying this condition is

$$\nabla_\alpha := \partial/\partial x^\alpha - iP_\alpha(x, \xi), \qquad \nabla^\gamma := \partial/\partial \xi_\gamma - iQ^\gamma(x, \xi). \tag{3.4.7}$$

We define the curvature of our U(1) connection as

$$R := -i \begin{pmatrix} \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha & \nabla_\alpha \nabla^\delta - \nabla^\delta \nabla_\alpha \\ \nabla^\gamma \nabla_\beta - \nabla_\beta \nabla^\gamma & \nabla^\gamma \nabla^\delta - \nabla^\delta \nabla^\gamma \end{pmatrix}. \tag{3.4.8}$$

It may seem that the entries of the $(2n) \times (2n)$ matrix (3.4.8) are differential operators. They are, in fact, operators of multiplication by "scalar functions". Namely, the more explicit form of (3.4.8) is

$$R = \begin{pmatrix} \frac{\partial P_\alpha}{\partial x^\beta} - \frac{\partial P_\beta}{\partial x^\alpha} & \frac{\partial P_\alpha}{\partial \xi_\delta} - \frac{\partial Q^\delta}{\partial x^\alpha} \\ \frac{\partial Q^\gamma}{\partial x^\beta} - \frac{\partial P_\beta}{\partial \xi_\gamma} & \frac{\partial Q^\gamma}{\partial \xi_\delta} - \frac{\partial Q^\delta}{\partial \xi_\gamma} \end{pmatrix}. \tag{3.4.9}$$

The $(2n) \times (2n)$ - component real quantity (3.4.9) is a rank 2 covariant antisymmetric tensor (2-form) on $T'M$. It is an analogue of the electromagnetic tensor.

Substituting (3.4.4) into (3.4.9) we get an expression for curvature in terms of the eigenvector of the principal symbol

$$R = i \begin{pmatrix} [v^{(j)}_{x^\beta}]^* v^{(j)}_{x^\alpha} - [v^{(j)}_{x^\alpha}]^* v^{(j)}_{x^\beta} & [v^{(j)}_{\xi_\delta}]^* v^{(j)}_{x^\alpha} - [v^{(j)}_{x^\alpha}]^* v^{(j)}_{\xi_\delta} \\ [v^{(j)}_{x^\beta}]^* v^{(j)}_{\xi_\gamma} - [v^{(j)}_{\xi_\gamma}]^* v^{(j)}_{x^\beta} & [v^{(j)}_{\xi_\delta}]^* v^{(j)}_{\xi_\gamma} - [v^{(j)}_{\xi_\gamma}]^* v^{(j)}_{\xi_\delta} \end{pmatrix}. \tag{3.4.10}$$

Examination of formula (3.4.10) shows that, as expected, curvature is invariant under the gauge transformation (3.2.16), (3.2.17).

It is natural to take the trace of the upper right block in (3.4.8) which, in the notation (3.2.14), gives us

$$-i(\nabla_\alpha \nabla^\alpha - \nabla^\alpha \nabla_\alpha) = -i\{[v^{(j)}]^*, v^{(j)}\}. \tag{3.4.11}$$

Thus, we have shown that the RHS of formula (3.2.18) is the scalar curvature of our U(1) connection.

We end this section by proving, as promised in Section 3.2, formula (3.2.20) without referring to microlocal analysis. In the following arguments we use our standard notation for the orthogonal projections onto the eigenspaces of the principal symbol, i.e. we write $P^{(k)} := v^{(k)}[v^{(k)}]^*$. We have $\operatorname{tr}\{P^{(j)}, P^{(j)}\} = 0$ and $\sum_l P^{(l)} = I$ which implies

$$\begin{aligned} 0 &= \sum_{l,j} \operatorname{tr}(P^{(l)}\{P^{(j)}, P^{(j)}\}) \\ &= \sum_j \operatorname{tr}(P^{(j)}\{P^{(j)}, P^{(j)}\}) + \sum_{l,j:\ l\neq j} \operatorname{tr}(P^{(l)}\{P^{(j)}, P^{(j)}\}). \end{aligned} \tag{3.4.12}$$

But, according to formula (4.15) from [22], for $l \neq j$ we have

$$\operatorname{tr}(P^{(l)}\{P^{(j)}, P^{(j)}\}) = -\operatorname{tr}(P^{(j)}\{P^{(l)}, P^{(l)}\}),$$

so the expression in the last sum in the RHS of (3.4.12) is antisymmetric in the indices l, j, which implies that this sum is zero. Hence, formula (3.4.12) can be

rewritten as

$$\sum_j \operatorname{tr}(P^{(j)}\{P^{(j)}, P^{(j)}\}) = 0.$$

It remains only to note that, according to formula (4.17) from [22],

$$\operatorname{tr}(P^{(j)}\{P^{(j)}, P^{(j)}\}) = \{[v^{(j)}]^*, v^{(j)}\}.$$

3.5 Spectral asymmetry

In this section we deal with the special case when the operator A is differential (as opposed to pseudodifferential). Our aim is to examine what happens when we change the sign of the operator. In other words, we compare the original operator A with the operator $\tilde{A} := -A$. In theoretical physics the transformation $A \mapsto -A$ would be interpreted as time reversal, see equation (3.2.3).

It is easy to see that for a differential operator the number m (number of equations in our system) has to be even and that the principal symbol has to have the same number of positive and negative eigenvalues. In the notation of Section 3.2 this fact can be expressed as $m = 2m^+ = 2m^-$.

It is also easy to see that the principal symbols of the two operators, A and $\tilde{A}$, and the eigenvalues and eigenvectors of the principal symbols are related as

$$A_1(x, \xi) = \tilde{A}_1(x, -\xi), \tag{3.5.1}$$

$$h^{(j)}(x, \xi) = \tilde{h}^{(j)}(x, -\xi), \tag{3.5.2}$$

$$v^{(j)}(x, \xi) = \tilde{v}^{(j)}(x, -\xi), \tag{3.5.3}$$

whereas the subprincipal symbols are related as

$$A_{\text{sub}}(x) = -\tilde{A}_{\text{sub}}(x). \tag{3.5.4}$$

Formulae (3.2.21), (3.2.22), (3.2.15), (3.2.14) and (3.5.1)–(3.5.4) imply

$$a(x) = \tilde{a}(x), \qquad b(x) = -\tilde{b}(x). \tag{3.5.5}$$

Substituting (3.5.5) into (3.2.8) and (3.2.9) we get

$$a = \tilde{a}, \qquad b = -\tilde{b}. \tag{3.5.6}$$

Formulae (3.2.7) and (3.5.6) imply that the spectrum of a generic first order differential operator is asymmetric about $\lambda = 0$. This phenomenon is known as *spectral asymmetry* [4, 5, 6, 7].

If we square our operator A and consider the spectral problem $A^2 v = \lambda^2 v$, then the terms $\pm b\lambda^{n-1}$ cancel out and the second asymptotic coefficient of the counting function (as well as the spectral function) of the operator A^2 turns to zero. This is in agreement with the known fact that for an even order semi-bounded matrix differential operator acting on a manifold without boundary the second asymptotic coefficient of the counting function is zero, see Section 6 of [80] and [68].

3.6 Bibliographic review

To our knowledge, the first publication on the subject of two-term spectral asymptotics for systems was Ivrii's 1980 paper [42] in Section 2 of which the author stated, without proof, a formula for the second asymptotic coefficient of the counting function. In a subsequent 1982 paper [43] Ivrii acknowledged that the formula from [42] was incorrect and gave a new formula, labelled (0.6), followed by a "proof". In his 1984 Springer Lecture Notes [44] Ivrii acknowledged on page 226 that both his previous formulae for the second asymptotic coefficient were incorrect and stated, without proof, yet another formula.

Roughly at the same time Rozenblyum [67] also stated a formula for the second asymptotic coefficient of the counting function of a first order system.

The formulae from [42], [43] and [67] are fundamentally flawed because they are proportional to the subprincipal symbol. As our formulae (3.2.9) and (3.2.22) show, the second asymptotic coefficient of the counting function may be nonzero even when the subprincipal symbol is zero. This illustrates, yet again, the difference between scalar operators and systems.

The formula on page 226 of [44] gives an algorithm for the calculation of the correction term designed to take account of the effect described in the previous paragraph. This algorithm requires the evaluation of a limit of a complicated expression involving the integral, over the cotangent bundle, of the trace of the symbol of the resolvent of the operator A constructed by means of pseudodifferential calculus. This algorithm was revisited in Ivrii's 1998 book, see formulae (4.3.39) and (4.2.25) in [45].

The next contributor to the subject was Safarov who, in his 1989 DSc Thesis [69], wrote down a formula for the second asymptotic coefficient of the counting function which was "almost" correct. This formula appears in [69] as formula (2.4). As explained in Section 3.2, Safarov lost only the curvature terms (3.2.23). Safarov's DSc Thesis [69] provides arguments which are sufficiently detailed and we were able to identify the precise point (page 163) at which the mistake occurred.

In 1998 Nicoll rederived [64] Safarov's formula (3.2.12) for the principal symbols of the propagator, using a method slightly different from [69], but stopped short of calculating the second asymptotic coefficient of the counting function.

In 2007 Kamotski and Ruzhansky [47] performed an analysis of the propagator of a first order elliptic system based on the approach of Rozenblyum [67], but stopped short of calculating the second asymptotic coefficient of the counting function.

Chapter 4

Microlocal analysis of first order systems II

In this chapter we present a detailed consideration of elliptic first order self-adjoint differential operators acting over a connected compact 3-manifold without boundary, with an eventual focus on the massless Dirac operator (1.1.1).

The motivation for this study is simple. The massless Dirac operator in three dimensions considered in Chapter 4 is a special case of the type of operator considered in Chapter 3. We are therefore in a position to apply the theory developed in Chapter 3 to the massless Dirac operator.

4.1 Main results

Consider a first order differential operator A acting on 2-columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued half-densities over a connected compact 3-dimensional manifold M without boundary. (cf. Section 1.3 for the definition of a half-density.) We assume the coefficients of the operator A to be infinitely smooth. We also

assume that the operator A is formally self-adjoint (symmetric):

$$\int_M w^*(Av)\,dx = \int_M (Aw)^* v\,dx \tag{4.1.1}$$

for all infinitely smooth $v, w : M \to \mathbb{C}^2$. Here and further on the superscript * in matrices, rows and columns indicates Hermitian conjugation in $\mathbb{C}^2$ and $dx := dx^1 dx^2 dx^3$, where $x = (x^1, x^2, x^3)$ are local coordinates on M.

Let $A_1(x, \xi)$ be the principal symbol of the operator A, i.e. matrix obtained by leaving in A only the leading (first order) derivatives and replacing each $\partial/\partial x^\alpha$ by $i\xi_\alpha$, $\alpha = 1, 2, 3$. Here $\xi = (\xi_1, \xi_2, \xi_3)$ is the variable dual to the position variable x. Our principal symbol $A_1(x, \xi)$ is a 2×2 Hermitian matrix-function on the cotangent bundle T^*M, linear in every fibre $T^*_x M$ (i.e. linear in ξ).

Throughout Chapter 4 we assume that the principal symbol $A_1(x, \xi)$ is trace-free for all $(x, \xi) \in T^*M$ and that

$$\det A_1(x, \xi) \neq 0, \qquad \forall (x, \xi) \in T'M, \tag{4.1.2}$$

where $T'M := T^*M \setminus \{\xi = 0\}$ (cotangent bundle with the zero section removed). The assumption (4.1.2) is a version of the ellipticity condition (cf. Subsection 3.1.3).

Under the above assumptions A is a self-adjoint operator in $L^2(M; \mathbb{C}^2)$ (Hilbert space of square integrable complex-valued column "functions") with domain $H^1(M; \mathbb{C}^2)$ (Sobolev space of complex-valued column "functions" which are square integrable together with their first partial derivatives) and the spectrum of A is discrete, with eigenvalues accumulating to $\pm\infty$. Let λ_k and $v_k = \begin{pmatrix} v_{k1}(x) & v_{k2}(x) \end{pmatrix}^T$ be the eigenvalues and eigenfunctions of the operator A. The eigenvalues λ_k are enumerated in increasing order with account of multiplicity, using a positive index $k = 1, 2, \ldots$ for positive λ_k and a nonpositive index $k = 0, -1, -2, \ldots$ for nonpositive λ_k.

We will be studying the spectral function (3.2.4) and the counting function (3.2.5) for the type of operator described above.

We aim to derive, under appropriate assumptions on Hamiltonian trajectories, two-term asymptotics for the spectral function (3.2.4) and the counting function (3.2.5), i.e. formulae of the type (3.2.6) and (3.2.7), respectively, as $\lambda \to +\infty$, where the real constants a, b appearing in (3.2.7) and real densities $a(x)$, $b(x)$ in (3.2.6) are related in accordance with (3.2.8) and (3.2.9) respectively. Note that, in Chapter 4, we always work in three dimensions, which reduces the complexity of the formulae from Chapter 3 somewhat.

In [22] (an overview of which is given in Chapter 3) we performed a comprehensive analysis of two-term spectral asymptotics for general first order elliptic systems. In doing this we showed that all previous publications on systems gave formulae for the second asymptotic coefficient that were either incorrect or incomplete (i.e. an algorithm for the calculation of the second asymptotic coefficient rather than an explicit formula), see Section 3.6 for the appropriate bibliographic review. The correct formula for the coefficient $b(x)$ was the main result presented in Chapter 3.

The problem examined in Chapter 4 is a special case of that from [22]. Namely, in Chapter 4 we make the following additional assumptions as compared to [22] and Chapter 3:

$$\text{our manifold has dimension 3,} \tag{4.1.3}$$

$$\text{the number of equations in our system is 2,} \tag{4.1.4}$$

$$\text{our operator is differential (as opposed to pseudodifferential),} \tag{4.1.5}$$

$$\text{the principal symbol is trace-free.} \tag{4.1.6}$$

The need for a detailed analysis of the special case (4.1.3)–(4.1.6) is driven by applications to the massless Dirac operator.

The additional assumptions (4.1.3)–(4.1.6) lead to the following simplifications as compared to Chapter 3.

- The subprincipal symbol A_{sub} does not depend on the dual variable ξ (momentum) and is a function of x (position) only. Recall that the subprincipal symbol is the zeroth order term of the full symbol of the first order operator A written in a way which makes it invariant under coordinate transformations, see formula (4.6.2) for formal definition and Subsection 2.1.3 in [72] for background material.

- The principal symbol A_1 admits a geometric description.

The first of these simplifications is trivial whereas the second is not. We list below the geometric objects encoded within the principal symbol.

Geometric object 1: the *metric.* Observe that the determinant of the principal symbol is a negative definite quadratic form

$$\det A_1(x,\xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta \tag{4.1.7}$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha,\beta = 1,2,3$, appearing in (4.1.7) can be interpreted as components of a (contravariant) Riemannian metric. This implies, in particular, that our Hamiltonian (positive eigenvalue of the principal symbol) takes the form

$$h^+(x,\xi) = \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta} \tag{4.1.8}$$

and the x-components of our Hamiltonian trajectories become geodesics.

Geometric object 2: the *Teleparallel connection.* This is an affine connection defined as follows. Suppose we have a covector ξ based at the point $x \in M$ and we want to construct a parallel covector $\tilde{\xi}$ based at the point $\tilde{x} \in M$. This is done by solving the linear system of equations

$$A_1(\tilde{x},\tilde{\xi}) = A_1(x,\xi). \tag{4.1.9}$$

Equation (4.1.9) is equivalent to a system of three real linear algebraic equations for the three real unknowns, components of the covector $\tilde{\xi}$, and it is easy

to see that this system has a unique solution. It is also easy to see that the affine connection defined by formula (4.1.9) preserves the Riemannian norm of covectors, i.e. $g^{\alpha\beta}(\tilde{x})\,\tilde{\xi}_\alpha\tilde{\xi}_\beta = g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta$, hence, it is metric compatible. The parallel transport defined by formula (4.1.9) does not depend on the curve along which we transport the (co)vector, so our connection has zero curvature. Recall that the word "Teleparallel" (parallel at a distance) is used in theoretical physics [39] to describe metric compatible affine connections with zero curvature. The origins of this terminology go back to the works of A. Einstein and É. Cartan [79, 73, 18], though Cartan preferred to use the term "absolute parallelism" rather than "Teleparallelism", see Section 2.1 for further details.

The Teleparallel connection coefficients $\Gamma^\alpha{}_{\beta\gamma}(x)$ can be written down explicitly in terms of the principal symbol, see formula (4.3.7), and this allows us to define yet another geometric object — the torsion tensor

$$T^\alpha{}_{\beta\gamma} := \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}\,. \tag{4.1.10}$$

Further on we raise and lower indices of the torsion tensor using the metric.

Geometric object 3: the *topological charge.* It turns out, see Section 4.3, that the existence of a principal symbol implies that our manifold M is parallelizable. Parallelizability implies orientability. Having chosen a particular orientation, we allow only changes of local coordinates x^α, $\alpha = 1, 2, 3$, which preserve orientation.

We define the topological charge as

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \mathrm{tr}\big((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\big), \tag{4.1.11}$$

with the subscripts ξ_α indicating partial derivatives. We show in Section 4.3 that the number $\mathbf{c}$ defined by formula (4.1.11) can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to the chosen orientation of local coordinates.

We have identified three geometric objects encoded within the principal symbol — metric, Teleparallel connection and topological charge. Consequently, one would expect the coefficient $b(x)$ from formula (3.2.6) to be expressed via these three geometric objects and the subprincipal symbol. This assertion is confirmed by the following theorem.

Theorem 4.1.1. *The coefficients in the two-term asymptotics (3.2.6) are given by the formulae*

$$a(x) = \frac{1}{6\pi^2}\sqrt{\det g_{\alpha\beta}(x)}\,, \tag{4.1.12}$$

$$b(x) = \frac{1}{8\pi^2}\big([\,3\,\mathbf{c}\,{*}T^{\mathrm{ax}} - 2\,\mathrm{tr}\,A_{\mathrm{sub}}\,]\,\sqrt{\det g_{\alpha\beta}}\,\big)(x)\,, \tag{4.1.13}$$

where

$$T^{\mathrm{ax}}_{\alpha\beta\gamma} := \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha}) \tag{4.1.14}$$

is axial torsion *(totally antisymmetric piece of the torsion tensor) and $*$ is the Hodge star (4.3.4).*

Remark 4.1.1. *The spectral and counting functions admit two-term asymptotic expansions (3.2.6) and (3.2.7) only under appropriate assumptions on geodesic loops and closed geodesics respectively, see Theorems 8.3 and 8.4 from [22]. However, one can easily reformulate asymptotic formulae (3.2.6) and (3.2.7) in such a way that they remain valid without assumptions on geodesics: this can easily be achieved, say, by taking a convolution with a function from Schwartz space $\mathcal{S}(\mathbb{R})$, see Theorems 7.1 and 7.2 in [22]. Thus, the second asymptotic coefficients of the spectral and counting functions are well-defined irrespective of how many geodesic loops or closed geodesics we have. We introduced the second asymptotic coefficients $b(x)$ and b via the unmollified asymptotic expansions (3.2.6) and (3.2.7) simply for the sake of clarity of presentation.*

The proof of Theorem 4.1.1 is given in Sections 4.2–4.5.

We now turn our attention to the massless Dirac operator. This operator is defined in Appendix 4.A, see formula (4.A.3), and it does not fit into our scheme because it is an operator acting on a 2-component complex-valued spinor (Weyl

spinor) rather than a pair of complex-valued half-densities. However, on a parallelizable manifold components of a spinor can be identified with half-densities. We call the resulting operator *the massless Dirac operator on half-densities*. The explicit formula for the massless Dirac operator on half-densities is (4.A.19).

The massless Dirac operator on half-densities is an operator of the type we are considering in Chapter 4, i.e. a self-adjoint first order elliptic differential operator acting on 2-columns of complex-valued half-densities and with a trace-free principal symbol. We address the question: is a given operator A a massless Dirac operator? The answer is given by the following theorem which the main result of Chapter 4.

Theorem 4.1.2. *The operator A is a massless Dirac operator on half-densities if and only if the following two conditions are satisfied at every point of the manifold M:*

a) the subprincipal symbol of the operator, $A_{\mathrm{sub}}(x)$, is proportional to the identity matrix, and

b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero.

Note that conditions a) and b) in Theorem 4.1.2 are invariant under special unitary transformations, i.e. transformations of the operator

$$A \mapsto RAR^*, \tag{4.1.15}$$

where $R : M \to \mathrm{SU}(2)$ is an arbitrary smooth special unitary matrix-function. The invariance of condition b) is obvious. In fact, condition b) is invariant under the action of a broader group: the unitary matrix-function $R(x)$ appearing in formula (4.1.15) does not have to be special. As to condition a), its invariance is established by examination of formula (9.3) from [22] with the use of the special commutation properties of trace-free Hermitian 2×2 matrices. The fact that the conditions of Theorem 4.1.2 are SU(2) invariant is not surprising as the massless

Dirac operator is designed around the concept of SU(2) invariance, see Property 4 in Appendix 4.A.

The proof of Theorem 4.1.2 is given in Sections 4.6 and 4.7.

Theorems 4.1.1 and 4.1.2 tell us that for the massless Dirac operator on half-densities formulae (3.2.6) and (3.2.7) read

$$e(\lambda, x, x) = \frac{\sqrt{\det g_{\alpha\beta}(x)}}{6\pi^2}\,\lambda^3 + o(\lambda^2), \tag{4.1.16}$$

$$N(\lambda) = \frac{\operatorname{Vol} M}{6\pi^2}\,\lambda^3 + o(\lambda^2), \tag{4.1.17}$$

where $\operatorname{Vol} M$ is the volume of the Riemannian 3-manifold M.

Remark 4.1.2. *The factor $\sqrt{\det g_{\alpha\beta}(x)}$ appears in the RHS of (4.1.16) because we are working with the massless Dirac operator on half-densities (4.A.19) rather than with the massless Dirac operator on spinors (4.A.3). For the massless Dirac operator on spinors the spectral function is a scalar field (as opposed to a density) and formula (4.1.16) reads $e(\lambda, x, x) = \frac{1}{6\pi^2}\lambda^3 + o(\lambda^2)$.*

4.2 Reduction from the general setting

As explained in Section 4.1, the problem considered in Chapter 4 is a special case of that from [22] and Chapter 3. Formulae (3.2.21) and (3.2.22) from Chapter 3 in this case read

$$a(x) = \int\limits_{h^+(x,\xi)<1} đ\xi\,, \tag{4.2.1}$$

$$b(x) = b_1(x) + b_2(x)\,, \tag{4.2.2}$$

where

$$b_1(x) = -3 \int\limits_{h^+(x,\xi)<1} ([v^+]^* A_{\mathrm{sub}} v^+)(x,\xi)\, đ\xi\,, \tag{4.2.3}$$

$$b_2(x) = \frac{3i}{2} \int\limits_{h^+(x,\xi)<1} \{[v^+]^*, A_1 - 2h^+ I, v^+\}(x,\xi)\, đ\xi\,. \tag{4.2.4}$$

Here $h^+(x,\xi)$ is the positive eigenvalue of the principal symbol (see also formula (4.1.8)), $v^+(x,\xi)$ is the corresponding normalized eigenvector (2-column), $đ\xi$ is shorthand for $đ\xi := (2\pi)^{-3}\,d\xi = (2\pi)^{-3}\,d\xi_1 d\xi_2 d\xi_3$ and I is the 2×2 identity matrix. Curly brackets in formula (4.1.13) denote the Poisson bracket on matrix-functions (3.2.14) and its further generalization (3.2.15) in three dimensions (rather than n dimensions as was seen in Chapter 3).

Put $P^+(x,\xi) := [v^+(x,\xi)][v^+(x,\xi)]^*$, which is the orthogonal projection onto the eigenspace $\operatorname{span} v^+$ of the principal symbol. We have $A_1 - 2h^+ I = 2h^+ P^+ - 3h^+ I$ and $\{[v^+]^*, P^+, v^+\} = 0$, so formula (4.2.4) can be rewritten as

$$b_2(x) = -\frac{9i}{2} \int\limits_{h^+(x,\xi)<1} (h^+\{[v^+]^*, v^+\})(x,\xi)\, đ\xi\,. \tag{4.2.5}$$

Our aim now is to evaluate the integrals (4.2.1), (4.2.3) and (4.2.5) explicitly.

Formulae (4.2.1) and (4.1.8) immediately imply (4.1.12).

In order to evaluate the integral (4.2.3) we rewrite this formula as

$$b_1(x) = -3 \int\limits_{h^+(x,\xi)<1} \operatorname{tr}(A_{\mathrm{sub}} P^+)(x,\xi)\, đ\xi$$

and use the fact that $P^+(x,\xi) = \frac{1}{2h^+(x,\xi)}(A_1(x,\xi) + h^+(x,\xi)\, I)$. We get

$$b_1(x) = -3 \int\limits_{h^+(x,\xi)<1} \frac{1}{2h^+(x,\xi)} \operatorname{tr}(A_{\mathrm{sub}}(A_1 + h^+ I))(x,\xi)\, đ\xi\,.$$

But A_{sub} does not depend on ξ whereas A_1 is an odd function of ξ, so the term $\frac{1}{2h^+}\operatorname{tr}(A_{\mathrm{sub}} A_1)$ integrates to zero, leaving us with

$$b_1(x) = -\frac{3}{2}(\operatorname{tr} A_{\mathrm{sub}})(x) \int\limits_{h^+(x,\xi)<1} đ\xi \;= -\frac{1}{4\pi^2}\big(\operatorname{tr} A_{\mathrm{sub}} \sqrt{\det g_{\alpha\beta}}\,\big)(x)\,. \tag{4.2.6}$$

In order to complete the proof of Theorem 4.1.1 we need to evaluate explicitly the integral (4.2.5). The next three sections deal with this nontrivial issue.

4.3 Teleparallel connection

We show in this section that the principal symbol generates a Teleparallel connection which allows us to reformulate the results of our spectral analysis in a much clearer geometric language.

Let us show first that the existence of a principal symbol implies that our manifold M is parallelizable. The principal symbol $A_1(x,\xi)$ is linear in ξ so it can be written as

$$A_1(x,\xi) = \sigma^\alpha(x)\,\xi_\alpha\,, \tag{4.3.1}$$

where $\sigma^\alpha(x)$, $\alpha = 1,2,3$, are some trace-free Hermitian 2×2 matrix-functions. Let us denote the elements of the matrices σ^α as $\sigma^\alpha{}_{\dot a b}$, where the dotted index, running through the values $\dot 1, \dot 2$, enumerates the rows and the undotted index, running through the values $1,2$, enumerates the columns; this notation is taken from [23]. Put

$$e_1{}^\alpha(x) := \operatorname{Re}\sigma^\alpha{}_{\dot 1 2}(x), \quad e_2{}^\alpha(x) := -\operatorname{Im}\sigma^\alpha{}_{\dot 1 2}(x), \quad e_3{}^\alpha(x) := \sigma^\alpha{}_{\dot 1 1}(x). \tag{4.3.2}$$

Formula (4.3.2) defines a triple of smooth real vector fields $e_j(x)$, $j=1,2,3$, on the manifold M. These vector fields are linearly independent at every point x of the manifold: this follows from formula (4.1.2) via routine calculations. Thus, the triple of vector fields e_j is a *frame*. The existence of a frame means that the manifold M is parallelizable.

Conversely, given a frame e_j we uniquely recover the principal symbol $A_1(x,\xi)$ via formulae (4.3.1), (4.A.1) and (4.A.2). Thus, a principal symbol is equivalent to a frame. Of course, this equivalence statement relies on our *a priori* assumptions (4.1.1), (4.1.2) and (4.1.3)–(4.1.6).

It is easy to see that the frame elements e_j are orthonormal with respect to the metric (4.1.7). Moreover, the metric can be determined directly from the frame as

$$g^{\alpha\beta} = \delta^{jk} e_j{}^{\alpha}\, e_k{}^{\beta}\,, \tag{4.3.3}$$

where the repeated frame indices j and k indicate summation over $j,k = 1,2,3$. The two definitions of the metric, (4.1.7) and (4.3.3), are equivalent.

Parallelizability implies orientability, see Proposition 13.5 in [52]. Having chosen a particular orientation, we allow only changes of local coordinates x^α, $\alpha = 1,2,3$, which preserve orientation and define the Hodge star in the standard way: the action of $*$ on a rank q antisymmetric tensor Q is

$$(*Q)_{\gamma_{q+1}\dots\gamma_3} := (q!)^{-1} \sqrt{\det g_{\alpha\beta}}\, Q^{\gamma_1\dots\gamma_q} \varepsilon_{\gamma_1\dots\gamma_3}\,, \tag{4.3.4}$$

where ε is the totally antisymmetric quantity, $\varepsilon_{123} := +1$, and g is the Riemannian metric (4.1.7). Here and further on we identify differential forms with covariant antisymmetric tensors. We raise and lower tensor indices using our metric.

Substituting formulae (4.3.1) and (4.3.2) into (4.1.11) we get

$$\mathbf{c} = \operatorname{sgn}\det e_j{}^{\alpha}. \tag{4.3.5}$$

Formula (4.3.5) provides an equivalent (and more natural) definition of topological charge. It also explains why the topological charge, initially defined in Section 4.1 in accordance with formula (4.1.11), can only take values $+1$ or -1.

The concept of a Teleparallel connection was already defined in Section 4.1 in accordance with formula (4.1.9). This connection can be equivalently defined via the frame as follows. Suppose we have a vector v based at the point $x \in M$ and we want to construct a parallel vector $\tilde{v}$ based at the point $\tilde{x} \in M$. We decompose the vector v with respect to the frame at the point x, $v = c^j e_j(x)$, and reassemble it with the same coefficients c^j at the point $\tilde{x}$, defining $\tilde{v} := c^j e_j(\tilde{x})$.

We now define the covariant derivative corresponding to the Teleparallel connection. Our Teleparallel connection is a special case of an affine connection, so we are looking at a covariant derivative acting on vector/covector fields in the usual manner

$$\nabla_\mu v^\alpha = \partial v^\alpha/\partial x^\mu + \Gamma^\alpha{}_{\mu\beta}\, v^\beta\,, \qquad \nabla_\mu w_\beta = \partial w_\beta/\partial x^\mu - \Gamma^\alpha{}_{\mu\beta}\, w_\alpha\,.$$

Note that in this thesis we use the symbol ∇ to denote two different covariant derivatives: one with respect to the Levi-Civita connection, and the other with respect to the Teleparallel connection. The connection with respect to which the covariant derivative is defined is always stated. As such, the meaning is clear from the context in which the symbol is used.

The Teleparallel connection coefficients are defined from the conditions

$$\nabla_\mu e_j{}^\alpha = 0\,, \tag{4.3.6}$$

where the e_j are elements of our frame. Formula (4.3.6) gives a system of 27 linear algebraic equations for the determination of 27 unknown connection coefficients. It is known (see, for example, formula (A2) in [15]), that the unique solution of this system is

$$\Gamma^\alpha{}_{\mu\beta} = e_k{}^\alpha(\partial e^k{}_\beta/\partial x^\mu)\,, \tag{4.3.7}$$

where

$$e^k{}_\beta := \delta^{kj} g_{\beta\gamma} e_j{}^\gamma. \tag{4.3.8}$$

The triple of covector fields e^k, $k = 1, 2, 3$, is called the *coframe*. The frame and coframe uniquely determine each other via the relation

$$e_j{}^\alpha e^k{}_\alpha = \delta_j{}^k. \tag{4.3.9}$$

One can check by performing explicit calculations that the Teleparallel connection has the following two important properties. Firstly

$$\nabla_\alpha g^{\beta\gamma} = 0. \tag{4.3.10}$$

This follows from (4.3.3) which gives

$$\begin{aligned}\nabla_\gamma g^{\alpha\beta} &= \nabla_\gamma \left(\delta^{jk} e_j{}^\alpha \, e_k{}^\beta\right) \\ &= \delta^{jk} \left(\nabla_\gamma e_j{}^\alpha\right) e_k{}^\beta + \delta^{jk} e_j{}^\alpha \left(\nabla_\gamma e_k{}^\beta\right) \\ &= 0\end{aligned}$$

from condition (4.3.6). This means that the connection is metric compatible. Secondly

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) v^\gamma = 0 \text{ for any vector field } v\,, \tag{4.3.11}$$

which means that the Riemann curvature tensor is zero; this follows from a lengthy calculation which is omitted here for brevity. Properties (4.3.10) and (4.3.11) are the defining properties of a Teleparallel connection: a Teleparallel connection is, by definition [39], an affine connection satisfying (4.3.10) and (4.3.11).

The tensor characterizing the "strength" of the Teleparallel connection is not the Riemann curvature tensor but the torsion tensor (4.1.10). The Teleparallel connection is, in a sense, the opposite of the more common Levi-Civita connection: the Levi-Civita connection has zero torsion but nonzero curvature, whereas the Teleparallel connection has nonzero torsion but zero curvature. In Chapter 4 we distinguish these two affine connections by using different notation for connection coefficients: we write the Teleparallel connection coefficients as $\Gamma^\alpha{}_{\beta\gamma}$ and the Levi-Civita connection coefficients (Christoffel symbols) as $\left\{ {\alpha \atop \beta\gamma} \right\}$, see formula (4.A.4). It is known, see formula (7.34) in [53], that the two sets of connection coefficients are related as

$$\Gamma^\alpha{}_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} + \frac{1}{2}(T^\alpha{}_{\beta\gamma} + T_\beta{}^\alpha{}_\gamma + T_\gamma{}^\alpha{}_\beta). \tag{4.3.12}$$

Note that in (4.3.12) the position of indices of the torsion tensor is extremely important. The indices are raised and lowered using the metric. For absolute clarity we relate the torsion tensor appearing in (4.3.12) to (4.1.10) in the following manner:

$$T_{\beta}{}^{\alpha}{}_{\gamma} = T^{\mu}{}_{\nu\gamma}\, g_{\mu\beta}\, g^{\nu\alpha},$$

see Chapter 7 of [53] for further details.

Substituting (4.3.7) into (4.1.10) we arrive at the following explicit formula for the torsion tensor of the Teleparallel connection

$$T = e_j \otimes de^j\,, \tag{4.3.13}$$

where the d stands for the exterior derivative. For the sake of clarity we rewrite formula (4.3.13) in more detailed form, retaining all tensor indices,

$$T^{\alpha}{}_{\beta\gamma} = e_j{}^{\alpha}(\partial e^j{}_{\gamma}/\partial x^{\beta} - \partial e^j{}_{\beta}/\partial x^{\gamma})\,. \tag{4.3.14}$$

As always, the repeated index j appearing in formulae (4.3.13) and (4.3.14) indicates summation over $j = 1, 2, 3$.

Torsion is a rank three tensor antisymmetric in the last two indices. Because we are working in dimension three, it is convenient, as in Chapter 2, to apply the Hodge star in the last two indices and deal with the rank two tensor

$$\overset{*}{T}{}^{\alpha}{}_{\beta} := \frac{1}{2}\, T^{\alpha\gamma\delta}\, \varepsilon_{\gamma\delta\beta}\, \sqrt{\det g_{\mu\nu}} \tag{4.3.15}$$

instead. Note that in (4.3.15) the position of the indices of the torsion tensor is extremely important. Recalling again that indices are raised and lowered using the metric, we see that the torsion tensor appearing in (4.3.15) is related to (4.1.10) in accordance with:

$$T^{\alpha\gamma\delta} = T^{\alpha}{}_{\mu\nu}\, g^{\mu\gamma}\, g^{\nu\delta}.$$

Substituting (4.3.13) into (4.3.15) we get

$$\overset{*}{T} = e_j \otimes \operatorname{curl} e^j \,, \tag{4.3.16}$$

where

$$(\operatorname{curl} e^j)_\beta := (*de^j)_\beta = \frac{1}{2}\,(de^j)^{\gamma\delta}\,\varepsilon_{\gamma\delta\beta}\,\sqrt{\det g_{\mu\nu}}\,. \tag{4.3.17}$$

4.4 Relation between curvature of the U(1) connection and torsion of the Teleparallel connection

This section is devoted to the examination of the integrand in formula (4.2.5). Recall that the curly brackets in this integrand denote the Poisson bracket on matrix-functions (3.2.14) in three dimensions.

As explained in Section 3.4, the expression $-i\{[v^+]^*, v^+\}$ is the scalar curvature of the U(1) connection generated by the eigenspace $\operatorname{span} v^+$ of the principal symbol. This curvature term appears in the general setting of a first order elliptic system. A feature of the particular case (4.1.3)–(4.1.6) considered in Chapter 4 is that the scalar curvature of the U(1) connection can be expressed via torsion of the Teleparallel connection. This is a substantial simplification. The Teleparallel connection is a simpler geometric object than the U(1) connection because the coefficients of the Teleparallel connection do not depend on the dual variable (momentum), i.e. they are "functions" on the base manifold M. The relationship between the two connections is established by the following lemma.

Lemma 4.4.1. *The scalar curvature of the* U(1) *connection is expressed via the torsion of the Teleparallel connection, metric and topological charge as*

$$-\,i\{[v^+]^*, v^+\}(x,\xi) = \frac{\mathbf{c}}{2}\,\frac{\overset{*}{T}{}^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}{(g^{\mu\nu}(x)\,\xi_\mu\xi_\nu)^{3/2}}\,. \tag{4.4.1}$$

Recall that the topological charge $\mathbf{c} = \pm 1$ is defined in accordance with formula (4.1.11) or, equivalently, in accordance with formula (4.3.5).

Proof of Lemma 4.4.1 We give the proof for the case

$$\mathbf{c} = +1\,. \tag{4.4.2}$$

There is no need to give a separate proof for the case $\mathbf{c} = -1$ as the two cases reduce to one another by means of a) the observation that torsion (4.3.13) is invariant under inversion of the frame and b) the identity

$$\{[v^+]^*, v^+\} + \{[v^-]^*, v^-\} = 0, \tag{4.4.3}$$

where $v^-(x,\xi)$ is the normalized eigenvector of the principal symbol corresponding to the negative eigenvalue. Formula (4.4.3) is a special case of formula (3.2.20) from Chapter 3.

We fix an arbitrary point $Q \in T'M$ and prove formula (4.4.1) at this point. As the LHS and RHS of (4.4.1) are invariant under changes of local coordinates x, it is sufficient to prove formula (4.4.1) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the projection of the point Q onto the base manifold, $g_{\mu\nu}(0) = \delta_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the formula we are proving involves only first partial derivatives in x, we may assume, without loss of generality, that

$$g_{\mu\nu}(x) = \delta_{\mu\nu} \tag{4.4.4}$$

for all x in some neighbourhood of the origin. In other words, it is sufficient to prove formula (4.4.1) for the case of Euclidean metric.

As both the LHS and RHS of (4.4.1) have the same degree of homogeneity in ξ, namely, -1, it is sufficient to prove formula (4.4.1) for ξ of norm 1. Moreover, by rotating our Cartesian coordinate system we can reduce the case of general ξ of norm 1 to the case

$$\xi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T. \tag{4.4.5}$$

There is one further simplification that can be made: we claim that it is sufficient to prove formula (4.4.1) for the case when

$$e_j{}^{\alpha}(0) = \delta_j{}^{\alpha}, \tag{4.4.6}$$

i.e. for the case when at the point $x = 0$ the elements of the frame are aligned with the coordinate axes. This claim follows from the observation that the LHS of formula (4.4.1) is invariant under rigid special unitary transformations of the column-function v^+, $v^+ \mapsto Rv^+$, where "rigid" refers to the fact that the matrix $R \in \mathrm{SU}(2)$ is constant. Of course, the column-function Rv^+ is no longer an eigenvector of the original principal symbol, but a new principal symbol obtained from the old one by the rigid special orthogonal transformation of the frame (4.A.14) with the 3×3 special orthogonal matrix O expressed in terms of the 2×2 special unitary matrix R in accordance with (4.A.15). One can always choose the special unitary matrix R so that at the point $x = 0$ the elements of the new frame are aligned with the coordinate axes (in fact, there are two possible choices of R which differ by sign). It remains only to note that direct inspection of formula (4.3.13) shows that torsion is also invariant under rigid special orthogonal transformations of the frame, and, hence, the tensor $\overset{*}{T}$ defined by formula (4.3.15) and appearing in the RHS of formula (4.4.1) is invariant under rigid special orthogonal transformations of the frame as well.

Having made the simplifying assumptions (4.4.4)–(4.4.6), we are now in a position to prove formula (4.4.1).

Let us calculate the RHS of (4.4.1) first. In view of (4.4.6) we have, in the linear approximation in x,

$$\begin{pmatrix} e_1{}^1(x) & e_1{}^2(x) & e_1{}^3(x) \\ e_2{}^1(x) & e_2{}^2(x) & e_2{}^3(x) \\ e_3{}^1(x) & e_3{}^2(x) & e_3{}^3(x) \end{pmatrix} = \begin{pmatrix} 1 & w^3(x) & -w^2(x) \\ -w^3(x) & 1 & w^1(x) \\ w^2(x) & -w^1(x) & 1 \end{pmatrix}, \tag{4.4.7}$$

where w is some smooth vector-function which vanishes at $x = 0$. Formula (4.4.7)

is the standard formula for the linearization of an orthogonal matrix about the identity; see also formula (2.10.1) in Chapter 2. Note that in Cosserat elasticity literature the vector-function w is called the *vector of microrotations*. Substituting (4.4.7) into (4.3.16) and (4.3.17) we get, at $x = 0$,

$$\overset{*}{T}_{\alpha\beta} = \partial w_\beta/\partial x^\alpha - \delta_{\alpha\beta}\,\mathrm{div}\, w, \tag{4.4.8}$$

which is formula (2.10.5) from Chapter 2. Here we freely lower and raise tensor indices using the fact that the metric is Euclidean (in the Euclidean case (4.4.4) it does not matter whether a tensor index comes as a subscript or a superscript). Substituting (4.4.8) and (4.4.5) into the RHS of (4.4.1) we get, at our point $Q \in T'M$,

$$\frac{1}{2}\,\frac{\overset{*}{T}^{\alpha\beta}\xi_\alpha\xi_\beta}{(g^{\mu\nu}\xi_\mu\xi_\nu)^{3/2}} = -\frac{1}{2}(\partial w^1/\partial x^1 + \partial w^2/\partial x^2)\,. \tag{4.4.9}$$

Let us now calculate the LHS of (4.4.1). The equation for the eigenvector $v^+(x,\xi)$ of the principal symbol is

$$\begin{pmatrix} e_3{}^\alpha\xi_\alpha - \|\xi\| & (e_1 - ie_2)^\alpha\xi_\alpha \\ (e_1 + ie_2)^\alpha\xi_\alpha & -e_3{}^\alpha\xi_\alpha - \|\xi\| \end{pmatrix} \begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} = 0\,. \tag{4.4.10}$$

In view of (4.4.5) and (4.4.6) the (normalized) solution of (4.4.10) at our point $Q \in T'M$ is $v^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Of course, our $v^+(x,\xi)$ is defined up to the gauge transformation

$$v^+ \mapsto e^{i\phi^+}v^+, \tag{4.4.11}$$

where

$$\phi^+ : T'M \to \mathbb{R} \tag{4.4.12}$$

is an arbitrary smooth function, however the LHS of (4.4.1) is invariant under this gauge transformation. We now perturb equation (4.4.10) about the point $Q \in T'M$, that is, about $x = 0$, $\xi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, making use of formula (4.4.7), which gives us the following equation for the increment δv^+ of the eigenvector

$v^+(x,\xi)$ of the principal symbol:

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \delta v_1^+ \\ \delta v_2^+ \end{pmatrix} + \begin{pmatrix} 0 & -w^2(x) - iw^1(x) \\ -w^2(x) + iw^1(x) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \delta\xi_1 - i\delta\xi_2 \\ \delta\xi_1 + i\delta\xi_2 & -2\delta\xi_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

or, equivalently,

$$\delta v_2^+ = \frac{1}{2}(-w^2(x) + iw^1(x) + \delta\xi_1 + i\delta\xi_2). \tag{4.4.13}$$

Formula (4.4.13) has to be supplemented by the normalization condition $\|v^+(x,\xi)\| = 1$, which in its linearized form reads

$$\operatorname{Re}\delta v_1^+ = 0. \tag{4.4.14}$$

Formulae (4.4.14) and (4.4.13) define δv^+ modulo an arbitrary $\operatorname{Im}\delta v_1^+$, with this degree of freedom being associated with the gauge transformation (4.4.11), (4.4.12). Without loss of generality we may assume that the gauge is chosen so that

$$\operatorname{Im}\delta v_1^+ = 0. \tag{4.4.15}$$

Combining formulae (4.4.14), (4.4.15) and (4.4.13) we get

$$\delta v^+ = \frac{1}{2} \begin{pmatrix} 0 \\ -w^2(x) + iw^1(x) + \delta\xi_1 + i\delta\xi_2 \end{pmatrix}. \tag{4.4.16}$$

Recall that the w appearing in this formula is some smooth vector-function which vanishes at $x = 0$.

Differentiation of (4.4.16) gives us

$$\frac{\partial v^+}{\partial x^\alpha} = \frac{1}{2} \begin{pmatrix} 0 \\ -\partial w^2/\partial x^\alpha + i\partial w^1/\partial x^\alpha \end{pmatrix}, \tag{4.4.17}$$

$$\frac{\partial v^+}{\partial \xi_1} = \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}, \qquad \frac{\partial v^+}{\partial \xi_2} = \frac{1}{2}\begin{pmatrix}0\\i\end{pmatrix}, \qquad \frac{\partial v^+}{\partial \xi_3} = 0. \tag{4.4.18}$$

Formulae (4.4.17) and (4.4.18) imply that at our point $Q \in T'M$

$$-i\{[v^+]^*, v^+\} = -\frac{1}{2}(\partial w^1/\partial x^1 + \partial w^2/\partial x^2). \tag{4.4.19}$$

Comparing formulae (4.4.9) and (4.4.19) and recalling (4.4.2), we arrive at the required result (4.4.1). □

4.5 Integration of the curvature term

Substituting (4.4.1) into (4.2.5) we get

$$b_2(x) = \frac{9\mathbf{c}}{4} \int\limits_{h^+(x,\xi)<1} \frac{\overset{*}{T}{}^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}{g^{\mu\nu}(x)\,\xi_\mu\xi_\nu}\, đ\xi\,. \tag{4.5.1}$$

Recall that $h^+(x,\xi)$ is given by formula (4.1.8).

The tensor $\overset{*}{T}$ can be decomposed into pure trace and trace-free pieces, i.e.

$$\overset{*}{T}{}^{\alpha\beta} = \frac{1}{3}g^{\alpha\beta}\overset{*}{T}{}^{\gamma}{}_{\gamma} + \left(\overset{*}{T}{}^{\alpha\beta} - \frac{1}{3}g^{\alpha\beta}\overset{*}{T}{}^{\gamma}{}_{\gamma}\right). \tag{4.5.2}$$

It is easy to see that the trace-free piece (second term in the RHS of (4.5.2)) does not contribute to the integral in (4.5.1), hence formula (4.5.1) becomes

$$b_2(x) = \frac{3\mathbf{c}}{4}\,\overset{*}{T}{}^{\gamma}{}_{\gamma}(x) \int\limits_{h^+(x,\xi)<1} đ\xi \;= \frac{\mathbf{c}}{8\pi^2}\big(\,\overset{*}{T}{}^{\gamma}{}_{\gamma}\,\sqrt{\det g_{\alpha\beta}}\,\big)(x)\,. \tag{4.5.3}$$

But formulae (4.1.14), (4.3.4) and (4.3.15) imply that

$$\overset{*}{T}{}^{\gamma}{}_{\gamma} = 3 * T^{\mathrm{ax}}. \tag{4.5.4}$$

Combining formulae (4.2.2), (4.2.6), (4.5.3) and (4.5.4) we arrive at formula (4.1.13). This completes the proof of Theorem 4.1.1.

4.6 The subprincipal symbol of the massless Dirac operator

In this section we calculate the subprincipal symbol of the massless Dirac operator, which prepares the ground for the proof of Theorem 4.1.2 in the next section. In view of Remark 2.1.10 from [72], defining the subprincipal symbol for the massless Dirac operator on spinors (4.A.3) is problematic, hence, we work with the massless Dirac operator on half-densities (4.A.19). For the sake of brevity we denote the massless Dirac operator on half-densities by A rather than by $W_{1/2}$.

Lemma 4.6.1. *The subprincipal symbol of the massless Dirac operator on half-densities (4.A.19) is*

$$A_{\mathrm{sub}}(x) = \frac{3\mathbf{c}}{4}\left(*T^{\mathrm{ax}}(x)\right) I\,, \tag{4.6.1}$$

where $\mathbf{c} = \pm 1$ *is the topological charge (4.3.5),* T^{ax} *is axial torsion (4.1.14),* $*$ *is the Hodge star (4.3.4) and* I *is the* 2×2 *identity matrix.*

Proof We give the proof of (4.6.1) for the case (4.4.2). There is no need to give a separate proof for the case $\mathbf{c} = -1$ as the two cases reduce to one another by inversion of the frame: the full symbol of the massless Dirac operator on half-densities changes sign under inversion of the frame and hence its subprincipal symbol changes sign under inversion of the frame, whereas torsion (4.3.13) is invariant under inversion of the frame.

According to formula (3.1.15) from Chapter 3 the subprincipal symbol is defined as

$$A_{\mathrm{sub}} := A_0 + \frac{i}{2}(A_1)_{x^\alpha \xi_\alpha}\,, \tag{4.6.2}$$

where $A_1(x,\xi)$ and $A_0(x)$ are the homogeneous (in ξ) components of the full symbol $A(x,\xi) = A_1(x,\xi) + A_0(x)$ of our first order differential operator, with

the subscript indicating degree of homogeneity. For the massless Dirac operator on half-densities (4.A.19) these homogeneous components read (4.3.1) and

$$A_0(x) = -\frac{i}{4}\sigma^\alpha\sigma_\beta\left(\frac{\partial\sigma^\beta}{\partial x^\alpha} + \left\{\begin{array}{c}\beta\\ \alpha\gamma\end{array}\right\}\sigma^\gamma\right) + \frac{i}{2}\sigma^\alpha\left\{\begin{array}{c}\beta\\ \alpha\beta\end{array}\right\} \tag{4.6.3}$$

respectively. Note that in writing down (4.6.3) we used the standard formula

$$\frac{1}{2\det g_{\kappa\lambda}}\frac{\partial\det g_{\mu\nu}}{\partial x^\alpha} = \left\{\begin{array}{c}\beta\\ \alpha\beta\end{array}\right\}.$$

Our task is to substitute (4.3.1) and (4.6.3) into (4.6.2).

We fix an arbitrary point $P \in M$ and prove formula (4.6.1) at this point. As the LHS and RHS of (4.6.1) are invariant under changes of local coordinates x, it is sufficient to check the identity (4.6.1) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the point P, $g_{\mu\nu}(0) = \delta_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the principal symbol is linear in ξ and the formula we are proving involves only first partial derivatives in x, we may assume, without loss of generality, that we have (4.4.4) for all x in some neighbourhood of the origin. In other words, it is sufficient to prove formula (4.6.1) for the case of Euclidean metric. Furthermore, by rotating our Cartesian coordinate system we can achieve (4.4.6), which opens the way to the use, in the linear approximation in x, of formula (4.4.7).

Substituting (4.4.7) into (4.A.1), we get, in the linear approximation in x,

$$\sigma^1 = \begin{pmatrix} w^2 & 1+iw^3 \\ 1-iw^3 & -w^2 \end{pmatrix} = \sigma_1\,,$$

$$\sigma^2 = \begin{pmatrix} -w^1 & -i+w^3 \\ i+w^3 & w^1 \end{pmatrix} = \sigma_2\,,$$

$$\sigma^3 = \begin{pmatrix} 1 & -iw^1-w^2 \\ iw^1-w^2 & -1 \end{pmatrix} = \sigma_3\,. \tag{4.6.4}$$

Recall that the w appearing in this formula is some smooth vector-function which vanishes at $x = 0$.

Substitution of (4.6.4) into (4.3.1) and (4.6.3) gives us

$$A_1(x,\xi) = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix} + \begin{pmatrix} w^2\xi_1 - w^1\xi_2 & iw^3\xi_1 + w^3\xi_2 + (-iw^1 - w^2)\xi_3 \\ -iw^3\xi_1 + w^3\xi_2 + (iw^1 - w^2)\xi_3 & -w^2\xi_1 + w^1\xi_2 \end{pmatrix}, \quad (4.6.5)$$

$$A_0(0) = -\frac{i}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial w^2/\partial x^1 & i\partial w^3/\partial x^1 \\ -i\partial w^3/\partial x^1 & -\partial w^2/\partial x^1 \end{pmatrix} + \dots . \quad (4.6.6)$$

Here formula (4.6.5) is written in the linear approximation in x, whereas formula (4.6.6) displays, for the sake of brevity, only one term out of nine (the one corresponding to $\alpha = \beta = 1$ in (4.6.3)), with the remaining eight terms concealed within the dots $\dots$. Note also that the Christoffel symbols disappeared because of our assumption that the metric is Euclidean.

Substituting (4.6.6) and (4.6.5) into (4.6.2), we get

$$A_{\text{sub}}(0) = -\frac{1}{2}(\operatorname{div} w)\, I. \quad (4.6.7)$$

But, according to (4.4.8),

$$\overset{*}{T}{}^{\gamma}{}_{\gamma}(0) = -2 \operatorname{div} w. \quad (4.6.8)$$

Formulae (4.6.7), (4.6.8), (4.5.4) and (4.4.2) imply formula (4.6.1) at $x = 0$. □

4.7 Proof of Theorem 4.1.2

As Theorem 4.1.2 is an if and only if theorem, our proof comes in two parts.

Part 1 of the proof Let A be a massless Dirac operator on half-densities. We need to prove that a) the subprincipal symbol of this operator, $A_{\rm sub}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero. The required result follows from Lemma 4.6.1 and Theorem 4.1.1.

Part 2 of the proof Let A be a differential operator such that a) the subprincipal symbol of this operator, $A_{\rm sub}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero. We need to prove that A is a massless Dirac operator on half-densities.

Theorem 4.1.1 implies that the subprincipal symbol of our operator A is given by formula (4.6.1). Let e_j be the frame corresponding to the principal symbol of the operator A, see formulae (4.3.1) and (4.3.2). Now, let B be the massless Dirac operator on half-densities corresponding to the same frame. Then the principal symbols of the operators A and B coincide. But Lemma 4.6.1 implies that the subprincipal symbols of the operators A and B coincide as well. A first order differential operator is determined by its principal and subprincipal symbols, hence, $A = B$. $\square$

4.8 Explicit formula for axial torsion

Torsion is a rank three tensor antisymmetric in the last two indices. It is known, see Subsection 2.4.2 from Chapter 2, that torsion has three irreducible pieces. Only one of the three irreducible pieces of torsion, namely, the piece which theoretical physicists label by the adjective "axial", appears in our spectral theoretic results, see Theorem 4.1.1 and Lemma 4.6.1. It is also interesting that axial torsion is the irreducible piece which is used when one models the massless neutrino [23] or the electron [16] by means of Cosserat elasticity.

Axial torsion is defined as the totally antisymmetric piece of the torsion tensor, see formula (4.1.14). This means that axial torsion is a 3-form. In view of the

importance of axial torsion, we give an explicit formula for its Hodge dual in terms of the principal symbol $A_1(x,\xi)$. Formulae (4.3.16), (4.3.17) and (4.5.4) imply

$$*\,T^{\mathrm{ax}} = \frac{\delta_{kl}}{3}\,\sqrt{\det g^{\alpha\beta}}\,\big[e^k{}_1\,\partial e^l{}_3/\partial x^2 + e^k{}_2\,\partial e^l{}_1/\partial x^3 + e^k{}_3\,\partial e^l{}_2/\partial x^1 - e^k{}_1\,\partial e^l{}_2/\partial x^3 - e^k{}_2\,\partial e^l{}_3/\partial x^1 - e^k{}_3\,\partial e^l{}_1/\partial x^2\big]. \quad (4.8.1)$$

Here the coframe e^k is determined from the principal symbol in accordance with formulae (4.3.1), (4.3.2) and (4.3.9), whereas the contravariant metric tensor $g^{\alpha\beta}$ is determined from the principal symbol in accordance with formula (4.1.7).

4.A The massless Dirac operator

Let M be a 3-dimensional connected compact oriented manifold equipped with a Riemannian metric $g_{\alpha\beta}$, $\alpha,\beta = 1,2,3$ being the tensor indices. Note that we are more prescriptive in this appendix than in the main text of Chapter 4: in the main text orientability emerged as a consequence of the existence of a principal symbol and the metric was defined via the principal symbol, whereas in this appendix orientability and metric are introduced *a priori*.

We work only in local coordinates with prescribed orientation.

It is known [76, 48] that a 3-dimensional oriented manifold is parallelizable, i.e. there exist smooth real vector fields e_j, $j = 1,2,3$, that are linearly independent at every point x of the manifold. (This fact is often referred to as *Steenrod's theorem*.) Each vector $e_j(x)$ has coordinate components $e_j{}^\alpha(x)$, $\alpha = 1,2,3$. Note that we use the Latin letter j for enumerating the vector fields (this is an *anholonomic* or *frame* index) and the Greek letter α for enumerating their components (this is a *holonomic* or *tensor* index). The triple of linearly independent vector fields e_j, $j = 1,2,3$, is called a *frame*. Without loss of generality we assume

further on that the vector fields e_j are orthonormal with respect to our metric: this can always be achieved by means of the Gram–Schmidt process.

Define Pauli matrices

$$\sigma^\alpha(x) := s^j \, e_j{}^\alpha(x)\,, \tag{4.A.1}$$

where

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1\,, \quad s^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = s_2\,, \quad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3\,. \tag{4.A.2}$$

In formula (4.A.1) summation is carried out over the repeated frame index $j = 1, 2, 3$, and $\alpha = 1, 2, 3$ is the free tensor index.

The massless Dirac operator is the matrix operator

$$W := -i\sigma^\alpha \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) \right), \tag{4.A.3}$$

where summation is carried out over $\alpha, \beta, \gamma = 1, 2, 3$, and

$$\left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} := \frac{1}{2} g^{\beta\delta} \left(\frac{\partial g_{\gamma\delta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\delta}}{\partial x^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x^\delta} \right) \tag{4.A.4}$$

are the Christoffel symbols. Here and throughout this appendix we raise and lower tensor indices using the metric. Note that we chose the letter "W" for denoting the massless Dirac operator because in theoretical physics literature it is often referred to as the *Weyl* operator.

Formula (4.A.3) is the formula from [23], only written in matrix notation (i.e. without spinor indices). Note that in the process of transcribing formulae from [23] into matrix notation we used the identity

$$\epsilon \sigma^\alpha \epsilon = (\sigma^\alpha)^T, \tag{4.A.5}$$

$\alpha = 1, 2, 3$, where

$$\epsilon := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.A.6}$$

is the "metric spinor". The identity (4.A.5) gives a simple way of raising/lowering spinor indices in Pauli matrices in the non-relativistic ($\alpha \neq 0$) setting.

Our definition (4.A.3) of the massless Dirac operator is a special case of the definition from [28]. The two definitions coincide when we work with a Spin connection as opposed to a Spin^c connection, see Propositions 2.14 and 2.15 in [28] for details.

Throughout Chapter 4 we work in dimension 3. The definition of the massless Dirac operator acting over a Riemannian manifold of arbitrary dimension can be found, for example, in [33, 30, 32].

Physically, our massless Dirac operator (4.A.3) describes a single massless neutrino living in a 3-dimensional compact universe M. The eigenvalues of the massless Dirac operator are the energy levels.

The massless Dirac operator (4.A.3) acts on 2-columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued scalar functions. In differential geometry this object is referred to as a (Weyl) spinor so as to emphasize the fact that v transforms in a particular way under transformations of the orthonormal frame e_j. However, as in our exposition the frame e_j is assumed to be chosen *a priori*, we can treat the components of the spinor as scalars. This issue will be revisited below when we state Property 4 of the massless Dirac operator.

We now list the main properties of the massless Dirac operator. We state these without proofs. The proofs can be found in Appendix 3.A of [20] or in [28].

Property 1. The massless Dirac operator is invariant under changes of local coordinates x, i.e. it maps 2-columns of smooth scalar functions $M \to \mathbb{C}^2$ to 2-columns of smooth scalar functions $M \to \mathbb{C}^2$ regardless of the choice of local coordinates.

Property 2. The massless Dirac operator is formally self-adjoint (symmetric) with respect to the inner product

$$\int_M w^* v \, \sqrt{\det g_{\alpha\beta}} \, dx \tag{4.A.7}$$

on 2-columns of smooth scalar functions $v, w : M \to \mathbb{C}^2$.

Property 3. The massless Dirac operator W commutes

$$\mathrm{C}(Wv) = W\mathrm{C}(v) \tag{4.A.8}$$

with the antilinear map

$$v \mapsto \mathrm{C}(v) := \epsilon \overline{v}, \tag{4.A.9}$$

where ϵ is the "metric spinor" (4.A.6). In theoretical physics the transformation (4.A.9) is referred to as *charge conjugation* [14, 28].

Formula (4.A.8) implies that v is an eigenfunction of the massless Dirac operator corresponding to an eigenvalue λ if and only if $\mathrm{C}(v)$ is an eigenfunction of the massless Dirac operator corresponding to the same eigenvalue λ. Hence, all eigenvalues of the massless Dirac operator have even multiplicity. Moreover, any eigenfunction v and its "partner" $\mathrm{C}(v)$ make the same contribution to the spectral function (3.2.4) at every point x of the manifold M.

If, as in [28], we introduce a magnetic field, then we lose the commutation property (4.A.8) and the double eigenvalues split up. This indicates that the double eigenvalues of the massless Dirac operator correspond to the two different spins.

Property 4. This property has to do with a particular behaviour under SU(2) transformations. Let $R : M \to \mathrm{SU}(2)$ be an arbitrary smooth special unitary matrix-function. Let us introduce new Pauli matrices

$$\tilde{\sigma}^\alpha := R\sigma^\alpha R^* \tag{4.A.10}$$

and a new operator $\tilde{W}$ obtained by replacing the σ in (4.A.3) by $\tilde{\sigma}$. It turns out (and this is Property 4) that the two operators, $\tilde{W}$ and W, are related in exactly the same way as the Pauli matrices, $\tilde{\sigma}$ and σ, that is,

$$\tilde{W} = RWR^*. \tag{4.A.11}$$

We now examine the geometric meaning of the transformation (4.A.10). Let us expand the new Pauli matrices $\tilde{\sigma}$ with respect to the basis (4.A.2):

$$\tilde{\sigma}^\alpha(x) = s^j \, \tilde{e}_j{}^\alpha(x). \tag{4.A.12}$$

Formulae (4.A.1), (4.A.12) and (4.A.10) give us the following identity relating the new vector fields $\tilde{e}_j$ and the old vector fields e_j:

$$Rs^kR^*e_k = s^j\tilde{e}_j \, . \tag{4.A.13}$$

Resolving (4.A.13) for $\tilde{e}_j$ we get

$$\tilde{e}_j = O_j{}^k e_k \, , \tag{4.A.14}$$

where the real scalars O_{jk} are given by the formula

$$O_j{}^k = \frac{1}{2} \operatorname{tr}(s_j R s^k R^*) \, . \tag{4.A.15}$$

Note that in writing formulae (4.A.13) and (4.A.14) we chose to hide the tensor index, i.e. we chose to hide the coordinate components of our vector fields. Say, formula (4.A.14) written in more detailed form reads $\tilde{e}_j{}^\alpha = O_j{}^k e_k{}^\alpha$.

The scalars (4.A.15) can be viewed as elements of a real 3×3 matrix-function O with the first index, j, enumerating rows and the second, k, enumerating columns. It is easy to check that this matrix-function O is special orthogonal. Hence, the new vector fields $\tilde{e}_j$ are orthonormal and have the same orientation as the old vector fields e_j. We have shown that the transformation (4.A.10) has the

geometric meaning of switching from our original oriented orthonormal frame e_j to a new oriented orthonormal frame $\tilde{e}_j$.

Formula (4.A.15) means that the special unitary matrix R is, effectively, a square root of the special orthogonal matrix O. It is easy to see that for a given matrix $O \in \mathrm{SO}(3)$ formula (4.A.15) defines the matrix $R \in \mathrm{SU}(2)$ uniquely up to sign. This observation allows us to view the issue of the geometric meaning of the transformation (4.A.10) the other way round: given a pair of orthonormal frames, e_j and $\tilde{e}_j$, with the same orientation (i.e. with $\operatorname{sgn}\det e_j{}^\alpha = \operatorname{sgn}\det \tilde{e}_j{}^\alpha$), we can recover the special orthogonal matrix-function $O(x)$ from formula (4.A.14) and then attempt finding a smooth special unitary matrix-function $R(x)$ satisfying (4.A.15). Unfortunately, this may not always be possible due to topological obstructions. We can only guarantee the absence of topological obstructions when the two frames, e_j and $\tilde{e}_j$, are sufficiently close to each other, which is equivalent to saying that we can only guarantee the absence of topological obstructions when the special orthogonal matrix-function $O(x)$ is sufficiently close to the identity matrix for all $x \in M$.

We illustrate the possibility of a topological obstruction by means of an explicit example. Consider the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π. The metric is assumed to be Euclidean. Define a pair of orthonormal frames

$$e_j{}^\alpha := \delta_j{}^\alpha \tag{4.A.16}$$

and

$$\tilde{e}_1{}^\alpha := \begin{pmatrix} \cos k_3 x^3 \\ \sin k_3 x^3 \\ 0 \end{pmatrix}, \qquad \tilde{e}_2{}^\alpha := \begin{pmatrix} -\sin k_3 x^3 \\ \cos k_3 x^3 \\ 0 \end{pmatrix}, \qquad \tilde{e}_3{}^\alpha := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{4.A.17}$$

where k_3 is an odd integer. Let W and $\tilde{W}$ be the massless Dirac operators corresponding to the frames (4.A.16) and (4.A.17) respectively. We claim that there does not exist a smooth matrix-function $R : \mathbb{T}^3 \to \mathrm{SU}(2)$ which would give

(4.A.15), where $O(x)$ is the special orthogonal matrix-function defined by formula (4.A.14). We justify this claim in two different ways.

Justification 1. Resolving the system (4.A.14)–(4.A.17) locally for R, we get

$$R(x^3) = \pm \begin{pmatrix} e^{\frac{i}{2}k_3 x^3} & 0 \\ 0 & e^{-\frac{i}{2}k_3 x^3} \end{pmatrix}, \tag{4.A.18}$$

and this solution is unique modulo choice of sign; here the freedom in the choice of sign is not surprising as SU(2) is the double cover of SO(3). Formula (4.A.18) defines a continuous single-valued matrix-function on the unit torus $\mathbb{T}^3$ if and only if the integer k_3 is even, which it is not.

Justification 2. It is sufficient to show that the two operators, W and $\tilde{W}$, have different spectra. Straightforward separation of variables shows that zero is an eigenvalue of the operator W but not an eigenvalue of the operator $\tilde{W}$.

One can generalize the above example by introducing rotations in three different directions, which leads to eight genuinely distinct parallelizations. See also [74] page 524 or [10] page 21.

Let us emphasize that the topological obstructions we were discussing have nothing to do with Stiefel–Whitney classes. We are working on a parallelizable manifold and the Stiefel–Whitney class of such a manifold is trivial. The topological issue at hand is that our parallelizable manifold may be equipped with different spin structures.

We say that two massless Dirac operators, W and $\tilde{W}$, are equivalent if there exists a smooth matrix-function $R : M \to \mathrm{SU}(2)$ such that the corresponding Pauli matrices, σ^α and $\tilde{\sigma}^\alpha$, are related in accordance with (4.A.10). In view of Property 4 (see formula (4.A.11)) all massless Dirac operators from the same equivalence class generate the same spectral function (3.2.4) and the same counting function (3.2.5), so for the purposes of Chapter 4 viewing such operators as equivalent is most natural.

As explained above, there may be many distinct equivalence classes of massless Dirac operators, the difference between which is topological. Studying the spectral theoretic implications of these topological differences is beyond the scope of Chapter 4. The two-term asymptotics (4.1.16) and (4.1.17) derived in the main text of Chapter 4 do not feel this topology.

In theoretical physics the SU(2) freedom involved in defining the massless Dirac operator is interpreted as a gauge degree of freedom. We do not adopt this point of view (at least explicitly) in order to fit the massless Dirac operator into the standard spectral theoretic framework.

We defined the massless Dirac operator (4.A.3) as an operator acting on 2-columns of scalar functions, i.e. on 2-columns of quantities which do not change under changes of local coordinates. This necessitated the introduction of the density $\sqrt{\det g_{\alpha\beta}}$ in the formula (4.A.7) for the inner product. In spectral theory it is more common to work with half-densities. Hence, we introduce the operator

$$W_{1/2} := (\det g_{\kappa\lambda})^{1/4}\, W\, (\det g_{\mu\nu})^{-1/4} \tag{4.A.19}$$

which maps half-densities to half-densities. We call the operator (4.A.19) *the massless Dirac operator on half-densities*.

4.B The spectrum for the torus and the sphere

In this appendix we examine the massless Dirac operator on the unit torus $\mathbb{T}^3$ and the unit sphere $\mathbb{S}^3$ and compare our asymptotic formulae (4.1.16) and (4.1.17) with known explicit formulae. The torus is assumed to be equipped with Euclidean metric (see also Appendix 4.A) whereas the sphere is assumed to be equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in Euclidean space $\mathbb{R}^4$. Note that in view of the obvious symmetries of the torus and the sphere the scalar function $e(\lambda, x, x)/\sqrt{\det g_{\alpha\beta}(x)}$ is constant (see also Remark 4.1.2), so

formulae (4.1.16) and (4.1.17) are in this case equivalent, in the sense that they follow from one another. Hence, we will be dealing with formula (4.1.17) only.

We have $\operatorname{Vol}\mathbb{T}^3 = (2\pi)^3$, so for the torus formula (4.1.17) reads

$$N(\lambda) = \frac{4}{3}\pi\lambda^3 + o(\lambda^2). \tag{4.B.1}$$

The nonperiodicity condition (see Definitions 8.3 and 8.4 from [22]) is fulfilled for the torus, so, according to Theorem 8.4 from [22], the asymptotic formula (4.B.1) holds as it is, without mollification.

In order to test formula (4.B.1) we calculate the spectrum of the massless Dirac operator on $\mathbb{T}^3$ explicitly. We do this first for the spin structure associated with the frame (4.A.16). Then the spectrum is as follows.

- Zero is an eigenvalue of multiplicity two.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $\|m\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $-\|m\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

Hence, $N(\lambda)+1$ is the number of integer lattice points inside a 2-sphere of radius λ in $\mathbb{R}^3$ centred at the origin. According to [38] the latter admits the asymptotic expansion

$$\frac{4}{3}\pi\lambda^3 + O_\varepsilon(\lambda^{21/16+\varepsilon}) \tag{4.B.2}$$

as $\lambda \to +\infty$, with ε being an arbitrary positive number. This agrees with our asymptotic formula (4.B.1).

Let us now consider the spin structure associated with the frame (4.A.17). Then the spectrum is as follows.

- For each $m \in \mathbb{Z}^3$ we have the eigenvalue $\|m - (0,0,1/2)\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

- For each $m \in \mathbb{Z}^3$ we have the eigenvalue $-\|m-(0,0,1/2)\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

Hence, $N(\lambda)$ is the number of integer lattice points inside a 2-sphere of radius λ in $\mathbb{R}^3$ centred at $(0,0,1/2)$. Here the sphere is shifted from the origin so one cannot apply the result from [38]. However, as the shift is rational, one can reduce the problem to counting integer lattice points in a rational ellipsoid centred at the origin, and an application of the result from [19] gives us for the shifted sphere the same asymptotic expansion (4.B.2) as for the sphere centred at the origin.

As explained in Appendix 4.A, the unit torus $\mathbb{T}^3$ admits a total of eight different spin structures. For each of these the problem of counting positive eigenvalues of the massless Dirac operator reduces to counting integer lattice points inside a 2-sphere of radius λ in $\mathbb{R}^3$ (possibly, shifted from the origin by a rational shift), so in all eight cases we do get (4.B.1). In fact, we can replace the remainder $o(\lambda^2)$ in (4.B.1) by $O_\varepsilon(\lambda^{21/16+\varepsilon})$ and this holds for all eight different spin structures.

In the remainder of this appendix we examine the massless Dirac operator on the unit sphere $\mathbb{S}^3$. We have $\operatorname{Vol}\mathbb{S}^3 = 2\pi^2$, so for the sphere formula (4.1.17) reads

$$N(\lambda) = \frac{\lambda^3}{3} + o(\lambda^2). \tag{4.B.3}$$

The nonperiodicity condition fails for the sphere because all geodesics are closed with period 2π, so formula (4.B.3) cannot be used in its original form and has to be mollified, see Remark 4.1.1. We will deal with the mollification issue later and give explicit formulae for the eigenvalues first.

It is known that $\mathbb{S}^3$ admits a unique spin structure, see Section 5 in [10]. The spectrum of the massless Dirac operator on $\mathbb{S}^3$ has been computed by different

authors using different methods [77, 78, 8, 10] and reads as follows: the eigenvalues are

$$\pm\left(k+\frac{1}{2}\right), \qquad k=1,2,\dots, \tag{4.B.4}$$

and their multiplicity is

$$k(k+1). \tag{4.B.5}$$

The mollification procedure from Section 7 of [22] goes as follows. Put $N(\lambda) := 0$ for $\lambda \leq 0$ and take an arbitrary real-valued even function $\rho(\lambda)$ from Schwartz space $\mathcal{S}(\mathbb{R})$ whose Fourier transform $\hat{\rho}(t)$ satisfies conditions $\hat{\rho}(0) = 1$ and $\operatorname{supp}\hat{\rho} \subset (-2\pi, 2\pi)$. Then, according to Theorem 7.2 from [22], the mollified version of formula (4.B.3) reads

$$\int N(\lambda-\mu)\,\rho(\mu)\,d\mu = \frac{\lambda^3}{3} + O(\lambda)$$

and this result holds notwithstanding the failure of the nonperiodicity condition. However, for the sphere there is a much simpler way of testing our asymptotic formula. Let $\lambda \geq 2$ be integer. Taking an integer λ puts us exactly in the middle of the gap between two consecutive clusters of eigenvalues, see formulae (4.B.4) and (4.B.5), and achieves the same averaging effect as convolution with a function from Schwartz space. For integer $\lambda \geq 2$ we get

$$N(\lambda) = \sum_{k=1}^{\lambda-1} k(k+1) = \frac{\lambda^3}{3} - \frac{\lambda}{3}$$

which agrees with our asymptotic formula (4.B.3).

Chapter 5

Spectral theoretic characterization of the massless Dirac action

5.1 Main result

Consider a first order differential operator A acting on 2-columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued half-densities over a connected compact 3-dimensional manifold M without boundary. We assume the coefficients of the operator A to be infinitely smooth. We also assume that the operator A is formally self-adjoint, see (4.1.1).

Let $A_1(x, \xi)$ be the principal symbol of the operator A. The principal symbol $A_1(x, \xi)$ is a 2×2 Hermitian matrix-function on the cotangent bundle T^*M, linear in every fibre $T^*_x M$. We assume that the operator A is elliptic in the sense described by (4.1.2).

We now make two additional assumptions:

Assumption 1: We assume the principal symbol to be trace-free.

Assumption 2: We assume the subprincipal symbol of the operator A to be zero (see Section 2.1.3 in [72] for background material pertaining to the subprincipal symbol).

The latter condition implies that our first order differential operator A is completely determined by its principal symbol. Namely, in local coordinates the full symbol $A(x,\xi)$ is expressed via the principal symbol $A_1(x,\xi)$ as

$$A(x,\xi) = A_1(x,\xi) - \frac{i}{2}(A_1)_{x^\alpha \xi_\alpha}(x) \tag{5.1.1}$$

where the subscripts x^α and ξ_α indicate partial derivatives and the repeated index α indicates summation over $\alpha = 1,2,3$.

We study the eigenvalue problem

$$Av = \lambda w v \tag{5.1.2}$$

where $w(x)$ is a given infinitely smooth positive scalar weight function. Obviously, the problem (5.1.2) has the same spectrum as the problem

$$w^{-1/2} A w^{-1/2} v = \lambda v \tag{5.1.3}$$

so it may appear that the weight function $w(x)$ is redundant. We will, however, work with the eigenvalue problem (5.1.2) rather than with (5.1.3) because we want our problem to possess a gauge degree of freedom (5.5.1). This gauge degree of freedom will eventually manifest itself as the conformal invariance of the massless Dirac action, see Section 5.5 for details.

The problem (5.1.2) has a discrete spectrum accumulating to $\pm\infty$. Recall the definition of the counting function from Chapter 3

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1, \tag{5.1.4}$$

i.e. the number of eigenvalues λ_k of the problem (5.1.2), with account of multiplicities, between zero and a positive λ.

Theorem 8.4 from [22] states that under appropriate assumptions on periodic trajectories our counting function admits a two-term asymptotic expansion

$$N(\lambda) = a\lambda^3 + b\lambda^2 + o(\lambda^2) \tag{5.1.5}$$

as $\lambda \to +\infty$. If one wishes to reformulate the asymptotic formula (5.1.5) in such a way that it remains valid without assumptions on periodic trajectories, this can easily be achieved, say, by taking a convolution with a function from Schwartz space $\mathcal{S}(\mathbb{R})$; see Theorem 7.2 in [22] and Chapter 3 for further details.

The objective of Chapter 5 is to establish the geometric meaning of the coefficient b. The logic behind restricting our analysis to the case when the manifold is 3-dimensional and A is a 2×2 matrix differential operator with trace-free principal symbol and zero subprincipal symbol is that this is the simplest[1] eigenvalue problem for a system of partial differential equations. Hence, it is ideal for the purpose of establishing the geometric meaning of the coefficient b. In addition, we are able to build our analysis on that presented in Chapter 3 and Chapter 4: the problem considered here is a special case of that considered in Chapter 4, the extra assumption being the vanishing subprincipal symbol.

In order to establish the geometric meaning of the coefficient b we first need to identify the geometric objects encoded within our eigenvalue problem (5.1.2).

Geometric object 1: the *metric.* Observe that the determinant of the principal symbol is a negative definite quadratic form in the dual variable ξ,

$$\det A_1(x,\xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta\,, \tag{5.1.6}$$

[1]The only way the eigenvalue problem (5.1.2) can be made simpler is by lowering the dimension of the manifold to two. Unfortunately, this raises geometric issues which we cannot currently resolve.

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, 2, 3$, appearing in (5.1.6) can be interpreted as the components of a (contravariant) Riemannian metric.

Geometric object 2: the *nonvanishing spinor field.* The determinant of the principal symbol does not determine the principal symbol uniquely. In order to identify a further geometric object encoded within the principal symbol $A_1(x, \xi)$ we will now start varying this principal symbol, assuming the metric g, defined by formula (5.1.6), to be fixed (prescribed).

Let us fix a reference principal symbol $B_1(x, \xi)$ corresponding to the prescribed metric g and look at all principal symbols $A_1(x, \xi)$ which correspond to the same prescribed metric g and are sufficiently close to the reference principal symbol. Restricting our analysis to principal symbols which are close to the reference principal symbol allows us to avoid dealing with certain topological issues; this restriction will be dropped in Section 5.4. It turns out, see Section 5.2, that the principal symbols $A_1(x, \xi)$ and $B_1(x, \xi)$ are related as

$$A_1(x, \xi) = R(x)\, B_1(x, \xi)\, R^*(x)\,, \tag{5.1.7}$$

where

$$R : M \to \mathrm{SU}(2) \tag{5.1.8}$$

is a unique infinitely smooth special unitary matrix-function which is close to the identity matrix. Thus, special unitary matrix-functions $R(x)$ provide a convenient parametrization of principal symbols with prescribed metric g.

Let B be the differential operator with principal symbol $B_1(x, \xi)$ and zero subprincipal symbol. It is important to emphasize that for the operators A and B themselves, as opposed to their principal symbols, we have, in general, the inequality $A \neq RBR^*$ because the substitution of (5.1.7) into (5.1.1) generates terms with derivatives of the matrix-function $R(x)$. Hence, the transformation of operators $B \mapsto A$ specified by formula (5.1.7) does, in general, change the spectrum.

The choice of reference principal symbol $B_1(x,\xi)$ in our construction is arbitrary, as long as this principal symbol corresponds to the prescribed metric g, i.e. as long as we have $\det B_1(x,\xi) = -g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta$ for all $(x,\xi) \in T^*M$. It is natural to ask the question: what happens if we choose a different reference principal symbol $B_1(x,\xi)$? The freedom in choosing the reference principal symbol $B_1(x,\xi)$ is a gauge degree of freedom in our construction and our results are invariant under changes of the reference principal symbol. This issue is addressed in Section 5.6.

In order to work effectively with special unitary matrices we need to choose coordinates on the 3-dimensional Lie group SU(2). It is convenient to describe a 2×2 special unitary matrix by means of a (Weyl) spinor ζ, i.e. a pair of complex numbers ζ^a, $a = 1,2$. The relationship between a matrix $R \in \mathrm{SU}(2)$ and a nonzero spinor ζ is given by the formula

$$R = \frac{1}{\|\zeta\|}\begin{pmatrix} \overline{\zeta^1} & \overline{\zeta^2} \\ -\zeta^2 & \zeta^1 \end{pmatrix}, \tag{5.1.9}$$

where the overline stands for complex conjugation and $\|\zeta\| := \sqrt{|\zeta^1|^2 + |\zeta^2|^2}$. Here $\overline{\zeta^a}$ is defined as follows: $\overline{\zeta^a} := \overline{\zeta}^{\dot{a}}$. Unlike in Chapter 2, we are able to avoid introducing dotted spinor indices using instead matrix notation throughout Chapter 5. This simplifies our calculations.

Formula (5.1.9) establishes a one-to-one correspondence between SU(2) matrices and nonzero spinors, modulo a rescaling of the spinor by an arbitrary positive real factor. We choose to specify the scaling of our spinor field $\zeta(x)$ in accordance with

$$\|\zeta(x)\| = w(x). \tag{5.1.10}$$

Remark 5.1.1. *In Chapter 4 we chose to work with a Teleparallel connection (metric compatible affine connection with zero curvature) rather than with a spinor field. These are closely related objects: locally a Teleparallel connection is equivalent to a normalized ($\|\zeta(x)\| = 1$) spinor field modulo rigid rotations (5.7.4) of the latter.*

Geometric object 3: the *topological charge.* It is known, see Section 4.3, that the existence of a principal symbol implies that our manifold M is parallelizable. Parallelizability, in turn, implies orientability. Having chosen a particular orientation, we allow only changes of local coordinates x^α, $\alpha = 1, 2, 3$, which preserve orientation.

We define the topological charge as

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \mathrm{tr}\big((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\big), \tag{5.1.11}$$

with the subscripts ξ_α indicating partial derivatives. As explained in Section 4.3, the number $\mathbf{c}$ defined by formula (5.1.11) can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to the chosen orientation of local coordinates.

We have identified three geometric objects encoded within the eigenvalue problem (5.1.2) — metric, nonvanishing spinor field and topological charge — defined in accordance with formulae (5.1.6)–(5.1.11). Consequently, one would expect the coefficients a and b from formula (5.1.5) to be expressed via these three geometric objects. This assertion is confirmed by the following theorem which is the main result of Chapter 5.

Theorem 5.1.1. *The coefficients in the two-term asymptotics (5.1.5) are given by the formulae*

$$a = \frac{1}{6\pi^2}\int_M \|\zeta\|^3\sqrt{\det g_{\alpha\beta}}\ dx\,, \tag{5.1.12}$$

$$b = \frac{S(\zeta)}{2\pi^2}\,, \tag{5.1.13}$$

where $S(\zeta)$ is the massless Dirac action (5.A.1) with Pauli matrices

$$\sigma^\alpha := (B_1)_{\xi_\alpha}, \qquad \alpha = 1, 2, 3. \tag{5.1.14}$$

Theorem 5.1.1 warrants the following remarks.

Firstly, recall that the B appearing in Theorem 5.1.1 is our reference operator which we need to describe all possible operators A with given metric g. It is natural to ask the question: what happens if we take $A = B$? In this case formula (5.1.13) holds with spinor field $\zeta^1(x) = w(x)$, $\zeta^2(x) = 0$. This in itself is a nontrivial result.

Secondly, the topological charge $\mathbf{c}$ does not appear explicitly in Theorem 5.1.1. Nevertheless, it is implicitly present in our Pauli matrices (5.1.14). Indeed, formula (5.1.7) implies that the integer quantity

$$-\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \mathrm{tr}\big((B_1)_{\xi_1}(B_1)_{\xi_2}(B_1)_{\xi_3}\big) \tag{5.1.15}$$

has the same value as (5.1.11).

Thirdly, it is tempting to apply Theorem 5.1.1 in the case when the operator A is itself a massless Dirac operator. This cannot be done because a massless Dirac operator acts on spinors, the components of which are invariant under changes of local coordinates, rather than on pairs of half-densities. This impediment can be overcome by switching to a massless Dirac operator on half-densities, see formula (4.A.3) in Appendix 4.A. However, we cannot take A to be a massless Dirac operator on half-densities either because, according to Lemma 4.6.1 from Chapter 4, the latter has a nontrivial subprincipal symbol.

Finally, Theorem 5.1.1 provides a fresh perspective on the history of the subject of two-term spectral asymptotics for first order systems, see Section 3.6 of Chapter 3 for details. Namely, Theorem 5.1.1 shows that even in the simplest case the second asymptotic coefficient for a first order system has a highly nontrivial geometric meaning. At a formal level, the application of microlocal techniques does not require the use of advanced differential geometric concepts. However, the calculations involved are so complicated that it is hard to avoid mistakes without an understanding of the differential geometric content of the spectral problem.

Chapter 5 is organized as follows. In Section 5.2 we explain the origins of formula (5.1.7) and in Section 5.3 we give the proof of Theorem 5.1.1. In Section 5.4 we introduce the concept of spin structure which allows us to drop the restriction that our principal symbol $A_1(x,\xi)$ is sufficiently close to the reference principal symbol $B_1(x,\xi)$. And in Sections 5.5–5.7 we show that our formula (5.1.13) is invariant under the action of certain gauge transformations.

5.2 Spinor representation of the principal symbol

Let $A_1(x,\xi)$ and $B_1(x,\xi)$ be a pair of trace-free Hermitian 2×2 principal symbols and let g be a prescribed Riemannian metric. Both $A_1(x,\xi)$ and $B_1(x,\xi)$ are assumed to be linear in ξ:

$$A_1(x,\xi) = A_1^{(\alpha)}(x)\,\xi_\alpha\,, \tag{5.2.1}$$

$$B_1(x,\xi) = B_1^{(\alpha)}(x)\,\xi_\alpha\,, \tag{5.2.2}$$

where $A_1^{(\alpha)}(x)$ and $B_1^{(\alpha)}(x)$, $\alpha = 1,2,3$, are some trace-free Hermitian 2×2 matrix-functions. The assumption that our principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ are linear in ξ means, of course, that we are dealing with differential operators as opposed to pseudodifferential operators. The principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ are assumed to satisfy

$$\det A_1(x,\xi) = \det B_1(x,\xi) = -g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta \tag{5.2.3}$$

for all $(x,\xi)\in T^*M$, and are also assumed to be sufficiently close in terms of the $C^\infty(M)$ topology applied to the matrix-functions $A_1^{(\alpha)}(x)$ and $B_1^{(\alpha)}(x)$, $\alpha = 1,2,3$.

Our task in this section is to show that there exists a unique infinitely smooth special unitary matrix-function (5.1.8) which is close to the identity matrix and

which relates the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ in accordance with formula (5.1.7).

We follow the convention of [23] and Chapter 4 in denoting the elements of the matrices $A_1^{(\alpha)}$ and $B_1^{(\alpha)}$ as $\big(A_1^{(\alpha)}\big)_{\dot a b}$ and $\big(B_1^{(\alpha)}\big)_{\dot a b}$ respectively, where the dotted index, running through the values $\dot 1, \dot 2$, enumerates the rows and the undotted index, running through the values $1, 2$, enumerates the columns.

Put

$$e_1{}^{\alpha} := \operatorname{Re}\big(A_1^{(\alpha)}\big)_{\dot 1 2}\,, \quad e_2{}^{\alpha} := -\operatorname{Im}\big(A_1^{(\alpha)}\big)_{\dot 1 2}\,, \quad e_3{}^{\alpha} := \operatorname{Re}\big(A_1^{(\alpha)}\big)_{\dot 1 1}\,, \tag{5.2.4}$$

$$f_1{}^{\alpha} := \operatorname{Re}\big(B_1^{(\alpha)}\big)_{\dot 1 2}\,, \quad f_2{}^{\alpha} := -\operatorname{Im}\big(B_1^{(\alpha)}\big)_{\dot 1 2}\,, \quad f_3{}^{\alpha} := \operatorname{Re}\big(B_1^{(\alpha)}\big)_{\dot 1 1}\,. \tag{5.2.5}$$

As explained in Section 4.3, formula (5.2.4) defines a *frame* — a triple of infinitely smooth real orthonormal vector fields $e_j(x)$, $j = 1, 2, 3$, on the manifold M — and, moreover, the principal symbol $A_1(x,\xi)$ is equivalent to the frame e_j in the sense that the principal symbol uniquely determines the frame and the frame uniquely determines the principal symbol. Similarly, formula (5.2.5) defines a frame f_j which is equivalent to the principal symbol $B_1(x,\xi)$.

Condition (5.2.3) implies that the frames e_j and f_j are orthonormal with respect to the same metric. Hence, this pair of orthonormal frames is related as

$$e_j{}^{\alpha} = O_j{}^{k} f_k{}^{\alpha}, \tag{5.2.6}$$

where $O(x)$ is a 3×3 orthogonal matrix-function with elements

$$O_j{}^{k} = \delta^{kl} g_{\alpha\beta}\, e_j{}^{\alpha} f_l{}^{\beta}.$$

As we assumed the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ to be close, the frames e_j and f_j are also close. Consequently, the matrix-function $O(x)$ is close to the identity matrix and, hence, special unitary.

It is well-known that the Lie group SO(3) is locally (in a neighbourhood of the identity) isomorphic to the Lie group SU(2). According to formulae (4.A.15) and (4.A.2) from Appendix 4.A, a 3×3 special orthogonal matrix O is expressed via a 2×2 special unitary matrix R as

$$O_j{}^k = \frac{1}{2} \operatorname{tr}(s_j R s^k R^*) \,, \tag{5.2.7}$$

where

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1 \,, \quad s^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = s_2 \,, \quad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3 \,. \tag{5.2.8}$$

Formula (5.2.7) tells us that a 3×3 special orthogonal matrix is, effectively, the square of a 2×2 special unitary matrix. Formula (5.2.7) provides a local diffeomorphism between neighbourhoods of the identity in SO(3) and in SU(2).

A straightforward calculation shows that formulae (5.2.1), (5.2.2) and (5.2.4)–(5.2.8) imply formula (5.1.7).

Let us now define Pauli matrices $\sigma^\alpha(x)$ in accordance with formula (5.1.14). Of course, we have

$$\sigma^\alpha := B_1^{(\alpha)}(x), \qquad \alpha = 1, 2, 3, \tag{5.2.9}$$

where the $B_1^{(\alpha)}$ are the matrix-functions from formula (5.2.2). We could stick with the notation $B_1^{(\alpha)}$ but we choose to switch to σ^α because this is how Pauli matrices are traditionally denoted in the subject.

It is easy to see that formula (5.2.3) implies

$$\sigma^\alpha \sigma^\beta + \sigma^\beta \sigma^\alpha = 2 I g^{\alpha\beta}, \tag{5.2.10}$$

where I is the 2×2 identity matrix. Formula (5.2.10) means that our σ^α satisfy the defining relation for Pauli matrices.

Formulae (5.2.6)–(5.2.8), (5.1.9), (5.2.5) and (5.2.9) allow us to express the frame e_j via the spinor field ζ and Pauli matrices σ^α. We took great care to choose coordinates on the Lie group SU(2) (i.e. structure of the matrix in the RHS of formula (5.1.9)) so that the resulting expressions agree with formulae (B.3), (B.4) and (B.1) from [23]. The only difference is in notation: the ϑ^j in Appendix B of [23] stands for $\vartheta^j{}_\alpha = \delta^{jk} g_{\alpha\beta}\, e_k{}^\beta$ (compare with formula (5.3.4)) and the spinor in Appendix B of [23] is denoted by ξ rather than by ζ. We do not denote the spinor by ξ in the current chapter because in microlocal analysis the letter ξ is traditionally reserved for denoting momentum.

The fact that our construction agrees with that in [23] will become important in the next section when we will make use of a particular formula from [23].

Remark 5.2.1. *As explained in Section 4.3, the topological charge, initially defined in accordance with formula (5.1.11), can be equivalently rewritten in terms of frames as*

$$\mathbf{c} = \operatorname{sgn} \det e_j{}^\alpha = \operatorname{sgn} \det f_j{}^\alpha. \tag{5.2.11}$$

The paper [23] was written under the assumption that

$$\mathbf{c} = +1\,, \tag{5.2.12}$$

see formula (A.1) in [23]. This means that care is required when using the results of [23]. Namely, in the next section we will first prove Theorem 5.1.1 for the case (5.2.12) and then provide a separate argument explaining why formula (5.1.13) remains true in the case

$$\mathbf{c} = -1\,. \tag{5.2.13}$$

5.3 Proof of Theorem 5.1.1

We prove Theorem 5.1.1 by examining the equivalent spectral problem (5.1.3). Note that transition from (5.1.2) to (5.1.3) is a special case of the gauge transformation (5.5.1). As explained in the beginning of Section 5.5, the gauge transformation (5.5.1) preserves the structure of our eigenvalue problem: the principal symbol of the operator $w^{-1/2}Aw^{-1/2}$ is trace-free and its subprincipal symbol is zero.

We now apply Theorem 4.1.1 from Chapter 4 to the eigenvalue problem (5.1.3).

Our formula (5.1.12) is an immediate consequence of formula (4.1.12) from Chapter 4 and our formulae (5.1.6) and (5.1.10). Here, of course, we use the fact that we are working in dimension three.

The proof of formula (5.1.13) is more delicate so we initially consider the case

$$w(x) = 1, \qquad \forall x \in M. \tag{5.3.1}$$

In this case, according to formulae (4.1.13) and (4.8.1) from Chapter 4, we have

$$b = \frac{3\mathbf{c}}{8\pi^2} \int_M {*T^{\mathrm{ax}}} \sqrt{\det g_{\alpha\beta}}\, dx\,, \tag{5.3.2}$$

where

$$\begin{aligned} *\,T^{\mathrm{ax}} = \frac{\delta_{kl}}{3} \sqrt{\det g^{\alpha\beta}}\, \big[& e^k{}_1\, \partial e^l{}_3/\partial x^2 + e^k{}_2\, \partial e^l{}_1/\partial x^3 + e^k{}_3\, \partial e^l{}_2/\partial x^1 \\ & - e^k{}_1\, \partial e^l{}_2/\partial x^3 - e^k{}_2\, \partial e^l{}_3/\partial x^1 - e^k{}_3\, \partial e^l{}_1/\partial x^2 \big], \end{aligned} \tag{5.3.3}$$

$$e^j{}_\alpha = \delta^{jk} g_{\alpha\beta}\, e_k{}^\beta. \tag{5.3.4}$$

The real scalar field ${*T^{\mathrm{ax}}}(x)$ has the geometric meaning of the Hodge dual of axial torsion of the Teleparallel connection, see Chapter 4 for details.

Let us now drop the assumption (5.3.1).

The introduction of a weight function is equivalent to a scaling of the principal symbol $A_1(x,\xi) \mapsto (w(x))^{-1} A_1(x,\xi)$, which, in view of formulae (5.2.4) and (5.1.6), leads to a scaling of the frame

$$e_j \mapsto w^{-1} e_j \tag{5.3.5}$$

and corresponding scaling of the metric

$$g^{\alpha\beta} \mapsto w^{-2} g^{\alpha\beta}. \tag{5.3.6}$$

Substituting (5.3.5) and (5.3.6) into (5.3.4) and (5.3.3) we see that the integrand in formula (5.3.2) scales as

$$* T^{\text{ax}} \sqrt{\det g_{\alpha\beta}} \;\mapsto\; w^2 * T^{\text{ax}} \sqrt{\det g_{\alpha\beta}}\;. \tag{5.3.7}$$

Here the remarkable fact is that we do not get derivatives of the weight function because these cancel out due to the antisymmetric structure of the RHS of formula (5.3.3). In other words, axial torsion, defined by formulae (4.1.14) and (4.3.13) from Chapter 4, has the remarkable property that it scales in a covariant manner under scaling of the frame. Note that the full torsion tensor, defined by formula (4.3.13) from Chapter 4, does not possess such a covariance property.

Formula (5.3.7) tells us that in order to accommodate an arbitrary weight function $w(x)$ we need to multiply the integrand in formula (5.3.2) by $(w(x))^2$, which gives us

$$b = \frac{3\mathbf{c}}{8\pi^2} \int_M w^2 * T^{\text{ax}} \sqrt{\det g_{\alpha\beta}}\; dx\,. \tag{5.3.8}$$

Let us emphasize that the metric and torsion appearing in formula (5.3.8) are the original, unscaled metric and torsion determined by the original, unscaled principal symbol $A_1(x,\xi)$. The scaling argument has been incorporated into the factor $(w(x))^2$.

We now need to express the integrand in (5.3.8) in terms of the spinor field ζ.

We already have a expression for the weight function in terms of the spinor field, see formula (5.1.10). So we only need to express the Hodge dual of axial torsion in terms of the spinor field. Formulae (5.2.4), (5.2.1), (5.2.2), (5.2.9), (5.1.7) and (5.1.9) allow us to express the frame e_j via the spinor field ζ and Pauli matrices σ^α. Hence one needs to combine all these formulae to get explicit expressions for the vector fields e_j, $j = 1, 2, 3$, and substitute these into (5.3.4) and (5.3.3). This is a massive calculation. Fortunately, for the case (5.2.12) this calculation was carried out in Appendix B of [23]: in the notation of the current chapter formula (B.5) from [23] reads

$$* T^{\text{ax}} = \frac{4 \operatorname{Re}(\zeta^* W \zeta)}{3 \|\zeta\|^2}, \tag{5.3.9}$$

where W is the massless Dirac operator (4.A.3).

Formulae (5.3.8), (5.1.10), (5.3.9) and (5.A.1) imply formula (5.1.13). This completes the proof of Theorem 5.1.1 for the case (5.2.12).

In order to prove formula (5.1.13) for the case (5.2.13), we invert coordinates ($x^\alpha \mapsto -x^\alpha$), which changes the sign of topological charge and allows us to use formula (5.1.13). We then invert coordinates again and use the facts that

- the integrand of the massless Dirac action (5.A.1) is invariant under inversion of coordinates and
- our spinor field ζ defined by formulae (5.1.7)–(5.1.10) is an anholonomic object, i.e. it does not depend on the choice of coordinates. □

5.4 Spin structure

In stating our results in Section 5.1 we assumed the principal symbols $A_1(x, \xi)$ and $B_1(x, \xi)$ to be sufficiently close. This was done in order to ensure that equation (5.1.7) could be resolved with respect to the special unitary matrix-function $R(x)$. The restriction of closeness of principal symbols can be overcome by means of the introduction of the concept of spin structure.

Definition 5.4.1. *We say that the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ have the same spin structure if there exists an infinitely smooth special unitary matrix-function (5.1.8) such that we have (5.1.7).*

Remark 5.4.1. *Principal symbols with the same spin structure form an equivalence class.*

The closeness of the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ is never used in the proof of Theorem 5.1.1. All that is needed for Theorem 5.1.1 to be true is for the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ to have the same spin structure, i.e. belong to the same equivalence class.

Hence, it would have been more logical to identify the spin structure as a separate geometric object from the very start, in Section 5.1, and avoid arguments relying on the closeness of the principal symbols. We chose not to proceed along this route in order to make the exposition in Section 5.1 as simple and clear as possible.

The only difference between the "local" setting (the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ are assumed to be close) and the "global" setting (the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ are assumed to have the same spin structure) is that we can no longer claim that the special unitary matrix-function $R(x)$ appearing in formula (5.1.7) is unique. In the "local" setting uniqueness was achieved by requiring $R(x)$ to be close to the identity matrix, whereas in the "global" setting $R(x)$ is defined modulo sign (not surprising as SU(2) is the double cover of SO(3)). This sign indeterminacy does not affect formula (5.1.13) because the massless Dirac action is quadratic in the spinor field.

The number of different spin structures (i.e. number of equivalence classes of principal symbols) depends on the topology of the manifold. Say, the torus $\mathbb{T}^3$ admits eight different spin structures, whereas the sphere $\mathbb{S}^3$ admits a unique spin structure. See Appendices 4.A and 4.B in Chapter 4 and further bibliographic references therein for more details.

It may seem that our Definition 5.4.1 is different from the definition of spin structure in differential geometric literature. Indeed, differential geometers do not operate with concepts such as the principal symbol, using frames instead. However, it has been shown in Section 4.3 that a principal symbol is equivalent to a frame, so our "microlocal" definition of spin structure can be easily recast in terms of frames, bringing it in agreement with the traditional differential geometric one.

Here it is important to emphasize that we do not claim to have redefined the notion of spin structure for the most general case. We work in the very specific setting of dimension three.

5.5 Conformal invariance

Let us transform the differential operator A and weight $w(x)$ as

$$A \mapsto e^{-\varphi/2} A e^{-\varphi/2}, \qquad w \mapsto e^{-\varphi} w, \tag{5.5.1}$$

where $\varphi : M \to \mathbb{R}$ is an arbitrary infinitely smooth real-valued scalar function. The transformation (5.5.1) does not change the spectrum of our eigenvalue problem (5.1.2) and, moreover, preserves its structure: the principal symbol remains trace-free and the subprincipal symbol remains zero (the latter follows from the well-known formula for the subprincipal symbol of a composition of pseudodifferential operators, see formula (1.4) in [27]).

In this section we examine how the gauge transformation (5.5.1) works its way into scalings of the metric, Pauli matrices and spinor field.

Formulae (5.1.6) and (5.5.1) imply that the metric transforms as

$$g^{\alpha\beta} \mapsto e^{-2\varphi} g^{\alpha\beta} \tag{5.5.2}$$

which means that we are looking at a conformal scaling of the metric.

We scale the reference principal symbol $B_1(x,\xi)$ the same way as the principal symbol $A_1(x,\xi)$, i.e. as

$$B_1 \mapsto e^{-\varphi} B_1, \tag{5.5.3}$$

because this way we maintain the condition (5.2.3). Formulae (5.1.14) and (5.5.3) imply that the Pauli matrices scale as

$$\sigma^\alpha \mapsto e^{-\varphi}\sigma^\alpha. \tag{5.5.4}$$

Of course, the scaling of Pauli matrices (5.5.4) agrees with the scaling of the metric (5.5.2) in the sense that the scaled Pauli matrices and metric satisfy the identity (5.2.10).

Formulae (5.1.10) and (5.5.1) imply that the spinor field scales as

$$\zeta \mapsto e^{-\varphi}\zeta. \tag{5.5.5}$$

Let us now examine what happens to the massless Dirac action (5.A.1) under the transformations (5.5.2) (5.5.4) and (5.5.5).

We first look at the expression $W\zeta$. Examination of formulae (4.A.3) and (4.A.4) show that the expression $W\zeta$ transforms as

$$W\zeta \mapsto e^{-2\varphi} W\zeta. \tag{5.5.6}$$

We see that the expression $W\zeta$ scales in a covariant way, with "covariant" meaning that the derivatives of φ do not appear in the RHS of (5.5.6). Of course, the covariance of the massless Dirac operator under conformal scalings of the metric is a know differential geometric fact: see Theorem 4.3 in [28].

Formulae (5.5.5) and (5.5.6) imply

$$\operatorname{Re}(\zeta^* W\zeta) \mapsto e^{-3\varphi} \operatorname{Re}(\zeta^* W\zeta). \tag{5.5.7}$$

Formula (5.5.2) implies $g_{\alpha\beta} \mapsto e^{2\varphi} g_{\alpha\beta}$ and, as we are working in dimension 3, this, in turn, implies that the Riemannian density scales as

$$\sqrt{\det g_{\alpha\beta}} \mapsto e^{3\varphi} \sqrt{\det g_{\alpha\beta}}\,. \tag{5.5.8}$$

Substituting formulae (5.5.7) and (5.5.8) into formula (5.A.1) we see that our massless Dirac action is invariant under the transformations (5.5.2) (5.5.4) and (5.5.5). This is, of course, in agreement with Theorem 5.1.1: as the gauge transformation (5.5.1) does not change the spectrum of our eigenvalue problem (5.1.2), it does not change the second asymptotic coefficient (5.1.13) of the counting function.

5.6 SU(2) invariance

Let us transform the reference principal symbol $B_1(x,\xi)$ as

$$B_1 \mapsto Q B_1 Q^*, \tag{5.6.1}$$

where $Q : M \to \mathrm{SU}(2)$ is an arbitrary infinitely smooth special unitary matrix-function. Formulae (5.1.14) and (5.6.1) imply

$$\sigma^\alpha \mapsto Q \sigma^\alpha Q^*. \tag{5.6.2}$$

Also, formulae (5.1.7) and (5.6.1) imply $R \mapsto RQ^*$, which can be equivalently rewritten as

$$R^* \mapsto Q R^*. \tag{5.6.3}$$

Examining the structure of the matrix R, see formula (5.1.9), we conclude that formula (5.6.3) is equivalent to the linear transformation of the spinor field

$$\zeta \mapsto Q\zeta. \tag{5.6.4}$$

Formulae (5.6.2), (5.6.4) and Property 4 from Appendix 4.A of Chapter 4 tell us that our massless Dirac action is invariant under the transformation (5.6.1). This is, of course, in agreement with Theorem 5.1.1: the choice of reference principal symbol does not affect the spectrum of our eigenvalue problem (5.1.2), hence, it does not affect the second asymptotic coefficient (5.1.13) of the counting function.

5.7 Invariance under rigid rotations

Let us transform the differential operator A as

$$A \mapsto \mathbf{Q}A\mathbf{Q}^{*} \tag{5.7.1}$$

where

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix} \tag{5.7.2}$$

is a *constant* special unitary matrix. The transformation (5.7.1) does not change the spectrum of our eigenvalue problem (5.1.2) and, moreover, preserves its structure: the principal symbol remains trace-free and the subprincipal symbol remains zero. We refer to the transformation (5.7.1) as a *rigid rotation* because it describes a rotation of the frame (5.2.4), with this rotation being the same at all points of the manifold M.

The transformation (5.7.1) is equivalent to the following transformation of the special unitary matrix-function $R(x)$ appearing in formula (5.1.7):

$$R \mapsto \mathbf{Q}R. \tag{5.7.3}$$

Formulae (5.1.9) and (5.1.10) give us a one-to-one correspondence between special unitary matrix-functions and weight functions on the one hand and nonvanishing spinor fields on the other. In terms of the spinor field the transformation (5.7.3)

reads

$$\begin{pmatrix}\zeta^1\\ \zeta^2\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{Q}_{21}\overline{\zeta^2}+\mathbf{Q}_{22}\zeta^1\\ -\mathbf{Q}_{21}\overline{\zeta^1}+\mathbf{Q}_{22}\zeta^2\end{pmatrix}. \tag{5.7.4}$$

Note that, unlike (5.6.3), this transformation is **not** linear because of the complex conjugation. The transformation (5.7.4) can be written as a sum of linear and antilinear transformations:

$$\zeta \mapsto \mathbf{Q}_{22}\zeta - \mathbf{Q}_{21}\mathrm{C}(\zeta) \tag{5.7.5}$$

where

$$\begin{pmatrix}\zeta^1\\ \zeta^2\end{pmatrix} \mapsto \mathrm{C}(\zeta) := \begin{pmatrix}-\overline{\zeta^2}\\ \overline{\zeta^1}\end{pmatrix} \tag{5.7.6}$$

is the charge conjugation operator, see formula (4.A.9) in Appendix 4.A.

Let us show, by performing explicit calculations, that the massless Dirac action (5.A.1) is invariant under the transformation (5.7.5). Using the fact that the massless Dirac operator commutes with the charge conjugation operator, see Property 3 in Appendix 4.A, we get

$$\begin{aligned}&(\mathbf{Q}_{22}\zeta - \mathbf{Q}_{21}\mathrm{C}(\zeta))^* W (\mathbf{Q}_{22}\zeta - \mathbf{Q}_{21}\mathrm{C}(\zeta))\\ &= |\mathbf{Q}_{22}|^2 \zeta^* W\zeta + |\mathbf{Q}_{21}|^2 (\mathrm{C}(\zeta))^*\mathrm{C}(W\zeta) - \overline{\mathbf{Q}_{22}}\mathbf{Q}_{21}\zeta^*\mathrm{C}(W\zeta) - \mathbf{Q}_{22}\overline{\mathbf{Q}_{21}}(\mathrm{C}(\zeta))^* W\zeta\\ &= |\mathbf{Q}_{22}|^2 \zeta^* W\zeta + |\mathbf{Q}_{21}|^2 \overline{\zeta^* W\zeta} + \big\{\overline{\mathbf{Q}_{22}}\mathbf{Q}_{21}(\mathrm{C}(\zeta))^T \overline{W\zeta} - \mathbf{Q}_{22}\overline{\mathbf{Q}_{21}}(\mathrm{C}(\zeta))^* W\zeta\big\}.\end{aligned}$$

But the expression in the curly brackets is purely imaginary, so

$$\begin{aligned}&\mathrm{Re}\big[(\mathbf{Q}_{22}\zeta - \mathbf{Q}_{21}\mathrm{C}(\zeta))^* W (\mathbf{Q}_{22}\zeta - \mathbf{Q}_{21}\mathrm{C}(\zeta))\big]\\ &= |\mathbf{Q}_{22}|^2 \mathrm{Re}\big[\zeta^* W\zeta\big] + |\mathbf{Q}_{21}|^2 \mathrm{Re}\big[\overline{\zeta^* W\zeta}\big] = \big(|\mathbf{Q}_{22}|^2 + |\mathbf{Q}_{21}|^2\big)\mathrm{Re}\big[\zeta^* W\zeta\big] = \mathrm{Re}\big[\zeta^* W\zeta\big].\end{aligned}$$

5.A The massless Dirac action

In this appendix we define, in a concise manner, the massless Dirac action. For more details concerning the massless Dirac operator, see Appendix 4.A.

In order to write down the massless Dirac action we need Pauli matrices, i.e. a triple of trace-free Hermitian 2×2 matrix-functions $\sigma^\alpha(x)$, $\alpha = 1, 2, 3$, satisfying the condition (5.2.10). In our case we have Pauli matrices $\sigma^\alpha(x)$ readily available: these are defined in accordance with formula (5.1.14), or, equivalently, in accordance with formulae (5.2.9) and (5.2.2). Covariant Pauli matrices are defined as $\sigma_\alpha := g_{\alpha\beta}\sigma^\beta$.

The massless Dirac operator is the matrix operator (4.A.3) which acts on 2-component complex-valued spinor fields $\zeta = \begin{pmatrix}\zeta^1 & \zeta^2\end{pmatrix}^T$ (such spinors are called *Weyl spinors*).

We define the massless Dirac action as

$$S(\zeta) := \int_M \mathrm{Re}(\zeta^* W\zeta)\,\sqrt{\det g_{\alpha\beta}}\;dx \tag{5.A.1}$$

where the star indicates Hermitian conjugation. This is the variational functional corresponding to the operator (4.A.3). Here, of course, we use the fact that in view of the self-adjointness of the operator W we have

$$\int_M \zeta^* W\zeta\,\sqrt{\det g_{\alpha\beta}}\;dx = \int_M (W\zeta)^*\zeta\,\sqrt{\det g_{\alpha\beta}}\;dx = \int_M \mathrm{Re}(\zeta^* W\zeta)\,\sqrt{\det g_{\alpha\beta}}\;dx\,.$$

5.B Example

In this appendix we consider an explicit example illustrating the use of Theorem 5.1.1.

Consider the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π. Let A be the differential operator with principal symbol

$$A_1(x,\xi) = \begin{pmatrix} \xi_3 & e^{2ix^3}(\xi_1 - i\xi_2) \\ e^{-2ix^3}(\xi_1 + i\xi_2) & -\xi_3 \end{pmatrix} \tag{5.B.1}$$

and zero subprincipal symbol. We examine below the eigenvalue problem (5.1.2) for this particular operator A and trivial weight function (5.3.1).

Substituting (5.B.1) into (5.1.6) we see that the above principal symbol generates the Euclidean metric

$$g^{\alpha\beta}(x) = \delta^{\alpha\beta}. \tag{5.B.2}$$

Hence, as the reference principal symbol it is natural to take

$$B_1(x,\xi) = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix}. \tag{5.B.3}$$

Substituting (5.B.3) into (5.1.14) we get standard Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.B.4}$$

It is not *a priori* obvious that the principal symbols $A_1(x,\xi)$ and $B_1(x,\xi)$ have the same spin structure. The only way to establish that they do indeed have the same spin structure is to resolve equation (5.1.7) with respect to the special unitary matrix-function $R(x)$. Straightforward calculations give

$$R(x) = \pm \begin{pmatrix} e^{ix^3} & 0 \\ 0 & e^{-ix^3} \end{pmatrix}. \tag{5.B.5}$$

Of course, the underlying reasons why in this particular case we do not encounter topological obstructions are that both principal symbols have the same (positive) topological charge and that the frame encoded in (5.B.1) makes an even number of turns (two turns) as x^3 runs from 0 to 2π. See Appendix 4.A for more details.

Formulae (5.1.9), (5.1.10), (5.3.1) and (5.B.5) give us the following expression for the spinor field:

$$\zeta(x) = \pm \begin{pmatrix} e^{-ix^3} \\ 0 \end{pmatrix}. \tag{5.B.6}$$

Substituting formulae (5.B.2), (5.B.4) and (5.B.6) into (4.A.3) and (5.A.1) we get $S(\zeta) = -(2\pi)^3$. Hence, Theorem 5.1.1 tells us that in our example the two-term asymptotics (5.1.5) takes the form

$$N(\lambda) = \frac{4}{3}\pi\lambda^3 - 4\pi\lambda^2 + o(\lambda^2) \tag{5.B.7}$$

as $\lambda \to +\infty$. Note that the nonperiodicity condition (see Definitions 8.3 and 8.4 in [22]) is fulfilled in our example, so, according to Theorem 8.4 from [22], the asymptotic formula (5.B.7) holds as it is, without mollification.

Observe now that in our example the spectrum of the operator A can be evaluated explicitly. Indeed, let B be the differential operator with principal symbol (5.B.3) and zero subprincipal symbol. In other words, let $B = B_1(x, -i\partial/\partial x)$. Put $\tilde{A} := RBR^*$, where R is the matrix-function (5.B.5) It is easy to check that the subprincipal symbol of the operator $\tilde{A}$ is $-I$, where I is the identity matrix. Hence,

$$A = RBR^* + I. \tag{5.B.8}$$

But the operator RBR^* is unitarily equivalent to the operator B and the spectrum of B is known, see Appendix 4.B. Using (5.B.8), we conclude that the eigenvalues of our operator A are as follows.

- 1 is an eigenvalue of multiplicity two.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $1 + \|m\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $1 - \|m\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

Thus, $N(\lambda) - 1$ is the number of integer lattice points inside a 2-sphere of radius $\lambda - 1$ in $\mathbb{R}^3$ centred at the origin. Applying the result from [38] we get

$$N(\lambda) = \frac{4}{3}\pi\lambda^3 - 4\pi\lambda^2 + O_\varepsilon(\lambda^{21/16+\varepsilon}) \tag{5.B.9}$$

as $\lambda \to +\infty$, with ε being an arbitrary positive number. The more advanced number theoretical result (5.B.9) agrees with our asymptotic formula (5.B.7).

Note that the calculations presented in this section remain unchanged if we replace everywhere $\xi_1 \mp i\xi_2$ by $\xi_1 \pm i\xi_2$. This is in agreement with the fact that the topological charge $\mathbf{c}$ does not appear in our formula (5.1.13).

Chapter 6

Spectral asymmetry of the massless Dirac operator

6.1 Introduction

Let M be a 3-dimensional connected compact oriented manifold without boundary equipped with a smooth Riemannian metric $g_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3$ being the tensor indices. Let W be the corresponding massless Dirac operator, see Appendix 4.A for definition. There are two basic examples when the spectrum of W can be calculated explicitly. The first is the unit torus $\mathbb{T}^3$ equipped with Euclidean metric. The second is the unit sphere $\mathbb{S}^3$ equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in Euclidean space $\mathbb{R}^4$. In both examples the spectrum turns out to be symmetric about zero, see Appendix 4.B for details. Physically, this means that in these two examples there is no difference between the properties of the particle (massless neutrino) and antiparticle (massless antineutrino).

As pointed out in [4, 5, 6, 7], for a general oriented Riemannian 3-manifold (M, g) there is no reason for the spectrum of the massless Dirac operator W to be symmetric. However, producing explicit examples of spectral asymmetry is a difficult task. To our knowledge, the only explicit example was constructed in [65], with the example based on the idea of choosing a 3-manifold with flat metric

but highly nontrivial topology. In this chapter we take a different route: we stick with the simplest possible topology (torus) and create spectral asymmetry by perturbing the metric.

Further on in this chapter we work on the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π.

Suppose first that the metric is Euclidean. Then the massless Dirac operator corresponding to the standard spin structure (see formula (4.A.16)) reads

$$W = -i \begin{pmatrix} \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} \end{pmatrix}. \tag{6.1.1}$$

The operator (6.1.1) admits separation of variables, i.e. one can seek its eigenfunctions in the form $v(x) = ue^{im_\alpha x^\alpha}$, $m \in \mathbb{Z}^3$, $u \in \mathbb{C}^2$, $u \neq 0$, and calculate the eigenvalues and eigenfunctions explicitly. The spectrum of the operator (6.1.1) is as follows.

- Zero is an eigenvalue of multiplicity two.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $\|m\|$ and unique (up to rescaling) eigenfunction of the form $ue^{im_\alpha x^\alpha}$.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $-\|m\|$ and unique (up to rescaling) eigenfunction of the form $ue^{im_\alpha x^\alpha}$.

We now perturb the metric, i.e. consider a metric $g_{\alpha\beta}(x;\epsilon)$ the components of which are smooth functions of coordinates x^α, $\alpha = 1, 2, 3$, and small real parameter ϵ, and which satisfies

$$g_{\alpha\beta}(x;0) = \delta_{\alpha\beta}. \tag{6.1.2}$$

One way of establishing spectral asymmetry of the perturbed problem is to compare the asymptotic distribution of large positive eigenvalues and large negative

eigenvalues. As explained in Section 10 of [22], for a generic first order differential operator this approach allows one to establish spectral asymmetry. Unfortunately, the massless Dirac operator is very special in that the second asymptotic coefficient of its counting function is zero, see formula (4.1.17), so in the first two approximations in powers of λ its large positive eigenvalues are distributed the same way as its large negative eigenvalues. Therefore, in order to demonstrate spectral asymmetry of the perturbed problem, we will, instead of dealing with large eigenvalues, deal with small eigenvalues.

6.2 Main result

Let $W(\epsilon)$ be the massless Dirac operator corresponding to the metric $g_{\alpha\beta}(x;\epsilon)$. The difficulty with applying standard perturbation techniques to the operator $W(\epsilon)$ is that all its eigenvalues have even multiplicity, this being a consequence of the fact that the massless Dirac operator $W(\epsilon)$ commutes with the antilinear operator of charge conjugation

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix} =: \mathrm{C}(v), \tag{6.2.1}$$

see Property 3 in Appendix 4.A. In order to overcome this difficulty we develop in Sections 6.3–6.5 a perturbation theory for the massless Dirac operator which accounts for this charge conjugation symmetry. We show that perturbation-wise the double eigenvalues of the massless Dirac operator can be treated as if they were simple eigenvalues: under perturbation a double eigenvalue remains[1] a double eigenvalue and all the usual formulae apply, with only one minor modification. The minor modification concerns the definition of the pseudoinverse of the unperturbed operator, see formulae (6.3.8)–(6.3.12). Namely, in the definition of

[1]Here, of course, it is important that we don't have a magnetic field. A magnetic field would split up a double eigenvalue, see [28]. The fact that the massless Dirac operator and the charge conjugation operator do not commute in the presence of a magnetic covector potential is well known in theoretical physics: see, for example, formula (2.5) in [41].

the pseudoinverse we separate out a two-dimensional eigenspace rather than a one-dimensional eigenspace.

Given a function $f : \mathbb{T}^3 \to \mathbb{C}$, we denote by

$$\hat{f}(m) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} e^{-im_\alpha x^\alpha} f(x)\, dx\,, \qquad m \in \mathbb{Z}^3, \tag{6.2.2}$$

its Fourier coefficients. Here $dx := dx^1 dx^2 dx^3$.

Let $\lambda_0(\varepsilon)$ be the eigenvalue of the massless Dirac operator with smallest modulus and let

$$h_{\alpha\beta}(x) := \left.\frac{\partial g_{\alpha\beta}}{\partial \epsilon}\right|_{\epsilon=0}. \tag{6.2.3}$$

Further on we raise and lower tensor indices using the Euclidean metric, which means that raising or lowering a tensor index doesn't change anything. A repeated tensor index always indicates summation over the values $1, 2, 3$.

The following theorem is the main result of this chapter.

Theorem 6.2.1. *We have*

$$\lambda_0(\epsilon) = c\,\epsilon^2 + O(\epsilon^3) \quad \textit{as} \quad \epsilon \to 0, \tag{6.2.4}$$

where the constant c is given by the formula

$$c = \frac{i}{16}\, \varepsilon_{\alpha\beta\gamma} \sum_{m \in \mathbb{Z}^3 \setminus \{0\}} \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2} \right) m_\alpha\, \hat{h}_{\beta\mu}(m)\, \overline{\hat{h}_{\gamma\nu}(m)}\,. \tag{6.2.5}$$

Here $\varepsilon_{\alpha\beta\gamma}$ is the totally antisymmetric quantity, $\varepsilon_{123} := +1$, and the overline stands for complex conjugation.

Theorem 6.2.1 warrants the following remarks.

- If the constant c defined by formula (6.2.5) is nonzero, then Theorem 6.2.1 tells us that for sufficiently small nonzero ϵ the spectrum of our massless Dirac operator is asymmetric about zero.

- Theorem 6.2.1 is in agreement with the established view [40, 9] that there are no topological obstructions preventing the shift of the zero eigenvalue of the massless Dirac operator.

- Theorem 6.2.1 is in agreement with the results of [2]. This chapter deals with the Dirac operator in the most general setting. When applied to the case of a compact oriented Riemannian 3-manifold (not necessarily a 3-torus with Euclidean metric) with specified spin structure the results of [2] tell us that if zero is an eigenvalue of the Dirac operator, then the metric can be perturbed so that the zero eigenvalue gets shifted. Furthermore, according to [1], the zero eigenvalue can be shifted by perturbing the metric on an arbitrarily small open set, which is also in agreement with our Theorem 6.2.1.

- Put

$$L_{\gamma\nu\beta\mu} := \frac{i\varepsilon_{\alpha\beta\gamma}}{(2\pi)^3} \sum_{m\in\mathbb{Z}^3\setminus\{0\}} \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2}\right) m_\alpha \int_{\mathbb{T}^3} e^{i(x-y)^\alpha m_\alpha} \,(\,\cdot\,)\, dy\,,$$

$$P_{\gamma\nu\beta\mu} := \frac{1}{4}(L_{\gamma\nu\beta\mu} + L_{\nu\gamma\beta\mu} + L_{\gamma\nu\mu\beta} + L_{\nu\gamma\mu\beta}).$$

  This gives us a first order pseudodifferential operator P acting in the vector space of rank two symmetric complex-valued tensor fields, $s_{\beta\mu} \mapsto P_{\gamma\nu\beta\mu} s_{\beta\mu}$. If we equip this vector space with the natural inner product $(r, s) := \int_{\mathbb{T}^3} r_{\alpha\beta}\, \overline{s_{\alpha\beta}}\, dx$ then it is easy to see that the operator P is formally self-adjoint and formula (6.2.5) can be rewritten as $c = \frac{1}{128\pi^3}(Ph, h)$, where h is defined in accordance with (6.2.3). This shows that our coefficient c has a nonlocal (global) nature, with the source of the nonlocality being the factor

$$\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2} \tag{6.2.6}$$

  in the symbol of the pseudodifferential operator P. In other words, formula (6.2.5) cannot be rewritten in terms of (linearized) local differential geometric quantities such as the curvature tensor and the Cotton tensor.

- The rank two tensor (6.2.6) can be identified with a linear map in $\mathbb{R}^3$, $p_\mu \mapsto \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2}\right) p_\nu$. This linear map is an orthogonal projection: it projects onto the plane orthogonal to the covector (momentum) m.

- Suppose that we are looking at a conformal scaling of the Euclidean metric, $g_{\alpha\beta}(x;\epsilon) = e^{2\epsilon\varphi(x)}\delta_{\alpha\beta}$, where $\varphi : \mathbb{T}^3 \to \mathbb{R}$. Then $h_{\alpha\beta}(x) = 2\varphi(x)\delta_{\alpha\beta}$ and formula (6.2.5) becomes

$$c = \frac{i}{4}\,\varepsilon_{\alpha\beta\gamma} \sum_{m\in\mathbb{Z}^3\setminus\{0\}} \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2}\right) m_\alpha \delta_{\beta\mu}\delta_{\gamma\nu}\,|\hat{\varphi}(m)|^2\ . \tag{6.2.7}$$

  The expression in the RHS of (6.2.7) is zero because the summand in $\sum_{m\in\mathbb{Z}^3\setminus\{0\}}$ is odd in m. (Another reason why the expression in the RHS of (6.2.7) is zero is that the summand is symmetric in β, γ.) This agrees with the well-known fact that the zero eigenvalue does not shift under a conformal scaling of the metric, see Theorem 4.3 in [28].

- Suppose that we replace the tensor $h_{\alpha\beta}(x)$ by the tensor $h_{\alpha\beta}(-x)$. Then $\hat{h}_{\alpha\beta}(m)$ is replaced by $-\hat{h}_{\alpha\beta}(-m)$ and, introducing a new summation index $n := -m$ in formula (6.2.5), we see that the coefficient c changes sign. Physically, this means that formula (6.2.5) feels the difference between "left" and "right", as one would expect of a formula describing a fermion.

The proof of Theorem 6.2.1 is given in Section 6.6. In Section 6.7 we treat the special case when the metric $g_{\alpha\beta}(x;\epsilon)$ is a function of the coordinate x^1 only. In Sections 6.8 and 6.9 we present families of metrics for which the eigenvalue $\lambda_0(\epsilon)$ can be evaluated explicitly. Finally, in Section 6.10 we examine the eta invariant of our ϵ-dependent massless Dirac operator.

6.3 Perturbation process I: preliminaries

Let M be a 3-dimensional connected compact oriented manifold without boundary equipped with a smooth Riemannian metric $g_{\alpha\beta}(x)$, $\alpha, \beta = 1, 2, 3$ being the

tensor indices and $x = (x^1, x^2, x^3)$ being local coordinates. The perturbation theory developed in this section and Sections 6.4–6.5 does not assume that the manifold is necessarily a 3-torus.

We perturb the metric in a smooth manner and denote the perturbed metric by $g_{\alpha\beta}(x;\epsilon)$, where ϵ is a small real parameter. Here we assume that $g_{\alpha\beta}(x;0)$ is the unperturbed metric described in the previous paragraph.

By $W_{1/2}(\epsilon)$ we denote the massless Dirac operator on half-densities corresponding to the metric $g_{\alpha\beta}(x;\epsilon)$, see Appendix 4.A for details. We choose to work with the massless Dirac operator on half-densities $W_{1/2}(\epsilon)$ rather than with the massless Dirac operator $W(\epsilon)$ because we do not want our Hilbert space to depend on ϵ. The difference between the operators $W(\epsilon)$ and $W_{1/2}(\epsilon)$ is explained in Appendix 4.A: compare formulae (4.A.3) and (4.A.19). The spectra of the operators $W(\epsilon)$ and $W_{1/2}(\epsilon)$ are the same.

The operator $W_{1/2}(\epsilon)$ is actually not a single operator, but an equivalence class of operators which differ by the transformation

$$W_{1/2}(\epsilon) \mapsto R\, W_{1/2}(\epsilon)\, R^*, \tag{6.3.1}$$

where $R(x;\epsilon)$ is an arbitrary smooth 2×2 special unitary matrix-function. See Property 4 in Appendix 4.A for a detailed discussion regarding the transformation (6.3.1), noting that the massless Dirac operator on half-densities $W_{1/2}(\epsilon)$ differs from the massless Dirac operator $W(\epsilon)$ only by "scalar" factors on the left and on the right — these "scalar" factors commute with matrix-functions $R(x;\epsilon)$ and $R^*(x;\epsilon)$. Obviously, the transformation (6.3.1) does not affect the spectrum. Later on, in Section 6.6, we will use this gauge degree of freedom to simplify calculations, see formula (6.6.11).

The operator $W_{1/2}(\epsilon)$ acts on 2-columns $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ of complex-valued half-densities. Our Hilbert space is $L^2(M;\mathbb{C}^2)$, which is the vector space of 2-columns

of square integrable half-densities equipped with inner product

$$\langle v, w\rangle := \int_M w^* v \, dx \,. \tag{6.3.2}$$

The domain of the operator $W_{1/2}(\epsilon)$ is $H^1(M;\mathbb{C}^2)$, which is the Sobolev space of 2-columns of half-densities that are square integrable together with their first partial derivatives. It is known that the operator $W_{1/2}(\epsilon) : H^1(M;\mathbb{C}^2) \to L^2(M;\mathbb{C}^2)$ is self-adjoint and that it has a discrete spectrum, with eigenvalues accumulating to $+\infty$ and $-\infty$. Note that here neither the Hilbert space nor the domain depend on ϵ. It is also known that the eigenfunctions of the operator $W_{1/2}(\epsilon)$ are infinitely smooth.

The antilinear operator of charge conjugation (6.2.1) maps any element of $L^2(M;\mathbb{C}^2)$ to an element of $L^2(M;\mathbb{C}^2)$ and any element of $H^1(M;\mathbb{C}^2)$ to an element of $H^1(M;\mathbb{C}^2)$. As the massless Dirac operator on half-densities $W_{1/2}(\epsilon)$ differs from the massless Dirac operator $W(\epsilon)$ only by real "scalar" factors on the left and on the right, it also commutes with the operator of charge conjugation:

$$\mathrm{C}(W_{1/2}(\epsilon)\, v) = W_{1/2}(\epsilon)\, \mathrm{C}(v)\,, \tag{6.3.3}$$

$\forall v \in H^1(M;\mathbb{C}^2)$. Note that the operator of charge conjugation does not itself depend on ϵ.

Observe that formulae (6.2.1) and (6.3.2) imply the following useful identities:

$$\mathrm{C}(\mathrm{C}(v)) = -v, \tag{6.3.4}$$

$$\langle v\,, \mathrm{C}(v)\rangle = 0, \tag{6.3.5}$$

$$\langle \mathrm{C}(v)\,, \mathrm{C}(w)\rangle = \langle w, v\rangle. \tag{6.3.6}$$

Let

$$W_{1/2}(\epsilon) = W^{(0)}_{1/2} + \epsilon W^{(1)}_{1/2} + \epsilon^2 W^{(2)}_{1/2} + \ldots \tag{6.3.7}$$

be the asymptotic expansion of the partial differential operator $W_{1/2}(\epsilon)$ in powers of the small parameter ϵ. Obviously, the operators $W_{1/2}^{(k)}$, $k = 0, 1, 2, \ldots$, are formally self-adjoint first order differential operators which commute with the antilinear operator of charge conjugation (6.2.1).

Suppose that $\lambda^{(0)}$ is a double eigenvalue of the operator $W_{1/2}^{(0)}$. As explained in Appendix 4.A, eigenvalues of the massless Dirac operator have even multiplicity, so a double eigenvalue is the "simplest" eigenvalue one can get.

Remark 6.3.1. *The spectrum of the massless Dirac operator on a 3-torus equipped with Euclidean metric was written down explicitly in Section 6.1. Examination of the relevant formulae shows that the only double eigenvalue is the eigenvalue zero as all others have multiplicity greater than or equal to six. However, in this section and Sections 6.4–6.5 we do not use the fact that $\lambda^{(0)} = 0$.*

Let $v^{(0)}$ be a normalized, $\|v^{(0)}\| = 1$, eigenfunction of the operator $W_{1/2}^{(0)}$ corresponding to the eigenvalue $\lambda^{(0)}$. Formula (6.3.3) and the fact that $\lambda^{(0)}$ is real imply that $\mathrm{C}(v^{(0)})$ is also an eigenfunction of the operator $W_{1/2}^{(0)}$ corresponding to the eigenvalue $\lambda^{(0)}$. Formula (6.3.6) implies that $\|\mathrm{C}(v^{(0)})\| = 1$, and, moreover, in view of formula (6.3.5), the eigenfunctions $v^{(0)}$ and $\mathrm{C}(v^{(0)})$ are orthogonal.

The argument presented in the previous paragraph shows that, when dealing with a double eigenvalue of the massless Dirac operator, it is sufficient to construct only one eigenfunction: the other one is obtained by charge conjugation. The argument is valid not only for the unperturbed operator $W_{1/2}^{(0)}$, but for the perturbed operator $W_{1/2}(\epsilon)$ as well, provided that ϵ is small enough (so that the multiplicity of the eigenvalue does not increase). Hence, in the perturbation process described in the next section we shall construct one eigenfunction only.

In the perturbation process that we will describe in the next section we will make use of the pseudoinverse of the unperturbed operator. This operator, which we denote by Q, is defined as follows. Consider the problem

$$(W_{1/2}^{(0)} - \lambda^{(0)})v = f \tag{6.3.8}$$

where $f \in L^2(M;\mathbb{C}^2)$ is given and $v \in H^1(M;\mathbb{C}^2)$ is to be found. Suppose that f satisfies the conditions

$$\langle f, v^{(0)} \rangle = \langle f, \mathrm{C}(v^{(0)}) \rangle = 0. \tag{6.3.9}$$

Then the problem (6.3.8) can be resolved for v, however this solution is not unique. We achieve uniqueness by imposing the conditions

$$\langle v, v^{(0)} \rangle = \langle v, \mathrm{C}(v^{(0)}) \rangle = 0 \tag{6.3.10}$$

and define Q as the linear operator mapping f to v,

$$Q : f \mapsto v. \tag{6.3.11}$$

Thus, Q is a bounded linear operator acting in the orthogonal complement of the eigenspace of the operator $W^{(0)}_{1/2}$ corresponding to the eigenvalue $\lambda^{(0)}$. We extend this operator to the whole Hilbert space $L^2(M;\mathbb{C}^2)$ in accordance with

$$Qv^{(0)} = Q\mathrm{C}(v^{(0)}) = 0. \tag{6.3.12}$$

It is clear from the above definition that the bounded linear operator Q is self-adjoint and commutes with the antilinear operator of charge conjugation (6.2.1). Note that our definition of the pseudoinverse Q of the unperturbed operator $W^{(0)}_{1/2} - \lambda^{(0)}$ is in agreement with Rellich's, see Chapter 2 Section 2 in [66].

Throughout our perturbation process we will have to deal with various formally self-adjoint linear operators which commute with the antilinear operator of charge conjugation (6.2.1). Such operators possess a special property which is the subject of the following lemma.

Lemma 6.3.1. *Let $L : C^\infty(M;\mathbb{C}^2) \to C^\infty(M;\mathbb{C}^2)$ be a (possibly unbounded) formally self-adjoint linear operator which commutes with the antilinear operator*

of charge conjugation (6.2.1). Then for any $v \in C^\infty(M;\mathbb{C}^2)$ we have

$$\langle Lv, \mathrm{C}(v)\rangle = 0. \tag{6.3.13}$$

Proof Take arbitrary $v, w \in C^\infty(M;\mathbb{C}^2)$. Using formula (6.3.6) and the fact that L is formally self-adjoint and commutes with C, we get

$$\langle L\,\mathrm{C}(w)\,, \mathrm{C}(v)\rangle = \langle \mathrm{C}(L\,w)\,, \mathrm{C}(v)\rangle = \langle v, Lw\rangle = \langle Lv, w\rangle. \tag{6.3.14}$$

For $w = \mathrm{C}(v)$ formula (6.3.14) reads

$$\langle L\,\mathrm{C}(\mathrm{C}(v))\,, \mathrm{C}(v)\rangle = \langle Lv, \mathrm{C}(v)\rangle. \tag{6.3.15}$$

But in view of (6.3.4) formula (6.3.15) can be rewritten as

$$-\langle L\,v\,, \mathrm{C}(v)\rangle = \langle L\,v\,, \mathrm{C}(v)\rangle,$$

which gives us the required identity (6.3.13). □

6.4 Perturbation process II: formal procedure

We now write down the formal perturbation process. A rigorous justification will be provided in the next section.

Further on in this section as well as in the two following sections (Sections 6.5 and 6.6) we write, for the sake of brevity, $A(\epsilon) = W_{1/2}(\epsilon)$ and $A^{(k)} = W^{(k)}_{1/2}$, $k = 0, 1, 2, \ldots$. In this new notation formula (6.3.7) reads

$$A(\epsilon) = A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \ldots. \tag{6.4.1}$$

We need to solve the eigenvalue problem

$$A(\epsilon)\, v(\epsilon) = \lambda(\epsilon)\, v(\epsilon)\,. \tag{6.4.2}$$

We seek the eigenvalue and eigenfunction of the perturbed operator $A(\epsilon)$ in the form of asymptotic expansions

$$\lambda(\epsilon) = \lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \dots, \tag{6.4.3}$$

$$v(\epsilon) = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \dots. \tag{6.4.4}$$

Note that we do not aim to preserve the normalization of our eigenfunction throughout the perturbation process.

Let us forget for a moment that we are dealing with a double eigenvalue and suppose that our eigenvalue is simple. Then the iterative procedure for the determination of $\lambda^{(k)}$ and $v^{(k)}$, $k = 1, 2, \dots$, is well known, see Chapter 2 Section 2 in [66]. At the kth step we get the equation

$$(A^{(0)} - \lambda^{(0)})v^{(k)} = f^{(k)}, \tag{6.4.5}$$

where

$$f^{(k)} := F^{(k)}v^{(0)}, \tag{6.4.6}$$

and $F^{(k)}$ is some linear operator. The explicit formula for the operator $F^{(k)}$ appearing in equations (6.4.5), (6.4.6) is written as follows. Put

$$D(\epsilon) := (B(0) - B(\epsilon))\left(I + \sum_{j=1}^{\infty}\big[Q\,(B(0) - B(\epsilon))\big]^j\right), \tag{6.4.7}$$

where I is the identity operator, $B(\epsilon) := A(\epsilon) - \lambda(\epsilon)I$ and the infinite sum is understood as an asymptotic series. The operator $D(\epsilon)$ can be expanded in

powers of the small parameter ϵ,

$$D(\epsilon) = \sum_{k=1}^{\infty} \epsilon^k F^{(k)}, \tag{6.4.8}$$

giving us the required $F^{(k)}$. The real number $\lambda^{(k)}$ is determined from the condition

$$\langle f^{(k)}, v^{(0)} \rangle = 0 \tag{6.4.9}$$

after which we resolve (6.4.5) by setting

$$v^{(k)} = Q f^{(k)}. \tag{6.4.10}$$

We claim that the above process carries over to the case of a double eigenvalue that we are dealing with. Indeed, the difference between the cases of a simple eigenvalue and a double eigenvalue is that at the kth step of the iterative process in addition to condition (6.4.9) we need to satisfy the condition

$$\langle f^{(k)}, \mathrm{C}(v^{(0)}) \rangle = 0. \tag{6.4.11}$$

The structure of the operator (6.4.7) is such that it is formally self-adjoint and commutes with the antilinear operator of charge conjugation (6.2.1), so the operator $F^{(k)}$ defined in accordance with formula (6.4.8) has the same properties and, hence, by Lemma 6.3.1, condition (6.4.11) is satisfied automatically and the asymptotic process continues as if the eigenvalue were simple.

We end this section by giving, for future reference, the explicit formulae for the coefficients $\lambda^{(1)}$ and $\lambda^{(2)}$ appearing in the asymptotic expansion (6.4.3):

$$\lambda^{(1)} = \langle A^{(1)} v^{(0)}, v^{(0)} \rangle, \tag{6.4.12}$$

$$\lambda^{(2)} = \langle A^{(2)} v^{(0)}, v^{(0)} \rangle - \langle (A^{(1)} - \lambda^{(1)})\, Q\, (A^{(1)} - \lambda^{(1)})\, v^{(0)}, v^{(0)} \rangle\,. \tag{6.4.13}$$

6.5 Perturbation process III: justification

Recall that by $\lambda^{(0)} = \lambda(0)$ we denote a double eigenvalue of the unperturbed operator $A^{(0)} = A(0)$ (the unperturbed massless Dirac operator on half-densities). Let us choose a $\delta > 0$ such that $\lambda^{(0)}$ is the only eigenvalue of the operator $A^{(0)}$ on the interval $[\lambda^{(0)} - \delta, \lambda^{(0)} + \delta]$.

In order to justify our perturbation process we will need the following lemma.

Lemma 6.5.1. *For sufficiently small ϵ the interval*

$$(\lambda^{(0)} - \delta, \lambda^{(0)} + \delta) \tag{6.5.1}$$

contains exactly one double eigenvalue of the operator $A(\epsilon)$ and no other eigenvalues.

Proof Denote $C_\delta := \{\mu \in \mathbb{C} \,|\, |\mu - \lambda^{(0)}| = \delta\}$ (circle in the complex plane) and $D_\delta := \{\mu \in \mathbb{C} \,|\, |\mu - \lambda^{(0)}| < \delta\}$ (open disc in the complex plane). Put $R_\mu^{(0)} := (A^{(0)} - \mu I)^{-1}$. Clearly, for $\mu \in C_\delta$ the operator $R_\mu^{(0)}$ is well-defined and, moreover, is a bounded operator acting from $L^2(M;\mathbb{C}^2)$ to $H^1(M;\mathbb{C}^2)$. Furthermore, the norm of the operator $R_\mu^{(0)} : L^2(M;\mathbb{C}^2) \to H^1(M;\mathbb{C}^2)$ is bounded uniformly over $\mu \in C_\delta$.

Let us now define the operator

$$R_\mu(\epsilon) := \left(I + \sum_{j=1}^{\infty} \left[-R_\mu^{(0)} (A(\epsilon) - A^{(0)}) \right]^j \right) R_\mu^{(0)}, \tag{6.5.2}$$

where $\mu \in C_\delta$. The operator $A(\epsilon) - A^{(0)}$ is a bounded operator acting from $H^1(M;\mathbb{C}^2)$ to $L^2(M;\mathbb{C}^2)$ and the norm of the operator $A(\epsilon) - A^{(0)} : H^1(M;\mathbb{C}^2) \to L^2(M;\mathbb{C}^2)$ tends to zero as ϵ tends to zero. Hence, the series in (6.5.2) converges for sufficiently small ϵ. Furthermore, it is easy to see that

$$R_\mu(\epsilon) \to R_\mu^{(0)} \quad \text{as} \quad \epsilon \to 0 \tag{6.5.3}$$

in the sense of the operator norm $L^2(M;\mathbb{C}^2) \to H^1(M;\mathbb{C}^2)$ and this convergence is uniform over $\mu \in C_\delta$.

Acting onto (6.5.2) with the operator $A(\epsilon)-\mu I$ we see that $(A(\epsilon)-\mu I)R_\mu(\epsilon) = I$, so $R_\mu(\epsilon) = (A(\epsilon) - \mu I)^{-1}$. Put

$$E(\epsilon) := \frac{1}{2\pi i}\int_{C_\delta} R_\mu(\epsilon)\, d\mu\,. \tag{6.5.4}$$

The operator $E(\epsilon)$ is the orthogonal projection onto the span of eigenvectors of the operator $A(\epsilon)$ corresponding to eigenvalues on the interval (6.5.1). In particular, $E(0) = E^{(0)}$ is the orthogonal projection onto the span of eigenvectors of the operator $A^{(0)}$ corresponding to the double eigenvalue $\lambda^{(0)}$.

Formulae (6.5.3) and (6.5.4) imply

$$\|E(\epsilon) - E^{(0)}\|_{\mathrm{op}} \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{6.5.5}$$

where $\|\cdot\|_{\mathrm{op}}$ stands for the operator norm in the Banach space of bounded linear operators $L^2(M;\mathbb{C}^2) \to L^2(M;\mathbb{C}^2)$. Formula (6.5.5) implies that for sufficiently small ϵ we have

$$\|E(\epsilon) - E^{(0)}\|_{\mathrm{op}} < 1. \tag{6.5.6}$$

Formula (6.5.6) and the fact that the orthogonal projections $E(\epsilon)$ and $E^{(0)}$ have finite rank imply that $\operatorname{rank} E(\epsilon) = \operatorname{rank} E^{(0)} = 2$. Thus, the operator $A(\epsilon)$ has two eigenvalues, counted with multiplicities, on the interval (6.5.1). We know, see Property 3 in Appendix 4.A, that the eigenvalues of the operator $A(\epsilon)$ have even multiplicity, so we are looking at one double eigenvalue on the interval (6.5.1). □

Let $\lambda(\epsilon)$ be the unique double eigenvalue of the operator $A(\epsilon)$ from the interval (6.5.1). Denote by $\sigma(\epsilon)$ the spectrum of the operator $A(\epsilon)$ and, for a given $\mu \in \mathbb{R}$, denote $\operatorname{dist}(\mu, \sigma(\epsilon)) = \min\limits_{\nu\in\sigma(\epsilon)} |\mu - \nu|$. Obviously, without additional information

on μ and on $\sigma(\epsilon)$ we can only guarantee the inequality

$$\operatorname{dist}(\mu, \sigma(\epsilon)) \leq |\mu - \lambda(\epsilon)|. \tag{6.5.7}$$

Choose an arbitrary natural k and denote

$$\tilde{\lambda}(\epsilon) = \lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \ldots + \epsilon^k\lambda^{(k)}, \tag{6.5.8}$$

$$\tilde{v}(\epsilon) = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \ldots + \epsilon^k v^{(k)}, \tag{6.5.9}$$

where the $\lambda^{(j)}$ and $v^{(j)}$, $j = 0, 1, \ldots, k$, are taken from (6.4.3) and (6.4.4). We have

$$\|\tilde{v}(\epsilon)\| = 1 + O(\epsilon), \tag{6.5.10}$$

$$\|(A(\epsilon) - \tilde{\lambda}(\epsilon))\tilde{v}(\epsilon)\| = O(\epsilon^{k+1}), \tag{6.5.11}$$

where $\|\cdot\|$ stands for the $L^2(M; \mathbb{C}^2)$ norm (see (6.3.2) for inner product). As our operator $A(\epsilon)$ is self-adjoint, formulae (6.5.10) and (6.5.11) imply

$$\operatorname{dist}(\tilde{\lambda}(\epsilon), \sigma(\epsilon)) \leq \frac{\|(A(\epsilon) - \tilde{\lambda}(\epsilon))\tilde{v}(\epsilon)\|}{\|\tilde{v}(\epsilon)\|} = O(\epsilon^{k+1}). \tag{6.5.12}$$

Formulae (6.5.8) and (6.5.12) and Lemma 6.5.1 imply that for sufficiently small ϵ,

$$\operatorname{dist}(\tilde{\lambda}(\epsilon), \sigma(\epsilon)) = |\tilde{\lambda}(\epsilon) - \lambda(\epsilon)|, \tag{6.5.13}$$

compare with (6.5.7). Combining formulae (6.5.12) and (6.5.13), we get $\lambda(\epsilon) = \tilde{\lambda}(\epsilon) + O(\epsilon^{k+1})$. This completes the justification of our perturbation process.

6.6 Proof of Theorem 6.2.1

The unperturbed massless Dirac operator on half-densities, which we denote by $A^{(0)}$, is given by the expression in the RHS of (6.1.1). The unperturbed eigenvalue,

$\lambda^{(0)}$, is zero and the corresponding normalized eigenfunction is

$$v^{(0)} = \frac{1}{(2\pi)^{3/2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6.6.1}$$

The pseudoinverse Q of the operator $A^{(0)}$ is given by the formula

$$\begin{aligned} Q &= \frac{1}{(2\pi)^3} \sum_{m \in \mathbb{Z}^3 \setminus \{0\}} e^{im_\alpha x^\alpha} \begin{pmatrix} m_3 & m_1 - im_2 \\ m_1 + im_2 & -m_3 \end{pmatrix}^{-1} \int_{\mathbb{T}^3} e^{-im_\alpha y^\alpha} (\,\cdot\,)\, dy \\ &= \frac{1}{(2\pi)^3} \sum_{m \in \mathbb{Z}^3 \setminus \{0\}} \frac{e^{im_\alpha x^\alpha}}{\|m\|^2} \begin{pmatrix} m_3 & m_1 - im_2 \\ m_1 + im_2 & -m_3 \end{pmatrix} \int_{\mathbb{T}^3} e^{-im_\alpha y^\alpha} (\,\cdot\,)\, dy\,, \end{aligned} \tag{6.6.2}$$

where $dy := dy^1 dy^2 dy^3$. The operator (6.6.2) is a self-adjoint pseudodifferential operator of order -1.

We have

$$\lambda(\epsilon) = \epsilon \lambda^{(1)} + \epsilon^2 \lambda^{(2)} + O(\epsilon^3), \tag{6.6.3}$$

where the coefficients $\lambda^{(1)}$ and $\lambda^{(2)}$ are given by formulae (6.4.12) and (6.4.13) respectively. Thus, in order to prove Theorem 6.2.1 we need to write down explicitly the differential operators $A^{(1)}$ and $A^{(2)}$ appearing in the asymptotic expansion of the perturbed massless Dirac operator on half-densities,

$$A(\epsilon) = A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + O(\epsilon^3). \tag{6.6.4}$$

In what follows we use terminology from microlocal analysis. In particular, we use the notions of the *principal* and *subprincipal* symbols of a differential operator, see Subsection 3.1.3 in the current document or Subsection 2.1.3 in [72] for details.

Let L be a first order 2×2 matrix differential operator. We denote its principal and subprincipal symbols by $L_1(x, \xi)$ and $L_{\mathrm{sub}}(x)$ respectively. Here $\xi = (\xi_1, \xi_2, \xi_3)$ is the variable dual to the position variable x. The subscript in $L_1(x, \xi)$ indicates the degree of homogeneity in ξ.

A first order differential operator L is completely determined by its principal and subprincipal symbols. Indeed, the principal symbol has the form

$$L_1(x,\xi) = M^{(\alpha)}(x)\,\xi_\alpha\,, \tag{6.6.5}$$

where $M^{(\alpha)}(x)$ are matrix-functions depending only on the position variable x. It is easy to see that the differential operator L is given by the formula

$$L = -\frac{i}{2}M^{(\alpha)}(x)\,\frac{\partial}{\partial x^\alpha} - \frac{i}{2}\frac{\partial}{\partial x^\alpha}\,M^{(\alpha)}(x) + L_{\mathrm{sub}}(x)\,. \tag{6.6.6}$$

Given a first order differential operator L, let us consider the expression $\langle Lv^{(0)}, v^{(0)}\rangle$, where $v^{(0)}$ is the constant column (6.6.1) and angular brackets indicate the inner product (6.3.2). Examination of formula (6.6.6) shows that

$$\langle Lv^{(0)}, v^{(0)}\rangle = \langle L_{\mathrm{sub}}v^{(0)}, v^{(0)}\rangle$$

because the terms coming from the principal symbol integrate to zero. Consequently formulae (6.4.12) and (6.4.13) simplify and now read

$$\lambda^{(1)} = \langle A^{(1)}_{\mathrm{sub}}v^{(0)}, v^{(0)}\rangle, \tag{6.6.7}$$

$$\lambda^{(2)} = \langle A^{(2)}_{\mathrm{sub}}v^{(0)}, v^{(0)}\rangle - \langle (A^{(1)}-\lambda^{(1)})\,Q\,(A^{(1)}-\lambda^{(1)})\,v^{(0)}, v^{(0)}\rangle\,. \tag{6.6.8}$$

We see that for the purpose of proving Theorem 6.2.1 we do not need to know the full operator $A^{(2)}$, only its subprincipal symbol $A^{(2)}_{\mathrm{sub}}$.

In order to write down explicitly the massless Dirac operator on half-densities $A(\epsilon)$ we need the concepts of *frame* and *coframe*. The differential geometric definition of coframe was given in Section 3 of [21]. However, as in the current chapter we are working in a specified coordinate system, we can adopt a somewhat simpler approach. For the purposes of the current chapter a coframe is a smooth

real-valued matrix-function $e^j{}_\alpha(x;\epsilon)$, $j,\alpha = 1,2,3$, satisfying the conditions

$$g_{\alpha\beta}(x;\epsilon) = \delta_{jk}\, e^j{}_\alpha(x;\epsilon)\, e^k{}_\beta(x;\epsilon)\,, \tag{6.6.9}$$

$$e^j{}_\alpha(x;0) = \delta^j{}_\alpha\,. \tag{6.6.10}$$

Here and further on when dealing with matrix-functions we use the convention that the first index (subscript or superscript) enumerates the rows and the second index (subscript or superscript) enumerates the columns. Say, in matrix notation the RHS of (6.6.9) reads as "product of coframe transposed and coframe".

Note that the reason we imposed condition (6.6.10) is so that our unperturbed operator has the form (6.1.1). See also formula (6.3.1) and associated discussion.

For a given metric $g_{\alpha\beta}(x;\epsilon)$ the coframe $e^j{}_\alpha(x;\epsilon)$ is not defined uniquely. We can multiply the matrix-function $e^j{}_\alpha(x;\epsilon)$ from the left by an arbitrary smooth 3×3 special orthogonal matrix-function $O(x;\epsilon)$ satisfying the condition $O(x;0) = I$, with I denoting the 3×3 identity matrix. This will give us a new coframe satisfying the defining relations (6.6.9) and (6.6.10). As explained in Appendix 4.A, this freedom in the choice of coframe is a gauge degree of freedom which does not affect the spectrum. In the current section we specify the gauge by requiring the matrix-function $e^j{}_\alpha(x;\epsilon)$ to be symmetric,

$$e^j{}_\alpha(x;\epsilon) = e^\alpha{}_j(x;\epsilon). \tag{6.6.11}$$

Condition (6.6.11) makes sense because we are working in a specified coordinate system. Looking ahead, let us point out the main advantage of the symmetric gauge (6.6.11): the asymptotic expansion of the subprincipal symbol of the massless Dirac operator on half-densities in powers of ϵ starts with a quadratic term and, moreover, the coefficient at ϵ^2 has an especially simple structure, see formulae (6.6.16) and (6.6.19).

In matrix notation condition (6.6.9) now reads "the symmetric positive matrix $g_{\alpha\beta}(x;\epsilon)$ is the square of the symmetric matrix $e^j{}_\alpha(x;\epsilon)$". Conversely, the

symmetric matrix $e^j{}_\alpha(x;\epsilon)$ is the square root of the symmetric positive matrix $g_{\alpha\beta}(x;\epsilon)$. We choose the branch of the square root so that the matrix $e^j{}_\alpha(x;\epsilon)$ is positive.

According to formulae (6.1.2) and (6.2.3) we have

$$g_{\alpha\beta}(x;\epsilon) = \delta_{\alpha\beta} + \epsilon h_{\alpha\beta}(x) + O(\epsilon^2), \tag{6.6.12}$$

hence, by Taylor's formula for $\sqrt{1+z}$,

$$e^j{}_\alpha(x;\epsilon) = \delta^j{}_\alpha + \frac{\epsilon}{2} h^j{}_\alpha(x) + O(\epsilon^2). \tag{6.6.13}$$

Here we follow the convention introduced in Section 6.2: we raise and lower indices in h using the Euclidean metric, which means that raising or lowering an index doesn't change anything. We also swap, when needed, tensor (Greek) indices for frame (Latin) indices, which is acceptable because we are working in a specified coordinate system.

The *frame* is the smooth real-valued matrix-function $e_j{}^\alpha(x;\epsilon)$, $j,\alpha = 1,2,3$, defined by the system of linear algebraic equations

$$e_j{}^\alpha(x;\epsilon)\, e^k{}_\alpha(x;\epsilon) = \delta_j{}^k. \tag{6.6.14}$$

Note the position of indices in formula (6.6.14). In matrix notation formula (6.6.14) reads as "the frame is the transpose of the inverse of the coframe". As we chose our coframe to be symmetric, our frame is also symmetric and is simply the inverse of the coframe. Formula (6.6.13) and Taylor's formula for $(1+z)^{-1}$ imply

$$e_j{}^\alpha(x;\epsilon) = \delta_j{}^\alpha - \frac{\epsilon}{2} h_j{}^\alpha(x) + O(\epsilon^2). \tag{6.6.15}$$

According to formulae (4.6.1), (4.3.5) and (4.8.1) the subprincipal symbol of the massless Dirac operator on half-densities is

$$A_{\mathrm{sub}}(x;\epsilon) = \frac{3}{4}\left(*T^{\mathrm{ax}}(x;\epsilon)\right) I\,, \tag{6.6.16}$$

where I is the 2×2 identity matrix and $*T^{\mathrm{ax}}(x;\epsilon)$ is the scalar function

$$\begin{aligned} *T^{\mathrm{ax}} = \frac{\delta_{kl}}{3}\sqrt{\det g^{\alpha\beta}}\,\big[&e^k{}_1\,\partial e^l{}_3/\partial x^2 + e^k{}_2\,\partial e^l{}_1/\partial x^3 + e^k{}_3\,\partial e^l{}_2/\partial x^1 \\ &- e^k{}_1\,\partial e^l{}_2/\partial x^3 - e^k{}_2\,\partial e^l{}_3/\partial x^1 - e^k{}_3\,\partial e^l{}_1/\partial x^2\big]. \end{aligned} \tag{6.6.17}$$

Note that formula (6.6.12) implies $g^{\alpha\beta}(x;\epsilon) = \delta_{\alpha\beta} - \epsilon h^{\alpha\beta}(x) + O(\epsilon^2)$, which, in turn, gives us

$$\sqrt{\det g^{\alpha\beta}(x;\epsilon)} = 1 - \frac{\epsilon}{2}\operatorname{tr} h(x) + O(\epsilon^2). \tag{6.6.18}$$

Substituting formulae (6.6.18) and (6.6.13) into (6.6.17) and using the symmetry condition (6.6.11), we get

$$\begin{aligned} &*T^{\mathrm{ax}}(x;\epsilon) = \\ &\frac{\epsilon^2\delta^{kl}}{12}\left[h_{k1}\frac{\partial h_{l3}}{\partial x^2} + h_{k2}\frac{\partial h_{l1}}{\partial x^3} + h_{k3}\frac{\partial h_{l2}}{\partial x^1} - h_{k1}\frac{\partial h_{l2}}{\partial x^3} - h_{k2}\frac{\partial h_{l3}}{\partial x^1} - h_{k3}\frac{\partial h_{l1}}{\partial x^2}\right] + O(\epsilon^3) \\ &\qquad = -\frac{\epsilon^2}{12}\varepsilon_{\beta\gamma\delta}h_{\alpha\beta}\frac{\partial h_{\alpha\gamma}}{\partial x^\delta} + O(\epsilon^3)\,. \end{aligned} \tag{6.6.19}$$

Formulae (6.6.16) and (6.6.19) imply

$$A^{(1)}_{\mathrm{sub}}(x) = 0, \tag{6.6.20}$$

$$A^{(2)}_{\mathrm{sub}}(x) = -\frac{1}{16}\,\varepsilon_{\beta\gamma\delta}\,h_{\alpha\beta}\,\frac{\partial h_{\alpha\gamma}}{\partial x^\delta}\,I\,. \tag{6.6.21}$$

Substituting (6.6.20) into (6.6.7) we get $\lambda^{(1)} = 0$. Formulae (6.6.3) and (6.6.8) now simplify and read

$$\lambda(\epsilon) = c\,\epsilon^2 + O(\epsilon^3), \tag{6.6.22}$$

$$c = \lambda^{(2)} = \langle A^{(2)}_{\mathrm{sub}} v^{(0)}, v^{(0)}\rangle - \langle A^{(1)}\,Q\,A^{(1)}\,v^{(0)}, v^{(0)}\rangle\,. \tag{6.6.23}$$

In order to complete our calculation we now need only to write down the principal symbol $A_1^{(1)}(x,\xi)$ of the differential operator $A^{(1)}$.

According to formulae (4.A.1)–(4.A.3) and (4.A.19) the principal symbol of the massless Dirac operator on half-densities is

$$A_1(x,\xi;\epsilon) = \begin{pmatrix} {e_3}^\alpha & {e_1}^\alpha - i{e_2}^\alpha \\ {e_1}^\alpha + i{e_2}^\alpha & -{e_3}^\alpha \end{pmatrix} \xi_\alpha \,. \tag{6.6.24}$$

Formulae (6.6.24) and (6.6.15) imply

$$A_1^{(1)}(x,\xi) = -\frac{1}{2} \begin{pmatrix} {h_3}^\alpha & {h_1}^\alpha - i{h_2}^\alpha \\ {h_1}^\alpha + i{h_2}^\alpha & -{h_3}^\alpha \end{pmatrix} \xi_\alpha \,. \tag{6.6.25}$$

Formulae (6.6.5), (6.6.6), (6.6.20) and (6.6.25) allow us to write down the differential operator $A^{(1)}$ explicitly:

$$A^{(1)} = \frac{i}{4} \begin{pmatrix} {h_3}^\alpha & {h_1}^\alpha - i{h_2}^\alpha \\ {h_1}^\alpha + i{h_2}^\alpha & -{h_3}^\alpha \end{pmatrix} \frac{\partial}{\partial x^\alpha} + \frac{i}{4} \frac{\partial}{\partial x^\alpha} \begin{pmatrix} {h_3}^\alpha & {h_1}^\alpha - i{h_2}^\alpha \\ {h_1}^\alpha + i{h_2}^\alpha & -{h_3}^\alpha \end{pmatrix} . \tag{6.6.26}$$

Substituting formulae (6.6.1), (6.6.2), (6.6.21) and (6.6.26) into (6.6.23) we arrive at (6.2.5). This completes the proof of Theorem 6.2.1.

6.7 Axisymmetric case

An important special case is when the metric $g_{\alpha\beta}(x;\epsilon)$ is a function of the coordinate x^1 only. In this case one can choose the coframe and frame so that they depend on the coordinate x^1 only and seek eigenfunctions in the form

$$v(x) = u(x^1) e^{i(m_2 x^2 + m_3 x^3)}, \qquad m_2, m_3 \in \mathbb{Z}.$$

We get separation of variables, i.e. the original eigenvalue problem for a partial differential operator reduces to an eigenvalue problem for an ordinary differential operator depending on the integers m_2 and m_3 as parameters. As we know the spectrum of the unperturbed operator (see Section 6.1) and as the eigenvalues of the original partial differential operator depend on the small parameter ϵ continuously, the eigenvalue with smallest modulus will come from the ordinary differential operator with $m_2 = m_3 = 0$. We call the case $m_2 = m_3 = 0$ the *axisymmetric case*.

The axisymmetric massless Dirac operator on half-densities reads

$$W_{1/2}(\epsilon) = \\ -\frac{i}{2}\begin{pmatrix} e_3{}^1 & e_1{}^1 - ie_2{}^1 \\ e_1{}^1 + ie_2{}^1 & -e_3{}^1 \end{pmatrix}\frac{d}{dx^1} - \frac{i}{2}\frac{d}{dx^1}\begin{pmatrix} e_3{}^1 & e_1{}^1 - ie_2{}^1 \\ e_1{}^1 + ie_2{}^1 & -e_3{}^1 \end{pmatrix} \\ + \frac{\delta_{jk}}{4\sqrt{\det g_{\alpha\beta}}}\left[e^j{}_3\left(\frac{de^k{}_2}{dx^1}\right) - e^j{}_2\left(\frac{de^k{}_3}{dx^1}\right)\right] I\,, \quad (6.7.1)$$

where I is the 2×2 identity matrix and

$$\sqrt{\det g_{\alpha\beta}} = \frac{1}{\sqrt{\det g^{\alpha\beta}}} = \det e^j{}_\alpha = \frac{1}{\det e_j{}^\alpha}\,.$$

Here $e^j{}_\alpha(x^1;\epsilon)$ and $e_j{}^\alpha(x^1;\epsilon)$ are the coframe and frame defined in accordance with formulae (6.6.9), (6.6.10) and (6.6.14).

Of course, for a given metric $g_{\alpha\beta}(x^1;\epsilon)$ the coframe $e^j{}_\alpha(x^1;\epsilon)$ and frame $e_j{}^\alpha(x^1;\epsilon)$ are not defined uniquely. We can multiply the matrix-functions $e^j{}_\alpha(x^1;\epsilon)$ and $e_j{}^\alpha(x^1;\epsilon)$ from the left by an arbitrary smooth 3×3 special orthogonal matrix-function $O(x^1;\epsilon)$ satisfying the condition $O(x^1;0) = I$, with I denoting the 3×3 identity matrix. This will give us a new coframe and a new frame satisfying the defining relations (6.6.9), (6.6.10) and (6.6.14). Note that in writing down formula (6.7.1) we did not assume a particular choice of gauge, compare with (6.6.11).

In the axisymmetric case formula (6.2.5) also simplifies and reads now

$$c = -\frac{1}{8} \sum_{m_1 \in \mathbb{N}} m_1 \operatorname{tr} \left[\begin{pmatrix} \hat{h}_{22} & \hat{h}_{23} \\ \hat{h}_{32} & \hat{h}_{33} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \hat{h}_{22} & \hat{h}_{23} \\ \hat{h}_{32} & \hat{h}_{33} \end{pmatrix}^* \right], \tag{6.7.2}$$

where $\hat{h}_{\alpha\beta} = \hat{h}_{\alpha\beta}(m_1)$ and the star stands for Hermitian conjugation.

6.8 Example of quadratic dependence on ϵ

Consider the metric

$$g_{\alpha\beta}(x^1;\epsilon)\, dx^\alpha dx^\beta = \big[dx^1\big]^2 + \big[\big(1+\epsilon\big(\cos x^1\big)\big)dx^2 + \epsilon\big(\sin x^1\big)dx^3\big]^2 + \big[\epsilon\big(\sin x^1\big)dx^2 + \big(1-\epsilon\big(\cos x^1\big)\big)dx^3\big]^2. \tag{6.8.1}$$

Then

$$e^j{}_\alpha(x^1;\epsilon) = \delta^j{}_\alpha + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos x^1 & \sin x^1 \\ 0 & \sin x^1 & -\cos x^1 \end{pmatrix} \tag{6.8.2}$$

is a coframe associated with the metric (6.8.1), see formulae (6.6.9), (6.6.10), and

$$e_j{}^\alpha(x^1;\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-\epsilon\cos x^1}{1-\epsilon^2} & -\frac{\epsilon\sin x^1}{1-\epsilon^2} \\ 0 & -\frac{\epsilon\sin x^1}{1-\epsilon^2} & \frac{1+\epsilon\cos x^1}{1-\epsilon^2} \end{pmatrix} \tag{6.8.3}$$

is the corresponding frame, see formula (6.6.14). Note that in writing formula (6.8.3) we used the fact that

$$\det e^j{}_\alpha(x;\epsilon) = 1-\epsilon^2 = \sqrt{\det g_{\alpha\beta}}\,. \tag{6.8.4}$$

Substituting formulae (6.8.2)–(6.8.4) into (6.7.1) we get

$$W(\epsilon) = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx^1} - \frac{\epsilon^2}{2(1-\epsilon^2)} I. \tag{6.8.5}$$

Note that in the LHS we dropped the subscript $1/2$: as the Riemannian density is constant, see (6.8.4), there is no need to distinguish the massless Dirac operator $W(\epsilon)$ and the massless Dirac operator on half-densities $W_{1/2}(\epsilon)$.

It is easy to see that the eigenvalues of the ordinary differential operator (6.8.5) subject to the (boundary) condition of 2π-periodicity are

$$\lambda_n(\epsilon) = n - \frac{\epsilon^2}{2(1-\epsilon^2)}, \qquad n \in \mathbb{Z},$$

and that all eigenvalues have multiplicity two. In particular, the eigenvalue with smallest modulus is

$$\lambda_0(\epsilon) = -\frac{\epsilon^2}{2(1-\epsilon^2)} = -\frac{\epsilon^2}{2} + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0. \tag{6.8.6}$$

Let us now test Theorem 6.2.1 by comparing the asymptotic formula from this theorem with formula (6.8.6). Substituting (6.8.1) into (6.2.3) we get

$$h_{\alpha\beta}(x^1) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos x^1 & \sin x^1 \\ 0 & \sin x^1 & -\cos x^1 \end{pmatrix}.$$

Application of the Fourier transform (6.2.2) gives us

$$\hat{h}_{\alpha\beta}(m_1) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & -1 \end{pmatrix} & \text{for} \quad m_1 = 1, \\ \\ 0 & \text{for} \quad m_1 = 2, 3, \ldots. \end{cases} \tag{6.8.7}$$

Substituting (6.8.7) into (6.7.2) we get $c = -\frac{1}{2}$, in agreement with (6.8.6).

6.9 Example of quartic dependence on ϵ

Consider the metric

$$g_{\alpha\beta}(x^1;\epsilon)\,dx^\alpha dx^\beta = \big[dx^1 + \epsilon\big(\cos x^1\big)dx^2 + \epsilon\big(\sin x^1\big)dx^3\big]^2 + \big[dx^2\big]^2 + \big[dx^3\big]^2. \quad (6.9.1)$$

Then

$$e^j{}_\alpha(x^1;\epsilon) = \delta^j{}_\alpha + \epsilon \begin{pmatrix} 0 & \cos x^1 & \sin x^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (6.9.2)$$

is a coframe associated with the metric (6.9.1), see formulae (6.6.9), (6.6.10), and

$$e_j{}^\alpha(x^1;\epsilon) = \delta_j{}^\alpha - \epsilon \begin{pmatrix} 0 & 0 & 0 \\ \cos x^1 & 0 & 0 \\ \sin x^1 & 0 & 0 \end{pmatrix} \quad (6.9.3)$$

is the corresponding frame, see formula (6.6.14). Note that in writing formula (6.9.3) we used the fact that

$$\det e^j{}_\alpha(x;\epsilon) = 1 = \sqrt{\det g_{\alpha\beta}}\,. \quad (6.9.4)$$

Substituting formulae (6.9.2)–(6.9.4) into (6.7.1) we get

$$W(\epsilon) = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx^1} - \frac{\epsilon^2}{4} I + \frac{i\epsilon}{2} \begin{pmatrix} \sin x^1 & -i\cos x^1 \\ i\cos x^1 & -\sin x^1 \end{pmatrix} \frac{d}{dx^1} + \frac{i\epsilon}{2} \frac{d}{dx^1} \begin{pmatrix} \sin x^1 & -i\cos x^1 \\ i\cos x^1 & -\sin x^1 \end{pmatrix}. \quad (6.9.5)$$

Note that in the LHS we dropped the subscript $1/2$: as the Riemannian density is constant, see (6.9.4), there is no need to distinguish the massless Dirac operator $W(\epsilon)$ and the massless Dirac operator on half-densities $W_{1/2}(\epsilon)$.

We shall now rewrite the ordinary differential operator (6.9.5) in a somewhat more convenient form. To this end, let us introduce the special unitary matrix

$$R := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{6.9.6}$$

and put

$$\tilde{W}(\epsilon) := R\, W(\epsilon)\, R^* , \tag{6.9.7}$$

compare with formula (6.3.1). Clearly, the operator $\tilde{W}(\epsilon)$ has the same spectrum as the operator $W(\epsilon)$. Substituting (6.9.5) and (6.9.6) into (6.9.7) we arrive at the following explicit formula for the ordinary differential operator $\tilde{W}(\epsilon)$:

$$\begin{aligned} \tilde{W}(\epsilon) = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx^1} - \frac{\epsilon^2}{4} I \\ + \frac{i\epsilon}{2} \begin{pmatrix} 0 & -ie^{-ix^1} \\ ie^{ix^1} & 0 \end{pmatrix} \frac{d}{dx^1} + \frac{i\epsilon}{2} \frac{d}{dx^1} \begin{pmatrix} 0 & -ie^{-ix^1} \\ ie^{ix^1} & 0 \end{pmatrix} . \end{aligned} \tag{6.9.8}$$

The coefficients of the ordinary differential operator (6.9.8) are trigonometric polynomials and one would not normally expect the eigenfunctions to be trigonometric polynomials. However, the operator (6.9.8) has a special structure which ensures that the eigenfunctions are trigonometric polynomials. Namely, put

$$\lambda_n(\epsilon) = -\frac{1}{2} - \frac{\epsilon^2}{4} + \sqrt{1+\epsilon^2} \left(n + \frac{1}{2} \right) , \qquad n \in \mathbb{Z}, \tag{6.9.9}$$

$$v^{(n)}(x^1;\epsilon) = \begin{pmatrix} \left(1 + \sqrt{1+\epsilon^2}\,\right) e^{inx^1} \\ -i\,\epsilon\, e^{i(n+1)x^1} \end{pmatrix} , \qquad n \in \mathbb{Z}. \tag{6.9.10}$$

It is easy to see that the column-functions (6.9.10) are eigenfunctions of the operator (6.9.8) corresponding to eigenvalues (6.9.9). Moreover, it is easy to

see that the charge conjugates, $\mathrm{C}(v^{(n)}(x^1;\epsilon))$, of the column-functions (6.9.10) are eigenfunctions of the operator (6.9.8) corresponding to the same eigenvalues (6.9.9). This means that the numbers (6.9.9) are eigenvalues of the operator (6.9.8) of multiplicity at least two. Finally, it is easy to see that

$$\begin{aligned}\operatorname{span}\big\{v^{(n)}(x^1;\epsilon),\ \mathrm{C}(v^{(n)}(x^1;\epsilon))\ \big|\ n\in\mathbb{Z}\big\}\\ =\operatorname{span}\big\{v^{(n)}(x^1;0),\ \mathrm{C}(v^{(n)}(x^1;0))\ \big|\ n\in\mathbb{Z}\big\},\end{aligned}\quad (6.9.11)$$

where $\operatorname{span} S$ denotes the linear span, i.e. set of all finite linear combinations of elements of a given set S. Formula (6.9.11) implies that we haven't missed any eigenvalues, that is, that the list (6.9.9) contains *all* the eigenvalues of the operator (6.9.8) and that each of these eigenvalues has multiplicity two.

Remark 6.9.1. *We do not fully understand the underlying reasons why the axisymmetric massless Dirac operator corresponding to the metric (6.9.1) admits an explicit evaluation of the eigenvalues and eigenfunctions. Somehow, this particular Dirac operator has properties similar to those of an integrable system.*

The eigenvalue (6.9.9) with smallest modulus is

$$\lambda_0(\epsilon)=\frac{2\sqrt{1+\epsilon^2}-2-\epsilon^2}{4}=-\frac{\epsilon^4}{16}+O(\epsilon^6)\quad\text{as}\quad\varepsilon\to 0. \qquad (6.9.12)$$

Let us now test Theorem 6.2.1 by comparing the asymptotic formula from this theorem with formula (6.9.12). Substituting (6.9.1) into (6.2.3) we get

$$h_{\alpha\beta}(x^1)=\begin{pmatrix}0 & \cos x^1 & \sin x^1\\ \cos x^1 & 0 & 0\\ \sin x^1 & 0 & 0\end{pmatrix}.$$

Application of the Fourier transform (6.2.2) gives us

$$\hat{h}_{\alpha\beta}(m_1) = \begin{cases} \begin{pmatrix} 0 & \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{i}{2} & 0 & 0 \end{pmatrix} & \text{for} \quad m_1 = 1, \\ \\ 0 & \text{for} \quad m_1 = 2, 3, \ldots . \end{cases} \tag{6.9.13}$$

Substituting (6.9.13) into (6.7.2) we get $c = 0$, in agreement with (6.9.12).

6.10 The eta invariant

Let H be a first order self-adjoint elliptic $m \times m$ matrix classical pseudodifferential operator acting on m-columns of complex-valued half-densities over a compact n-dimensional manifold M without boundary. Here ellipticity is understood as the nonvanishing of the determinant of the principal symbol of H, see Chapter 4. The *eta function* of H is defined as

$$\eta_H(s) := \sum \frac{\operatorname{sign} \lambda}{|\lambda|^s}\,, \tag{6.10.1}$$

where summation is carried out over all nonzero eigenvalues λ of H, and $s \in \mathbb{C}$ is the independent variable. Asymptotic formulae for the counting function imply that the series (6.10.1) converges absolutely for $\operatorname{Re} s > n$ and defines a holomorphic function in this half-plane. It is known [7] that the eta function extends meromorphically to the whole s-plane. Moreover, it is known, see Theorem 4.5 in [7], that if the dimension n is odd, then the eta function is holomorphic at $s = 0$. This justifies, for odd n, the definition of the *eta invariant* as the real number $\eta_H(0)$. The eta invariant $\eta_H(0)$ is the traditional measure of spectral asymmetry of the operator H.

If we have only a finite number of eigenvalues (i.e. if we are looking at an Hermitian matrix rather than a differential operator) then the eta invariant is an integer number: it is the number of positive eigenvalues minus the number of negative eigenvalues. However, in the case of a differential operator there is no reason for the eta invariant to be integer. The basic example [4] is that of the scalar ordinary differential operator $H(\epsilon) := -i\frac{d}{dx^1} + \epsilon$ acting on the unit circle parameterized by the cyclic coordinate x^1 of period 2π, with ϵ being a real parameter. It is known [4] that the eta invariant $\eta_{H(\epsilon)}(0)$ of this ordinary differential operator is the odd 1-periodic function defined by the formula $\eta_{H(\epsilon)}(0) = 1 - 2\epsilon$ for $\epsilon \in (0, 1)$. In particular, we have $\eta_{H(0)}(0) = 0$ and $\lim\limits_{\epsilon\to 0^{\pm}} \eta_{H(\epsilon)}(0) = \pm 1$.

The current state of affairs (from an analyst's perspective) in the subject area of zeta/eta functions of elliptic operators is described in detail in the two papers [36, 37]. Let us highlight a few facts.

- The key results are Theorem 2.7 from [36] and Proposition 2.9 from [37]. Arguing along the lines of [7] one can recover from these results, in a rigorous analytic fashion, properties of the eta function.
- The eta function is holomorphic at $s = 0$ in any dimension $n \in \mathbb{N}$ (i.e. without the assumption of n being odd). This fact was proved by P. B. Gilkey [31].
- The seminal paper of R. T. Seeley [75] contained a small mistake: see page 482 in [36] or Remark 2.6 on page 39 in [37] for details.

The more recent survey papers [34, 35] provide an overview of the subject.

Let us denote our massless Dirac operator on half-densities by $A(\epsilon)$, where $\epsilon \in \mathbb{R}$ is the small parameter appearing in our metric $g_{\alpha\beta}(x;\epsilon)$. Theorem 6.2.1 implies the following corollary.

Corollary 6.10.1. *Suppose that the coefficient c defined by formula (6.2.5) is nonzero. Then*

$$\lim_{\epsilon\to 0} \eta_{A(\epsilon)}(0) = 2\operatorname{sign} c\,. \tag{6.10.2}$$

Note that we have $\eta_{A(0)}(0) = 0$, so formula (6.10.2) implies that the function $\eta_{A(\epsilon)}(0)$ is discontinuous at $\epsilon = 0$.

Proof of Corollary 6.10.1 Put $f(\epsilon, t) := \mathrm{Tr}\left[A(\epsilon)\, e^{-t(A(\epsilon))^2}\right]$, where $t > 0$ and Tr is the operator (as opposed to pointwise) trace[2]. Having fixed ϵ, let us examine the behaviour of $f(\epsilon, t)$ as $t \to 0^+$. For a generic first order pseudodifferential operator $A(\epsilon)$ we would have $f(\epsilon, t) = O(t^{-2})$. However, as explained in Chapter II of [11], the Dirac operator in odd dimensions is very special and there are a lot of cancellations when one computes the asymptotic expansion for $f(\epsilon, t)$ as $t \to 0^+$. Namely, it was shown in [11] that

- $f(\epsilon, t) = O(\sqrt{t}\,)$ as $t \to 0^+$,
- $\eta_{A(\epsilon)}(s)$ is holomorphic in the half-plane $\mathrm{Re}\, s > -2$, and
- $$\eta_{A(\epsilon)}(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^{+\infty} t^{(s-1)/2}\, f(\epsilon, t)\, dt \quad \text{for} \quad \mathrm{Re}\, s > -2\,. \tag{6.10.3}$$

See also Section 1 in [82].

Formula (6.10.3) implies

$$\eta_{A(\epsilon)}(0) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{f(\epsilon, t)}{\sqrt{t}}\, dt\,, \tag{6.10.4}$$

so in order to prove Corollary 6.10.1 we need to examine the behaviour of the integral (6.10.4) as $\epsilon \to 0$.

Let us denote by $\lambda_0(\epsilon)$ the eigenvalue of the operator $A(\epsilon)$ with smallest modulus and by $E_0(\epsilon)$ the orthogonal projection onto the corresponding 2-dimensional

[2] The paper [11] to which we are about to refer to actually deals with pointwise estimates, i.e. the trace in [11] is understood as the matrix trace of the integral kernel on the diagonal at a given point x of the manifold. We do not need pointwise estimates for the proof of Corollary 6.10.1.

eigenspace. Put

$$A_0(\epsilon) := \lambda_0(\epsilon)\, E_0(\epsilon)\,, \qquad \tilde{A}(\epsilon) := A(\epsilon) - A_0(\epsilon),$$

$$f_0(\epsilon, t) := \operatorname{Tr}\left[A_0(\epsilon)\, e^{-t(A_0(\epsilon))^2}\right] = 2\,\lambda_0(\epsilon)\, e^{-t(\lambda_0(\epsilon))^2},$$
$$\tilde{f}(\epsilon, t) := \operatorname{Tr}\left[\tilde{A}(\epsilon)\, e^{-t(\tilde{A}(\epsilon))^2}\right] = f(\epsilon, t) - f_0(\epsilon, t).$$

Then formula (6.10.4) can be rewritten as

$$\eta_{A(\epsilon)}(0) = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{f(\epsilon, t)}{\sqrt{t}}\, dt + \frac{1}{\sqrt{\pi}} \int_1^{+\infty} \frac{\tilde{f}(\epsilon, t)}{\sqrt{t}}\, dt + \frac{2}{\sqrt{\pi}} \int_1^{+\infty} \frac{\lambda_0(\epsilon)\, e^{-t(\lambda_0(\epsilon))^2}}{\sqrt{t}}\, dt\,. \quad (6.10.5)$$

The three terms in the RHS of (6.10.5) are functions of the parameter ϵ and we shall now examine how they depend on ϵ.

The first term in the RHS of (6.10.5) is continuous at $\epsilon = 0$ because asymptotic formulae for $f(\epsilon, t)$ as $t \to 0^+$ are uniform in ϵ. This follows from the construction of heat kernel type asymptotics for $t \to 0^+$: the algorithm is straightforward and examination of this algorithm shows that the asymptotic coefficients and remainder term depend on additional parameters in a continuous fashion.

The second term in the RHS of (6.10.5) is continuous at $\epsilon = 0$ because the eigenvalues of the operator $\tilde{A}(\epsilon)$ depend on ϵ continuously and because all these eigenvalues, bar one double eigenvalue, are uniformly separated from zero. The double eigenvalue in question is identically zero as a function of ϵ and does not contribute to the second term in the RHS of (6.10.5).

Thus, the proof of formula (6.10.2) reduces to the proof of the statement

$$\frac{2}{\sqrt{\pi}} \lim_{\epsilon \to 0} \int_1^{+\infty} \frac{\lambda_0(\epsilon)\, e^{-t(\lambda_0(\epsilon))^2}}{\sqrt{t}}\, dt = 2 \operatorname{sign} c\,. \quad (6.10.6)$$

But formula (6.10.6) is an immediate consequence of formula (6.2.4). □

The geometric meaning of the eta invariant of the Dirac operator acting over a compact oriented Riemannian manifold of dimension $4k-1$, $k \in \mathbb{N}$, has been extensively studied in [4, 5, 6, 7]. We are, however, unaware of publications dealing specifically with the Dirac operator on a 3-torus, though certain 2-torus bundles over a circle were examined in [3].

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