Mathematika 62 (2016) 701–718
doi:10.1112/S0025579315000509

SPECTRAL THEORETIC CHARACTERIZATION OF THE MASSLESS DIRAC ACTION

ROBERT J. DOWNES AND DMITRI VASSILIEV

Abstract. We consider an elliptic self-adjoint first-order differential operator L acting on pairs (2-columns) of complex-valued half-densities over a connected compact three-dimensional manifold without boundary. The principal symbol of the operator L is assumed to be trace-free and the subprincipal symbol is assumed to be zero. Given a positive scalar weight function, we study the weighted eigenvalue problem for the operator L. The corresponding counting function (number of eigenvalues between zero and a positive λ) is known to admit, under appropriate assumptions on periodic trajectories, a two-term asymptotic expansion as $\lambda \to +\infty$ and we have recently derived an explicit formula for the second asymptotic coefficient. The purpose of this paper is to establish the geometric meaning of the second asymptotic coefficient. To this end, we identify the geometric objects encoded within our eigenvalue problem—metric, non-vanishing spinor field and topological charge—and express our asymptotic coefficients in terms of these geometric objects. We prove that the second asymptotic coefficient of the counting function has the geometric meaning of the massless Dirac action.

§1. *Main result.* Consider a first-order differential operator L acting on 2-columns $v = (v_1 \quad v_2)^T$ of complex-valued half-densities over a connected compact three-dimensional manifold M without boundary. We assume the coefficients of the operator L to be infinitely smooth. We also assume that the operator L is formally self-adjoint (symmetric): $\int_M u^* L v \, dx = \int_M (Lu)^* v \, dx$ for all infinitely smooth $u, v : M \to \mathbb{C}^2$. Here, and further on, the superscript * in matrices, rows and columns indicates Hermitian conjugation in $\mathbb{C}^2$ and $dx := dx^1 dx^2 dx^3$, where $x = (x^1, x^2, x^3)$ are local coordinates on M.

Let $L_{\text{prin}}(x, p)$ be the principal symbol of the operator L, that is, the matrix obtained by leaving in L only the leading (first-order) derivatives and replacing each $\partial/\partial x^\alpha$ by $\mathrm{i}p_\alpha$, $\alpha = 1, 2, 3$. Here $p = (p_1, p_2, p_3)$ is the variable dual to the position variable x; in physics literature the p would be referred to as *momentum*. Our principal symbol $L_{\text{prin}}(x, p)$ is a 2×2 Hermitian matrix-function on the cotangent bundle T^*M, linear in every fibre T_x^*M (that is, linear in p).

Received 25 April 2015.
MSC (2010): 35P20 (primary), 35J46, 35R01, 35Q41 (secondary).
The second author was supported by EPSRC grant EP/M000079/1.

We assume that $\det L_{\mathrm{prin}}(x,p)\neq 0$ for all $(x,p)\in T'M:=T^*M\setminus\{p=0\}$ (cotangent bundle with the zero section removed), which is a version of the ellipticity condition.

Remark 1.1. The tradition in microlocal analysis is to denote momentum by ξ. We choose to denote it by p instead because we will need the letter ξ for the spinor.

We now make two additional assumptions:

- we assume the principal symbol to be trace-free; and
- we assume the subprincipal symbol of the operator L to be zero (see Appendix A for the definition of the subprincipal symbol).

The latter condition implies that our differential operator L is completely determined by its principal symbol. Namely, in local coordinates our operator reads

$$L=-\mathrm{i}[(L_{\mathrm{prin}})_{p_\alpha}(x)]\frac{\partial}{\partial x^\alpha}-\frac{\mathrm{i}}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x),$$

where the subscripts indicate partial derivatives and the repeated index α indicates summation over $\alpha=1,2,3$. Of course, the above formula is a special case of formula (A.5).

We study the eigenvalue problem

$$Lv=\lambda wv, \tag{1.1}$$

where $w(x)$ is a given infinitely smooth positive scalar weight function. Obviously, the problem (1.1) has the same spectrum as the problem

$$w^{-1/2}Lw^{-1/2}v=\lambda v, \tag{1.2}$$

so it may appear that the weight function $w(x)$ is redundant. We will, however, work with the eigenvalue problem (1.1) rather than with (1.2) because we want our problem to possess a gauge degree of freedom (5.1). This gauge degree of freedom will eventually manifest itself as the conformal invariance of the massless Dirac action (see §5 for details).

The problem (1.1) has a discrete spectrum accumulating to $\pm\infty$. We define the counting function $N(\lambda):=\sum_{0<\lambda_k<\lambda}1$ as the number of eigenvalues λ_k of the problem (1.1), taking account of multiplicities, between zero and a positive λ. Ref. [**3**, Theorem 8.4] states that, under appropriate assumptions on periodic trajectories, our counting function admits a two-term asymptotic expansion

$$N(\lambda)=a\lambda^3+b\lambda^2+o(\lambda^2) \tag{1.3}$$

as $\lambda\to+\infty$. If one wishes to reformulate the asymptotic formula (1.3) in such a way that it remains valid without assumptions on periodic trajectories, this can easily be achieved, say, by taking a convolution with a function from Schwartz space $\mathcal{S}(\mathbb{R})$; see [**3**, Theorem 7.2] for details.

Alternatively, one can look at the eta function $\eta(s):=\sum|\lambda|^{-s}\operatorname{sgn}\lambda$, where summation is carried out over all non-zero eigenvalues λ and $s\in\mathbb{C}$ is the independent variable. The series converges absolutely for $\operatorname{Re} s > 3$ and defines a holomorphic function in this half-plane. Moreover, it is known [1] that the eta function extends meromorphically to the whole s-plane with simple poles. Formula (10.6) from [3] implies that the eta function does not have a pole at $s = 3$ and that the residue at $s = 2$ is $4b$, where b is the coefficient from (1.3).

There is an extensive bibliography devoted to the subject of two-term spectral asymptotics for first-order systems. This bibliography spans a period of over three decades. Unfortunately, all publications prior to [3] gave formulae for the second asymptotic coefficient that were either incorrect or incomplete (that is, an algorithm for the calculation of the second asymptotic coefficient rather than an actual formula). The appropriate bibliographic review is presented in [3, §11]. The correct explicit formula for the coefficient b is given in [3, §1].

The objective of this paper is to establish the geometric meaning of the coefficient b. The logic behind restricting our analysis to the case when the manifold is three-dimensional and L is a 2×2 matrix differential operator with trace-free principal symbol and zero subprincipal symbol is that this is the simplest eigenvalue problem for a system of partial differential equations. Hence, it is ideal for the purpose of establishing the geometric meaning of the coefficient b.

In order to establish the geometric meaning of the coefficient b we first need to identify the geometric objects encoded within our eigenvalue problem (1.1).

Geometric object 1: the metric. Observe that the determinant of the principal symbol is a negative definite quadratic form in the dual variable (momentum) p,

$$\det L_{\text{prin}}(x, p) = -g^{\alpha\beta} p_\alpha p_\beta, \tag{1.4}$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, 2, 3$, appearing in (1.4) can be interpreted as the components of a (contravariant) Riemannian metric.

Geometric object 2: the non-vanishing spinor field. The determinant of the principal symbol does not determine the principal symbol uniquely. In order to identify a further geometric object encoded within the principal symbol $L_{\text{prin}}(x, p)$ we will now start varying this principal symbol, assuming the metric g, defined by formula (1.4), to be fixed (prescribed).

Let us fix a reference principal symbol $\mathring{L}_{\text{prin}}(x, p)$ corresponding to the prescribed metric g and look at all principal symbols $L_{\text{prin}}(x, p)$ which correspond to the same prescribed metric g and are sufficiently close to the reference principal symbol. Restricting our analysis to principal symbols which are close to the reference principal symbol allows us to avoid dealing with certain topological issues; this restriction will be dropped in §4. It turns out (see §2) that the principal symbols $L_{\text{prin}}(x, p)$ and $\mathring{L}_{\text{prin}}(x, p)$ are related as

$$L_{\text{prin}}(x, p) = R(x)\, \mathring{L}_{\text{prin}}(x, p)\, R^*(x), \tag{1.5}$$

where

$$R : M \to \mathrm{SU}(2) \tag{1.6}$$

is a unique infinitely smooth special unitary matrix-function which is close to the identity matrix. Thus, special unitary matrix-functions $R(x)$ provide a convenient parametrization of principal symbols with prescribed metric g.

Let $\mathring{L}$ be the differential operator with principal symbol $\mathring{L}_{\text{prin}}(x, p)$ and zero subprincipal symbol. It is important to emphasize that for the operators L and $\mathring{L}$ themselves, as opposed to their principal symbols, we have, in general, the inequality

$$L \neq R\mathring{L}R^* \tag{1.7}$$

because according to [3, formula (9.3)] the operator $R\mathring{L}R^*$ has non-trivial subprincipal symbol $(\mathrm{i}/2)(R_{x^\alpha}(\mathring{L}_{\text{prin}})_{p_\alpha}R^* - R(\mathring{L}_{\text{prin}})_{p_\alpha}R^*_{x^\alpha})$. Hence, the transformation of operators $\mathring{L} \mapsto L$ specified by formula (1.5) and the conditions that the subprincipal symbols of L and $\mathring{L}$ are zero does, in general, change the spectrum.

The choice of reference principal symbol $\mathring{L}_{\text{prin}}(x, p)$ in our construction is arbitrary, as long as this principal symbol corresponds to the prescribed metric g: that is, as long as we have $\det \mathring{L}_{\text{prin}}(x, p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta$ for all $(x, p) \in T^*M$. It is natural to ask the question: what happens if we choose a different reference principal symbol $\mathring{L}_{\text{prin}}(x, p)$? The freedom in choosing the reference principal symbol $\mathring{L}_{\text{prin}}(x, p)$ is a gauge degree of freedom in our construction and our results are invariant under changes of the reference principal symbol. This issue will be addressed in §6.

In order to work effectively with special unitary matrices, we need to choose coordinates on the three-dimensional Lie group SU(2). It is convenient to describe a 2×2 special unitary matrix by means of a spinor ξ: that is, a pair of complex numbers ξ^a, $a = 1, 2$. The relationship between a matrix $R \in \mathrm{SU}(2)$ and a non-zero spinor ξ is given by the formula

$$R = \frac{1}{\|\xi\|}\begin{pmatrix} \overline{\xi^1} & \overline{\xi^2} \\ -\xi^2 & \xi^1 \end{pmatrix}, \tag{1.8}$$

where the overline stands for complex conjugation and $\|\xi\| := \sqrt{|\xi^1|^2 + |\xi^2|^2}$.

Formula (1.8) establishes a one-to-one correspondence between SU(2) matrices and non-zero spinors, modulo a rescaling of the spinor by an arbitrary positive real factor. We choose to specify the scaling of our spinor field $\xi(x)$ in accordance with

$$\|\xi(x)\| = w(x). \tag{1.9}$$

Remark 1.2. In [4], we chose to work with a teleparallel connection (metric compatible affine connection with zero curvature) rather than with a spinor field. These are closely related objects: locally a teleparallel connection is equivalent to a normalized ($\|\xi(x)\| = 1$) spinor field modulo rigid rotations (7.3) of the latter.

Geometric object 3: the topological charge. It is known, see [**4**, §3], that the existence of a principal symbol implies that our manifold M is parallelizable. Parallelizability, in turn, implies orientability. Having chosen a particular orientation, we allow only changes of local coordinates x^α, $\alpha = 1, 2, 3$, which preserve orientation.

We define the topological charge as

$$\mathbf{c} := -\frac{\mathrm{i}}{2}\sqrt{\det g_{\alpha\beta}}\ \operatorname{tr}((L_{\mathrm{prin}})_{p_1}(L_{\mathrm{prin}})_{p_2}(L_{\mathrm{prin}})_{p_3}), \tag{1.10}$$

with the subscripts p_α indicating partial derivatives. As explained in [**4**, §3], the number $\mathbf{c}$ defined by formula (1.10) can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to the chosen orientation of local coordinates.

Formula (1.10) defines the topological charge in a purely analytic fashion. However, later we will give an equivalent definition which is more geometrical (see formula (2.11)). The frame $e_j{}^\alpha$ appearing in formula (2.11) is related to the metric as $g^{\alpha\beta} = \delta^{jk} e_j{}^\alpha e_k{}^\beta$, so it can be interpreted as the square root of the contravariant metric tensor. Hence, the topological charge can be loosely described as the sign of the determinant of the square root of the metric tensor.

We have identified three geometric objects encoded within the eigenvalue problem (1.1)—metric, non-vanishing spinor field and topological charge—defined in accordance with formulae (1.4)–(1.10). Consequently, one would expect the coefficients a and b from formula (1.3) to be expressed via these three geometric objects. This assertion is confirmed by the following theorem which is the main result of our paper.

THEOREM 1.1. *The coefficients in the two-term asymptotics (1.3) are given by the formulae*

$$a = \frac{1}{6\pi^2}\int_M \|\xi\|^3 \sqrt{\det g_{\alpha\beta}}\, dx, \tag{1.11}$$

$$b = \frac{S(\xi)}{2\pi^2}, \tag{1.12}$$

where $S(\xi)$ is the massless Dirac action (B.3) with Pauli matrices

$$\sigma^\alpha := (\mathring{L}_{\mathrm{prin}})_{p_\alpha}, \quad \alpha = 1, 2, 3. \tag{1.13}$$

Theorem 1.1 warrants the following remarks.

Firstly, recall that the $\mathring{L}$ appearing in Theorem 1.1 is our reference operator which we need to describe all possible operators L with given metric g. What happens if we take $L = \mathring{L}$? In this case formula (1.12) holds with spinor field $\xi^1(x) = w(x)$, $\xi^2(x) = 0$. This, on its own, is a non-trivial result.

Secondly, the topological charge $\mathbf{c}$ does not appear explicitly in Theorem 1.1. Nevertheless, it is implicitly present in our Pauli matrices (1.13). Indeed, formula

(1.5) implies that the integer quantity

$$-\frac{\mathrm{i}}{2}\sqrt{\det g_{\alpha\beta}}\ \operatorname{tr}((\mathring{L}_{\mathrm{prin}})_{p_1}(\mathring{L}_{\mathrm{prin}})_{p_2}(\mathring{L}_{\mathrm{prin}})_{p_3})$$

has the same value as (1.10).

Thirdly, it is tempting to apply Theorem 1.1 in the case when the operator L is itself a massless Dirac operator. This cannot be done because a massless Dirac operator acts on spinors rather than on pairs of half-densities. This impediment can be overcome by switching to a massless Dirac operator on half-densities (see formula [**4**, (A.19)]). However, we cannot take L to be a massless Dirac operator on half-densities either because, according to [**4**, Lemma 6.1], the latter has a non-trivial subprincipal symbol. Furthermore, it is known [**2**, **4**] that for the massless Dirac operator the coefficient b is zero.

Finally, Theorem 1.1 provides a fresh perspective on the history of the subject of two-term spectral asymptotics for first-order systems (see [**3**, §11] for details). Namely, Theorem 1.1 shows that, even in the simplest case, the second asymptotic coefficient for a first-order system has a highly non-trivial geometric meaning. At a formal level, the application of microlocal techniques does not require the use of advanced differential geometric concepts. However, the calculations involved are so complicated that it is hard to avoid mistakes without an understanding of the differential geometric content of the spectral problem.

It is also worth noting that we use the term "Pauli matrices" in a more general sense than in traditional quantum mechanics. The traditional definition is the one from formula (2.8) and it corresponds to flat space, whereas our definition is adapted to curved space. For us, Pauli matrices σ^α are trace-free Hermitian 2×2 matrices satisfying the identity (2.10). It might have been more appropriate to call our matrices σ^α, $\alpha = 1, 2, 3$, Pauli matrices of Riemannian metric $g^{\alpha\beta}$, but, as this expression is too long, we call them simply Pauli matrices.

The paper is organized as follows. In §2 we explain the origins of formula (1.5) and, in §3, we give the proof of Theorem 1.1. In §4, we introduce the concept of spin structure which allows us to drop the restriction that our principal symbol $L_{\mathrm{prin}}(x, p)$ is sufficiently close to the reference principal symbol $\mathring{L}_{\mathrm{prin}}(x, p)$. Finally, in §§5–7, we show that our formula (1.12) is invariant under the action of certain gauge transformations.

§2. *Spinor representation of the principal symbol.* Let $L_{\mathrm{prin}}(x, p)$ and $\mathring{L}_{\mathrm{prin}}(x, p)$ be a pair of trace-free Hermitian 2×2 principal symbols and let g be a prescribed Riemannian metric. Both $L_{\mathrm{prin}}(x, p)$ and $\mathring{L}_{\mathrm{prin}}(x, p)$ are assumed to be linear in p: that is,

$$L_{\mathrm{prin}}(x, p) = L_{\mathrm{prin}}^{(\alpha)}(x)\, p_\alpha, \tag{2.1}$$

$$\mathring{L}_{\mathrm{prin}}(x, p) = \mathring{L}_{\mathrm{prin}}^{(\alpha)}(x)\, p_\alpha, \tag{2.2}$$

where $L_{\mathrm{prin}}^{(\alpha)}(x)$ and $\mathring{L}_{\mathrm{prin}}^{(\alpha)}(x)$, $\alpha = 1, 2, 3$, are some trace-free Hermitian 2×2 matrix-functions. The assumption that our principal symbols $L_{\mathrm{prin}}(x, p)$ and

$\mathring{L}_{\mathrm{prin}}(x,p)$ are linear in p means, of course, that we are dealing with differential operators as opposed to pseudodifferential operators. The principal symbols $L_{\mathrm{prin}}(x,p)$ and $\mathring{L}_{\mathrm{prin}}(x,p)$ are assumed to satisfy

$$\det L_{\mathrm{prin}}(x,p)=\det \mathring{L}_{\mathrm{prin}}(x,p)=-g^{\alpha\beta}(x)\,p_\alpha p_\beta \tag{2.3}$$

for all $(x,p)\in T^*M$, and are also assumed to be sufficiently close in terms of the $C^\infty(M)$ topology applied to the matrix-functions $L^{(\alpha)}_{\mathrm{prin}}(x)$ and $\mathring{L}^{(\alpha)}_{\mathrm{prin}}(x)$, $\alpha=1,2,3$.

Our task in this section is to show that there exists a unique infinitely smooth special unitary matrix-function (1.6) which is close to the identity matrix and which relates the principal symbols $L_{\mathrm{prin}}(x,p)$ and $\mathring{L}_{\mathrm{prin}}(x,p)$ in accordance with formula (1.5).

We follow the convention of [**4**, **5**] in denoting the elements of the matrices $L^{(\alpha)}_{\mathrm{prin}}$ and $\mathring{L}^{(\alpha)}_{\mathrm{prin}}$ as $(L^{(\alpha)}_{\mathrm{prin}})_{\dot{a}b}$ and $(\mathring{L}^{(\alpha)}_{\mathrm{prin}})_{\dot{a}b}$, respectively, where the dotted index, running through the values $\dot{1},\dot{2}$, enumerates the rows and the non-dotted index, running through the values $1,2$, enumerates the columns.

Put

$$e_1{}^\alpha := \mathrm{Re}(L^{(\alpha)}_{\mathrm{prin}})_{\dot{1}2},\qquad e_2{}^\alpha := -\mathrm{Im}(L^{(\alpha)}_{\mathrm{prin}})_{\dot{1}2},\qquad e_3{}^\alpha := \mathrm{Re}(L^{(\alpha)}_{\mathrm{prin}})_{\dot{1}1}, \tag{2.4}$$

$$\mathring{e}_1{}^\alpha := \mathrm{Re}(\mathring{L}^{(\alpha)}_{\mathrm{prin}})_{\dot{1}2},\qquad \mathring{e}_2{}^\alpha := -\mathrm{Im}(\mathring{L}^{(\alpha)}_{\mathrm{prin}})_{\dot{1}2},\qquad \mathring{e}_3{}^\alpha := \mathrm{Re}(\mathring{L}^{(\alpha)}_{\mathrm{prin}})_{\dot{1}1}. \tag{2.5}$$

As explained in [**4**, §3], formula (2.4) defines a *frame*—a triple of infinitely smooth real orthonormal vector fields $e_j(x)$, $j=1,2,3$, on the manifold M—and, moreover, the principal symbol $L_{\mathrm{prin}}(x,p)$ is equivalent to the frame e_j in the sense that the principal symbol uniquely determines the frame and the frame uniquely determines the principal symbol. Similarly, formula (2.5) defines a frame $\mathring{e}_j$ which is equivalent to the principal symbol $\mathring{L}_{\mathrm{prin}}(x,p)$.

Condition (2.3) implies that the frames e_j and $\mathring{e}_j$ are orthonormal with respect to the same metric. Hence, the relationship between this pair of orthonormal frames is

$$e_j{}^\alpha = O_j{}^k\,\mathring{e}_k{}^\alpha, \tag{2.6}$$

where $O(x)$ is a 3×3 orthogonal matrix-function with elements

$$O_j{}^k=\delta^{kl}g_{\alpha\beta}\,e_j{}^\alpha\,\mathring{e}_l{}^\beta.$$

As we assumed the principal symbols $L_{\mathrm{prin}}(x,p)$ and $\mathring{L}_{\mathrm{prin}}(x,p)$ to be close, the frames e_j and $\mathring{e}_j$ are also close. Consequently, the matrix-function $O(x)$ is close to the identity matrix and, hence, special orthogonal.

It is well known that the Lie group SO(3) is locally (in a neighbourhood of the identity) isomorphic to the Lie group SU(2). According to [**4**, formulae (A.15) and (A.2)], a 3×3 special orthogonal matrix O is expressed via a 2×2 special unitary matrix R as

$$O_j{}^k=\tfrac12\,\mathrm{tr}(s_jRs^kR^*), \tag{2.7}$$

where

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1, \quad s^2 := \begin{pmatrix} 0 & -\mathrm{i} \\ i & 0 \end{pmatrix} = s_2, \quad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3. \tag{2.8}$$

Formula (2.7) tells us that a 3×3 special orthogonal matrix is, effectively, the square of a 2×2 special unitary matrix. Formula (2.7) provides a local diffeomorphism between neighbourhoods of the identity in SO(3) and in SU(2).

A straightforward calculation shows that formulae (2.1), (2.2) and (2.4)–(2.8) imply formula (1.5).

Let us now define Pauli matrices σ^α in accordance with formula (1.13). Of course,

$$\sigma^\alpha(x) = \mathring{L}^{(\alpha)}_{\text{prin}}(x), \quad \alpha = 1, 2, 3, \tag{2.9}$$

where the $\mathring{L}^{(\alpha)}_{\text{prin}}$ are the matrix-functions from formula (2.2). We could stick with the notation $\mathring{L}^{(\alpha)}_{\text{prin}}$, but we choose to switch to σ^α because this is how Pauli matrices are traditionally denoted in the subject.

It is easy to see that formula (2.3) implies

$$\sigma^\alpha \sigma^\beta + \sigma^\beta \sigma^\alpha = 2I g^{\alpha\beta}, \tag{2.10}$$

where I is the 2×2 identity matrix. Formula (2.10) means that our σ^α satisfy the defining relation for Pauli matrices.

Formulae (2.6)–(2.8), (1.8), (2.5) and (2.9) allow us to express the frame e_j via the spinor field ξ and Pauli matrices σ^α. We took great care to choose coordinates on the Lie group SU(2) (that is, the structure of the matrix in the right-hand side of formula (1.8)) so that the resulting expressions agree with formulae [**5**, (B.3), (B.4) and (B.1)]. The only difference is in notation: the ϑ^j in [**5**, Appendix B] stands for $\vartheta^j{}_\alpha = \delta^{jk} g_{\alpha\beta}\, e_k{}^\beta$ (compare with formula (3.4)).

The fact that our construction agrees with that in [**5**] will become important in the next section when we will make use of a particular formula from [**5**].

Remark 2.1. As explained in [**4**, §3], the topological charge, initially defined in accordance with formula (1.10), can be equivalently rewritten in terms of frames as

$$\mathbf{c} = \operatorname{sgn}\det e_j{}^\alpha = \operatorname{sgn}\det \mathring{e}_j{}^\alpha. \tag{2.11}$$

The paper [**5**] was written under the assumption that

$$\mathbf{c} = +1 \tag{2.12}$$

(see formula [**5**, (A.1)]). This means that care is required when using the results of [**5**]. Namely, in the next section we will first prove Theorem 1.1 for the case (2.12) and then provide a separate argument explaining why formula (1.12) remains true in the case

$$\mathbf{c} = -1. \tag{2.13}$$

§3. *Proof of Theorem 1.1.* We prove Theorem 1.1 by examining the equivalent spectral problem (1.2). Note that transition from (1.1) to (1.2) is a special case of the gauge transformation (5.1) with $\varphi = \ln w$. As explained in the beginning of §5, the transformation (5.1) preserves the structure of our eigenvalue problem: the principal symbol of the operator $w^{-1/2}Lw^{-1/2}$ is trace-free and its subprincipal symbol is zero.

We now apply [**4**, Theorem 1.1] to the eigenvalue problem (1.2).

Our formula (1.11) is an immediate consequence of [**4**, formula (1.18)] and our formulae (1.4) and (1.9). Here, of course, we use the fact that we are working in dimension three.

The proof of formula (1.12) is more delicate, so we initially consider the case

$$w(x) = 1 \quad \text{for all } x \in M. \tag{3.1}$$

In this case, according to [**4**, formulae (1.19) and (8.1)],

$$b = \frac{3\mathbf{c}}{8\pi^2} \int_M *T^{\mathrm{ax}} \sqrt{\det g_{\alpha\beta}}\, dx, \tag{3.2}$$

where

$$\begin{aligned} *T^{\mathrm{ax}} = \frac{\delta_{kl}}{3} \sqrt{\det g^{\alpha\beta}}\, [&{e^k}_1\, \partial {e^l}_3/\partial x^2 + {e^k}_2\, \partial {e^l}_1/\partial x^3 + {e^k}_3\, \partial {e^l}_2/\partial x^1 \\ &- {e^k}_1\, \partial {e^l}_2/\partial x^3 - {e^k}_2\, \partial {e^l}_3/\partial x^1 - {e^k}_3\, \partial {e^l}_1/\partial x^2], \end{aligned} \tag{3.3}$$

$${e^j}_\alpha = \delta^{jk} g_{\alpha\beta}\, {e_k}^\beta. \tag{3.4}$$

The real scalar field $*T^{\mathrm{ax}}(x)$ has the geometric meaning of the Hodge dual of axial torsion of the teleparallel connection (see [**4**] for details).

Let us now drop the assumption (3.1).

The introduction of a weight function is equivalent to a scaling of the principal symbol $L_{\mathrm{prin}}(x,p) \mapsto (w(x))^{-1} L_{\mathrm{prin}}(x,p)$, which, in view of formulae (2.4) and (1.4), leads to a scaling of the frame

$$e_j \mapsto w^{-1} e_j \tag{3.5}$$

and corresponding scaling of the metric

$$g^{\alpha\beta} \mapsto w^{-2} g^{\alpha\beta}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) and (3.3) we see that the integrand in formula (3.2) scales as

$$*T^{\mathrm{ax}} \sqrt{\det g_{\alpha\beta}} \;\mapsto\; w^2 *T^{\mathrm{ax}} \sqrt{\det g_{\alpha\beta}}\,. \tag{3.7}$$

Here the remarkable fact is that we do not get derivatives of the weight function because these cancel out due to the antisymmetric structure of the right-hand side of formula (3.3). In other words, axial torsion, defined by [**4**, formulae (1.20) and

(3.12)], has the remarkable property that it scales in a covariant manner under scaling of the frame. Note that the full torsion tensor, defined by [**4**, formula (3.12)], does not possess such a covariance property.

Formula (3.7) tells us that, in order to accommodate an arbitrary weight function $w(x)$, we need to multiply the integrand in formula (3.2) by $(w(x))^2$, which gives

$$b = \frac{3\mathbf{c}}{8\pi^2} \int_M w^2 *T^{\mathrm{ax}} \sqrt{\det g_{\alpha\beta}}\, dx. \tag{3.8}$$

Let us emphasize that the metric and torsion appearing in formula (3.8) are the original, unscaled metric and torsion determined by the original, unscaled principal symbol $L_{\mathrm{prin}}(x,p)$. The scaling argument has been incorporated into the factor $(w(x))^2$.

We now need to express the integrand in (3.8) in terms of the spinor field ξ.

We already have an expression for the weight function in terms of the spinor field (see formula (1.9)). So we only need to express the Hodge dual of axial torsion in terms of the spinor field. Formulae (2.4), (2.1), (2.2), (2.9), (1.5) and (1.8) allow us to express the frame e_j via the spinor field ξ and Pauli matrices σ^α. Hence one needs to combine all these formulae to get explicit expressions for the vector fields e_j, $j = 1, 2, 3$, and substitute these into (3.4) and (3.3). This is a massive calculation. Fortunately, for the case (2.12), this calculation was carried out in [**5**, Appendix B]: [**5**, formula (B.5)] reads

$$*T^{\mathrm{ax}} = \frac{4\,\mathrm{Re}(\xi^* W \xi)}{3\|\xi\|^2}, \tag{3.9}$$

where W is the massless Dirac operator (B.1).

Formulae (3.8), (1.9), (3.9) and (B.3) imply formula (1.12). This completes the proof of Theorem 1.1 for the case (2.12).

In order to prove formula (1.12) for the case (2.13), we invert coordinates ($x^\alpha \mapsto -x^\alpha$), which changes the sign of topological charge and allows us to use formula (1.12). We then invert coordinates again and use the facts that:

- the integrand of the massless Dirac action (B.3) is invariant under inversion of coordinates; and
- our spinor field ξ defined by formulae (1.5)–(1.9) is an anholonomic object, that is, it does not depend on the choice of coordinates.

§4. *Spin structure.* In stating our results in §1 we assumed the principal symbols $L_{\mathrm{prin}}(x,p)$ and $\mathring{L}_{\mathrm{prin}}(x,p)$ to be sufficiently close. This was done in order to ensure that equation (1.5) could be resolved with respect to the special unitary matrix-function $R(x)$. The restriction of closeness of principal symbols can be overcome by means of the introduction of the concept of spin structure.

Definition 4.1. We say that the principal symbols $L_{\mathrm{prin}}(x,p)$ and $\mathring{L}_{\mathrm{prin}}(x,p)$ have the same spin structure if there exists an infinitely smooth special unitary matrix-function (1.6) such that we have (1.5).

Remark 4.1. Principal symbols with the same spin structure form an equivalence class.

The closeness of the principal symbols $L_{\text{prin}}(x, p)$ and $\mathring{L}_{\text{prin}}(x, p)$ was never used in the proof of Theorem 1.1. All that is needed for Theorem 1.1 to be true is for the principal symbols $L_{\text{prin}}(x, p)$ and $\mathring{L}_{\text{prin}}(x, p)$ to have the same spin structure, that is, belong to the same equivalence class.

Hence, it would have been more logical to identify the spin structure as a separate geometric object from the very start, in §1, and avoid arguments relying on the closeness of the principal symbols. We chose not to proceed along this route in order to make the exposition in §1 as simple and clear as possible.

The only difference between the "local" setting (the principal symbols $L_{\text{prin}}(x, p)$ and $\mathring{L}_{\text{prin}}(x, p)$ are assumed to be close) and the "global" setting (the principal symbols $L_{\text{prin}}(x, p)$ and $\mathring{L}_{\text{prin}}(x, p)$ are assumed to have the same spin structure) is that we can no longer claim that the special unitary matrix-function $R(x)$ appearing in formula (1.5) is unique. In the "local" setting, uniqueness was achieved by requiring $R(x)$ to be close to the identity matrix, whereas, in the "global" setting, $R(x)$ is defined modulo sign (not surprising as SU(2) is the double cover of SO(3)). This sign indeterminacy does not affect formula (1.12) because the massless Dirac action is quadratic in the spinor field.

The number of different spin structures (that is, the number of equivalence classes of principal symbols) depends on the topology of the manifold. Say, the torus $\mathbb{T}^3$ admits eight different spin structures, whereas the sphere $\mathbb{S}^3$ admits a unique spin structure. See [**4**, Appendices A and B] and further bibliographic references therein for more details.

It may seem that our Definition 4.1 is different from the definition of spin structure in differential geometric literature. Indeed, differential geometers do not operate with concepts such as the principal symbol, using frames instead. However, it has been shown in [**4**, §3] that a principal symbol is equivalent to a frame, so our "microlocal" definition of spin structure can be easily recast in terms of frames, bringing it into agreement with the traditional differential geometric one.

Here it is important to emphasize that we do not claim to have redefined the notion of spin structure for the most general case. We work in the very specific setting of dimension three.

§5. *Conformal invariance.* Let us transform the differential operator L and weight $w(x)$ as

$$L \mapsto e^{-\varphi/2} L e^{-\varphi/2}, \qquad w \mapsto e^{-\varphi} w, \tag{5.1}$$

where $\varphi : M \to \mathbb{R}$ is an arbitrary infinitely smooth real-valued scalar function. The transformation (5.1) does not change the spectrum of our eigenvalue problem (1.1) and, moreover, preserves its structure: the principal symbol remains trace-free and the subprincipal symbol remains zero. The fact that the subprincipal symbol remains zero is established by using formula [**3**, (9.3)].

Here, of course, it is important that we are conjugating the operator by a real-valued scalar function $e^{-\varphi/2}$ rather than a complex-valued matrix-function R.

In this section we examine how the gauge transformation (5.1) works its way into scalings of the metric, Pauli matrices and spinor field.

Formulae (1.4) and (5.1) imply that the metric transforms as

$$g^{\alpha\beta} \mapsto e^{-2\varphi} g^{\alpha\beta}, \tag{5.2}$$

which means that we are looking at a conformal scaling of the metric.

We scale the reference principal symbol $\mathring{L}_{\text{prin}}(x, p)$ in the same way as the principal symbol $L_{\text{prin}}(x, p)$: that is, as

$$\mathring{L}_{\text{prin}} \mapsto e^{-\varphi} \mathring{L}_{\text{prin}}, \tag{5.3}$$

because, this way, we maintain the condition (2.3). Formulae (1.13) and (5.3) imply that the Pauli matrices scale as

$$\sigma^\alpha \mapsto e^{-\varphi} \sigma^\alpha. \tag{5.4}$$

Of course, the scaling of Pauli matrices (5.4) agrees with the scaling of the metric (5.2) in the sense that the scaled Pauli matrices and metric satisfy the identity (2.10).

Formulae (1.9) and (5.1) imply that the spinor field scales as

$$\xi \mapsto e^{-\varphi} \xi. \tag{5.5}$$

Let us now examine what happens to the massless Dirac action (B.3) under the transformations (5.2), (5.4) and (5.5).

We first look at the expression $W\xi$. Examination of formulae (B.1) and (B.2) shows that the expression $W\xi$ transforms as

$$W\xi \mapsto e^{-2\varphi} W\xi. \tag{5.6}$$

We see that the expression $W\xi$ scales in a covariant way, with "covariant" meaning that the derivatives of φ do not appear in the right-hand side of (5.6). Of course, the covariance of the massless Dirac operator under conformal scaling of the metric is a known differential geometric fact (see [7, Theorem 4.3]).

Formulae (5.5) and (5.6) imply that

$$\operatorname{Re}(\xi^* W\xi) \mapsto e^{-3\varphi} \operatorname{Re}(\xi^* W\xi). \tag{5.7}$$

Formula (5.2) implies that $g_{\alpha\beta} \mapsto e^{2\varphi} g_{\alpha\beta}$ and, as we are working in dimension three, this, in turn, implies that the Riemannian density scales as

$$\sqrt{\det g_{\alpha\beta}} \mapsto e^{3\varphi} \sqrt{\det g_{\alpha\beta}}. \tag{5.8}$$

Substituting formulae (5.7) and (5.8) into formula (B.3), we see that our massless Dirac action is invariant under the transformations (5.2), (5.4) and (5.5). This is, of course, in agreement with Theorem 1.1: as the gauge transformation (5.1) does not change the spectrum of our eigenvalue problem (1.1), it does not change the second asymptotic coefficient (1.12) of the counting function.

§6. SU(2) *invariance.* Let us transform the reference principal symbol $\mathring{L}_{\mathrm{prin}}(x,p)$ as

$$\mathring{L}_{\mathrm{prin}} \mapsto Q\mathring{L}_{\mathrm{prin}}Q^*, \tag{6.1}$$

where $Q : M \to \mathrm{SU}(2)$ is an arbitrary infinitely smooth special unitary matrix-function. Formulae (1.13) and (6.1) imply that

$$\sigma^\alpha \mapsto Q\sigma^\alpha Q^*. \tag{6.2}$$

Also, formulae (1.5) and (6.1) imply that $R \mapsto RQ^*$, which can be, equivalently, rewritten as

$$R^* \mapsto QR^*. \tag{6.3}$$

Examining the structure of the matrix R (see formula (1.8)), we conclude that formula (6.3) is equivalent to the linear transformation of the spinor field

$$\xi \mapsto Q\xi. \tag{6.4}$$

Formulae (6.2), (6.4) and Property 4 from Appendix A of [**4**] tell us that our massless Dirac action is invariant under the transformation (6.1). This is, of course, in agreement with Theorem 1.1: the choice of reference principal symbol does not affect the spectrum of our eigenvalue problem (1.1), and hence it does not affect the second asymptotic coefficient (1.12) of the counting function.

§7. *Invariance under rigid rotations.* Let us transform the differential operator L as

$$L \mapsto \mathbf{Q}L\mathbf{Q}^*, \tag{7.1}$$

where $\mathbf{Q} = \begin{pmatrix}\mathbf{Q}_{11} & \mathbf{Q}_{12}\\ \mathbf{Q}_{21} & \mathbf{Q}_{22}\end{pmatrix}$ is a *constant* special unitary matrix. The transformation (7.1) does not change the spectrum of our eigenvalue problem (1.1) and, moreover, preserves its structure: the principal symbol remains trace-free and the subprincipal symbol remains zero. We refer to the transformation (7.1) as a *rigid rotation* because it describes a rotation of the frame (2.4), with this rotation being the same at all points of the manifold M.

The transformation (7.1) is equivalent to the following transformation of the special unitary matrix-function $R(x)$ appearing in formula (1.5): that is,

$$R \mapsto \mathbf{Q}R. \tag{7.2}$$

Formulae (1.8) and (1.9) give us a one-to-one correspondence between special unitary matrix-functions and weight functions on the one hand and non-vanishing spinor fields on the other. In terms of the spinor field, the transformation (7.2) reads

$$\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} \mapsto \begin{pmatrix}\mathbf{Q}_{21}\overline{\xi^2} + \mathbf{Q}_{22}\xi^1\\ -\mathbf{Q}_{21}\overline{\xi^1} + \mathbf{Q}_{22}\xi^2\end{pmatrix}. \tag{7.3}$$

Note that, unlike (6.4), this transformation is *not* linear because of the complex conjugation. The transformation (7.3) can be written as a sum of linear and antilinear transformations: that is,

$$\xi \mapsto \mathbf{Q}_{22}\xi - \mathbf{Q}_{21}\mathrm{C}(\xi) \tag{7.4}$$

where $\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix} \mapsto \mathrm{C}(\xi) := \begin{pmatrix}-\overline{\xi^2}\\ \overline{\xi^1}\end{pmatrix}$ is the charge conjugation operator (see formula [4, (A.9)]).

Let us show, by performing explicit calculations, that the massless Dirac action (B.3) is invariant under the transformation (7.4). Using the fact that the massless Dirac operator commutes with the charge conjugation operator (see Property 3 in [4, Appendix A]),

$$\begin{aligned}
&(\mathbf{Q}_{22}\xi - \mathbf{Q}_{21}\mathrm{C}(\xi))^*\, W\, (\mathbf{Q}_{22}\xi - \mathbf{Q}_{21}\mathrm{C}(\xi))\\
&\quad = |\mathbf{Q}_{22}|^2\, \xi^* W\xi + |\mathbf{Q}_{21}|^2\, (\mathrm{C}(\xi))^*\mathrm{C}(W\xi)\\
&\qquad - \overline{\mathbf{Q}_{22}}\mathbf{Q}_{21}\xi^*\mathrm{C}(W\xi) - \mathbf{Q}_{22}\overline{\mathbf{Q}_{21}}(\mathrm{C}(\xi))^* W\xi\\
&\quad = |\mathbf{Q}_{22}|^2\, \xi^* W\xi + |\mathbf{Q}_{21}|^2\, \overline{\xi^* W\xi}\\
&\qquad + \{\overline{\mathbf{Q}_{22}}\mathbf{Q}_{21}(\mathrm{C}(\xi))^T\, \overline{W\xi} - \mathbf{Q}_{22}\overline{\mathbf{Q}_{21}}(\mathrm{C}(\xi))^* W\xi\}.
\end{aligned}$$

But the expression in the curly brackets is purely imaginary, so

$$\begin{aligned}
&\mathrm{Re}[(\mathbf{Q}_{22}\xi - \mathbf{Q}_{21}\mathrm{C}(\xi))^*\, W\, (\mathbf{Q}_{22}\xi - \mathbf{Q}_{21}\mathrm{C}(\xi))]\\
&\quad = |\mathbf{Q}_{22}|^2\, \mathrm{Re}[\xi^* W\xi] + |\mathbf{Q}_{21}|^2\, \mathrm{Re}[\,\overline{\xi^* W\xi}\,]\\
&\quad = (|\mathbf{Q}_{22}|^2 + |\mathbf{Q}_{21}|^2)\, \mathrm{Re}[\xi^* W\xi] = \mathrm{Re}[\xi^* W\xi].
\end{aligned}$$

A. Appendix. Invariant analytic description of a first-order differential operator. Let L be a formally self-adjoint first-order linear differential operator acting on m-columns $v = \begin{pmatrix}v_1 \dots v_m\end{pmatrix}^T$ of complex-valued half-densities over a connected n-dimensional manifold M without boundary.

In local coordinates $x = (x^1, \dots, x^n)$ our operator reads

$$L = P^\alpha(x)\frac{\partial}{\partial x^\alpha} + Q(x), \tag{A.1}$$

where $P^\alpha(x)$ and $Q(x)$ are some $m \times m$ matrix-functions and summation is carried out over $\alpha = 1, \dots, n$. The full symbol of the operator L is the matrix-function

$$L(x, p) := \mathrm{i}P^\alpha(x)\, p_\alpha + Q(x), \tag{A.2}$$

where $p = (p_1, \dots, p_n)$ is the dual variable (momentum).

The problem with the full symbol (A.2) is that it is not invariant under changes of local coordinates. The standard analytic way of overcoming this problem is by introducing the principal and subprincipal symbols in accordance with formulae

$$L_{\text{prin}}(x,p) := \mathrm{i}P^\alpha(x)\,p_\alpha, \tag{A.3}$$

$$L_{\text{sub}}(x) := Q(x) + \frac{\mathrm{i}}{2}(L_{\text{prin}})_{x^\alpha p_\alpha}(x), \tag{A.4}$$

where the subscripts indicate partial derivatives. It is known that L_{prin} and L_{sub} are invariantly defined matrix-functions on T^*M and M, respectively (see [**9**, §2.1.3] for details). As we assumed our operator L to be formally self-adjoint, the matrix-functions L_{prin} and L_{sub} are Hermitian.

The definition of the subprincipal symbol (A.4) originates from the classical paper Duistermaat and Hörmander [**6**] (see formula (5.2.8) in that paper). Unlike [**6**], we work with matrix-valued symbols, but this does not affect the formal definition of the subprincipal symbol.

A peculiar feature of first-order differential operators, as opposed to pseudodifferential operators and higher-order differential operators, is that the principal and subprincipal symbols uniquely determine the operator. Namely, examination of formulae (A.1), (A.3) and (A.4) gives, in local coordinates, the following expression for the operator in terms of its principal and subprincipal symbols: that is,

$$L = -\mathrm{i}[(L_{\text{prin}})_{p_\alpha}(x)]\frac{\partial}{\partial x^\alpha} - \frac{\mathrm{i}}{2}(L_{\text{prin}})_{x^\alpha p_\alpha}(x) + L_{\text{sub}}(x). \tag{A.5}$$

B. Appendix. Massless Dirac action. In this appendix, we define, in a concise manner, the massless Dirac action. For more details see [**4**, Appendix A].

In order to write down the massless Dirac action we need Pauli matrices, that is, a triple of trace-free Hermitian 2×2 matrix-functions $\sigma^\alpha(x)$, $\alpha = 1,2,3$, satisfying the condition (2.10). In our case, we have Pauli matrices $\sigma^\alpha(x)$ readily available: these are defined in accordance with formula (1.13), or, equivalently, in accordance with formulae (2.9) and (2.2). Covariant Pauli matrices are defined as $\sigma_\alpha := g_{\alpha\beta}\sigma^\beta$.

The massless Dirac operator is the matrix operator

$$W := -\mathrm{i}\sigma^\alpha\left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta\left(\frac{\partial\sigma^\beta}{\partial x^\alpha} + \left\{\begin{matrix}\beta\\ \alpha\gamma\end{matrix}\right\}\sigma^\gamma\right)\right), \tag{B.1}$$

where summation is carried out over $\alpha,\beta,\gamma = 1,2,3$, and

$$\left\{\begin{matrix}\beta\\ \alpha\gamma\end{matrix}\right\} := \frac{1}{2}g^{\beta\delta}\left(\frac{\partial g_{\gamma\delta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\delta}}{\partial x^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x^\delta}\right) \tag{B.2}$$

are the Christoffel symbols. The operator (B.1) acts on a 2-component complex-valued spinor field ξ, which we write as a 2-column, $\xi = \begin{pmatrix}\xi^1 & \xi^2\end{pmatrix}^T$.

We chose the letter "W" to denote the massless Dirac operator because in theoretical physics literature it is often referred to as the *Weyl* operator. Note that one should really be referring here to the *static* Weyl operator because we have excluded time, which is natural in the setting of spectral theory.

We define the massless Dirac action as

$$S(\xi) := \int_M \operatorname{Re}(\xi^* W \xi)\,\sqrt{\det g_{\alpha\beta}}\,dx, \tag{B.3}$$

where the star indicates Hermitian conjugation. This is the variational functional corresponding to the operator (B.1). Here, of course, we use the fact that, in view of the self-adjointness of the operator W,

$$\begin{aligned}\int_M \xi^*(W\xi)\,\sqrt{\det g_{\alpha\beta}}\,dx &= \int_M (W\xi)^*\xi\,\sqrt{\det g_{\alpha\beta}}\,dx\\ &= \int_M \operatorname{Re}(\xi^* W\xi)\,\sqrt{\det g_{\alpha\beta}}\,dx.\end{aligned}$$

C. Appendix. Example. In this appendix we consider an explicit example illustrating the use of Theorem 1.1.

Consider the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π. Let L be the differential operator with principal symbol

$$L_{\mathrm{prin}}(x,p) = \begin{pmatrix} p_3 & e^{2ix^3}(p_1 - ip_2)\\ e^{-2ix^3}(p_1 + ip_2) & -p_3\end{pmatrix} \tag{C.1}$$

and zero subprincipal symbol. Below, we examine the eigenvalue problem (1.1) for this particular operator L and trivial weight function (3.1).

Substituting (C.1) into (1.4) we see that the above principal symbol generates the Euclidean metric

$$g^{\alpha\beta}(x) = \delta^{\alpha\beta}. \tag{C.2}$$

Hence, as the reference principal symbol, it is natural to take

$$\mathring{L}_{\mathrm{prin}}(x,p) = \begin{pmatrix} p_3 & p_1 - ip_2\\ p_1 + ip_2 & -p_3\end{pmatrix}. \tag{C.3}$$

Substituting (C.3) into (1.13), we get standard Pauli matrices

$$\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{C.4}$$

It is not *a priori* obvious that the principal symbols $L_{\mathrm{prin}}(x,p)$ and $\mathring{L}_{\mathrm{prin}}(x,p)$ have the same spin structure. The only way to establish that they do indeed have the same spin structure is to resolve equation (1.5) with respect to the special unitary matrix-function $R(x)$. Straightforward calculations give

$$R(x) = \pm\begin{pmatrix} e^{ix^3} & 0\\ 0 & e^{-ix^3}\end{pmatrix}. \tag{C.5}$$

Of course, the underlying reasons why, in this particular case, we do not encounter topological obstructions are that both principal symbols have the same (positive) topological charge and that the frame encoded in (C.1) makes an even number of turns (two turns) as x^3 runs from zero to 2π. See [4, Appendix A] for more details.

Formulae (1.8), (1.9), (3.1) and (C.5) give the expression for the spinor field as

$$\xi(x) = \pm \begin{pmatrix} e^{-ix^3} \\ 0 \end{pmatrix}. \tag{C.6}$$

Substituting formulae (C.2), (C.4) and (C.6) into (B.1)–(B.3) we conclude that $S(\xi) = -(2\pi)^3$. Hence, Theorem 1.1 tells us that, in our example, the two-term asymptotics (1.3) takes the form

$$N(\lambda) = \tfrac{4}{3}\pi\lambda^3 - 4\pi\lambda^2 + o(\lambda^2) \tag{C.7}$$

as $\lambda \to +\infty$. Note that the non-periodicity condition (see [3, Definitions 8.3 and 8.4]) is fulfilled in our example, so, according to [3, Theorem 8.4], the asymptotic formula (C.7) holds as it is, without mollification.

Now observe that, in our example, the spectrum of the operator L can be evaluated explicitly. Indeed, let $\mathring{L}$ be the differential operator with principal symbol (C.3) and zero subprincipal symbol. In other words, let $\mathring{L} = \mathring{L}_{\mathrm{prin}}(x, -i\partial/\partial x)$. Then consider the operator $R\mathring{L}R^*$, where R is the matrix-function (C.5). It is easy to check that the subprincipal symbol of the operator $R\mathring{L}R^*$ is $-I$, where I is the 2×2 identity matrix. Hence,

$$L = R\mathring{L}R^* + I \tag{C.8}$$

(compare with formula (1.7)). But the operator $R\mathring{L}R^*$ is unitarily equivalent to the operator $\mathring{L}$ and the spectrum of $\mathring{L}$ is known (see [4, Appendix B]). Using (C.8), we conclude that the eigenvalues of our operator L are as follows.

- The number 1 is an eigenvalue of multiplicity two.
- For each $m \in \mathbb{Z}^3\backslash\{0\}$ we have the eigenvalue $1 + \|m\|$ and a unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.
- For each $m \in \mathbb{Z}^3\backslash\{0\}$ we have the eigenvalue $1 - \|m\|$ and a unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

Thus, $N(\lambda) - 1$ is the number of integer lattice points inside a 2-sphere of radius $\lambda - 1$ in $\mathbb{R}^3$ centred at the origin. Applying the result from [8] we get

$$N(\lambda) = \tfrac{4}{3}\pi\lambda^3 - 4\pi\lambda^2 + O_\varepsilon(\lambda^{21/16+\varepsilon}) \tag{C.9}$$

as $\lambda \to +\infty$, with ε being an arbitrary positive number. The more advanced number theoretic result (C.9) agrees with our asymptotic formula (C.7).

Note that the calculations presented in this section remain unchanged if we replace everywhere $p_1 \mp ip_2$ by $p_1 \pm ip_2$. This is in agreement with the fact that the topological charge $\mathbf{c}$ does not appear in our formula (1.12).

References

1. M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry III. *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 71–99.

2. J.-M. Bismut and D. S. Freed, The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem. *Commun. Math. Phys.* **107** (1986), 103–163.

3. O. Chervova, R. J. Downes and D. Vassiliev, The spectral function of a first order elliptic system. *J. Spectr. Theory* **3** (2013), 317–360.

4. O. Chervova, R. J. Downes and D. Vassiliev, Spectral theoretic characterization of the massless Dirac operator. *J. Lond. Math. Soc.* (2) **89** (2014), 301–320.

5. O. Chervova and D. Vassiliev, The stationary Weyl equation and Cosserat elasticity. *J. Phys. A* **43** (2010), 335203.

6. J. J. Duistermaat and L. Hörmander, Fourier integral operators II. *Acta Math.* **128** (1972), 183–269.

7. L. Erdős and J. P. Solovej, The kernel of Dirac operators on $\mathbb{S}^3$ and $\mathbb{R}^3$. *Rev. Math. Phys.* **13** (2001), 1247–1280.

8. D. R. Heath-Brown, Lattice points in the sphere. In *Number Theory in Progress: Proceedings of the International Conference on Number Theory held in Honor of the 60th Birthday of Andrzej Schinzel*, (eds K. Győry, H. Iwaniec and J. Urbanowicz), Walter de Gruyter (Berlin, 1999), 883–892.

9. Yu. Safarov and D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, American Mathematical Society (Providence, RI, 1997).

Robert J. Downes,
Department of War Studies,
King's College London,
Strand,
London WC2R 2LS,
U.K.
E-mail: robert.downes@kcl.ac.uk

Dmitri Vassiliev,
Department of Mathematics,
University College London,
Gower Street,
London WC1E 6BT,
U.K.
E-mail: D.Vassiliev@ucl.ac.uk
http://www.homepages.ucl.ac.uk/~ucahdva/