

# All Admissible Meromorphic Solutions of Hayman's Equation

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We find all nonrational meromorphic solutions of the equation  $ww'' - (w')^2 = \alpha(z)w + \beta(z)w' + \gamma(z)$ , where  $\alpha$ ,  $\beta$ , and  $\gamma$  are rational functions of  $z$ . In so doing, we answer a question of Hayman by showing that all such solutions have finite order. Apart from special choices of the coefficient functions, the general solution is not meromorphic and contains movable branch points. For some choices for the coefficient functions, the equation admits a one-parameter family of nonrational meromorphic solutions. Nevanlinna theory is used to show that all such solutions have been found and allows us to avoid issues that can arise from the fact that resonances can occur at arbitrarily high orders. We actually solve the more general problem of finding all meromorphic solutions that are admissible in the sense of Nevanlinna theory, where the coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$  are meromorphic functions.

## 1 Introduction

Local series methods often provide strong necessary conditions for the general solution of an ordinary differential equation (ODE) to have a meromorphic general solution. The existence of a meromorphic general solution (or more generally, that an ODE has the Painlevé property, see, e.g., [1]) is often used as a way to identify equations that are integrable, that is, in some sense exactly solvable. We wish to extend this idea to that

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of finding all sufficiently complicated meromorphic solutions of an ODE, even when the general solution is not meromorphic. We are effectively using singularity structure to find integrable sectors of the solution space of the equation under consideration.

In this paper, we will find all *admissible* meromorphic solutions of the differential equation

$$ww'' - w'^2 = \alpha(z)w + \beta(z)w' + \gamma(z), \tag{1}$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are meromorphic functions. Heuristically a meromorphic solution is admissible if it is more complicated than the coefficients that appear in the equation. In particular, if the coefficients are rational functions, then any transcendental (i.e., non-rational) meromorphic solution is admissible. If the coefficients are constants, then any nonconstant meromorphic solution is admissible. The precise definition of an admissible meromorphic solution  $w$  of Equation (1) is that  $w$  satisfies

$$T(r, \alpha) + T(r, \beta) + T(r, \gamma) = S(r, w), \tag{2}$$

where  $T$  is the Nevanlinna characteristic and  $S(r, w)$  is used to denote any function of  $r$  that is  $o(T(r, w))$  as  $r \rightarrow \infty$  outside of some possible exceptional set of finite linear measure.

The main result of this paper is the following.

**Theorem 1.1.** Suppose that  $w$  is a meromorphic solution of Equation (1), where the meromorphic coefficients  $\alpha(z)$ ,  $\beta(z)$ , and  $\gamma(z)$  satisfy Equation (2). Then,  $w$  is one of the solutions described in the following list, where  $c_1$  and  $c_2$  are constants.

- (1) If  $\beta \equiv \gamma \equiv 0$  and  $k_1 = \alpha \neq 0$  is a constant, then  $w = \frac{k_1}{c_1} \{1 + \cosh(c_1 z + c_2)\}$  or  $w = -\frac{k_1}{2} (z + c_2)^2$ .
- (2) If  $\gamma \equiv 0$ ,  $\beta \neq 0$  and  $k_1 = -\alpha/\beta$  is a constant, then  $w(z) = c_1 e^{k_1 z}$ .
- (3) If  $\gamma \equiv 0$  and  $\alpha + \beta' \equiv 0$ , then  $w(z) = e^{c_1 z} \{c_2 - \int \beta(z) e^{-c_1 z} dz\}$ .
- (4) If  $\gamma \neq 0$  and there is a constant  $k_1$  and a meromorphic function  $h$  satisfying  $h^2 + \beta h + \gamma = 0$  and  $h' - k_1 h = \alpha + k_1 \beta$ , then  $w = e^{k_1 z} (c_1 + \int h(z) e^{-k_1 z} dz)$ .
- (5) Suppose that  $\gamma \neq 0$  and  $A = \frac{\beta(\alpha + \beta') - \gamma'}{\gamma}$  is a constant.
  - (a) If  $A = 0$  and there is a nonzero constant  $k_1$  such that  $k_1^2 \beta + \beta'' + 2\alpha' = 0$ , then

$$k_2^2 = \frac{1}{k_1^2} \left\{ \frac{1}{4k_1^2} (\beta' + 2\alpha)^2 + \left( \gamma - \frac{\beta^2}{4} \right) \right\}$$

is also a constant. If  $k_2 \neq 0$ , then  $w = \pm k_2 \cosh(k_1 z + c_1) + \frac{\beta' + 2\alpha}{2k_1^2}$ .

- (b) If  $k_1^2 = \frac{(\frac{\beta}{2}A - \beta' - 2\alpha)^2}{\beta^2 - 4\gamma}$  is a nonzero constant, then  $w = c_1 e^{(-\frac{A}{2} \pm k_1)z} - \frac{1}{2k_1^2}(\frac{\beta}{2}A - \beta' - 2\alpha)$ .
- (c) If  $\alpha$  and  $\gamma$  are nonzero constants and  $\beta = 0$ , then  $w(z) = -\frac{\alpha}{2}(z + c_1)^2 - \frac{\gamma}{2\alpha}$ .
- (d) If  $k_1^2 = \beta^2/4 - \gamma \neq 0$  is a constant and  $A = 0$ , then  $w(z) = \pm k_1 z + c_1 - \frac{1}{2} \int \beta \, dz$ .
- (e) If  $\beta^2/4 - \gamma \equiv 0$ , then  $w = e^{-Az/2} \{c_1 - \int \frac{\beta}{2} e^{Az/2} \, dz\}$ . □

We have used  $k_1$  and  $k_2$  to denote constants that appear in constraints on the coefficient functions. The constants  $c_1$  and  $c_2$  are parameters in families of solutions of Equation (1), that is, they are integration constants.

The case in which  $\alpha$ ,  $\beta$ , and  $\gamma$  are constants was solved in [3]. In this case, any nonconstant solution is admissible. Since it is trivial to find the constant solutions, all meromorphic solutions were found.

In [10], Hayman conjectured that all entire solutions of

$$ff' - f'^2 = \kappa_0 + \kappa_1 f + \kappa_2 f' + \kappa_3 f'' \tag{3}$$

have finite order, where  $\kappa_0, \dots, \kappa_3$  are rational functions of  $z$ . If we let  $w = f - \kappa_3$ , then  $w$  solves Equation (1) with  $\alpha = \kappa_1 - \kappa_3'$ ,  $\beta = \kappa_2 + \kappa_3'$  and  $\gamma = \kappa_0 + \kappa_1\kappa_3 + \kappa_2\kappa_3' + (\kappa_3')^2$ . This provided the initial motivation for studying the meromorphic solutions of Equation (1). However, the problem of the explicit determination of all meromorphic solutions soon became the main problem of interest. Nevertheless, Hayman’s question is answered by the following elementary corollary of Theorem 1.1.

**Corollary 1.2.** If  $\alpha$ ,  $\beta$ , and  $\gamma$  are rational functions, then any transcendental meromorphic solution  $w$  of Equation (1) is of order 1, exponential type. □

This corollary follows immediately on noting that any *meromorphic* function that can be expressed as an integral of the form  $\int \beta e^{Az}$ , for some constant  $A$ , is itself of the form  $B(z) e^{Az}$ , for some rational function  $B$ . This can be seen by decomposing  $\beta$  into partial fractions, using integration by parts, and noting that the coefficients of terms of the form  $\int (z - c)^{-1} e^{Az}$ , where  $c$  is constant, must vanish. In [2], Barsegian et al. obtained some estimates for the number of poles of meromorphic solutions of Equation (1) in the case in which  $\alpha$ ,  $\beta$ , and  $\gamma$  are polynomials.

In some sense, Equation (3) is the simplest differential equation which is neither covered by the results of Steinmetz [13, Theorem 12.2; 15] nor Hayman [10, Theorem C].

Both these results generalize the classical Gol'dberg Theorem [8] that all meromorphic solutions of the first-order ODE  $\Omega(z, f, f') = 0$ , where  $\Omega$  is polynomial in all its arguments, are of finite order.

Equation (1) is singular when  $w = 0$ . Suppose that  $w$  has a zero at  $z = z_0$ , which is neither a zero nor a pole of the coefficients, and substitute the expansion

$$w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+p}$$

in Equation (1), where  $a_0 \neq 0$  and  $p$  is a positive integer. If  $\gamma \neq 0$ , then  $p = 1$  and there are (generally) two possible values for  $a_0$  given by  $a_0^2 + \beta(z_0)a_0 + \gamma(z_0) = 0$ . For each choice of  $a_0$ , we have a recurrence relation of the form

$$(n+1)(n-r)a_0 a_n = P_n(a_0, \dots, a_{n-1}), \quad (4)$$

where for each  $n$ ,  $P_n$  is a polynomial in its arguments. For Equation (1),  $r$  depends on  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $a_0$ .

If  $r$  is not a positive integer, then all of the coefficients  $a_n$  are determined by the choice of  $a_0$ . In this case, there are at most two solutions with a zero at  $z_0$ . This is the so-called *finiteness property* that has been used by several authors to characterize meromorphic solutions of equations [4–7, 12]. It is particularly effective for constant coefficient equations as it can be used to deduce periodicity of solutions.

If  $r$  is a positive integer, then only  $a_1, \dots, a_{r-1}$  are determined by  $a_0$ . Equation (4) shows that there is a necessary (*resonance*) condition,  $P(a_0, \dots, a_{r-1}) = 0$ , which must be satisfied. Subject to this constraint, all remaining coefficients,  $a_{r+1}, \dots$  are determined by  $a_r$  and  $a_0$ . This is very useful for identifying equations that admit meromorphic solutions (see, e.g., [16]). One of the main difficulties with Equation (1) is that the location of the resonance depends on the coefficients:  $r = (\beta(z_0)/a_0) + 2$ . So even in the constant coefficient case considered in [3], we can choose  $\beta$  and  $\gamma$  so that there is a positive integer resonance at an arbitrary high coefficient in the expansion for  $w$ , implying that high-order derivatives of  $w$  at a zero of  $w$  are not determined by the equation and leading-order term (cf. [14]).

In the present paper, we bypass issues related to resonance by using at most the first two terms in the series expansion for  $w$  at zeros to construct a small (in the sense of Nevanlinna theory) function of  $w$  and  $w'$ , the coefficient functions  $\alpha$ ,  $\beta$ ,  $\gamma$  and their derivatives. In this way, we construct first-order equations that we can solve for  $w$ .

2 Proof of Theorem 1.1

If  $\alpha(z) \equiv \beta(z) \equiv \gamma(z) \equiv 0$ , then Equation (1) becomes  $(w'/w)' = 0$ , which has the general solution  $w(z) = c_2 e^{c_1 z}$ . This is a special case of part (3) of Theorem 1.1. From now on, we take at least one of  $\alpha, \beta, \gamma$  to be nonzero.

For any meromorphic function  $f$ , we define the set  $\Phi_f$  as follows. If  $f \equiv 0$ , then  $\Phi_f = \emptyset$ . If  $f \not\equiv 0$ , then  $\Phi_f$  is the set of all zeros and poles of  $f$ . Let  $\Phi = \Phi_\alpha \cup \Phi_\beta \cup \Phi_\gamma$ . Let  $w \not\equiv 0$  be a meromorphic solution of Equation (1) and let  $z_0 \in \Omega := \mathbb{C} \setminus \Phi$  be either a zero or a pole of  $w$ . Then,  $w$  has a Laurent series expansion of the form

$$w(z) = a_0 \zeta^p + a_1 \zeta^{p+1} + O(\zeta^{p+2}),$$

where  $\zeta = z - z_0, a_0 \neq 0$  and  $p \in \mathbb{Z} \setminus \{0\}$ . Equation (1) then becomes

$$-p a_0^2 \zeta^{2p-2} + \dots = \alpha(z_0)(a_0 \zeta^p + \dots) + \beta(z_0)(a_0 p \zeta^{p-1} + \dots) + (\gamma(z_0) + \dots). \tag{5}$$

It follows that if  $\beta \equiv \gamma \equiv 0$ , then  $p = 2$ . Otherwise  $p = 1$ . In particular,  $w$  is analytic on  $\Omega$ .

Throughout this proof, we will use the standard notation from Nevanlinna theory (see, e.g., Hayman [9] or Laine [13]). In particular, for any meromorphic function  $f$ , we denote the (integrated) counting function with multiplicities by  $N(r, f)$  and without multiplicities by  $\bar{N}(r, f)$ . Furthermore, we will denote by  $N_\Phi(r, f)$  and  $\bar{N}_\Phi(r, f)$  the counting functions (with and without multiplicities, respectively) where we only count the poles of  $f$  in the set  $\Phi$ . In particular, it follows that if  $w$  is a meromorphic solution of Equation (1), then  $N(r, w) = N_\Phi(r, w)$ . Now for any meromorphic function  $f$ ,  $\bar{N}_\Phi(r, f) \leq \bar{N}(r, \alpha) + \bar{N}(r, 1/\alpha) + \bar{N}(r, \beta) + \bar{N}(r, 1/\beta) + \bar{N}(r, \gamma) + \bar{N}(r, 1/\gamma) = S(r, w)$ , where if  $\alpha \equiv 0$  we take  $\bar{N}(r, 1/\alpha) = 0$ , etc. So  $\bar{N}(r, w) = \bar{N}_\Phi(r, w) = S(r, w)$ .

When the coefficient functions  $\alpha, \beta$  and  $\gamma$  are rational functions, then  $\Phi$  is a finite set and  $N_\Phi(r, w) = S(r, w)$ . However, for transcendental coefficients this does not follow immediately.

Case 1:  $\alpha \neq 0, \beta \equiv \gamma \equiv 0$ .

Substituting  $w(z) = a_0 \zeta^2 + a_1 \zeta^3 + O(\zeta^4)$  in Equation (1) shows that about any  $z_0 \in \Omega$  such that  $w(z_0) = 0$ , we have

$$w(z) = -\frac{\alpha(z_0)}{2} \zeta^2 - \frac{\alpha'(z_0)}{2} \zeta^3 + O(\zeta^4).$$

Together with the fact that  $w$  is analytic on  $\Omega$ , it follows that

$$f(z) := \left( \frac{w'}{w} - \frac{\alpha'}{\alpha} \right)^2 + 2 \frac{\alpha}{w} \tag{6}$$

is also analytic on  $\Omega$ . Using Equation (1) with  $\beta \equiv \gamma \equiv 0$ , we see that

$$f(z) = \left( \frac{w'}{w} - \frac{\alpha'}{\alpha} \right)^2 + 2 \left( \frac{w'}{w} \right)'. \quad (7)$$

Hence

$$\begin{aligned} N(r, f) &= N_\phi(r, f) \leq 2N_\phi \left( r, \frac{w'}{w} \right) + 2N_\phi \left( r, \frac{\alpha'}{\alpha} \right) + N_\phi \left( r, \left( \frac{w'}{w} \right)' \right) \\ &= 4 \left\{ \bar{N}_\phi(r, w) + \bar{N}_\phi \left( r, \frac{1}{w} \right) \right\} + 2N_\phi \left( r, \frac{\alpha'}{\alpha} \right) = S(r, w). \end{aligned}$$

Furthermore, applying the Lemma on the Logarithmic Derivative to Equation (7) gives  $m(r, f) = S(r, w)$ . So  $T(r, f) = S(r, w)$ .

Differentiating Equation (6) and using Equation (1) to eliminate  $w''$  gives

$$f' = 2 \left( \frac{\alpha'}{\alpha} \right)' \left( \frac{\alpha'}{\alpha} - \frac{w'}{w} \right). \quad (8)$$

When  $\left( \frac{\alpha'}{\alpha} \right)' \neq 0$ , we obtain

$$\frac{w'}{w} = \frac{\alpha'}{\alpha} - \frac{f'}{2} \left[ \left( \frac{\alpha'}{\alpha} \right)' \right]^{-1}.$$

Substituting this into Equation (6) gives

$$f = \frac{f'^2}{4} \left[ \left( \frac{\alpha'}{\alpha} \right)' \right]^{-2} + 2 \frac{\alpha}{w}. \quad (9)$$

Applying Nevanlinna's First Fundamental Theorem to Equation (9), we obtain  $T(r, w) = S(r, w)$ , a contradiction. Therefore  $(\alpha'/\alpha)' \equiv 0$ , so from Equation (8),  $f' \equiv 0$ . Thus,

$$\alpha(z) \equiv k_1 e^{k_2 z} \quad \text{and} \quad f(z) \equiv c_1^2,$$

where  $k_1 \neq 0$ ,  $k_2$ , and  $c_1$  are constants.

In terms of  $u = w/\alpha$ , Equation (6) becomes

$$u^2 = c_1^2 u^2 - 2u.$$

If  $c_1 = 0$ , this gives  $u = -\frac{1}{2}(z + c_2)^2$ . When  $k_2 \neq 0$ , we arrive at the contradiction  $T(r, w) = T(r, \alpha) + S(r, \alpha) = S(r, w)$ . Thus,  $k_2 = 0$ , so  $\alpha = k_1$  and  $w = -\frac{k_1}{2}(z + c_2)^2$ . For the case  $c_1 \neq 0$ ,  $u = c_1^{-2} \{ \cosh(c_1 z + c_2) + 1 \}$ , where  $c_2$  is a constant. This gives part (1) of Theorem 1.1.

Case 2:  $\beta \neq 0$ ,  $\gamma \equiv 0$ .

Recall that in this case  $w$  has only simple zeros in  $\Omega$ . Substituting  $w(z) = a_0\zeta + a_1\zeta^2 + O(\zeta^3)$  in Equation (1) yields, at the leading order  $a_0 = -\beta(z_0)$  and at the next-to-leading order we find the constraint  $\alpha(z_0) + \beta'(z_0) = 0$ .

Case 2a:  $\alpha + \beta' \neq 0$ ,  $\gamma \equiv 0$ .

Let  $f = w'/w$ . If  $z_0$  is a pole of  $w$ , then  $z_0 \in \Phi$ . If  $z_0$  is a zero of  $w$ , then either  $z_0 \in \Phi$  or  $\alpha(z_0) + \beta'(z_0) = 0$ . Hence

$$N(r, f) = N\left(r, \frac{w'}{w}\right) \leq \bar{N}_\Phi(r, w) + \bar{N}_\Phi\left(r, \frac{1}{w}\right) + \bar{N}\left(r, \frac{1}{\alpha + \beta'}\right) = S(r, w).$$

It then follows from the Lemma on the Logarithmic Derivative that  $T(r, f) = T(r, w'/w) = S(r, w)$ . Substituting  $w' = fw$  and  $w'' = (f' + f^2)w$  in Equation (1) with  $\gamma \equiv 0$  yields

$$f'w = \alpha + f\beta.$$

Since  $T(r, f') = S(r, w)$  and  $T(r, \alpha + f\beta) = S(r, w)$ , we must have  $f' \equiv \alpha + f\beta \equiv 0$ , thus  $f \equiv k_1$  is a constant. Hence there exists a constants  $k_1$  such that  $w(z) = c_1 e^{k_1 z}$  and  $\alpha(z) = -k_1\beta(z)$ , giving part (1.1) of the theorem.

Case 2b:  $\alpha + \beta' \equiv 0$ ,  $\gamma \equiv 0$ .

Equation (1) takes the form  $((w' + \beta)/w)' = 0$ , which has the general solution  $w = e^{c_1 z}\{c_2 - \int \beta(z)e^{-c_1 z} dz\}$ , where  $c_2$  is a constant. This gives part (3) of the theorem.

Case 3:  $\gamma \neq 0$ .

Recall that in this case  $w$  is analytic in  $\Omega$  and any zero  $z_0$  of  $w$  in  $\Omega$  is simple. On substituting  $w(z) = a_0\zeta + a_1\zeta^2 + O(\zeta^3)$  in Equation (1), we find that

$$a_0^2 + \beta(z_0)a_0 + \gamma(z_0) = 0 \quad \text{and} \quad a_1 = \frac{1}{2\gamma(z_0)}\{\gamma'(z_0) - \beta(z_0)(\alpha(z_0) + \beta'(z_0))\}a_0 - \frac{1}{2}(\alpha(z_0) + \beta'(z_0)).$$

Let

$$f(z) = \frac{(w')^2 + \beta w' + \gamma}{w^2}.$$

If  $f$  has a pole at  $z_0 \in \Omega$ , then  $w(z_0) = 0$ . From Equation (1),  $f(z) = (w'' - \alpha)/w$ , so in a neighborhood of  $z_0$ ,

$$f(z) = \frac{2a_1 - \alpha(z_0)}{a_0\zeta} + O(1) = \left\{ \frac{\gamma'(z_0) - \beta(z_0)[\alpha(z_0) + \beta'(z_0)]}{\gamma(z_0)\zeta} - \frac{2\alpha(z_0) + \beta'(z_0)}{a_0\zeta} \right\} + O(1).$$

Therefore

$$g(z) = \frac{(w')^2 + \beta w' + \gamma}{w^2} + A \frac{w'}{w} + \frac{2\alpha + \beta'}{w}, \quad A = \frac{\beta(\alpha + \beta') - \gamma'}{\gamma} \tag{10}$$

is analytic on  $\Omega$ .

Rewriting Equation (1) as

$$\frac{1}{w^2} = \frac{1}{\gamma} \left\{ \left( \frac{w'}{w} \right)' - \frac{1}{w} \left( \alpha + \beta \frac{w'}{w} \right) \right\}, \tag{11}$$

we see that

$$2N_\phi \left( r, \frac{1}{w} \right) \leq N_\phi \left( r, \frac{1}{\gamma} \right) + N_\phi \left( r, \left( \frac{w'}{w} \right)' \right) + N_\phi \left( r, \frac{1}{w} \right) + N_\phi(r, \alpha) + N_\phi(r, \beta) + N_\phi \left( r, \frac{w'}{w} \right).$$

Hence

$$N_\phi \left( r, \frac{1}{w} \right) \leq 3 \left\{ \bar{N}_\phi(r, w) + \bar{N}_\phi \left( r, \frac{1}{w} \right) \right\} + S(r, w) = S(r, w).$$

So from Equation (10), we have

$$N(r, g) = N_\phi(r, g) \leq 2\bar{N}_\phi(r, w) + 2N_\phi \left( r, \frac{1}{w} \right) + S(r, w) = S(r, w).$$

Taking the proximity function of both sides of Equation (11), we obtain

$$\begin{aligned} 2m \left( r, \frac{1}{w} \right) &\leq m \left( r, \frac{1}{\gamma} \right) + m \left( r, \left( \frac{w'}{w} \right)' \right) + m \left( r, \frac{1}{w} \right) + m(r, \alpha) + m(r, \beta) + m \left( r, \frac{w'}{w} \right) \\ &= m \left( r, \frac{1}{w} \right) + S(r, w). \end{aligned}$$

Hence  $m(r, g) = S(r, w)$ . So  $T(r, g) = m(r, g) + N(r, g) = S(r, w)$ .

Differentiating  $w^2 \times$  Equation (10) and using Equation (1) to eliminate  $w''$  and Equation (10) to eliminate  $(w')^3$  and then  $(w')^2$ , we have

$$A'w' = g'w - B, \tag{12}$$

where  $B = \beta g + \alpha A + 2\alpha' + \beta''$ .

Case 3a:  $A' \neq 0$ .

Using Equation (12) to eliminate  $w'$  from Equation (10) gives

$$(g'^2 - gA'^2 + g'AA')w^2 + ([2\alpha + \beta']A'^2 + \beta g'A' - AA'B - 2g'B)w + (B^2 - \beta A'B + \gamma A'^2) = 0.$$

Since the coefficients of the different powers of  $w$  are all  $S(r, w)$ , we must have that each coefficient vanishes identically. In particular, the coefficient of  $w^2$  gives  $g'^2 - gA'^2 +$



$g'AA' = 0$ . It follows that  $G = g + (A^2/4)$  satisfies  $(G')^2 = (A')^2G$ . Hence either  $G = 0$  (i.e.,  $g = -A^2/4$ ) or  $G = (A/2 + k_1)^2$  (i.e.,  $g = k_1A + k_1^2$ ), where  $k_1$  is a constant.

*Case 3a(i):*  $g = k_1A + k_1^2$ .

Equation (12) now has the form

$$w' = k_1w + h, \quad (13)$$

where  $h = -B/A'$ . Hence  $w'' = k_1^2w + (h' + k_1h)$  and we see that any solution of Equation (13) solves Equation (1) if and only if

$$(h' - k_1h - \alpha - k_1\beta)w = h^2 + \beta h + \gamma,$$

so  $h^2 + \beta h + \gamma = 0$  and  $h' - k_1h = \alpha + k_1\beta$ . This corresponds to part (4) of the theorem.

*Case 3a(ii):*  $g = -A^2/4$ .

Equation (12) becomes  $w' = -(A/2)w + h$ , where  $h = -B/A'$ . Hence  $w'' = [(A^2/4) - (A'/2)]w + h' - hA/2$ . Using these expressions to eliminate the first and second derivatives in Equation (1) leads to

$$-\frac{A'}{2}w^2 + \left(\frac{Ah}{2} + h' - \alpha + \frac{\beta A}{2}\right)w = h^2 + \beta h + \gamma$$

with coefficients that are  $S(r, w)$ . By the Valiron–Mokhon'ko Theorem [17, Theorem 1.13], we have  $2T(r, w) = S(r, w)$ , which is impossible.

*Case 3b:*  $A' \equiv 0$ , that is,  $A$  is a constant.

It follows from Equation (12) that  $g$  is also a constant and  $B = 0$ . Equation (10) can be rewritten as

$$\left(w' + \frac{1}{2}[Aw + \beta]\right)^2 = \left(g + \frac{A^2}{4}\right)w^2 + \left(\frac{\beta}{2}A - \beta' - 2\alpha\right)w + \left(\frac{\beta^2}{4} - \gamma\right). \quad (14)$$

Let  $h(z) = (\frac{\beta}{2}A - \beta' - 2\alpha)e^{Az/2}$ . Then

$$\left(\left[\frac{\beta^2}{4} - \gamma\right]e^{Az}\right)' = \frac{\beta}{2}e^{Az/2}h \quad (15)$$

and the condition  $B = 0$  is equivalent to

$$h' = \left(g + \frac{A^2}{4}\right)\beta e^{Az/2}. \quad (16)$$

Clearly, if  $g = -A^2/4$ , then  $h$  is constant.

*Case 3b(i):*  $g + \frac{A^2}{4} \neq 0$ .

So  $k_1^2 = g + \frac{A^2}{4}$  is a nonzero constant. It follows from Equations (15) and (16) that

$$\left( \left[ \frac{\beta^2}{4} - \gamma \right] e^{Az} \right)' = \frac{1}{2k_1^2} hh'.$$

Integration shows that

$$k_2^2 = \frac{1}{k_1^2} \left\{ \frac{h^2}{4k_1^2} + \left( \gamma - \frac{\beta^2}{4} \right) e^{Az} \right\} = \frac{1}{k_1^2} \left\{ \frac{1}{4k_1^2} \left( \frac{\beta}{2} A - \beta' - 2\alpha \right)^2 + \left( \gamma - \frac{\beta^2}{4} \right) \right\} e^{Az} \tag{17}$$

is a constant. Let

$$u = w e^{Az/2} + \frac{h}{2k_1^2}.$$

Then, Equation (14) becomes

$$(u')^2 = k_1^2 (u^2 - k_2^2). \tag{18}$$

When  $k_2 \neq 0$ , we have

$$w = \left( \pm k_2 \cosh(k_1 z + c_1) - \frac{h}{2k_1^2} \right) e^{-Az/2},$$

where  $c_1$  is a constant. Therefore,  $T(r, w) \leq K_1 r + S(r, w)$  for some  $K_1 > 0$ . When  $A \neq 0$ , Equation (17) shows that  $r \leq K_2 T(r, e^{Az}) = S(r, w)$ , which gives the contradiction  $T(r, w) = S(r, w)$ . Hence  $A = 0$  if  $k_2 \neq 0$ . This is part (5)(a) of the theorem. Part (5)(b) corresponds to the case in which  $k_2 = 0$ , where

$$w = e^{-Az/2} \left( c_1 e^{\pm k_1 z} - \frac{h}{2k_1^2} \right) = c_1 e^{(-\frac{A}{2} \pm k_1)z} - \frac{1}{2k_1^2} \left( \frac{\beta}{2} A - \beta' - 2\alpha \right).$$

Case 3b(ii):  $g = -A^2/4, h \neq 0$ .

Let  $\lambda = \int \frac{\beta}{2} e^{Az/2} dz$ . It follows from Equation (15) and Equation (16) that  $h$  and

$$C := \frac{1}{h} \left( \frac{\beta^2}{4} - \gamma \right) e^{Az} - \lambda$$

are constants. Let  $u = w e^{Az/2} + \lambda$ . Then, Equation (14) becomes  $(u')^2 = h(u + C)$ , which has the general solution  $u = \frac{h}{4}(z + c_1)^2 - C$ . Hence

$$w = \frac{h}{4} e^{-Az/2} (z + c_1)^2 - \frac{1}{h} \left( \frac{\beta^2}{4} - \gamma \right) e^{Az/2} = \frac{1}{4} \left( \frac{\beta}{2} A - \beta' - 2\alpha \right) (z + c_1)^2 - \frac{\frac{\beta^2}{4} - \gamma}{\frac{\beta}{2} A - \beta' - 2\alpha}. \tag{19}$$

So  $T(r, w) = O(r) + S(r, w)$ . Recall that  $h$  is a nonzero constant. Now if  $A \neq 0$ , we have  $r \leq K_1 T(r, e^{Az/2}) = K_1 T(r, \frac{\beta}{2} A - \beta' - 2\alpha) + O(1) = S(r, w)$ , a contradiction. Therefore  $A = 0$ .

Now Equation (19) with  $A = 0$  shows that  $T(r, w) = 2 \log r + S(r, w)$ . Hence  $w$  is admissible if and only if the coefficients  $\alpha$ ,  $\beta$ , and  $\gamma$  are constants. This gives part (5)(c).

Case 3b(iii):  $g = -A^2/4$ ,  $h = 0$ .

It follows from Equations (14) and (15) that  $w' + \frac{1}{2}[Aw + \beta] = k_1 e^{-Az/2}$ , where  $k_1^2 = (\frac{\beta^2}{4} - \gamma) e^{Az}$  is a constant. Hence  $(w e^{Az/2})' = k_1 - \frac{1}{2}\beta e^{Az/2}$ , giving

$$w = e^{-Az/2} \left\{ k_1 z + c_1 - \int \frac{\beta}{2} e^{Az/2} dz \right\}.$$

To study the admissibility of this solution, we will use the following theorem from Hayman and Miles [11].

**Theorem 2.1.** Let  $f(z)$  be a transcendental meromorphic function and  $K > 1$  be a real number. Then, there exists a set  $M(K)$  of upper logarithmic density at most  $d(K) = 1 - (2e^{K-1} - 1)^{-1} > 0$  such that for every positive integer  $k$ , we have

$$\limsup_{r \rightarrow \infty, r \notin M(K)} \frac{T(r, f)}{T(r, f^{(k)})} \leq 3eK. \quad \square$$

Furthermore, note that if  $f$  is any nonconstant rational function other than a degree 1 polynomial, then  $T(r, f) \leq KT(r, f')$  for some  $K > 0$ . Therefore, if  $w e^{Az/2}$  is not a constant or a degree 1 polynomial and  $k_1 \neq 0$ , it follows that there is a sequence of values of  $r \rightarrow \infty$  such that for some  $K_1 > 0$ ,

$$\begin{aligned} T(r, w) &\leq T(r, w e^{Az/2}) + T(r, e^{-Az/2}) \\ &\leq K_1 T(r, (w e^{Az/2})') + T(r, e^{-Az/2}) \\ &= K_1 T\left(r, \frac{1}{2}\beta e^{Az/2} - k_1\right) + T(r, e^{-Az/2}) \\ &= K_1 T\left(r, \frac{1}{2}k_1\beta \left(\frac{\beta^2}{4} - \gamma\right)^{-1/2} - k_1\right) + T\left(r, k_1^{-1} \left(\frac{\beta^2}{4} - \gamma\right)^{1/2}\right) \\ &= o(T(r, w)), \end{aligned}$$

which is a contradiction. If  $w e^{Az/2}$  is at most a degree 1 polynomial, then  $\beta = k_2 e^{-Az/2}$  and  $w$  is only admissible if  $A = 0$ . Now  $w$  is a polynomial of degree no more than 1, so  $\alpha$ ,  $\beta$ , and  $\gamma$  are constants. It follows from  $h = 0$  that  $\alpha = 0$ . At the same time,  $A = 0$  and  $\alpha = 0$  implies that  $g = 0$ , so we have  $w^2 + \beta w' + \gamma = 0$ . This corresponds to part (5)(d) of the theorem. Otherwise, we have  $k_1 = 0$ , that is,  $\gamma = \frac{\beta^2}{4}$ , which corresponds to part (5)(e).

### 3 Discussion

The proof provided in Section 2 would have been significantly shorter had we restricted ourselves to the rational coefficient case. In the first instance, the fact that  $N(r, 1/w) = \bar{N}(r, 1/w) + S(r, w)$  would have followed immediately from Equation (5). Also, many of the subcases considered in the proof could be eliminated or simplified because they require that in general certain rational functions of the coefficient functions be an exponential in  $z$ .

When we allowed some of the coefficients to be transcendental, we generated many exact solutions only to discard them later because these solutions grow at the same rate as the coefficients. From the point of view of using the existence of meromorphic solutions as a detector of exactly solvable cases, this suggests that perhaps a weaker notion of "admissibility" would be more fruitful. These are all perfectly good solutions and it is undesirable merely to discard them or even to search for more efficient methods to avoid considering them in the first place. It seems wasteful not to modify the problem so that such solutions will appear in the final classification. We hope to explore this problem in future work.

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