Height growth of solutions and a discrete Painlevé equation

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Abstract

Consider the discrete equation

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2},$$

where the right side is of degree two in y_n and where the coefficients a_n, b_n and c_n are rational functions of n with rational coefficients. Suppose that there is a solution such that for all sufficiently large n, $y_n \in \mathbb{Q}$ and the height of y_n dominates the height of the coefficient functions a_n, b_n and c_n. We show that if the logarithmic height of y_n grows no faster than a power of n then either the equation is a well known discrete Painlevé equation $\mathrm{dP_{II}}$ or its autonomous version or y_n is also an admissible solution of a discrete Riccati equation. This provides further evidence that slow height growth is a good detector of integrability.

1 Introduction

For discrete equations, integrability appears to be closely related to the slow growth of various measures of complexity [20]. For example, algebraic entropy [6, 14, 5] measures the degree growth of iterates as a function of the initial conditions, while in the Nevanlinna approach [2, 12], one considers the order of growth of meromorphic solutions. A discrete equation on a number field is said to be *Diophantine integrable* if the logarithmic height of solutions grows polynomially [10]. The last two approaches are connected by Vojta's

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dictionary [21], which relates ideas from Nevanlinna theory (the value distribution of meromorphic functions) to those in Diophantine approximation. See [9] for a survey of different approaches to detecting integrability in discrete systems.

This paper concerns a Diophantine analogue of a classification result of Halburd and Korhonen [11] using Nevanlinna theory. Specifically, we will study discrete equations of the form

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2}, \tag{1}$$

where a_n, b_n and c_n are in $\mathbb{Q}(n)$ and the degree of the right side of equation (1) is two. The logarithmic height of a rational number $x = a/b$, where a and b have no common factors, is $h(x) = \log H(x)$, where $H(x) = \max\{|a|, |b|\}$ is the height. A discrete equation such as (1) is said to be *Diophantine integrable* if the logarithmic height of its solution y_n over a number field grows no faster than a power of n [10]. Abarenkova *et al* [1] used height growth to estimate the entropy of a map. Slow height growth has been used as an efficient numerical test in [15, 4, 17, 16, 7].

The purpose of this paper is to prove the following.

Theorem 1 *Let r_0 be sufficiently large and let $(y_n)_{n \geq r_0} \subset \mathbb{Q} \setminus \{-1, 1\}$ be a solution of (1), where a_n, b_n and c_n are rational functions of n with coefficients in $\mathbb{Q}$ and the right side of (1) is of degree two in y_n. If*

$$\sum_{n=r_0}^{r} \max\{1, h(a_n), h(b_n), h(c_n)\} = o\left(\sum_{n=r_0}^{r} h(y_n)\right) \tag{2}$$

as $r \to \infty$, then either

1. *$a_n = \alpha n + \beta$, $b_n = \gamma$, $c_n = 0$ for constants α, β, γ; or*
2. *y_n also solves the discrete Riccati equation*

$$y_{n+1} = \frac{1/2(a_n + \theta b_n - 2\theta) + y_n}{1 - \theta y_n}, \text{ where } \theta = -1 \text{ or } 1; \text{ or} \tag{3}$$

3. $\displaystyle \limsup_{r \to \infty} \frac{\log \log \sum_{n=r_0}^{r} h(y_n)}{\log r} \geq 1.$

This result first appeared in the PhD thesis of the first author.

A solution of equation (1) satisfying (2) will be called *admissible*. Of the three possible outcomes described in Theorem 1, the first says that equation (1) is the discrete Painlevé equation $\mathrm{dP_{II}}$ (see Nijhoff and Papageorgiou [18]) or its autonomous version ($\alpha = 0$), the second says that y_n solves a well known linearisable discrete Riccati equation and the third implies that $h(y_n)$ grows faster than any power of n. If equation (1) has more than two one-parameter families of admissible solutions, they cannot both solve discrete Riccati equations of the form described by the second conclusion unless the equation is $\mathrm{dP_{II}}$. In case 1 with $\alpha = 0$, equation (1) can be derived from the addition law on an elliptic curve, for which it is known that the logarithmic height grows quadratically.

The full version of $\mathrm{dP_{II}}$ allows a_n to have the more general form $a_n = \alpha n + \beta + \delta(-1)^n$, where δ is another constant. We do not capture this form as we have assumed that the coefficients a_n, b_n and c_n are rational functions of n. This assumption simplifies some of the arguments.

Diophantine integrability is a property of all solutions, not just those that are admissible. Our method involves working with one solution at a time, so an admissibility-type condition is necessary to avoid counterexamples in which a_n, b_n and y_n are chosen arbitrarily and then c_n is determined by equation (1).

Of central importance in our proof of Theorem 1 is the fact that there is a simple relationship between the height of a rational number x and a certain sum over all non-trivial absolute values of x. For a fixed prime p, the p-adic absolute value of a non-zero rational number x is $|x|_p = p^{-r}$, where $x = \frac{m}{n}p^r$ for integers m, n and r such that $p \nmid mn$. The p-adic absolute values are non-Archimedean, which means that they satisfy the stronger triangle inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. The usual absolute value, denoted by $|\cdot|_\infty$, is Archimedean. Ostrovski's Theorem says that, up to equivalence, the only non-trivial absolute values on $\mathbb{Q}$ are the p-adic absolute values, $|\cdot|_p$ and the usual absolute value $|\cdot|_\infty$. In terms of these absolute values, we have the important identity

$$h(x) = \sum_{p \leq \infty} \log^+ |x|_p, \tag{4}$$

where the sum is taken over all finite primes and $p = \infty$ (the "prime at infinity") and $\log^+ y := \max\{0, \log y\}$.

One of the first properties of discrete equations to be used to identify discrete Painlevé equations was singularity confinement [8, 19], which involves the behaviour of solutions as one iterates through a singularity of the equation. For equation (1), one needs to examine the singular values $y = 1$ and $y = -1$. In order to resolve indeterminacies that arise in future iterates, we consider the initial conditions $y_{k-1} = \kappa$, $y_k = \theta + \epsilon$, where κ is arbitrary, $\theta^2 = 1$ and ϵ is small (as we will take the limit $\epsilon \to 0$ after a finite number of steps). Generically we find that, after taking the limit $\epsilon \to 0$, $y_m = \infty$ for infinitely many $m > k$. However, for certain choices of a_n, b_n and c_n, the singularity appears to be confined to a finite number of iterates.

At the heart of the proof of Theorem 1 are some calculations that look very much like those described above for singularity confinement. The main difference is that we have to consider not only the case in which $1 - \theta y_k$ is small with respect to the usual absolute value, but also cases in which it is small with respect to p-adic absolute values. The identity (4) is eventually used to convert certain statements about absolute values into statements about logarithmic heights. The following theorem is an example of such an expression of singularity (non-)confinement in terms of absolute values. It should be stressed, however, that we do not make assumptions about the long term behaviour of solutions or whether they are eventually confined. For each absolute value $|\cdot|_p$, $\epsilon_k \equiv \epsilon_{k,p}$, which is defined precisely in equation (16), determines a length scale in terms of the coefficients a_k, b_k, c_k and a finite number of their shifts.

Theorem 2 *Let* $(y_n)_{n=k-1}^{k+3} \subset \mathbb{Q} \setminus \{-1, 1\}$ *satisfy*

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2},$$

where k is sufficiently large and the right hand side of the equation is irreducible. Assume that for a fixed absolute value $|.|_p$ ($p \leq \infty$) we have $|y_{k-1}|_p \leq |1 - \theta y_k|_p^{-1/2}$ for $\theta = 1$ or -1. Furthermore, for sufficiently small $\delta > 0$, if $|1 - \theta y_k|_p < \epsilon_k$ (where ϵ_k is defined in (16)), then

(i) $y_{k+1} = \dfrac{a_k + \theta b_k}{2(1 - \theta y_k)} + A_k$, *where* $|A_k|_p \leq |1 - \theta y_k|_p^{-1/2}$ *if* $p < \infty$ *and if* $p = \infty$, $|A_k|_\infty \leq \frac{11}{10} \cdot |1 - \theta y_k|_\infty^{-1/2}$.

(ii) $y_{k+2} = -\theta + \left(\dfrac{\theta a_k + b_k - 2b_{k+1}}{a_k + \theta b_k} \right) (1 - \theta y_k) + B_k$,

where $|B_k|_p \leq |1-\theta y_k|_p^{3/2-5\delta}$ *if* $p<\infty$ *and if* $p=\infty$, $|B_k|_\infty \leq \frac{1}{2}\cdot|1-\theta y_k|_\infty^{3/2-5\delta}$.

(iii) $y_{k+3} = \dfrac{\big(a_{k+2}-\theta b_{k+2}-\theta(\theta a_k + b_k - 2b_{k+1})\big)}{2(1+\theta y_{k+2})} + C_k$, *where* $|C_k|_p \leq |1+\theta y_{k+2}|_p^{-(2/3+2\delta)}$ *if* $p<\infty$ *and if* $p=\infty$, $|C_k|_\infty \leq 2|1+\theta y_{k+2}|_\infty^{-(2/3+2\delta)}$.

Theorems 1 and 2 can easily be extended to arbitrary number fields (finite field extensions of the rationals) as there is a simple analogue of the identity (4) in this case.

In [10], it was shown that if an equation of the form

$$y_{n+1} + y_{n-1} = R(n, y_n), \tag{5}$$

where R is rational, has an admissible solution, then $\deg_y R(n,y) \leq 2$. The case $R(n,y_n) = (a_n + b_n y_n + c_n y_n^2)/y_n^2$ was studied in [13]. There are essential technical difficulties which distinguish the two cases and consequently the analysis used to treat each of them. The fact that, for certain a_n, b_n and c_n, there are solutions of equation (1) that also satisfy discrete Riccati equations requires a more subtle analysis.

2 Proof of Theorem 1

First we consider the case in which $c_n \not\equiv -2$, 0 or 2. We introduce a quantity ϵ_n, which provides a scale with respect to which we measure distances between iterates and certain singular values. For any finite set of rational functions $\{f_1,\ldots,f_m\}$ of n, there exists $K\in\mathbb{N}$ such that for every function f_j that is not identically zero, $f_j(n)$ is finite and nonzero for all $n \geq K$. Throughout this paper we will refer to such an integer K, which may need to be increased a finite number of times, without further comment. Since the right side of equation (1) is of degree 2, neither of the rational functions $a_n+b_n+c_n$ nor $a_n-b_n+c_n$ vanishes identically. For $n>K$, define the sets $X_{n,0} = \left\{1/2, b_n, c_n, c_{n-1}^{-1}, c_{n+1}^{-1}\right\}$, $X_{n,\pm} = \{(a_n \pm b_n + c_n)^{-1}, (a_{n+1}\pm b_{n+1}+c_{n+1})/2, (a_{n-1}\pm b_{n-1}+c_{n-1})/2, (c_{n+1}\pm 2)^{-1}, (c_{n-1}\pm 2)^{-1}\}$ and let $X_n = X_{n,+}\cup X_{n,0}\cup X_{n,-}$. For a fixed sufficiently small $\delta>0$ we define ϵ_n ($\forall n > K$) by

$$\epsilon_n^{-\delta} = \kappa_p \max_{x\in X_n}\{|x|_p\}, \tag{6}$$

where $\kappa_p = 1$ for $p < \infty$ and $\kappa_\infty = 10$. Equation (6) allows us to estimate certain combinations of the coefficients a_n, b_n, c_n in terms of ϵ_n. For example, for $p < \infty$, we have $|c_n|_p \leq \epsilon_n^{-\delta} \Rightarrow \epsilon_n^{\delta} \leq |c_n|_p^{-1}$. For the Archimedean absolute value ($p = \infty$), we have $10|c_n|_\infty \leq \epsilon_n^{-\delta} \Rightarrow \epsilon_n^{\delta} \leq \frac{1}{10}|c_n|_\infty^{-1}$.

Lemma 3 *Let $(y_n) \subset \mathbb{Q} \setminus \{-1, 1\}$ be a solution of equation (1) where a_n, b_n and c_n are in $\mathbb{Q}(n)$ and c_n is a rational function not identically 0 or ± 2. Furthermore, assume that the numerator and the denominator of (1) are coprime. For a fixed prime $p \leq \infty$ and $k > K$, let ϵ_k be as defined in (6). If $|1 - \theta y_k|_p < \epsilon_k$ for $\theta = 1$ or -1, then either*

$$|y_{k+1}|_p \geq \frac{1}{|1 - \theta y_k|_p^{1-\delta}} \quad \text{and} \quad |1 \pm \theta y_{k+2}|_p \geq \epsilon_{k+2},$$

or

$$|y_{k-1}|_p \geq \frac{1}{|1 - \theta y_k|_p^{1-\delta}} \quad \text{and} \quad |1 \pm \theta y_{k-2}|_p \geq \epsilon_{k-2}.$$

Proof : The definition of $\epsilon_n^{-\delta}$ in (6) implies that $\epsilon_n \leq 1$. Furthermore when $p = \infty$ or $p = 2$, $\epsilon_n < 1$. First we consider the non-Archimedean case for a fixed prime $p < \infty$. Let $|1 - \theta y_k|_p < \epsilon_k$ for some $k > K$, where $\theta = 1$ or -1. From equation (1) we have

$$(y_{k+1} + y_{k-1})(1 + \theta y_k) = \frac{a_k + \theta b_k + c_k}{1 - \theta y_k} - \theta b_k - c_k(1 + \theta y_k). \qquad (7)$$

So from equations (6) and (7),

$$\begin{aligned} & |1 - \theta y_k|_p^{-(1-\delta)} < \tfrac{\epsilon_k^{\delta}}{|1-\theta y_k|_p} \leq \tfrac{|a_k + \theta b_k + c_k|_p}{|1-\theta y_k|_p} \\ \leq \quad & |(y_{k+1} + y_{k-1})(1 + \theta y_k) + \theta b_k + c_k(1 + \theta y_k)|_p \\ \leq \quad & \max\{|y_{k+1} + y_{k-1}|_p \cdot |1 + \theta y_k|_p, |b_k|_p, |c_k|_p \cdot |1 + \theta y_k|_p\}. \end{aligned} \qquad (8)$$

From (6), $|b_k|_p \leq \epsilon_k^{-\delta} \leq \epsilon_k^{-(1-\delta)} < |1 - \theta y_k|_p^{-(1-\delta)}$. Similarly, $|c_k|_p < |1 - \theta y_k|_p^{-(1-\delta)}$. Also, $|1 + \theta y_k|_p = |2 - (1 - \theta y_k)|_p \leq \max\{|2|_p, |1 - \theta y_k|_p\} \leq \max\{1, \epsilon_k\} = 1$. Therefore, (8) reduces to $|1 - \theta y_k|_p^{-(1-\delta)} \leq |y_{k+1} + y_{k-1}|_p \leq \max\{|y_{k+1}|_p, |y_{k-1}|_p\}$. Without loss of generality, we choose the maximum to be $|y_{k+1}|_p$ and for the rest of the proof we use $|y_{k+1}|_p \geq |1 - \theta y_k|_p^{-(1-\delta)}$.

From $|1-\theta y_k|_p < \epsilon_k$ and $|y_{k+1}|_p \geq |1-\theta y_k|_p^{-(1-\delta)}$, we have

$$\epsilon_k^{-(1-\delta)} < |y_{k+1}|_p \leq \max\{1, |1 \pm y_{k+1}|_p\} = |1 \pm y_{k+1}|_p.$$

Rewriting equation (1) as $y_{k+2}+c_{k+1}+\theta = (a_{k+1}+b_{k+1}+c_{k+1})/[2(1-y_{k+1})]+(a_{k+1}-b_{k+1}+c_{k+1})/[2(1+y_{k+1})]+\theta(1-\theta y_k)$, we have $|y_{k+2}+c_{k+1}+\theta|_p < \epsilon_k^{1-2\delta}$. From (6),

$$\begin{aligned}\epsilon_k &\leq \epsilon_k^\delta \leq |c_{k+1}+\theta \pm \theta|_p = |(y_{k+2}+c_{k+1}+\theta) \pm \theta(1 \mp \theta y_{k+2})|_p \\ &\leq \max\{|y_{k+2}+c_{k+1}+\theta|_p, |1 \mp \theta y_{k+2}|_p\} = |1 \mp \theta y_{k+2}|_p,\end{aligned}$$

for $\delta < 1/3$, which proves the lemma for non-Archimedean absolute values ($p < \infty$).

The Archimedean case ($p = \infty$) is similar. We have

$$\begin{aligned} 10|1-\theta y_k|_\infty^{-(1-\delta)} &< \frac{10\epsilon_k^\delta}{|1-\theta y_k|_\infty} \leq \frac{|a_k+\theta b_k+c_k|_\infty}{|1-\theta y_k|_\infty} \\ &\leq |(y_{k+1}+y_{k-1})(1+\theta y_k)+\theta b_k + c_k(1+\theta y_k)|_\infty \\ &\leq |y_{k+1}+y_{k-1}|_\infty \cdot |1+\theta y_k|_\infty + |b_k|_\infty + |c_k|_\infty \cdot |1+\theta y_k|_\infty. \qquad (9)\end{aligned}$$

Also $|1+\theta y_k|_\infty \leq |1-\theta y_k|_\infty + |2|_\infty < \epsilon_k + 2 < 3$. Finally we have from (6) that $|b_k|_\infty < |1-\theta y_k|_\infty^{-(1-\delta)}$ and $|c_k|_\infty < |1-\theta y_k|_\infty^{-(1-\delta)}$. So (9) gives $10|1-\theta y_k|_\infty^{-(1-\delta)} < 3|y_{k+1}+y_{k-1}|_\infty + |b_k|_\infty + 3|c_k|_\infty < 3|y_{k+1}+y_{k-1}|_\infty + 4|1-\theta y_k|_\infty^{-(1-\delta)}$. Therefore, $2|1-\theta y_k|_\infty^{-(1-\delta)} \leq |y_{k+1}+y_{k-1}|_\infty \leq |y_{k+1}|_\infty + |y_{k-1}|_\infty$. Hence, either $|y_{k+1}|_\infty \geq |1-\theta y_k|_\infty^{-(1-\delta)}$ or $|y_{k-1}|_\infty \geq |1-\theta y_k|_\infty^{-(1-\delta)}$, which proves the first assertion of the lemma. Without loss of generality, we take $|y_{k+1}|_\infty \geq |1-\theta y_k|_\infty^{-(1-\delta)}$.

We have $|1 \pm y_{k+1}|_\infty^{-1} < \frac{5}{4}\epsilon_k^{(1-\delta)}$, so $|y_{k+2}+c_{k+1}+\theta|_\infty \leq |a_{k+1}+b_{k+1}+c_{k+1}|_\infty/(2|1-y_{k+1}|_\infty) + |a_{k+1}-b_{k+1}+c_{k+1}|_\infty/(2|1+y_{k+1}|_\infty) + |1-\theta y_k|_\infty < \frac{1}{8}\epsilon_k^{1-2\delta} + \frac{1}{8}\epsilon_k^{1-2\delta} + \epsilon_k < 9\epsilon_k^\delta$. This gives $10\epsilon_k^\delta \leq |c_{k+1}+\theta \pm \theta|_\infty = |(y_{k+2}+c_{k+1}+\theta) \pm \theta(1 \mp \theta y_{k+2})|_\infty < 9\epsilon_k^\delta + |1 \mp \theta y_{k+2}|_\infty$. Hence, $|y_{k+2}+c_{k+1}+\theta|_\infty \leq 9\epsilon_k^\delta < 9|1 \mp \theta y_{k+2}|_\infty$. Now $10\epsilon_{k+2} < 10\epsilon_{k+2}^\delta \leq |c_{k+1}+\theta \pm \theta|_\infty = |(y_{k+2}+c_{k+1}+\theta) \pm \theta(1 \mp \theta y_{k+2})|_\infty \leq 10|1 \mp \theta y_{k+2}|_\infty$. □

We are now ready to prove the following.

Theorem 4 *Let $(y_n) \subset \mathbb{Q} \setminus \{-1, 1\}$ be an admissible solution of the equation (1), where a_n, b_n and c_n are rational functions of n with $c_n \neq 0$ or ± 2 and the right hand side of (1) is of degree 2 in y_n for all sufficiently large n. Then* *there exists an integer r_0 such that for all $r \geq r_0$ and $F < 2$, the summed logarithmic height*

$$h_r(y_n) = \sum_{n=r_0}^{r} h(y_n) = \sum_{n=r_0}^{r} \sum_{p \leq \infty} \log^+ |y_n|_p,$$

satisfies $h_r(y_n) \geq F^r D$ for some $D > 0$.

Proof : We will show that there is a number $\tau < 2$ such that for each absolute value $|.|_p$ ($\forall p \leq \infty$) on $\mathbb{Q}$ and for all $r \geq r_0$,

$$\sum_{n=r_0}^{r} \left(\log^+ \frac{1}{|1 - y_n|_p} + \log^+ \frac{1}{|1 + y_n|_p} \right) \leq \tau \sum_{n=r_0}^{r+1} \log^+ |y_n|_p. \qquad (10)$$

We can then sum this inequality over all absolute values to show that the summed logarithmic height grows exponentially.

Fix a prime $p \leq \infty$ and an integer $r_0 > K$ and define the four sets

$$\begin{aligned} A_r^{\pm} &= \{n \,:\, r_0 \leq n \leq r \text{ and } |1 \mp y_n|_p < \epsilon_n\}, \\ B_r^{\pm} &= \{n \,:\, r_0 \leq n \leq r \text{ and } \epsilon_n \leq |1 \mp y_n|_p < 1\}, \end{aligned}$$

where ϵ_n is given by equation (6). We now show that $A_r^+ \cap A_r^- = \emptyset$. For any $n \in A_r^+$ we have $|1 - y_n| < \epsilon_n$. If $p < \infty$,

$$\epsilon_n \leq \epsilon_n^{\delta} \leq |2|_p \leq \max\{|1 - y_n|_p, |1 + y_n|_p\} = |1 + y_n|_p,$$

so $n \notin A_r^-$. The same conclusion holds in the Archimedean case since $\epsilon_n < 1$ and so

$$1 < |2|_\infty - \epsilon_n \leq |1 - y_n|_\infty + |1 + y_n|_\infty - \epsilon_n < |1 + y_n|_\infty.$$

Lemma 3 shows that for each $n \in A_r^{\pm}$, we can define $\sigma_n^{\pm} = -1$ or 1 such that $|y_{n+\sigma_n^{\pm}}|_p \geq \frac{1}{|1 \mp y_n|_p^{1-\delta}}$. Lemma 3 also shows that $\{n + \sigma_n^+ | n \in A_r^+\} \cap \{n + \sigma_n^- | n \in A_r^-\} = \emptyset$ and that

$$\begin{aligned} & \sum_{k \in A_r^+} \log^+ \frac{1}{|1 - y_k|_p} + \sum_{k \in A_r^-} \log^+ \frac{1}{|1 + y_k|_p} \\ \leq \;& \frac{1}{1-\delta} \Big(\sum_{k \in A_r^+} \log^+ |y_{k+\sigma_k^+}|_p + \sum_{k \in A_r^-} \log^+ |y_{k+\sigma_k^-}|_p \Big) \\ \leq \;& \frac{1}{1-\delta} \sum_{k=r_0-1}^{r+1} \log^+ |y_k|_p. \qquad (11) \end{aligned}$$

Recalling the definition of ϵ_k in (6), we have

$$\sum_{k\in B_r^\pm} \log^+ \frac{1}{|1\pm y_k|_p} \le \sum_{k\in B_r^\pm} \log^+ \epsilon_k^{-1} = \frac{1}{\delta}\sum_{k\in B_r^\pm} \log^+ \left(\kappa_p \max_{x\in X_k}\{|x|_p\}\right)$$

$$\le \frac{1}{\delta}\sum_{k=r_0}^{r}\left(\log^+\kappa_p + \sum_{x\in X_k}\log^+|x|_p\right) =: M_p. \tag{12}$$

So from the inequalities (11) and (12) we see that for all primes $p \le \infty$,

$$\sum_{k=r_0}^{r}\left(\log^+\frac{1}{|1-y_k|_p} + \log^+\frac{1}{|1+y_k|_p}\right) \le \frac{1}{1-\delta}\sum_{k=r_0-1}^{r+1}\log^+|y_k|_p + 2M_p.$$

To get the height, we sum over all the primes ($p \le \infty$) which yields

$$\sum_{k=r_0}^{r} h\left(\frac{1}{1-y_k}\right) + \sum_{k=r_0}^{r} h\left(\frac{1}{1+y_k}\right) \le \frac{1}{1-\delta}\sum_{k=r_0-1}^{r+1} h(y_k) + R_r, \tag{13}$$

where

$$R_r = \frac{2}{\delta}\sum_{k=r_0}^{r}\sum_{x\in X_k} h(x) + \frac{r-r_0}{2}\log 10 = o\left(\sum_{k=r_0-1}^{r+1} h(y_k)\right),$$

where the second equality follows from our admissibility condition (2). Furthermore, we have $|h\left((1-\theta y_k)^{-1}\right) - h(y_k)|_\infty \le \log 2$, where $\theta = 1$ or -1. So we see that the summed logarithmic height satisfies $h_{r+1}(y_k) \ge 2(1-\delta)h_r(y_k) + o(h_{r+1}(y_k))$ and hence for any $\nu > 0$ there is a constant $D > 0$ such that

$$h_r(y_k) \ge \left(\frac{2(1-\delta)}{1+\nu}\right)^r D. \tag{14}$$

For sufficiently small δ, ν, $1 < F = \frac{2(1-\delta)}{1+\nu} < 2$, which proves the theorem. $\square$

Now we consider the case in which c_n vanishes identically, i.e.

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2}. \tag{15}$$

Our strategy is again to prove an inequality of the form (10) with $\tau < 2$. The integer K is chosen such that for all $n > K$, $a_n + b_n$ and $a_n - b_n$ are nonzero and each of the expressions $\pm a_n + b_n - 2b_{n+1}$, $\pm a_n + b_n - 2b_{n-1}$ and $a_n \pm b_n \pm (\pm a_{n-2} + b_{n-2} - 2b_{n-1})$ is either identically zero or for all $n > K$ it is nonzero.

In the following definition of ϵ_n we take the maximum over a set for which we ignore those elements that are undefined (or infinite) and take the maximum of all the remaining finite elements. For sufficiently small $\delta > 0$ and for all $n > K$ we define ϵ_n by

$$\begin{aligned}\epsilon_n^{-\delta} = \kappa_p \max\Big\{ \quad & |2|_p^{-1}, |1/2|_p \cdot |a_n \pm b_n|_p, |1/2|_p^{-1} \cdot |a_n \pm b_n|_p^{-1}, |a_{n+1}|_p, |b_{n+1}|_p, \\ & |a_{n-1}|_p, |b_{n-1}|_p, |1/2|_p \cdot |a_{n+2} \pm b_{n+2}|_p, |1/2|_p \cdot |a_{n-2} \pm b_{n-2}|_p, \\ & | \pm a_n + b_n - 2b_{n+1}|_p, | \pm a_n + b_n - 2b_{n-1}|_p, |a_n \pm b_n|_p^{-1}, \\ & | \pm a_n + b_n - 2b_{n+1}|_p^{-1}, | \pm a_n + b_n - 2b_{n-1}|_p^{-1}, |a_n \pm b_n|_p, \\ & |1/2|_p^{-1} \cdot |a_n \mp b_n \mp (\pm a_{n-2} + b_{n-2} - 2b_{n-1})|_p^{-1} \Big\}, \end{aligned} \tag{16}$$

where $\kappa_p = 1$ if $p < \infty$ and $\kappa_\infty = 10$. It is evident from the definition that $\epsilon_n \leq 1$ when $p < \infty$ and $p \neq 2$, while $\epsilon_n < 1$ when $p = \infty$ or $p = 2$.

We again define the sets $A_r^{\pm}$ and $B_r^{\pm}$ as in (11). The points of $A_r^{\pm}$ will be called ± 1 points (since y_n is close to ± 1 with respect to the absolute value). As in the proof of Theorem 4, it can be shown that if $|1 - \theta y_n|_p < \epsilon_n$ for $\theta = 1$ or -1, then $|1 + \theta y_n|_p \geq \epsilon_n$. Hence $A_r^+ \cap A_r^- = \emptyset$. The admissibility of y_n then implies that

$$\sum_{n=r_0}^{r} \left(\log^+ \frac{1}{|1 - y_n|_p} + \log^+ \frac{1}{|1 + y_n|_p} \right) = \sum_{n \in A_r^+} \log^+ \frac{1}{|1 - y_n|_p} + \sum_{n \in A_r^-} \log^+ \frac{1}{|1 + y_n|_p} + \Phi_r,$$

where $\sum_{p \leq \infty} \Phi_r = o(h_{r+1}(y_n))$.

We construct a number of disjoint subintervals containing only 1 points, -1 points and points where y_n is sufficiently large to make a significant contribution to the right hand side of the inequality (10).

Definition 5 *Suppose that $|1 - \theta y_k|_p < \epsilon_k$, for some $k \in \mathbb{Z}$ and $\theta = 1$ or $\theta = -1$. Then the* <u>*oscillating sequence*</u> *S containing k is the longest interval in $\mathbb{Z}$ (possibly unbounded) satisfying the following conditions.*

1. *If $k + 2l \in S$ then $|1 - (-1)^l \theta y_{k+2l}|_p < \epsilon_{k+2l}$;*

2. *If $\{k+2l-1, k+2l\} \subseteq S$, then $|y_{k+2l-1}|_p \geq |1 - (-1)^l \theta y_{k+2l}|_p^{-(1-\delta)}$; and*

3. If $\{k+2l, k+2l+1\} \subseteq S$, *then* $|y_{k+2l+1}|_p \geq |1-(-1)^l \theta y_{k+2l}|_p^{-(1-\delta)}$.

If $|1-\theta y_n|_p < \epsilon_n$ then either $|y_{n+1}|_p \geq |1-\theta y_n|_p^{-(1-\delta)}$ or $|y_{n-1}|_p \geq |1-\theta y_n|_p^{-(1-\delta)}$, so every ± 1 point lies in an oscillating sequence containing at least two elements. For a fixed oscillating sequence S and $r \geq r_0$, we will now obtain a suitable upper bound for

$$\sum_{n \in S \cap A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in S \cap A_r^-} \log^+ \frac{1}{|1+y_n|_p}. \tag{17}$$

Case 1: Let $m+1$ be the total number of 1 points and -1 points in $S \cap [r_0, r]$ and assume that $m \geq 2$. Let I be the shortest subinterval of $S \cap [r_0, r]$ containing these ± 1 points. Let k be the first term in I, so that $|1-\theta y_k|_p < \epsilon_k$ for some choice of $\theta = -1$ or 1. Then $I = \{k, k+1, \ldots, k+2m\}$ and contains exactly m points on which y_n is big in the sense that $|y_{k+1}|_p \geq |1-\theta y_k|_p^{-(1-\delta)}$, $|y_{k+2m-1}|_p \geq |1-(-1)^m \theta y_{k+2m}|_p^{-(1-\delta)}$ and $|y_{k+2l+1}|_p \geq \max\{|1-(-1)^l \theta y_{k+2l}|_p^{-(1-\delta)}, |1-(-1)^{l+1}\theta y_{k+2l+2}|_p^{-(1-\delta)}\}$, for all $l = 1, \ldots, m-2$. Hence

$$\begin{aligned}
&\sum_{n \in S \cap A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in S \cap A_r^-} \log^+ \frac{1}{|1+y_n|_p} \\
&= \sum_{l=0}^{m} \log^+ \frac{1}{|1-(-1)^l \theta y_{k+2l}|_p} \\
&= \sum_{l=1}^{m} \frac{l}{m} \log^+ \frac{1}{|1-(-1)^l \theta y_{k+2l}|_p} + \sum_{l=0}^{m-1} \frac{m-l}{m} \log^+ \frac{1}{|1-(-1)^l \theta y_{k+2l}|_p} \\
&= \sum_{l=0}^{m-1} \frac{l+1}{m} \log^+ \frac{1}{|1-(-1)^{l+1} \theta y_{k+2l+2}|_p} + \sum_{l=0}^{m-1} \frac{m-l}{m} \log^+ \frac{1}{|1-(-1)^l \theta y_{k+2l}|_p} \\
&\leq \frac{1}{1-\delta} \sum_{l=0}^{m-1} \left(\frac{l+1}{m} + \frac{m-l}{m} \right) \log^+ |y_{k+2l+1}|_p \\
&= \frac{m+1}{(1-\delta)m} \sum_{l=0}^{m-1} \log^+ |y_{k+2l+1}|_p = \frac{m+1}{(1-\delta)m} \sum_{n \in S \cap [r_0, r]} \log^+ |y_n|_p \\
&\leq \frac{3}{2(1-\delta)} \sum_{n \in S \cap [r_0, r]} \log^+ |y_n|_p,
\end{aligned}$$

where the last inequality follows from $m \geq 2$.

Case 2: There are exactly two ± 1 points in $S \cap [r_0, r]$. Define k such that these points are k and $k+2$. That is, for some choice of $\theta = \pm 1$, we have $|1-\theta y_k|_p < \epsilon_k$ and $|1+\theta y_{k+2}|_p < \epsilon_{k+2}$. We will use the following corollary of Theorem 2.

Corollary 6 *Fix a prime $p \leq \infty$. For some $k > K$ let ϵ_k be given (16) and suppose that for $\theta = 1$ or -1, $|1-\theta y_k|_p < \epsilon_k$, $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$ and $|1+\theta y_{k+2}|_p < \epsilon_{k+2}$. Assume that $a_k - \theta b_k - \theta(\theta a_{k-2} + b_{k-2} - 2b_{k-1}) \not\equiv 0$, then $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-1/2}$.*

Proof: We begin with the non-Archimedean case $p < \infty$. From part *(iii)* of Theorem 2 and the definition (16) , we have

$$\begin{aligned} |1+\theta y_{k+2}|_p^{-(1-\delta)} &< \frac{\epsilon_{k+2}^{\delta}}{|1+\theta y_{k+2}|_p} \leq \frac{|a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1})|_p}{|2|_p \cdot |1+\theta y_{k+2}|_p} \\ &= |y_{k+3} - C_k|_p \leq \max\{|y_{k+3}|_p, |C_k|_p\}. \end{aligned} \tag{18}$$

From Theorem 2 we have that for sufficiently small $\delta > 0$, $|C_k|_p \leq |1+\theta y_{k+2}|_p^{-2/3-2\delta} \leq |1+\theta y_{k+2}|_p^{-(1-\delta)}$. So (18) reduces to $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-(1-\delta)} \geq |1+\theta y_{k+2}|_p^{-1/2}$.

For the Archimedean absolute value, $\kappa_\infty = 10$ in (16), giving

$$\begin{aligned} &10|1+\theta y_{k+2}|_p^{-(1-\delta)} < \frac{10\epsilon_{k+2}^{\delta}}{|1+\theta y_{k+2}|_p} \\ \leq\ & \frac{|a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1})|_p}{|2|_p \cdot |1+\theta y_{k+2}|_p} = |y_{k+3} - C_k|_p \leq |y_{k+3}|_p + |C_k|_p \\ \leq\ & |y_{k+3}|_p + 2|1+\theta y_{k+2}|_p^{-2/3-2\delta} \leq |y_{k+3}|_p + 9|1+\theta y_{k+2}|_p^{-(1-\delta)}. \end{aligned}$$

So $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-(1-\delta)} \geq |1+\theta y_{k+2}|_p^{-1/2}$ which completes the proof. □

Hence if $a_k - \theta b_k - \theta(\theta a_{k-2} + b_{k-2} - 2b_{k-1}) \not\equiv 0$, then either $|y_{k-1}|_p > |1-\theta y_k|_p^{-1/2}$ or $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-1/2}$. This says that, even if neither $k-1$ nor $k+3$ is in S, at least one of y_{k-1} or y_{k+3} has to be moderately large. Without loss of generality, we assume that $|y_{k-1}|_p > |1-\theta y_k|_p^{-1/2}$. For $\eta > 0$, we have

$$\begin{aligned}
&\sum_{n\in S\cap A_r^+}\log^+\frac{1}{|1-y_n|_p}+\sum_{n\in S\cap A_r^-}\log^+\frac{1}{|1+y_n|_p}\\
=\;&\log^+\frac{1}{|1-\theta y_k|_p}+\log^+\frac{1}{|1+\theta y_{k+2}|_p}\\
=\;&\eta\log^+\frac{1}{|1-\theta y_k|_p}+(1-\eta)\log^+\frac{1}{|1-\theta y_k|_p}+\log^+\frac{1}{|1+\theta y_{k+2}|_p}\\
\le\;&2\eta\log^+|y_{k-1}|_p+\frac{1-\eta}{1-\delta}\log^+|y_{k+1}|_p+\frac{1}{1-\delta}\log^+|y_{k+1}|_p\\
=\;&2\eta\log^+|y_{k-1}|_p+\frac{2-\eta}{1-\delta}\log^+|y_{k+1}|_p. \qquad (19)
\end{aligned}$$

So we can reduce the coefficient of $\log^+|y_{k+1}|_p$ by introducing a contribution from y_{k-1}. If $k-1\in S$, this is not problematic and an upper bound for (19) is

$$\max\left(\frac{2-\eta}{1-\delta},2\eta\right)\sum_{n\in S}\log^+|y_n|_p.$$

However, if $k-1\notin S$ then we need to be careful because we will later sum our estimates for (17) over all oscillating sequences. When we do this we might need to "share" the term $k-1$ with another oscillating sequence, in which case it will appear twice in the upper bound and we will need to sum the contributions. Note that the term $k-1$ here cannot be part of a subinterval I of the type considered in case 1 above as such subintervals of oscillating sequences have only ± 1 points as endpoints. There could, however, be two adjacent oscillating sequences S_1 and S_2 both of the type considered in the present case (case 2) that need to share the contribution from y_{k-1}. If so, then summing over the contributions for both oscillating sequences would give the upper bound

$$\frac{2-\eta}{1-\delta}\log^+|y_{k-3}|_p+4\eta\log^+|y_{k-1}|_p+\frac{2-\eta}{1-\delta}\log^+|y_{k+1}|_p$$

which, in turn, is bounded from above by

$$\max\left(\frac{2-\eta}{1-\delta},4\eta\right)\sum_{n=k-3}^{k+1}\log^+|y_n|_p.$$

Note that $k-1$ could also be part of an oscillating sequence of the type we are about to consider in case 3.

Case 3: There is exactly one $k_1 \in S \cap [r_0, r]$ such that $|1-\theta y_{k_1}|_p < \epsilon_{k_1}$ for $\theta = -1$ or 1. Since S has at least two points, we know that either $|y_{k_1-1}|_p \geq |1-\theta y_{k_1}|_p^{-(1-\delta)}$ or $|y_{k_1+1}|_p \geq |1-\theta y_{k_1}|_p^{-(1-\delta)}$. Without loss of generality, we assume the latter. Note that since $k_1 \in S \cap [r_0, r]$, then $k_1 + 1 \in S \cap [r_0, r+1]$. So

$$\sum_{n \in S \cap A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in S \cap A_r^-} \log^+ \frac{1}{|1+y_n|_p} = \log^+ \frac{1}{|1-\theta y_{k_1}|_p} \leq \frac{1}{1-\delta} \log^+ |y_{k_1+1}|_p.$$

It is conceivable that k_1+1 is adjacent to, or part of, a sequence of the type considered in case 2 in such a way that it plays the role of $k-1$ in the analysis above of that case. In other words, summing over the contributions of these two oscillating sequences in the left side of (10) leads to a term of the form

$$\left(\frac{1}{1-\delta} + 2\eta\right) \log^+ |y_{k_1+1}|_p$$

on the right hand side.

If both $a_k - b_k - a_{k-2} - b_{k-2} + 2b_{k-1}$ and $a_k + b_k - a_{k-2} + b_{k-2} - 2b_{k-1}$ are nonzero then combining our results from the above cases, we have

$$\sum_{n=r_0}^{r} \left(\log^+ \frac{1}{|1-y_n|_p} + \log^+ \frac{1}{|1+y_n|_p}\right) \leq \tau \sum_{n=r_0}^{r+1} \log^+ |y_n|_p + \Phi_r,$$

where

$$\tau = \max\left(\frac{3}{2(1-\delta)}, \frac{2-\eta}{1-\delta}, 2\eta, 4\eta, \frac{1}{1-\delta} + 2\eta\right).$$

In particular, choosing $\eta = 3/8$ and δ sufficiently small, we have $\tau = 3/4 + (1-\delta)^{-1} < 2$. Since $\sum_{p \leq \infty} \Phi_r = o(h_{r+1}(y_n))$, we see that

$$h_r(y_n) \;\; \leq \;\; \frac{\tau}{2} h_{r+1}(y_n) + o(h_{r+1}(y_n)), \qquad (20)$$

so $h_r(y)$ grows exponentially with r.

The argument above is based on the fact that Corollary 6 guarantees that if $a_k - \theta b_k - \theta(\theta a_{k-2} + b_{k-2} - 2b_{k-1}) \not\equiv 0$ then there can be no *special oscillating sequences* as defined below.

Definition 7 *The special oscillating sequence S_p starting with k is $S_p = \{k, k+1, k+2\}$. It is an oscillating sequence of length 3 starting with k in $\mathbb{Z}$ such that $|1-\theta y_k|_p < \epsilon_k$, $|y_{k+1}|_p \geq \max\left\{|1-\theta y_k|_p^{-(1-\delta)}, |1+\theta y_{k+2}|_p^{-(1-\delta)}\right\}$ and $|1+\theta y_{k+2}|_p < \epsilon_{k+2}$. Also, we have $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$ and $|y_{k+3}|_p \leq |1+\theta y_{k+2}|_p^{-1/2}$.*

Note that there are two types of special oscillating sequences depending on whether $\theta = 1$ or $\theta = -1$. In order for $h_r(y_n)$ to grow sub-exponentially, there must be infinitely many special oscillating sequences. If there are infinitely many special oscillating sequences of both types then both $a_k - b_k - (a_{k-2} + b_{k-2} - 2b_{k-1})$ and $a_k + b_k + (-a_{k-2} + b_{k-2} - 2b_{k-1})$ must vanish, which characterises part *(i)* of the theorem. The rest of this section will be a careful analysis of the case in which there are infinitely many special oscillating sequences of one kind only, corresponding to a fixed value of $\theta = \pm 1$. For the rest of this section when we refer to special oscillating sequences we mean those sequences of the form θ, ∞, $-\theta$, for this fixed value of θ (where "∞" refers to a large term).

We define f_n by

$$f_n = (1-\theta y_n)y_{n+1} - y_n. \tag{21}$$

So $y_{n+1} = (f_n + y_n)/(1-\theta y_n)$, $y_{n-1} = (y_n - f_{n-1})/(1+\theta y_n)$, and (15) yield

$$y_{n+1} + y_{n-1} = \frac{(f_n - f_{n-1}) + (2 + \theta f_n + \theta f_{n-1})y_n}{1-y_n^2} = \frac{a_n + b_n y_n}{1-y_n^2}.$$

Hence

$$(b_n - 2 - \theta f_n - \theta f_{n-1})y_n = f_n - f_{n-1} - a_n. \tag{22}$$

If for all n, $b_n - 2 - \theta f_n - \theta f_{n-1} = 0$, then $f_n - f_{n-1} - a_n = 0$ and

$$f_n = \frac{1}{2\theta}(\theta a_n + b_n - 2).$$

This shows that y_n solves the discrete Riccati equation (3).

Next consider the case $b_n - 2 - \theta f_n - \theta f_{n-1} \neq 0$, $\forall n > K$. From (22) we have

$$y_n = \frac{f_n - f_{n-1} - a_n}{b_n - 2 - \theta f_n - \theta f_{n-1}}. \tag{23}$$

Taking the logarithmic height of both sides of (23) and using some elementary properties of heights, we have

$$\begin{aligned} h(y_n) &= h\left(\frac{f_n - f_{n-1} - a_n}{b_n - 2 - \theta f_n - \theta f_{n-1}}\right) \\ &\leq h(f_n - f_{n-1} - a_n) + h\left(\frac{1}{b_n - 2 - \theta f_n - \theta f_{n-1}}\right) \\ &= h(f_n - f_{n-1} - a_n) + h(b_n - 2 - \theta f_n - \theta f_{n-1}) \\ &\leq 2h(f_n) + 2h(f_{n-1}) + h(a_n) + h(b_n) + \log 24. \end{aligned}$$

Summing both sides of the inequality above and using the fact that $h_r(f_n)$ is a non-decreasing function of n, we have

$$h_r(y_n) \leq 4h_{r+1}(f_n) + h_r(a_n) + h_r(b_n) + (r - r_0 + 1)\log 24. \tag{24}$$

From (21) we have

$$f_n + \theta = \theta(1 - \theta y_n)(1 + \theta y_{n+1}). \tag{25}$$

For every prime $p \leq \infty$, we define a set $C_p \subset \mathbb{Z}$ such that it consists of all the big terms in special oscillating sequences i.e. the terms ∞s in the form: θ, ∞, $-\theta$. For a fixed prime p and sufficiently large r_0, we have

$$\begin{aligned} \sum_{n=r_0}^{r} \log^+ \frac{1}{|f_n + \theta|_p} = & \sum\nolimits^r_{\substack{n = r_0 \\ n \in C_p}} \log^+ \tfrac{1}{|f_n+\theta|_p} + \sum\nolimits^r_{\substack{n = r_0 \\ n+1 \in C_p}} \log^+ \tfrac{1}{|f_n+\theta|_p} \\ & \sum\nolimits^r_{\substack{n = r_0 \\ n \notin C_p \text{ and } n+1 \notin C_p}} \log^+ \tfrac{1}{|f_n+\theta|_p}. \end{aligned} \tag{26}$$

In the above inequality we split the interval $[r_0, r]$ into points that are in special oscillating sequences (where n, $n+1 \in C_p$) and points in any other oscillating sequence that is not special. Note that for $n \in C_p$, we have $|1 + \theta y_{n+1}|_p^{-(1-\delta)} \leq |y_n|_p$. Therefore, for $n \in C_p$ we have

$$\begin{aligned} \log^+ \frac{1}{|f_n + \theta|_p} &= \log^+ \frac{1}{|1 - \theta y_n|_p \cdot |1 + \theta y_{n+1}|_p} \leq \log^+ |1 - \theta y_n|_p^{-1} \cdot |y_n|_p^{\frac{1}{1-\delta}} \\ &= \log^+ |1 - \theta y_n|_p^{-1} \cdot |y_n|_p^{\frac{\delta+1-\delta}{1-\delta}} = \log^+ |1 - \theta y_n|_p^{-1} \cdot |y_n|_p \cdot |y_n|_p^{\frac{\delta}{1-\delta}} \\ &\leq \frac{\delta}{1-\delta} \log^+ |y_n|_p + \log^+ \left|\frac{y_n}{1 - \theta y_n}\right|_p. \end{aligned}$$

Since $|y_n|_p$ is big, it is away from θ and $-\theta$. If $p < \infty$, then $|y_n|_p = |\theta - \theta(1-\theta y_n)|_p \le \max\{1, |1-\theta y_n|_p\} = |1-\theta y_n|_p$, since $|y_n|_p > 1$. Hence, the term $\log^+ \left|\frac{y_n}{1-\theta y_n}\right|_p$ vanishes. For $p = \infty$ we have the following relation $\epsilon_{n+1}^{-\delta} < \epsilon_{n+1}^{-(1-\delta)} < |1+\theta y_{n+1}|_\infty^{-(1-\delta)} \le |y_n|_\infty \le 1 + |1-\theta y_n|_\infty$ which yields $\epsilon_{n+1}^{-\delta} - 1 \le |1-\theta y_n|_\infty$. Consequently, $\frac{1}{|1-\theta y_n|_\infty} \le \frac{1}{\epsilon_{n+1}^{-\delta}-1}$. Starting with $|y_n|_\infty \le 1+|1-\theta y_n|_\infty$ then dividing both sides by $|1-\theta y_n|_\infty$ implies $\frac{|y_n|_\infty}{|1-\theta y_n|_\infty} \le \frac{1}{|1-\theta y_n|_\infty} + 1 \le \frac{1}{\epsilon_{n+1}^{-\delta}-1} + 1$. Therefore, $\left|\frac{y_n}{1-\theta y_n}\right|_\infty \le \frac{1}{4} + 1 = \frac{5}{4}$ since $5 \le \epsilon_{n+1}^{-\delta}$, giving

$$\sum_{p\le\infty} \sum_{\substack{n = r_0 \\ n \in C_p}}^{r} \log^+ \frac{1}{|f_n+\theta|_p} \le \frac{\delta}{1-\delta} h_r(y_n) + (r - r_0 + 1)\log(5/4). \qquad (27)$$

Similarly,

$$\sum_{p\le\infty} \sum_{\substack{n = r_0 \\ n+1 \in C_p}}^{r} \log^+ \frac{1}{|f_n+\theta|_p} \le \frac{\delta}{1-\delta} h_{r+1}(y_n) + (r - r_0 + 1)\log(5/4). \qquad (28)$$

Summing over all $p \le \infty$ in (26) and using (25), (27) and (28) yields

$$h_r(f_n) - (r-r_0+1)\log 2 \le h_r\left(\frac{1}{f_n+\theta}\right) \le \frac{2\delta}{1-\delta} h_{r+1}(y_n) + 2(r-r_0+1)\log(5/4)$$
$$+ \sum_{p\le\infty}\left\{ \sum_{\substack{n = r_0 \\ n+1 \notin C_p}}^{r} \log^+ \frac{1}{|1-\theta y_n|_p} + \sum_{\substack{n = r_0 \\ n \notin C_p}}^{r} \log^+ \frac{1}{|1+\theta y_{n+1}|_p} \right\}.$$

Therefore,

$$h_r(f_n) \le \frac{2\delta}{1-\delta} h_{r+1}(y_n) + (r-r_0+1)\log(25/8) + B_{r+1}, \qquad (29)$$

where

$$B_r = \sum_{p\le\infty}\left\{ \sum_{\substack{n = r_0 \\ n+1 \notin C_p}}^{r} \log^+ \frac{1}{|1-\theta y_n|_p} + \sum_{\substack{n = r_0 \\ n-1 \notin C_p}}^{r} \log^+ \frac{1}{|1+\theta y_n|_p} \right\}. \qquad (30)$$

From our previous analysis of oscillating sequences that are not special, it follows from (20) that

$$B_r \leq \tau \sum_{p\leq\infty} \sum_{\substack{n = r_0 \\ n \notin C_p}}^{r+1} \log^+ |y_n|_p + R_r. \tag{31}$$

Recall that $\tau < 2$ and R_r is an expression that involves the summed logarithmic heights of the coefficients a_n and b_n. Applying the shift $r \to r+1$ in (29) and (31), then using the result in (24) yields

$$h_r(y_n) \leq \frac{8\delta}{1-\delta} h_{r+2}(y_n) + 4\tau \sum_{p\leq\infty} \sum_{\substack{n = r_0 \\ n \notin C_p}}^{r+3} \log^+ |y_n|_p + \widehat{R}_{r+2}, \tag{32}$$

where $\widehat{R}_{r+2} = o(h_{r+2}(y_n))$.

Now we consider the following inequality

$$\sum_{p\leq\infty} \sum_{n=r_0}^{r} \left\{ \log^+ \frac{1}{|1-y_n|_p} + \log^+ \frac{1}{|1+y_n|_p} \right\}$$

$$\leq \sum_{p\leq\infty} \left\{ \sum_{\substack{n = r_0 \\ n+1 \in C_p}}^{r} \log^+ \frac{1}{|1-\theta y_n|_p} + \sum_{\substack{n = r_0 \\ n-1 \in C_p}}^{r} \log^+ \frac{1}{|1+\theta y_n|_p} \right\} + B_r.$$

Recall that if $n+1 \in C_p$ (or $n-1 \in C_p$), then $|y_{n+1}|_p \geq |1-\theta y_n|_p^{-(1-\delta)}$ (or $|y_{n-1}|_p \geq |1+\theta y_n|_p^{-(1-\delta)}$). Using this fact and (31) we have

$$\begin{aligned}
&\sum_{p\leq\infty}\sum_{n=r_0}^{r}\left\{\log^+\frac{1}{|1-y_n|_p}+\log^+\frac{1}{|1+y_n|_p}\right\}\\
\leq\ & \frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\in C_p}}^{r+1}\log^+|y_n|_p+\tau\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+R_r\\
=\ & \frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\in C_p}}^{r+1}\log^+|y_n|_p+\frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p\\
-\ & \frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+\tau\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+R_r\\
=\ & \frac{2}{1-\delta}h_{r+1}(y_n)-\left(\frac{2}{1-\delta}-\tau\right)\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+R_r. \qquad (33)
\end{aligned}$$

This implies that

$$2h_r(y_n)\leq\frac{2}{1-\delta}h_{r+1}(y_n)-\left(\frac{2}{1-\delta}-\tau\right)\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+\widetilde{R_r}, \qquad (34)$$

where $\widetilde{R_r}=o(h_r(y_n))$ as $r\to\infty$.

Considering the two inequalities in (32) and (34), we have two cases to consider depending on whether the expression $\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p$ is very small compared to $h_{r+1}(y_n)$ on a large set. In either case we obtain an inequality of the form $h_{r+s}(y_n)\geq\alpha h_r(y_n)$, for some $\alpha<1$ and $s>0$, on a set of infinite logarithmic measure, which implies conclusion *(iii)* of the theorem.

Case 1: Assume that there is a sufficiently small constant $c>0$ such that

$$\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p\leq ch_{r+1}(y_n),$$

on a set of infinite discrete logarithmic measure. Then (32) implies

$$h_r(y_n) \le \left(\frac{8\delta}{1-\delta} + 4\tau c\right) h_{r+3}(y_n) + \widehat{R}_{r+2},$$

on a set of infinite discrete logarithmic measure.

Case 2: Assume that

$$\sum_{p\le\infty} \sum_{\substack{n = r_0 \\ n \notin C_p}}^{r+1} \log^+ |y_n|_p > c h_{r+1}(y_n),$$

on a set of infinite discrete logarithmic measure. Using this inequality in (34) yields

$$2h_r(y_n) \le \left[\frac{2}{1-\delta} - \left(\frac{2}{1-\delta} - \tau\right) c\right] h_{r+1}(y_n) + \widetilde{R}_r.$$

Since $\left[\frac{2}{1-\delta} - (\frac{2}{1-\delta} - \tau)c\right] < 2$ for sufficiently small δ.

Conclusion *(iii)* of the theorem follows from the following with $w_r = h_r(y_n)$.

Lemma 8 *Let $(w_n)_{n\ge n_0}$ $(n_0 > 0)$ be a non-decreasing sequence of positive numbers. For a fixed real number $\alpha > 1$ and a fixed positive integer s we define*

$$F = \{n \ge n_0 : \ w_{n+s} \ge \alpha w_n\}. \tag{35}$$

If F has infinite discrete logarithmic measure, i.e. $\sum_{n\in F} \frac{1}{n} = \infty$, then

$$\limsup_{r\to\infty} \frac{\log\log w_r}{\log r} \ge 1. \tag{36}$$

Proof: Define a sequence (r_n) using induction as follows. Let $r_0 = \min(F)$ and for all $n > 0$, define $r_n = \min(F \cap [r_{n-1} + s, \infty))$. Hence, $r_{n+1} \ge r_n + s$ and $F \subseteq \cup_{n=0}^{\infty} [r_n, r_n + s]$. This yields $w_{r_{n+1}} \ge w_{r_n+s} \ge \alpha w_{r_n}$ for all $n \ge 0$. Iterating this relation recursively yields

$$w_{r_n} \ge \alpha^n w_{r_0}. \tag{37}$$

We use the notation $\lfloor x \rfloor$ to denote the integer part of x in the following chain of inequalities. Assume that there is a constant $\varepsilon > 0$ and an integer $m > 1$ such that $r_n \geq n^{1+\varepsilon}$ for all $n > m$. Then there is a constant E such that

$$\begin{aligned}\sum_{j \in F} \frac{1}{j} &\leq E + \sum_{n=m}^{\infty} \sum_{k=\lfloor n^{1+\varepsilon} \rfloor}^{\lfloor n^{1+\varepsilon} \rfloor + s} \frac{1}{k} \leq E + \sum_{n=m}^{\infty} \int_{n^{1+\varepsilon}-2}^{n^{1+\varepsilon}+s} \frac{\mathrm{d}t}{t} \\ &\leq E + \sum_{n=m}^{\infty} \left((s+2) n^{-(1+\varepsilon)} + O(n^{-2(1+\varepsilon)}) \right) < \infty.\end{aligned}$$

But this is a contradiction to our assumption that F has infinite discrete logarithmic measure. Therefore, there exists a subsequence (r_{n_k}) such that $r_{n_k} < n_k^{1+\varepsilon}$ for all $k \geq 0$. From (37) we have $w_{r_{n_k}} \geq \alpha^{n_k} w_{r_0}$. Hence,

$$\begin{aligned}\limsup_{r \longrightarrow \infty} \frac{\log \log w_r}{\log r} &\geq \limsup_{k \longrightarrow \infty} \frac{\log \log w_{r_{n_k}}}{\log r_{n_k}} \geq \limsup_{k \longrightarrow \infty} \frac{\log \log \alpha^{n_k} w_{r_0}}{\log n_k^{1+\varepsilon}} \\ &= \limsup_{k \longrightarrow \infty} \frac{\log (n_k \log \alpha + \log w_{r_0})}{(1+\varepsilon) \log n_k} \geq \frac{1}{1+\varepsilon}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary small number, this proves the lemma. □

This concludes the case in which $c_n \equiv 0$. The cases $c_n \equiv \pm 2$ are similar except that the existence of Riccati solutions is prohibited by the assumption in the theorem that the degree of the right side of equation (1) is two.

3 Proof of Theorem 2

For a fixed absolute value $|.|_p$, assume that $|1 - \theta y_k|_p < \epsilon_k$ for some $k > K$, where $\theta = -1$ or 1. First we rewrite equation (15) as

$$y_{k+1} + y_{k-1} = \frac{1/2(a_k + \theta b_k)}{1 - \theta y_k} + \frac{1/2(a_k - \theta b_k)}{1 + \theta y_k}. \tag{38}$$

It follows that

$$A_k := y_{k+1} - \frac{1/2(a_k + \theta b_k)}{1 - \theta y_k} = \frac{1/2(a_k - \theta b_k)}{1 + \theta y_k} - y_{k-1}. \tag{39}$$

We begin by considering non-Archimedean absolute values ($p < \infty$). In this case

$$|A_k|_p \leq \max \left\{ \frac{|1/2|_p \cdot |a_k - \theta b_k|_p}{|1 + \theta y_k|_p}, |y_{k-1}|_p \right\}. \tag{40}$$

Note that $\epsilon_k^\delta \leq |2|_p \leq \max\{|1-\theta y_k|_p, |1+\theta y_k|_p\}$. If $|1-\theta y_k|_p > |1+\theta y_k|_p$, then we obtain the contradiction $\epsilon_k^\delta < \epsilon$. Hence $\epsilon_k^\delta \leq \max\{|1-\theta y_k|_p, |1+\theta y_k|_p\} = |1+\theta y_k|_p$. This implies $|1+\theta y_k|_p{}^{-1} \leq \epsilon_k^{-\delta} < |1-\theta y_k|_p^{-\delta}$. Using this relation and (16) in (40) yields $|A_k|_p \leq \max\left\{|1-\theta y_k|_p^{-2\delta}, |1-\theta y_k|_p^{-1/2}\right\} = |1-\theta y_k|_p^{-1/2}$.

Consider

$$B_k := (y_{k+2}+\theta) - \left(\theta - \frac{2b_{k+1}}{a_k+\theta b_k}\right)(1-\theta y_k). \tag{41}$$

Incrementing equation (15) we obtain

$$\begin{aligned} B_k = \frac{a_{k+1}}{(1-y_{k+1})(1+y_{k+1})} &+ \frac{b_{k+1}}{(1-y_{k+1})(1+y_{k+1})(y_{k+1}-A_k)} \\ &- \frac{b_{k+1}A_k y_{k+1}}{(1-y_{k+1})(1+y_{k+1})(y_{k+1}-A_k)}. \end{aligned} \tag{42}$$

Now

$$|1-\theta y_k|_p{}^{-(1-\delta)} < \frac{\epsilon_k^\delta}{|1-\theta y_k|_p} \leq \frac{|1/2|_p \cdot |a_k+\theta b_k|_p}{|1-\theta y_k|_p} = |y_{k+1}-A_k|_p. \tag{43}$$

So $|1-\theta y_k|_p^{-(1-\delta)} < \max\{|y_{k+1}|_p, |1-\theta y_k|_p^{-1/2}\} \leq \max\{|y_{k+1}|_p, |1-\theta y_k|_p^{-(1-\delta)}\}$ $= |y_{k+1}|_p$. Hence, $\epsilon_k^{-(1-\delta)} < |1-\theta y_k|_p^{-(1-\delta)} \leq |y_{k+1}|_p = |1-(1\pm y_{k+1})|_p \leq \max\{1, |1\pm y_{k+1}|_p\} \leq \max\{\epsilon^{-(1-\delta)}, |1\pm y_{k+1}|_p\}$, giving

$$|1\pm y_{k+1}|_p^{-1} \leq |1-\theta y_k|_p^{1-\delta}. \tag{44}$$

Moreover, we have from the first part of the theorem that

$$|y_{k+1}|_p \leq \max\left\{\frac{|1/2|_p \cdot |a_k+\theta b_k|_p}{|1-\theta y_k|_p}, |A_k|_p\right\} \leq |1-\theta y_k|_p^{-(1+\delta)}. \tag{45}$$

Taking the p-adic absolute value of equation (42) and using the estimates above, we get

$$|B_k|_p \leq \max\left\{|1-\theta y_k|_p^{2-3\delta}, |1-\theta y_k|_p^{3-4\delta}, |1-\theta y_k|_p^{3/2-5\delta}\right\}.$$

Hence $|B_k|_p \leq |1-\theta y_k|_p^{3/2-5\delta}$, as required.

Next we have

$$C_k := y_{k+3} - \frac{(a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1}))}{2(1+\theta y_{k+2})}.$$

Incrementing equation (15) twice and eliminating y_{k+3} from the above yields

$$C_k = \frac{1/2(a_{k+2} + \theta b_{k+2})}{1 - \theta y_{k+2}} - \frac{a_k + \theta b_k}{2(1-\theta y_k)} + \frac{\theta(\theta a_k + b_k - 2b_{k+1})}{2(1+\theta y_{k+2})} - A_k.$$

Combining the two middle terms and using part (ii) in the numerator gives

$$C_k = \frac{1/2(a_{k+2} + \theta b_{k+2})}{1 - \theta y_{k+2}} - \frac{B_k(a_k + \theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)} - A_k. \tag{46}$$

From part (ii) of the theorem, we have

$$\begin{aligned} |1+\theta y_{k+2}|_p &\leq \max\left\{\frac{|\theta a_k + b_k - 2b_{k+1}|_p}{|a_k + \theta b_k|_p}|1-\theta y_k|_p, |B_k|_p\right\} \\ &\leq \max\{|1-\theta y_k|_p^{1-2\delta}, |1-\theta y_k|_p^{3/2-5\delta}\} < \epsilon_k^{1-2\delta}, \end{aligned} \tag{47}$$

where we have used (16). Also, $\epsilon_k^{\delta} \leq |2|_p \leq \max\{|1+\theta y_{k+2}|_p, |1-\theta y_{k+2}|_p\} \leq \max\{\epsilon_k^{1-2\delta}, |1-\theta y_{k+2}|_p\} = |1-\theta y_{k+2}|_p$. Hence

$$|1-\theta y_{k+2}|_p^{-1} \leq \epsilon_k^{-\delta} < |1-\theta y_k|_p^{-\delta}. \tag{48}$$

Note that if $|\theta a_k + b_k - 2b_{k+1}|_p \not\equiv 0$, then

$$\begin{aligned} &|1-\theta y_k|_p^{1+2\delta} = |1-\theta y_k|_p \cdot |1-\theta y_k|_p^{2\delta} < |1-\theta y_k|_p \epsilon_k^{2\delta} \\ \leq\ &\tfrac{|\theta a_k + b_k - 2b_{k+1}|_p}{|a_k + \theta b_k|_p}|1-\theta y_k|_p = |(1+\theta y_{k+2}) - B_k|_p \leq \max\{|1+\theta y_{k+2}|_p, |B_k|_p\} \\ &\leq \max\left\{|1+\theta y_{k+2}|_p, |1-\theta y_k|_p^{3/2-5\delta}\right\} = |1+\theta y_{k+2}|_p. \end{aligned}$$

So $|1+\theta y_{k+2}|_p^{-1} < |1-\theta y_k|_p^{-(1+2\delta)}$.

If $|\theta a_k + b_k - 2b_{k+1}|_p \not\equiv 0$ then the second term in (46) satisfies

$$\left|\frac{B_k(a_k + \theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p \leq |1-\theta y_k|_p^{1/2-6\delta} \cdot |1+\theta y_{k+2}|_p^{-1} \leq |1-\theta y_k|_p^{-1/2-8\delta}, \tag{49}$$

where we have used (16). From equation (46) we have

$$|C_k|_p \leq \max\left\{\frac{|1/2|_p\cdot|a_{k+2}+\theta b_{k+2}|_p}{|1-\theta y_{k+2}|_p}, \left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p, |A_k|_p\right\}$$
$$\leq \max\left\{|1-\theta y_k|_p^{-2\delta}, |1-\theta y_k|_p^{-1/2-8\delta}, |1-\theta y_k|_p^{-1/2}\right\} = |1-\theta y_k|_p^{-1/2-8\delta}, \quad (50)$$

where we have used (48), (49) and the first part of the theorem. From (47) we have $|1+\theta y_{k+2}|_p \leq |1-\theta y_k|_p^{1-2\delta}$. So for sufficiently small δ

$$|C_k|_p \leq |1-\theta y_k|_p^{-(1/2+8\delta)} \leq |1+\theta y_{k+2}|_p^{-1/2-10\delta} \leq |1+\theta y_{k+2}|_p^{-2/3-2\delta}.$$

Now if $|\theta a_k + b_k - 2b_{k+1}|_p \equiv 0$, then the upper bound on the second term in (46) is

$$\left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p = \frac{|1+\theta y_{k+2}|_p\cdot|a_k+\theta b_k|_p}{|2|_p\cdot|1+\theta y_{k+2}|_p\cdot|1-\theta y_k|_p} \leq |1-\theta y_k|_p^{-(1+\delta)}.$$

Consequently,

$$\begin{aligned}
|C_k|_p &\leq \max\left\{\frac{|1/2|_p\cdot|a_{k+2}+\theta b_{k+2}|_p}{|1-\theta y_{k+2}|_p}, \left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p, |A_k|_p\right\} \\
&\leq \max\{|1-\theta y_k|_p^{-2\delta}, |1-\theta y_k|_p^{-(1+\delta)}, |1-\theta y_k|_p^{-1/2}\} = |1-\theta y_k|_p^{-(1+\delta)}.
\end{aligned}$$

Since $|1+\theta y_{k+2}|_p = |B_k|_p \leq |1-\theta y_k|_p^{3/2-5\delta}$, it yields that $|1-\theta y_k|_p^{-(1+\delta)} \leq |1+\theta y_{k+2}|_p^{\frac{-(1+\delta)}{3/2-5\delta}} \leq |1+\theta y_{k+2}|_p^{-2/3-2\delta}$. Hence, $|C_k|_p \leq |1+\theta y_{k+2}|_p^{-2/3-2\delta}$ which proves the last part of the theorem for the non-Archimedean absolute value.

Estimates for the Archimedean case ($p=\infty$) are similar to the above. Here we will derive the estimate for $|A_k|_p$ only. Since $2 = |2|_\infty \leq |1-\theta y_k|_\infty + |1+\theta y_k|_\infty < \epsilon_k + |1+\theta y_k|_\infty < 1 + |1+\theta y_k|_\infty$, we have $|1+\theta y_k|_\infty^{-1} < 1$. So equation (39) gives

$$\begin{aligned}
|A_k|_\infty &\leq \frac{|1/2|_\infty\cdot|a_k-\theta b_k|_\infty}{|1+\theta y_k|_\infty} + |y_{k-1}|_\infty \leq \frac{1}{10}\epsilon_k^{-\delta}\cdot 1 + |1-\theta y_k|_\infty^{-1/2} \\
&\leq \frac{1}{10}|1-\theta y_k|_\infty^{-\delta} + |1-\theta y_k|_\infty^{-1/2} \leq \frac{11}{10}|1-\theta y_k|_\infty^{-1/2},
\end{aligned}$$

for sufficiently small δ, which proves the first part of the theorem for $p=\infty$. □

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