

# Speculative Runs on Interest Rate Pegs

## Appendix for Online Publication

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Equation numbers (1) through (41) refer to the published paper.

## A Verification of the Transversality and no-Ponzi conditions

The following proposition covers all the equilibrium paths that we discuss in the paper, and proves that the transversality and no-Ponzi conditions hold.

**Proposition 3** *Let a sequence  $\{P_t, Q_{t+1}, T_t, R_t, C_t, Y_t, A_{t+1}, B_t, M_t\}_{t=0}^{\infty}$  satisfy equations (10), (11), (15), (22), (3), (5), (14) and either (12) or (21), depending on the set  $\mathcal{B}_t$ , and let fiscal policy satisfy Assumption 3. Assume also that either Assumption 1 or Assumption 2 holds. Then equations (6) and (16) hold.*

We prove this proposition in 3 steps. First, we prove that  $\underline{A}_{t+1}$ , as defined in (6), is well defined. Second, we prove that (16) holds, and finally that (6) holds.

### A.1 $\underline{A}_{t+1}$ is well defined.

We work backwards on the individual components of the sum defining  $\underline{A}_{t+1}$  in equation (6). From (12) or (21) we obtain<sup>3</sup>

$$\max_{\hat{b} \in \mathcal{B}_t} \left[ \hat{b} \left( E_{t+j} Q_{t+j+1} - \frac{1}{1 + R_{t+j}} \right) \right] = 0. \quad (42)$$

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<sup>3</sup>If the borrowing limit is not 0, the expression in (42) would not be 0, but it can be proven that  $\underline{A}_{t+1}$  is nonetheless well defined.

From either Assumption 1 or Assumption 2, it follows that the marginal disutility of work evaluated at the optimal choice is uniformly bounded; let  $\tilde{u}_y$  be the bound. We then use (14) to get

$$\begin{aligned}
E_{t+1} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) P_{t+j} \right\} &\leq \tilde{u}_y E_{t+1} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) \frac{P_{t+j}}{\hat{u}_y(\hat{R}_{t+j})} \right\} = \\
\tilde{u}_y E_{t+1} \left\{ \left( \prod_{v=1}^{j-1} Q_{t+v+1} \right) \frac{P_{t+j}}{\hat{u}_y(\hat{R}_{t+j})} E_{t+j} Q_{t+j+1} \right\} &= \\
\tilde{u}_y E_{t+1} \left\{ \left( \prod_{v=1}^{j-1} Q_{t+v+1} \right) \frac{P_{t+j}}{\hat{u}_y(\hat{R}_{t+j})(1 + \hat{R}_{t+j})} \right\} &= \tag{43} \\
\beta \tilde{u}_y E_{t+1} \left\{ \left( \prod_{v=1}^{j-2} Q_{t+v+1} \right) \frac{P_{t+j-1}}{\hat{u}_y(\hat{R}_{t+j-1})(1 + \hat{R}_{t+j-1})} \right\} &= \\
\beta^{j-1} \frac{\tilde{u}_y P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1 + \hat{R}_{t+1})} &
\end{aligned}$$

Equation (43) implies<sup>4</sup>

$$E_{t+1} \sum_{j=1}^{\infty} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) P_{t+j} \right\} \leq \frac{\tilde{u}_y P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1 + \hat{R}_{t+1})(1 - \beta)}, \tag{44}$$

which proves that the second piece of the infinite sum defining  $\underline{A}_{t+1}$  is well defined. From Assumption 3, we have  $|T_{t+j+1}| \leq \bar{T} P_{t+j} + |B_{t+j}|$ , and so

$$\left| E_{t+1} \sum_{j=1}^{\infty} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) T_{t+j+1} \right\} \right| \leq \sum_{j=1}^{\infty} E_{t+1} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) [P_{t+j} \bar{T} + |B_{t+j}|] \right\}. \tag{45}$$

We analyze equation (45) in pieces. Using (44), we have

$$\bar{T} \sum_{j=1}^{\infty} E_{t+1} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) P_{t+j} \right\} \leq \frac{\bar{T} \tilde{u}_y P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1 + \hat{R}_{t+1})(1 - \beta)}. \tag{46}$$

To work on the sum of debt, notice first that equation (1) continues to hold even if we replace  $R_t$  by  $\hat{R}_t$ . This is because  $B_t = 0$  in the periods and states of nature in which  $\hat{R}_t > R_t$ . If Assumption 1 is retained, define  $\bar{S} := \max_{R \in [0, \bar{R}]} [\hat{c}(R)(1 + R)]$ ; alternatively, if Assumption 2 is

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<sup>4</sup>We can interchange the order of the sum and the expectations since all elements of the sum have the same sign.

adopted instead, define  $\bar{S} := \max_{R \in [0, \infty]} [\hat{c}(R)(1 + R)]$ . Under Assumption 1 it is immediate that  $\bar{S} < \infty$ . Under Assumption 2,

$$\lim_{R \rightarrow \infty} \hat{c}(R)(1 + R) = \lim_{R \rightarrow \infty} -\hat{c}(R) \frac{u_c(\hat{c}(R), \hat{c}(R) + \bar{G})}{u_y(\hat{c}(R), \hat{c}(R) + \bar{G})}.$$

Since  $u(0, \bar{G})$  is finite, both  $\lim_{c \rightarrow 0} cu_c(c, c + \bar{G})$  and  $\lim_{c \rightarrow 0} u_y(c, c + \bar{G})$  must also be finite.

Finally, notice that Assumption 3 implies

$$|T_{t+j} - B_{t+j-1}| \leq P_{t+j-1}(\bar{T} + \bar{B}) + (1 - \alpha)|B_{t+j-1}|. \quad (47)$$

We can then use (1), (15), (14), and (47) to get

$$\begin{aligned} E_{t+1} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) |B_{t+j}| \right\} &= E_{t+1} \left\{ \left( \prod_{v=1}^{j-1} Q_{t+v+1} \right) \left| \left[ P_{t+j-1} \bar{G} - \right. \right. \right. \\ &T_{t+j} + B_{t+j-1} + \hat{c}(\hat{R}_{t+j-1})P_{t+j-1} - \hat{c}(\hat{R}_{t+j})P_{t+j} \left. \right] \left. \right\} = \\ E_{t+1} \left\{ \left( \prod_{v=1}^{j-1} Q_{t+v+1} \right) \left| \left[ P_{t+j-1} \bar{G} - T_{t+j} + B_{t+j-1} + \hat{c}(\hat{R}_{t+j-1})P_{t+j-1} - \right. \right. \right. \\ &\left. \left. \frac{\beta P_{t+j-1} \hat{c}(\hat{R}_{t+j})(1 + \hat{R}_{t+j}) \hat{u}_y(\hat{R}_{t+j})}{\hat{u}_y(\hat{R}_{t+j-1})} \right] \right| \right\} \leq \\ E_{t+1} \left\{ \left( \prod_{v=1}^{j-1} Q_{t+v+1} \right) \left[ \left( \bar{G} + \bar{T} + \bar{B} + \frac{\beta \tilde{u}_y \bar{S}}{\hat{u}_y(\hat{R}_{t+j-1})} + \hat{c}(0) \right) P_{t+j-1} + \right. \right. \\ &\left. \left. (1 - \alpha) |B_{t+j-1}| \right] \right\}. \end{aligned} \quad (48)$$

Using (43) and (48), we obtain (for  $j > 1$ )

$$\begin{aligned} E_{t+1} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) |B_{t+j}| \right\} &\leq E_{t+1} \left\{ \sum_{s=2}^j (1 - \alpha)^{j-s} \left[ \left( \prod_{v=1}^{s-1} Q_{t+v+1} \right) \cdot \right. \right. \\ &\left. \left[ \left( \bar{G} + \bar{T} + \bar{B} + \frac{\beta \tilde{u}_y \bar{S}}{u_y(\hat{R}_{t+s-1})} + \hat{c}(0) \right) P_{t+s-1} \right] \right\} + (1 - \alpha)^{j-1} \frac{|B_{t+1}|}{1 + \hat{R}_{t+1}} \leq \\ &\frac{\tilde{u}_y P_{t+1} (\bar{G} + \bar{T} + \bar{B} + \beta \bar{S} + \hat{c}(0))}{\hat{u}_y(\hat{R}_{t+1})(1 + \hat{R}_{t+1})} \sum_{s=2}^j [\beta^{s-2} (1 - \alpha)^{j-s}] + (1 - \alpha)^{j-1} \frac{|B_{t+1}|}{1 + \hat{R}_{t+1}} = \\ &\frac{\tilde{u}_y P_{t+1} [(1 - \alpha)^{j-1} - \beta^{j-1}] (\bar{G} + \bar{T} + \bar{B} + \beta \bar{S} + \hat{c}(0))}{\hat{u}_y(\hat{R}_{t+1})(1 + \hat{R}_{t+1})(1 - \alpha - \beta)} + (1 - \alpha)^{j-1} \frac{|B_{t+1}|}{1 + \hat{R}_{t+1}}. \end{aligned} \quad (49)$$

Using (49) we get

$$\begin{aligned} \sum_{j=1}^{\infty} E_{t+1} \left\{ \left( \prod_{v=1}^j Q_{t+v+1} \right) |B_{t+j}| \right\} \leq \\ \frac{\tilde{u}_y P_{t+1} (\overline{G} + \overline{T} + \overline{B} + \beta \overline{S} + \hat{c}(0))}{\hat{u}_y(\hat{R}_{t+1})(1 + \hat{R}_{t+1})\alpha(1 - \beta)} + \frac{|B_{t+1}|}{\alpha(1 + \hat{R}_{t+1})} \end{aligned} \quad (50)$$

Collecting all terms, equations (44), (46), and (50) imply

$$\begin{aligned} |A_{t+1}| \leq \frac{\tilde{u}_y P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})(1 + \hat{R}_{t+1})(1 - \beta)} \left[ 1 + \overline{T} + \right. \\ \left. \left( \frac{1}{\alpha} \right) (\overline{G} + \overline{T} + \overline{B} + \beta \overline{S} + \hat{c}(0)) \right] + \\ \frac{|B_{t+1}|}{\alpha(1 + \hat{R}_{t+1})} + P_t [1 + \hat{c}(0) + \overline{T}] + |B_t|. \end{aligned} \quad (51)$$

## A.2 Equation (16) holds.

Use (49) to obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} E_0 \left[ \left( \prod_{j=1}^{t+1} Q_j \right) |B_t| \right] \leq \\ \frac{\tilde{u}_y P_0 (\overline{G} + \overline{T} + \overline{B} + \beta \overline{S} + \hat{c}(0))}{\hat{u}_y(\hat{R}_0)(1 + \hat{R}_0)(1 - \alpha - \beta)} \lim_{t \rightarrow \infty} [(1 - \alpha)^t - \beta^t] + \\ \frac{|B_0|}{1 + \hat{R}_0} \lim_{t \rightarrow \infty} (1 - \alpha)^t = 0. \end{aligned} \quad (52)$$

We then use (14), (51), and (52) to prove

$$\begin{aligned} \lim_{t \rightarrow \infty} E_0 \left[ \left( \prod_{j=1}^{t+1} Q_j \right) |A_{t+1}| \right] \leq \frac{\tilde{u}_y}{1 - \beta} \left[ 1 + \overline{T} + \right. \\ \left. \left( \frac{1}{\alpha} \right) (\overline{G} + \overline{T} + \overline{B} + \beta \overline{S} + \hat{c}(0)) \right] \lim_{t \rightarrow \infty} E_0 \left[ \left( \prod_{j=1}^{t+2} Q_j \right) \frac{P_{t+1}}{\hat{u}_y(\hat{R}_{t+1})} \right] + \\ \frac{1}{\alpha} \lim_{t \rightarrow \infty} E_0 \left[ \left( \prod_{j=1}^{t+2} Q_j \right) |B_{t+1}| \right] + \tilde{u}_y [1 + \hat{c}(0) + \overline{T}] \lim_{t \rightarrow \infty} E_0 \left[ \left( \prod_{j=1}^{t+1} Q_j \right) \frac{P_t}{\hat{u}_y(\hat{R}_t)} \right] + \\ \lim_{t \rightarrow \infty} E_0 \left[ \left( \prod_{j=1}^{t+1} Q_j \right) |B_t| \right] = \frac{\tilde{u}_y P_0}{(1 + \hat{R}_0)\hat{u}_y(\hat{R}_0)} \left\{ \frac{\beta}{1 - \beta} \left[ 1 + \overline{T} + \right. \right. \\ \left. \left. \left( \frac{1}{\alpha} \right) (\overline{G} + \overline{T} + \overline{B} + \beta \overline{S} + \hat{c}(0)) \right] + 1 + \hat{c}(0) + \overline{T} \right\} \lim_{t \rightarrow \infty} \beta^t = 0. \end{aligned} \quad (53)$$

Equations (11), (52), and (53) imply (16).

### A.3 Equation (6) holds.

The same steps used to prove (52) can also be used to prove

$$\lim_{j \rightarrow \infty} E_t \left\{ \left( \prod_{v=1}^{j+1} Q_{t+v} \right) | B_{t+j} \right\} = 0. \quad (54)$$

As previously noted, equation (1) continues to hold even if we replace  $R_t$  with  $\hat{R}_t$ , since the two values only differ when  $B_t = 0$ . We can then iterate (1) forward, taking expectations conditional on time- $t + 1$  information, and use (54) to obtain

$$B_t = M_{t+1} - M_t - T_{t+1} - P_t \bar{G} + E_{t+1} \left\{ \sum_{s=1}^{\infty} \left[ \left( \prod_{v=1}^s Q_{t+v+1} \right) \cdot \right. \right. \\ \left. \left. (M_{t+s+1} - M_{t+s} + T_{t+s+1} - P_{t+s} \bar{G}) \right] \right\} > \underline{A}_{t+1}, \quad (55)$$

which completes the proof. Equation (55) relies on  $\bar{G} < 1$  (government spending must be less than the maximum producible output) and on

$$E_{t+s} [M_{t+s}(1 - Q_{t+s+1})] = \frac{\hat{R}_{t+s} M_{t+s}}{1 + \hat{R}_{t+s}} \geq 0.$$

This completes the proof of proposition 3.

## B Sufficient Conditions for the Existence of Equilibria with Positive Debt

**Proposition 4** *Let Assumption 3 and either Assumption 1 or Assumption 2 hold. Then equations (23), (24), and (25) are sufficient to obtain  $B_t > 0$  in all periods  $t \geq 0$  in the deterministic equilibrium constructed in Section 3 in which  $\hat{R}_t = R_t$ . Furthermore, the same conditions ensure the existence of sunspot equilibria in which  $\hat{R}_t = R_t$  and  $B_t > 0$  with probability 1 in all periods  $t \geq 0$ .*

**Proof.** In period 0, consumption will be equal to  $\hat{c}(R_0)$ . With  $R_0 > 0$ , the cash in advance constraint holds as an equality. The household budget constraint (5) and (24), together with market clearing, imply  $B_0 > 0$ . Equation (24) imposes an upper bound on the initial price level  $P_0$ . Equation (23) ensures that this upper bound is positive, so that there are values of  $P_0$  that satisfy equation (24).

In subsequent periods ( $t > 0$ ), using once again the cash-in-advance constraint, the household budget constraint, and market clearing,  $B_t$  will be strictly positive in each period  $t > 0$  if and only if the following condition is satisfied

$$T_t < P_{t-1}\bar{G} + B_{t-1} + M_{t-1} - P_t C_t \quad (56)$$

In the case of a deterministic equilibrium,  $C_t = \hat{c}(R_t)$  and equation (14) can be used to substitute out  $P_t$  as a function of past information and current policy:

$$P_t = \beta(1 + R_t)P_{t-1} \frac{\hat{u}_y(R_t)}{\hat{u}_y(R_{t-1})}. \quad (57)$$

With this substitution, equation (56) yields (25), which is thus a sufficient condition on fiscal policy expressed purely in terms of predetermined variables.<sup>5</sup>

The proof above can be used to show that  $B_t > 0$  in any equilibrium in which there is no run in either period  $t - 1$  or period  $t$  and in which the time- $t$  allocation and price level does not depend on the period- $t$  sunspot; the allocation can still depend on sunspots up to period  $t - 1$ . We exploit this fact to construct sunspot equilibria in which debt is strictly positive with probability 1. This works by continuity. In each period  $t - 1$ , conditional on the past, we know from the construction above that we can construct a deterministic continuation for period  $t$  that satisfies the equilibrium conditions and positive debt. Equation (14) implies that  $P_t$  will be close to the value in (57) even when it depends on the sunspot  $s_t$ , provided the support of its distribution is sufficiently tight. Equation (25) will then be sufficient to ensure that (56) holds. This allows us to recursively construct sunspot equilibria where, at each period  $t$ ,  $P_t$  depends on the sunspot  $s_t$ . QED.

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<sup>5</sup>We assume that  $R_t$  is a deterministic function of the past, so that it is known when  $T_t$  is set.

When either Assumption 1 or Assumption 2 holds, an example of a fiscal policy rule that satisfies (27) (and thus (25)) and Assumption 3 is the following: choose  $T_t = \alpha B_{t-1} + \hat{T}_t$ , with  $\alpha \in (0, 1)$  and

$$\underline{T}P_{t-1} < \hat{T}_t < P_{t-1}\bar{G} + \hat{c}(R_{t-1})P_{t-1} - \frac{P_{t-1}\beta\bar{u}_y}{\underline{u}_y(0)} \quad (58)$$

for some value  $\underline{T}$ . Since the right-hand side of (58) (divided by  $P_{t-1}$ ) is bounded under either Assumption 1 or Assumption 2, a lower bound  $\underline{T}$  that is uniformly smaller than the right-hand side can always be found.