THE HYPERBOLIC AX-LINDEMANN-WEIERSTRASS CONJECTURE

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1. Introduction.

1.1. **The hyperbolic Ax-Lindemann-Weierstraß conjecture.** Around 1885 Lindemann and Weierstraß proved that if $\alpha_1, \dots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic numbers then $e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$ ([12], [32]). This classical transcendence result has the following functional "flat" analogue, which is a particular case of a result of Ax [2]: *Define* $\pi = (\exp, \dots, \exp) : \mathbb{C}^n \longrightarrow (\mathbb{C}^*)^n$. *Let* $V \subset (\mathbb{C}^*)^n$ *be an algebraic subvariety. Any maximal complex irreducible algebraic subvariety* $Y \subset \pi^{-1}(V)$ *is a translate of a rational linear subspace.* Another "flat" Ax-Lindemann-Weierstraß theorem is obtained when studying the uniformizing map of an abelian variety: *Let* $\pi : \mathbb{C}^n \longrightarrow A$ *be the uniformizing map of a complex abelian variety of dimension* n*. Let* $V \subset A$ *be an algebraic subvariety. Any maximal complex irreducible algebraic subvariety* $Y \subset \pi^{-1}(V)$ *is the preimage of a translate of an abelian subvariety contained in* V*.*

The main result of this paper is a proof of a similar statement, the hyperbolic Ax-Lindemann-Weierstraß conjecture, for any *arithmetic variety* $S := \Gamma\backslash X$. Here X denotes a Hermitian symmetric domain and Γ is any *arithmetic subgroup* of the real adjoint Lie group G of biholomorphisms of X. This means that there exists a semisimple $\mathbb{Q}$-algebraic group $\mathbf{G}$ and a surjective morphism with compact kernel $p : \mathbf{G}(\mathbb{R}) \longrightarrow G$ such that Γ is commensurable with the projection $p(\mathbf{G}(\mathbb{Z}))$ (cf. section 2.1 for the definition of $\mathbf{G}(\mathbb{Z})$ and [13] for a general reference on arithmetic lattices).

While X is not a complex algebraic variety it admits a canonical realisation as a bounded symmetric domain $\mathcal{D} \subset \mathbb{C}^N$ (with $N = \dim_{\mathbb{C}} X$) (cf. [27, §II.4]). We will say that a subset $Y \subset \mathcal{D}$ is an *irreducible algebraic subvariety* of $\mathcal{D}$ if Y is an irreducible component of the analytic set $\mathcal{D} \cap \widetilde{Y}$ where $\widetilde{Y}$ is an algebraic subset of $\mathbb{C}^N$. An algebraic subvariety of $\mathcal{D}$ is then defined as a finite union of irreducible algebraic subvarieties. On the other hand the arithmetic variety S admits a natural structure of complex quasi-projective variety via the Baily-Borel embedding [3]. Recall that an irreducible algebraic subvariety of S is said *weakly special* if its smooth locus is totally geodesic in S endowed with its canonical Hermitian metric.

The uniformization map $\pi\colon \mathcal{D} \longrightarrow S = \Gamma\backslash\mathcal{D}$ is highly transcendental with respect to these algebraic structures (in the simplest case where $\mathcal{D}$ is the Poincaré disk and S is the modular curve, the map $\pi : \mathcal{D} \longrightarrow S$ is the usual j-invariant seen on the disk). The hyperbolic Ax-Lindemann-Weierstraß conjecture is the following statement:

Theorem 1.1. *(The hyperbolic Ax-Lindemann-Weierstraß conjecture.) Let $S = \Gamma\backslash\mathcal{D}$ be an arithmetic variety with uniformising map $\pi : \mathcal{D} \longrightarrow S$. Let V be an algebraic subvariety of S. Maximal irreducible algebraic subvarieties of $\pi^{-1}V$ are precisely the irreducible components of the preimages of maximal weakly special subvarieties contained in V.*

Remarks 1.1.
(a) In [31] Ullmo and Yafaev proved the theorem 1.1 in the special case where S is compact. In [25] Pila and Tsimerman proved theorem 1.1 in the special case $S = \mathcal{A}_g$, the moduli space of principally polarised abelian varieties of dimension g.
(b) Mok has a very nice, entirely complex-analytic, approach to the hyperbolic Ax-Lindemann-Weierstraß conjecture. In the rank 1 case his approach should extend some of the results of this text to the non-arithmetic case. We refer to [15], [16] for partial results.
(c) We defined algebraic subvarieties of X using the Harish-Chandra realisation $\mathcal{D}$ of X but we could have used as well any other *realisation* of X in the sense of [29, section 2.1]. Indeed morphisms of realisations are necessarily semi-algebraic, thus X admits a canonical semi-algebraic structure and a canonical notion of algebraic subvarieties (cf. appendix B for details). Hence one can replace $\mathcal{D}$ in theorem 1.1 by any other realisation of X, for example the Borel realisation (cf. [14, p.52]).

1.2. **Motivation: the André-Oort conjecture.** Let $(\mathbf{G}, X_{\mathbf{G}})$ be a Shimura datum. Let X be a connected component of $X_{\mathbf{G}}$ (hence X is a Hermitian symmetric domain). We denote by $\mathbf{G}(\mathbb{Q})_+$ the stabiliser of X in $\mathbf{G}(\mathbb{Q})$. Let K_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$, where $\mathbb{A}_f$ denotes the finite adèles of $\mathbb{Q}$ and let $\Gamma := \mathbf{G}(\mathbb{Q})_+ \cap K_f$ be the corresponding congruence arithmetic lattice of $\mathbf{G}(\mathbb{Q})$.

Then the arithmetic variety $S := \Gamma\backslash X$ is a component of the complex quasi-projective Shimura variety

$$\mathrm{Sh}_K(\mathbf{G}, X) := \mathbf{G}(\mathbb{Q})_+\backslash X \times \mathbf{G}(\mathbb{A}_f)/K_f \quad .$$

The variety S contains the so-called special points and special subvarieties (these are the weakly special subvarieties of S containing one special point, we refer to [5] or [17] for the detailed definitions). One of the main motivations for studying the Ax-Lindemann-Weierstraß conjecture is the André-Oort conjecture predicting that irreducible subvarieties of S containing Zariski dense sets of special points are precisely the special subvarieties. The André-Oort conjecture has been proved under the assumption of the Generalised Riemann Hypothesis (GRH) by the authors of this paper ([30], [11]). Recently Pila and Zannier [26] came up with a new proof of the Manin-Mumford conjecture for abelian varieties using the flat Ax-Lindemann-Weierstraß theorem. This gave hope to prove the André-Oort conjecture unconditionally with the same strategy. In [22] Pila succeeded in applying this strategy to the case where S is a product of modular curves. Roughly speaking, the strategy consists of two main ingredients: the first is the problem of bounding below the sizes of Galois orbits of special points and the second is the hyperbolic Ax-Lindemann-Weierstraß conjecture (cf. [28]).

We refer to [29] for details on how the hyperbolic Ax-Lindemann-Weierstraß conjecture and a good lower bound on the sizes of Galois orbits of special points imply the André-Oort conjecture. As a direct corollary of theorem 1.1 and the proof of [29, theor.5.1] one obtains:

Corollary 1.2. *The André-Oort conjecture holds for $\mathcal{A}_6^n$ for any positive integer n.*

A new proof of the André-Oort conjecture under the GRH is a consequence of theorem 1.1 and an upper bound for the height of special points in Siegel sets. This last step is currently studied by C.Daw and M.Orr.

1.3. **Strategy of the proof of theorem 1.1.** Our strategy for proving theorem 1.1 is as follows:

(i) Let $S := \Gamma\backslash X$ and $\pi\colon X \longrightarrow S$ be the uniformising map. Even though the map π is transcendental, it still enables us to relate the semi-algebraic structures on X and S through a larger o-minimal structure. We refer to [31, section 3], [6], [7] for details on o-minimal structures. Recall that a fundamental set for the action of Γ on X is a connected open subset $\mathcal{F}$ of X such that $\Gamma\overline{\mathcal{F}} = X$ and such that the set $\{\gamma \in \Gamma \ |\gamma\mathcal{F} \cap \mathcal{F} \neq \emptyset\}$ is finite. We prove in section 4 the following result:

Theorem 1.2. *There exists a semi-algebraic fundamental set $\mathcal{F}$ for the action of Γ on X such that the restriction $\pi_{|\mathcal{F}}\colon \mathcal{F} \longrightarrow S$ is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$.*

Remarks 1.3.
(a) The special case of theorem 1.2 when S is compact is much easier and was proven in [31], Proposition 4.2. In this case, the map $\pi_{|\mathcal{F}}$ is even definable in $\mathbb{R}_{\mathrm{an}}$. Theorem 1.2 in the case where $X = \mathcal{H}_g$ is the Siegel upper half plane of genus g was proven by Peterzil and Starchenko (see [21] and [20]) and is a crucial ingredient in [25]. Notice that this particular case implies theorem 1.2 for any special subvariety S of $\mathcal{A}_g$ (see Proposition 2.5 of [29]).
(b) Our proof of theorem 1.2 does not use [21] or [20] but relies on the general theory of compactifications of arithmetic varieties (cf. [1]).

(ii) Choose a semi-algebraic fundamental set $\mathcal{F}$ for the action of Γ as in the theorem 1.2 above. The choice of a reasonable representation $\rho : \mathbf{G} \longrightarrow \mathbf{GL}(E)$ (cf. section 2.1) allows us to define a *height function* $H : \Gamma \longrightarrow \mathbb{R}$ (cf. definition 2.2). In section 5 we show the following result:

Theorem 1.3. *Let Y be a positive dimensional irreducible algebraic subvariety of X. Define*

$$N_Y(T) = |\{\gamma \in \Gamma : H(\gamma) \leq T, Y \cap \gamma\mathcal{F} \neq \emptyset\}| \ .$$

Then there exists a positive constant c_1 such that for all T large enough:

$$N_Y(T) \geq T^{c_1} \ .$$

Remark 1.4. When S is compact Ullmo and Yafaev proved (cf. [31]) that the length function grows exponentially and theorem 1.3 follows in this case. We were not able to obtain such a result on the length in the general case.

(iii) In section 6, applying the counting result above and some strong form of Pila-Wilkie's theorem [23], we prove:

Theorem 1.4. *Let V be an algebraic subvariety of S and Y a maximal irreducible algebraic subvariety of $\pi^{-1}V$. Let Θ_Y denotes the stabiliser of Y in $\mathbf{G}(\mathbb{R})$ and define $\mathbf{H}_Y$ as the connected component of the identity of the Zariski closure of $\mathbf{G}(\mathbb{Z}) \cap \Theta_Y$. Then $\mathbf{H}_Y$ is a non-trivial $\mathbb{Q}$-subgroup of $\mathbf{G}$, such that $\mathbf{H}_Y(\mathbb{R})$ is non-compact.*

(iv) Without loss of generality one can assume that V is the smallest algebraic subvariety of S containing $\pi(Y)$. With this assumption we show in section 7 that $\widetilde{V}$ is invariant under $\mathbf{H}_Y(\mathbb{Q})$, where $\widetilde{V}$ is an analytic irreducible component of $\pi^{-1}V$ containing Y, and then conclude that $\pi(Y) = V$ is weakly special using monodromy arguments.

2. Preliminaries

2.1. **Notations.** In the rest of the text:

- X denotes a Hermitian symmetric domain (not necessarily irreducible).
- G is the adjoint semi-simple real algebraic group, whose set of real points, also denoted by G, is the group of biholomorphisms of X; hence $X = G/K$ where K is a maximal compact subgroup of G.
- $\Gamma \subset G$ is an arithmetic lattice. This means (cf. [13]) that there exists a semi-simple linear algebraic group $\mathbf{G}$ over $\mathbb{Q}$ and $p : \mathbf{G}(\mathbb{R}) \longrightarrow G$ a surjective morphism with compact kernel such that Γ is commensurable with $p(\mathbf{G}(\mathbb{Z}))$. Here we recall that two subgroups of a group are commensurable if their intersection is of finite index in both of them; moreover $\mathbf{G}(\mathbb{Z})$ denotes $\mathbf{G}(\mathbb{Q}) \cap \rho^{-1}(\mathbf{GL}(E_{\mathbb{Z}}))$ for some faithful representation $\rho : \mathbf{G} \longrightarrow \mathbf{GL}(E)$, where E is a finite-dimensional $\mathbb{Q}$-vector space and $E_{\mathbb{Z}}$ is a $\mathbb{Z}$-lattice in E; the commensurability of Γ and $p(\mathbf{G}(\mathbb{Z}))$ is independant of the choice of ρ and $E_{\mathbb{Z}}$.
- One easily checks that theorem 1.1 holds for Γ if and only if it holds for any Γ' commensurable with Γ. In particular without loss of generality one can and will assume that the group $\mathbf{G}(\mathbb{Z})$ is neat (meaning that for any $\gamma \in \mathbf{G}(\mathbb{Z})$ the group generated by the eigenvalues of $\rho(\gamma)$ is torsion-free) and the group Γ coincides with $p(\mathbf{G}(\mathbb{Z}))$ and is torsion-free.
- Without loss of generality we can and will assume that *the group $\mathbf{G}$ is of adjoint type*. Indeed let $\lambda \colon \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ denotes the natural algebraic morphism to the adjoint group $\mathbf{G}^{\mathrm{ad}}$ of $\mathbf{G}$ (quotient by the centre). As the Lie group G is adjoint

the morphism $p : \mathbf{G}(\mathbb{R}) \longrightarrow G$ factorises through

$$\begin{tikzcd} \mathbf{G}(\mathbb{R}) \arrow[r, "\lambda"] \arrow[dr, "p"'] & \mathbf{G}^{\mathrm{ad}}(\mathbb{R}) \arrow[d, "p^{\mathrm{ad}}"] \\ & G \end{tikzcd}$$

and Γ is commensurable with $p^{\mathrm{ad}}(\mathbf{G}^{\mathrm{ad}}(\mathbb{Z}))$.

- Without loss of generality we can and will assume that *each* $\mathbb{Q}$*-simple factor of* $\mathbf{G}$ *is* $\mathbb{R}$*-isotropic*. Indeed let $\mathbf{H}$ be the quotient of $\mathbf{G}$ by its $\mathbb{R}$-anisotropic $\mathbb{Q}$-factors. Again, the morphism $p : \mathbf{G}(\mathbb{R}) \longrightarrow G$ factorises through $\mathbf{H}(\mathbb{R})$ and Γ is commensurable with the projection of $\mathbf{H}(\mathbb{Z})$.
- $K_\infty = p^{-1}K$ is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. Hence $X = \mathbf{G}(\mathbb{R})/K_\infty$. We denote by x_0 the base-point eK_∞ of X.
- The quotient $S := \Gamma\backslash X$ is a smooth complex quasi-projective variety. We denote by $\pi : X \longrightarrow S$ the uniformization map.
- We fix a basis $(e_1, \dots, e_n)$ of $E_\mathbb{Z}$. For each prime number p we define the non-archimedean norm $\|\cdot\|_p : E_{\mathbb{Q}_p} \longrightarrow \mathbb{R}$ by $\|\sum_{i=1}^r x_i e_i\|_p = \max_i |x_i|_p$, where $|\cdot|_p$ denotes the usual norm on $\mathbb{Q}_p$ normalised by $|p|_p = 1/p$. At the infinite place we choose $\|\cdot\|_\infty : E_\mathbb{R} \longrightarrow \mathbb{R}$ a Euclidean norm which is $\rho(K_\infty)$-invariant.
- We denote by $\mathcal{P}$ the set of all places of $\mathbb{Q}$.
- We denote by $\mathcal{X}$ any realization of X (cf. appendix B).

2.2. **Norm, distance, height.** Let $*$ be the adjunction on $E_\mathbb{R}$ associated to the Hilbert structure $\|\cdot\|_\infty$ on $E_\mathbb{R}$. The restriction of the bilinear form $(u, v) \mapsto \mathrm{tr}(u^*v)$ to the Lie algebra $\mathrm{Lie}(\mathbf{G}(\mathbb{R}))$ defines a $\mathbf{G}(\mathbb{R})$-invariant Kähler metric g_X on X. We denote by $d : X \times X \longrightarrow \mathbb{R}$ the associated distance and by ω the associated Kähler form.

For each place v of $\mathbb{Q}$ we still denote by $\|\cdot\|_v$ the operator norm on $\mathrm{End}\, E_v$ associated to $\|\cdot\|_v$ on E_v:

$$\forall\, \varphi \in \mathrm{End}\, E_v, \quad \|\varphi\|_v = \sup_{\|x\|_v \leq 1} \|\varphi(x)\|_v \ .$$

By restriction we also denote by $\|\cdot\|_v : \mathbf{G}(\mathbb{Q}_v) \longrightarrow \mathbb{R}$ the function $\|\cdot\|_v \circ \rho$.

Remark 2.1. As K_∞ preserves the norm $\|\cdot\|_\infty$ on $E_\mathbb{R}$, the function $\|\cdot\|_\infty : \mathbf{G}(\mathbb{R}) \longrightarrow \mathbb{R}$ is K_∞-bi-invariant, in particular descends to a function $\|\cdot\|_\infty : X \longrightarrow \mathbb{R}$.

Definition 2.2. *We define the (multiplicative) height function* $H : \mathrm{End}\, E \longrightarrow \mathbb{R}$ *as*

$$\forall\, \varphi \in \mathrm{End}\, E, \;\; H(\varphi) = \prod_{v \in \mathcal{P}} \max(1, \|\varphi\|_v) \ .$$

Remark 2.3. When $\dim_\mathbb{Q} E = 1$, this height function coincides with the usual multiplicative height function on rational numbers.

By restriction, we also denote by $H : \mathbf{G}(\mathbb{Q}) \longrightarrow \mathbb{R}$ the function $H \circ \rho$. As usual the height is particularly simple on $\mathbf{G}(\mathbb{Z})$:

$$\forall\, \varphi \in \mathbf{G}(\mathbb{Z}),\quad H(\varphi) = \max(1, \|\varphi\|_\infty) \enspace .$$

Notice that $\|\varphi\|_\infty$ is the square root of the largest eigenvalue of the positive definite matrix $\varphi^*\varphi$. As φ is integral it follows that $\|\varphi\|_\infty$ is at least 1, hence

$$\forall\, \varphi \in \mathbf{G}(\mathbb{Z}),\quad H(\varphi) = \|\varphi\|_\infty \enspace .$$

It follows from remark 2.1 that the height function on $\mathbf{G}(\mathbb{Z})$ factorizes through $H : \Gamma \longrightarrow \mathbb{R}$.

3. Compactification of arithmetic varieties

3.1. **Siegel sets.** First we recall the definition of Siegel sets for Γ. We refer to [4, §12] for details. We follow Borel's conventions, except that for us the group G acts on X on the left.

Let $\mathbf{P}$ be a minimal $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$ such that $K_\infty \cap \mathbf{P}(\mathbb{R})$ is a maximal compact subgroup of $\mathbf{P}(\mathbb{R})$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$ and let $\mathbf{A}$ be a maximal split torus of $\mathbf{P}$. We denote by $\mathbf{S}$ a maximal split torus of $\mathbf{GL}(E)$ containing $\rho(\mathbf{A})$. We denote by $\mathbf{M}$ the maximal anisotropic subgroup of the connected centralizer $\mathbf{Z}(\mathbf{A})^0$ of $\mathbf{A}$ in $\mathbf{P}$ and by Δ the set of simple roots of $\mathbf{G}$ with respect to $\mathbf{A}$ and $\mathbf{P}$. We denote by $A \subset \mathbf{S}(\mathbb{R})$ the real torus $\mathbf{A}(\mathbb{R})$. For any real number $t > 0$ we let

$$A_t := \{a \in A \mid a^\alpha \geq t \text{ for any } \alpha \in \Delta\} \enspace .$$

A Siegel set for $\mathbf{G}(\mathbb{R})$ for the data $(K_\infty, \mathbf{P}, \mathbf{A})$ is a product:

$$\Sigma'_{t,\Omega} := \Omega \cdot A_t \cdot K_\infty \subset \mathbf{G}(\mathbb{R})$$

where Ω is a compact neighborhood of e in $\mathbf{M}^0(\mathbb{R}) \cdot \mathbf{U}(\mathbb{R})$.

The image

$$\Sigma_{t,\Omega} := \Omega \cdot A_t \cdot x_o \subset \mathcal{X}$$

of $\Sigma'_{t,\Omega}$ in $\mathcal{X}$ is called a Siegel set in $\mathcal{X}$.

Theorem 3.1. [4, theor.13.1] *Let X, G, $\mathbf{G}$, Γ, $\mathbf{P}$, $\mathbf{A}$, K_∞, and $\mathcal{X}$ be as above. Then for any Siegel set $\Sigma_{t,\Omega}$, the set $\{\gamma \in \Gamma \mid \gamma\Sigma_{t,\Omega} \cap \Sigma_{t,\Omega} \neq \emptyset\}$ is finite. There exist a Siegel set (called a Siegel set for Γ) $\Sigma_{t_0,\Omega}$ and a finite subset J of $\mathbf{G}(\mathbb{Q})$ such that $\mathcal{F} := J \cdot \Sigma_{t_0,\Omega}$ is a fundamental set for the action of Γ on $\mathcal{X}$.*

When Ω is chosen to be semi-algebraic the Siegel set $\Sigma_{t,\Omega}$ and the fundamental set $\mathcal{F}$ are semi-algebraic as by definition of a complex realisation (cf. appendix B) the action of $\mathbf{G}(\mathbb{R})$ on $\mathcal{X}$ is semi-algebraic and the subset $\Omega \cdot A_t$ of $\mathbf{G}(\mathbb{R})$ is semi-algebraic.

We will only consider semi-algebraic Siegel sets in the rest of the text.

3.2. **Boundary components.** General references for this section and the next one are [18] and [1].

Let $\mathcal{D} \hookrightarrow \mathbb{C}^N$ be the Harish-Chandra realisation of X as a bounded symmetric domain. The action of G extends to the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in $\mathbb{C}^N$. The boundary $\partial\mathcal{D} := \overline{\mathcal{D}} \backslash \mathcal{D}$ is a smooth manifold which decomposes into a (continuous) union of *boundary components*, which are defined as maximal complex analytic submanifolds of $\partial\mathcal{D}$ (or alternatively as holomorphic path components of $\partial\mathcal{D}$). Explicitly, let us say that a real affine hyperplane $H \subset \mathbb{C}^N$ is a supporting hyperplane if $H \cap \overline{\mathcal{D}}$ is nonempty but $H \cap \mathcal{D}$ is empty. Let H be a supporting hyperplane and let $\overline{F} = H \cap \overline{\mathcal{D}} = H \cap \partial\mathcal{D}$. Let L be the smallest affine subspace of $\mathbb{C}^N$ which contains $\overline{F}$. Then $\overline{F}$ is the closure of a nonempty open subset $F \subset L$ which is then a single boundary component of $\mathcal{D}$ (cf. [27, §III.8.11]). The boundary component F turns out to be a bounded symmetric domain in L.

Fix a boundary component F. The normaliser $N(F) := \{g \in G \mid gF = F\}$ turns out to be a proper parabolic subgroup of G. The Levi decomposition $N(F) = R(F) \cdot W(F)$ (where $W(F)$ denotes the unipotent radical of $N(F)$ and $R(F)$ is the unique reductive Levi factor stable under the Cartan involution corresponding to K) can be refined into

$$N(F) = (G_h(F) \cdot G_l(F) \cdot M(F)) \cdot V(F) \cdot U(F) \ , \tag{3.1}$$

where:

- $U(F)$ is the centre of $W(F)$. It is a real vector space;

- $V(F) = W(F)/U(F)$ turns out to be abelian. It is a real vector space of even dimension $2l$, and we get a decomposition $W(F) = V(F) \cdot U(F)$ using "exp";

- $G_l(F) \cdot M(F) \cdot V(F) \cdot U(F)$ acts trivially on F and $G_h(F)$ modulo a finite center is $\mathrm{Aut}^0(F)$;

- $G_h(F) \cdot M(F) \cdot V(F) \cdot U(F)$ commutes with $U(F)$ and $G_l(F)$ modulo a finite central group acts faithfully on $U(F)$ by inner automorphisms;

- $M(F)$ is compact.

The boundary component F is said to be *rational* if $\Gamma_F := \Gamma \cap N(F)$ is an arithmetic subgroup of $N(F)$. There are only finitely many Γ-orbits of rational boundary components, we choose representatives $F_1, \ldots, F_r$ for these Γ-orbits. Then the Baily-Borel compactification of S is

$$\overline{S}^{BB} = S \cup \bigcup_{i=1}^{r} (\Gamma_{F_i} \backslash F_i)$$

with a suitable analytic structure.

3.3. **Toroidal compactifications and local coordinates.** Let $X^\vee$ be the compact dual of X and $\mathcal{D} \hookrightarrow X^\vee$ be the Borel embedding. Recall that $X^\vee$ has an algebraic action by $G_{\mathbb{C}}$. Given a boundary component F of $\mathcal{D}$ we define, following [18, section 3], an open subset $\mathcal{D}_F$ of $X^\vee$ containing $\mathcal{D}$ as follows:

$$\mathcal{D}_F = \bigcup_{g \in U(F)_{\mathbb{C}}} g \cdot \mathcal{D} \ .$$

The embedding of $\mathcal{D}$ in $\mathcal{D}_F$ is Piatetskii-Shapiro's realisation of $\mathcal{D}$ as Siegel Domain of the third kind. In fact there is a canonical holomorphic isomorphism (we refer to the proof of lemma 4.2 for a precise description of this isomorphism):

$$\mathcal{D}_F \overset{j}{\simeq} U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \quad .$$

This biholomorphism defines complex coordinates (x, y, t) on $\mathcal{D}_F$, such that

$$\mathcal{D} \overset{j}{\simeq} \{(x, y, t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \mid \mathrm{Im}(x) + l_t(y, y) \in C(F)\} \subset \mathcal{D}_F$$

where $\mathrm{Im}(x)$ is the imaginary part of x, $C(F) \subset U(F)$ is a self-adjoint convex cone homogeneous under the $G_l(F)$-action on $U(F)$ and $l_t \colon \mathbb{C}^l \times \mathbb{C}^l \longrightarrow U(F)$ is a symmetric $\mathbb{R}$-bilinear form varying real-analytically with $t \in F$. The group $U(F)_{\mathbb{C}}$ acts on $\mathcal{D}_F$ and in these coordinates the action of $a \in U(F)(\mathbb{C})$ is given by:

$$(x, y, t) \longrightarrow (x + a, y, t).$$

From now on we fix a Γ-admissible collection of polyhedra $\boldsymbol{\sigma} = (\sigma_\alpha)$ (cf. [1, definition 5.1]) such that the associated toroidal compactification $\overline{S} = \overline{S}_{\boldsymbol{\sigma}}$ constructed in [1] is smooth projective and the complement $\overline{S} \setminus S$ is a divisor with normal crossings. We refer to [1] for details and we just recall what is needed for our purposes.

The compactification $\overline{S}$ is covered by a finite set of coordinates charts constructed as follows (cf. [18, p.255-256]):

(a) Take a rational boundary component F of $\mathcal{D}$;

(b) We may choose some complex coordinates $x = (x_1, \ldots, x_k)$ on $U(F)_{\mathbb{C}}$ (depending on the choice of Σ) such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook] \arrow[d] & \mathcal{D}_F \overset{j}{\simeq} U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \arrow[d, "\exp_F"] & \\
\exp_F(\mathcal{D}) = \Gamma \cap U_F \backslash \mathcal{D} \arrow[r, hook] \arrow[d, "\pi_F"'] & \mathbb{C}^{*k} \times \mathbb{C}^l \times F \arrow[r, hook] & \mathbb{C}^k \times \mathbb{C}^l \times F \\
S & &
\end{tikzcd} \tag{3.2}$$

where $\exp_F : U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \to \mathbb{C}^{*k} \times \mathbb{C}^l \times F$ is given by

$$(x, y, t) \mapsto (\exp(2i\pi x), y, t), \text{ where } \exp(2i\pi x) = (\exp(2i\pi x_1), \ldots, \exp(2i\pi x_k)) \quad . \tag{3.3}$$

(c) Define the "partial compactification of $\exp_F(\mathcal{D})$ in the direction F" to be the set $\exp_F(\mathcal{D})^\vee$ of points P in $\mathbb{C}^k \times \mathbb{C}^l \times F$ having a neighborhood Θ such that

$$\Theta \cap \mathbb{C}^{*k} \times \mathbb{C}^l \times F \subset \exp_F(\mathcal{D}) \quad .$$

Then there exists an integer m, $1 \leq m \leq k$, such that $\exp_F(\mathcal{D})^\vee$ contains

$$S(F, \boldsymbol{\sigma}) = \cup_{i=1}^m \{(z, y, t) | z = (z_1, \ldots, z_k), z_i = 0\}.$$

(d) The basic property of $\overline{S}$ is that the covering map $\pi_F : \exp_F(\mathcal{D}) \to S$ extends to a local homeomorphism $\overline{\pi_F} : \exp_F(\mathcal{D})^\vee \to \overline{S}$ making the diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[d, "\exp_F"] \arrow[dd, bend right=60, "\pi"'] & \\
\exp_F(\mathcal{D}) \arrow[r, hook] \arrow[d, "\pi_F"] & \exp_F(\mathcal{D})^\vee \arrow[d, "\overline{\pi}_F"] \\
S \arrow[r, hook] & \overline{S}
\end{tikzcd} \tag{3.4}$$

commutative. Moreover every point P of $\overline{S} - S$ is of the form $\overline{\pi}_F((z, y, t))$ with $z_i = 0$ for some $i \leq m$, for some F.

The following proposition summarizes what we will need:

Proposition 3.2. *Let $\Sigma = \Sigma_{t,\Omega} \subset \mathcal{D}$ be a Siegel set for the action of Γ. Then Σ is covered by a finite number of open subsets Θ having the following properties. For each Θ there is a rational boundary component F, a simplicial cone $\sigma \in \boldsymbol{\sigma}$ with $\sigma \subset \overline{C(F)}$, a point $a \in C(F)$, relatively compact subsets U', Y' and F' of $U(F)$, $\mathbb{C}^l$ and F respectively such that the set Θ is of the form*

$$\Theta \overset{j}{\simeq} \{(x, y, t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F,\, \mathrm{Re}(x) \in U', y \in Y', t \in F' \mid \mathrm{Im}(x) + l_t(y, y) \in \sigma + a\}$$
$$\subset U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \overset{j^{-1}}{\simeq} \mathcal{D}_F \ .$$

Proof. Let us provide a proof of this proposition, essentially stated without proof in [18, p.259]. Let $\mathcal{D} \overset{\Psi}{\simeq} W(F) \times C(F) \times F$ be the real-analytic isomorphism deduced from the group-theoretic isomorphism (3.1) constructed in [1, p.233]. Following [1, p.266, corollary of proof], the Siegel set Σ is covered by a finite number of sets Θ of the form

$$\Theta \overset{\Psi}{\simeq} \omega_F \times (C_0 \cap \sigma_\alpha^F) \times E \ ,$$

where $E \subset F$ and $\omega_W \subset W(F)$ are compact, $C_0 \subset C(F)$ is a rational core and σ_α^F is one of the polyhedra in our decomposition of $C(F)$.

Considering $C(F)$ as a cone in $\sqrt{-1} \cdot U(F)$ and decomposing $W(F)$ as $U(F) \cdot V(F)$, the isomorphism Ψ extends to the real-analytic isomorphism $\mathcal{D}_F \overset{\Psi}{\simeq} U(F)_{\mathbb{C}} \times V(F) \times F$ constructed in [1, p.235]. Hence the Siegel set Σ is covered by a finite number of sets Θ of the form

$$\Theta \overset{\Psi}{\simeq} \Psi(\mathcal{D}) \cap \{(x, s, t) \in U(F)_{\mathbb{C}} \times V(F) \times F \quad | \quad \mathrm{Re}(x) \in U', s \in S', t \in F'\} \tag{3.5}$$

where $F' \subset F$, $U' \subset U(F)$ and $S' \subset V(F)$ are relatively compact.

Using the definition of j given in [33, §7] and recalled in the proof of lemma 4.2 below, it follows, as stated in [1, p.238], that the diffeomorphism $j \circ \Psi^{-1} : U(F)_{\mathbb{C}} \times V(F) \times F \simeq$

$U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$ is a change of trivialisation of the real-analytic bundle

$$\begin{tikzcd} \mathcal{D}_F \arrow[d, "\pi'_F"] \arrow[dd, bend right=60, "\pi_F"'] \\ \mathcal{D}'_F \arrow[d, "p_F"] \\ F \end{tikzcd}$$

studied in [1, p.237]. Here the map π'_F is a $U(F)_{\mathbb{C}}$-principal homogeneous space, the map p_F is a $V(F)$-principal homogeneous space, and the map $j \circ \Psi^{-1}$ is $U(F)_{\mathbb{C}}$-equivariant and respects the fibrations over F. These two properties ensure that $j \circ \Psi^{-1}$ identifies the set $\Psi(\Theta)$ of (3.5) to a set of the required form

$$\Theta \overset{j}{\simeq} \{(x, y, t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F,\ \mathrm{Re}(x) \in U', y \in Y', t \in F' \mid \mathrm{Im}(x) + l_t(y, y) \in \sigma + a\}$$
$$\subset U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \ .$$

□

4. Definability of the uniformisation map: proof of theorem 1.2.

First notice that, although the variety S does not canonically embed into some $\mathbb{R}^n$, the statement of theorem 1.2 makes sense as S has a canonical structure of real algebraic manifold, hence of $\mathbb{R}_{\text{an,exp}}$-manifold: cf. appendix A.

By theorem 3.1 there exist a semi-algebraic Siegel set Σ and a finite subset J of $\mathbf{G}(\mathbb{Q})$ such that $\mathcal{F} := J \cdot \Sigma$ is a (semi-algebraic) fundamental set for the action of Γ on $\mathcal{D}$. Hence theorem 1.2 follows from the following more precise result.

Theorem 4.1. *The restriction $\pi_{|\Sigma} : \Sigma \longrightarrow S$ of the uniformising map $\pi \colon \mathcal{D} \longrightarrow S$ is definable in $\mathbb{R}_{\text{an,exp}}$.*

Proof. By the proposition 3.2 we know that Σ is covered by a finite union of open subsets Θ with the following properties. For each Θ there is a rational boundary component F, a simplicial cone $\sigma \in \boldsymbol{\sigma}$ with $\sigma \subset \overline{C(F)}$, a point $a \in C(F)$, relatively compact subsets U', Y' and F' of $U(F)$, $\mathbb{C}^l$ and F respectively such that the set Θ is of the form

(4.1)
$$\Theta \overset{j}{\simeq} \{(x, y, t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F,\ \mathrm{Re}(x) \in U', y \in Y', t \in F' \mid \mathrm{Im}(x) + l_t(y, y) \in \sigma + a\}$$
$$\subset U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \ .$$

We first prove that the holomorphic coordinates we introduced on $\mathcal{D}_F$ are definable:

Lemma 4.2. *The canonical isomorphism $j : \mathcal{D}_F \simeq U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$ is semi-algebraic.*

Proof. The isomorphism j was studied in [21] and in full generality in [33, §7] (cf. [3, §1.6] for a survey). To keep the amount of definitions at a reasonable level we follow in this proof (and this proof only) the notations of Wolf and Koranyi in [33]. For example our X, resp. $X^\vee$ is denoted by M, resp. M^*.

Let $\xi\colon \mathfrak{p}^- = \mathbb{C}^N \longrightarrow M^*$ be the Harish-Chandra morphism defined by $\xi(E) = \exp(E)\cdot x$ (cf. [33, p.901]; in the notations of Wolf and Koranyi x is the base point of M^*). This is a holomorphic embedding onto a dense open subset of M^*. Notice that the map ξ is real algebraic: indeed $\mathfrak{p}^-$ is a nilpotent sub-algebra of $\mathfrak{g}^{\mathbb{C}}$ hence the exponential is polynomial in restriction to $\mathfrak{p}^-$. The bounded symmetric domain $\mathcal{D}$ is $\xi^{-1}(G^0(x))$.

Let Δ be a maximal set of strongly orthogonal positive non-compact roots of $\mathfrak{g}^{\mathbb{C}}$ as in [33, p.901]. For any $\alpha \in \Delta$ let $c_\alpha \in G$ be the partial Cayley transform of M associated to α (cf. [33, p.902], recall that with the notations of Wolf and Koranyi G is the compact form of the complexified group $\mathbf{G}^{\mathbb{C}}$!). For a subset $\theta \subset \Delta$ we denote by $c_\theta := \prod_{\alpha\in\theta} c_\alpha$ the partial Cayley transform associated with θ (cf. [33, §4.1]).

Following [33, theor. 4.8] there exists a unique subset $\theta \subset \Delta$ such that $F = \xi^{-1} c_{\Delta-\theta} M_\theta$, where $M_\theta = G^0_\theta(x)$ is defined in [33, p.912]. Let $\mathfrak{p}_\theta^{-1} \subset \mathfrak{p}^-$ be defined as in [33, p.912], let $\mathfrak{p}^-_{\Delta-\theta,1}$ be the $(+1)$-eigenspace of $\mathrm{ad}(c^4_{\Delta-\theta})$ on $\mathfrak{p}^-_{\Delta-\theta}$ and $\mathfrak{p}_2^{\theta,-}$ be the (-1)-eigenspace of $\mathrm{ad}(c^4_{\Delta-\theta})$ on $\mathfrak{p}^-$. One has a canonical decomposition (cf. [33, p.933]):

$$\mathfrak{p}^- = \mathfrak{p}^-_{\Delta-\theta,1} \oplus \mathfrak{p}_2^{\theta,-} \oplus \mathfrak{p}^-_\theta \quad . \tag{4.2}$$

The decomposition (3.1) of the normalizer $N(F) = B^\theta$ (cf. [33, remark 3 p.932]) is proven in [33, theorem 6.8]. In particular it follows that $\exp_{\Delta-\theta} := \exp\circ\,\mathrm{ad}\, c_{\Delta-\theta} : \mathfrak{p}^-_{\Delta-\theta,1} \longrightarrow U(F)_{\mathbb{C}}$ and $\exp : \mathfrak{p}_2^{\theta,-} \longrightarrow \mathbb{C}^l$ are polynomial isomorphisms, while $F \subset \mathfrak{p}^-$ is a bounded symmetric domain of $\mathfrak{p}^-_\theta$.

Following [33, §7.6 and §7.7] the map $j : \mathcal{D} \longrightarrow U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \subset U(F)_{\mathbb{C}} \times \mathbb{C}^l \times \mathfrak{p}^-_\theta$ is the composition of the semi-algebraic holomorphic maps

$$\mathcal{D} \xrightarrow{\xi^{-1}c_{\Delta-\theta}\xi} \mathfrak{p}^- = \mathfrak{p}^-_{\Delta-\theta,1} \oplus \mathfrak{p}_2^{\theta,-} \oplus \mathfrak{p}^-_\theta \xrightarrow{(\exp_{\Delta-\theta},\exp,\mathrm{Id})} U(F)_{\mathbb{C}} \times \mathbb{C}^l \times \mathfrak{p}^-_\theta$$

which finishes the proof of lemma 4.2. □

The previous lemma enables us to forget about the definable biholomorphism j. From now on and for simplicity of notations we simply write $\mathcal{D}_F = U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$.

In the description (4.1) we may and do assume that U', Y' and F' are semi-algebraic subsets respectively of $U(F)_{\mathbb{C}}$, $\mathbb{C}^l$ and F. Then the set Θ is definable in $\mathbb{R}_{\mathrm{an}}$ because:

- the function $\psi : Y' \times F' \to U(F)$ defined by $\psi(y,t) = l_t(y,y)$ is analytic and defined on a compact semi-algebraic set.
- the cone σ is polyhedral, hence semi-algebraic.

Hence the restriction $\pi_{|\Sigma} : \Sigma \longrightarrow S$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$ if and only if the restriction $\pi_{|\Theta} : \Theta \longrightarrow S$ to any set Θ appearing in the proposition 3.2 is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Fix such a set

$$\Theta = \{(x,y,t), y \in Y', t \in F', \mathrm{Re}(x) \in U' | \mathrm{Im}(x) + l_t(y,y) \in \sigma + a\}$$

associated to a rational boundary component $F \in \{F_1, \ldots, F_r\}$.

Consider the left-hand side of the diagram (3.4):

$$\begin{tikzcd} \mathcal{D} \arrow[d, "\exp_F"] \arrow[dd, bend right=60, "\pi"'] \\ \exp_F(\mathcal{D}) \arrow[d, "\pi_F"] \\ S \end{tikzcd}$$

Recall that $\exp_F : \mathcal{D}_F \to \mathbb{C}^{*k} \times \mathbb{C}^l \times F$ is given by

$$(x, y, t) \mapsto (\exp(2i\pi x, y, t), \text{ where } \exp(2i\pi x) = (\exp(2i\pi x_1), \ldots, \exp(2i\pi x_k)) \ \ .$$

The function $\mathrm{Re}(x_i)$, $1 \leq i \leq k$, is bounded on Θ hence the restriction to Θ of the map $x \mapsto \exp(2i\pi \mathrm{Re}(x))$ is definable in $\mathbb{R}_{\mathrm{an}}$. On the other hand the restriction to Θ of the function $x \mapsto \exp(-2\pi \mathrm{Im}(x))$ is definable in $\mathbb{R}_{\exp}$ by definition of $\mathbb{R}_{\exp}$. Thus the restriction to Θ of the map $\exp_F$ is definable in $\mathbb{R}_{\mathrm{an},\exp}$ and we are reduced to showing that $\pi_F : \exp_F(\Theta) \longrightarrow S$ is definable in $\mathbb{R}_{\mathrm{an},\exp}$.

Consider the lower part of the diagram (3.4):

$$\begin{tikzcd} \exp_F(\mathcal{D}) \arrow[r, hook] \arrow[d, "\pi_F"] & \exp_F(\mathcal{D})^\vee \arrow[d, "\overline{\pi}_F"] \\ S \arrow[r, hook] & \overline{S} \end{tikzcd} \ .$$

As U', V', F' are relatively compact and the imaginary part of x has a lower bound on Θ, the closure $\overline{\exp_F(\Theta)}$ of $\exp_F(\Theta)$ is compact in $\exp_F(\mathcal{D})^\vee$. Hence $\pi_F : \exp_F(\Theta) \longrightarrow S$, which is the restriction of the analytic map $\overline{\pi}_F : \exp_F(\mathcal{D})^\vee \longrightarrow \overline{S}$ to the relatively compact subset $\exp_F(\Theta)$ of $\exp_F(\mathcal{D})^\vee$, is definable in $\mathbb{R}_{\mathrm{an}}$.

□

5. Growth of the height: proof of theorem 1.3

In this section we prove theorem 1.3.

5.1. **Comparing norm and distance.**

Lemma 5.1. *For any $g \in \mathbf{G}(\mathbb{R})$ the following inequality holds:*

$$\log \|g\|_\infty \leq d(g \cdot x_0, x_0) \ \ .$$

Proof. Let $\mathbf{G}(\mathbb{R}) = K_\infty \cdot A_\infty \cdot K_\infty$ be a Cartan decomposition of $\mathbf{G}(\mathbb{R})$ associated to K_∞, where A_∞ is a maximal split real torus of G containing A. Let $g \in \mathbf{G}(\mathbb{R})$ and write $g = k_1 \cdot a \cdot k_2$ its Cartan decomposition, with k_1, k_2 in K_∞ and $a \in A_\infty$. As $\| \cdot \|_\infty$ is K_∞-bi-invariant and d is $\mathbf{G}(\mathbb{R})$-equivariant the equalities $\log \|g\|_\infty = \log \|a\|_\infty$ and $d(g \cdot x_0, x_0) = d(a \cdot x_0, x_0)$ do hold.

The torus A_∞ is diagonalisable in an orthonormal basis $(f_1, \dots, f_n)$ of $E_{\mathbb{R}}$. Write $a = \mathrm{diag}(a_1, \dots, a_n)$ in this basis, then:

$$\log \|a\|_\infty = \max_i \log |a_i| \quad \text{and} \quad d(a \cdot x_0, x_0) = \sqrt{\sum_{i=1}^{n} (\log |a_i|)^2}$$

hence the result. □

5.2. **Comparing norm and height.**

Lemma 5.2. *There exists a constant $B > 0$ such that the following holds. Let $\gamma \in \mathbf{G}(\mathbb{Z})$ and $u \in \gamma\mathcal{F}$. Then*

$$H(\gamma) \leq B \cdot \|u\|_\infty \ .$$

Proof. Write $u = \gamma \cdot j \cdot x$ with $j \in J$ and $x = \omega \cdot a \cdot k \in \Sigma'_{t_0,\Omega} = \Omega \cdot A_{t_0} \cdot K_\infty$. Hence $\gamma = u \cdot k^{-1} \cdot a^{-1} \cdot \omega^{-1} \cdot j^{-1}$. As the operator norm $\| \cdot \|$ is sub-multiplicative, one obtains:

$$H(\gamma) = \|\gamma\|_\infty \leq \|u\|_\infty \cdot \|a^{-1}\|_\infty \cdot \|\omega^{-1} \cdot j^{-1}\|_\infty \ ,$$

where we used that $\| \cdot \|_\infty$ is constant equal to 1 on K_∞. As Ω is compact and J is finite, the norm of $\omega^{-1} \cdot j^{-1}$ is bounded by a constant independant of γ. As a belongs to $A_{t_0} \subset \mathbf{A}(\mathbb{R})$ the inequality $(a^{-1})^\alpha \leq \max(1, 1/t_0)$ holds for any simple root α of $\mathbf{GL}(E)$ associated to $\mathbf{A}$ and a Borel subgroup of $\mathbf{GL}(E)$ containing $\mathbf{P}$. Hence $\|a^{-1}\|_\infty$, which is the maximal absolute value of an eigenvalue of a^{-1}, is bounded independently of γ. The result follows.

□

5.3. **Lower bound for the volume of an algebraic curve.**

In [10, Corollary 3 p.1227], Hwang and To prove the following lower bound for the area of any complex analytic curve in $\mathcal{D}$:

Theorem 5.1 (Hwang and To)**.** *Let C be a complex analytic curve in $\mathcal{D}$. For any point $x_0 \in C$ there exist positive constants a_1, c_1 such that for any positive real number R one has :*

$$\mathrm{Vol}_C(C \cap B(x_0, R)) \geq a_1 \exp(c_1 \cdot R) \ . \tag{5.1}$$

Here Vol_C denotes the area for the Riemanian metric on C restriction of the metric g_X on $\mathcal{D}$ and $B(x_0, R)$ denotes the geodesic ball of $\mathcal{D}$ with center x_0 and radius R.

5.4. **Upper bound for the volume of algebraic curves on Siegel sets.**

Lemma 5.3. (i) *There exists a constant $A_0 > 0$ such that for any algebraic curve $C \subset \mathcal{D}$ of degree d we have the bound*

$$\mathrm{Vol}_C(C \cap \Sigma) \leq A_0 \cdot d \ .$$

(ii) *There exists a constant $A > 0$ such that for any algebraic curve $C \subset \mathcal{D}$ of degree d we have the bound*

$$\mathrm{Vol}_C(C \cap \mathcal{F}) \leq A \cdot d \ .$$

Proof. We first prove (i). Recall that Σ is covered by a finite union of open subsets Θ described in proposition 3.2: there is a rational boundary component F, a simplicial cone $\sigma \in \Sigma$ with $\sigma \subset \overline{C(F)}$, a point $a \in C(F)$, relatively compact subsets U', Y' and F' of $U(F)$, $\mathbb{C}^l$ and F respectively such that the set Θ is of the form

$$\Theta = \{(x,y,t) \in \mathcal{D}_{\mathcal{F}}, y \in Y', t \in F', \mathrm{Re}(x) \in U' | \mathrm{Im}(x) + l_t(y,y) \in \sigma + a\} \subset \mathcal{D}_F = U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \ .$$

Recall that ω denotes the natural Kähler form on X. As $C \subset X$ is a complex analytic curve, one has:

$$\mathrm{Vol}_C(C \cap \Theta) = \int_{C \cap \Theta} \omega \ .$$

On the other hand let $\omega_{\mathcal{D}_F}$ be the Poincaré metric on $\mathcal{D}_F$ defined in the Siegel coordinates by:

$$\omega_{\mathcal{D}_F} = \sum \frac{dx_i \wedge d\overline{x}_i}{\mathrm{Im}(x_i)^2} + \sum dy_j \wedge d\overline{y}_j + \sum df_k \wedge d\overline{f}_k.$$

Mumford [18, Theor.3.1] proved that there exists a positive constant c such that on $\mathcal{D}$:

$$\omega \leq c \cdot \omega_{\mathcal{D}_F} \ .$$

Hence:

$$\mathrm{Vol}_C(C \cap \Theta) \leq c \int_{C \cap \Theta} \omega_{\mathcal{D}_F} \ .$$

Let p_{x_i}, p_{y_j} and p_{f_k} be the projections on $\mathcal{D}_F$ to the coordinates x_i, y_j and f_k.

As the curve C has degree d the restriction of these maps to $C \cap \Theta$ are either constant or at most d to 1, hence

$$\mathrm{Vol}_C(C \cap \Theta) \leq c \cdot d \cdot (\sum \int_{p_{x_i}(\Theta)} \frac{dx_i \wedge d\overline{x}_i}{\mathrm{Im}(x_i)^2} + \sum \int_{p_{y_j}(\Theta)} dy_j \wedge d\overline{y}_j + \sum \int_{p_{f_k}(\Theta)} df_k \wedge d\overline{f}_k).$$

Let i be such that the map p_{x_i} is not constant. In view of the description of Θ the projection $p_{x_i}(\Theta)$ is contained in a usual fundamental set of the upper-half plane, of finite hyperbolic area.

Let w be a coordinate y_j, f_k and p_w be the associated projection on the w axis. By the definition of Θ the projection $p_w(\Theta)$ is a relatively compact open set of the plane, hence of finite Euclidean area.

This finishes the proof of (i).

Let us prove (ii). As $C \cap \mathcal{F} = C \cap J \cdot \Sigma$, one has the inequality:

$$\mathrm{Vol}_C(C \cap \mathcal{F}) \leq \sum_{j \in J} \mathrm{Vol}_C(C \cap j \cdot \Sigma) = \sum_{j \in J} \mathrm{Vol}_{j^{-1}C}(j^{-1}C \cap \Sigma) \leq |J| \cdot A_0 \cdot d$$

where we used part (i) applied to the algebraic curves $j^{-1}C$ of $\mathcal{D}$, $j \in J$, which are of degree d.

This finishes the proof of lemma 5.3. $\square$

5.5. **Proof of theorem 1.3.** Choose $C \subset Y$ an irreducible algebraic curve. To prove theorem 1.3 for Y it is enough to prove it for C.

Consider the set

$$C(T) := \{z \in C \text{ and } \|z\|_\infty \leq T\} \ .$$

As $\mathcal{F}$ is a fundamental domain for the action of Γ one has on the one hand:

$$\begin{aligned} C(T) &= \bigcup_{\substack{\gamma \in \Gamma \\ \gamma\mathcal{F} \cap C \neq \emptyset}} \{u \in \gamma\mathcal{F} \cap C \text{ and } \|u\|_\infty \leq T\} \\ &\subset \bigcup_{\substack{\gamma \in \Gamma \\ \gamma\mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T}} \{u \in \gamma\mathcal{F} \cap C\} \quad \text{by lemma 5.2} \ . \end{aligned}$$

Taking volumes:

$$\mathrm{Vol}_C(C(T)) \leq \sum_{\substack{\gamma \in \Gamma \\ \gamma\mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T}} \mathrm{Vol}_C(\mathcal{F} \cap \gamma^{-1} C)$$

hence

$$\mathrm{Vol}_C(C(T)) \leq (A \cdot d) \cdot N_C(B \cdot T) \tag{5.2}$$

where we applied lemma 5.3(ii) to the algebraic curves $\gamma^{-1}C$, $\gamma \in \Gamma$, which are all of degree d.

On the other hand if follows from lemma 5.1 that

$$C \cap B(x_0, \log T) \subset C(T) \ ,$$

hence

$$\mathrm{Vol}_C(C \cap B(x_0, \log T)) \leq \mathrm{Vol}_C(C(T)) \ . \tag{5.3}$$

The result now follows from inequalities (5.2), (5.3) and theorem 5.1.

6. Stabilisers of a maximal algebraic subset: proof of theorem 1.4.

6.1. Pila-Wilkie theorem.

Definition 6.1. *The classical height $H_{\text{class}}(x)$ of a point $x = (x_1, \ldots, x_m) \in \mathbb{Q}^m$ is defined as*

$$H_{\text{class}}(x) = \max(H(x_1), \ldots, H(x_m))$$

where H is the usual multiplicative height of a rational number.

Let $Z \subset \mathbb{R}^m$ be a subset and $T \geq 0$ a real number, we define:

$$\Psi_{\text{class}}(Z, T) := \{x \in Z \cap \mathbb{Q}^m : H(x) \leq T\}$$

and

$$N_{\text{class}}(Z, T) := |\Psi(Z, T)| \ .$$

For $Z \subset \mathbb{R}^m$ a definable set in a o-minimal structure we define the algebraic part Z^{alg} of Z to be the union of all positive dimensional semi-algebraic subsets of Z.

Recall (cf. definition 3.3 of [31]), that a semi-algebraic block of dimension w in $\mathbb{R}^m$ is a connected definable set $W \subset \mathbb{R}^m$ of dimension w, regular at every point, such that there exists a semi-algebraic set $A \subset \mathbb{R}^m$ of dimension w, regular at every point with $W \subset A$.

The following result is a strong form, proven by Pila [22, theor.3.6], of the original theorem of Pila and Wilkie [23]:

Theorem 6.1 (Pila-Wilkie). *Let $Z \subset \mathbb{R}^m$ be a definable set in a o-minimal structure. For every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$N_{\mathrm{class}}(Z \backslash Z^{\mathrm{alg}}, T) < C_\epsilon T^\epsilon$$

and the set $\Psi_{\mathrm{class}}(Z,T)$ is contained in the union of at most $C_\epsilon T^\epsilon$ semi-algebraic blocks.

6.2. **Comparison of heights.**

Lemma 6.2. *Let H_{class} be the classical height defined on* $\operatorname{End} E$ *using the basis $(e_i^* \otimes e_j)_{i,j}$. There exists a constant $C > 1$ such that, if H is the height on* $\operatorname{End} E$ *as in definition 2.2, then:*

$$\forall \varphi \in \operatorname{End} E, \quad \frac{1}{C} \cdot H_{\mathrm{class}}(\varphi) \leq H(\varphi) \leq C \cdot H_{\mathrm{class}}(\varphi) \quad .$$

Proof. Let $v \in \mathcal{P}$ and define the classical norm $|\cdot|_v$ on $\operatorname{End} E_v$ as follows. Given $\varphi \in \operatorname{End} E_v$ and $(\varphi)_{ij}$ its matrix in the $\mathbb{Q}_v$-basis $(e_1, \dots, e_r)$ or E_v, one defines $|\varphi|_v := \max_{i,j} |\varphi_{ij}|_v$.

The lemma follows immediately from the classical fact that for any finite $v \in \mathcal{P}$ and any $\varphi \in \operatorname{End} E_v$ one has:

$$|\varphi|_v \leq \|\varphi\|_v \leq (\dim_{\mathbb{Q}} E) \cdot |\varphi|_v \quad .$$

□

As a corollary, theorem 6.1 still holds if one replaces H_{class} by H:

Corollary 6.2. *Let $Z \subset \operatorname{End} E_{\mathbb{R}}$ be a definable set in a o-minimal structure. Define $\Psi(Z,T) := \{x \in Z \cap \operatorname{End} E : H(x) \leq T\}$ and $N(Z,T) := |\Psi(Z,T)|$. For every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*

$$N(Z \backslash Z^{\mathrm{alg}}, (T) < C_\epsilon T^\epsilon$$

and the set $\Psi(Z,T)$ is contained in the union of at most $C_\epsilon T^\epsilon$ semi-algebraic blocks.

6.3. **Proof of theorem 1.4.** Let V be an algebraic subvariety of S and Y a maximal irreducible algebraic subvariety of $\pi^{-1}V$. Let Θ_Y be the stabiliser of Y in $\mathbf{G}(\mathbb{R})$ and $\mathbf{H}_Y$ be the neutral component of the Zariski-closure of $\mathbf{G}(\mathbb{Z}) \cap \Theta_Y$ in $\mathbf{G}$. We want to show that $\mathbf{H}_Y$ is a non-trivial subgroup of $\mathbf{G}$, acting non-trivially on X.

Via $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(E)$, we view $\mathbf{G}(\mathbb{R})$ as a semi-algebraic (and hence definable) subset of $\operatorname{End} E_{\mathbb{R}}$. As $\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow S$ is definable by theorem 1.2, lemmas 5.1 and 5.2 of [31] show the following:

Proposition 6.3. *Let us define*

$$\Sigma(Y) = \{g \in \mathbf{G}(\mathbb{R}) : \dim(gY \cap \pi^{-1}V \cap \mathcal{F}) = \dim(Y)\}$$
$$\text{and} \quad \Sigma'(Y) = \{g \in \mathbf{G}(\mathbb{R}) : g^{-1}\mathcal{F} \cap Y \neq \emptyset\}.$$

The following properties hold:

1. *The set $\Sigma(Y)$ is definable and for all $g \in \Sigma(Y)$, $gY \subset \pi^{-1}V$.*
2. *For all $\gamma \in \Sigma(Y) \cap \mathbf{G}(\mathbb{Z})$, γY is a maximal algebraic subset of $\pi^{-1}V$.*
3. *The following equality holds:*

$$\Sigma(Y) \cap \mathbf{G}(\mathbb{Z}) = \Sigma'(Y) \cap \mathbf{G}(\mathbb{Z}) \ .$$

It follows that the number $N_Y(T)$ defined in theorem 1.3 coincide with $|\Theta(Y,T)|$, where

$$\Theta(Y,T) := \mathbf{G}(\mathbb{Z}) \cap \Psi(\Sigma(Y),T) \ .$$

We can now finish the proof of the theorem 1.4 in exactly the same way as the proof of theorem 5.4 of [31]. For the sake of completeness, we reproduce it here. As $\Theta(Y,T) \subset \Psi(\Sigma(Y),T)$ it follows from the version 6.2 of Pila-Wilkie's theorem, that for T large enough, the set $\Theta(Y,T^{\frac{1}{2n}})$ is contained in at most $T^{\frac{c_1}{4n}}$ semi-algebraic blocks. As $|\Theta(Y,T^{\frac{1}{2n}})| = N_Y(T^{\frac{1}{2n}}) \geq T^{\frac{c_1}{2n}}$ by theorem 1.3, we see that there is a semi-algebraic block W in $\Sigma(Y)$ containing at least $T^{\frac{c_1}{4n}}$ elements $\gamma \in \Sigma(Y) \cap \mathbf{G}(\mathbb{Z})$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$.

Using lemma 5.5 of [30] which applies verbatim in our case, we see that there exists an element σ in $\Sigma(Y)$ such that $\sigma\Theta_Y$ contains at least $T^{\frac{c_1}{4n}}$ elements $\gamma \in \Sigma(Y) \cap \mathbf{G}(\mathbb{Z})$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$.

Let γ_1 and γ_2 be two elements of $\sigma\Theta_Y \cap \mathbf{G}(\mathbb{Z})$ such that $H(\gamma) \leq T^{\frac{1}{2n}}$.

Let $\gamma := \gamma_2^{-1}\gamma_1 \in \mathbf{G}(\mathbb{Z}) \cap \Theta_Y$. Using elementary properties of heights, we see that $H(\gamma) \leq c_n T^{1/2}$ where c_n is a constant depending on n only. It follows that for all T large enough, Θ_Y contains at least $T^{\frac{c_1}{4n}}$ elements $\gamma \in \mathbf{G}(\mathbb{Z})$ with $H(\gamma) \leq T$. Hence the connected component of the identity $\mathbf{H}_Y$ of the Zariski closure of $\mathbf{G}(\mathbb{Z}) \cap \Theta_Y$ in $\mathbf{G}$ is a positive dimensional algebraic subgroup of $\mathbf{G}$ contained in Θ_Y. This finishes the proof of the theorem 1.4.

7. End of the proof of theorem 1.1.

Let V be an algebraic subvariety of S. Our aim is to show that maximal irreducible algebraic subvarieties Y of $\pi^{-1}V$ are precisely the irreducible components of the preimages of maximal weakly special subvarieties contained in V.

Using Deligne's interpretation of Hermitian symmetric spaces in terms of Hodge theory the representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}(E)$ defines a polarized $\mathbb{Z}$-variation of Hodge structure on S. We refer to [17, section 2] for the definition of the Hodge locus of X and S. Recall that an irreducible analytic subvariety M of X or S is said to be Hodge generic if it is not contained in the Hodge locus. If M is not irreducible we say that M is Hodge generic if all the irreducible components of M are Hodge generic.

Let $V' \subset V$ be the Zariski closure of $\pi(Y)$, as Y is analytically irreducible it easily follows that V' is irreducible. Replacing V by V' we can without loss of generality assume that $\pi(Y)$ is not contained in a proper algebraic subvariety of V. We now have to show that $\pi(Y) = V$ and V is an arithmetic subvariety of S.

Since the group $\mathbf{G}$ is adjoint, it is a direct product

$$\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$$

where the $\mathbf{G}_i$'s are the $\mathbb{Q}$-simple factors of $\mathbf{G}$. This induces decompositions

$$G = \prod_{i=1}^{r} G_i, \quad X = \prod_{i=1}^{r} X_i, \quad \mathbf{G}(\mathbb{Z}) = \prod_{i=1}^{r} \mathbf{G}_i(\mathbb{Z}), \quad \Gamma = \prod_{i=1}^{r} \Gamma_i, \quad S = \prod_{i=1}^{r} S_i,$$

where G_i is a group of Hermitian type, X_i its associated Hermitian symmetric domain, Γ_i is an arithmetic lattice in G_i, $S_i := \Gamma_i \backslash X_i$ is the associated arithmetic variety and $\pi_i : X_i \longrightarrow S_i$ the associated uniformization map.

Our main theorem 1.1 is then a consequence of the following:

Theorem 7.1. *Let $\widetilde{V}$ be the an analytic irreducible component of $\pi^{-1}V$ containing Y. In the situation described above, after, if necessary, reordering the factors, one has*

$$\widetilde{V} = X_1 \times \widetilde{V_{>1}}$$

where $\widetilde{V_{>1}}$ is an analytic subvariety of $X_2 \times \cdots \times X_r$ (in particular if $r = 1$ then $\widetilde{V} = X_1 = X$).

We first show:

Proposition 7.1. *Theorem 7.1 implies the main theorem 1.1.*

Proof. Let t, $1 \leq t \leq r$, be the largest integer such that, after reordering the factors if necessary, we have:

$$\widetilde{V} = X_1 \times \cdots \times X_t \times \widetilde{V_{>t}}$$

with $\widetilde{V_{>t}}$ an analytic irreducible subvariety of $X_{t+1} \times \cdots \times X_r$ which does not (after reordering the factors if necessary) decompose into a product $X_{t+1} \times V_{>t+1}$.

In this case necessarily one has:

$$Y = X_1 \times \cdots \times X_t \times Y_{>t}$$

where $Y_{>t}$ is a maximal algebraic subset of $\widetilde{V_{>t}}$.

Suppose that $\dim_{\mathbb{C}}(\widetilde{V_{>t}}) > 0$. Let $x_{\leq t}$ be a special point on $X_1 \times \cdots \times X_t$ and $x_{>t}$ be a Hodge generic point of $Y_{>t}$. Let $\mathbf{H} \subset \mathbf{G}$ be the Mumford-Tate group of the point $(x_{\leq t}, x_{>t})$ of X and let $X_H \subset X$ be the $\mathbf{H}(\mathbb{R})$-orbit of x. Replace G by H the group of biholomorphisms of X_H, X by X_H, $\mathbf{G}$ by $\mathbf{H}^{\mathrm{ad}}$, Γ by Γ_H the projection of $\mathbf{H}(\mathbb{Z})$ on H, S by $S_H := \Gamma_H \backslash X_H$, $\pi : X \longrightarrow S$ by $\pi_H : X_H \longrightarrow S_H$, V by $V_H := \pi_H(x_{\leq t} \times \widetilde{V_{>t}})$ and Y by $x_{\leq t} \times Y_{>t}$ and apply theorem 7.1 for these new data: this shows that there exists $t' > t+1$ such that $\widetilde{V_{>t}} = X_{t+1} \times \cdots \times X_{t'} \times \widetilde{V_{>t'}}$. This contradicts the maximality of t.

Hence $\widetilde{V_{>t}}$ is a point $(x_{t+1}, \dots, x_r)$. Thus

$$\widetilde{V} = X_1 \times \cdots \times X_t \times (x_{t+1}, \dots, x_r)$$

is weakly special, in particular algebraic, hence by maximality

$$Y = \widetilde{V} = X_1 \times \cdots \times X_t \times (x_{t+1}, \dots, x_r)$$

and Y is weakly special.

□

Let us prove theorem 7.1. Let $\mathbf{H}_Y$ be the maximal connected $\mathbb{Q}$-subgroup in the stabiliser of Y in $\mathbf{G}(\mathbb{R})$. By theorem 1.4 the group $\mathbf{H}_Y$ is a non-trivial algebraic subgroup of $\mathbf{G}$.

Lemma 7.2. *The group $\mathbf{H}_Y(\mathbb{Q})$ stabilises $\widetilde{V}$.*

Proof. Suppose there exists $h \in \mathbf{H}_Y(\mathbb{Q})$ such that

$$\widetilde{V} \neq h\widetilde{V} \ \ .$$

As Y is contained in $\widetilde{V} \cap h\widetilde{V}$ and Y is irreducible, we can choose an analytic irreducible component $\widetilde{V'}$ of $\widetilde{V} \cap h\widetilde{V}$ containing Y. Notice that $\pi(\widetilde{V'})$ is an irreducible component, say V', of $V \cap T_h(V)$. As $\dim_{\mathbb{C}}(\widetilde{V'}) < \dim_{\mathbb{C}}(\widetilde{V})$, we have that $\dim_{\mathbb{C}}(V') < \dim_{\mathbb{C}}(V)$.

As $\pi(Y) \subset V'$, this contradicts the assumption that $\pi(Y)$ is Zariski dense in V. □

Choose a Hodge generic point z of V^{sm} (smooth locus of V) and a point $\widetilde{z}$ of $\widetilde{V}$ lying over z. Let

$$\rho^{\mathrm{mon}} \colon \pi_1(V^{\mathrm{sm}}, z) \longrightarrow \mathbf{GL}(E_{\mathbb{Z}})$$

be the corresponding monodromy representation. We let $\Gamma_V \subset \mathbf{G}(\mathbb{Z})$ be the image of ρ. By usual topological Galois theory the group Γ_V is the subgroup of $\mathbf{G}(\mathbb{Z})$ stabilising $\widetilde{V}$ (cf. section 3 of [17]), in particular Γ_V contains $\mathbf{H}_Y(\mathbb{Z})$.

By Deligne's monodromy theorem (see Theorem 1.4 of [17]), the connected component of the identity $\mathbf{H}^{\mathrm{mon}}$ of the Zariski closure $\overline{\Gamma_V}^{\mathrm{Zar},\mathbb{Q}}$ of Γ_V in $\mathbf{G}$ is a normal subgroup of $\mathbf{G}$. As $\mathbf{G}$ is semi-simple of adjoint type, after reordering the factors we may assume that $\mathbf{H}^{\mathrm{mon}}$ coincides with $\mathbf{G}_1 \times \cdots \times \mathbf{G}_t \times \{1\}$ for some integer $t \geq 1$. In particular $\mathbf{H}_Y \subset \mathbf{G}_1 \times \cdots \times \mathbf{G}_t \times \{1\}$.

We claim that Γ_V normalises $\mathbf{H}_Y$. Let $\gamma \in \Gamma_V$. Consider the $\mathbb{Q}$-algebraic group $\mathbf{F}$ generated by $\mathbf{H}_Y$ and $\gamma \mathbf{H}_Y \gamma^{-1}$. Then $\mathbf{F}(\mathbb{R})^+ \cdot \widetilde{V} = \widetilde{V}$, where $\mathbf{F}(\mathbb{R})^+$ denotes the connected component of the identity of $\mathbf{F}(\mathbb{R})$. Hence $\mathbf{F}(\mathbb{R})^+ \cdot Y \subset \widetilde{V}$. By lemma B.2 there exists an irreducible (complex) algebraic subvariety $\tilde{Y}$ of $\tilde{V}$ containing U, hence Y. By maximality of Y one has $\tilde{Y} = Y$ hence

$$\mathbf{F}(\mathbb{R})^+ \cdot Y = Y.$$

By maximality of $\mathbf{H}_Y$, we have $\mathbf{F} = \mathbf{H}_Y$. This proves the claim.

As $\mathbf{H}_Y$ is normalised by Γ_V, it is normalised by $\mathbf{H}^{\mathrm{mon}} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_t \times \{1\}$. It follows that (after possibly reordering factors) $\mathbf{H}_Y$ contains $\mathbf{G}_1 \times \{1\}$.

The fact that $\mathbf{H}_Y(\mathbb{R})$ stabilises $\widetilde{V}$ shows (by taking the $\mathbf{H}_Y(\mathbb{R})$-orbit of any point of $\widetilde{V}$) that $\widetilde{V} = X_1 \times \widetilde{V}_{>1}$. This concludes the proof of theorem 7.1 and hence of theorem 1.1.

Appendix A. Definability

A.1. **About theorem 1.2.** Let $\mathcal{R}$ be any fixed o-minimal expansion of $\mathbb{R}$ (in our case $\mathcal{R} = \mathbb{R}_{\text{an,exp}}$). Recall [6, chap.10] that a *definable manifold* of dimension n is an equivalence class (for the usual relation) of triple $(X, X_i, \phi_i)_{i\in I}$ where $\{X_i : i \in I\}$ is a finite cover of the set X and for each $i \in I$:

(i) we have injective maps $\phi_i : X_i \longrightarrow \mathbb{R}^n$ such that $\phi_i(X_i)$ is an open, definably connected, definable set.
(ii) each $\phi(X_i \cap X_j)$ is an open definable subset of $\phi_i(X_i)$.
(iii) the map $\phi_{ij} : \phi_i(X_i \cap X_j) \longrightarrow \phi_j(X_i \cap X_j)$ given by $\phi_{ij} = \phi_j \cap \phi_i^{-1}$ is a definable homeomorphism for all $j \in I$ such that $X_i \cap X_j \neq \emptyset$.

We say that a subset $Z \subset X$ is definable (resp. open or closed) if $\phi_i(Z \cap X_i)$ is a definable (resp. open or closed) subset of $\phi_i(X_i)$ for all $i \in I$. A definable map between abstract definable manifolds is a map whose graph is a definable subset of the definable product manifold.

Notice in particular that $X = \mathbb{P}^n\mathbb{C}$ has a canonical structure of a definable manifold (for any $\mathcal{R}$): take $X_i = \mathbb{C}^n = \{[z_o, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n] \in \mathbb{P}^n\mathbb{C}\}$, $0 \leq i \leq n$ where we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. As a corollary any complex quasi-projective variety is canonically a definable manifold. This apply in particular to S. In particular the statement of theorem 1.2 has an intrinsic meaning.

Appendix B. Algebraic subvarieties of X

Recall from [29, section 2.1] that a realisation $\mathcal{X}$ of X for $\mathbf{G}$ is any analytic subset of a complex quasi-projective variety $\widetilde{\mathcal{X}}$, with a transitive holomorphic action of $\mathbf{G}(\mathbb{R})$ on $\mathcal{X}$ such that for any $x_0 \in \mathcal{X}$ the orbit map $\psi_{x_0} : \mathbf{G}(\mathbb{R}) \longrightarrow \mathcal{X}$ mapping g to $g \cdot x_0$ is semi-algebraic and identifies $\mathbf{G}(\mathbb{R})/K_\infty$ with X. A morphism of realisations is a $\mathbf{G}(\mathbb{R})$-equivariant biholomorphism. By [29, lemma 2.1] any realisation of X has a canonical semi-algebraic structure and any morphism of realisations is semi-algebraic. Hence X has a canonical semi-algebraic structure.

Let $\mathcal{X}$ be a realisation of X for $\mathbf{G}$. A subset $Y \subset \mathcal{X}$ is called an *irreducible algebraic subvariety* of $\mathcal{X}$ if Y is an irreducible component of the analytic set $\mathcal{X} \cap \widetilde{Y}$ where $\widetilde{Y}$ is an algebraic subset of $\widetilde{\mathcal{X}}$. By [9, section 2] the set Y has only finitely many analytic irreducible components and these components are semi-algebraic. An *algebraic subvariety* of $\mathcal{X}$ is defined to be a finite union of irreducible algebraic subvarieties of $\mathcal{X}$.

Lemma B.1. *A subset Y of $\mathcal{X}$ is algebraic if and only if Y is a closed complex analytic subvariety of $\mathcal{X}$ and semi-algebraic in $\mathcal{X}$.*

Proof. Let $Y \subset X$ be a closed complex analytic subvariety of $\mathcal{X}$, semi-algebraic in $\mathcal{X}$. Without loss of generality we can assume that Y is irreducible as an analytic subvariety, of dimension d. Consider the real Zariski-closure $\widetilde{Y}$ of Y in the real algebraic variety $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\widetilde{\mathcal{X}}$, where $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ denotes the Weil restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$. Let us show that $\widetilde{Y}_{\mathbb{R}}$ has a canonical structure of a complex subvariety of $\widetilde{\mathcal{X}}$. Choose an affine open cover $(\widetilde{\mathcal{X}}_i)_{i\in I} \subset \mathbb{A}^{n_i}$ of $\widetilde{\mathcal{X}}$ and denote by $\widetilde{Y}_i$ the intersection $\widetilde{Y} \cap \widetilde{\mathcal{X}}_i$. Let $i \in I$ such that $\widetilde{Y}_i$ is non-empty. As Y is semi-algebraic, Y is open in $\widetilde{Y}$ for the Hausdorff topology, hence $Y_i := Y \cap \widetilde{\mathcal{X}}_i$ is non-empty and open in $\widetilde{Y}_i$ for the Hausdorff topology. Consider the Gauss map φ_i from the smooth part $\widetilde{Y}_i^{\mathrm{sm}}$ of $\widetilde{Y}_i$ to the real Grassmannian $\mathbf{Gr}^{2d,2n_i}$ of real $2d$-planes of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{A}^{n_i}$ associating to a point its tangent space. The map φ_i is real analytic and its restriction to the open subset Y_i^{sm} of $\widetilde{Y}_i^{\mathrm{sm}}$ takes values in the closed real analytic subvariety $\mathbf{Gr}_{\mathbb{C}}^{d,n_i} \subset \mathbf{Gr}^{2d,2n_i}$ of complex d-planes of $\mathbb{A}^{n_i}_{\mathbb{C}}$. By analytic continuation φ_i takes values in $\mathbf{Gr}_{\mathbb{C}}^{d,n_i}$. Hence $\widetilde{Y}_i$ is a complex algebraic subvariety of $\mathbb{A}^{n_i}$. As this is true for all $i \in I$, $\widetilde{Y}$ is a complex algebraic subvariety of $\widetilde{\mathcal{X}}$. As $Y \subset \widetilde{Y}$ is open and Y is closed analytically irreducible in $\mathcal{X}$, it follows that Y is an irreducible component of $\mathcal{X} \cap \widetilde{Y}$, hence algebraic.

The other implication is clear. □

As any morphism of realisations is an analytic biholomorphism and semi-algebraic the previous lemma implies immediately:

Corollary B.1. *Let $\varphi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ be a morphism of realisations of X. A subset Y_1 of $\mathcal{X}_1$ is algebraic if and only if its image $Y_2 := \varphi(Y_1) \subset \mathcal{X}_2$ is algebraic.*

This defines the notion of algebraic subsets of X.

Lemma B.2. *Let $\mathcal{X}$ be a realisation of a Hermitian symmetric domain X. Let $Z \subset \mathcal{X} \subset \mathbb{C}^n$ be a complex analytic subvariety and $W \subset Z$ a semi-algebraic set. There exists an irreducible complex algebraic subvariety $Y \subset \mathbb{C}^n$ such that*

$$W \subset Y \cap X \subset Z$$

Proof. This is a consequence of the proof of [24, lemma 4.1]. □

References

[1] A. Ash, D. Mumford, M. Rapoport, Y. Tai, *Smooth Compactification of Locally Symmetric varieties*, Lie Groups: History Frontiers and applications vol. **4**. Math Sci Press. (1975).

[2] J. Ax, *On Schanuel's conjecture*, Annals of Math. **93** (1971), 1-24

[3] W.L. Baily, A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Annals of Math., **84** (1966), 442-528

[4] A. Borel, *Introduction aux groupes arithmétiques*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341 Hermann, Paris (1969)

[5] P. Deligne, *Variétés de Shimura: interprétation modulaire et techniques de construction de modèles canoniques*, dans *Automorphic Forms, Representations, and L-functions* part. **2**; Editeurs: A. Borel et W Casselman; Proc. of Symp. in Pure Math. **33**, American Mathematical Society (1979) p. 247-290.

[6] L. van den Dries, *Tame Topology and o-minimal structures.* LMS lecture note series, 248. Cambridge University Press, 1998.

[7] L. van den Dries, C. Miller *On the real exponential field with restricted analytic functions*, Israel J. Math. **85** (1994), 19–56.

[8] B. Edixhoven, A. Yafaev, *Subvarieties of Shimura varieties*, Annals of Mathematics (2) **157** (2003) 621-645

[9] E. Fortuna, S. Lojasiewicz, *Sur l'algébricité des ensembles analytiques complexes*, J. Reine Angew. Math. **329** (1981) 215-220

[10] J.M. Hwang, W.K. To, *Volumes of complex analytic subvarieties of Hermitian symmetric spaces*, Amer. J. Math., **124** (2002), 1221-1246

[11] B. Klingler, A. Yafaev ,*The André-Oort conjecture*, preprint, available on first author's web-page.

[12] F. Lindemann, *Über di Zahl* π, Math. Ann. **20** (1882), 213-225

[13] Margulis G.A, Discrete subgroups of semisimple Lie groups, *Ergebnisse der Mathematik und ihrer Grengebiete* 17, Springer-Verlag, (1991)

[14] N. Mok, *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Math. **6**. World Scientific (1989).

[15] N. Mok, *On the Zariski closure of a germ of totally geoedesic complex submanifold on a subvariety of a complex hyperbolic space form of finite volume*, Complex Analysis, 2-79-300, Trends Math., Birkhäuser Springer (2010).

[16] N. Mok, *Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric*, J. Eur. Math. Soc. **14** (2012), 1617-1656

[17] B. Moonen, *Linearity properties of Shimura varieties. I*, J. Algebraic Geom. **7** (1998), 539-567.

[18] D. Mumford, *Hirzebruch's proportionality theorem in the non-cocompact case*, Inventiones Math. **42**, 239-272, 1979

[19] Y. Peterzil, S. Starchenko, *Definability of restricted theta functions and families of abelian varieties*, Duke Math. J. **162**, (2013), 731-765

[20] Y. Peterzil, S. Starchenko *Tame complex analysis and o-minimality.* Proceedings of the ICM, Hyderabad, 2010. Available on first author's web-page.

[21] I.I. Pyateskii-Shapiro, *Automorphic functions and the geometry of classical domains*, translated from the Russian, Mathematics and its Applications, Vol. 8, Gordon and Breach Science Publishers (1969)

[22] J. Pila, *O-minimality and the Andre-Oort conjecture for* $\mathbb{C}^n$, Annals Math. **173** (2011), 1779-1840.

[23] J. Pila, A. Wilkie, *The rational points on a definable set*, Duke Math. Journal **133**, (2006) 591-616.

[24] J. Pila, J. Tsimerman, *The André-Oort conjecture for the moduli space of abelian surfaces*, preprint, 2011, available on ArXiv.

[25] J. Pila, J. Tsimerman, *Ax-Lindemann for* $\mathcal{A}_g$. Preprint, 2012

[26] J. Pila, U. Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **19** (2008), no. 2, 149-162.

[27] I. Satake, *Algebraic structures of symmetric domains*, Kanu Memorial Lectures 4, Iwanami Shoten and Princeton University Press (1980)

[28] T. Scanlon, *O-minimality as an approach to the André-Oort conjecture*, preprint 2012. Available on author's web-page.

[29] E. Ullmo, *Quelques applications du théorème d'Ax-Lindemann hyperbolique*, preprint 2012, Available on author's web-page.

[30] E. Ullmo, A. Yafaev, *Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture,* preprint, Available on author's web-page.

[31] E. Ullmo, A. Yafaev, *Hyperbolic Ax-Lindemann theorem in the cocompact case,* preprint 2011

[32] K. Weierstraß, *Zu Lindemanns Abhandlung: "Über die Ludolph'sche Zahl"*, Berl. Ber. (1885) 1067-86

[33] J.A. Wolf, A. Korányi, *Generalized Cailey transformations of bounded symmetric domains*, Amer.J.Math. **87** (1965), 899-939

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