Metaplectic cusp forms on the group $SL_2(\mathbb{Q}(i))$

by

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Declaration

I, Nenna Campbell-Platt, confirm that the work in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

The aim of this thesis is to contribute to the understanding of genuine cusp forms on the group $SL_2/\mathbb{Q}(i)$, from a computational point of view. We show, via the generalised Eichler-Shimura-Harder isomorphism, that a genuine cusp form of cohomological type exists at level $SL_2(\mathbb{Z}[i], 4)SL_2(\mathbb{Z})$. We show, by calculating cohomology groups, that such a form exists at weight $(2, 2)$. Finally, we compute the genuine quotient of the Hecke algebra acting on representations of $\overline{SL}_2(\mathbb{Q}_2(i))$ containing non-zero $SL_2(\mathbb{Z}_2[i], 4)SL_2(\mathbb{Z}_2)$-fixed vectors. When such a representation $\overline{\varpi}$ corresponds to an unramified representation of $SL_2(\mathbb{Q}_2(i))$, we show that the space of $SL_2(\mathbb{Z}_2[i], 4)SL_2(\mathbb{Z}_2)$-fixed vectors in $\overline{\varpi}$ is a sum of two 1-dimensional components. We determine which 1-dimensional representations arise in this way.

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It is my Mum, my life coach and inspiration, to whom this thesis is dedicated.

Contents

Chapter 0

Introduction

The aim of this thesis is to contribute to the understanding of genuine cusp forms on $SL_2/\mathbb{Q}(i)$, from a computational point of view. These so-called "Bianchi modular forms" are of half-integer weight, and their level is a subgroup of finite index of $SL_2(\mathbb{Z}[i], 4\mathfrak{n})SL_2(\mathbb{Z})$, for some non-zero ideal $\mathfrak{n} \subseteq \mathbb{Z}[i]$, where $SL_2(\mathbb{Z}[i], 4\mathfrak{n})$ is the group of matrices in $SL_2(\mathbb{Z}[i])$ which are congruent to the identity modulo $4\mathfrak{n}$. It is of interest to determine the (finite) dimension of the space of such forms, for given level and weight, and in particular, how small $\mathfrak{n}$ needs to be for the space to be of positive dimension. In the first chapter of this thesis, we prove that $\mathfrak{n} = \mathbb{Z}[i]$ will suffice: there is a non-trivial genuine cusp form at level $SL_2(\mathbb{Z}[i], 4)SL_2(\mathbb{Z})$, for some weight.

In chapter two, we show, computationally, that there is a non-trivial cusp form of level $SL_2(\mathbb{Z}[i], 4)SL_2(\mathbb{Z})$ at weight $(2, 2)$. Our approach is via the generalised Eichler-Shimura-Harder isomorphism: we calculate the cohomology of $SL_2(\mathbb{Z}[i], 4)SL_2(\mathbb{Z})$ with certain non-trivial coefficients. This well-known method involves plugging information from the fundamental domain for the group $SL_2(\mathbb{Z}[i])$ into a spectral sequence. We use this method to compute the rational cohomology of some congruence subgroups, as well as the integral cohomology in some cases.

Jacquet and Langlands [24] re-wrote the theory of integral weight automorphic forms in the language of representation theory: it was found that an automorphic form generates the space of a "representation" of the adèle group of GL_2, or of SL_2. Gelbart and Piatetski-Shapiro [18, 20] generalised this theory to half-integral weight forms; in this case, however, one obtains a representation of the adèle group of $\overline{SL}_2$, the two-fold cover of SL_2. The notion of "level" of a Bianchi modular form extends to these representations: in the case of the number field $\mathbb{Q}(i)$, the level of such a representation is a compact open subgroup of

$$\prod_{v \text{ finite}} SL_2(\mathbb{Z}[i]_v).$$

The local component of these representations at the even prime, $1+i$, has an action of the group $\overline{SL}_2(\mathbb{Q}_2(i))$. Suppose that its subspace of $SL_2(\mathbb{Z}_2[i], 4\mathfrak{n})SL_2(\mathbb{Z}_2)$-fixed vectors is not trivial. There is no clear picture of what these representations should look like. The corresponding question over $\mathbb{Q}$, however, has been solved fairly recently by Loke and Savin [28].

In the third chapter of this thesis, we completely determine the (compact part of the) Hecke algebra acting on the even component of a representation. We compute the action explicitly on a representation when $\mathfrak{n}$ is the trivial ideal, that is, when the level is $SL_2(\mathbb{Z}_2[i], 4)SL_2(\mathbb{Z}_2) \times \prod_{v \text{ finite, odd}} SL_2(\mathbb{Z}[i]_v)$. We find that the subspace of $SL_2(\mathbb{Z}_2[i], 4)SL_2(\mathbb{Z}_2)$-fixed vectors is two-dimensional: it is the sum of two one-dimensional eigenspaces for the action of a Hecke operator. We hope this goes a long way towards a good definition of an "unramified" representation of $\overline{SL}_2(\mathbb{Q}_2(i))$.

Background

If d is a square-free, positive integer, let $F_{-d} = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic number field with integer ring $\mathcal{O}_{-d}$. Groups of the form $GL_2(\mathcal{O}_{-d})$ and $SL_2(\mathcal{O}_{-d})$ are called "Bianchi groups"; living inside $GL_2(\mathbb{C})$, they are discrete, and act properly discontinuously on 3-dimensional real upper-half space $\mathbb{H} = \{(z, r) \mid z \in \mathbb{C}, r > 0\}$. Explicitly, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{C})$, define

$$g \cdot (z, r) = (z', r')$$

where

$$z' = \frac{(az+b)\overline{(cz+d)} + a\bar{c}r^2}{|cz+d|^2 + |cr|^2}, \qquad r' = \frac{|ad-bc|r}{|cz+d|^2 + |cr|^2}. \tag{0.1}$$

and $|z| = z\bar{z}$ is the usual norm in $\mathbb{C}$.

Let $\mathfrak{H}$ be the ring of quaternions. It is 4-dimensional as a vector space over $\mathbb{R}$, with basis $1, i, j, k$. If we identify $\mathbb{H}$ with a subset of $\mathfrak{H}$ in the following way:

$$\begin{aligned} \mathbb{H} &\longrightarrow \mathfrak{H} \\ (z, r) = (x + iy, r) &\longmapsto q = x + iy + jr, \end{aligned} \tag{0.2}$$

then the action (0.1) takes the more aesthetically pleasing form:

$$g \cdot q = (aq + b)(cq + d)^{-1}. \tag{0.3}$$

The stabilizer of the point $j \in \mathfrak{H}$ is the group

$$\mathbb{R}^\times \cdot SU(2) = \left\{ \begin{pmatrix} \bar{d} & -\bar{c} \\ c & d \end{pmatrix} \,\middle|\, c, d \in \mathbb{C},\ (c, d) \neq (0, 0) \right\}.$$

Now fix $d = 1$. Then $F_{-1} = \mathbb{Q}(i)$, and $\mathcal{O}_{-1} = \mathbb{Z}[i]$. Suppose that Υ is a finite index subgroup of the Bianchi group $SL_2(\mathbb{Z}[i])$. Let $L^2(\Upsilon \backslash SL_2(\mathbb{C}))$ denote the space of complex-valued square integrable functions on $\Upsilon \backslash SL_2(\mathbb{C})$; we'll regard this as a representation of $SL_2(\mathbb{C})$. The space $L^2(\Upsilon \backslash SL_2(\mathbb{C}))$ is a direct sum of the continuous spectrum $L^2_c(\Upsilon \backslash SL_2(\mathbb{C}))$ and the discrete spectrum $L^2_d(\Upsilon \backslash SL_2(\mathbb{C}))$ [4,21]. By a well-known result of Gelfand and Piatetski-Shapiro, the latter space is a Hilbert direct sum of irreducible subspaces with finite multiplicities:

$$L^2_d(\Upsilon \backslash SL_2(\mathbb{C})) = \widehat{\bigoplus_{\varpi}} m(\varpi, \Upsilon) H_\varpi \tag{0.4}$$

where ϖ ranges over the set of equivalence classes of irreducible unitary representations of $SL_2(\mathbb{C})$.

Write $L^2_0(\Upsilon \backslash SL_2(\mathbb{C}))^\infty$ for the subspace of smooth cuspidal functions in $L^2_d(\Upsilon \backslash SL_2(\mathbb{C}))$, and let M be a finite-dimensional, irreducible representation of $SL_2(\mathbb{C})$. It was Borel [4] who established that the map

$$j : H^q_{\text{cts}}(SL_2(\mathbb{C}), L^2_0(\Upsilon \backslash SL_2(\mathbb{C}))^\infty \otimes M) \longrightarrow H^q(\Upsilon, M) \tag{0.5}$$

is injective for all non-negative integers q. The image of j is called the cuspidal cohomology and is written $H^q_{\text{cusp}}(\Upsilon, M)$. Results (0.4) and (0.5) can be used to show that

$$H^q_{\text{cusp}}(\Upsilon, M) = \bigoplus_{\varpi} m(\varpi, \Upsilon) H^q_{\text{cts}}(SL_2(\mathbb{C}), H^\infty_\varpi \otimes M) \text{ for } q = 1, 2$$

where H_{ϖ}^{∞} means the subspace of smooth vectors in H_{ϖ}, and ϖ ranges over the irreducible unitary representations of $SL_2(\mathbb{C})$. Suppose now that Υ is a congruence subgroup. Let $\mathbb{A}$ be the adèle ring of $\mathbb{Q}(i)$, let $\mathbb{A}_f$ be its finite adèles, and let $K_f(\Upsilon)$ be the closure of Υ in $SL_2(\mathbb{A}_f)$. There is an isomorphism

$$\Upsilon\backslash SL_2(\mathbb{C}) \cong SL_2(\mathbb{Q}(i))\backslash SL_2(\mathbb{A})/K_f(\Upsilon).$$

Let $L^2(SL_2(\mathbb{Q}(i))\backslash SL_2(\mathbb{A})/K_f(\Upsilon))$ be the space of square integrable functions on $SL_2(\mathbb{Q}(i))\backslash SL_2(\mathbb{A})/K_f(\Upsilon)$. It is the direct sum of a continuous spectrum and a discrete spectrum $L^2_d(SL_2(\mathbb{Q}(i))\backslash SL_2(\mathbb{A})/K_f(\Upsilon))$. There are results analogous to (0.4), (0.5), and there is a decomposition, for $q = 1, 2$,

$$H^q_{\text{cusp}}(\Upsilon, M) = \bigoplus_{\varpi} m(\varpi, \Upsilon) H^q_{\text{cts}}(SL_2(\mathbb{C}), H^{\infty}_{\varpi_\infty} \otimes M) \otimes H^{K_f(\Upsilon)}_{\varpi_f} \tag{0.6}$$

where the sum is taken over the set of cuspidal automorphic representations $\varpi = \varpi_f \otimes \varpi_\infty$ of $SL_2(\mathbb{A})$ of level $K_f(\Upsilon)$. Equation (0.6) is called the generalised Eichler-Shimura-Harder isomorphism.

Suppose that μ_2 is the multiplicative group of square roots of unity in $\mathbb{Q}(i) \subset \mathbb{C}$. Consider the central extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{SL_2}(\mathbb{A}) \longrightarrow SL_2(\mathbb{A}) \longrightarrow 1 \tag{0.7}$$

by which the metaplectic group $\overline{SL_2}(\mathbb{A})$ is defined. The extension splits on $SL_2(\mathbb{Q}(i))$ and on $SL_2(\mathbb{C})K_f(\Gamma')$, where Γ' is a congruence subgroup of $SL_2(\mathbb{Z}[i])$, and it therefore splits in two ways on the intersection $\Gamma' = SL_2(\mathbb{Q}(i)) \cap SL_2(\mathbb{C})K_f(\Gamma')$. Let $\overline{SL_2}(\mathbb{C})$ be the pre-image of $SL_2(\mathbb{C})$ in $\overline{SL_2}(\mathbb{A})$; it occurs in an extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{SL_2}(\mathbb{C}) \longrightarrow SL_2(\mathbb{C}) \longrightarrow 1. \tag{0.8}$$

Dividing one splitting of (0.7) by the other on Γ', we get a homomorphism

$$\kappa : \Gamma' \longrightarrow \mu_2$$

and a bijection between Γ' and a subgroup $\widehat{\Gamma'}$ of $\overline{SL_2}(\mathbb{C})$. We can regard κ as a one-dimensional $\mathbb{Q}$-representation $\kappa_{\mathbb{Q}}$ of Γ'.

In this thesis, we shall be concerned with "genuine" objects on $\overline{SL_2}(\mathbb{A})$. We say that a function ϕ on $\overline{SL_2}(\mathbb{A})$ is *genuine* if $\phi(\xi g) = \xi\phi(g)$ for all $g \in \overline{SL_2}(\mathbb{A})$ and for all $\xi \in \mu_2$ (note that μ_2 is contained in the centre of $\overline{SL_2}(\mathbb{A})$). In particular, if V is a complex vector space, an irreducible representation $\bar{\rho} : \overline{SL_2}(\mathbb{A}) \to GL(V)$ is genuine if the subgroup μ_2 acts faithfully.

We shall write $\widehat{SL_2}(\mathbb{Q}(i))$ (resp. $\widehat{K_f}(\Gamma'))$ for the lift of the subgroup $SL_2(\mathbb{Q}(i))$ (resp. $K_f(\Gamma'))$ to $\overline{SL_2}(\mathbb{A})$. Denote by $L^2(\widehat{SL_2}(\mathbb{Q}(i))\backslash\overline{SL_2}(\mathbb{A})/\widehat{K_f}(\Gamma'))$ the space of genuine, square integrable functions on $\widehat{SL_2}(\mathbb{Q}(i))\backslash\overline{SL_2}(\mathbb{A})/\widehat{K_f}(\Gamma')$; it has a continuous and a discrete part. The bijection

$$\widehat{\Gamma}'\backslash\overline{SL_2}(\mathbb{C}) \cong \widehat{SL_2}(\mathbb{Q}(i))\backslash\overline{SL_2}(\mathbb{A})/\widehat{K_f}(\Gamma')$$

allows us to replace $SL_2(\mathbb{C})$ with $\overline{SL_2}(\mathbb{C})$ in (0.4) and (0.5), and we obtain

$$H^q_{\mathrm{cusp}}(\Gamma', \kappa_{\mathbb{Q}} \otimes M) = \bigoplus_{\overline{\varpi}} m(\overline{\varpi}, \Gamma') H^q_{\mathrm{cts}}(\overline{SL_2}(\mathbb{C}), \overline{H}^{\infty}_{\overline{\varpi}_\infty} \otimes \kappa_{\mathbb{Q}} \otimes M) \otimes \overline{H}^{\widehat{K_f}(\Gamma')}_{\overline{\varpi}_f} \tag{0.9}$$

for $q = 1, 2$, where the sum is taken over the set of genuine, cuspidal automorphic representations $\overline{\varpi} = \overline{\varpi}_f \otimes \overline{\varpi}_\infty$ of $\overline{SL_2}(\mathbb{A})$. The expression (0.9) is the metaplectic version of the generalised Eichler-Shimura-Harder isomorphism (0.6).

Computer-aided calculation

Many of the computations in this thesis are large. Notably, some cohomology calculations and the determination of the genuine quotient of the Hecke algebra in Chapter three. To mitigate the risk of human error, we have used Sage to carry out most of this work. We have chosen to explain the algorithm in the body of the text where we think necessary, and have otherwise relegated the code to the appendices so as not to disrupt the discussion.

Chapter 1

Existence of a genuine cusp form

In this chapter, we shall show that a genuine cusp form, of level one and of cohomological type, exists on the group $SL_2/\mathbb{Q}(i)$ (Corollary 1.4.17). In particular, there is a non-negative integer k such that

$$H^2_{\text{cusp}}(\Gamma', \kappa_{\mathbb{Q}} \underset{\mathbb{C}}{\otimes} E_{k,k}(\mathbb{C})) \neq 0.$$

The notation $E_{k,k}(\mathbb{C})$ shall be defined below.

Our method of proof rests on a theorem of Flicker [17], who formulates his result in terms of the group GL_2. Despite the fact that we shall only apply Flicker's result to the group SL_2, it will be necessary to give the background for the larger group GL_2. For this reason, the reader is warned that the notation in this chapter is cumbersome, and many of the definitions may seem superfluous.

Sections 1.1 through to 1.4.2 form the necessary background: Sections 1.1 and 1.2 define the local and global metaplectic groups, on which these forms live. In Propositions 1.2.3 and 1.2.7, we give an explicit description of the homomorphism

$$\kappa : \Gamma' \longrightarrow \mu_2$$

which is of central importance in this thesis.

In Section 1.3, we sketch the interpretation of integral and half-integral weight modular forms as functions on the group $GL_2(\mathbb{A})$ and its two-fold cover. In Sections 1.4.1 through to 1.4.2, we develop the local and global representation theory needed to define a cuspidal automorphic representation, and we describe, in particular, those representations of $GL_2(\mathbb{C})$ which have cohomology (Section 1.4.1.2). Finally, in Section 1.4.3, we give a modification of Flicker's correspondence (Definition 1.4.19) which enables us to derive our main result.

1.1 The local metaplectic groups

Once and for all, fix $F = F_{-1}$ to be the imaginary quadratic field $\mathbb{Q}(i)$ and $\mathcal{O} = \mathcal{O}_{-1}$ to be its ring of integers $\mathbb{Z}[i]$. We shall denote by v a place of F, and by F_v the completion of F at v. In particular, if v is the infinite place, then F_v is archimedean, and $F_v = \mathbb{C}$. If v is finite, then F_v is non-archimedean, and is a finite algebraic extension of the p-adic field $\mathbb{Q}_p$. There is a unique *even* place of F, namely $(1+i)$; by an abuse of notation, we shall write π both for the place, and the prime in $\mathcal{O}$. If F_v is non-archimedean, let $\mathcal{O}_v$ denote the ring of integers of F_v and $\mathcal{O}_v^\times$ its group of units. If x is an element of $F_v^\times$, we shall define its order by $x = uv^{\mathrm{ord}_v(x)}$ for a unit $u \in \mathcal{O}_v^\times$.

Throughout the thesis, the group $\mu_2 = \{\pm 1\} \subset F$ will be the multiplicative group of square roots of 1; the algebraic group GL_2 will be denoted by G until further notice. If R is a ring, we shall denote the R points of G by $G(R)$, unless $R = F_v$, in which case we shall denote these points by G_v.

In general, if H is a (multiplicative) locally compact group, a two-cocycle σ on H is a Borel-measurable map $\sigma : H \times H \to \mu_2$ with the properties

$$\sigma(h_1, h_2h_3)\sigma(h_2, h_3) = \sigma(h_1h_2, h_3)\sigma(h_1, h_2) \text{ and } \sigma(1, h) = \sigma(h, 1) = 1 \tag{1.1}$$

We call the cocycle *trivial* if there is a map $s : H \to \mu_2$ such that

$$\sigma(h_1, h_2) = s(h_1)s(h_2)s(h_1h_2)^{-1} \text{ for all } h_1, h_2 \in H.$$

In this thesis, we shall be concerned with a specific two-cocycle on G (Theorem 1.1.2 below) whose formula was given by Kubota [25]. Such a cocycle σ determines an exact sequence of groups

$$1 \longrightarrow \mu_2 \longrightarrow \overline{H} \longrightarrow H \longrightarrow 1 \tag{1.2}$$

where $\overline{H}$ is realised as the set of pairs $\{h, \xi\}$ with $h \in H$ and $\xi \in \mu_2$ with multiplication given by

$$\{h_1, \xi_1\}\{h_2, \xi_2\} = \{h_1h_2, \xi_1\xi_2\sigma(h_1, h_2)\}$$

The extension is called *central* because $\xi \mapsto \{1, \xi\}$ is an injective homomorphism from μ_2 to the centre of $\overline{H}$. The group $\overline{H}$ has a natural locally compact topology and the map $\overline{H} \to H$ is a covering map.

If K is any subgroup of H, $\overline{K}$ will denote its full inverse image in $\overline{H}$. We say that the extension (1.2) *splits* over K if there is a map $s : K \to \mu_2$ which satisfies $\sigma(k_1, k_2) = s(k_1)s(k_2)s(k_1k_2)^{-1}$ for all $k_1, k_2 \in K$. Equivalently, $\overline{K}$ is the direct product $\overline{K} = \widehat{K} \times \mu_2$ for some subgroup $\widehat{K} \subset \overline{H}$ isomorphic to K. The extension (1.2) is called *trivial* if it splits over H itself. Finally, we shall write $\overline{K}^n$ for the inverse image of K^n in $\overline{H}$, and not for the set of n-th powers of elements of $\overline{K}$.

The quadratic Hilbert symbol $(\ ,\ )_v$ is a symmetric bilinear map from $F_v^\times \times F_v^\times$ to μ_2 which takes $(x, y)_v$ to 1 iff x in $F_v^\times$ is a norm from $F_v(\sqrt{y})$. In particular, $(x, y)_v$ is identically 1 if y is a square. Thus $(\ ,\ )_v$ is trivial on $(F_v^\times)^2 \times (F_v^\times)^2$ for every F_v and trivial on $F_v^\times \times F_v^\times$ itself if $F_v = \mathbb{C}$.

Some properties of the quadratic Hilbert symbol which we shall use repeatedly are collected below.

Proposition 1.1.1. *1. For each F_v, $(\ ,\ )_v$ satisfies*

$$(a, b)_v = (a, -ab)_v = (a, (1-a)b)_v, \tag{1.3}$$

and

$$(a, b)_v = (-ab, a+b)_v; \tag{1.4}$$

2. If v is odd, $(u, u')_v$ is identically 1 on $\mathcal{O}_v^\times \times \mathcal{O}_v^\times$;

3. If $v = \pi$, and u in $\mathcal{O}_\pi$ is such that $u \equiv 1 \pmod{\pi^4}$, then $(u, u')_\pi$ is identically 1 on $\mathcal{O}_\pi^\times$;

4. If $v = \pi$, and $a \equiv a' \pmod{\pi^5}$ for $a \in \mathcal{O}_\pi^\times$, then $(a, b)_\pi = (a', b)_\pi$ for all $b \in F_\pi$;

5. If $a, b \in F^\times$, then

$$\prod_v (a, b)_v = 1, \tag{1.5}$$

the product being over all places v of F;

6. *If v lies above the rational prime p, and if $a \in \mathbb{Q}_p^\times$, then*

$$(a,b)_v = (a, N(b))_p$$

where $(\cdot,\cdot)_p$ denotes the Hilbert symbol in $\mathbb{Q}_p$, and N is the Norm map $N : F_v^\times \to \mathbb{Q}_p^\times$.

Parts (2) and (3) of Proposition 1.1.1 can be found in O'Meara [31, § 63] and Artin-Tate [1, Chapter 12]. Part (4) is a consequence of Hensel's Lemma for complete local rings: if $a \equiv 1 \pmod{\pi^5}$, then a is a square in F_π. Parts (1), (5) and (6) can be found in [32, pp. 101–102].

Now suppose $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and set $x(h)$ equal to c if $c \neq 0$ and equal to d if $c = 0$.

Theorem 1.1.2 (Kubota)**.** *The map $\omega_v : SL_2(F_v) \times SL_2(F_v) \to \mu_2$ defined by*

$$\omega_v(h_1, h_2) = (x(h_1), x(h_2))_v(-x(h_1)x(h_2), x(h_1h_2))_v \tag{1.6}$$

is a two-cocycle on $SL_2(F_v)$. Moreover, this cocycle is trivial if and only if $F_v = \mathbb{C}$.

The exact sequence of locally compact groups determined by ω_v is

$$1 \longrightarrow \mu_2 \longrightarrow \overline{SL_2}(F_v) \longrightarrow SL_2(F_v) \longrightarrow 1.$$

The group $\overline{SL_2}(F_v)$ is also called a *two-fold cover* of $SL_2(F_v)$. The first non-trivial two-fold cover of $SL_2(k)$ for k a non-archimedean field was given by Weil [39] and named the "metaplectic group".

There is an extension of ω_v to G_v: if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to G_v, write $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \det(g) \end{smallmatrix}\right) p(g)$ where

$$p(g) = \begin{pmatrix} a & b \\ \frac{c}{\det(g)} & \frac{d}{\det(g)} \end{pmatrix} \in SL_2(F_v) \tag{1.7}$$

For g_1, g_2 in G_v, define

$$\sigma_v(g_1, g_2) = \omega_v(p(g_1)^{\det(g_2)}, p(g_2))v(\det(g_2), p(g_1)) \tag{1.8}$$

where

$$h^y = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}^{-1} h \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \tag{1.9}$$

and

$$v(y,h) = \begin{cases} 1 & \text{if } c \neq 0 \\ (y,d)_v & \text{otherwise} \end{cases} \tag{1.10}$$

if $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proposition 1.1.3. *The cocycle σ_v determines a non-trivial central extension of G_v by μ_2 whose restriction to $SL_2(F_v) \times SL_2(F_v)$ is ω_v.*

For a proof, see [20].

The group $\overline{G}_v$ is thus realised as the set of pairs $\{g, \xi\}$ with multiplication given by $\{g_1, \xi_1\}\{g_2, \xi_2\} = \{g_1 g_2, \xi_1 \xi_2 \sigma_v(g_1, g_2)\}$.

Let T_v, N_v and B_v denote the following subgroups of G_v:

$$T_v = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \middle| \, a_i \in F_v^\times \right\};$$

$$N_v = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in F_v \right\};$$

$$B_v = \left\{ \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \middle| \, a_i \in F_v^\times, \ b \in F_v \right\} = N_v T_v = T_v N_v.$$

Further, let K_v be the maximal compact subgroup of G_v ($U(2)$ if $F_v = \mathbb{C}$, and $G(\mathcal{O}_v)$ otherwise). In the case that v is the infinite place, write $T_\mathbb{C}, N_\mathbb{C}$ and $B_\mathbb{C}$ for T_v, N_v and B_v respectively, and let $Z_\mathbb{C}$ denote the centre of $G_\mathbb{C}$.

Remark 1.1.1. The reader is warned that we shall use the same notation (T_v, N_v, B_v, K_v) when we mean the corresponding subgroups of $SL_2(F_v)$. We hope any ambiguity shall be eliminated by the context.

Note that the ideals (π^4) and (4) in $\mathcal{O}_\pi$ are equal. If $a \in \mathcal{O}_\pi$, $\tilde{a}$ shall denote its reduction modulo (4). Define a compact open subgroup $K_\pi(4)$ of G_π by

$$K_\pi(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{Z}/4\mathbb{Z}, \ ad - bc \in \mathcal{O}_\pi^\times \right\}, \tag{1.11}$$

and for any v, consider the extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_v \longrightarrow G_v \longrightarrow 1. \tag{1.12}$$

Proposition 1.1.4 (Kubota). *The extension* (1.12) *splits over* K_v *when* v *is finite and odd, and splits over* $K_\pi(4)$ *when* $v = \pi$. *More precisely, for any finite* v, *if* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* G_v, *set*

$$\kappa_v\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} (c, d\ det(g))_v & \text{if } cd \neq 0 \text{ and } ord_v(c) \text{ is odd} \\ 1 & \text{otherwise} \end{cases} \tag{1.13}$$

Then, for all $h_1, h_2 \in K_v$ *or* $K_\pi(4)$,

$$\sigma_v(h_1, h_2) = \kappa_v(h_1)\kappa_v(h_2)\kappa_v(h_1h_2)^{-1} \tag{1.14}$$

Proof. When v is finite and odd, this is Kubota's result [26, Theorem 2]. When v is even, Kubota's result applies to the group

$$G(\mathcal{O}_\pi, 4) := \left\{ \gamma \in G(\mathcal{O}_\pi) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 4 \right\}.$$

Observe that $K_\pi(4) = G(\mathcal{O}_\pi, 4)G(\mathbb{Z}_2)$. Hence, to prove the proposition, we must show that when $v = \pi$, the extension (1.12) splits on the group $G(\mathbb{Z}_2)$.

Suppose that $g_1, g_2 \in G(\mathbb{Z}_2)$. Then both $\sigma_\pi(g_1, g_2)$ and $\kappa_\pi(g_1)\kappa_\pi(g_2)\kappa_\pi(g_1g_2)^{-1}$ are a product of Hilbert symbols of the form

$$(x, y)_\pi \text{ for some } x, y \in \mathbb{Z}_2.$$

But since $x, y \in \mathbb{Z}_2$, each of these Hilbert symbols is 1 as a consequence of part (6) of Proposition 1.1.1. Therefore, both $\kappa_\pi(g_1)\kappa_\pi(g_2)\kappa_\pi(g_1g_2)^{-1}$ and $\sigma_\pi(g_1, g_2)$ are 1, so they are equal.

□

Definition 1.1.1. *When* $F_v = \mathbb{C}$, *let* $\kappa_v(g)$ *be the function which is identically* 1 *on* G_v. *If* F_v *is non-archimedean, let* κ_v *be as in (1.13), and in general, let* $\beta_v(g_1, g_2)$ *denote the* 2*-cocycle* $\beta_v(g_1, g_2) = \sigma_v(g_1, g_2)\kappa_v(g_1)\kappa_v(g_2)\kappa_v(g_1g_2)^{-1}$.

The cocycle β_v determines an equivalent extension to that determined by σ_v, but β_v has the added advantage that its restriction to $K_v \times K_v$ (respectively, $K_\pi(4) \times K_\pi(4)$) is identically 1, so K_v (resp. $K_\pi(4)$) lifts to a subgroup of $\overline{G}_v$ via $k \mapsto \{k, 1\}$, which we shall denote by $\widehat{K}_v$ (resp. $\widehat{K}_\pi(4)$).

We collect here some properties of the cocycle β_v and the extension (1.12) which will be useful later.

Lemma 1.1.5. *Let*

$$g_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} \in B_v, \quad i = 1, 2$$

Then $\beta_v(g_1, g_2) = (a_1, c_2)_v$

Proof. Observe that

$$\begin{aligned} \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & a_i c_i \end{pmatrix} \begin{pmatrix} a_i & b_i \\ 0 & a_i^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \det(g_i) \end{pmatrix} \begin{pmatrix} a_i & b_i \\ 0 & a_i^{-1} \end{pmatrix} \end{aligned}$$

It follows that

$$\begin{aligned} \sigma_v(g_1, g_2) &= \omega_v \left(\begin{pmatrix} a_1 & a_2 c_2 b_1 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & a_2^{-1} \end{pmatrix} \right) v \left(a_2 c_2, \begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{pmatrix} \right) \\ &= (a_1^{-1}, a_2^{-1})_v (a_1^{-1} a_2^{-1}, a_1^{-1})_v (a_1^{-1} a_2^{-1}, a_2^{-1})_v (a_2 c_2, a_1^{-1})_v \\ &= (a_1, c_2)_v \text{ by (1.3).} \end{aligned}$$

The proof is concluded by observing that

$$\kappa_v \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = 1 \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B_v.$$

□

Corollary 1.1.6. *The extension* (1.12) *splits on the subgroups* N_v *and* T_v^2*, where*

$$T_v^2 = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \middle| \, a_i \in F_v^{\times 2} \right\}.$$

It splits uniquely on N_v *and canonically on* T_v^2*.*

Corollary 1.1.7. *The centre of* $\overline{G}_v$ *is*

$$Z(\overline{G}_v) = \left\{ \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \xi \right\} \middle| \, z \in F_v^{\times 2}, \ \xi \in \mu_2 \right\}.$$

We refer to [20, § 2.1] for the proof.

Remark 1.1.2. The group $\overline{T}_v \subset \overline{G}_v$ is not abelian. However, when $\overline{T}_v$ is considered as a subgroup of $\overline{SL_2}(F_v)$, it is abelian. On the other hand, $\overline{T}_v^2 \subset \overline{G}_v$ is abelian. We shall use these facts in Chapter 3.

1.2 The global metaplectic group

Let $\mathbb{A}$ be the adèle ring of F, and $\mathbb{A}^\times$ its group of idèles, so that $G_{\mathbb{A}} := GL_2(\mathbb{A})$ is the restricted direct product $\prod_v' G_v$ with respect to the subgroups K_v. We write $\mathbb{A}_f$ for $\prod_{v<\infty}' F_v$, and $G(\mathbb{A}_f)$ for $\prod_{v<\infty}' G_v$.

According to Proposition 1.1.4, if $g = (g_v), g' = (g'_v) \in G_{\mathbb{A}}$, then $\beta_v(g_v, g'_v) = 1$ for almost every v. Thus we can define $\beta_{\mathbb{A}}(g, g') = \prod_v \beta_v(g_v, g'_v)$ which is a two-cocycle giving rise to the extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_{\mathbb{A}} \longrightarrow G_{\mathbb{A}} \longrightarrow 1. \tag{1.15}$$

Note that $\overline{G}_{\mathbb{A}}$ is *not* a restricted direct product of the local groups $\overline{G}_v$ [18], but rather a quotient of it, by the subgroup

$$\left\{ \prod_v \epsilon_v \in \prod_v (\mu_2)_v \;\middle|\; \epsilon_v = 1 \text{ for all but an even number of } v \right\}.$$

Consider the following subgroups of $G_{\mathbb{A}}$:

$$K'_f := K_\pi(4) \times \prod_{v<\infty,\mathrm{odd}} K_v$$

$$G_F := GL_2(F) \text{ embedded diagonally.}$$

Proposition 1.2.1. *The extension* (1.15) *splits on the subgroup K'_f of $G_{\mathbb{A}}$ via the map $k_f \mapsto \{k_f, 1\}$.*

Proof. This is just Proposition 1.1.4. □

In line with the notation above, we'll denote this subgroup of $\overline{G}_{\mathbb{A}}$ by $\widehat{K}'_f$.

Proposition 1.2.2. *For $h \in G_F$, let*

$$\kappa_{\mathbb{A}}(h) = \prod_v \kappa_v(h) \tag{1.16}$$

the product extending over all (finite) primes of F. Then the map

$$h \mapsto \{h, \kappa_{\mathbb{A}}(h)\} \tag{1.17}$$

provides an isomorphism between G_F and a subgroup $\widehat{G}_F$ of $\overline{G}_{\mathbb{A}}$.

Proof. First note that if $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_F$, then c is a unit for almost all v, and consequently $\kappa_v(h) = 1$ for almost all v. Therefore, the product appearing in (1.16) is well-defined.

To prove the proposition it will suffice to show that $\kappa_{\mathbb{A}}(h_1)\kappa_{\mathbb{A}}(h_2)\beta_{\mathbb{A}}(h_1, h_2) = \kappa_{\mathbb{A}}(h_1 h_2)$ for all $h_1, h_2 \in G_F$. Indeed, for such h_1, h_2, and for almost all v, all of the entries of h_1, h_2 are units, so $\sigma_v(h_1, h_2) = 1$ by (2) of Proposition 1.1.1. For the remaining v, we have

$$\sigma_v(h_1, h_2) = (r_1, r_2)_v (r_3, r_4)_v$$

for r_1, r_2, r_3, r_4 in F, and so by (5) of Proposition 1.1.1, $\prod_v \sigma_v(h_1, h_2) = 1$. Thus we've shown that

$$\kappa_{\mathbb{A}}(h_1)\kappa_{\mathbb{A}}(h_2)\beta_{\mathbb{A}}(h_1, h_2) = \kappa_{\mathbb{A}}(h_1 h_2). \tag{1.18}$$

□

Proposition 1.2.2 allows us to make sense out of the space $\widehat{G}_F \backslash \overline{G}_{\mathbb{A}}$.

Consider now the congruence subgroup $\Gamma(4)$ of G_F defined by

$$\Gamma(4) = \left\{ \gamma \in G(\mathcal{O}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 4 \right\},$$

and observe that

$$\Gamma(4)G(\mathbb{Z}) = G_F \cap G(\mathbb{C})K'_f.$$

We shall examine the splitting $\kappa_{\mathbb{A}}$ on $\Gamma(4)G(\mathbb{Z})$.

Let $a, b \in F^\times$ such that b is relatively prime to a and 2. Put $(b) = \prod_v v^{\mathrm{ord}_v(b)}$, the product extending over all places v of F. Of course, for almost all v, $\mathrm{ord}_v(b) = 0$. The v occuring in the product which satisfy $\mathrm{ord}_v(b) \neq 0$ will be relatively prime to a and 2. Define the local quadratic power residue symbol $\left(\frac{a}{v}\right)$ to be 1 if $a = \delta^2$ for some $\delta \in \mathcal{O}_v$ and -1 otherwise. The (global) quadratic power residue symbol $\left(\frac{a}{b}\right)_F$ is defined to be the product

$$\left(\frac{a}{b}\right)_F = \prod_{v<\infty,\ \mathrm{odd}} \left(\frac{a}{v}\right)^{\mathrm{ord}_v(b)} \tag{1.19}$$

We have

Proposition 1.2.3 (Kubota)**.** *Define a function*

$$\kappa : \Gamma(4) \longrightarrow \mu_2$$
$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\frac{c}{d}\right)_F$$

Then

$$\kappa(\gamma) = \kappa_{\mathbb{A}}(\gamma).$$

This is one of the main results of [26]. In fact, the proof of Proposition 1.2.3 rests on the fact that

$$\left(\frac{a}{b}\right)_F = \prod_{v|a} (a, b)_v.$$

Corollary 1.2.4. *The function $\kappa : \Gamma(4) \longrightarrow \mu_2$ is a character.*

For a proof, see [20, p. 27].

We shall collect some properties of the quadratic power residue symbol here for later use.

Lemma 1.2.5 (Properties of the quadratic power residue symbol)**.** *Let $(\ ) := (\ )_F$ denote the quadratic power residue symbol for the field F and let $b, b' \in \mathcal{O}$ be odd and relatively prime (respectively) to $a, a' \in \mathcal{O}$. Then,*

1. $$\left(\frac{aa'}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{a'}{b}\right).\ \textit{In particular,}\ \left(\frac{a^2}{b}\right) = 1,$$

2. $$\left(\frac{a}{bb'}\right) = \left(\frac{a}{b}\right)\left(\frac{a}{b'}\right),$$

3. *If $a \equiv a'$ (mod b) then*
$$\left(\frac{a}{b}\right) = \left(\frac{a'}{b}\right),$$

4. *If $b \equiv b'$ (mod a) and $4|a$ then*
$$\left(\frac{a}{b}\right) = \left(\frac{a}{b'}\right),$$

5. *If* $aa' \equiv 1 \pmod{b}$ *then*
$$\left(\frac{a}{b}\right) = \left(\frac{a'}{b}\right),$$

6. *If, additionally,* b *is in* $\mathbb{Q}$*, then*
$$\left(\frac{a}{b}\right) = \left(\frac{N(a)}{b}\right)_{\mathbb{Q}},$$
where $N : F \to \mathbb{Q}$ *is the norm map, and* $\left(\frac{\cdot}{\cdot}\right)_{\mathbb{Q}}$ *is the quadratic residue symbol in* $\mathbb{Q}$*,*

7. *If* a *and* b *are in* $\mathbb{Q}$*, then*
$$\left(\frac{a}{b}\right) = 1,$$

8. *If* $a, b \in \mathcal{O}$ *are coprime, and if* b *is prime and odd, then*
$$\left(\frac{a}{b}\right) \equiv a^{\frac{N(b)-1}{2}} \pmod{b}.$$

See [27, p. 112] for parts (1), (3) and (6), and [32, p. 100] for parts (2), (5) and (8). We shall prove part (7). By part (6), if $b \in \mathbb{Q}$, then
$$\left(\frac{a}{b}\right) = \left(\frac{N(a)}{b}\right)_{\mathbb{Q}}.$$
However, since $a \in \mathbb{Q}$, we have $N(a) = a^2$, and
$$\left(\frac{a^2}{b}\right)_{\mathbb{Q}} = 1.$$

Lemma 1.2.6 (Quadratic reciprocity law)**.** *Let* $a = a_1 + ia_2, b = b_1 + ib_2$ *be two primes in* $\mathcal{O}$ *which are relatively prime and congruent to* $1 \pmod{2+2i}$*. Then,*
$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)^{-1} = (-1)^{\frac{Na-1}{2}\frac{Nb-1}{2}}$$
In particular, if either a *or* b *is* $\equiv 1 \pmod{4}$*, then*
$$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right).$$
The supplementary laws are given by:
$$\left(\frac{i}{a}\right) = (-1)^{\frac{a_1^2+a_2^2-1}{4}}$$
$$\left(\frac{1+i}{a}\right) = (-1)^{\frac{a_1-a_2-a_2^2-1}{4}}$$

See [27, p. 195] for the analogous result for the quartic residue symbol in F (the quadratic residue symbol is the square of the quartic residue symbol in F).

The function κ has an extension to $\Gamma' = \Gamma(4)G(\mathbb{Z})$ which we shall also call κ.

Proposition 1.2.7. *Define*

$$\kappa : \Gamma' \longrightarrow \mu_2 \quad \textit{by}$$
$$\kappa(\gamma h) = \kappa(\gamma) \ \textit{where}\ \gamma \in \Gamma(4), h \in G(\mathbb{Z})$$

Then κ is a character of Γ'.

Proof. Observe that part (7) of Lemma 1.2.5 says that κ is trivial on $G(\mathbb{Z})$. If $\gamma_1, \gamma_2 \in \Gamma(4)$, $h_1, h_2 \in G(\mathbb{Z})$, and $\gamma_1 h_1 = \gamma_2 h_2$ then $\kappa(\gamma_2^{-1}\gamma_1) = \kappa(h_2 h_1^{-1}) = 1$ so $\kappa(\gamma_1) = \kappa(\gamma_2)$. Hence κ is well-defined.

We must show that $\kappa(\gamma_1 h_1 \gamma_2 h_2) = \kappa(\gamma_1)\kappa(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma(4)$ and all $h_1, h_2 \in G(\mathbb{Z})$. But, $\kappa(\gamma_1 h_1 \gamma_2 h_2) = \kappa(\gamma_1(h_1\gamma_2 h_1^{-1})h_1 h_2) = \kappa(\gamma_1)\kappa(h_1\gamma_2 h_1^{-1})$, so it will suffice to show that

$$\kappa(h\gamma h^{-1}) = \kappa(\gamma) \text{ for all } \gamma \in \Gamma(4) \text{ and all } h \text{ in a set of generators for } G(\mathbb{Z}).$$

A set of generators for $G(\mathbb{Z})$ is given by [6]:

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

However,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so a different set of generators is:

$$\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $h_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\kappa(h_1 \gamma h_1^{-1}) = \kappa\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \kappa\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix} \right).$$

Write $ad - bc = i^\delta$ for some $\delta \in \{0, 1, 2, 3\}$. Suppose that $\delta = 0$ or 2. Then $b \equiv \pm c^{-1} \pmod a$, and using $(1), (2), (3)$ and (5) of Lemma 1.2.5, we have

$$\left(\frac{b}{a}\right) = \left(\frac{-c}{a}\right) = \left(\frac{c}{a}\right) = \left(\frac{c}{ad}\right)\left(\frac{c}{d}\right). \tag{1.20}$$

Thus,

$$\kappa(h_1 \gamma h_1^{-1}) = \kappa(\gamma) \Leftrightarrow \left(\frac{c}{ad}\right) = 1.$$

Any element $x \in \mathcal{O}$ has a factorization $x = \pi^\alpha i^\beta x'$ for $x' \equiv 1 \pmod{\pi^3}$, where $\alpha \geq 0$, $\beta \in \{0, 1, 2, 3\}$ (this is essentially due to the fact that the homomorphism $\mathcal{O}^\times \to (\mathcal{O}/\pi^3)^\times$ is bijective [16]). If we write $c = \pi^\alpha i^\beta c'$ then since $c \equiv 0 \pmod 4$, we have $\alpha \geq 4$. Now,

$$\left(\frac{c}{ad}\right) = \left(\frac{\pi}{a}\right)\left(\frac{\pi}{d}\right)\left(\frac{i}{a}\right)\left(\frac{i}{d}\right)\left(\frac{c'}{ad}\right).$$

Put $a = a_1 + a_2 i, d = d_1 + d_2 i$. The congruence $a \equiv d \equiv 1 \pmod 4$, implies that $a_1 \equiv d_1 \equiv 1 \pmod 4$ and $a_2 \equiv d_2 \equiv 0 \pmod 4$. By Lemma 1.2.6, we have

$$\begin{aligned}\left(\frac{i}{a}\right) &= (-1)^{\frac{a_1^2 + a_2^2 - 1}{4}} \\ &= 1 \text{ since } a_1^2 + a_2^2 - 1 \equiv 0 \pmod 8 \\ &= (-1)^{\frac{d_1^2 + d_2^2 - 1}{4}} \text{ since } d_1^2 + d_2^2 - 1 \equiv 0 \pmod 8 \\ &= \left(\frac{i}{d}\right)\end{aligned} \tag{1.21}$$

Similarly,

$$\begin{aligned}\left(\frac{\pi}{a}\right) &= (-1)^{\frac{a_1 - a_2 - a_2^2 - 1}{4}} \\ &= 1 \text{ since } a_1 - a_2 - a_2^2 - 1 \equiv 0 \pmod 8 \\ &= (-1)^{\frac{d_1 - d_2 - d_2^2 - 1}{4}} \text{ since } d_1 - d_2 - d_2^2 - 1 \equiv 0 \pmod 8 \\ &= \left(\frac{\pi}{d}\right)\end{aligned}$$

Hence by (1) of Lemma 1.2.5, and the fact that $ad \equiv 1 \pmod 4$,

$$\left(\frac{c}{ad}\right) = \left(\frac{c'}{ad}\right) = \left(\frac{ad}{c'}\right) \tag{1.22}$$

Since $ad \equiv \pm 1 \pmod{c'}$,

$$\left(\frac{ad}{c'}\right) = \left(\frac{1}{c'}\right) = 1. \tag{1.23}$$

Thus we have shown that when $\delta = 0$ or 2, by (1.20), (1.22) and (1.23), that $\kappa(h_1\gamma h_1^{-1}) = \kappa(\gamma)$.

If $\delta = 1$ or 3, $b \equiv \mp\, ic^{-1}$, and again, using (1), (2), (3) and (5) of Lemma 1.2.5, we have

$$\left(\frac{b}{a}\right) = \left(\frac{ic}{a}\right) = \left(\frac{ic}{ad}\right)\left(\frac{ic}{d}\right).$$

Now,

$$\kappa(h_1\gamma h_1^{-1}) = \kappa(\gamma) \Leftrightarrow \left(\frac{ic}{ad}\right)\left(\frac{i}{d}\right) = 1.$$

However,

$$\left(\frac{ic}{ad}\right)\left(\frac{i}{d}\right) = \left(\frac{i}{a}\right)\left(\frac{c}{ad}\right)$$

so by the case above with $\delta = 0, 2$ and (1.21), we have our result.

Let $h_2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(4)$,

$$\begin{aligned} h_2\gamma h_2^{-1} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \end{aligned}$$

We are required to show that

$$\left(\frac{-c}{d}\right) = \left(\frac{c}{d}\right),$$

but this is clear.

If $h_3 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ then

$$\kappa\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right) = \left(\frac{c}{d-c}\right),$$

but $c \equiv 0 \pmod 4$ hence

$$\left(\frac{c}{d-c}\right) = \left(\frac{c}{d}\right) \text{ by (4) of Lemma 1.2.5.}$$

□

When convenient, we shall think of κ as a 1-dimensional representation of the group Γ'. When we require this representation to be rational, we shall denote it by $\kappa_{\mathbb{Q}}$; explicitly,

$$\kappa_{\mathbb{Q}} : \Gamma' \longrightarrow GL_1(\mathbb{Q}).$$

1.3 Modular forms over $\mathbb{Q}(i)$

There are several ways to view modular forms on $G := GL_2$ when the base field is imaginary quadratic. For example, they can be described as functions on upper-half space $\mathbb{H}$; functions on the adèle group $G_{\mathbb{A}}$; "representations" of the group $G_{\mathbb{A}}$; and cohomology classes of congruence subgroups of $SL_2(\mathcal{O})$. In this thesis, we shall content ourselves with the description of modular forms both as "representations" of the group $G_{\mathbb{A}}$, and, via the generalised Eichler-Shimura-Harder isomorphism, as cohomology classes.

We shall, however, give a rather loose interpretation of them as functions on the adèle group $G_{\mathbb{A}}$ since we feel this description highlights some of the salient points of the theory.

Half-integral weight forms have similar interpretations, except the group $G_{\mathbb{A}}$ is replaced with its two-fold cover $\overline{G}_{\mathbb{A}}$. We give the same treatment for half-integral weight forms.

1.3.1 Integral weight forms

The material in this subsection is based on Bygott's thesis [9] and Kudla's paper [3, Chapter 7]; we refer the reader there for much more detail. If H is a locally compact multiplicative group, by a *quasi-character* χ of H we mean a continuous homomorphism of groups $\chi : H \to \mathbb{C}^\times$, and by a *unitary* character we mean a quasi-character whose image is contained in S^1, the set of complex numbers of norm 1. Let $Z(G_{\mathbb{A}})$ denote the centre of $G_{\mathbb{A}}$; observe that

$$Z(G_{\mathbb{A}})/(Z(G_{\mathbb{A}}) \cap G_F) \cong \mathbb{A}^\times/F^\times.$$

Let $\psi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ be a quasi-character, whose restriction to $F_\infty^\times = \mathbb{C}^\times$, denoted ψ_∞, is trivial. The space $\mathcal{A}_0(\psi)$ of *cuspidal automorphic forms* on $G_{\mathbb{A}}$ with central character ψ is the space of functions $\Phi : G_{\mathbb{A}} \to \mathbb{C}^{(3)}$ subject to the following conditions:

(A) $\Phi(\gamma g) = \Phi(g)$ for all $\gamma \in G_F$ and $g \in G_{\mathbb{A}}$;

(B) $\Phi(g\zeta) = \Phi(g)\psi(\det(\zeta))$ for all $g \in G_{\mathbb{A}}$ and all $\zeta \in Z(G_{\mathbb{A}})$;

(C) $\Phi(gk) = \Phi(g)$ for all $g \in G_{\mathbb{A}}$ and $k \in L_f$, a compact open subgroup of $G(\mathbb{A}_f)$;

(D) The induced function $\Phi_\infty : G_{\mathbb{C}} \to \mathbb{C}^{(3)}$ is smooth, and is an eigenfunction of the Casimir elements Δ, Δ';

(E) Φ is slowly increasing;

(F) The space spanned by the right translates of Φ by elements of K_∞ is finite-dimensional (we say that Φ is *right* K_∞*-finite*);

(G) Φ satisfies the cuspidal condition:

$$\int_{\mathbb{A}/F} \Phi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g\right) db = 0 \text{ for almost all } g \in G_{\mathbb{A}}.$$

Condition (D) has the following meaning. Let $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_2(\mathbb{C})$ be the complexified Lie algebra of $G_{\mathbb{C}}$. If $\Phi \in \mathcal{A}_0(\psi)$, then $\mathfrak{g}$ acts on the induced function Φ_∞ by right translation, and this action extends to the universal enveloping algebra of $\mathfrak{g}$. The centre of the universal enveloping algebra is generated by the centre of $\mathfrak{g}$ and the Casimir elements Δ, Δ', and the centre of $\mathfrak{g}$ acts trivially by (B) and by our assumption that ψ_∞ is trivial on $\mathbb{C}^\times$. For the definition of (E), we refer to [3, Chapter 7].

1.3.2 Half-integral weight forms

Recall the global extension (1.15)

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_{\mathbb{A}} \longrightarrow G_{\mathbb{A}} \longrightarrow 1$$

defining $\overline{G}_{\mathbb{A}}$. This extension splits over K'_f (Proposition 1.2.1). Observe that

$$Z(\overline{G}_{\mathbb{A}})/(Z(\overline{G}_{\mathbb{A}}) \cap \widehat{G}_F) \cong \mathbb{A}^{\times 2}/F^{\times 2} \oplus \mu_2;$$

let $\psi : Z(\overline{G}_{\mathbb{A}})/(Z(\overline{G}_{\mathbb{A}}) \cap \widehat{G}_F) \to \mathbb{C}^\times$ be a *genuine* quasi-character: that is,

$$\psi(\xi g) = \xi \psi(g) \text{ for all } g \in Z(\overline{G}_{\mathbb{A}}), \xi \in \mu_2.$$

The space $\overline{\mathcal{A}}_0(\psi)$ of *genuine cuspidal automorphic forms* on $\overline{G}_{\mathbb{A}}$ with central character ψ is the space of functions $\Phi : \overline{G}_{\mathbb{A}} \to \mathbb{C}^{(3)}$ subject to the following conditions:

(A′) $\Phi(\gamma g) = \Phi(g)$ for all $\gamma \in \widehat{G}_F$ and $g \in \overline{G}_{\mathbb{A}}$;

(B′) $\Phi(g\zeta) = \Phi(g)\psi(\zeta)$ for all $g \in \overline{G}_{\mathbb{A}}$ and all $\zeta \in Z(\overline{G}_{\mathbb{A}})$;

(C′) $\Phi(gk) = \Phi(g)$ for all $g \in \overline{G}_{\mathbb{A}}$ and $k \in L'_f \subseteq \widehat{K}'_f$, a compact open subgroup of $\overline{G}(\mathbb{A}_f)$;

(D′) The induced function $\Phi_\infty : \overline{G}_{\mathbb{C}} \to \mathbb{C}^{(3)}$ is genuine, smooth when restricted to $G_{\mathbb{C}}$, and is an eigenfunction of the Casimir elements Δ, Δ';

(E′) Φ is slowly increasing;

(F′) The space spanned by the right translates of Φ by elements of $\overline{K}_\infty$ is finite-dimensional (we say that Φ is *right $\overline{K}_\infty$-finite*);

(G′) Φ satisfies the cuspidal condition:

$$\int_{\mathbb{A}/F} \Phi\left(\widehat{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}} g\right) db = 0 \text{ for almost all } g \in \overline{G}_{\mathbb{A}},$$

where $\widehat{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)}$ is the image of $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ under the unique splitting of (1.15) on the group $\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$.

Remark 1.3.1. There is a function $f_\Phi : \mathbb{H} \to \mathbb{C}^{(3)}$ associated to each such Φ. If $L'_f = \widehat{K}'_f$ in (C′) above, then one can show that f_Φ must have the following transformation property:

$$f_\Phi(\gamma\tau) = \kappa(\gamma) f_\Phi(\tau) \text{ for all } \gamma \in \Gamma', \tau \in \mathbb{H}.$$

1.4 A modification of Flicker's correspondence

In 1973, in his seminal paper [36], Shimura associated to each classical $\mathbb{Q}$-cusp form of half-integer weight $\frac{k}{2}$, (odd $k \geq 3$) and Dirichlet character χ, a modular form of weight $k-1$ and character χ^2. In a series of open-ended questions at the end of his report, Shimura suggested that his correspondence might lend itself to be understood via representation theory. In a sequence of papers, Gelbart and Piatetski-Shapiro [18] did just this. They showed that a cusp form of half-integer weight could be thought of as a "genuine automorphic representation" of $\overline{G}_{\mathbb{A}}$. Moreover, their results were valid over any number field. It was Flicker who, in 1980, gave a comprehensive correspondence: not only for *all* "automorphic representations" of $\overline{G}_{\mathbb{A}}$ (not just the cuspidal ones), but also for n-fold covering groups of GL_2. His work is based on character theory: he uses the Selberg trace formula for both $G_{\mathbb{A}}$ and its cover $\overline{G}_{\mathbb{A}}$. We won't delve into the details of his approach; we'll simply collect the representation theory we require and then describe the correspondence.

1.4.1 Local representation theory

Let v be a finite or infinite place, and recall that we've used the notation G_v for $GL_2(F_v)$. The group G_v is a locally compact Hausdorff space, and it is totally disconnected (such groups are of 'td-type' in the terminology of [11]). By a representation of G_v, we shall mean a pair (ρ, V) where V is a complex vector space, and ρ is a homomorphism from G_v to the invertible linear maps in V; the representation shall be called *continuous* if the map ρ is continuous. Our notation for (ρ, V) shall vary between ρ, V or the pair (ρ, V) according to what best suits the situation; we hope this does not cause confusion. If H is a subgroup of G_v, we'll write V^H for the subspace of vectors $v \in V$ which satisfy $\rho(h)(v) = v$ for all $h \in H$.

We call (ρ, V) *uniterisable* if there exists a positive definite Hermitian form on V which is preserved by $\rho(g)$ for all $g \in G_v$. One can then take the completion of V with respect to the inner product defined by the form to obtain a unitary representation of G_v on a Hilbert space $\mathcal{H}$.

If $\overline{H}$ is a group which contains μ_2 in the centre, recall that by a *genuine* representation of $\overline{H}$, we mean a representation (ρ, V) such that

$$\rho(\xi h) = \xi\rho(h) \text{ for all } \xi \in \mu_2, h \in \overline{H}.$$

1.4.1.1 The archimedean place

Let (ρ, V) be a continuous representation of $G_{\mathbb{C}}$ on a Hilbert space V. Such a representation is called *irreducible* if it has no non-trivial closed subrepresentation;

it is called *unitary* if its restriction to $U(2)$ is a Hilbert direct sum of irreducible representations, each of which has finite multiplicity. We want to restrict the class of representations in what follows.

Definition 1.4.1. *The representation (ρ, V) is called admissible if its restriction to $SU(2)$ decomposes into finite-dimensional representations with finite multiplicities.*

Recall the subgroups $T_{\mathbb{C}}, N_{\mathbb{C}}, B_{\mathbb{C}}$ of $G_{\mathbb{C}}$. Let (χ_1, χ_2) be a pair of quasi-characters $\mathbb{C}^\times \to \mathbb{C}^\times$. The pair (χ_1, χ_2) defines a character of the (abelian) group $T_{\mathbb{C}}$ if we put

$$\begin{aligned} T_{\mathbb{C}} &\longrightarrow \mathbb{C}^\times \\ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} &\longmapsto \chi_1(a_1)\chi_2(a_2). \end{aligned}$$

Let $(\tau, \mathbb{C})$ be the 1-dimensional complex representation of the group $B_{\mathbb{C}}$ given by:

$$\tau\left(\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}\right) = |a_1 a_2^{-1}|_{\mathbb{C}}^{\frac{1}{2}} \, \chi_1(a_1)\chi_2(a_2).$$

Consider the vector space $B(\chi_1, \chi_2)$ of measurable functions f on $G_{\mathbb{C}}$ which satisfy

$$\begin{aligned} &f\,(bg) = \tau(b) f(g) \text{ for all } b \in B_{\mathbb{C}}, \text{ and} \\ &\int_{SU(2)} |f(k)|_{\mathbb{C}}^2 \, dk < \infty. \end{aligned}$$

By the Iwasawa decomposition, $G_{\mathbb{C}} = B_{\mathbb{C}} SU(2)$, and so the functions f are completely determined by their restriction to $SU(2)$.

The group $G_{\mathbb{C}}$ acts on these functions via $(fg)(y) = f(gy)$ and therefore $B(\chi_1, \chi_2)$ is the space of a representation of $G_{\mathbb{C}}$ *induced* from that of $(\tau, \mathbb{C})$. For any choice of quasi-characters (χ_1, χ_2), we denote by $\rho(\chi_1, \chi_2)$ the representation of $G_{\mathbb{C}}$ whose space is $B(\chi_1, \chi_2)$.

The representations $\rho(\chi_1, \chi_2)$ are admissible, and in fact, every irreducible admissible representation of $G_{\mathbb{C}}$ is a subquotient of some such $\rho(\chi_1, \chi_2)$.

Schur's Lemma [8] implies that if (ρ, V) is any irreducible admissible representation of the locally compact totally disconnected group G_v, then the centre $F_v^\times$ of G_v acts by scalars on V. That is, there is a quasi-character $\chi : F_v^\times \to \mathbb{C}^\times$, called the *central character* of (ρ, V), such that

$$\rho\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right)(v) = \chi(z)(v) \text{ for all } z \in F_v^\times \text{ and } v \in V.$$

The central character is called *odd* (resp. *even*) if $\chi(-1) = -1$ (resp. $\chi(-1) = 1$), and the representation (ρ, V) is called *odd* (resp. *even*) if its central character is odd (resp. even). The centre of $G_{\mathbb{C}}$ is the set of scalar matrices $Z_{\mathbb{C}} = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in \mathbb{C}^\times\} \subset B_{\mathbb{C}}$ and therefore the central character of $\rho(\chi_1, \chi_2)$ is easily seen to be $\chi(\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)) = \chi_1(a)\chi_2(a)$.

Let $B(\rho_n)$ be the subspace of $B(\chi_1, \chi_2)$ which transforms according to the unique representation of $SU(2)$ of dimension $n+1$. Recall that if $z \in \mathbb{C}$, its complex conjugate is denoted by $\bar{z}$. See [20] and the references therein for proof of the following:

Proposition 1.4.1 (Jacquet-Langlands)**.** *1.* $\rho(\chi_1, \chi_2)$ *is irreducible if* $\chi_1(a_1)\chi_2(a_2) \neq (a_1 a_2^{-1})^{\pm p}(\bar{a}_1 \bar{a}_2^{-1})^{\pm q}$ *where* $p, q \in \mathbb{Z}^{\geq 1}$*. If* $\rho(\chi_1, \chi_2)$ *is irreducible, it will be denoted by* $\varpi_\infty(\chi_1, \chi_2)$*.*

2. If $\chi_1(a_1)\chi_2(a_2) = (a_1 a_2^{-1})^p(\bar{a}_1 \bar{a}_2^{-1})^q$ *with* $p, q \geq 1$*, let*

$$B_s(\chi_1, \chi_2) = \sum_{n \geq p+q, n \equiv p+q \ (2)} B(\rho_n).$$

Then $B_s(\chi_1, \chi_2)$ *is the only proper invariant subspace of* $B(\chi_1, \chi_2)$*. Let* $\sigma_\infty(\chi_1, \chi_2)$ *denote the representation of* $G_{\mathbb{C}}$ *whose space is* $B_s(\chi_1, \chi_2)$*, and let* $\varpi_\infty(\chi_1, \chi_2)$ *denote the representation of* $G_{\mathbb{C}}$ *with quotient space* $B/B_s(\chi_1, \chi_2)$*.*

3. If $\chi_1(a_1)\chi_2(a_2) = (a_1 a_2^{-1})^{-p}(\bar{a}_1 \bar{a}_2^{-1})^{-q}$ *with* $p, q \geq 1$*, let*

$$B_f(\chi_1, \chi_2) = \sum_{|p-q| \leq n < p+q, n \equiv p+q \ (2)} B(\rho_n).$$

Then $B_f(\chi_1, \chi_2)$ *is the only proper invariant subspace of* $B(\chi_1, \chi_2)$*. Let* $\varpi_\infty(\chi_1, \chi_2)$ *denote the representation of* $G_{\mathbb{C}}$ *whose space is* $B_f(\chi_1, \chi_2)$*, and let* $\sigma_\infty(\chi_1, \chi_2)$ *denote the representation with quotient space* $B/B_f(\chi_1, \chi_2)$*.*

4. $\varpi_\infty(\chi_1, \chi_2)$ *and* $\varpi_\infty(\chi_1', \chi_2')$ *are equivalent iff* $(\chi_1, \chi_2) = (\chi_1', \chi_2')$ *or* $(\chi_1, \chi_2) = (\chi_2', \chi_1')$*,*

5. *If* $\sigma_\infty(\chi_1, \chi_2)$ *and* $\sigma_\infty(\chi_1', \chi_2')$ *are defined, then they are equivalent iff* $(\chi_1, \chi_2) = (\chi_1', \chi_2')$ *or* $(\chi_1, \chi_2) = (\chi_2', \chi_1')$*,*

6. *If* $\chi_1(a_1)\chi_2(a_2) \neq (a_1 a_2^{-1})^p (\bar{a}_1 \bar{a}_2^{-1})^q$*, then there exists a pair* (ν_1, ν_2) *of quasi-characters such that* $\sigma_\infty(\chi_1, \chi_2) = \varpi_\infty(\nu_1, \nu_2)$*.*

When these representations are irreducible, we shall denote them by $\varpi_\infty(\chi_1, \chi_2)$ and refer to them as representations of the *principal series.*

We shall ultimately want our representations to be unitary, and for this we need the following:

Proposition 1.4.2. *Each irreducible unitarisable representation of* $G_\mathbb{C}$ *is one of the following type:*

- $\varpi_\infty(\chi_1, \chi_2)$ *with* χ_1, χ_2 *unitary. Such representations are called continuous series.*

- $\varpi_\infty(\chi_1, \chi_2)$ *in which* $\chi_1(a_1)\chi_2(a_2)^{-1} = |a_1 a_2|_\mathbb{C}^{2s}$ *with* $0 < s < 1$*. These are called complementary series representations.*

- *A* 1*-dimensional representation of the form* $g \mapsto \omega(det(g))$ *for some unitary character* $\omega : \mathbb{C}^\times \to S^1$*.*

Remark. We will consider representations of $SL_2(\mathbb{C})$ below. Suffice it to say that every irreducible unitary representation of $G_\mathbb{C}$ is constructed from one of $SL_2(\mathbb{C})$ by extending the central character to the whole of $\mathbb{C}^\times$. In addition, unitarity holds for $G_\mathbb{C}$ if and only if it holds for $SL_2(\mathbb{C})$.

1.4.1.2 An interlude: representations with cohomology

If R is a commutative ring, let $M_2(R)$ be the set of 2×2 matrices with entries in R. Consider the following modules: if k is a non-negative integer, let $R_k[x, y]$ be the ring of homogeneous polynomials $P(x, y)$ in x, y of degree k with coefficients in R. Let $E_k(R) = R_k[x, y]$ and let $E_k(R) \otimes \det(v)$, for an integer v, be $E_k(R)$ as a set.

As an $M_2(R)$-module, $E_k(R) \otimes \det(v)$ has a (left) action given by:

$$\begin{aligned}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)(x,y) =& P\left(\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^v \\ =& P(ax+cy, bx+dy)\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^v\end{aligned}$$

If $R = \mathbb{Z}, \mathcal{O}, F$ or $\mathbb{C}$, define the following $M_2(R)$-module

$$E_{k,l,v,w}(R) := (E_k(R) \otimes \det(v)) \underset{R}{\otimes} (\overline{E_l(R) \otimes \det(w)})$$

where the overline on the second factor means the action is twisted with complex conjugation. That is, as a module, $\overline{E_l(R) \otimes \det(w)} = E_l(R) \otimes \det(w)$, but the action of $M_2(R)$ is given by:

$$\begin{aligned}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)(x,y) &= P\left(\begin{pmatrix} x & y \end{pmatrix}\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right)\left(\overline{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right)^v \\ &= P(\bar{a}x+\bar{c}y, \bar{b}x+\bar{d}y)\left(\overline{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\right)^v,\end{aligned}$$

and we define $\overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$ to be $\left(\begin{smallmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{smallmatrix}\right)$.

Recall that $\mathfrak{g}$ is the complexified Lie algebra of $G_{\mathbb{C}}$ and $K_\infty = U(2)$.

Definition 1.4.2. *By a $(\mathfrak{g}, K_\infty)$-module, we mean a complex vector space W with actions of $\mathfrak{g}$ and K_∞, such that all vectors in W are K_∞-finite, and such that the two actions are compatible.*[1] *A $(\mathfrak{g}, K_\infty)$-module W is called admissible if, for every irreducible representation σ_∞ of K_∞, the multiplicity of σ_∞ in W is finite.*

Now let $R = \mathbb{C}$, and put $M = E_{k,l,v,w}(\mathbb{C})$. Let (ρ, V) be an infinite-dimensional, irreducible, unitarisable representation of $G_{\mathbb{C}}$. Let $V(K_\infty)$ be the subspace of K_∞-finite, smooth vectors in V. Then the cohomology of $G_{\mathbb{C}}$ is given by its $(\mathfrak{g}, K_\infty)$-cohomology [5]:

$$H^q_{\text{cts}}(G_{\mathbb{C}}, V \underset{\mathbb{C}}{\otimes} M) = H^q(\mathfrak{g}, K_\infty, V(K_\infty) \underset{\mathbb{C}}{\otimes} M) \text{ for } q \in \mathbb{Z}^{\geq 0} \tag{1.24}$$

[1] For more detail, see [3, p. 140].

Proposition 1.4.1 tells us that such a (ρ, V) is of the form $(\varpi_\infty(\chi_1, \chi_2), B(\chi_1, \chi_2))$ for a pair of quasi-characters (χ_1, χ_2). If we further require that the restriction of (ρ, V) to $SL_2(\mathbb{C})$ is unitary, Harder [23] has shown that there is only one (equivalence class) of such (ρ, V) for which the right-hand side of (1.24) does not vanish. Indeed, put ρ equal to $\varpi(||_{\mathbb{C}}^{-\frac{1}{2}}\,\eta_1, ||_{\mathbb{C}}^{\frac{1}{2}}\,\eta_2)$ with the pair (η_1, η_2) defined below.

Let $\eta_1, \eta_2 : \mathbb{C}^\times \to \mathbb{C}^\times$ be two characters which satisfy

$$\begin{aligned}\eta_1(a_1) =& a_1^{1-v}\bar{a}_1^{-l-w} \text{ and}\\ \eta_2(a_2) =& a_2^{-1-k-v}\bar{a}_2^{-w}.\end{aligned}$$

Define the pair (ν_1, ν_2) so that $\varpi(\nu_1, \nu_2) = \varpi(||_{\mathbb{C}}^{-\frac{1}{2}}\,\eta_1, ||_{\mathbb{C}}^{\frac{1}{2}}\,\eta_2)$. Then

$$\begin{aligned}\nu_1(a_1) =& a_1^{\frac{1}{2}-v}\bar{a}_1^{-\frac{1}{2}-l-w} \text{ and}\\ \nu_2(a_2) =& a_2^{-\frac{1}{2}-k-v}\bar{a}_2^{\frac{1}{2}-w}.\end{aligned}$$

Note that the restriction of the representation $\varpi(\nu_1, \nu_2)$ to $SL_2(\mathbb{C})$ is unitary if and only if $k = l$. When this is the case, let $(\varpi_\infty(\nu_1, \nu_2), B(\nu_1, \nu_2))$ denote the unitary completion of the representation of $G_{\mathbb{C}}$ which is induced from the characters (ν_1, ν_2). We have

$$H^q_{\mathrm{cts}}(G_{\mathbb{C}}, B(\nu_1, \nu_2) \underset{\mathbb{C}}{\otimes} M) = \begin{cases}\mathbb{C} & \text{if } q = 1, 2\\ 0 & \text{otherwise}\end{cases}$$

1.4.1.3 Back to the archimedean place

Recall that the extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_{\mathbb{C}} \longrightarrow G_{\mathbb{C}} \longrightarrow 1$$

splits. That is, $\overline{G}_{\mathbb{C}} = \widehat{G}_{\mathbb{C}} \oplus \mu_2$ and $\widehat{G}_{\mathbb{C}} \cong G_{\mathbb{C}}$. If $\overline{\varpi}$ is an irreducible, genuine, uniterisable representation of $\overline{G}_{\mathbb{C}}$ and if ϵ is the non-trivial character of μ_2, then there are unitary characters χ_1, and χ_2 of $\mathbb{C}^\times$ such that $\overline{\varpi} = \varpi(\chi_1, \chi_2) \otimes \epsilon$. We shall only be interested in the case where $\varpi(\chi_1, \chi_2) = \varpi_\infty(\nu_1, \nu_2)$ has cohomology in the above sense.

Definition 1.4.3. *By a $(\mathfrak{g}, \overline{K}_\infty)$-module, we mean a complex vector space $\overline{W}$ with genuine actions of $\mathfrak{g}$ and $\overline{K}_\infty$, such that all vectors in $\overline{W}$ are $\overline{K}_\infty$-finite, and such* *that the two actions are compatible. A $(\mathfrak{g}, \overline{K}_\infty)$-module $\overline{W}$ is called admissible if, for every irreducible genuine representation σ_∞ of $\overline{K}_\infty$, the multiplicity of σ_∞ in $\overline{W}$ is finite.*

Suppose that $(\bar{\rho}, \overline{V})$ is a genuine, irreducible, uniterisable representation of $\overline{G}_{\mathbb{C}}$, and let (ρ, V) be the representation of $G_{\mathbb{C}}$ which gives rise to $\overline{V}$: that is, $\overline{V} = V \otimes \epsilon$. Write $\overline{V}(\overline{K}_\infty)$ for the subspace of $\overline{K}_\infty$-finite vectors in $\overline{V}$; let $M = E_{k,l,v,w}(\mathbb{C})$ be the representation of $G_{\mathbb{C}}$ given above, whose extension to $\overline{G}_{\mathbb{C}}$ is given by the value -1 on the non-trivial element of μ_2. We have

$$H^q_{\text{cts}}(\overline{G}_{\mathbb{C}}, \overline{V} \underset{\mathbb{C}}{\otimes} M) = H^q(\mathfrak{g}, \overline{K}_\infty, \overline{V}(\overline{K}_\infty) \underset{\mathbb{C}}{\otimes} M) \text{ for } q \in \mathbb{Z}^{\geq 0}.$$

It follows that

$$H^q_{\text{cts}}(\overline{G}_{\mathbb{C}}, \overline{V} \underset{\mathbb{C}}{\otimes} M) \neq 0 \Leftrightarrow H^q_{\text{cts}}(G_{\mathbb{C}}, V \underset{\mathbb{C}}{\otimes} M) \neq 0.$$

We shall denote by $(\overline{\varpi}_\infty(\nu_1, \nu_2), \overline{B}(\nu_1, \nu_2))$ the representation of $\overline{G}_{\mathbb{C}}$ with the property $H^q_{\text{cts}}(\overline{G}_{\mathbb{C}}, \overline{B}(\nu_1, \nu_2) \underset{\mathbb{C}}{\otimes} M) \neq 0$ for $q = 1, 2$.

1.4.1.4 The nonarchimedean places

Let v be a non-archimedean place. We'll first recall the representation theory of G_v, as the theory for $\overline{G}_v$ only requires slight modification. The main result is that every irreducible 'admissible' (in a new sense) representation of G_v is either 'supercuspidal', or it is equivalent to a subquotient of the principal series.

Definition 1.4.4. *A representation (ρ, V) of G_v is said to be admissible if:*

- *For every $w \in V$, the stabiliser in G_v of w is an open subgroup of G_v, and*
- *For every compact open subgroup H in G_v, the space V^H is finite-dimensional.*

Recall that the group G_v is a locally compact Hausdorff space. Any such group has a translation invariant measure.

Definition 1.4.5. *An irreducible representation (ρ, V) of G_v is said to be supercuspidal if for every vector w in V there is an open compact subgroup U of N_v for which*

$$\int_U \rho(n) w \, dn = 0$$

Definition 1.4.6. *An irreducible admissible representation (ρ, V) is said to be square-integrable if there is a non-zero w in V and a non-zero u in $\check{V}$ such that*

$$\int_{Z_v \backslash G_v} |\langle \rho(g)w, u\rangle|_v^2 |\chi(det\ g)|_v^{-1} dg < \infty$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing between V and its contragredient $\check{V}$, Z_v denotes the centre of G_v, and χ is the central character of ρ.

We shall not make use of Definitions 1.4.5 and 1.4.6 other than in Theorem 1.4.13 below.

Suppose that (ρ, V) is an irreducible, admissible representation which is *not* supercuspidal. Then, by a theorem of Jacquet and Harish-Chandra [19], (ρ, V) is equivalent to a subquotient of an induced representation of G_v of the following form. Let τ be the 1-dimensional representation of B_v given by:

$$\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} \mapsto |a_1 a_2^{-1}|_v^{\frac{1}{2}} \chi_1(a_1)\chi_2(a_2)$$

for a pair (χ_1, χ_2) of quasi-characters $F_v^\times \to \mathbb{C}^\times$. Consider the space $B(\chi_1, \chi_2)$ of complex-valued locally constant functions f on G_v which satisfy

$$f\left(\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} g\right) = |a_1 a_2^{-1}|_v^{\frac{1}{2}} \chi_1(a_1)\chi_2(a_2) f(g) \text{ and} \tag{1.25}$$

$$\int_{K_v} |f(k)|_v^2\, dk < \infty \tag{1.26}$$

The group G_v acts on these functions via $(fg)(h) = f(gh)$ and hence $B(\chi_1, \chi_2)$ is the space of a representation of G_v induced from that of τ which we denote by $(\rho(\chi_1, \chi_2), B(\chi_1, \chi_2))$.

The following result [24] of Jacquet-Langlands tells us when the representations $(\rho(\chi_1, \chi_2), B(\chi_1, \chi_2))$ are irreducible and uniterisable.

Theorem 1.4.3. *1. $\rho(\chi_1, \chi_2)$ is irreducible if $\chi_1(a_1)\chi_2^{-1}(a_2) \neq |a_1 a_2|_v^{\pm 1}$.*

2. If $\chi_1(a_1)\chi_2^{-1}(a_2) = |a_1 a_2|_v^{-1}$, then $\rho(\chi_1, \chi_2)$ has a unique irreducible subrepresentation $\varpi_v(\chi_1, \chi_2)$ which is 1-dimensional. The corresponding quotient is irreducible, square-integrable and denoted by $\sigma_v(\chi_1, \chi_2)$.

3. *If* $\chi_1(a_1)\chi_2^{-1}(a_2) = |a_1a_2|_v$, *then* $\rho(\chi_1, \chi_2)$ *also has a unique irreducible subrepresentation which we denote by* $\sigma_v(\chi_1, \chi_2)$ *which is always infinite-dimensional. The corresponding* 1*-dimensional quotient is denoted* $\varpi_v(\chi_1, \chi_2)$.

4. *The representations in* (1) *are uniterisable if either:*

   - *Both characters* χ_1 *and* χ_2 *are unitary. These are called the continuous series. Or,*

   - $\chi_1(a_1) = \overline{\chi_2^{-1}(a_2)}$ *and* $\chi_1(a_1)\chi_2^{-1}(a_2) = |a_1a_2|_v^{\alpha}$ *with* $0 < \alpha < 1$ *.This gives the complementary series.*

   The representations in (2) *and* (3) *are uniterisable iff* $\chi_1\chi_2$ *is unitary.*

5. $\varpi_v(\chi_1, \chi_2)$ *is equivalent to* $\varpi_v(\chi_1', \chi_2')$ *iff either* $(\chi_1, \chi_2) = (\chi_1', \chi_2')$ *or* $(\chi_1, \chi_2) = (\chi_2', \chi_1')$.

When the representations $\rho(\chi_1, \chi_2)$ are irreducible, we shall denote them by $\varpi_v(\chi_1, \chi_2)$ and refer to them as representations of the *principal series*. The irreducible subquotients that we have denoted by $\sigma_v(\chi_1, \chi_2)$ are called *special* representations.

We shall also need the following:

Definition 1.4.7. *An irreducible, admissible representation* (ρ, V) *is said to be level* 1*, or unramified, if it contains a* K_v*-invariant non-zero vector. Or, equivalently, if it contains the identity representation of* K_v *at least once.*

We can extend this notion of 'level' in the following way [19].

Theorem 1.4.4. *Let* (ρ, V) *denote any irreducible admissible representation of* G_v*. Then there is a largest subgroup* L_v *of* K_v *such that the space of vectors* v *with*

$$\rho(l)(v) = v \text{ for all } l \in L_v$$

is not trivial. Furthermore, this space has dimension one.

We shall call a quasi-character $\chi : F_v^\times \to \mathbb{C}^\times$ *unramified* if it is trivial on $\mathcal{O}_v^\times$. We shall require the following [19].

Theorem 1.4.5. *An irreducible admissible representation (ρ, V) of G_v is level 1 if and only if $\rho = \varpi(\chi_1, \chi_2)$ for some pair of unramified characters χ_1, χ_2 of $F_v^\times$, and ρ is not a special representation. In this case, the identity representation of K_v is contained exactly once in (ρ, V).*

To conclude our discussion of the irreducible, admissible, uniterisable representations of G_v, it remains to observe that the irreducible supercuspidal representations with unitary central character are uniterisable. We shall say no more than they do exist, and can be constructed using the 'Weil' representation [19].

Consider the group $\overline{G}_v$.

Definition 1.4.8. *A representation $(\overline{\rho}, \overline{V})$ of $\overline{G}_v$ is said to be admissible if:*

- *For every $w \in \overline{V}$, the stabiliser in $\overline{G}_v$ of w is an open subgroup of $\overline{G}_v$, and*
- *For every compact open subgroup $\overline{H}$ in $\overline{G}_v$, the space of vectors stabilized by $\overline{H}$ is finite dimensional.*

Definition 1.4.9. *An irreducible representation $(\overline{\rho}, \overline{V})$ of $\overline{G}_v$ is said to be supercuspidal if for every vector w in $\overline{V}$ there is an open compact subgroup U of $\widehat{N}_v$ for which*

$$\int_U \overline{\rho}(n) w \, dn = 0$$

Note that this makes sense because the extension defining $\overline{G}_v$ always splits uniquely over N_v.

It turns out [17] that every irreducible, genuine, admissible, non-supercuspidal representation of $\overline{G}_v$ is equivalent to a subquotient of an induced representation. We turn to the description of the induced representation.

Suppose that we have a genuine irreducible 1-dimensional representation τ_0 of the group $\overline{T}_v^2$. By Remark 1.1.2, the group $\overline{T}_v^2$ is abelian, and by Corollary 1.1.6, it lifts to a subgroup $\widehat{T}_v^2$ in $\overline{G}_v$. Thus, τ_0 is a pair of quasi-characters (χ_1, χ_2) of $F_v^{\times 2}$. Extend τ_0 to a maximal abelian (finite index) subgroup $\overline{T}_v^0$ of $\overline{T}_v$, and then extend it again to the group $\overline{T}_v^0 \widehat{N}_v$ by the value 1 on $\widehat{N}_v$. Write τ for the representation τ_0 induced to $\overline{B}_v$. Note that the dimension of τ is 4 when v is odd, and 16 otherwise.

Consider the space $\overline{B}(\chi_1, \chi_2)$ of complex-valued locally constant functions on the group $\overline{G}_v$ which satisfy:

$$f\left(\left\{\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}, \xi\right\} g\right) = \xi\, \tau\left(\left\{\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}, 1\right\}\right) |a_1 a_2^{-1}|_v^{\frac{1}{2}} f(g)$$

$$\text{for all } \left\{\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}, \xi\right\} \in \overline{B}_v, \text{ and all } g \in \overline{G}_v \text{ and}$$

$$\int_{\overline{K}_v} |f(k)|_v^2 \, dk < \infty$$

The group $\overline{G}_v$ acts on this space via right translation; we write $(\bar{\rho}(\chi_1, \chi_2), \overline{B}(\chi_1, \chi_2))$, or Ind $^{\overline{G}_v}_{\overline{B}_v}(\chi_1, \chi_2)$, for the induced representation of $\overline{G}_v$.

See [20, p. 115] for the next result.

Theorem 1.4.6. *1.* $\bar{\rho}(\chi_1, \chi_2)$ *is irreducible if* $\chi_1(a_1)\chi_2^{-1}(a_2) \neq |a_1 a_2|_v^{\pm\frac{1}{2}}$.

2. If $\chi_1(a_1)\chi_2^{-1}(a_2) = |a_1 a_2|_v^{-\frac{1}{2}}$, *then* $\bar{\rho}(\chi_1, \chi_2)$ *has a unique irreducible subrepresentation denoted* $\overline{\varpi}_v(\chi_1, \chi_2)$*;*

3. If $\chi_1(a_1)\chi_2^{-1}(a_2) = |a_1 a_2|_v^{\frac{1}{2}}$, *then* $\bar{\rho}(\chi_1, \chi_2)$ *has a unique irreducible subrepresentation which we denote by* $\overline{\sigma}_v(\chi_1, \chi_2)$*;*

4. The representations in (1) *are uniterisable if either:*

- *Both characters* χ_1 *and* χ_2 *are unitary (these are called the continuous series), or*
- $\chi_1(a_1) = \overline{\chi_2^{-1}}(a_2)$ *and* $\chi_1(a_1)\chi_2^{-1}(a_2) = |a_1 a_2|_v^{\alpha}$ *with* $0 < \alpha < \frac{1}{2}$ *(this gives the complementary series).*

The representations in (2) *and* (3) *are uniterisable iff* $\chi_1\chi_2$ *is unitary.*

5. $\overline{\varpi}_v(\chi_1, \chi_2)$ *is equivalent to* $\overline{\varpi}_v(\chi_1', \chi_2')$ *iff either* $(\chi_1, \chi_2) = (\chi_1', \chi_2')$ *or* $(\chi_1, \chi_2) = (\chi_2', \chi_1')$.

When the representations $\bar{\rho}(\chi_1, \chi_2)$ are irreducible, we shall denote them by $\overline{\varpi}_v(\chi_1, \chi_2)$ and refer to them as representations of the *principal series*. The representation we have denoted by $\overline{\sigma}_v(\chi_1, \chi_2)$ is called *special*.

Definition 1.4.10. *If v is an odd place, an irreducible, admissible representation $(\bar{\rho}, \overline{V})$ of $\overline{G}_v$ is said to be level* 1*, or unramified, if it contains the representation*

$$\{k, \xi\} \mapsto \xi$$

of $\overline{K}_v$ at least once, or equivalently, if $\overline{V}^{\widehat{K}_v} \neq 0$.

An irreducible, admissible representation $(\bar{\rho}, \overline{V})$ of $\overline{G}_\pi$ is said to be level 1 *if it contains the representation*

$$\{k, \xi\} \mapsto \xi$$

of $\overline{K}_\pi(4)$ at least once.

In analogy with Theorem 1.4.5, we have [18]:

Theorem 1.4.7. *An irreducible, admissible representation $(\bar{\rho}, \overline{V})$ of $\overline{G}_v$ is unramified if and only if it is of the form $\overline{\varpi}_v(\chi_1, \chi_2)$ with χ_1^2 and χ_2^2 unramified and $\chi_1(a_1)\chi_2^{-1}(a_2) \neq |a_1 a_2|^{\frac{1}{2}}$: that is, it is not special. In this case, it contains the identity representation of $\overline{K}_v$ exactly once.*

The representation theory of $\overline{G}_v$ is concluded by noting that the supercuspidal representations again exist, but since we have no use for them in this thesis, we simply remark that they are associated to the 'Weil' representation [20].

1.4.2 Global representation theory

The global theory is built from the local theory as we shall see below. This section is based on [3] and [20].

Recall the definition (1.4.2) of a $(\mathfrak{g}, K_\infty)$-module.

Definition 1.4.11. *A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module (ρ, H) is a $(\mathfrak{g}, K_\infty)$-module with a 'smooth' action of $G(\mathbb{A}_f)$. The action of $G(\mathbb{A}_f)$ is smooth if every vector $x \in H$ is fixed by some compact open subgroup $L_f \subset G(\mathbb{A}_f)$.*

The point of this definition is that, for a fixed quasi-character $\psi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$, the space $\mathcal{A}_0(\psi)$ of cuspidal automorphic forms is a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module .

Definition 1.4.12. *A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module (ρ, H) is called admissible, if, for every irreducible representation σ of $K_\infty \times K_f$, the multiplicity of σ in H is finite.*

In this subsection, we shall use the notation (ρ_v, H_v) for an irreducible, admissible representation of G_v.

Theorem 1.4.8. *i. Suppose that (ρ_∞, H_∞) is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module, and suppose for that for each finite place v, (ρ_v, H_v) is an irreducible admissible representation of G_v. Suppose, moreover, that for almost all finite places, the representation is unramified and a basis vector $x_v^0 \in H_v^{K_v}$ is given. Then the restricted tensor product*

$$H = \bigotimes_v{}' H_v$$

with respect to the vectors x_v^0 is an irreducble admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module.

ii. Conversely, if (ρ, H) is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module, then there is a collection $\{(\rho_v, H_v)\}$ as in (i) and an isomorphism

$$\bigotimes_v{}' H_v \longrightarrow H$$

of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules.

It is customary to write $\rho = \bigotimes_v \rho_v$ for the pair (ρ, H). Let us make precise the meaning of the restricted tensor product $\bigotimes_v{}' H_v$. Let S be the finite set of places, including ∞, for which (ρ_v, H_v) is not unramified. If $v \notin S$, choose a K_v-fixed unit vector, x_v^0. For every finite set $S' \supset S$, let

$$H_{S'} = \bigotimes_{v \in S'} H_v$$

If $S'' \supset S'$, define the embedding $H_{S'} \to H_{S''}$ by $x \mapsto x \otimes \bigotimes_{v \in S'' \setminus S'} x_v^0$. Taking the direct limit, we obtain a Hilbert space $H = \bigotimes_v{}' H_v = \varinjlim H_{S'}$.

Consider again the $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module $\mathcal{A}_0(\psi)$ of cuspidal automorphic forms, whose central character ψ is fixed. The following theorem is of central importance.

Theorem 1.4.9. *1. The space $\mathcal{A}_0(\psi)$ is a direct sum of irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules,*

$$\mathcal{A}_0(\psi) = \bigoplus_\rho \mathcal{A}_0(\psi, \rho),$$

where $\mathcal{A}_0(\psi, \rho)$ is a summand isomorphic to ρ.

Motivated by this, we make the following definition:

Definition 1.4.13. *A cuspidal automorphic representation of $G_{\mathbb{A}}$ is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module which is equivalent to a module ρ which occurs in $\mathcal{A}_0(\psi)$.*

We want the $(\mathfrak{g}, K_\infty)$-module ρ_∞ to have cohomology in the sense of Section 1.4.1.2, and for this we make the following definition:

Definition 1.4.14. *A cuspidal automorphic representation of $G_{\mathbb{A}}$ of cohomological type is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module which occurs in $\mathcal{A}_0(\psi)$ and whose component ρ_∞ at infinity is equivalent to $\varpi_\infty(\nu_1, \nu_2)$.*

Remark 1.4.1. Recall that a function $\Phi \in \mathcal{A}_0(\psi)$ is fixed by some compact open subgroup $L_f \subset G(\mathbb{A}_f)$. We'll refer to L_f as the *level* of the representation ρ generated by Φ. This is compatible with the notion of 'level' that we used for local representations; if $L_f = \prod_{v<\infty} L_v$ then the level of the local component ρ_v is L_v. In particular, we'll say that the representation ρ is *level* 1 if $L_f = K_f$.

There are analogous versions of Theorems 1.4.8 and 1.4.9 for the metaplectic group $\overline{G}_{\mathbb{A}}$. To state them, we need the relevant notions from the representation theory of $\overline{G}_{\mathbb{A}}$.

Recall the definition (1.4.3) of a $(\mathfrak{g}, \overline{K}_\infty)$-module.

Definition 1.4.15. *i. A $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-module $\overline{H}$ is a $(\mathfrak{g}, \overline{K}_\infty)$-module with a genuine, 'smooth' action of $\overline{G}(\mathbb{A}_f)$. The action of $\overline{G}(\mathbb{A}_f)$ is smooth if every vector $x \in \overline{H}$ is fixed by some compact open subgroup $\overline{L}_f \subset \overline{G}(\mathbb{A}_f)$;*

ii. A $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-module $(\bar{\rho}, \overline{H})$ is called admissible, if, for every irreducible representation σ of $\overline{K}_\infty \times \widehat{K}'_f$, the multiplicity of σ in $\overline{H}$ is finite.

Again, we write $(\bar{\rho}_v, \overline{H}_v)$ for an irreducible, admissible representation of $\overline{G}_v$.

Theorem 1.4.10. *i. Suppose that $(\bar{\rho}_\infty, \overline{H}_\infty)$ is an irreducible admissible $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-module, and suppose that for each finite place v, $(\bar{\rho}_v, \overline{H}_v)$ is an irreducible admissible representation of $\overline{G}_v$. Suppose, moreover, that for almost all finite* *places, the representation is unramified and a basis vector* $x_v^0 \in \overline{H}_v^{\widehat{K}_v}$ *is given. Then the restricted tensor product*

$$\overline{H} = \bigotimes_v{}' \, \overline{H}_v$$

with respect to the vectors x_v^0 *is an irreducble admissible* $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$*-module.*

ii. *Conversely, if* $(\bar{\rho}, \overline{H})$ *is an irreducible admissible* $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$*-module, then there is a collection* $\{(\bar{\rho}_v, \overline{H}_v)\}$ *as in (i) and an isomorphism*

$$\bigotimes_v{}' \, \overline{H}_v \longrightarrow \overline{H}$$

of $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$*-modules.*

We shall write $\bar{\rho}$ for the pair $(\bar{\rho}, \overline{H})$. To make sense of a product of local genuine representations of $\overline{G}_v$, it will be necessary to view these representations as projective representations of G_v. Let $\mathcal{H}$ denote a Hilbert space and $U(\mathcal{H})$ the unitary operators in $\mathcal{H}$. A projective representation of G_v is a measurable map $\phi : G_v \to U(\mathcal{H})/\mathbb{C}^\times$. Any such map can be lifted to a map $\phi' : G_v \to U(\mathcal{H})$, but not uniquely. In fact, the 'representation' ϕ' will have the properties:

1. $\phi'(1) = 1$

2. $\phi'(g_1)\phi'(g_2) = \theta_v(g_1, g_2)\phi'(g_1 g_2)$

for all $g_1, g_2 \in G_v$. Here, θ_v is a function from $G_v \times G_v$ to $\mathbb{C}^\times$ which (by the associative law in G_v) is a 2-cocycle. We call ϕ' a θ_v-*representation*, or a *multiplier*-representation of G_v; each such representation arises from a projective representation as above.

Let $(\overline{\rho}_v, \overline{H}_v)$ be a genuine representation of $\overline{G}_v$, in which multiplication is defined by the cocycle β_v, and consider the section $s : G_v \to \overline{G}_v$, $g \mapsto (g, 1)$. Define $\rho_v' := \overline{\rho}_v \circ s$. Then [20]:

Lemma 1.4.11. *There is a bijective correspondence between genuine representations* $(\overline{\rho}_v, \overline{H}_v)$ *of* $\overline{G}_v$ *and* β_v*-representations given by* $\overline{\rho}_v \mapsto \rho_v' = \overline{\rho}_v \circ s$*, and this correspondence preserves direct sums and unitary equivalence.*

Suppose that for each v, $(\overline{\rho}_v, \overline{H}_v)$ is an irreducible, admissible, genuine representation of $\overline{G}_v$, and that for almost every v, $(\overline{\rho}_v, \overline{H}_v)$ is unramified. Let $\{(\overline{\rho}'_v, \overline{H}'_v)\}$ be the collection of β_v-representations that they determine, and form the tensor products $(\rho', H') = (\bigotimes_v' \overline{\rho}'_v, \bigotimes_v' \overline{H}'_v)$ as above. Then (ρ', H') is an irreducible, admissible $\beta_{\mathbb{A}}$-representation of $G_{\mathbb{A}}$, which determines a genuine representation of $\overline{G}_{\mathbb{A}}$.

Fix a genuine quasi-character $\psi : Z(\overline{G}_{\mathbb{A}})/(Z(\overline{G}_{\mathbb{A}}) \cap \widehat{G}_F) \to \mathbb{C}^\times$. Recall the space $\overline{\mathcal{A}}_0(\psi)$ of genuine cuspidal automorphic forms. It is a $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-module, and we have the following important result:

Theorem 1.4.12. *1. The space $\overline{\mathcal{A}}_0(\psi)$ is a direct sum of irreducible genuine admissible $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-modules,*

$$\overline{\mathcal{A}}_0(\psi) = \bigoplus_{\bar{\rho}} \overline{\mathcal{A}}_0(\psi, \bar{\rho}),$$

where $\overline{\mathcal{A}}_0(\psi, \bar{\rho})$ is a summand isomorphic to $\bar{\rho}$.

Motivated by this, we make the following definition:

Definition 1.4.16. *A cuspidal genuine automorphic representation of $\overline{G}_{\mathbb{A}}$ is an irreducible admissible $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-module which is equivalent to a module $\bar{\rho}$ which occurs in $\overline{\mathcal{A}}_0(\psi)$.*

We want the $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-module $\bar{\rho}_\infty$ to have cohomology in the sense of Section 1.4.1.3, hence:

Definition 1.4.17. *A genuine cuspidal automorphic representation of $\overline{G}_{\mathbb{A}}$ of cohomological type is an irreducible admissible $(\mathfrak{g}, \overline{K}_\infty) \times \overline{G}(\mathbb{A}_f)$-module which occurs in $\overline{\mathcal{A}}_0(\psi)$ and whose component $\bar{\rho}_\infty$ at infinity is equivalent to $\overline{\varpi}_\infty(\nu_1, \nu_2)$.*

Recall that a function $\Phi \in \overline{\mathcal{A}}_0(\psi)$ is fixed by some subgroup $\widehat{L}_f \subset \widehat{K}'_f$. As in Remark 1.4.1, we shall call $\widehat{L}_f$ the *level* of the representation $\bar{\rho}$ generated by Φ. Again, this is compatible with the local notion of level, and we shall say that $\bar{\rho}$ is *level one* if $\widehat{L}_f = \widehat{K}'_f$.

1.4.3 The correspondence

Flicker has defined a correspondence, which we'll call S:

$$\left\{\begin{array}{c}\text{Irreducible, admissible, genuine}\\ \text{representations of } \overline{G}_{\mathbb{A}}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Irreducible, admissible}\\ \text{representations of } G_{\mathbb{A}}\end{array}\right\}$$

The global correspondence S can be thought of as a product of local correspondences, each defined S_v:

$$\left\{\begin{array}{c}\text{Irreducible, admissible, genuine}\\ \text{representations of } \overline{G}_v\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Irreducible, admissible}\\ \text{representations of } G_v\end{array}\right\}$$

Suppose $(\overline{\rho}_v, \overline{H}_v)$ is an irreducible, admissible, genuine representation of the local group $\overline{G}_v$, with central character χ. Then

$$\begin{aligned}&S_v(\overline{\rho}_v) \text{ is a representation of } G_v \text{ with central character } \chi',\\ &\text{where } \chi'(z) := \chi(z^2) \text{ for } z \in F_v^\times\end{aligned}$$

This holds at all places v of F.

In Section 1.4.1, we saw that there are four types of irreducible admissible, genuine representations of $\overline{G}_v$: the principal series representations, special representations, one-dimensional representations and the supercuspidal representations.

In particular, if $\overline{\varpi}_v(\chi_1, \chi_2)$ is a genuine irreducible principal series representation of $\overline{G}_v$ whose characters χ_1, χ_2 are unramified, recall that the central character is given by $\chi = \chi_1\chi_2$. Then,

$$S_v(\overline{\varpi}_v(\chi_1, \chi_2)) = \varpi_v(\chi_1', \chi_2') \text{ where } \chi'(z) := \chi_1'\chi_2'(z) = \chi(z^2) = \chi_1\chi_2(z^2) \text{ for } z \in F_v^\times,$$

and $\varpi_v(\chi_1', \chi_2')$ is the principal series representation of G_v whose central character is χ'. In other words, the image of a principal series representation under S_v is a principal series representation.

Remark 1.4.2. If $(\rho_v(\chi'), H_v)$ is in the image of S_v, then χ' is even since $\chi'(-1) = \chi((-1)^2) = 1$.

Theorem 1.4.13 (The local correspondence)**.** *Every irreducible, admissible, genuine representation of* $\overline{G}_v$ *corresponds to an irreducible, admissible representation of*

G_v. All supercuspidal representations whose central character is even are obtained by the correspondence from supercuspidal representations of $\overline{G}_v$. Any even special representation $\sigma_v(\chi')$ is obtained from the square-integrable subquotient $\overline{\sigma}_v(\chi)$ of the induced representation $\overline{\rho}_v(\chi)$, hence any even 1-dimensional representation $\varpi_v(\chi')$ is obtained from the quotient $\overline{\varpi}_v(\chi)$ of $\overline{\rho}_v(\chi)$ by $\overline{\sigma}_v(\chi)$. Any odd special representation is obtained from a supercuspidal representation.

The proof of Theorem 1.4.13, as well as Corollary 1.4.14 and Theorem 1.4.15 below, can be found in Flicker [17].

Corollary 1.4.14. *For all places v, the correspondence S_v is one-to-one, and takes level 1 representations to level 1 representations and square-integrable to square-integrable representations. There are supercuspidal representations which correspond to (odd) special representations (which are not supercuspidal).*

Remark 1.4.3. We identified in Subsection 1.4.1.3 that the only representation of $\overline{G}_{\mathbb{C}}$ whose restriction to $\overline{SL}_2(\mathbb{C})$ is unitary and which has cohomology is the genuine continuous series representation we denoted by $\overline{\varpi}_\infty(\nu_1, \nu_2)$. It turns out that its image under S_∞ does not have cohomology. To see this, write $\overline{\varpi}_\infty(k, l, v, w)$ for $\overline{\varpi}_\infty(\nu_1, \nu_2)$. Then S_∞ takes $\overline{\varpi}_\infty(k, l, v, w)$ to $\varpi_\infty(1 + 2k, 1 + 2l, -\frac{1}{2} + 2v, -\frac{1}{2} + 2w)$. But $\varpi_\infty(1+2k, 1+2l, -\frac{1}{2}+2v, -\frac{1}{2}+2w)$ has no cohomology since, if $v, w \in \mathbb{Z}$, then $-\frac{1}{2} + 2v, -\frac{1}{2} + 2w \notin \mathbb{Z}$.

Definition 1.4.18. *The correspondence S takes $(\bar{\rho} = \otimes_v \bar{\rho}_v, \overline{H} = \otimes_v \overline{H}_v)$ to the constituent $(\rho = \otimes_v \rho_v, H = \otimes_v H_v)$ if $(\bar{\rho}_v, \overline{H}_v)$ corresponds to (ρ_v, H_v) for all v.*

Thus we can formulate the global correspondence.

Theorem 1.4.15 (The global correspondence)**.** *Every irreducible, admissible genuine representation $(\bar{\rho}, \overline{H})$ of $\overline{G}_{\mathbb{A}}$ corresponds to an irreducible, admissible representation (ρ, H) of $G_{\mathbb{A}}$. The correspondence is one-to-one and its image consists of all $\rho = \otimes_v \rho_v$ such that ρ_v has even central character for all v and such that if $\rho_v = \varpi(\chi_1', \chi_2')$ then both χ_1' and χ_2' are even. Moreover, the pre-image of a cuspidal automorphic representation is cuspidal.*

It follows from Remark 1.4.3 that if $\bar{\rho}$ is an irreducible, genuine, automorphic cuspidal representation of cohomological type, then its image under S will *not* be of cohomological type.

We'd like to modify S so that it takes representations of cohomological type to representations which are still of cohomological type. To this end, consider the correspondence $\widetilde{S}$ given locally by

$$\widetilde{S}_v(\bar{\rho}) = S_v(\bar{\rho}) \underset{F_v}{\otimes} |\det|_v^{\frac{1}{2}} \tag{1.27}$$

where 'det' is the 1-dimensional representation of G_v which takes g to $\det(g) \in F_v^{\times}$.

Definition 1.4.19. *The correspondence $\widetilde{S} = S \otimes |det|_{\mathbb{A}}^{\frac{1}{2}}$ takes $(\bar{\rho} = \otimes_v \bar{\rho}_v, \overline{H} = \otimes_v \overline{H}_v)$ to the constituent $(\rho = \otimes_v \rho_v, H = \otimes_v H_v)$ if $(\bar{\rho}_v, \overline{H}_v)$ corresponds to (ρ_v, H_v) for all v.*

We claim

Proposition 1.4.16. *If $\widetilde{S}(\bar{\rho}, \overline{H})$ is an irreducible, admissible, cuspidal representation of $G_{\mathbb{A}}$ of cohomological type, then $(\bar{\rho}, \overline{H})$ is an irreducible, admissible, genuine, cuspidal representation of $\overline{G}_{\mathbb{A}}$ of cohomological type.*

Proof. We must show that the $\widetilde{S}$-pre-image of a cuspidal automorphic representation is a cuspidal automorphic representation. By Theorem 1.4.15, this is true for S; thus it will suffice to show that tensoring with $|\det|_{\mathbb{A}}^{\frac{1}{2}}$ preserves this condition.

Suppose V is a subspace of cusp forms isomorphic to a representation ρ. Put

$$W = \left\{ f(g)\ |\det(g)|_{\mathbb{A}}^{\frac{1}{2}} \ : \ f \in V \right\}.$$

Then it is trivial to check that W is a space of cusp forms, and is isomorphic to $\rho \otimes |\det|_{\mathbb{A}}^{\frac{1}{2}}$.

To check that the $\widetilde{S}$-pre-image of a representation with cohomology still has cohomology, we only need to check the pre-image under the local component $\widetilde{S}_\infty$. Recall from Remark 1.4.3 that $\overline{\varpi}_\infty(\nu_1, \nu_2)$ is denoted by $\overline{\varpi}_\infty(k, l, v, w)$. A straightforward computation shows that $\widetilde{S}_\infty(\overline{\varpi}_\infty(k, l, v, w)) = \varpi_\infty(1 + 2k, 1 + 2l, -1 + 2v, -1 + 2w)$. Thus, if $k', l' \geq 1$ are odd integers, and if $v', w' \in \mathbb{Z}$ are odd, then the pre-image of $\varpi_\infty(k', l', v'.w')$ is of the form $\overline{\varpi}_\infty(\frac{k'-1}{2}, \frac{l'-1}{2}, \frac{v'+1}{2}, \frac{w'+1}{2})$, and $\frac{k'-1}{2}, \frac{l'-1}{2}$ are non-negative integers, while $\frac{v'+1}{2}, \frac{w'+1}{2}$ are integers.

□

Remark 1.4.4. When $\varpi_\infty(k,l,v,w)$ (resp. $\overline{\varpi}_\infty(k,l,v,w)$) is the infinite component of an automorphic cuspidal representation ρ (resp. $\bar{\rho}$) of $G_{\mathbb{A}}$ (resp. $\overline{G}_{\mathbb{A}}$), we shall refer to the pair (k,l) as the *weight* of ρ (resp. $\bar{\rho}$).

Observe that $\widetilde{S}_\infty$ preserves the unitarity of $\overline{\varpi}_\infty(k,l,v,w)$ when restricted to $\overline{SL}_2(\mathbb{C})$. Indeed, if $k=l$ then $1+2k=1+2l$. Furthermore, twisting by the character $g \mapsto |\det(g)|_v^{\frac{1}{2}}$ does not change the level of the representation: suppose that $l \in L_v \subset K_v$ for some v. Then $|\det(l)|_v^{\frac{1}{2}} = 1$ because $\det(l) \in \mathcal{O}_v^\times$. It follows that

$$\rho_v^{L_v} \neq 0 \Leftrightarrow \left(\rho_v \otimes |\det|_v^{\frac{1}{2}}\right)^{L_v} \neq 0.$$

That is, the level of $S(\bar{\rho})$ agrees with that of $\widetilde{S}(\bar{\rho})$.

Finally, the conclusion we wish to draw:

Corollary 1.4.17. *For all non-negative integers k,*

$$H^2_{cusp}(SL_2(\mathcal{O}), E_{2k+1,2k+1}(\mathbb{C})) \neq 0 \Rightarrow H^2_{cusp}(\Gamma', \kappa_{\mathbb{Q}} \underset{\mathbb{C}}{\otimes} E_{k,k}(\mathbb{C})) \neq 0.$$

Proof. Suppose

$$H^2_{\text{cusp}}(SL_2(\mathcal{O}), E_{2k+1,2k+1}(\mathbb{C})) \neq 0$$

for some non-negative integer k. By the generalised Eichler-Shimura-Harder isomorphism (0.6), there is a cuspidal automorphic representation ϖ of $SL_2(\mathbb{A})$ of level 1 (see Remark 1.4.1), whose infinite component is equivalent to $\varpi_\infty(2k+1, 2k+1, v, w)$ for some odd integers v, w. Since ϖ is of level 1, its local constituents ϖ_v are of level 1, and by Theorem 1.4.5, they are principal series representations with an associated pair of unramified characters. An unramified character is even, therefore the representations are even. Thus by Theorem 1.4.15, ϖ is in the image of $\widetilde{S}$.

Let $\overline{\varpi}$ be the $\widetilde{S}$-pre-image of ϖ. By Proposition 1.4.16, $\overline{\varpi}$ is a genuine cuspidal automorphic representation of $\overline{SL}_2(\mathbb{A})$ of cohomological type. We shall determine the level of $\overline{\varpi}$.

At each odd, finite prime, the local constituent $\overline{\varpi}_v$ is unramified, and therefore of level 1. We claim that at the even prime π,

$$\overline{\varpi}_\pi^{\widehat{K}_\pi(4)} \neq 0.$$

We calculate this space in Section 3.4, and we show that it is non-zero. Thus $\overline{\varpi}$ is of level 1 (see the paragraph immediately after Definition 1.4.17).

By the last paragraph of the proof of Proposition 1.4.16, the infinite component $\overline{\varpi}_\infty$ of $\overline{\varpi}$ is equivalent to $\overline{\varpi}_\infty(k, k, \frac{v+1}{2}, \frac{w+1}{2})$. Therefore, by the metaplectic generalised Eichler-Shimura-Harder isomorphism (0.9),

$$H^2_{\text{cusp}}(\Gamma', \kappa_{\mathbb{Q}} \underset{\mathbb{C}}{\otimes} E_{k,k}(\mathbb{C})) \neq 0,$$

and the Corollary is proved. □

By invoking Corollary 1.5 of [13, p. 4], we see that for large k,

$$H^2_{\text{cusp}}(SL_2(\mathcal{O}), E_{2k+1,2k+1}(\mathbb{C})) \neq 0.$$

Chapter 2

Second cohomology

There is a natural action (0.3) of $SL_2(\mathcal{O})$ on $\mathbb{H}$. One method for calculating the cohomology of $SL_2(\mathcal{O})$ or a finite index subgroup $\Upsilon \subset SL_2(\mathcal{O})$ is to build it from the cohomology of $\Upsilon\backslash\mathbb{H}$: there is a spectral sequence relating $H^*(\Upsilon\backslash\mathbb{H}, -)$ to $H^*(\Upsilon, -)$. The space $\Upsilon\backslash\mathbb{H}$, however, is not optimal: not only is it not compact, but the dimension of $\mathbb{H}$ is strictly larger than the groups' virtual cohomological dimension, which is 2. In 1980, Mendoza [30] produced a smaller space $D \subset \mathbb{H}$ of dimension 2, called a "spine", on which the group still acts properly, and such that $\Upsilon\backslash D$ is compact. Moreover, there is a deformation retract $\Upsilon\backslash\mathbb{H} \to \Upsilon\backslash D$ which verifies the isomorphism

$$H^*(\Upsilon\backslash\mathbb{H}, \mathcal{W}) \cong H^*(\Upsilon\backslash D, \mathcal{W})$$

for all Υ-modules W associated to the local system $\mathcal{W}$.

The principal conclusion of Chapter 1 was that

$$H^2_{\text{cusp}}(\Gamma', \kappa_{\mathbb{Q}} \underset{\mathbb{C}}{\otimes} E_{k,k}(\mathbb{C})) \neq 0 \text{ for some } k \geq 0;$$

indeed, finding a non-trivial cohomology class amounts to finding a non-trivial genuine automorphic cuspidal representation of $\overline{SL}_2(\mathbb{A})$. In this chapter, we shall show that there is a non-trivial cohomology class when $k = 2$. Our main result (Proposition 2.4.1) is that

$$\dim_{\mathbb{C}} H^2_{\text{cusp}}(\Gamma, \text{Ind}\,{}^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{Q}}) \underset{\mathbb{C}}{\otimes} E_{2,2}(\mathbb{C})) \geq 8. \tag{2.1}$$

Expression (2.1) says that there is a non-trivial genuine cusp form of level one and weight $(2, 2)$.

In Section 2.1, we shall recall the spectral sequence used to calculate the cohomology of Υ and define the subset D and $\Upsilon\backslash D$ in the case that $\Upsilon = SL_2(\mathcal{O})$. In Section 2.2, we shall utilise D to exhibit the cohomology of $SL_2(\mathcal{O})$ and some of its congruence subgroups. In particular, we give the $\mathbb{Q}$-dimensions of $H^2(\Gamma_0(\mathfrak{a}), \mathbb{Q})$ and $H^2(\Gamma_1(\mathfrak{a}), \mathbb{Q})$ for some non-zero ideals $\mathfrak{a} \subset \mathcal{O}$, and we show (Proposition 2.2.2) that

$$H^2(\Gamma', \kappa_{\mathbb{Q}}) \cong \mathbb{Q}^{(5)}. \tag{2.2}$$

Section 2.3 is concerned with torsion in the integral second cohomology of $SL_2(\mathcal{O})$ and Γ'. We compute

$$H^2(SL_2(\mathcal{O}), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

and we calculate $H^2(\Gamma', \kappa_{\mathbb{Z}})$ up to extension: Proposition 2.3.5 says that there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^{(5)} \longrightarrow H^2(\Gamma', \kappa_{\mathbb{Z}}) \longrightarrow (\mathbb{Z}/2)^{(35)} \oplus (\mathbb{Z}/4)^{(16)} \oplus (\mathbb{Z}/12)^{(9)} \longrightarrow 0.$$

In the last section, 2.4, as well as our main result Proposition 2.4.1 mentioned above, we show that when $\mathfrak{a} = (1 + 2i), (1 + 4i)$ or $(3 + 2i)$, the cohomology $H^2(\Gamma_0(\mathfrak{a}), \mathbb{Q})$ is entirely Eisenstein: that is, $H^2_{\text{cusp}}(\Gamma_0(\mathfrak{a}), \mathbb{Q}) = 0$ for these ideals $\mathfrak{a}$. Moreover, we show that (2.2) is Eisenstein: we prove, both algebraically and geometrically, that

$$H^2_{\text{cusp}}(\Gamma', \kappa_{\mathbb{Q}}) = 0.$$

Notation shall be described below.

2.1 The tools

Suppose that Υ is a group which acts cellularly (on the left) on a contractible CW-complex X of finite dimension. For each cell δ of X, let Υ_δ be the stabilizer subgroup $\Upsilon_\delta = \{\gamma \in \Upsilon \mid \gamma\delta = \delta\}$ and let X_p be a set of representatives for the Υ-orbits of p-cells of X. Then there is a natural equivariant spectral sequence:

$$E_1^{p,q}(M) = \bigoplus_{\delta \in X_p} H^q(\Upsilon_\delta, M) \Rightarrow H^{p+q}(\Upsilon, M) \tag{2.3}$$

for any Υ-module M. For the derivation of this sequence, see [7] or (for a homological version) [34].

Fix, once and for all, the notation $\Gamma = SL_2(\mathcal{O})$. We call a subset $D \subset \mathbb{H}$ a *spine* for Γ if it is a Γ-equivariant deformation retract of $\mathbb{H}$ of dimension 2.

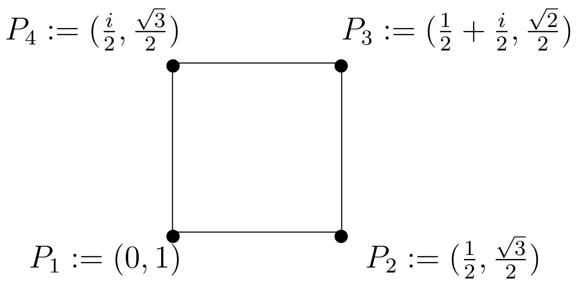
Theorem 2.1.1. *There exists a spine D for Γ with the following properties:*

- *D is naturally endowed with the structure of a locally finite regular CW-complex;*
- *The action of Γ on D is cellular;*
- *The quotient $\Gamma \backslash D$ is a finite CW-complex.*

A more general version of Theorem 2.1.1 can be found in [40]. See also [34].

Definition 2.1.1. *A finite subcomplex $D' \subset D$ is called a fundamental cellular domain for Γ if $D = \Gamma D'$ and if points in open 2-cells are not Γ-equivalent. If we denote by "$\sim$" the cellular equivalence relation on D' induced by identification of 0 or 1-cells under Γ, then it follows that $\sim \backslash D'$ and $\Gamma \backslash D$ are isomorphic as CW-complexes.*

The following picture shows the fundamental cellular domain $\sim \backslash D'$ for Γ in $\mathbb{H}$, as seen from $(0, \infty)$:



The domain $\sim \backslash D'$ is contained in the unit hemisphere $\{(z, r) \in \mathbb{H} \ : \ |z|^2 + r^2 = 1\}$ centred at the origin of $\mathbb{H}$. The 4 vertices (shown on the diagram as dots) are the 0-cells, the 4 lines are the 1-cells and the single face is a 2-cell. We can apply the spectral sequence (2.3) to the pair $X = \sim \backslash D', \Upsilon = \Gamma$. Let

$$a := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}, \quad d := \begin{pmatrix} i & 0 \\ 1+i & -i \end{pmatrix}, e := \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix}.$$

Let Γ_i denote the stabilizer of the 0-cell P_i, and Γ_{ij} the stabilizer of the edge P_iP_j. Then:

$$\begin{aligned} \Gamma_{12} &=< a >\cong C_4\ ; & \Gamma_1 &=< a, b >\cong Q_8 \\ \Gamma_{23} &=< ce^3 >\cong C_6\ ; & \Gamma_2 &=< a, ce^3 >\cong Q_{12} \\ \Gamma_{34} &=< c >\cong C_6\ ; & \Gamma_3 &=< c, d, e >\cong T_{24} \\ \Gamma_{41} &=< b >\cong C_4\ ; & \Gamma_4 &=< b, c >\cong Q_{12} \end{aligned}$$

where C_l is the cyclic group of order l, $Q_8 = \langle\ x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\rangle\ = \langle\ x, y, z \mid x^2 = y^2 = z^2 = xyz = -1\rangle$ is the group of Quaternions of order 8, Q_{12} is part of the family of the generalised Quaternion groups $Q_{4k} = \langle\ x, y \mid x^{2k} = y^4 = 1, x^k = y^2, y^{-1}xy = x^{-1}\rangle$ of order $4k$, where k is an integer ≥ 2, and $T_{24} = \langle\ r, s, t \mid r^2 = s^3 = t^3 = rst\rangle$ is the binary tetrahedral group of order 24.

We shall only prove that $\Gamma_4 =< b, c >\cong Q_{12}$; the proof exhibits the salient features of the rest of the calculations of stabilizer subgroups.

Proof of $\Gamma_4 =< b, c >\cong Q_{12}$.

We shall use the identification (0.2) to think of points $(z, r) \in \mathbb{H}$ as elements $q = z + rj \in \mathfrak{H}$. Accordingly, let $P = \frac{i}{2} + \frac{\sqrt{3}}{2}j$. Note that if $w = x + iy$ is a complex number, and $\bar{w} = x - iy$ denotes its conjugate, then we have the identity:

$$Pw = \bar{w}P + \frac{i}{2}(w - \bar{w}). \tag{2.4}$$

Recall the action (0.3) of Γ on $\mathfrak{H}$. Suppose that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in Γ_4. Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P = P \Leftrightarrow$$

$$\begin{aligned} aP + b =& P(cP + d) \\ &= (\bar{c}P + \frac{i}{2}(c - \bar{c}))P + \bar{d}P + \frac{i}{2}(d - \bar{d}) \text{ by (2.4)} \\ &= \bar{c}P^2 + (\frac{i}{2}(c - \bar{c}) + \bar{d})P + \frac{i}{2}(d - \bar{d}) \\ &= (\frac{i}{2}(c - \bar{c}) + \bar{d})P + (\frac{i}{2}(d - \bar{d}) - \bar{c}) \text{ since } P^2 = -1. \end{aligned}$$

Hence, $a = \frac{i}{2}(c - \bar{c}) + \bar{d}$ and $b = \frac{i}{2}(d - \bar{d}) - \bar{c}$. Let $N : F \to \mathbb{Q}$ denote the norm map $N(w) = w\bar{w}$. We have

$$\begin{aligned} 1 = ad - bc =& (\frac{i}{2}(c - \bar{c}) + \bar{d})d - (\frac{i}{2}(d - \bar{d}) - \bar{c})c \\ =& \frac{i}{2}((c - \bar{c})d - (d - \bar{d})c) + N(c) + N(d) \\ =& \frac{i}{2}(c\bar{d} - \bar{c}d) + N(c) + N(d). \end{aligned}$$

Since $i(d\bar{c} - c\bar{d}) = N(c + id) - N(c) - N(d)$, we can write $\frac{i}{2}(c\bar{d} - \bar{c}d) = \frac{1}{2}(N(c) + N(d) - N(c + id))$, hence

$$\begin{aligned} 1 =& \frac{1}{2}(N(c) + N(d) - N(c + id)) + N(c) + N(d) \\ =& \frac{1}{2}(3N(c) + 3N(d) - N(c + id)) \Leftrightarrow \\ 2 =& 3N(c) + 3N(d) - N(c + id). \end{aligned}$$

The three possible solutions for $N(c), N(d), N(c + id)$ are therefore

$$\begin{aligned} &N(c) = 0, N(d) = 1, N(c + id) = 1, \text{ or} \\ &N(c) = 1, N(d) = 0, N(c + id) = 1, \text{ or} \\ &N(c) = 1, N(d) = 1, N(c + id) = 4. \end{aligned}$$

Let $c = w + ix$ and $d = y + iz$ for $w, x, y, z \in \mathbb{Z}$.

$$\begin{aligned} 3N(c) + 3N(d) - N(c + id) =& 3w^2 + 3x^2 + 3y^2 + 3z^2 - (c + id)(\bar{c} - i\bar{d}) \\ =& 3w^2 + 3x^2 + 3y^2 + 3z^2 - ((w - z)^2 + (x + y)^2) \\ =& 3w^2 + 3x^2 + 3y^2 + 3z^2 - (w^2 - 2wz + z^2 + x^2 + 2xy + y^2) \Leftrightarrow \\ 2 =& 2(w^2 + x^2 + y^2 + z^2) + 2(zw - xy) \\ 2 =& 2(w^2 + zw + z^2) + 2(x^2 - xy + y^2). \end{aligned}$$

We can write this as two quadratic forms: one in x, y and one in w, z. Writing both in matrix form and row reducing:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix},$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 \\ 0 & \frac{3}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix},$$

we see that both forms are positive definite. That is,

$$1 = (w^2 + zw + z^2) + (x^2 - xy + y^2)$$

and both $w^2 + zw + z^2$ and $x^2 - xy + y^2$ are non-negative. This implies that, either

1. $w^2 + zw + z^2 = 1$ and $x = y = 0$, or
2. $x^2 - xy + y^2 = 1$ and $w = z = 0$.

If case (1), then $(w + \frac{1}{2}z)^2 + \frac{3}{4}z^2 = 1$ so $|z| \leq 1$ and $z \in \mathbb{Z}$ means that $z = 0, 1$ or -1. In fact,

$$\begin{aligned} z = 0 &\Rightarrow w = \pm 1, \\ z = 1 &\Rightarrow w = 0 \text{ or } -1, \\ z = -1 &\Rightarrow w = 0 \text{ or } 1. \end{aligned}$$

Similarly, if case (2), then $(x - \frac{1}{2}y)^2 + \frac{3}{4}y^2 = 1$ so $|x| \leq 1$ and $x \in \mathbb{Z}$ means that $x = 0, 1$ or -1. We have

$$\begin{aligned} x = 0 &\Rightarrow y = \pm 1, \\ x = 1 &\Rightarrow y = 0 \text{ or } 1, \\ x = -1 &\Rightarrow y = 0 \text{ or } -1. \end{aligned}$$

Hence Γ_4 consists of the following 12 matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -i & -1 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ -1 & i \end{pmatrix}, \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & -1 \end{pmatrix}.$$

Put

$$b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}.$$

Then, $c^6 = b^4 = 1$, and

$$\begin{aligned} b^{-1}cb &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix} \\ &= c^{-1}. \end{aligned}$$

Therefore, $\Gamma_4 \cong Q_{12} = \langle\, b, c \rangle$. □

The stabilizer subgroups are also calculated in [34] for the action of $PSL_2(\mathcal{O}) = SL_2(\mathcal{O})/\{\pm 1\}$ on $\mathbb{H}$. In this case, the fundamental cellular domain $\sim \backslash D'$ is the same, but the stabilizer subgroups are half as big: they are the quotient of ours by the group $\{\pm 1\}$.

For later use, we shall collect some information about the stabilizer groups.

We can view Q_8 as two distinct extensions:

$$1 \longrightarrow C_2 \longrightarrow Q_8 \longrightarrow C_2 \times C_2 \longrightarrow 1$$
$$1 \longrightarrow C_4 \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

The first is central, and in the second, the quotient C_2 acts on the kernel $C_4 = \langle\, x\rangle$ by the automorphism $x \mapsto x^{-1}$. There are several ways to view Q_{12}. We can write it, as below, as 3 distinct extensions. The first is split and the others are non-split:

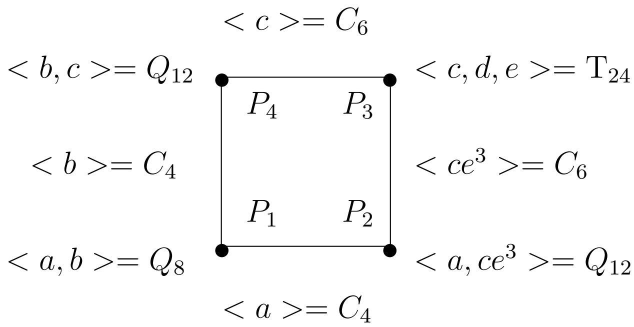
$$1 \longrightarrow C_3 \longrightarrow Q_{12} \longrightarrow C_4 \longrightarrow 1$$
$$1 \longrightarrow C_2 \longrightarrow Q_{12} \longrightarrow D_6 \longrightarrow 1$$
$$1 \longrightarrow C_6 \longrightarrow Q_{12} \longrightarrow C_2 \longrightarrow 1$$

The group $D_6 = \langle\, x, y \mid x^3 = y^2 = 1, yxy = x^{-1}\rangle$ is the Dihedral group of order 6. In the first and third extension, the quotient acts on the kernel by the non-trivial automorphism, that is, by sending the generator of the kernel to its inverse. The second extension is central, so D_6 acts trivially on C_2. The group T_{24} can be written as 2 extensions:

$$1 \longrightarrow C_2 \longrightarrow T_{24} \longrightarrow A_4 \longrightarrow 1$$
$$1 \longrightarrow Q_8 \longrightarrow T_{24} \longrightarrow C_3 \longrightarrow 1$$

where $A_4 = \langle\ x, y \mid x^3 = y^2 = (xy)^3 = 1\rangle$ is the alternating group of order 12. The first extension is non-split and central, and the second is non-central and split. The quotient C_3 acts on Q_8 by rotating the 3 generators of order 4.

Returning to the fundamental cellular domain $\sim \backslash D'$, we see that the stabilizer of the only 2-cell is $\{\pm 1\} \cong C_2$. In pictorial form, the stabilizers are



The spectral sequence (2.3) simplifies if M is a Γ-module over $\mathbb{Z}[\frac{1}{6}]$. In this case, since the primes above 2 and 3 are inverted, the cohomology of the (finite) stabilizers vanish in degree greater than 0. Hence we have $E_1^{p,q}(M) = 0$ for all $q > 0$. Therefore the spectral sequence is concentrated on the horizontal axis $q = 0$ and the cohomology of the cochain complex

$$E_1^{0,0}(M) \xrightarrow{d_1^{0,0}} E_1^{1,0}(M) \xrightarrow{d_1^{1,0}} E_1^{2,0}(M) \tag{2.5}$$

gives $H^*(\Gamma, M)$. That is,

$$H^0(\Gamma, M) = \mathrm{Ker}(d_1^{0,0}),\ H^1(\Gamma, M) = \mathrm{Ker}(d_1^{1,0})/\mathrm{Im}(d_1^{0,0}),\quad H^2(\Gamma, M) = M^{\{\pm \mathrm{Id}\}}/\mathrm{Im}(d_1^{1,0}).$$

With the appropriate substitutions, (2.5) reads

$$\bigoplus_{0-cell\ P_i} H^0(\Gamma_i, M) \xrightarrow{d_1^{0,0}} \bigoplus_{1-cell\ P_{ij}} H^0(\Gamma_{ij}, M) \xrightarrow{d_1^{1,0}} H^0(\{\pm \mathrm{Id}\}, M) \tag{2.6}$$

Or,

$$M^{\Gamma_1} \oplus M^{\Gamma_2} \oplus M^{\Gamma_3} \oplus M^{\Gamma_4} \xrightarrow{d_1^{0,0}} M^{\Gamma_{12}} \oplus M^{\Gamma_{23}} \oplus M^{\Gamma_{34}} \oplus M^{\Gamma_{41}} \xrightarrow{d_1^{1,0}} M^{\pm \mathrm{Id}}$$

To determine H^2, clearly it remains to describe the differential $d_1^{1,0}$. In fact, since the boundary of the 2-cell in the fundamental cellular domain $\sim \backslash D'$ is the sum of the 1-cells each counted once, $d_1^{1,0}$ is simply the map $d_1^{1,0}((m_1, m_2, m_3, m_4)) = m_1 + m_2 + m_3 + m_4$. If we make the assumption that -1 acts trivially on M then H^2 is described by

$$H^2(\Gamma, M) = M/(M^{\Gamma_{12}} + M^{\Gamma_{23}} + M^{\Gamma_{34}} + M^{\Gamma_{41}}) \tag{2.7}$$

If, on the other hand, 6 is not invertible in M - for example, if $M = \mathbb{Z}$ - then, in order to calculate in the spectral sequence (2.3), we shall need the following Lemma.

Lemma 2.1.2. *The integral cohomology of the finite groups C_n $(n > 0), Q_8, Q_{12}, T_{24}$ is given, respectively, by:*

$$H^q(C_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \ odd \\ \mathbb{Z}/n\mathbb{Z} & q \ even\ , q > 0 \end{cases}$$

$$H^q(Q_8, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \equiv 1 \ (mod\ 4) \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & q \equiv 2 \ (mod\ 4) \\ 0 & q \equiv 3 \ (mod\ 4) \\ \mathbb{Z}/8\mathbb{Z} & q \equiv 0 \ (mod\ 4), q > 0 \end{cases}$$

$$H^q(Q_{12},\mathbb{Z}) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q\equiv 1\ (mod\ 4) \\ \mathbb{Z}/4\mathbb{Z} & q\equiv 2\ (mod\ 4) \\ 0 & q\equiv 3\ (mod\ 4) \\ \mathbb{Z}/12\mathbb{Z} & q\equiv 0\ (mod\ 4), q>0 \end{cases}$$

$$H^q(T_{24},\mathbb{Z}) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q\equiv 1\ (mod\ 4) \\ \mathbb{Z}/3\mathbb{Z} & q\equiv 2\ (mod\ 4) \\ 0 & q\equiv 3\ (mod\ 4) \\ \mathbb{Z}/24\mathbb{Z} & q\equiv 0\ (mod\ 4), q>0 \end{cases}$$

Proof. First, note that if $\mathfrak{H} = \mathbb{R}\oplus\mathbb{R}i\oplus\mathbb{R}j\oplus\mathbb{R}k$ is the Hamiltonian quaternion algebra, and if G is a subgroup of the multiplicative group $\mathfrak{H}^\times$, then it is well known [7] that G has periodic cohomology of period 4. This applies to $G = C_n, Q_8, Q_{12}$ and T_{24}. Moreover, for such G,

$$H^4(G,\mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}, \text{ and } H^{\mathrm{odd}}(G,\mathbb{Z}) = 0. \tag{2.8}$$

Consider the cohomology of Q_8. It suffices to prove that

$$H^q(Q_8,\mathbb{Z}) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q=1 \\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & q=2 \\ 0 & q=3 \\ \mathbb{Z}/8\mathbb{Z} & q=4 \end{cases}$$

Consider the Hochschild-Serre spectral sequence related to the group extension given above,

$$1 \longrightarrow C_4 \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1,$$

in which C_2 acts on C_4 non-trivially. Indeed,

$$E_2^{p,q} = H^p(C_2, H^q(C_4, \mathbb{Z})) \Longrightarrow H^{p+q}(Q_8, \mathbb{Z}) \tag{2.9}$$

To list the terms on the E_2 sheet, we must calculate the structure of $H^{\cdot}(C_4, \mathbb{Z})$ as a C_2-module. With the operation of cup product, $H^{\cdot}(C_4, \mathbb{Z}) = \mathbb{Z}[x]/ < 4x >$ is a ring [38], where $\deg(x) = 2$ and the action of C_2 on x is $x \mapsto -x$. Hence, in $H^2(C_4, \mathbb{Z})$, under the action of C_2,

$$\begin{aligned} x &\longmapsto -x \\ 2x &\longmapsto 2x \\ -x &\longmapsto x \\ 4x &\longmapsto 4x \end{aligned}$$

This shows that $H^2(C_4, \mathbb{Z})^{C_2} \cong \mathbb{Z}/2\mathbb{Z}$. In degree 4, C_2 sends x^2 to x^2, and hence we have $H^4(C_4, \mathbb{Z})^{C_2} \cong \mathbb{Z}/4\mathbb{Z}$. It is not hard to show that

$$\begin{aligned} H^{\text{odd}}(C_2, \mathbb{Z}/4\mathbb{Z}) &\cong \mathbb{Z}/2\mathbb{Z} \text{ and} \\ H^{\text{even}}(C_2, \mathbb{Z}/4\mathbb{Z}) &\cong \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

The first 5 rows and 5 columns of the E_2 sheet of the spectral sequence read:

q					
	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
	0	0	0	0	0
	$\boxed{\mathbb{Z}/2\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$	$\boxed{\mathbb{Z}/2\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
	0	0	0	0	0
	$\boxed{\mathbb{Z}}$	0	$\boxed{\mathbb{Z}/2\mathbb{Z}}$	0	$\mathbb{Z}/2\mathbb{Z}$
					p

The boxed terms are stable. Since we know (by (2.8)) that $H^3(Q_8, \mathbb{Z}) = 0$, $E_2^{1,2}$ must vanish. So there must be an injective map $E_3^{1,2} \to E_3^{4,0}$ which is also an isomorphism. We know (again, by (2.8)) that $H^4(Q_8, \mathbb{Z}) \cong \mathbb{Z}/8\mathbb{Z}$, hence $E_3^{0,4} = \mathbb{Z}/4\mathbb{Z}$ must be stable from the E_3 sheet onwards, so the map $E_3^{0,4} \to E_3^{3,2} \cong \mathbb{Z}/2\mathbb{Z}$ must be trivial. Given that $H^5(Q_8, \mathbb{Z}) = 0$, this forces the map $E_3^{3,2} \to E_3^{6,0} \cong \mathbb{Z}/2\mathbb{Z}$ to be an isomorphism.

The terms we are interested in stabilize at the E_4 sheet, which takes the form:

$$
\begin{array}{c|ccccc}
q & & & & & \\
 & \mathbb{Z}/4\mathbb{Z} & 0 & & & \\
 & 0 & 0 & 0 & & \\
 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \\
 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & 0 \\
\hline
 & & & & & p
\end{array}
$$

Therefore, $H^0(Q_8, \mathbb{Z}) \cong \mathbb{Z}, H^4(Q_8, \mathbb{Z}) \cong \mathbb{Z}/8\mathbb{Z}$ and $H^2(Q_8, \mathbb{Z})$ occurs in an exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow H^2(Q_8, \mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Observe that $H^2(C_2, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. This means that $H^2(Q_8, \mathbb{Z})$ is either $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (the split extension), or $\mathbb{Z}/4\mathbb{Z}$ (the non-split extension). To determine which, consider the short exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

and the associated long exact sequence in group cohomology:

$$\cdots \to H^1(Q_8, \mathbb{Z}) \to H^1(Q_8, \mathbb{Q}) \to H^1(Q_8, \mathbb{Q}/\mathbb{Z}) \to H^2(Q_8, \mathbb{Z}) \to H^2(Q_8, \mathbb{Q})$$
$$\to H^2(Q_8, \mathbb{Q}/\mathbb{Z}) \to \cdots$$

Since $\mathbb{Q}$ is torsion-free, $H^1(Q_8, \mathbb{Q}) = H^2(Q_8, \mathbb{Q}) = 0$, so we have an isomorphism $H^2(Q_8, \mathbb{Z}) \cong H^1(Q_8, \mathbb{Q}/\mathbb{Z})$. Let $[Q_8, Q_8] = \{xyx^{-1}y^{-1} \mid x, y \in Q_8\}$ denote the subgroup of commutators of Q_8. Now,

$$\begin{aligned} H^1(Q_8, \mathbb{Q}/\mathbb{Z}) =& \mathrm{Hom}(Q_8, \mathbb{Q}/\mathbb{Z}), \\ =& \mathrm{Hom}(Q_8/\left[Q_8, Q_8\right], \mathbb{Q}/\mathbb{Z}) \text{ since } \mathbb{Q}/\mathbb{Z} \text{ is abelian}, \\ =& \mathrm{Hom}(C_2 \times C_2, \mathbb{Q}/\mathbb{Z}), \\ =& \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

This proves that $H^2(Q_8, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

See [34, p. 592] for $H^{\cdot}(C_n, \mathbb{Z})$ and $H^{\cdot}(T_{24}, \mathbb{Z})$. For $H^{\cdot}(Q_{12}, \mathbb{Z})$, see [10, p. 254].

□

The differential maps $E_1^{p,q} \to E_1^{p+1,q}$ on the E_1 sheet of the spectral sequence (2.3) involve restriction maps from group cohomology. We determine these maps in the following Lemmata.

First, we need a Lemma from finite group cohomology.

Lemma 2.1.3. *For any finite group G and positive integer n, there is an isomorphism*

$$H^n(G, \mathbb{Z}) \cong H^n(G, \widehat{\mathbb{Z}})$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$.

Proof. Let n be an integer > 0. Consider the tensor product of abelian groups $H^n(G, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \widehat{\mathbb{Z}}$ and the map

$$H^n(G, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \widehat{\mathbb{Z}} \longrightarrow H^n(G, \widehat{\mathbb{Z}}). \tag{2.10}$$

Since $\widehat{\mathbb{Z}}$ is torsion-free as an abelian group, it is flat, and therefore (2.10) is an isomorphism. On the other hand, since $n > 0$, $H^n(G, \mathbb{Z})$ is finite, and

$$H^n(G, \mathbb{Z}) \cong H^n(G, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \widehat{\mathbb{Z}}.$$

□

The proceeding series of Lemmata give the maps induced on cohomology by the inclusion maps of the cyclic groups into Q_8, Q_{12} and T_{24}. If $\mathbb{Z}/l\mathbb{Z}$ (respectively, $\mathbb{Z}$) is a finite (respectively, infinite) cyclic group, we shall denote by 1 its generator.

Consider Q_8. Note that the matrices a, b and ab generate the three cyclic subgroups of order 4.

Lemma 2.1.4. *An inclusion $i : C_4 \to Q_8$ induces the following restriction maps on cohomology:*

1. *In degree* 0*, the map* $\mathbb{Z} \to \mathbb{Z}$ *is* $1 \mapsto 1$*;*

2. *In degrees congruent to* 2 *mod* 4*, the three inclusions give three restriction maps as follows:*

$$\begin{aligned}
&\text{The map } \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to H^2(\langle\ a\rangle\ ,\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \text{ is}\\
&\quad (1,0) \mapsto 2, \quad (0,1) \mapsto 0\\
&\text{The map } \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to H^2(\langle\ b\rangle\ ,\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \text{ is}\\
&\quad (1,0) \mapsto 0, \quad (0,1) \mapsto 2\\
&\text{The map } \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to H^2(\langle\ ab\rangle\ ,\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \text{ is}\\
&\quad (1,0) \mapsto 2, \quad (0,1) \mapsto 2.
\end{aligned}$$

   Here, we are identifying $H^2(Q_8, \mathbb{Z})$ *with* $Hom(C_2 \times C_2, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *using the isomorphism* $Q_8/[Q_8, Q_8] \cong C_2 \times C_2$*;*

3. *In positive degrees congruent to* 0 *mod* 4*, the map* $\mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ *is* $1 \mapsto 1$ *or* $1 \mapsto 3$*;*

4. *In all other degrees, the map is trivial.*

Proof. Our proof will be based on the proof of Lemma 2.1.2. Indeed, recall the E_2 sheet of the spectral sequence (2.9).

Consider the (higher) 5-term exact sequence:

$$0 \longrightarrow E_2^{2,0} \xrightarrow{\text{inf}} H^2(Q_8, \mathbb{Z}) \xrightarrow{\text{res}} E_2^{0,2} \longrightarrow E_2^{3,0} \longrightarrow H^3(Q_8, \mathbb{Z}) \longrightarrow 0 \tag{2.11}$$

where 'inf' and 'res' are the inflation and restriction maps, respectively. Given that $E_2^{3,0} = 0$, sequence (2.11) reduces to an exact sequence

$$0 \longrightarrow H^2(C_2, \mathbb{Z}) \xrightarrow{\text{inf}} H^2(Q_8, \mathbb{Z}) \xrightarrow{\text{res}} H^2(C_4, \mathbb{Z})^{C_2} \longrightarrow 0$$

which, with the appropriate substitutions, reads:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

This implies that the map res: $H^2(Q_8, \mathbb{Z}) \to H^2(C_4, \mathbb{Z})$ must be non-trivial.

Recall the isomorphism $H^2(Q_8, \mathbb{Z}) \cong \operatorname{Hom}(Q_8/[Q_8, Q_8], \mathbb{Q}/\mathbb{Z})$ from the proof of Lemma 2.1.2. Considering the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

and the long exact sequence in cohomology, we arrive at a similar isomorphism $H^2(C_4, \mathbb{Z}) \cong \operatorname{Hom}(C_4, \mathbb{Q}/\mathbb{Z})$.

Let $\langle a \rangle$, $\langle b \rangle$, and $\langle ab \rangle$ be the three distinct cyclic subgroups of Q_8 of order 4. The following isomorphisms are canonical:

$$\begin{aligned} H^1(C_4, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(\langle a \rangle, \mathbb{Q}/\mathbb{Z}) &\longrightarrow \mathbb{Z}/4\mathbb{Z} \\ \phi &\longmapsto \phi(a) \\ H^1(C_4, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(\langle b \rangle, \mathbb{Q}/\mathbb{Z}) &\longrightarrow \mathbb{Z}/4\mathbb{Z} \\ \phi &\longmapsto \phi(b) \\ H^1(C_4, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(\langle ab \rangle, \mathbb{Q}/\mathbb{Z}) &\longrightarrow \mathbb{Z}/4\mathbb{Z} \\ \phi &\longmapsto \phi(ab) \end{aligned}$$

and, if x denotes a, b or ab, then for each copy of $C_4 = \langle x \rangle$, we have a commutative diagram

$$\begin{array}{ccccc} H^2(Q_8, \mathbb{Z}) & \xrightarrow{=} & \operatorname{Hom}(\langle a \rangle \times \langle b \rangle, \mathbb{Q}/\mathbb{Z}) & & \phi \\ \Big\downarrow{\scriptstyle \text{res}} & & \Big\downarrow{\scriptstyle \text{res}} & & \Big\downarrow \\ H^2(C_4, \mathbb{Z}) & \xrightarrow{=} & \operatorname{Hom}(\langle x \rangle, \mathbb{Q}/\mathbb{Z}) & & \phi \end{array}$$

This means that the three maps res: $H^2(Q_8, \mathbb{Z}) \to H^2(C_4, \mathbb{Z})$ map the three different copies of $\mathbb{Z}/2\mathbb{Z}$ inside $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ in each case.

Consider the map res: $H^4(Q_8, \mathbb{Z}) \to H^4(C_4, \mathbb{Z})$ in degree 4. Looking at the E_4 sheet of the spectral sequence (2.9) from the proof of Lemma 2.1.2, and owing to the fact that $E_4^{3,2} = 0$, we have a short exact sequence

$$0 \longrightarrow H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \longrightarrow H^4(Q_8, \mathbb{Z}) \longrightarrow H^4(C_4, \mathbb{Z}) \longrightarrow 0$$

That is,

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow 0,$$

which shows that the map res: $H^4(Q_8, \mathbb{Z}) \longrightarrow H^4(C_4, \mathbb{Z})$ is surjective, hence the generator 1 maps to 1 or 3.

That the map $H^0(Q_8,\mathbb{Z}) \to H^0(C_4,\mathbb{Z})$ is an isomorphism is clear. Since $H^1(Q_8,\mathbb{Z}) = H^3(Q_8,\mathbb{Z}) = 0$, it is also clear that the map is trivial in odd degrees.

It is sufficient to prove the Lemma in degrees $0, 1, 2, 3$ and 4 as the following argument shows. Let x generate $H^4(Q_8,\mathbb{Z})$, and let y be the image of x under res: $H^4(Q_8,\mathbb{Z}) \longrightarrow H^4(C_4,\mathbb{Z})$. Then y generates $H^4(C_4,\mathbb{Z})$, and the diagram

$$\begin{array}{ccc} H^{4+k}(Q_8,\mathbb{Z}) & \xrightarrow{\text{res}} & H^{4+k}(C_4,\mathbb{Z}) \\ \cong \downarrow \cup x & & \cong \downarrow \cup y \\ H^{k}(Q_8,\mathbb{Z}) & \xrightarrow{\text{res}} & H^{k}(C_4,\mathbb{Z}) \end{array}$$

commutes. □

Lemma 2.1.5. *An inclusion* $i : C_6 \to Q_{12}$ *induces the following maps on cohomology:*

1. *In degree* 0, *the map* $\mathbb{Z} \to \mathbb{Z}$ *is* $1 \mapsto 1$;
2. *In degrees congruent to* 2 *mod* 4, *the map* $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ *is* $1 \mapsto (1, 0)$;
3. *In positive degrees congruent to* 0 *mod* 4, *the map* $\mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ *is surjective: that is,* $1 \mapsto (1, 1)$ *or* $1 \mapsto (1, 2)$;
4. *In all other degrees, the map is trivial.*

An inclusion $i : C_4 \to Q_{12}$ *induces the following maps on cohomology:*

1. *In degree* 0, *the map* $\mathbb{Z} \to \mathbb{Z}$ *is* $1 \mapsto 1$;
2. *In degrees congruent to* 2 *mod* 4, *the map* $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ *is surjective: that is,* $1 \mapsto 1$ *or* $1 \mapsto 3$;
3. *In positive degrees congruent to* 0 *mod* 4, *the map* $\mathbb{Z}/12\mathbb{Z} = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ *is surjective: that is, either* $(1, 0) \mapsto 1, (0, 1) \mapsto 0$ *or* $(1, 0) \mapsto 3, (0, 1) \mapsto 0$;
4. *In all other degrees, the map is trivial.*

Proof. As explained in the proof of Lemma 2.1.4, it suffices to prove the Lemma in degrees $0, 1, 2, 3$ and 4. Consider the non-central extension

$$1 \longrightarrow C_6 \longrightarrow Q_{12} \longrightarrow C_2 \longrightarrow 1$$

and the associated Hochschild-Serre spectral sequence with coefficients in the 2-adic integers $\mathbb{Z}_2$:

$$E_2^{p,q} = H^p(C_2, H^q(C_6, \mathbb{Z}_2)) \Longrightarrow H^{p+q}(Q_{12}, \mathbb{Z}_2) \tag{2.12}$$

Note that

$$H^p(C_2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \ \text{ if } p = 0 \\ 0 \text{ if } p \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} \text{ if } p > 0, \text{ is even} \end{cases}$$

$$H^q(C_6, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 \ \text{ if } q = 0 \\ 0 \text{ if } q \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} \text{ if } q > 0, \text{ is even} \end{cases}$$

Moreover, $H^q(C_2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for all q, and the action of C_2 on $H^{\cdot}(C_6, \mathbb{Z}_2)$ is trivial. The following terms of the spectral sequence stabilize at the E_4 sheet, which takes the form

$$\begin{array}{c|lllll}
q & & & & & \\
 & \mathbb{Z}/2\mathbb{Z} & & & & \\
 & 0 & 0 & & & \\
 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & & \\
 & 0 & 0 & 0 & 0 & \\
 & \mathbb{Z}_2 & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & 0 \\
\hline
 & & & & & p
\end{array}$$

We can therefore read off the exact sequence

$$0 \longrightarrow H^2(C_2, \mathbb{Z}_2) \longrightarrow H^2(Q_{12}, \mathbb{Z}_2) \longrightarrow H^2(C_6, \mathbb{Z}_2)^{C_2} \longrightarrow 0$$

That is,

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

from which we see that the degree 2 map at the prime 2, is

$$\begin{aligned} \text{res } : H^2(Q_{12}, \mathbb{Z}_2) &\to H^2(C_6, \mathbb{Z}_2) \\ 1 &\mapsto 1 \end{aligned}$$

Furthermore, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow H^4(Q_{12}, \mathbb{Z}_2) \xrightarrow{\text{res}} H^4(C_6, \mathbb{Z}_2) \longrightarrow 0$$

That is,

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

from which we see that the degree 4 map at the prime 2, is

$$\begin{aligned} \text{res } : H^4(Q_{12}, \mathbb{Z}_2) &\to H^4(C_6, \mathbb{Z}_2) \\ 1 &\mapsto 1 \end{aligned}$$

Next, consider the spectral sequence (2.12) at the prime 3, and note the following results:

$$H^p(C_2, \mathbb{Z}_3) = \begin{cases} \mathbb{Z}_3 & \text{if } p = 0 \\ 0 \text{ if } p \neq 0 \end{cases}$$

$$H^q(C_6, \mathbb{Z}_3) = \begin{cases} \mathbb{Z}_3 & \text{if } q = 0 \\ 0 \text{ if } q \text{ is odd} \\ \mathbb{Z}/3\mathbb{Z} \otimes \chi^{\frac{q}{2}} \text{ if } q > 0, \text{ is even} \end{cases}$$

where $\chi^{\frac{q}{2}}$ is the C_2-module with action given by the non-trivial character $\chi : C_2 \to (\mathbb{Z}/3\mathbb{Z})^\times$. However, since $H^p(C_2, H^q(C_6, \mathbb{Z}_3)) = 0$ for $p > 0$, only the first column of the spectral sequence is non-zero, so that the edge maps

$$\text{res } : H^q(Q_{12}, \mathbb{Z}_3) \longrightarrow H^q(C_6, \mathbb{Z}_3)^{C_2}$$

are isomorphisms.

Observe that by Lemma 2.1.3, we can decompose

$$H^n(Q_{12}, \mathbb{Z}) \cong H^n(Q_{12}, \mathbb{Z}_2) \oplus H^n(Q_{12}, \mathbb{Z}_3) \oplus H^n(Q_{12}, \prod_{p \neq 2,3} \mathbb{Z}_p) \text{ for } n > 0$$

However, since 12 is invertible in $\prod_{p \neq 2,3} \mathbb{Z}_p$, the last summand on the right is 0, and we are left with

$$H^n(Q_{12}, \mathbb{Z}) \cong H^n(Q_{12}, \mathbb{Z}_2) \oplus H^n(Q_{12}, \mathbb{Z}_3) \text{ for } n > 0.$$

The degree 0 map, $\mathbb{Z} \to \mathbb{Z}$ is clearly an isomorphism. If $n > 0$ we can put the information at the primes 2 and 3 together to conclude that the degree 2 map is non-trivial: that is,

$$\begin{aligned} H^2(Q_{12}, \mathbb{Z}) &\longrightarrow H^2(C_6, \mathbb{Z}) \\ 1 &\mapsto 3 \end{aligned}$$

and the degree 4 map is surjective: that is,

$$\begin{aligned} H^4(Q_{12}, \mathbb{Z}) &\longrightarrow H^4(C_6, \mathbb{Z}) \\ 1 &\mapsto (1,1), \text{ or} \\ 1 &\mapsto (1,2). \end{aligned}$$

To prove the second part of the Lemma, consider an inclusion $C_4 \hookrightarrow Q_{12}$ and recall the extension:

$$1 \longrightarrow C_3 \longrightarrow Q_{12} \longrightarrow C_4 \longrightarrow 1$$

where the generator of C_4 gives the non-trivial automorphism of C_3. The restriction maps in degrees $0, 1$ and 3 are clear. Consider the Hochschild-Serre spectral sequence associated to this extension, with 2-adic coefficients $\mathbb{Z}_2$:

$$H^p(C_4, H^q(C_3, \mathbb{Z}_2)) \Longrightarrow H^{p+q}(Q_{12}, \mathbb{Z}_2) \tag{2.13}$$

Given that $H^q(C_3, \mathbb{Z}_2) = 0$ for $q > 0$, the spectral sequence collapses to yield an isomorphism

$$\inf : H^n(C_4, \mathbb{Z}_2) \cong H^n(Q_{12}, \mathbb{Z}_2)$$

in all dimensions. On the other hand, one can check that the composition $\text{res} \circ \inf$ is the identity on cocycles, hence

$$\text{res} : H^n(Q_{12}, \mathbb{Z}_2) \cong H^n(C_4, \mathbb{Z}_2) \tag{2.14}$$

is an isomorphism for all n.

Consider (2.13) with coefficients in the 3-adic integers $\mathbb{Z}_3$. Only the first column of the E_2-sheet is nonzero, so again, the spectral sequence collapses to yield

$$H^n(Q_{12}, \mathbb{Z}_3) \cong H^n(C_3, \mathbb{Z}_3)^{C_4} \tag{2.15}$$

Putting (2.14) and (2.15) together, we conclude that the restriction map in degree 2 is an isomorphism:

$$\begin{gathered} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \\ 1 \mapsto 1, \text{ or} \\ 1 \mapsto 3, \end{gathered}$$

and in degree 4, is surjective:

$$\begin{gathered} \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \\ (1,0) \mapsto 1, \\ (0,1) \mapsto 0, \text{ or} \\ (1,0) \mapsto 3, \\ (0,1) \mapsto 0. \end{gathered}$$

□

Lemma 2.1.6. *An inclusion $i : C_6 \to T_{24}$ induces the following maps on cohomology:*

1. *In degree 0, the map $\mathbb{Z} \to \mathbb{Z}$ is $1 \mapsto 1$;*
2. *In degrees congruent to 2 mod 4, the map $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ is $1 \mapsto 2$ or $1 \mapsto 4$;*
3. *In positive degrees congruent to 0 mod 4, the map $\mathbb{Z}/12\mathbb{Z} = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ is surjective: that is, $(1,0) \mapsto 3, (0,1) \mapsto 4$ or $(1,0) \mapsto 3, (0,1) \mapsto 2$;*
4. *In all other degrees, the map is trivial.*

Proof. By Lemma 2.1.3,

$$H^{\cdot}(T_{24}, \mathbb{Z}) = H^{\cdot}(T_{24}, \mathbb{Z}_2) \oplus H^{\cdot}(T_{24}, \mathbb{Z}_3).$$

Consider the subgroup $C_6 = C_3 \times C_2 \subset T_{24}$, and the spectral sequence at the prime 2:

$$E_2^{p,q} = H^p(C_2, H^q(C_3, \mathbb{Z}_2)) \Longrightarrow H^{p+q}(C_6, \mathbb{Z}_2) \tag{2.16}$$

Since $H^q(C_3, \mathbb{Z}_2) = 0$ for $q > 0$, the spectral sequence collapses, and the edge maps give isomorphisms

$$\text{res} \ : H^{\cdot}(C_6, \mathbb{Z}_2) \cong H^{\cdot}(C_2, \mathbb{Z}_2).$$

It shall therefore suffice, at the prime 2, to consider the restriction map $H^{\cdot}(T_{24}, \mathbb{Z}_2) \to H^{\cdot}(C_2, \mathbb{Z}_2)$. But now C_2 is a normal subgroup, so we can use the spectral sequence associated to the extension

$$1 \longrightarrow C_2 \longrightarrow T_{24} \longrightarrow A_4 \longrightarrow 1,$$

namely,

$$E_2^{p,q} = H^p(A_4, H^q(C_2, \mathbb{Z}_2)) \Longrightarrow H^{p+q}(T_{24}, \mathbb{Z}_2).$$

The integral homology of A_4 is given in [34]. Using this, and the Universal Coefficients Theorem, one can show that

$$H^q(A_4, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & q = 0 \\ 0 & q = 1 \\ \mathbb{Z}/2\mathbb{Z} & q = 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & q = 3 \\ \mathbb{Z}/2\mathbb{Z} & q = 4 \end{cases}$$

Similarly, one can calculate $H^q(A_4, \mathbb{Z}_2)$ for $0 \leq q \leq 4$. The following terms of the spectral sequence stabilize at the E_4 sheet, which reads:

$$
\begin{array}{c|ccccc}
q & & & & & \\
 & \mathbb{Z}/2\mathbb{Z} & & & & \\
 & 0 & 0 & & & \\
 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} & & \\
 & 0 & 0 & 0 & 0 & \\
 & \mathbb{Z}_2 & 0 & 0 & 0 & \mathbb{Z}/2\mathbb{Z} \\
\hline
 & & & & & p
\end{array}
$$

Since $H^2(T_{24}, \mathbb{Z}_2)$ is itself trivial, the map in degree 2:

$$H^2(T_{24}, \mathbb{Z}_2) \longrightarrow H^2(C_2, \mathbb{Z}_2) \text{ is trivial.}$$

There is a filtration on $H^4(T_{24}, \mathbb{Z}_2)$:

$$\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z} \subset H^4(T_{24}, \mathbb{Z}_2) = \mathbb{Z}/8\mathbb{Z},$$

where the top quotient is $E_4^{4,0} = H^4(C_2, \mathbb{Z}_2) \cong \mathbb{Z}/2\mathbb{Z}$; hence there is an exact sequence

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow H^4(T_{24}, \mathbb{Z}_2) \longrightarrow H^4(C_2, \mathbb{Z}_2) \longrightarrow 0.$$

That is,

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

from which we can conclude that the degree 4 map is

$$H^4(T_{24}, \mathbb{Z}_2) \longrightarrow H^4(C_2, \mathbb{Z}_2)$$
$$1 \mapsto 1.$$

Next consider the spectral sequence (2.16) at the prime 3. Owing to the fact that $H^p(C_2, \mathbb{Z}_3) = 0$ for $p > 0$, the edge maps give isomorphisms

$$\text{res} \ : H^{\cdot}(C_6, \mathbb{Z}_3) \cong H^{\cdot}(C_3, \mathbb{Z}_3).$$

However, C_3 is not a normal subgroup of T_{24}, so the above method does not apply. The subgroup C_3 is instead the Sylow 3-subgroup, and a well-known result [10, p. 259] gives

$$H^{\cdot}(T_{24}, \mathbb{Z}_3) \cong H^{\cdot}(C_3, \mathbb{Z}_3)^{N(C_3)}.$$

The set $H^{\cdot}(C_3, \mathbb{Z}_3)^{N(C_3)}$ denotes the cohomology classes which are invariant under the action of the normalizer $N(C_3)$ of C_3 in T_{24}. The cohomology $H^{\cdot}(C_3, \mathbb{Z}_3)$ is a ring $\mathbb{Z}_3[x]/ < 3x >$ in which $\deg(x) = 2$, and the action of any element of $N(C_3)$ is either trivial, or else takes x to $-x$. A quick check using Sage gives

$$N(C_3) = C_3 \cup \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix} \right\},$$

and each element in $N(C_3)$ acts trivially on C_3. Therefore,

$$H^n(C_3, \mathbb{Z}_3)^{N(C_3)} = \begin{cases} \mathbb{Z}_3 & \text{for } n = 0 \\ 0 & \text{for } n \text{ odd} \\ \mathbb{Z}/3\mathbb{Z} & \text{for } n \text{ even}, > 0 \end{cases}$$

The restriction map at the prime 3, for both degrees 2 and 4, is an isomorphism:

$$H^i(T_{24}, \mathbb{Z}_3) \xrightarrow{\cong} H^i(C_3, \mathbb{Z}_3) \text{ for } i = 2, 4.$$

Combining the information at both primes, and making use of Lemma 2.1.3, we arrive at the desired result.

□

In the sequel, we shall calculate the second cohomology of some congruence subgroups of Γ. Rather than find a fundamental cellular domain for each subgroup, we shall employ a fundamental tool called “Shapiro's Lemma”. Suppose that Υ is a subgroup of Γ and that M is a right $R[\Upsilon]$-module under ρ for some commutative ring R. We can regard $R[\Gamma]$ as a $R[\Gamma] - R[\Upsilon]$ bi-module: that is, a left $R[\Gamma]$-module and a right $R[\Upsilon]$-module. Then $\mathrm{Coind}_{\Upsilon}^{\Gamma}(M) := \mathrm{Hom}_{R[\Upsilon]}(R[\Gamma], M) = \{f : \Gamma \to M \mid f(\gamma h) = \rho(h) f(\gamma)\ \forall \gamma \in \Gamma, h \in \Upsilon\}$ is a left Γ-module called the *coinduced* Γ*-module*. The action of Γ is given by $(\gamma f)(\gamma') = f(\gamma^{-1}\gamma')$.

Lemma 2.1.7 (Shapiro)**.** *If Υ is a subgroup of Γ and M is an $R[\Upsilon]$-module then there is an isomorphism*

$$H^i(\Upsilon, M) \cong H^i(\Gamma, \mathit{Coind}\,_{\Upsilon}^{\Gamma}(M)) \text{ for all } i \geq 0$$

For a proof, see [38, p. 171].

Remark 2.1.1. If Υ has finite index in Γ, then $\mathrm{Coind}_{\Upsilon}^{\Gamma}(M) \cong \mathrm{Ind}_{\Upsilon}^{\Gamma}(M)$.

Remark 2.1.2. In the case that M is a trivial $R[\Upsilon]$-module, $\mathrm{Coind}_{\Upsilon}^{\Gamma}(M)$ can be identified with the Γ-module of functions $\{f : \Gamma/\Upsilon \to M\}$.

2.2 Examples

Let R be a commutative ring, and recall the $M_2(R)$-modules $E_k(R) \otimes \det(v)$ from Subsection 1.4.1.2. Write $E_k(R)$ for this module when considered as a representation of $SL_2(R)$. Note that $E_k(R)$ has $\{x^i y^{k-i} \ : \ 0 \le i \le k\}$ as an R-basis. Furthermore, write $E_{k,l}(R)$ for $E_{k,l,v,w}(R)$ considered as an $SL_2(R)$-module. We shall only be interested in the case when $k = l$; in particular, this ensures that $-\mathrm{Id}$ acts trivially. It is useful to remark that $E_k(R) \cong \mathrm{Sym}^k(R^{(2)})$ and $E_{k,l}(R) \cong \mathrm{Sym}^k(R^{(2)}) \underset{R}{\otimes} \overline{\mathrm{Sym}^l(R^{(2)})}$ as $SL_2(R)$-modules, where $\mathrm{Sym}^i(R^{(2)})$ is the i^{th} symmetric power of the standard representation of $SL_2(R)$ on $R^{(2)}$ and the overline on the second factor means it is twisted with complex conjugation.

Consider the following congruence subgroups of Γ:

$$\Gamma(\mathfrak{a}) := \left\{ \gamma \in \Gamma \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{a}} \right\}$$
$$\Gamma_1(\mathfrak{a}) := \left\{ \gamma \in \Gamma \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{a}} \right\}$$
$$\Gamma_0(\mathfrak{a}) := \left\{ \gamma \in \Gamma \;\middle|\; \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \pmod{\mathfrak{a}} \right\}$$

for some non-zero ideal $\mathfrak{a}$ of $\mathcal{O}$ called the *level*. For simplicity, we shall assume that $\mathfrak{a}$ is a power of a single prime ideal $\mathfrak{p}$. Note the inconsistency in notation with Chapter 1; to avoid using too many symbols, we have used the notation $\Gamma(\mathfrak{a})$ for both the congruence subgroup of $GL_2(F)$ and of $SL_2(F)$.

We shall make use of Lemma 2.1.7 in the case that M is a trivial $\mathbb{Q}[\Upsilon]$-module.

First note the following two set bijections:

$$\Gamma/\Gamma_1(\mathfrak{a}) \longleftrightarrow \text{ primitive vectors in } (\mathcal{O}/\mathfrak{a})^2$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, c)$$
$$\Gamma/\Gamma_0(\mathfrak{a}) \longleftrightarrow \mathbb{P}^1(\mathcal{O}/\mathfrak{a})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} a \\ c \end{bmatrix} \text{ where } \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda c \end{bmatrix} \text{ if } \lambda \in (\mathcal{O}/\mathfrak{a})^\times$$

We call a vector (x, y) in $(\mathcal{O}/\mathfrak{a})^2$ *primitive* if x and y are coprime in $\mathcal{O}/\mathfrak{a}$; since $\mathfrak{a} = \mathfrak{p}^n$, this is equivalent to either x or y being a unit in $\mathcal{O}/\mathfrak{a}$. The set $\mathbb{P}^1(\mathcal{O}/\mathfrak{a}) = \mathbb{P}^1(\mathcal{O}/\mathfrak{p}^n)$ is:

$$\begin{aligned}\mathbb{P}^1(\mathcal{O}/\mathfrak{p}^n) := & \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} : x \in \mathcal{O}/\mathfrak{p}^n \right\} \cup \left\{ \begin{bmatrix} x \\ \mathfrak{p} \end{bmatrix} : x \in (\mathcal{O}/\mathfrak{p}^n)^\times/(1 + \mathfrak{p}^{n-1}) \right\} \\ & \cup \left\{ \begin{bmatrix} x \\ \mathfrak{p}^2 \end{bmatrix} : x \in (\mathcal{O}/\mathfrak{p}^n)^\times/(1 + \mathfrak{p}^{n-2}) \right\} \cup \cdots \\ & \cup \left\{ \begin{bmatrix} x \\ \mathfrak{p}^{n-1} \end{bmatrix} : x \in (\mathcal{O}/\mathfrak{p}^n)^\times/(1 + \mathfrak{p}) \right\} \cup \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}\end{aligned}$$

To calculate $H^2(\Gamma_0(\mathfrak{a}), M)$, the general method is to list the elements of $\mathbb{P}^1(\mathcal{O}/\mathfrak{a})$, calculate $\mathrm{Ind}_{\Gamma_0(\mathfrak{a})}^{\Gamma}(M)^{\Gamma_{ij}}$ for each 1-cell P_{ij} and then determine the quotient as in (2.7). Let s_i be a lift to Γ of the i^{th} element x_i of $\mathbb{P}^1(\mathcal{O}/\mathfrak{a})$: that is, $\{s_i\}$ is a set of coset representatives for $\Gamma/\Gamma_0(\mathfrak{a})$. Let $f \in \mathrm{Ind}_{\Gamma_0(\mathfrak{a})}^{\Gamma}(M)$. We have defined the action of Γ so that if $g \in \Gamma$, then

$$(gf)(s_i) = f(g^{-1}s_i) = f(s_j h) = f(s_j) \text{ when } g^{-1}s_i = s_j h \text{ for some } h \in \Gamma_0(\mathfrak{a})$$

A $\mathbb{Q}$-basis for $\mathrm{Ind}_{\Gamma_0(\mathfrak{a})}^{\Gamma}(M)$ is given by $\{f_{x_i} \mid x_i \in \mathbb{P}^1(\mathcal{O}/\mathfrak{a})\}$ where $f_{x_i}(x_j) = 1$ if $i = j$ and 0 otherwise. The action of Γ on a basis element f_{x_i} is

$$gf_{x_i} = f_{x_j} \text{ if } \exists\, h \in \Gamma_0(\mathfrak{a}) \text{ so that } g^{-1}x_i = x_j h$$

Proposition 2.2.1.

$$dim_{\mathbb{Q}}(H^2(\Gamma_0(3), \mathbb{Q})) = 0.$$

Proof. A set of representatives for the quotient $\mathcal{O}/(3)$ is given by

$$\{0, 1, i, 2, 1+i, 2+i, 2i, 1+2i, 2+2i\}.$$

Therefore,

$$\mathbb{P}^1(\mathcal{O}/(3)) = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} 2+i \\ 1 \end{bmatrix}, \begin{bmatrix} 2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}, \begin{bmatrix} 2+2i \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Let γ_i denote the generator of the stabilizer Γ_{ij} of the 1-cell P_{ij}. That is, define

$$\gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}, \gamma_4 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Note that $\gamma_1 = a, \gamma_2 = b, \gamma_3 = c$ and $\gamma_4 = ce^3$.

For $i = 1, \cdots, 4$, we shall calculate the orbit of each element in $\mathbb{P}^1(\mathcal{O}/(3))$ under γ_i.

Observe that if $x \in \mathcal{O}/(3)$, then

$$\begin{aligned} \gamma_1^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} &= \begin{bmatrix} -i \\ -ix \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ x \end{bmatrix} \\ &= \begin{bmatrix} x^{-1} \\ 1 \end{bmatrix}. \end{aligned}$$

For simplicity, let x denote the element $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathbb{P}^1(\mathcal{O}/(3))$, and write ∞ for $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. The set of orbits under the action of γ_1 is:

$$\{\{0, \infty\}, \{1\}, \{2\}, \{i, 2i\}, \{1+i, 2+i\}, \{1+2i, 2+2i\}\}.$$

Let us write M for $\mathrm{Ind}_{\Gamma_0(3)}^{\Gamma}(\mathbb{Q})$ and f_x for the basis element in M which satisfies $f_x(y) = 1$ if $x = y \in \mathbb{P}^1(\mathcal{O}/(3))$. Then

$$M^{\Gamma_1} = \mathrm{span}_{\mathbb{Q}}\{f_0 + f_\infty, f_1, f_2, f_i + f_{2i}, f_{1+i} + f_{2+i}, f_{1+2i} + f_{2+2i}\}$$

Similarly, the set of orbits under the action of γ_2 is

$$\{\{0, \infty\}, \{1, 2\}, \{i\}, \{2i\}, \{1+i, 1+2i\}, \{2+2i, 2+i\}\},$$

hence

$$M^{\Gamma_2} = \text{span}_{\mathbb{Q}}\{f_0 + f_\infty, f_1 + f_2, f_i, f_{2i}, f_{1+i} + f_{1+2i}, f_{2+i} + f_{2+2i}\}.$$

The set of orbits under the action of γ_3 is

$$\{\{0, \infty, i\}, \{1, 2+2i, 1+i\}, \{2+i, 1+2i, 2\}, \{2i\}\},$$

hence

$$M^{\Gamma_3} = \text{span}_{\mathbb{Q}}\{f_0 + f_\infty + f_i, f_1 + f_{2+2i} + f_{1+i}, f_{2+i} + f_{1+2i} + f_2\}.$$

The set of orbits under the action of γ_4 is

$$\{\{0, \infty, 1\}, \{2\}, \{i, 2+2i, 1+i\}, \{2+i, 1+2i, 2i\}\},$$

thus

$$M^{\Gamma_4} = \text{span}_{\mathbb{Q}}\{f_0 + f_\infty + f_1, f_2, f_i + f_{2+2i} + f_{1+i}, f_{2+i} + f_{1+2i} + f_{2i}\}.$$

Using row reduction, we find easily that

$$\begin{aligned} M^{\Gamma_1} + M^{\Gamma_2} + M^{\Gamma_3} + M^{\Gamma_4} =& \text{span}_{\mathbb{Q}}\{f_0 + f_1 + f_\infty, f_2, f_{1+i} + f_{2+2i}, f_{2+i} + f_{1+2i}, f_{2i}, f_i, \\ & f_{1+i} + f_{2+i}, f_{1+2i} + f_{2+2i}, f_{1+i} + f_{1+2i}, f_{2+i} + f_{2+2i}\} \end{aligned}$$

Note that $-1 \in \Gamma$ acts trivially on $\mathbb{P}^1(\mathcal{O}/(3))$ and hence on M. The space $H^2(\Gamma_0(3), \mathbb{Q})$ is therefore given by the quotient (2.7). Since $\dim_{\mathbb{Q}}(M^{\Gamma_1} + M^{\Gamma_2} + M^{\Gamma_3} + M^{\Gamma_4}) = \dim_{\mathbb{Q}}(M) = 10$, it follows that

$$\dim_{\mathbb{Q}}(M/(M^{\Gamma_1} + M^{\Gamma_2} + M^{\Gamma_3} + M^{\Gamma_4})) = 0.$$

□

Our result is compatible with that of Adem and Naffah, who calculate that

$$H^2(\Gamma_0(3), \mathbb{Z}) \cong \mathbb{Z}/6.$$

The following table summarises the range of our calculations:

$\mathfrak{a}$	$\dim_{\mathbb{Q}}(H^2(\Gamma_0(\mathfrak{a}),\mathbb{Q}))$	$\dim_{\mathbb{Q}}(H^2(\Gamma_1(\mathfrak{a}),\mathbb{Q}))$
$(1+i)$	1	
$(1+i)^2$	2	
$(1+i)^3$	3	
(3)	0	
$(1+2i)$	1	12
$(2+i)$	1	
$(1+4i)$	7	
$(3+2i)$	1	

In [12], Şengün calculates the rank of a certain subspace of $H^2(\Gamma_0(\mathfrak{a}),\mathcal{O})$ for many ideals $\mathfrak{a}$. The rank of $H^2(\Gamma_0(\mathfrak{a}),\mathcal{O})$ is equal to the dimension of $H^2(\Gamma_0(\mathfrak{a}),\mathbb{Q})$. Şengün reports that approximately ninety percent of the time, the rank of this subspace is 0. Our results above are compatible with his in so far that the rank of his subspace is never larger than 7. See Remark 2.4.2 below.

Consider the congruence subgroup $\Gamma' = \Gamma(4)SL_2(\mathbb{Z})$. Note that $\Gamma' \subset \Gamma$, but we used the same symbol for the subgroup $\Gamma(4)G(\mathbb{Z}) \subset G(\mathcal{O})$ in Chapter 1. Recall (Proposition 1.2.7) the 1-dimensional representation of Γ' denoted $\kappa_{\mathbb{Q}}$.

Proposition 2.2.2.

$$H^2(\Gamma',\kappa_{\mathbb{Q}}) \cong \mathbb{Q}^{(5)}$$

Let $V = \mathrm{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}}) = \{f : \Gamma \to \mathbb{Q} \mid f(x\gamma') = \kappa_{\mathbb{Q}}(\gamma')f(x)\}$. Since $\dim_{\mathbb{Q}} V = [\Gamma : \Gamma'] = 64$, the calculation is carried out using Sage. The proof of Proposition 2.2.2 shall be an explanation of the Sage algorithm used. First, we need a Lemma.

If a generates the ideal $\mathfrak{a} \subset \mathcal{O}$, we shall write $\Gamma(a)$ as shorthand for $\Gamma(\mathfrak{a})$. If the ring R is not $\mathcal{O}$, we'll write $SL_2(R,b)$ for the group of matrices in $SL_2(R)$ which are congruent to the identity modulo the ideal $\mathfrak{b} \subset R$ (generated by b).

Lemma 2.2.3. *Let $\{a_i\}$ be the elements of $SL_2(\mathcal{O}/4, 1+i)/SL_2(\mathcal{O}/4, 2)$ and let $\{b_j\}$ be the elements of $SL_2(\mathcal{O}/4, 2+2i)$. Choose lifts $\{\hat{a}_i\}$ and $\{\hat{b}_j\}$ to Γ. Then*

$\{\hat{a}_i\hat{b}_j$ for $1 \leq i, j \leq 8\}$ is a set of (left and right) coset representatives for the group Γ' in Γ.

Proof. First note that

$$\Gamma/\Gamma' \cong SL_2(\mathcal{O}/4)/SL_2(\mathbb{Z}/4\mathbb{Z}). \tag{2.17}$$

(This is a set bijection, since neither side is a group).

Consider the filtration

$$SL_2(\mathcal{O}/4) \supset SL_2(\mathcal{O}/4, 1+i) \supset SL_2(\mathcal{O}/4, 2) \supset SL_2(\mathcal{O}/4, 2+2i) \supset 1,$$

and note that, for $1 \leq n \leq 3$, there is an isomorphism

$$\begin{aligned} SL_2(\mathcal{O}/4, \pi^n)/SL_2(\mathcal{O}/4, \pi^{n+1}) &\longrightarrow \mathfrak{sl}_2(\mathbb{Z}/2\mathbb{Z}) \\ 1 + \pi^n x &\longmapsto x \end{aligned}$$

where $\mathfrak{sl}_2$ is the Lie algebra of SL_2. The top quotient $SL_2(\mathcal{O}/4)/SL_2(\mathcal{O}/4, 1+i)$ is isomorphic to $SL_2(\mathbb{F}_2)$. This means that

$$|SL_2(\mathcal{O}/4)| = 6 * 8^3.$$

On the other hand, there is a filtration

$$SL_2(\mathbb{Z}/4) \supset SL_2(\mathbb{Z}/4, 2) \supset 1,$$

and a similar argument shows that

$$|SL_2(\mathbb{Z}/4)| = 6 * 8.$$

Consequently, we must find 8^2 representatives.

From above, it follows

$$SL_2(\mathcal{O}/4) = \{g'(1+\pi x')(1+2y')(1+\pi^3 z') \ : \ g \in SL_2(\mathbb{Z}/2\mathbb{Z}); x, y, z \in \mathfrak{sl}_2(\mathbb{Z}/2\mathbb{Z})\}$$

where $g' \in SL_2(\mathbb{Z}/4\mathbb{Z})$ and $g' \equiv g \pmod 2$, and $x', y', z' \in \mathfrak{sl}_2(\mathbb{Z}/4\mathbb{Z})$, such that $x \equiv x' \pmod 2, y \equiv y' \pmod 2, z \equiv z' \pmod 2$. But $(1+2y')$ and $(1+\pi^3 z')$ commute, so we have

$$SL_2(\mathcal{O}/4) = \{(1+\pi x')(1+\pi^3 z')(1+2y')g' \ : \ g \in SL_2(\mathbb{Z}/2\mathbb{Z}); x, y, z \in \mathfrak{sl}_2(\mathbb{Z}/2\mathbb{Z})\},$$

and $(1+2y')g' \in SL_2(\mathbb{Z}/4\mathbb{Z})$.

The set of 64 representatives $\{(1+\pi x')(1+\pi^3 z')\}$ cover all the cosets, so there is exactly one representative in each coset.

□

A set of lifts to Γ of representatives for the top quotient $SL_2(\mathcal{O}/4)/SL_2(\mathcal{O}/4, 1+i) = SL_2(\mathbb{Z}/2\mathbb{Z})$ is given by:

$$SL_2(\mathcal{O}/4)/SL_2(\mathcal{O}/4,1+i) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\}$$

Likewise, the second quotient $SL_2(\mathcal{O}/4, 1+i)/SL_2(\mathcal{O}/4, 2)$ is $SL_2(\mathcal{O}/2, 1+i)$ and a set of lifts of representatives is:

$$SL_2(\mathcal{O}/4,1+i)/SL_2(\mathcal{O}/4,2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix}, \begin{pmatrix} i & \pi \\ 0 & -i \end{pmatrix}, \begin{pmatrix} i & 0 \\ \pi & -i \end{pmatrix}, \begin{pmatrix} 1 & \pi \\ \pi & 1+2i \end{pmatrix}, \begin{pmatrix} i & \pi \\ \pi & 2-i \end{pmatrix} \right\}$$

The lifts of the third and fourth quotient are given respectively by:

$$SL_2(\mathcal{O}/4,2)/SL_2(\mathcal{O}/4,2+2i) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} -5 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

$$SL_2(\mathcal{O}/4,2+2i) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3-6i & 4 \\ -4 & -1-2i \end{pmatrix}, \begin{pmatrix} 1 & 2+2i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2+2i & 1 \end{pmatrix}, \begin{pmatrix} -5+2i & 4 \\ 2+2i & -1-2i \end{pmatrix}, \begin{pmatrix} -5+2i & 2+2i \\ 4 & -1-2i \end{pmatrix}, \begin{pmatrix} 1 & 2+2i \\ 2+2i & 1+8i \end{pmatrix}, \begin{pmatrix} -1+2i & 2+2i \\ 2+2i & 3-2i \end{pmatrix} \right\}$$

We have used the fact that $SL_2(\mathcal{O}/4,2)/SL_2(\mathcal{O}/4,2+2i) = SL_2(\mathcal{O}/(2+2i),2)$.

We turn now to the algorithm used to calculate $H^2(\Gamma', \kappa_{\mathbb{Q}})$. Consider the following Sage environment, which shall be the setting of all algorithms in this thesis:

```
F.<i> = NumberField(x^2+1)
R = F.ring_of_integers()
pi = F.ideal(1+i)
k = R.residue_field(pi,'b')
kk = R.quotient_ring(2,'b')
kkk = R.quotient_ring(2*pi,'b')
kkkk = R.quotient_ring(4,'b')
kkkkk = R.quotient_ring(4*pi,'b')
M = MatrixSpace(F,2)
m = MatrixSpace(k,2)
mm = MatrixSpace(kk,2)
mmm = MatrixSpace(kkk,2)
mmmm = MatrixSpace(kkkk,2)
```

Recall formula (2.7). To calculate the space $V^{\Gamma_{12}} + V^{\Gamma_{23}} + V^{\Gamma_{34}} + V^{\Gamma_{41}}$, we must determine the action of Γ on V.

For $r \in \{\hat{a}_i\hat{b}_j \text{ for } 1 \leq i,j \leq 8\}$, define

$$\delta_r(x) = \begin{cases} \kappa_{\mathbb{Q}}(h) \text{ if } x = rh,\ h \in \Gamma' \\ 0 \text{ otherwise} \end{cases}$$

Then $\{\delta_r \ : r \in \{\hat{a}_i\hat{b}_j \text{ for } 1 \leq i,j \leq 8\}\}$ forms a $\mathbb{Q}$-basis for the vector space $V = \mathrm{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}})$. If $\gamma \in \Gamma$,

$$(\gamma\delta_r)(x) = \delta_r(\gamma^{-1}x) = \begin{cases} \kappa_{\mathbb{Q}}(h) \text{ if } \gamma^{-1}x = rh,\ h \in \Gamma' \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} \kappa_{\mathbb{Q}}(h')\delta_s(x) \text{ if } x = \gamma rh = sh'h, \text{ for } h, h' \in \Gamma' \\ 0 \text{ otherwise} \end{cases}$$

It follows that $\gamma\delta_r = \kappa_{\mathbb{Q}}(h)\delta_s$ if $\gamma r = sh$. The matrix of the action of γ on V is a 64×64 matrix whose $(r,s)^{\text{th}}$ entry is $\kappa_{\mathbb{Q}}(h)$ if $\gamma r = sh$ and 0 otherwise. Clearly, we must be able to find the representative $r \in \{\hat{a}_i\hat{b}_j \text{ for } 1 \leq i,j \leq 8\}$ of any given $\gamma \in \Gamma$.

Define:

```
quotient0 = [M([1,0,0,1]),M([0,1,-1,0]),M([1,1,0,1]),M([1,0,1,1]),
M([1,-1,1,0]), M([0,-1,1,1])]

quotient1 = [M([1,0,0,1]),M([i,0,0,-i]), M([1,1+i,0,1]),
M([1,0,1+i,1]), M([i,1+i,0,-i]), M([i,0,1+i,-i]),
M([1,1+i,1+i,1+2*i]), M([i,1+i,1+i,2-i])]

quotient2 = [M([1,0,0,1]),M([1,2,0,1]), M([1,0,2,1]),
M([-1,0,0,-1]), M([-1,2,0,-1]), M([-1,0,2,-1]), M([1,2,2,5]),
M([-5,2,2,-1])]

quotient3 = [M([1,0,0,1]),M([3-6*i,4,-4,-1-2*i]), M([1,2+2*i,0,1]),
M([1,0,2+2*i,1]), M([-5+2*i,2+2*i,4,-1-2*i]),
M([-5+2*i,4,2+2*i,-1-2*i]), M([1,2+2*i,2+2*i,1+8*i]),
M([-1+2*i,2+2*i,2+2*i,3-2*i])]

quotient0m = [m([1,0,0,1]),m([0,1,1,0]),m([1,1,0,1]),m([1,0,1,1]),
m([1,1,1,0]),m([0,1,1,1])]

quotient1mm = [mm([1,0,0,1]), mm([i,0,0,-i]), mm([1,1+i,0,1]),
mm([1,0,1+i,1]), mm([i,1+i,0,-i]), mm([i,0,1+i,-i]),
mm([1,1+i,1+i,1+2*i]), mm([i,1+i,1+i,2-i])]

quotient2mmm = [mmm([1,0,0,1]), mmm([1,2,0,1]),
mmm([1,0,2,1]), mmm([-1,0,0,-1]),
 mmm([-1,2,0,-1]), mmm([-1,0,2,-1]),
mmm([1,2,2,5]), mmm([-5,2,2,-1])]
```

```
quotient3mmmm = [mmmm([1,0,0,1]), mmmm([3-6*i,4,-4,-1-2*i]),
mmmm([1,2+2*i,0,1]), mmmm([1,0,2+2*i,1]),
mmmm([-5+2*i,2+2*i,4,-1-2*i]), mmmm([-5+2*i,4,2+2*i,-1-2*i]),
mmmm([1,2+2*i,2+2*i,1+8*i]), mmmm([-1+2*i,2+2*i,2+2*i,3-2*i])]

representatives = [a*b for a in quotient1 for b in quotient3]
```

The input of the following algorithm is an arbitrary element $\gamma \in \Gamma$, and the output is the factorisation $\gamma = rep1 * rep3 * \gamma_5 * rep2 * rep0$ for $rep1 \in \Gamma(1+i)/\Gamma(2)$ (called 'quotient1' above), $rep3 \in \Gamma(2+2i)/\Gamma(4)$ (called 'quotient3' above) , $\gamma_5 \in \Gamma(4)$ and $rep2 * rep0 \in SL_2(\mathbb{Z})$.

```
def Decomposition(gamma):
    r = quotient0m.index(m(gamma))
    rep0 = quotient0[r]
    gamma2 = gamma*rep0.inverse()
    a = quotient1mm.index(mm(gamma2))
    rep1 = quotient1[a]
    gamma3 = rep1^-1*gamma2
    u = quotient2mmm.index(mmm(gamma3))
    rep2 = quotient2[u]
    gamma4 = gamma3*rep2^-1
    b = quotient3mmmm.index(mmmm(gamma4))
    rep3 = quotient3[b]
    gamma5 = rep3^-1*gamma4
    return([rep1,rep3,gamma5,rep2*rep0])
```

We must implement the character $\kappa_{\mathbb{Q}}$.

```
def residuesymbol(x,y):
    K = R.residue_field(y)
    xbar = K(x)
    answer = xbar^((norm(y)-1)/2)
```

```
        if answer == K(1):
            return 1
        elif answer == K(-1):
            return -1
        elif answer == K(0):
            return 0


    def legendresymbol(x,y):
        factors = F.factor(y)
        answer = prod([residuesymbol(x,p[0]) for p in factors if p[1]%2])
        return answer


    def kappa(A):
        c = A[1][0]
        d = A[1][1]
        return legendresymbol(c,d)
```

It is sufficient to calculate the action on V of each γ which generates the stabilizer Γ_{ij} of each 1-cell P_{ij}: that is, $\gamma_1 = a, \gamma_2 = b, \gamma_3 = c, \gamma_4 = ce^3$.

```
gammainverselist = [M([0,-i,-i,0]),M([0,1,-1,0]), M([1,-i,-i,0]),
M([1,-1,1,0])]


matrixlist = []
for kk in range(4):
    gamma = gammainverselist[kk]
    for ii in range(64):
        r = representatives[ii]
        answer = Decomposition(gamma*r)
        newrep = answer[0]*answer[1]
        jj = representatives.index(newrep)
        kappavalue = kappa(answer[2])
        matrixlist.append([ii,jj,kk,kappavalue])
```

```
def func(ii,jj,kk):
    for entry in matrixlist:
        if entry[0]==ii and entry[1]==jj and entry[2]==kk:
            return entry[3]
    return 0
```

'gammaactionK' is a list of the four 64×64 matrices which give the action of $\gamma_1, \gamma_2, \gamma_3$ and γ_4 on V.

```
gammaactionK = [Matrix([[func(ii,jj,kk) for ii in range(64)]
for jj in range(64)]) for kk in range(4)]
```

Finally, we use formula (2.7), together with Lemma 2.1.7 to calculate H^2 as the quotient $V/(V^{\Gamma_{12}} + V^{\Gamma_{23}} + V^{\Gamma_{34}} + V^{\Gamma_{41}})$:

```
V = QQ^64

kernels = [(1-gammaaction[kk]).right_kernel() for kk in range(4)]

generators = []
for W in kernels:
    generators=generators+W.basis()

subspace=V.span(generators)
H2=V.quotient(subspace)
```

The output is that 'H2' is 5-dimensional. This concludes the proof of Proposition 2.2.2.

We can regard κ as a 1-dimensional representation over $\mathbb{C}$. Indeed, define

$$\kappa_{\mathbb{C}} : \Gamma' \longrightarrow GL_1(\mathbb{Q} \underset{\mathbb{Q}}{\otimes} \mathbb{C}).$$

Since $\mathbb{C}$ is flat as a $\mathbb{Q}$-vector space,

$$H^2(\Gamma', \kappa_{\mathbb{C}}) = H^2(\Gamma', \kappa_{\mathbb{Q}} \underset{\mathbb{Q}}{\otimes} \mathbb{C}) = H^2(\Gamma', \kappa_{\mathbb{Q}}) \underset{\mathbb{Q}}{\otimes} \mathbb{C} \cong \mathbb{C}^{(5)}.$$

For convenience, we shall regard κ as a representation over $\mathbb{C}$ in the following results.

Consider the Γ-modules $E_2(\mathbb{C})$ and $E_{2,2}(\mathbb{C})$ defined at the beginning of this section. Below, we shall modify the above programme to calculate $H^2(\Gamma, E_{2,2}(\mathbb{C}))$, $H^2(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_2(\mathbb{C}))$ and $H^2(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_{2,2}(\mathbb{C}))$.

Proposition 2.2.4.

$$H^2(\Gamma, \mathit{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_2(\mathbb{C})) \cong \mathbb{C}^{(3)}$$

Recall that $E_2(\mathbb{C})$ has a $\mathbb{C}$-basis of 3 elements. Let e_i denote the matrix of the action of γ_i on $E_2(\mathbb{C})$. Then

$$e_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \qquad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$e_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 2i \\ -1 & i & 1 \end{pmatrix}; \qquad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

We define the set $\{e_i\}$ in Sage as ‘gammaactionS’:

```
E = MatrixSpace(F,3)
gammaactionS = [E([0,0,-1,0,-1,0,-1,0,0]),E([0,0,1,0,-1,0,1,0,0]),
E([0,0,-1,0,-1,2*i,-1,i,1]),E([0,0,1,0,-1,2,1,-1,1])]
```

We take the tensor product of the representations (‘gammaactionKS’ below) and calculate the quotient H^2 as above:

```
gammaactionKS = []
for ff in range(4):
    temp = gammaactionS[ff]
    for gg in range(4):
        hh = gammaactionK[gg]
        if ff==gg:
            answer = temp.tensor_product(hh)
            gammaactionKS.append(answer)
```

```
KS_space1 = MatrixSpace(F,192)
KS_space2 = F^192
KS_kernels = [KS_space1(1-gammaactionKS[p]).right_kernel()
for p in range(4)]

KS_generators = []
for W in KS_kernels:
    KS_generators=KS_generators+W.basis()

KS_subspace = KS_space2.span(KS_generators)

H2_KS = KS_space2.quotient(KS_subspace)
```

The output is that H^2 is 3-dimensional.

Next, let $\bar{e}_i$ denote the matrix of the action of γ_i on $\overline{E_2(\mathbb{C})}$. Then

$$\bar{e}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \qquad \bar{e}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\bar{e}_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & i-1 & 2i \\ 2i & i+1 & 1 \end{pmatrix}; \qquad \bar{e}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

We define the set $\{\bar{e}_i\}$ in Sage as 'gammaactionSC':

```
gammaactionSC = [E([0,0,-1,0,-1,0,-1,0,0]),E([0,0,1,0,-1,0,1,0,0]),
E([0,0,-1,0,i-1,2*i,2*i,i+1,1]),E([0,0,1,0,-1,2,1,-1,1])]
```

The following 'gammaactionSSC' is the action of Γ on the representation $E_{2,2}(\mathbb{C})$ $(= E_2(\mathbb{C}) \bigotimes_{\mathbb{C}} \overline{E_2(\mathbb{C})})$.

```
gammaactionSSC = []
for r in range(4):
    temp1 = gammaactionS[r]
    for s in range(4):
```

```
            temp2 = gammaactionSC[s]
            if r == s:
                answer = temp1.tensor_product(temp2)
                gammaactionSSC.append(answer)
```

Analogously, we denote the action of Γ on $\text{Ind}\,^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_{2,2}(\mathbb{C})$ as 'gammaactionKSSC', and thereafter the cohomology is calculated as before.

```
gammaactionKSSC = []
for r in range(4):
    temp1 = gammaactionSSC[r]
    for s in range(4):
        temp2 = gammaactionK[s]
        if r == s:
            answer = temp1.tensor_product(temp2)
            gammaactionKSSC.append(answer)
```

Proposition 2.2.5.

$$H^2(\Gamma, E_{2,2}(\mathbb{C})) \cong \mathbb{C} \quad \textit{and}$$
$$H^2(\Gamma, \textit{Ind}\,^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_{2,2}(\mathbb{C})) \cong \mathbb{C}^{(13)}$$

Remark 2.2.1. Şengün has calculated [12] that, as an $\mathcal{O}$-module, $H^2(\Gamma/\{\pm 1\}, E_{2,2}(\mathcal{O}))$ has rank 1 and contains 2-torsion. He also conjectures that $H^2(\Gamma/\{\pm 1\}, E_{k,k}(\mathcal{O}))$ contains 2-torsion except when $k = 0$.

2.3 Integral cohomology

The integral cohomology of groups is in general much harder to determine than its counterpart over a field, owing to the possible torsion in the group. The homology groups of the Bianchi groups have been completely determined [34] as has the integral ring structure [2]. In this subsection, we shall calculate

$$H^2(\Gamma, \mathbb{Z})$$

and determine, up to extension,

$$H^2(\Gamma', \kappa_{\mathbb{Z}})$$

for the integral representation

$$\kappa_{\mathbb{Z}} : \Gamma' \longrightarrow GL_1(\mathbb{Z}).$$

Recall the spectral sequence (2.3) with $M = \mathbb{Z}$. Using the integral cohomology calculations of Section 2.1, we can write down the first 3 rows of the E_1-sheet:

$$\begin{array}{l|ccccc}
q & H^2(\Gamma_1,\mathbb{Z}) \oplus \cdots \oplus H^2(\Gamma_4,\mathbb{Z}) & \overset{d_1^{0,2}}{\longrightarrow} & H^2(\Gamma_{12},\mathbb{Z}) \oplus \cdots \oplus H^2(\Gamma_{41},\mathbb{Z}) & \overset{d_1^{1,2}}{\longrightarrow} & H^2(C_2,\mathbb{Z}) \\
 & 0 & & 0 & & 0 \\
 & \mathbb{Z}^{(4)} & \overset{d_1^{0,0}}{\longrightarrow} & \mathbb{Z}^{(4)} & \overset{d_1^{1,0}}{\longrightarrow} & \mathbb{Z} \\
\hline
 & & & & & p
\end{array}$$

If P_iP_j is an edge with vertices P_i and P_j, and an orientation directed from P_i to P_j, then Γ_{ij} is a subgroup of both Γ_i and Γ_j, and there are restriction maps in group cohomology:

$$\mathrm{res}^{\Gamma_i}_{\Gamma_{ij}} : H^{\cdot}(\Gamma_i, \mathbb{Z}) \longrightarrow H^{\cdot}(\Gamma_{ij}, \mathbb{Z}) \text{ and } \mathrm{res}^{\Gamma_j}_{\Gamma_{ij}} : H^{\cdot}(\Gamma_j, \mathbb{Z}) \longrightarrow H^{\cdot}(\Gamma_{ij}, \mathbb{Z}),$$

and the map

$$\bigoplus_{0-\text{cells } P_i} H^{\cdot}(\Gamma_i, \mathbb{Z}) \longrightarrow H^{\cdot}(\Gamma_{ij}, \mathbb{Z}) \quad \text{is}$$

$$(x_1, x_2, x_3, x_4) \mapsto \mathrm{res}^{\Gamma_j}_{\Gamma_{ij}}(x_j) - \mathrm{res}^{\Gamma_i}_{\Gamma_{ij}}(x_i).$$

For ease of notation, write $H^0(\Gamma_i)$ (respectively, $H^2(\Gamma_i)$) for $H^0(\Gamma_i, \mathbb{Z})$ (respectively, $H^2(\Gamma_i, \mathbb{Z})$), and $\mathbb{Z}/m$ for $\mathbb{Z}/m\mathbb{Z}$. We shall choose an (anti-clockwise) orientation of the cell complex $\sim \backslash D'$ so that the map $d_1^{0,0}$ is given by

$$H^0(\Gamma_1) \oplus H^0(\Gamma_2) \oplus H^0(\Gamma_3) \oplus H^0(\Gamma_4) \to H^0(\Gamma_{12}) \oplus H^0(\Gamma_{23}) \oplus H^0(\Gamma_{34}) \oplus H^0(\Gamma_{41})$$

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$(w_1, w_2, w_3, w_4) \to (w_2 - w_1, w_3 - w_2, w_4 - w_3, w_1 - w_4)$$

Hence $\ker(d_1^{0,0}) = \mathbb{Z}$.

Recall, from Lemmata 2.1.4, 2.1.5 and 2.1.6, that for some i and j, an inclusion $\Gamma_{ij} \hookrightarrow \Gamma_i$ induces an isomorphism $H^2(\Gamma_i) \cong H^2(\Gamma_{ij})$. In such cases, we shall choose the isomorphism which maps the generator 1, to 1. Then $d_1^{0,2}$ is the map

$$H^2(\Gamma_1) \oplus H^2(\Gamma_2) \oplus H^2(\Gamma_3) \oplus H^2(\Gamma_4) \to H^2(\Gamma_{12}) \oplus H^2(\Gamma_{23}) \oplus H^2(\Gamma_{34}) \oplus H^2(\Gamma_{41})$$
$$(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/4 \to \mathbb{Z}/4 \oplus (\mathbb{Z}/2 \oplus \mathbb{Z}/3) \oplus (\mathbb{Z}/2 \oplus \mathbb{Z}/3) \oplus \mathbb{Z}/4$$
$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_3 - 2x_1, -x_3, x_4, x_5, -x_4, 2x_2 - x_5)$$

Hence $\ker(d_1^{0,2}) = \mathrm{span}\{(1,1,2,0,2), (0,1,0,0,2)\}$. Both generators of the kernel have order 2, and they are linearly independent, so $\ker(d_1^{0,2}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

We shall need the following restriction maps:

Lemma 2.3.1.

The inclusion $C_2 \hookrightarrow C_4$ induces the map $H^2(C_4, \mathbb{Z}) \to H^2(C_2, \mathbb{Z}),\ 1 \mapsto 1$.

The inclusion $C_2 \hookrightarrow C_6$ induces the map $H^2(C_6, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to H^2(C_2, \mathbb{Z})$,

$$(1,0) \mapsto 1, (0,1) \mapsto 0.$$

Proof. In both cases, the Hochschild-Serre spectral sequences stabilize at the E_2 sheet, giving, in the first case, an exact sequence:

$$0 \longrightarrow H^2(C_2, \mathbb{Z}) \longrightarrow H^2(C_4, \mathbb{Z}) \xrightarrow{\text{res}} H^2(C_2, \mathbb{Z}) \longrightarrow 0,$$

and in the second case, an exact sequence:

$$0 \longrightarrow H^2(C_3, \mathbb{Z}) \longrightarrow H^2(C_6, \mathbb{Z}) \xrightarrow{\text{res}} H^2(C_2, \mathbb{Z}) \longrightarrow 0.$$

□

Therefore, $d_1^{1,2}$ is the map

$$H^2(\Gamma_{12}) \oplus H^2(\Gamma_{23}) \oplus H^2(\Gamma_{34}) \oplus H^2(\Gamma_{41}) \to H^2(C_2)$$
$$(y_1, y_2, y_3, y_4, y_5, y_6) \mapsto y_1 + y_2 + y_4 + y_6.$$

We have $\ker(d_1^{1,2}) = \{(y_1, y_2, y_4, y_6) \ :\ y_1 + y_2 + y_4 + y_6 \text{ is even } \} \oplus (\mathbb{Z}/3)^2$. The set $\{(y_1, y_2, y_4, y_6) \ :\ y_1 + y_2 + y_4 + y_6 \text{ is even } \}$ is generated by

$$(1,1,0,0), (0,0,1,1), (0,1,1,0).$$

The first and second elements are 4-torsion, and the third element is 2-torsion. Hence,

$$\ker(d_1^{1,2}) = (\mathbb{Z}/4)^2 \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2.$$

It is clear that $\operatorname{Im}(d_1^{1,2}) = \mathbb{Z}/2$.

The $E_2 = E_\infty$ sheet of the spectral sequence has first 3 rows:

q

$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$((\mathbb{Z}/4)^2 \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2)/\operatorname{Im}(d_1^{0,2})$	0
0	0	0
$\mathbb{Z}$	0	0

p

Hence,

Proposition 2.3.2.

$$H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Next we turn our attention to the calculation of $H^2(\Gamma', \kappa_{\mathbb{Z}})$. For this, we need

Proposition 2.3.3. *For all* 1*-cells* P_iP_j*,*

$$H^1(\Gamma_{ij}, \mathit{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) = 0$$

Proof. Fix i, j (i.e. P_iP_j) and consider $\operatorname{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})$ as a representation of Γ_{ij}. There is a Γ_{ij} - isomorphism,

$$\operatorname{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}}) \cong \bigoplus_{\Gamma_{ij}x\Gamma'} M_x$$

where the sum is taken over double cosets $\Gamma_{ij}x\Gamma' \subset \Gamma$ and $M_x := \{f : \Gamma_{ij}x\Gamma' \to \mathbb{Z} \mid f(x\gamma') = \kappa_{\mathbb{Z}}(\gamma')f(x)\}$.

For each $x \in \Gamma_{ij}\backslash\Gamma/\Gamma'$, let $\Gamma_{ij,x} = \{\gamma \in \Gamma_{ij} \mid \gamma x\Gamma' = x\Gamma'\}$. Then $M_x = \operatorname{Ind}_{\Gamma_{ij,x}}^{\Gamma_{ij}}(\kappa_{\mathbb{Z}} \circ \operatorname{ad}(x))$ where $\kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)$ is the representation of $\Gamma_{ij,x}$ given by $\kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)(\gamma) = \kappa_{\mathbb{Z}}(x^{-1}\gamma x)$ when $\gamma \in \Gamma_{ij,x}$. Then by Lemma 2.1.7,

$$H^1(\Gamma_{ij}, \operatorname{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) = \bigoplus_{x \in \Gamma_{ij}\backslash\Gamma/\Gamma'} H^1(\Gamma_{ij,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \tag{2.18}$$

If $\kappa(x^{-1}\gamma x) = 1$ for each double coset representative x and each $\gamma \in \Gamma_{ij,x}$, then

$$H^1(\Gamma_{ij,x}, \kappa_{\mathbb{Z}} \circ \mathrm{ad}(x)) = H^1(\Gamma_{ij,x}, \mathbb{Z})$$

and one can check that indeed this is the case. It remains to observe that $H^1(G, \mathbb{Z}) = 0$ for all finite groups G. □

A near identical proof of Proposition 2.3.3 works for the 0-cells and for the 2-cell. That is, for all 0-cells P_i, and for the stabilizer of the 2-cell $\{\pm 1\}$,

$$H^1(\Gamma_i, \mathrm{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) = H^1(\{\pm 1\}, \mathrm{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) = 0.$$

Hence,

Corollary 2.3.4. *The map*

$$H^2(\sim \backslash D', \mathit{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) \to H^2(\Gamma, \mathit{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}}))$$

is injective.

Proof. Recall the spectral sequence (2.3) with $M := \mathrm{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})$. On the E_1 sheet, the cohomology of the bottom row $E_1^{\cdot,0}$ gives the cohomology of the cell complex $\sim \backslash D'$. By Proposition 2.3.3, the first three columns of the row $E_1^{p,1}$ are 0. The E_2 sheet therefore takes the form

$$\begin{array}{c|lll}
q & & & \\
 & Y & & \\
 & 0 & 0 & 0 \\
 & H^0(\sim \backslash D', M) & H^1(\sim \backslash D', M) & H^2(\sim \backslash D', M) \\
\hline
 & & & p
\end{array}$$

The exact sequence of low-dimensional terms reads

$$0 \longrightarrow H^2(\sim \backslash D', M) \longrightarrow H^2(\Gamma, M) \longrightarrow Y \longrightarrow 0$$

and the result follows. □

Remark 2.3.1.

$$Y := \ker \left\{ d_1^{0,2} : \bigoplus_{0-\text{cell } P_i} H^2(\Gamma_i, \text{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) \to \bigoplus_{1-\text{cell } P_iP_j} H^2(\Gamma_{ij}, \text{Ind}\,_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) \right\}$$

We have shown that $H^2(\Gamma', \kappa_{\mathbb{Z}})$ is an extension

$$0 \longrightarrow H^2(\sim \backslash D', \kappa_{\mathbb{Z}}) \longrightarrow H^2(\Gamma', \kappa_{\mathbb{Z}}) \longrightarrow Y \longrightarrow 0 \tag{2.19}$$

Our next task is to understand the map $d_1^{0,2}$. To this end, we shall use the decomposition (2.18). We require a precise list of the groups $\Gamma_{ij,x}$ for all 1-cells P_{ij} and all double cosets $x_k \in \Gamma_{ij} \backslash \Gamma / \Gamma'$. For $0 \leq k \leq 63$, let $\{x_k\}$ denote the (ordered) set of representatives Γ / Γ'.

	$\lvert\Gamma_{ij} \backslash \Gamma / \Gamma'\rvert$	Equivalent reps	$\Gamma_{ij,x}$
P_{12}	63	$0 \sim 8$	$\Gamma_{12,x_k} = C_2$ for all k
P_{23}	62	$2 \sim 3 \sim 7$	$\Gamma_{23,x_k} = \begin{cases} C_6 \text{ for } k = 0,1,8,9 \\ C_2 \text{ otherwise} \end{cases}$
P_{34}	62	$0 \sim 32 \sim 40$	$\Gamma_{34,x_k} = \begin{cases} C_4 \text{ for } k = 54,55,58,60 \\ C_2 \text{ otherwise} \end{cases}$
P_{41}	63	$2 \sim 3$	$\Gamma_{41,x_k} = \begin{cases} C_4 \text{ for } k = 0,1,6,7,8,9,14,15 \\ C_2 \text{ otherwise} \end{cases}$

Then,

$$\begin{aligned}
H^*(\Gamma_{12}, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x \in \Gamma_{12} \backslash \Gamma / \Gamma'} H^*(\Gamma_{12,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_2, \mathbb{Z})^{(63)} \\
H^*(\Gamma_{23}, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x \in \Gamma_{23} \backslash \Gamma / \Gamma'} H^*(\Gamma_{12,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_6, \mathbb{Z})^{(4)} \oplus H^*(C_2, \mathbb{Z})^{(58)} \\
H^*(\Gamma_{34}, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x \in \Gamma_{34} \backslash \Gamma / \Gamma'} H^*(\Gamma_{34,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_4, \mathbb{Z})^{(4)} \oplus H^*(C_2, \mathbb{Z})^{(58)} \\
H^*(\Gamma_{41}, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x \in \Gamma_{41} \backslash \Gamma / \Gamma'} H^*(\Gamma_{41,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_4, \mathbb{Z})^{(8)} \oplus H^*(C_2, \mathbb{Z})^{(55)}
\end{aligned}$$

We do the same for the 0-cells.

	$\|\Gamma_i\backslash\Gamma/\Gamma'\|$	Equivalent reps	$\Gamma_{i,x}$
P_1	62	$0\sim 8, 2\sim 3$	$\Gamma_{1,x_k} = \begin{cases} C_4 \text{ for } k = 0,1,6,7,9, \\ \qquad 14,15 \\ C_2 \qquad \text{otherwise} \end{cases}$
P_2	61	$0\sim 8, 2\sim 3\sim 7$	$\Gamma_{2,x_k} = \begin{cases} C_6 \text{ for } k = 0,1,9 \\ C_2 \qquad \text{otherwise} \end{cases}$
P_3	59	$0\sim 32\sim 40\sim 55, 2\sim 3\sim 7$	$\Gamma_{3,x_k} = \begin{cases} C_6 \text{ for } k = 0,1,8,9,17, \\ \qquad 27,29,33 \\ C_4 \text{ for } k = 54,60 \\ C_2 \qquad \text{otherwise} \end{cases}$
P_4	61	$0\sim 32\sim 40, 2\sim 3$	$\Gamma_{4,x_k} = \begin{cases} C_4 \text{ for } k = 0,1,6,7,8,9, \\ \qquad 14,15,16,17, \\ \qquad 19,21,27,29, \\ \qquad 30,31,33,35, \\ \qquad 37,41,42,44 \\ C_2 \qquad \text{otherwise} \end{cases}$

$$
\begin{aligned}
H^*(\Gamma_1, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x\in\Gamma_1\backslash\Gamma/\Gamma'} H^*(\Gamma_{1,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_4,\mathbb{Z})^{(7)} \oplus H^*(C_2,\mathbb{Z})^{(55)} \\
H^*(\Gamma_2, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x\in\Gamma_2\backslash\Gamma/\Gamma'} H^*(\Gamma_{2,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_6,\mathbb{Z})^{(3)} \oplus H^*(C_2,\mathbb{Z})^{(58)} \\
H^*(\Gamma_3, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x\in\Gamma_3\backslash\Gamma/\Gamma'} H^*(\Gamma_{3,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_6,\mathbb{Z})^{(8)} \oplus H^*(C_4,\mathbb{Z})^{(2)} \oplus H^*(C_2,\mathbb{Z})^{(49)} \\
H^*(\Gamma_4, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &= \bigoplus_{x\in\Gamma_4\backslash\Gamma/\Gamma'} H^*(\Gamma_{4,x}, \kappa_{\mathbb{Z}} \circ \operatorname{ad}(x)) \\
&= H^*(C_4,\mathbb{Z})^{(22)} \oplus H^*(C_2,\mathbb{Z})^{(39)}
\end{aligned}
$$

In particular,

$$
\begin{aligned}
E_1^{0,2} &= \bigoplus_{0-\text{cell } P_i} H^2(\Gamma_i, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &&\cong \mathbb{Z}/2\mathbb{Z}^{(212)} \oplus \mathbb{Z}/3\mathbb{Z}^{(11)} \oplus \mathbb{Z}/4\mathbb{Z}^{(31)} \\
E_1^{1,2} &= \bigoplus_{1-\text{cell } P_{ij}} H^2(\Gamma_{ij}, \operatorname{Ind}{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Z}})) &&\cong \mathbb{Z}/2\mathbb{Z}^{(238)} \oplus \mathbb{Z}/3\mathbb{Z}^{(4)} \oplus \mathbb{Z}/4\mathbb{Z}^{(12)}
\end{aligned}
$$

Observe that, for fixed i,j and $x \in \Gamma_i\backslash\Gamma/\Gamma'$, either $\Gamma_i x\Gamma' = \Gamma_{ij}x\Gamma'$, or $\Gamma_i x\Gamma' = \bigcup_y \Gamma_{ij}y\Gamma'$, for $y \in \Gamma_{ij}\backslash\Gamma_i x\Gamma'$, and indeed, this information is given in the tables above. For example, since $\Gamma_2 x_2\Gamma' = \Gamma_2 x_3\Gamma' = \Gamma_2 x_7\Gamma'$, but $x_2\Gamma', x_3\Gamma'$ and $x_7\Gamma'$ are not equivalent under Γ_{12}, it follows that $\Gamma_2 x_2\Gamma' = \Gamma_{12}x_2\Gamma' \cup \Gamma_{12}x_3\Gamma' \cup \Gamma_{12}x_7\Gamma'$. For fixed x and y, the map

$$\operatorname{res}^{i,x}_{ij,y} : H^{\cdot}(\Gamma_{i,x},\mathbb{Z}) \longrightarrow H^{\cdot}(\Gamma_{ij,y},\mathbb{Z})$$

is the restriction map in group cohomology, and if the vertex i has edge ij entering it, and edge il leaving it, then, again for fixed x, the map

$$
\begin{aligned}
H^{\cdot}(\Gamma_{i,x},\mathbb{Z}) \longrightarrow \bigoplus_{y_1\in\Gamma_{ij}\backslash\Gamma_i x\Gamma'} H^{\cdot}(\Gamma_{ij,y_1},\mathbb{Z}) \;\oplus \bigoplus_{y_2\in\Gamma_{il}\backslash\Gamma_i x\Gamma'} H^{\cdot}(\Gamma_{il,y_2},\mathbb{Z}) \text{ is} \\
z \mapsto \left(\operatorname{res}^{i,x}_{ij,y_1}(z), -\operatorname{res}^{i,x}_{il,y_2}(z)\right).
\end{aligned}
$$

Summing over all 0 and 1-cells, we have the map:

$$d_1^{0,2} : \bigoplus_{\tau \in (\sim \backslash D')_0} \bigoplus_{x \in \Gamma_\tau \backslash \Gamma / \Gamma'} H^2(\Gamma_{\tau,x}, \mathbb{Z}) \longrightarrow \bigoplus_{\delta \in (\sim \backslash D')_1} \bigoplus_{y \in \Gamma_\delta \backslash \Gamma / \Gamma'} H^2(\Gamma_{\delta,y}, \mathbb{Z})$$

We shall write $d_1^{0,2}$ in a more convenient form. Consider the following lattices Λ_i and Λ_{ij}:

$$\begin{aligned} \Lambda_1 &:= (4\mathbb{Z})^{(7)} \oplus (2\mathbb{Z})^{(55)} & \Lambda_{12} &:= (2\mathbb{Z})^{(63)} \\ \Lambda_2 &:= (6\mathbb{Z})^{(3)} \oplus (2\mathbb{Z})^{(58)} & \Lambda_{23} &:= (6\mathbb{Z})^{(4)} \oplus (2\mathbb{Z})^{(58)} \\ \Lambda_3 &:= (6\mathbb{Z})^{(8)} \oplus (2\mathbb{Z})^{(45)} \oplus (4\mathbb{Z})^{(2)} \oplus (2\mathbb{Z})^{(4)} & \Lambda_{34} &:= (2\mathbb{Z})^{(53)} \oplus (4\mathbb{Z})^{(4)} \oplus (2\mathbb{Z})^{(5)} \\ \Lambda_4 &:= (4\mathbb{Z})^{(22)} \oplus (2\mathbb{Z})^{(39)} & \Lambda_{41} &:= (4\mathbb{Z})^{(8)} \oplus (2\mathbb{Z})^{(55)} \end{aligned}$$

Define:

$$\Lambda_\bullet = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus \Lambda_4 \text{ and } \Lambda_{\bullet\bullet} = \Lambda_{12} \oplus \Lambda_{23} \oplus \Lambda_{34} \oplus \Lambda_{41}$$

Then,

$$d_1^{0,2} : \mathbb{Z}^{(243)}/\Lambda_\bullet \longrightarrow \mathbb{Z}^{(250)}/\Lambda_{\bullet\bullet}$$

Consider the map $d_1^{0,2} - \Lambda_{\bullet\bullet} : \mathbb{Z}^{(243)} \oplus \mathbb{Z}^{(250)} \to \mathbb{Z}^{(250)}$, and if $W \subset \mathbb{Z}^{(243)} \oplus \mathbb{Z}^{(250)}$, let $\mathrm{pr}_{\mathbb{Z}^{(243)}}(W)$ denote the projection of W onto the space $\mathbb{Z}^{(243)}$. We have

$$\ker(d_1^{0,2}) = \mathrm{pr}_{\mathbb{Z}^{(243)}}(\ker\ (d_1^{0,2} - \Lambda_{\bullet\bullet}))/\mathrm{Im}\ \Lambda_\bullet$$

Recall the underlying Sage environment given in Subsection 2.2. We use the following component maps to determine the kernel of the map $d_1^{0,2}$:

$$\begin{aligned} A &: \mathbb{Z}^{(62)}/\Lambda_1 \to \mathbb{Z}^{(63)}/\Lambda_{12} & B &: \mathbb{Z}^{(61)}/\Lambda_2 \to \mathbb{Z}^{(63)}/\Lambda_{12} \\ C &: \mathbb{Z}^{(61)}/\Lambda_2 \to \mathbb{Z}^{(62)}/\Lambda_{23} & D &: \mathbb{Z}^{(59)}/\Lambda_3 \to \mathbb{Z}^{(62)}/\Lambda_{23} \\ E &: \mathbb{Z}^{(59)}/\Lambda_3 \to \mathbb{Z}^{(62)}/\Lambda_{34} & FF &: \mathbb{Z}^{(61)}/\Lambda_4 \to \mathbb{Z}^{(62)}/\Lambda_{34} \\ G &: \mathbb{Z}^{(61)}/\Lambda_4 \to \mathbb{Z}^{(63)}/\Lambda_{41} & H &: \mathbb{Z}^{(62)}/\Lambda_1 \to \mathbb{Z}^{(63)}/\Lambda_{41} \end{aligned}$$

We write the same letter (A,B,C,D,E,FF,G,H) for the corresponding matrix. The matrix of $d_1^{0,2}$ takes the form

$$\begin{pmatrix} A & B & 0 & 0 \\ 0 & C & D & 0 \\ 0 & 0 & E & FF \\ G & 0 & 0 & H \end{pmatrix}$$

Proposition 2.3.5. *There is a short exact sequence*

$$0 \longrightarrow \mathbb{Z}^{(5)} \longrightarrow H^2(\Gamma', \kappa_{\mathbb{Z}}) \longrightarrow (\mathbb{Z}/2)^{(35)} \oplus (\mathbb{Z}/4)^{(16)} \oplus (\mathbb{Z}/12)^{(9)} \longrightarrow 0$$

Proof. Recall the exact sequence (2.19). A very slight modification of the Sage programme (Appendix A.1) used to calculate $H^2(\Gamma', \kappa_{\mathbb{Q}})$ in the proof of Proposition 2.2.2 gives

$$H^2(\sim \backslash D', \kappa_{\mathbb{Z}}) \cong \mathbb{Z}^{(5)}.$$

The programme described above yields

$$\ker(d_1^{0,2}) \cong (\mathbb{Z}/2)^{(35)} \oplus (\mathbb{Z}/4)^{(16)} \oplus (\mathbb{Z}/12)^{(9)}.$$

□

2.4 Cuspidal cohomology

Suppose that $\Upsilon \subseteq \Gamma$ is a subgroup of finite index. It is well known [22] that the quotient 3-fold $Y_\Upsilon := \Upsilon \backslash \mathbb{H}$ is never compact. We can compactify Y_Υ by adding a boundary component at each cusp in such a way that the compactification is a homotopy equivalence. If c is a cusp of Υ, then the subgroup Υ_c, which stabilizes the cusp c, has an action on the complex numbers by translations and rotations. The boundary component for the cusp c is the quotient $\Upsilon_c \backslash \mathbb{C}$. This kind of compactification is known as the Borel-Serre method. Let X_Υ denote the compactification of Y_Υ.

Let $\mathbb{P}^1(F)$ denote the projective line over F. The group Γ acts naturally on $F^{(2)}$ and hence on $\mathbb{P}^1(F)$. It is a classical result which was first observed by Bianchi and proved by Hurwitz, in 1892, that if F_{-d} is an imaginary quadratic field, then the size of the orbit space $\Gamma\backslash\mathbb{P}^1(F_{-d})$ is equal to the class number of F_{-d}. Hence the set $\Upsilon\backslash\mathbb{P}^1(F)$ is finite, and called the set of *cusps* of Υ. The number of boundary components forming ∂X_Υ is indexed by the cusps of Υ. Following the notation of [12], for an element $D \in \mathbb{P}^1(F)$, let B_D denote its stabilizer in $SL_2(F)$. If c is a cusp of Υ, choose a representative $D_c \in F^{(2)}$, and define the groups

$$\Upsilon_c = B_{D_c} \cap \Upsilon$$

and

$$U(\Upsilon) = \bigoplus_{c \in \Upsilon\backslash\mathbb{P}^1(F)} \Upsilon_c \tag{2.20}$$

They are independent of the chosen representatives D_c. In general, $\Upsilon_c/\{\pm 1\}$ will not be torsion-free. If it is, it is free abelian of rank 2 [35, p. 507].

Consider the long exact sequence in compactly-supported group cohomology associated to the pair $(\Upsilon, U(\Upsilon))$:

$$\cdots \to H^i_c(\Upsilon, M) \to H^i(\Upsilon, M) \to H^i(U(\Upsilon), M) \to H^{i+1}_c(\Upsilon, M) \to \cdots \tag{2.21}$$

where $H^i_c(\Upsilon, M) = H^i(\Upsilon, U(\Upsilon), M)$, and M is any Υ-module.

Definition 2.4.1. *The cuspidal cohomology*[1]*,* $H^2_{cusp}(\Upsilon, M)$*, is defined to be the kernel of the restriction map* $H^2(\Upsilon, M) \to H^2(U(\Upsilon), M)$*, or equivalently, is defined to be the image of the map* $H^2_c(\Upsilon, M) \to H^2(\Upsilon, M)$.

Definition 2.4.2. *The Eisenstein cohomology,* $H^2_{Eis}(\Upsilon, M)$*, is defined to be the quotient* $H^2(\Upsilon, M)/H^2_{cusp}(\Upsilon, M)$ *and is isomorphic to the image of the restriction map* $H^2(\Upsilon, M) \to H^2(U(\Upsilon), M)$.

Sequence (2.21) has a natural analogue in topology, and under favourable conditions - rather than being an analogue - it is exactly the same. The analogue is the long exact sequence in relative cohomology associated to the pair $(X_\Upsilon, \partial X_\Upsilon)$:

$$\cdots \to H^i_c(X_\Upsilon, \mathcal{M}) \to H^i(X_\Upsilon, \mathcal{M}) \to H^i(\partial X_\Upsilon, \mathcal{M}) \to H^{i+1}_c(X_\Upsilon, \mathcal{M}) \to \cdots \tag{2.22}$$

where, again, $H^i_c(X_\Upsilon, M) = H^i(X_\Upsilon, \partial X_\Upsilon, M)$, and $\mathcal{M}$ is the local system on X_Υ induced by the representation M. Recall that by construction, the embedding $Y_\Upsilon \hookrightarrow$

[1] *See Appendix A.3 for why this definition is equivalent to the definition of cusp cohomology as the image of the map* (0.5) *given in the Introduction.*

X_Υ is a homotopy invariance, and there is a spectral sequence (2.3) relating the cohomology of Y_Υ (and thus of X_Υ) to the cohomology of Υ. In particular, if 6 is invertible in M, then

$$H^i(X_\Upsilon, \mathcal{M}) \cong H^i(Y_\Upsilon, \mathcal{M}) \cong H^i(\Upsilon, M).$$

If, for each cusp c of Υ, we knew that $\Upsilon_c/\{\pm 1\}$ were torsion-free, then the boundary component S_c in ∂X_Υ associated to c would be an Eilenberg-Maclane space for $\Upsilon_c/\{\pm 1\}$, and we would be able to make the correspondence

$$H^i(\partial X_\Upsilon, \mathcal{M}) = \bigoplus_c H^i(S_c, \mathcal{M}) \cong \bigoplus_c H^i(\Upsilon_c/\{\pm 1\}, M) = H^i(U(\Upsilon), M)$$

as long as 2 is invertible in M. In the general case, each Υ_c acts with fixed points on the space S_c and the spectral sequence (2.3) relates the cohomology of one with the other. In the sequel, we shall assume that as well as 6, the size of the stabilizer subgroups $\Upsilon_{c,\delta}$ of cells δ in S_c are invertible in the coefficients M, in order to guarantee that the exact sequences (2.21) and (2.22) will be the same. In particular, we shall make use of the isomorphism

$$H^i(\partial X_\Upsilon, \mathcal{M}) \cong H^i(U(\Upsilon), M) \text{ for all } i \geq 0$$

so that

$$H^2_{\text{cusp}}(\Upsilon, M) \cong \ker\{H^2(X_\Upsilon, \mathcal{M}) \longrightarrow H^2(\partial X_\Upsilon, \mathcal{M})\}. \tag{2.23}$$

2.4.1 Level one

Since the class number of F is one, Γ has one cusp; we can choose the representative of this unique cusp $c \in \Gamma\backslash\mathbb{P}^1(F)$ to be $\infty := \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Then

$$\Gamma_\infty = B_\infty \cap \Gamma = \left\{ \begin{pmatrix} i^a & b \\ 0 & i^{-a} \end{pmatrix} : a \in \mathbb{Z}, b \in \mathcal{O} \right\}$$

In the proof of Lemma 4 in the paper mentioned above [22], it is shown that, in this case,

$$X_\Gamma = Y_\Gamma \cup \Gamma_\infty\backslash\mathbb{C}$$

That is,

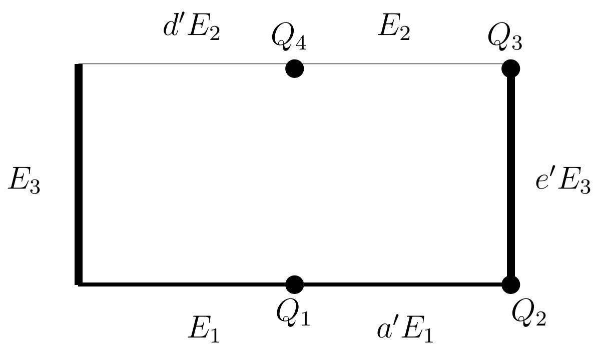
$$\partial X_\Gamma = \Gamma_\infty\backslash\mathbb{C} \tag{2.24}$$

and $\Gamma_\infty \backslash \mathbb{C}$ is a 2-sphere, which we shall label S_∞.

Let $\mathbb{H}_r = \{(z, r) \mid z \in \mathbb{C}\}$ be a plane in $\mathbb{H}$, and identify it with $\mathbb{C}$. The group Γ_∞ acts on $\mathbb{H}_r$ and Bianchi-Humbert theory [16] gives the following rectangle for a fundamental domain:

$$\left\{ z \in \mathbb{C} \mid 0 \leq |\text{Re } z|_{\mathbb{C}} \leq \frac{1}{2},\ 0 \leq \text{Im } z \leq \frac{1}{2} \right\}$$

In particular, Γ_∞ acts (with fixed points) on $\mathbb{H}_\infty = \partial(\mathbb{H} \cup \mathbb{P}^1(\mathbb{C}))$ and a fundamental *cellular* domain is given by S_∞:

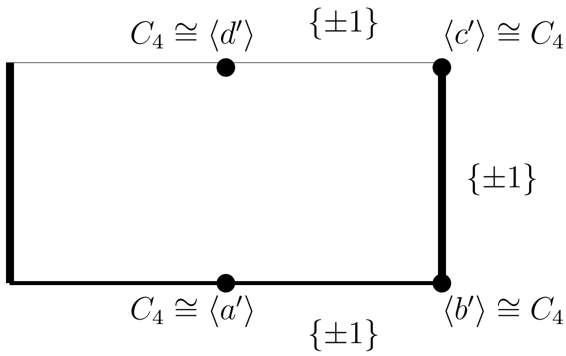


where,

$$a' := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, b' := \begin{pmatrix} i & -i \\ 0 & -i \end{pmatrix}, c' := \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix}, d' := \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}, e' := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Up to Γ_∞-equivalence, there are four 0-cells, three 1-cells and one 2-cell. The 4 inequivalent vertices (shown on the diagram as dots) are given by $Q_1 := (0, \infty), Q_2 := (\frac{1}{2}, \infty), Q_3 := (\frac{1}{2} + \frac{i}{2}, \infty), Q_4 := (\frac{i}{2}, \infty)$. The two horizontal edges along the top, E_2 and $d'E_2$, are identified via d' (which fixes the vertex in the middle) and thus have opposite orientation. The far left vertical edge E_3 is identified with the far right vertical edge $e'E_3$, and thus these edges have the same orientation. The two horizontal edges along the bottom, E_1 and $a'E_1$, are identified via a' (which fixes the vertex in the middle) and thus have opposite orientation.

The picture below shows the stabilizers of the cells. Note that the stabilizer of the 2-cell is $\{\pm 1\}$.



Let M be an $R[\Gamma]$-module (hence an $R[\Gamma_\infty]$-module) for some commutative ring R in which 2 and 3 are invertible. We shall make the additional assumption that $-\mathrm{Id}$ acts trivially on M. We can use the spectral sequence

$$E_1^{p,q}(M) = \bigoplus_{\delta \in (S_\infty)_p} H^q(\Gamma_{\infty,\delta}, M) \Rightarrow H^{p+q}(\Gamma_\infty, M) \tag{2.25}$$

to calculate $H^2(\Gamma_\infty, M)$. In this case, $E_1^{p,q}(M) = 0$ for $q > 0$ and the cohomology of Γ_∞ is given by the cohomology of the complex

$$M^{<a'>} \oplus M^{<b'>} \oplus M^{<c'>} \oplus M^{<d'>} \overset{d_1^{0,0}}{\to} M^{<-1>} \oplus M^{<-1>} \oplus M^{<-1>} \overset{d_1^{1,0}}{\to} M^{<-1>} \tag{2.26}$$

With our chosen orientation, the boundary of the 2-cell is given by

$$E_1(1-a') + E_2(1-d') + E_3(1-e').$$

This implies that $d_1^{1,0}$ is the map $d_1^{1,0}(m_1 + m_2 + m_3) = (1-a')m_1 + (1-d')m_2 + (1-e')m_3$, so we have

$$\begin{aligned} H^2(\Gamma_\infty, M) =& M^{\{\pm 1\}}/((1-a')M + (1-d')M + (1-e')M) \\ =& M/((1-a')M + (1-d')M + (1-e')M) \end{aligned} \tag{2.27}$$

In particular, if $M = \mathbb{C}$ is the trivial module, then $H^2(\Gamma_\infty, \mathbb{C}) = \mathbb{C}$.

The cohomology of the complex (2.26) is also the cohomology of the space $S_\infty = \partial X_\Gamma$, so we have an isomorphism $H^i(\partial X_\Gamma, \mathcal{M}) \cong H^i(\Gamma_\infty, M)$. The map

$$H^2(\Gamma, M) \longrightarrow H^2(\Gamma_\infty, M) \tag{2.28}$$

can be calculated using the original fundamental domain for Γ in $\mathbb{H}$. Suppose that $\mathcal{F}$ is defined by

$$\mathcal{F} = \left\{ (z, r) \in \mathbb{H} \ : \ 0 \leq |\mathrm{Re}(z)|_{\mathbb{C}} \leq \frac{1}{2}, 0 \leq |\mathrm{Im}(z)|_{\mathbb{C}} \leq \frac{1}{2}, |z|^2_{\mathbb{C}} + r^2 \geq 1 \right\}$$

Then $\mathcal{F}$ is a fundamental domain for Γ.

The intersection of $\mathcal{F}$ with the unit hemisphere, inside $\mathbb{H}$, is the region enclosed by the 4 points $(-\frac{1}{2} - \frac{i}{2}, \frac{\sqrt{2}}{2}), (\frac{1}{2} - \frac{i}{2}, \frac{\sqrt{2}}{2}), (\frac{1}{2} + \frac{i}{2}, \frac{\sqrt{2}}{2}), (-\frac{1}{2} + \frac{i}{2}, \frac{\sqrt{2}}{2})$. The intersection of $\mathcal{F}$ with the boundary of $\mathbb{H} \cup \mathbb{P}^1(\mathbb{C})$ at ∞ gives the fundamental cellular domain S_∞ for Γ_∞.

Observe that the projection of S_∞ onto the bottom face of the fundamental domain gives the region bounded by the points $(-\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{1}{2} + \frac{i}{2}, \frac{\sqrt{2}}{2}), (-\frac{1}{2} + \frac{i}{2}, \frac{\sqrt{2}}{2})$, and the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ maps one half of this rectangle to the other half, leaving the line through the points $(0, 1)$ and $(\frac{i}{2}, \frac{\sqrt{2}}{2})$ fixed. The resulting one quarter of the bottom face is the fundamental cellular domain $\sim \backslash D'$ given in Section 2.1.

As a consequence, one can show that the map (2.28) is given by

$$\begin{aligned} H^2(\Gamma, M) &\longrightarrow H^2(\Gamma_\infty, M) \\ m + (M^{\Gamma_{12}} + M^{\Gamma_{23}} + M^{\Gamma_{34}} + M^{\Gamma_{41}}) &\mapsto \left(1 - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) m \\ &\quad + ((1 - a')M + (1 - d')M + (1 - e')M). \end{aligned}$$

Proposition 2.4.1.

$$\begin{aligned} &\mathit{dim}_{\mathbb{C}}\ H^2_{cusp}(\Gamma, E_{2,2}(\mathbb{C})) = 0 \ \mathit{or}\ 1, \ \mathit{and} \\ &\mathit{dim}_{\mathbb{C}}\ H^2_{cusp}(\Gamma, \mathit{Ind}\,^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_{2,2}(\mathbb{C})) \geq 8. \end{aligned}$$

Proof. Using Sage, we can calculate that:

$$\begin{aligned} H^2(\Gamma_\infty, E_{2,2}(\mathbb{C})) &\cong \mathbb{C}, \ \text{and} \\ H^2(\Gamma_\infty, \mathrm{Ind}\,^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_{2,2}(\mathbb{C})) &\cong \mathbb{C}^{(5)} \end{aligned}$$

See Appendix A.2 for the code used to do this.

Recall (Proposition 2.2.5) that $H^2(\Gamma, E_{2,2}(\mathbb{C})) \cong \mathbb{C}$, and $H^2(\Gamma, \mathrm{Ind}\,^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{C}}) \underset{\mathbb{C}}{\otimes} E_{2,2}(\mathbb{C})) \cong \mathbb{C}^{(13)}$. The result follows.

□

Remark 2.4.1. It is of interest to note that if $H^2_{\mathrm{cusp}}(\Gamma, E_{2,2}(\mathbb{C})) \neq 0$, the 1-dimensional space of Bianchi modular forms this would give rise to is exhausted by lifts of (twists of) classical elliptic modular forms (defined over $\mathbb{Q}$), or by forms which arise from a quadratic extension of F via automorphic induction (see [33] for details). In fact, Rahm and Şengün [33] have found this to be true for almost all Bianchi groups $SL_2(\mathcal{O}_{-d})$, and for almost all weights $E_{k,k}(\mathbb{C})$.

Remark 2.4.2. Recall that we calculated $H^2(\Gamma_0(\mathfrak{p}), \mathbb{Q})$ for some ideals $\mathfrak{p} \subset \mathcal{O}$. It would be interesting to know whether these cohomology classes are cuspidal or Eisenstein (when they are nonzero). Şengün has determined the dimension of $H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{Q})$ in the following way.

Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$ of residue degree 1. Consider the exact sequence (see (2.21)):

$$H^1(\Gamma_0(\mathfrak{p}), \mathbb{Q}) \to H^1(U(\Gamma_0(\mathfrak{p})), \mathbb{Q}) \to H^2_c(\Gamma_0(\mathfrak{p}), \mathbb{Q}) \to H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{Q}) \to 0.$$

The group $\Gamma_0(\mathfrak{p})$ has two cusps. Şengün has shown that for each cusp c, $H^1(\Gamma_0(\mathfrak{p})_c, \mathbb{Q}) = 0$. This implies that $H^1(U(\Gamma_0(\mathfrak{p})), \mathbb{Q}) = 0$, which in turn means that there is an isomorphism

$$H^2_c(\Gamma_0(\mathfrak{p}), \mathbb{Q}) \cong H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{Q}).$$

By Lefschetz duality, $H^2_c(\Gamma_0(\mathfrak{p}), \mathbb{Q}) \cong H_1(\Gamma_0(\mathfrak{p}), \mathbb{Q})$, and the latter group is the abelianisation of $\Gamma_0(\mathfrak{p})$, denoted $\Gamma_0(\mathfrak{p})^{\mathrm{ab}}$. That is,

$$\dim_{\mathbb{Q}}(H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{Q})) = \mathrm{rank}(\Gamma_0(\mathfrak{p})^{\mathrm{ab}}).$$

In [15, p. 51], it is shown that the prime ideal $\mathfrak{p}$ of residue degree 1 with the smallest norm which has an infinite abelianisation is $\mathfrak{p} = (11 + 4i)$. This means that for each prime ideal $\mathfrak{p} = (1 + 2i), (1 + 4i), (3 + 2i)$ in our table in Section 2.2,

$$\dim_{\mathbb{Q}}(H^2_{\mathrm{cusp}}(\Gamma_0(\mathfrak{p}), \mathbb{Q})) = 0.$$

We have calculated the cohomology of the group Γ_∞ geometrically. It can also be done algebraically. We have an explicit description of the group Γ_∞. Namely, Γ_∞ is the group of matrices of the form $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ where $\alpha \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}$. If we let U_∞ be the unipotent subgroup of Γ_∞ (i.e. $U_\infty = U_{D_\infty} \cap \Gamma_\infty$), then we have an extension

$$1 \longrightarrow U_\infty \longrightarrow \Gamma_\infty \longrightarrow \mu_4 \longrightarrow 1$$

(where μ_4 is the multiplicative group of 4^{th} roots of unity) and a Hochschild-Serre spectral sequence

$$H^p(\mu_4, H^q(U_\infty, M)) \Longrightarrow H^{p+q}(\Gamma_\infty, M)$$

Again, since $2^{-1} \in M$, the spectral sequence collapses and the edge maps gives isomorphisms

$$H^i(\Gamma_\infty, M) \cong H^i(U_\infty, M)^{\mu_4} \quad i \geq 0$$

The group U_∞ is free abelian and generated by e' and $f' := \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right)$. The tensor product of the two resolutions

$$0 \longrightarrow R[\langle e' \rangle] \xrightarrow{1-e'} R[\langle e' \rangle] \xrightarrow{\epsilon} R \longrightarrow 0$$
$$0 \longrightarrow R[\langle f' \rangle] \xrightarrow{1-f'} R[\langle f' \rangle] \xrightarrow{\epsilon} R \longrightarrow 0$$

in which ϵ is the augmentation map, gives a resolution of U_∞, and then it is clear that the second cohomology is described by

$$H^2(U_\infty, M) \cong M/((1-e')M + (1-f')M) \tag{2.29}$$

Note that equation (2.27) can be re-written as $H^2(\Gamma_\infty, M) \cong M_{\Gamma_\infty}$; similarly, (2.29) can be written as $H^2(U_\infty, M) \cong M_{U_\infty}$. Hence, to show that (2.27) and $H^2(U_\infty, M)^{\mu_4}$ are the same, it remains to observe that for any finite group G, and any representation V of G over $\mathbb{Q}$,

$$V^G \cong V_G.$$

2.4.2 Level four

Theorem 2.4.2. *Let* Γ' *be the congruence subgroup* $\Gamma(4)SL_2(\mathbb{Z})$. *Then*

$$H^2_{cusp}(\Gamma', \kappa_{\mathbb{Q}}) = 0.$$

Theorem 2.4.2 shall be proved in a number of steps. The first step is the calculation of $H^2(U(\Gamma'), \kappa_{\mathbb{Q}})$ where $U(\Gamma') = \bigoplus_{c \in \Gamma' \backslash \mathbb{P}^1(F)} \Gamma'_c$ as defined in (2.20) above. To do this, we must determine the cusps of Γ'.

Lemma 2.4.3. *A set of representatives for the cusps of* Γ' *is given by*

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}, \begin{bmatrix} 2i \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1+2i \end{bmatrix}$$

Proof. Since the class number of F is 1, Γ acts transitively on $\mathbb{P}^1(F)$, and we can make the identification

$$\Gamma/\Gamma_\infty \longleftrightarrow \mathbb{P}^1(F)$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto g(\infty) = \begin{bmatrix} a \\ c \end{bmatrix}. \tag{2.30}$$

We have seen that there is a set bijection:

$$\Gamma' \backslash \Gamma \cong SL_2(\mathbb{Z}/4\mathbb{Z}) \backslash SL_2(\mathcal{O}/4) \tag{2.31}$$

given by reducing the coefficients of a matrix in Γ modulo 4. The cusps of Γ' are therefore in one-to-one correspondence with elements in the double coset space

$$SL_2(\mathbb{Z}/4\mathbb{Z}) \backslash SL_2(\mathcal{O}/4) / \left\{ \begin{pmatrix} i^\alpha & x \\ 0 & i^{-\alpha} \end{pmatrix} \ : \ \alpha \in \mathbb{Z}/4\mathbb{Z}, x \in \mathcal{O}/4 \right\},$$

which we shall write as

$$< i > SL_2(\mathbb{Z}/4\mathbb{Z}) \backslash SL_2(\mathcal{O}/4) / \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ : \ x \in \mathcal{O}/4 \right\}.$$

Note the identification

$$SL_2(\mathcal{O}/4) / \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ : \ x \in \mathcal{O}/4 \right\} =$$
$$\left\{ \begin{pmatrix} u \\ z \end{pmatrix} \ : \ u \in (\mathcal{O}/4)^\times, z \in \mathcal{O}/4 \right\} \cup \left\{ \begin{pmatrix} \pi z \\ u \end{pmatrix} \ : \ z \in \mathcal{O}/(2+2i), u \in (\mathcal{O}/4)^\times \right\}.$$

That is, since $\mathcal{O}/4 = (\mathcal{O}/4)^\times \cup \{\pm\pi, \pm i\pi\} \cup \{\pi^2, i\pi^2\} \cup \{\pi^3\} \cup \{0\}$, we must have that $|SL_2(\mathcal{O}/4)/\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \ : \ x \in \mathcal{O}/4\}| = 8 * 16 + 8 * 8 = 192$, and $|SL_2(\mathcal{O}/4)/\{\left(\begin{smallmatrix} i^\alpha & x \\ 0 & i^{-\alpha} \end{smallmatrix}\right) :$ $\alpha \in \mathbb{Z}/4\mathbb{Z}, x \in \mathcal{O}/4\}| = 192/4 = 48$.

Let $v = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in (\mathcal{O}/4)^2$, and consider the stabilizer of v under the action of $SL_2(\mathbb{Z}/4) < i >$, where an element $i^\alpha \in < i >$ acts as $v \mapsto i^\alpha v$, and $SL_2(\mathbb{Z}/4)$ acts in the usual way. Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i^\alpha \end{pmatrix} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}/4)$$
$$\Leftrightarrow b = 0 \text{ and } a = d = \pm 1.$$

That is, $\mathrm{stab}_{SL_2(\mathbb{Z}/4\mathbb{Z})<i>}(v) \cong \{\pm\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) : c \in \mathbb{Z}/4\mathbb{Z}\}$, and has 8 elements, and so the orbit of v has $|SL_2(\mathbb{Z}/4) < i > |/8 = 24$ elements.

Next consider $\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right) \in (\mathcal{O}/4)^2$. Similarly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i^\alpha \\ i^{\alpha+1} \end{pmatrix} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}/4)$$
$$\Leftrightarrow ai + b = i^\alpha \text{ and } ci + d = i^{\alpha+1}.$$

That is, $\mathrm{stab}_{SL_2(\mathbb{Z}/4\mathbb{Z})<i>}(v) \cong \{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$, and has 4 elements, so the orbit of v has $|SL_2(\mathbb{Z}/4) < i > |/4 = 48$ elements.

Using the same method, one can show that the orbit of $\left(\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right)$ has 48 elements, the orbit of $\left(\begin{smallmatrix} 2i \\ 1 \end{smallmatrix}\right)$ has 48 elements, and that the orbit of $\left(\begin{smallmatrix} 0 \\ 1+2i \end{smallmatrix}\right)$ has 24 elements. Furthermore, one can check that the elements $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2i \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1+2i \end{smallmatrix}\right)$ are inequivalent under the action of $SL_2(\mathbb{Z}/4\mathbb{Z}) < i >$. The preimage of $\left(\begin{smallmatrix} 0 \\ 1+2i \end{smallmatrix}\right)$ under the map (2.31) is $\left(\begin{smallmatrix} -4 \\ 1+2i \end{smallmatrix}\right)$. □

For each representative $D_c \in \{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2i \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -4 \\ 1+2i \end{smallmatrix}\right]\}$, we must now determine the group Γ'_c. For notational convenience, let us denote the cusp $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ by x and $\left[\begin{smallmatrix} -4 \\ 1+2i \end{smallmatrix}\right]$ by $1 + 2i$.

Claim.

$$\Gamma'_i = \left\{ \pm \begin{pmatrix} 1 - ia & -a \\ -a & 1 + ia \end{pmatrix}, \pm \begin{pmatrix} i - ib & 2 - b \\ -b & -i + ib \end{pmatrix} \;\middle|\; a \equiv 0 \pmod 4,\ b \equiv 1 \pmod 4 \right\}$$

The group Γ'_i is generated by

$$\left\{ \begin{pmatrix} 1 - 4i & -4 \\ -4 & 1 + 4i \end{pmatrix}, \begin{pmatrix} 5 & -4i \\ -4i & -3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Proof of claim. The stabilizer of each cusp c of Γ' is of the form $\gamma\Gamma_\infty\gamma^{-1} \cap \Gamma'$ where $\gamma \cdot \infty = D_c$ and Γ_∞ is the stabilizer of ∞ in Γ. For $D_c = \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]$, such a γ equals

$\left(\begin{smallmatrix} i & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\left(\begin{smallmatrix} \mu & \alpha \\ 0 & \mu^{-1} \end{smallmatrix}\right)$ be in Γ_∞, so $\mu \in \{1, -1, i, -i\}$ and $\alpha \in \mathcal{O}$. Then

$$\gamma \begin{pmatrix} \mu & \alpha \\ 0 & \mu^{-1} \end{pmatrix} \gamma^{-1} = \begin{pmatrix} \mu^{-1} - i\alpha & i\mu - \alpha - i\mu^{-1} \\ -\alpha & \mu + i\alpha \end{pmatrix}$$

If $\mu = \pm 1$, then

$$\gamma \begin{pmatrix} \mu & \alpha \\ 0 & \mu^{-1} \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 - i\alpha & -\alpha \\ -\alpha & 1 + i\alpha \end{pmatrix}$$

and if $\left(\begin{smallmatrix} 1-i\alpha & -\alpha \\ -\alpha & 1+i\alpha \end{smallmatrix}\right)$ is in Γ', then we must have $\alpha \equiv 0 \pmod 4$.

If $\mu = i$, then

$$\gamma \begin{pmatrix} \mu & \alpha \\ 0 & \mu^{-1} \end{pmatrix} \gamma^{-1} = \begin{pmatrix} -i - i\alpha & 2 - \alpha \\ -\alpha & i + i\alpha \end{pmatrix}$$

and if $\left(\begin{smallmatrix} -i-i\alpha & 2-\alpha \\ -\alpha & i+i\alpha \end{smallmatrix}\right)$ is in Γ', then we must have $1 + \alpha \equiv 0 \pmod 4$, i.e. $\alpha \equiv -1 \pmod 4$.

For $\mu = -i$, the calculation is similar. The second part of the claim follows easily.

□

Lemma 2.4.4. *The group Γ'_{-i} is generated by*

$$\left\{ \begin{pmatrix} 1+4i & -4 \\ -4 & 1-4i \end{pmatrix}, \begin{pmatrix} -3 & -4i \\ -4i & 5 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The group Γ'_{2i} is generated by

$$\left\{ -Id, \begin{pmatrix} 1-8i & -16 \\ -4 & 1+8i \end{pmatrix}, \begin{pmatrix} 9 & -16i \\ -4i & -7 \end{pmatrix}, \begin{pmatrix} 1-4i & -8 \\ -2 & 1+4i \end{pmatrix} \right\}.$$

The group Γ'_{1+2i} is generated by

$$\left\{ -Id, \begin{pmatrix} 5+8i & 16 \\ 3-4i & -3-8i \end{pmatrix}, \begin{pmatrix} -31+16i & 64i \\ 16+12i & 33-16i \end{pmatrix} \right\}.$$

Let us now make a new definition.

Definition 2.4.3. *We call a cusp c for Γ' essential if $\kappa_{\mathbb{Q}}|_{\Gamma'_c} = 1$.*

Proposition 2.4.5. *The dimension, over $\mathbb{Q}$, of $H^2(U(\Gamma'), \kappa_{\mathbb{Q}})$ is the number of essential cusps of Γ'.*

Proof. We observed above that for each cusp c, Γ'_c acts on S_c and the spectral sequence (2.3) relates their cohomology. In particular, since 2 is invertible in the representation $\kappa_{\mathbb{Q}}$, we have

$$H^i(\Gamma'_c, \kappa_{\mathbb{Q}}) \cong H^i(S_c, \kappa_{\mathbb{Q}}) \text{ for all } i \geq 0$$

Poincaré Duality for closed, orientable 2-dimensional manifolds implies that

$$H^i(S_c, \kappa_{\mathbb{Q}}) \cong H_{2-i}(S_c, \kappa_{\mathbb{Q}})$$

This means we have an isomorphism

$$H^2(\Gamma'_c, \kappa_{\mathbb{Q}}) \cong H_0(\Gamma'_c, \kappa_{\mathbb{Q}}).$$

Hence, we can identify H^2 with the Γ'_c-coinvariants in $\kappa_{\mathbb{Q}}$. That is,

$$H^2(\Gamma'_c, \kappa_{\mathbb{Q}}) = \begin{cases} \mathbb{Q} & \text{if } \kappa_{\mathbb{Q}}|_{\Gamma'_c} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Taking the union over c, we find that the $\mathbb{Q}$-dimension of $H^2(U(\Gamma'), \kappa_{\mathbb{Q}})$ is the number of essential cusps.

□

Our next step, then, is to determine which cusps of Γ' are essential.

Lemma 2.4.6. *All of the cusps of Γ' are essential.*

Proof. Consider the cusp i. If $\gamma \in \Gamma'_i$, we must show that $\kappa_{\mathbb{Q}}(\gamma) = 1$. Since $\kappa_{\mathbb{Q}}$ is a homomorphism, it is sufficient to show that $\kappa_{\mathbb{Q}}$ is trivial on the generators of Γ'_i.

Now,

$$\begin{aligned} \kappa_{\mathbb{Q}}\left(\begin{pmatrix} 1-4i & -4 \\ -4 & 1+4i \end{pmatrix}\right) &= \left(\frac{-4}{1+4i}\right)_F \\ &= \left(\frac{-1}{1+4i}\right)_F \left(\frac{4}{1+4i}\right)_F \text{ by Proposition 1.2.5 (1)} \\ &= \left(\frac{i}{1+4i}\right)_F^2 \left(\frac{2}{1+4i}\right)_F^2 \text{ by Proposition 1.2.5 (1)} \\ &= 1. \end{aligned}$$

Furthermore,

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}5 & -4i\\ -4i & -3\end{pmatrix}\right) &= \left(\frac{-4i}{3}\right)_F\\ &= \left(\frac{-1}{3}\right)_F\left(\frac{i}{3}\right)_F\left(\frac{4}{3}\right)_F \text{ by Proposition 1.2.5 (1)}\\ &= \left(\frac{i}{3}\right)_F \text{ by Proposition 1.2.5 (1)}\\ &= (-1)^{\frac{9-1}{4}} \text{ by Lemma 1.2.6}\\ &= 1.\end{aligned}$$

Clearly,

$$\kappa_{\mathbb{Q}}\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) = 1.$$

Thus, the cusp i is essential.

Consider the cusp $-i$.

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}1+4i & -4\\ -4 & 1-4i\end{pmatrix}\right) &= \left(\frac{-4}{1-4i}\right)_F\\ &= \left(\frac{-1}{1-4i}\right)_F\left(\frac{4}{1-4i}\right)_F \text{ by Proposition 1.2.5 (1)}\\ &= \left(\frac{i}{1-4i}\right)_F^2\left(\frac{2}{1-4i}\right)_F^2 \text{ by Proposition 1.2.5 (1)}\\ &= 1.\end{aligned}$$

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}-3 & -4i\\ -4i & 5\end{pmatrix}\right) &= \left(\frac{-4i}{5}\right)_F\\ &= \left(\frac{-1}{5}\right)_F\left(\frac{4}{5}\right)_F\left(\frac{i}{5}\right)_F \text{ by Proposition 1.2.5 (1)}\\ &= \left(\frac{i}{5}\right)_F \text{ by Proposition 1.2.5 (1)}\\ &= (-1)^{\frac{25-1}{4}} \text{ by Lemma 1.2.6}\\ &= 1.\end{aligned}$$

Obviously, $\kappa_{\mathbb{Q}}\left(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right)$.

Next, consider the cusp $2i$. Clearly, $\kappa_{\mathbb{Q}}(-\mathrm{Id}) = 1$. We have

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}1-8i & -16\\ -4 & 1+8i\end{pmatrix}\right) &= \left(\frac{-4}{1+8i}\right)_F\\ &= \left(\frac{-1}{1+8i}\right)_F\left(\frac{4}{1+8i}\right)_F\left(\frac{i}{5}\right)_F \text{ by Proposition 1.2.5 (1)}\\ &= 1.\end{aligned}$$

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}9 & -16i\\ -4i & -7\end{pmatrix}\right) &= \left(\frac{-4i}{-7}\right)_F\\ &= \left(\frac{i}{-7}\right)_F \text{ by Proposition 1.2.5 (1)}\\ &= (-1)^{\frac{49-1}{4}} \text{ by Lemma 1.2.6}\\ &= 1.\end{aligned}$$

Note that $\left(\begin{smallmatrix}1-4i & -8\\ -2 & 1+4i\end{smallmatrix}\right) \notin \Gamma(4)$. Rather,

$$\begin{pmatrix}1-4i & -8\\ -2 & 1+4i\end{pmatrix} = \begin{pmatrix}17-4i & -8\\ -4-8i & 1+4i\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

where $\left(\begin{smallmatrix}17-4i & -8\\ -4-8i & 1+4i\end{smallmatrix}\right) \in \Gamma(4)$, and $\left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Thus,

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}17-4i & -8\\ -4-8i & 1+4i\end{pmatrix}\right) &= \left(\frac{-4-8i}{1+4i}\right)_F\\ &= \left(\frac{-4}{1+4i}\right)_F\left(\frac{1+2i}{1+4i}\right)_F\\ &= \left(\frac{1+2i}{1+4i}\right)_F\\ &= \left(\frac{-2i}{1+4i}\right)_F \quad \text{by Proposition 1.2.5 (3)}\\ &= \left(\frac{-1}{1+4i}\right)_F\left(\frac{1+i}{1+4i}\right)_F^2\\ &= 1.\end{aligned}$$

Finally, take $1+2i$. We can write

$$\begin{pmatrix}5+8i & 16\\ 3-4i & -3-8i\end{pmatrix} = \begin{pmatrix}21+8i & 16\\ -12i & -3-8i\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix},$$

and,

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}21+8i & 16\\ -12i & -3-8i\end{pmatrix}\right) &= \left(\frac{-12i}{-3-8i}\right)_F\\ &= \left(\frac{3i}{-3-8i}\right)_F\\ &= \left(\frac{3}{-3-8i}\right)_F\left(\frac{i}{-3-8i}\right)_F\\ &= \left(\frac{-8i}{-3-8i}\right)_F\\ &= \left(\frac{1+i}{-3-8i}\right)_F^6\\ &= 1.\end{aligned}$$

Moreover,

$$\begin{aligned}\kappa_{\mathbb{Q}}\left(\begin{pmatrix}-31+16i & 64i\\ 16+12i & 33-16i\end{pmatrix}\right) &= \left(\frac{16+12i}{33-16i}\right)_F\\ &= \left(\frac{-21i}{33-16i}\right)_F\\ &= \left(\frac{-i}{33-16i}\right)_F\left(\frac{7}{33-16i}\right)_F\left(\frac{3}{33-16i}\right)_F\\ &= (-1)^{\frac{1345-1}{4}}\left(\frac{7}{N(33-16i)}\right)_{\mathbb{Q}}\left(\frac{3}{N(33-16i)}\right)_{\mathbb{Q}}\\ &= \left(\frac{7}{5}\right)_{\mathbb{Q}}\left(\frac{7}{269}\right)_{\mathbb{Q}}\left(\frac{3}{5}\right)_{\mathbb{Q}}\left(\frac{3}{269}\right)_{\mathbb{Q}}\\ &= \left(\frac{7}{269}\right)_{\mathbb{Q}}\left(\frac{3}{269}\right)_{\mathbb{Q}}\\ &= 1.\end{aligned}$$

This concludes the proof. □

Recall the exact sequence (2.21):

$$\cdots \longrightarrow H_c^2(\Gamma', \kappa_{\mathbb{Q}}) \longrightarrow H^2(\Gamma', \kappa_{\mathbb{Q}}) \xrightarrow{\text{res}} H^2(U(\Gamma'), \kappa_{\mathbb{Q}}) \longrightarrow H_c^3(\Gamma', \kappa_{\mathbb{Q}}) \longrightarrow \cdots$$

We are interested in $H^2_{\text{cusp}}(\Gamma', \kappa_{\mathbb{Q}}) := \ker\{\text{res} : H^2(\Gamma', \kappa_{\mathbb{Q}}) \to H^2(U(\Gamma'), \kappa_{\mathbb{Q}})\}$. In Section 2.2 we proved that the dimension of $H^2(\Gamma', \kappa_{\mathbb{Q}})$ is 5 (Proposition 2.2.2), and we have just shown that $H^2(U(\Gamma'), \kappa_{\mathbb{Q}})$ is of the same dimension. It remains to observe, by Poincaré duality for the 3-dimensional manifold $X_{\Gamma'}$, that $H_c^3(\Gamma', \kappa_{\mathbb{Q}}) = 0$, to show that the restriction map is a surjection, and therefore an injection. That is, $H^2_{\text{cusp}}(\Gamma', \kappa_{\mathbb{Q}}) = 0$. This concludes the proof of Theorem 2.4.2.

There is an alternative, geometric, way to prove Theorem 2.4.2, using the following proposition. First we need a geometric version of Shapiro's Lemma.

Lemma 2.4.7.

$$H^2(\Gamma_\infty, Ind\,{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}})) \cong \bigoplus_{c\,\in \Gamma'\backslash \mathbb{P}^1(F)} H^2(\Gamma'_c, \kappa_{\mathbb{Q}}).$$

Proof. Consider $\text{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}})$ as a Γ_∞-module. There is a Γ_∞-isomorphism:

$$\text{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}}) \cong \bigoplus_{g\in \Gamma'\backslash\Gamma/\Gamma_\infty} \text{Ind}_{\Gamma_\infty\cap g\Gamma' g^{-1}}^{\Gamma_\infty}(\kappa_{\mathbb{Q}} \circ \text{ad}(g)),$$

where $\text{ad}(g)$ is the automorphism of Γ' given by

$$\text{ad}(g)(\gamma') := g^{-1}\gamma' g.$$

Therefore,

$$\begin{aligned} H^2(\Gamma_\infty, \text{Ind}\,{}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}})) \cong & H^2(\Gamma_\infty, \bigoplus_{g\in \Gamma'\backslash\Gamma/\Gamma_\infty} \text{Ind}_{\Gamma_\infty\cap g\Gamma' g^{-1}}^{\Gamma_\infty}(\kappa_{\mathbb{Q}} \circ \text{ad}(g))) \\ \cong & \bigoplus_{g\in \Gamma'\backslash\Gamma/\Gamma_\infty} H^2(\Gamma_\infty, \text{Ind}_{\Gamma_\infty\cap g\Gamma' g^{-1}}^{\Gamma_\infty}(\kappa_{\mathbb{Q}} \circ \text{ad}(g))). \end{aligned} \tag{2.32}$$

Since there are isomorphisms [7]

$$\text{Ind}_K^G(M) \cong \text{Ind}_{gKg^{-1}}^G(gM) \text{ for groups } G, K \text{ and a } K\text{-module } M, \text{ and,}$$
$$H^2(gKg^{-1}, M) \cong H^2(K, M) \text{ for any } g \in G, K \subset G, \text{ and } G\text{-module } M,$$

we have

$$\text{Ind}_{\Gamma_\infty\cap g\Gamma' g^{-1}}^{\Gamma_\infty}(\kappa_{\mathbb{Q}} \circ \text{ad}(g)) \cong \text{Ind}_{g^{-1}\Gamma_\infty g\cap\Gamma'}^{g^{-1}\Gamma_\infty g}(\kappa_{\mathbb{Q}}) \text{ and,} \tag{2.33}$$

$$H^2(\Gamma_\infty, \text{Ind}_{g^{-1}\Gamma_\infty g\cap\Gamma'}^{g^{-1}\Gamma_\infty g}(\kappa_{\mathbb{Q}})) \cong H^2(g^{-1}\Gamma_\infty g, \text{Ind}_{g^{-1}\Gamma_\infty g\cap\Gamma'}^{g^{-1}\Gamma_\infty g}(\kappa_{\mathbb{Q}})). \tag{2.34}$$

Putting (2.32), (2.33), and (2.34) together yields:

$$\begin{aligned} H^2(\Gamma_\infty, \mathrm{Ind}\,^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{Q}})) &\cong \bigoplus_{g\in\Gamma'\backslash\Gamma/\Gamma_\infty} H^2(\Gamma_\infty, \mathrm{Ind}^{g^{-1}\Gamma_\infty g}_{g^{-1}\Gamma_\infty g\cap\Gamma'}(\kappa_{\mathbb{Q}})) \\ &\cong \bigoplus_{g\in\Gamma'\backslash\Gamma/\Gamma_\infty} H^2(g^{-1}\Gamma_\infty g, \mathrm{Ind}^{g^{-1}\Gamma_\infty g}_{g^{-1}\Gamma_\infty g\cap\Gamma'}(\kappa_{\mathbb{Q}})) \\ &\cong \bigoplus_{g\in\Gamma'\backslash\Gamma/\Gamma_\infty} H^2(g^{-1}\Gamma_\infty g\cap\Gamma', \kappa_{\mathbb{Q}}) \text{ by Lemma 2.1.7} \\ &\cong \bigoplus_{g\in\Gamma'\backslash\Gamma/\Gamma_\infty} H^2(\Gamma'_c, \kappa_{\mathbb{Q}}) \text{ where } g^{-1}\infty = c \\ &\cong \bigoplus_{c\in\Gamma'\backslash\mathbb{P}^1(F)} H^2(\Gamma'_c, \kappa_{\mathbb{Q}}). \end{aligned}$$

To see the last isomorphism, recall the identification (2.30). This implies that there is a bijective map

$$\begin{aligned} \Gamma'\backslash\Gamma/\Gamma_\infty &\longrightarrow \Gamma'\backslash\mathbb{P}^1(F) = \{\text{cusps of } \Gamma'\} \\ g^{-1} &\longmapsto g^{-1}\infty = \begin{bmatrix} a \\ c \end{bmatrix} \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

□

Proposition 2.4.8.

$$H^2_{cusp}(\Gamma', \kappa_{\mathbb{Q}}) \cong Ker(H^2(X_\Gamma, Ind(\kappa_{\mathbb{Q}})) \longrightarrow H^2(\partial X_\Gamma, Ind(\kappa_{\mathbb{Q}}))).$$

Proof. By Lemma 2.4.7, there is a commutative square:

$$\begin{array}{ccc} H^2(\Gamma_\infty, \mathrm{Ind}^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{Q}})) & \xrightarrow{\cong} & \bigoplus_{c\,\in\Gamma'\backslash\mathbb{P}^1(F)} H^2(\Gamma'_c, \kappa_{\mathbb{Q}}) \\ \downarrow\cong & & \downarrow\cong \\ H^2(\partial X_\Gamma, \mathrm{Ind}^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{Q}})) & \longrightarrow & H^2(\partial X_{\Gamma'}, \kappa_{\mathbb{Q}}) \end{array}$$

in which the vertical arrows and the top horizontal arrows are isomorphisms, hence

$$H^2(\partial X_\Gamma, \mathrm{Ind}^{\Gamma}_{\Gamma'}(\kappa_{\mathbb{Q}})) \longrightarrow H^2(\partial X_{\Gamma'}, \kappa_{\mathbb{Q}}) \tag{2.35}$$

is an isomorphism.

Recall the formula (2.23) for any finite index subgroup $\Upsilon \subset SL_2(\mathcal{O})$. In particular,

$$H^2_{\mathrm{cusp}}(\Gamma', \kappa_{\mathbb{Q}}) \cong \ker\{H^2(X_{\Gamma'}, \kappa_{\mathbb{Q}}) \longrightarrow H^2(\partial X_{\Gamma'}, \kappa_{\mathbb{Q}})\}. \tag{2.36}$$

But, since 6 is invertible in $\kappa_{\mathbb{Q}}$, and by Lemma 2.1.7,

$$H^2(X_{\Gamma'}, \kappa_{\mathbb{Q}}) \cong H^2(\Gamma', \kappa_{\mathbb{Q}}) \cong H^2(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}})) \cong H^2(X_{\Gamma}, \mathrm{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}})). \tag{2.37}$$

Equations (2.35), (2.36) and (2.37) give

$$H^2_{\mathrm{cusp}}(\Gamma', \kappa_{\mathbb{Q}}) \cong \mathrm{Ker}(H^2(X_{\Gamma}, \mathrm{Ind}(\kappa_{\mathbb{Q}})) \longrightarrow H^2(\partial X_{\Gamma}, \mathrm{Ind}(\kappa_{\mathbb{Q}}))).$$

□

Proposition 2.4.8 allows us to check our calculation of $H^2_{\mathrm{cusp}}(\Gamma', \kappa_{\mathbb{Q}})$ using a Sage programme. We did this (Appendix C) and the result agrees with our original finding that $H^2_{\mathrm{cusp}}(\Gamma', \kappa_{\mathbb{Q}}) = 0$.

Chapter 3

A ramified genuine Hecke algebra

Recall the definition (1.11) of $K_\pi(4)$. We want to consider $K_\pi(4)$ as a subgroup of $SL_2(F_\pi)$. Thus, if $a \in \mathcal{O}_\pi$, let $\tilde{a}$ be its reduction modulo (4), and put

$$K_\pi(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{Z}/4\mathbb{Z},\ ad - bc = 1 \right\}.$$

Let $\widehat{K}_\pi(4)$ be the lift of $K_\pi(4)$ to $\overline{SL}_2(F_\pi)$.

Let ϖ be an automorphic representation of $\overline{SL}_2(\mathbb{A})$ containing a non-zero vector which is fixed by $\widehat{K}_\pi(4)$. The Hecke algebra $\mathcal{H}(\overline{SL}_2(F_\pi), \widehat{K}_\pi(4))$ acts on the subspace of $\widehat{K}_\pi(4)$-fixed vectors. The complicated part of this Hecke algebra is the finite-dimensional subalgebra $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$. The aim of this chapter is twofold.

We shall completely describe the subalgebra $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$ and list its genuine representations. Moreover, in the case that ϖ corresponds, via Theorem 1.4.15, to a level one automorphic cuspidal representation of $SL_2(\mathbb{A})$, we shall determine the action of $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$ on the $\widehat{K}_\pi(4)$-fixed vectors.

In brief summary, our results are as follows.

The "genuine quotient" of $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$ is 14-dimensional as a vector space over $\mathbb{C}$. As a ring, it is isomorphic to $\mathbb{C}^{(6)} \times M_2(\mathbb{C})^{(2)}$ (Theorems 3.3.7 and 3.3.8). In particular, it has six 1-dimensional representations and two irreducible 2-dimensional representations. We describe all these representations explicitly. If ϖ corresponds to a level one cuspidal representation of $SL_2(\mathbb{A})$, then the subspace of $\widehat{K}_\pi(4)$-fixed vectors is 2-dimensional: it is a sum of two 1-dimensional representations of $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$ (Theorem 3.4.4). We find which 1-dimensional representations arise in this way.

Analogous results (over $\mathbb{Q}$ rather than $\mathbb{Q}(i)$) have been obtained by Loke and Savin [28]; our approach is based on their method. However, since the calculations in our case are rather long, we have used Sage, whereas their work is done entirely by hand. One fundamental difference between the case over $\mathbb{Q}$ and over $\mathbb{Q}(i)$ is that there are two classes of "unramified" representation of $\overline{SL}_2(\mathbb{Q}_2)$ and only one class of "unramified" representation of $\overline{SL}_2(F_\pi)$. More precisely, since the centre of $\overline{SL}_2(\mathbb{Q}_2)$ is C_4 with generator $\left\{\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right), 1\right\}$, there are two distinct genuine central characters of an unramified representation. On the other hand, the centre of $\overline{SL}_2(F_\pi)$ is $C_2 \times C_2$ with generators $\left\{\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right), 1\right\}$ and $\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), -1\}$. If ϖ_π is a level one principal series representation, then the value of the central character of ϖ_π is 1 on $\left\{\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right), 1\right\}$ since this element is in $\widehat{K}_\pi(4)$. Thus there is a unique genuine central character (cf. Remark 3.4.1 below).

3.1 Preliminaries

Suppose, for the moment, that G is a locally compact group, and H is a compact open subgroup.

Definition 3.1.1. *Let $\mathcal{H}(G, H)$ denote the abelian group of functions given by:*

$$\mathcal{H}(G,H) = \{f : G \to \mathbb{C} \mid f(hgh') = f(g) \text{ for all } h, h' \in H, g \in G; f \text{ is locally constant and supported on only finitely many double cosets } HgH\}$$

It follows from the definition that every $f \in \mathcal{H}(G, H)$ is a finite sum of characteristic functions $\mathbb{1}_{HgH}$ of double cosets HgH. In what follows, we will blur the distinction between the function $\mathbb{1}_{HgH}$ and the double coset HgH.

We say that G *commensurates* H (written $G \sim H$) if for every $g \in G$, $gHg^{-1} \cap H$ has finite index in both gHg^{-1} and H. If H is compact and open in G, then G commensurates H (for any $g \in G$, $gHg^{-1} \cap H$ is compact and open in H, so $[H : gHg^{-1} \cap H] < \infty$, and $[gHg^{-1} : gHg^{-1} \cap H] = [H : gHg^{-1} \cap H]$).

Lemma 3.1.1. *If $g \in G$, there are disjoint coset decompositions*

$$HgH = \bigcup_{i=1}^{d} H\alpha_i \text{ with } d = [H : H \cap g^{-1}Hg]$$

$$HgH = \bigcup_{j=1}^{e} \beta_j H \text{ with } e = [H : H \cap gHg^{-1}]$$

Note that $d = e$ if G is unimodular.

Proof. Since $[H : gHg^{-1} \cap H] < \infty$, we can write

$$H = \bigcup_{i=1}^{d} (H \cap gHg^{-1})h_i$$

Therefore,

$$(g^{-1}Hg)H = \bigcup_{i=1}^{d} (g^{-1}Hg)h_i \quad \text{and}$$

$$HgH = \bigcup_{i=1}^{d} Hgh_i$$

To prove that the single coset decomposition is disjoint, suppose that $Hgh_i = Hgh_j$. Then $h_i h_j^{-1} \in g^{-1}Hg \cap H$ so $i = j$. The second relation is proved in the same way.

□

Lemma 3.1.1 allows us to define a multiplication on the group $\mathcal{H}(G, H)$. If $H\alpha H, H\beta H \in \mathcal{H}(G, H)$, write $H\alpha H = \cup_{i=1}^{d} H\alpha_i$ and $H\beta H = \cup_{j=1}^{e} H\beta_j$. Then we define $\mathcal{H}(G, H) \times \mathcal{H}(G, H) \to \mathcal{H}(G, H)$, $(H\alpha H, H\beta H) \mapsto H\alpha H * H\beta H$ by

$$H\alpha H * H\beta H = \sum_{\gamma} c(\gamma) H\gamma H \text{ where}$$

$$c(\gamma) = \text{number of pairs } (i, j) \text{ such that } H\alpha_i\beta_j = H\gamma \text{ for a fixed } \gamma, \tag{3.1}$$

and the sum is extended over all double cosets $H\gamma H \subset H\alpha H\beta H$. Note that $c(\gamma)$ is independent of the choice of representatives α_i, β_j and γ. We can define a multiplication on the whole of $\mathcal{H}(G, H)$ by extending $\mathbb{C}$-linearly. One can show that the multiplication is associative and that the trivial double coset $H1H$ is the identity.

Definition 3.1.2. *The degree of a double coset, deg($H\alpha H$), is the number of single cosets $H\alpha_i$ inside $H\alpha H$.*

We can extend this notion to the $\mathbb{C}$-algebra $\mathcal{H}(G,H)$: for a general element $\sum_k c_k H\alpha_k H$, let

$$\deg\left(\sum_k c_k H\alpha_k H\right) = \sum_k c_k \deg(H\alpha_k H)$$

Proposition 3.1.2. *The function deg: $\mathcal{H}(G,H) \to \mathbb{C}$ is an algebra homomorphism.*

We call $\mathcal{H}(G,H)$ the *Hecke algebra* of G with respect to H.

Suppose that H_1, H_2 are two subgroups of G, such that $H_1 \triangleleft H_2$. If $g \in G$, we can write an H_2-double coset H_2gH_2 as a (not necessarily disjoint) union of H_1-double cosets. First, write $H_2 = \cup_{i=1}^n H_1 g_i = \cup_{i=1}^n g_i H_1$. Then,

$$\begin{aligned} H_2gH_2 &= \bigcup_{i=1}^n H_1 g_i g H_2 \\ &= \bigcup_{j=1}^n \bigcup_{i=1}^n H_1 g_i g g_j H_1. \end{aligned}$$

Thus we can define a map

$$\mathcal{H}(G,H_2) \longrightarrow \mathcal{H}(G,H_1) \tag{3.2}$$

$$H_2gH_2 \longmapsto \sum_{j=1}^n \sum_{i=1}^n H_1 g_i g g_j H_1. \tag{3.3}$$

The Hecke algebra we shall be interested in is the case where v is a finite place of F, $G = \overline{SL}_2(F_v)$ and $H = \widehat{K}_v$ is a compact open subgroup. In the case that $\widehat{K}_v$ is maximal, we already know the structure of $\mathcal{H}(\overline{SL}_2(F_v), \widehat{K}_v)$.

Proposition 3.1.3. *If $\widehat{K}_v = \widehat{SL}_2(\mathcal{O}_v)$, then the "genuine quotient" of $\mathcal{H}(\overline{SL}_2(F_v), \widehat{K}_v)$ is a polynomial ring with one generator. In particular, the genuine quotient of $\mathcal{H}(\overline{SL}_2(F_v), \widehat{K}_v)$ is finitely generated and commutative.*

Proposition 3.1.3 is a special case of [29, Theorem 10.1].

Recall that if v is finite and odd, the extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{SL}_2(F_v) \longrightarrow SL_2(F_v) \longrightarrow 1 \tag{3.4}$$

splits on the maximal compact subgroup $SL_2(\mathcal{O}_v)$. By Proposition 3.1.3, the Hecke algebra $\mathcal{H}(\overline{SL}_2(F_v), \widehat{SL}_2(\mathcal{O}_v))$ is completely understood; in particular, its irreducible representations are 1-dimensional. On the other hand, if $v = \pi$, recall that (3.4) does not split over $SL_2(\mathcal{O}_v)$, but rather over the smaller group $K_\pi(4)$. Thus we can form $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$, and apply the following:

Proposition 3.1.4 (Bernstein)**.** *If $\widehat{K}_v$ is an arbitrary compact open subgroup, then $\mathcal{H}(\overline{SL}_2(F_v), \widehat{K}_v)$ is finitely generated as a module over its centre.*

We shall spend Sections 3.2 and 3.3 calculating $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$.

3.2 Calculation of $\mathcal{H}(SL_2(\mathcal{O}_\pi), K_\pi(4))$

As a first approximation to $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$, we shall calculate the algebra $\mathcal{H} = \mathcal{H}(SL_2(\mathcal{O}_\pi), K_\pi(4))$.

For the remainder of this thesis, let G denote the group $SL_2(\mathcal{O}/4)$, let H denote the group $SL_2(\mathbb{Z}/4)$, let $T = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in (\mathcal{O}/4)^\times\}$ be the diagonal subgroup of G and let $Z = \{\left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right) \mid b \in (\mathcal{O}/4)^\times, b^2 = 1\}$ be its centre. Note that G was used to denote the group GL_2 in Chapter 1 and an arbitrary locally compact group in Section 3.1; we hope the overlap in notation will not cause confusion.

Lemma 3.2.1. *There is an isomorphism of Hecke algebras*

$$\mathcal{H} \cong \mathcal{H}(G, H).$$

Proof. We saw (2.17) that there is a set bijection

$$SL_2(\mathcal{O})/SL_2(\mathcal{O}, 4)SL_2(\mathbb{Z}) \cong SL_2(\mathcal{O}/4)/SL_2(\mathbb{Z}/4).$$

On the other hand, it is clear that

$$SL_2(\mathcal{O})/SL_2(\mathcal{O}, 4)SL_2(\mathbb{Z}) \cong SL_2(\mathcal{O}_\pi)/SL_2(\mathcal{O}_\pi, 4)SL_2(\mathbb{Z}_2).$$

□

There is a normal chain of groups

$$H \lhd ZH \lhd TH. \tag{3.5}$$

Define the elements

$$t = \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix} \in T, \text{ and } z = \begin{pmatrix} 1+2i & 0 \\ 0 & 1+2i \end{pmatrix} \in Z,$$

Lemma 3.2.2. *There are disjoint coset decompositions*

$$TH = ZH \cup ZHt \qquad \text{and} \qquad ZH = H \cup Hz.$$

Proof. First note that $|H| = 6 * 8, |T| = 8$ and $|Z| = 4$. Thus,

$$\left|\frac{TH}{ZH}\right| = \frac{|TH|}{|ZH|} = \frac{(|T|\,|H|\,/\,|T \cap H|)}{(|Z|\,|H|\,/\,|Z \cap H|)} = \frac{(|T|\,/\,|T \cap H|)}{(|Z|\,/\,|Z \cap H|)}$$
$$= \frac{8 * 2}{4 * 2} = 2.$$

We have

$$T = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}, \begin{pmatrix} \pm(1+2i) & 0 \\ 0 & \pm(1+2i) \end{pmatrix}, \begin{pmatrix} \pm(2+i) & 0 \\ 0 & \mp(2-i) \end{pmatrix} \right\}$$
$$= \quad Z \quad \cup \quad \left\{ \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}, \begin{pmatrix} \pm(2+i) & 0 \\ 0 & \mp(2-i) \end{pmatrix} \right\},$$

and

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}^{-1} = \begin{pmatrix} -1+2i & 0 \\ 0 & -1+2i \end{pmatrix} \in Z.$$

Hence,

$$Z \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = Z \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}.$$

For the second part of the lemma, observe that

$$\left|\frac{ZH}{H}\right| = \left|\frac{Z}{H \cap Z}\right| = \frac{|Z|}{|H \cap Z|} = \frac{4}{2} = 2,$$

and

$$ZH = \quad H \quad \cup \quad \left\{ \begin{pmatrix} \pm(1+2i) & 0 \\ 0 & \pm(1+2i) \end{pmatrix} \right\}.$$

Note that z is the only non-trivial representative of ZH/H since

$$H \begin{pmatrix} 1+2i & 0 \\ 0 & 1+2i \end{pmatrix} = H \begin{pmatrix} -(1+2i) & 0 \\ 0 & -(1+2i) \end{pmatrix}.$$

□

The chain (3.5) induces maps (cf. (3.2))

$$\mathcal{H}(G, TH) \longrightarrow \mathcal{H}(G, ZH)$$
$$THgTH \mapsto ZHgZH \cup ZHtgZH \cup ZHgtZH \cup ZHtgtZH \text{ and} \tag{3.6}$$
$$\mathcal{H}(G, ZH) \longrightarrow \mathcal{H}(G, H)$$
$$ZHgZH \mapsto HgH \cup HgzH \tag{3.7}$$

It follows that the structure of $\mathcal{H}(G, H)$ as a vector space can be determined if we know that of $\mathcal{H}(G, TH)$ and $\mathcal{H}(G, ZH)$. This is the method we employ.

In what follows, if $x \in G$ is a matrix, we shall write $\hat{x}$ for a double coset. Whether this double coset is $THxTH$, $ZHxZH$ or HxH, shall be clear from the context.

Proposition 3.2.3. *The Hecke algebra $\mathcal{H}(G, TH)$ is commutative. It is 4-dimensional as a vector space over $\mathbb{C}$, with basis $\hat{1}, \hat{x}, \hat{y}, \hat{u}$ given by the following matrices:*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad x = \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}; \quad y = \begin{pmatrix} 1 & \pi^2 \\ 0 & 1 \end{pmatrix}; \quad u = \begin{pmatrix} 1 & \pi \\ \pi^3 & 1 \end{pmatrix}.$$

Its multiplication table is:

	1	x	y	u
1	1	x	y	u
x	x	6+2x+2y+2u	x+2u	2x+4y+2u
y	y	x+2u	3+2y	2x+u
u	u	2x+4y+2u	2x+u	6+2x+2y+2u

Proof of Proposition 3.2.3. Recall from Lemma 2.2.3 that a set of left coset representatives for G/H is given by the set $\{a_i b_j \text{ for } 1 \leq i, j \leq 8\}$ where $\{a_i\}$ are the elements of $SL_2(\mathcal{O}/4, 1+i)/SL_2(\mathcal{O}/4, 2)$ and $\{b_j\}$ are the elements of $SL_2(\mathcal{O}/4, 2+2i)$. Taking equivalence classes under T, one can easily show that a set of coset representatives for the quotient G/TH is given by $\{c_i d_j \text{ for } 1 \leq i, j \leq 4\}$ where $c_i \in$ 'THquotient1mmmm' below, and $d_j \in$ 'THquotient3mmmm' below.

```
THquotient1mmmm = [mmmm([1,0,0,1]), mmmm([1,1+i,0,1]),
mmmm([1,1+i,1+i,1+2*i]), mmmm([1,0,1+i,1])]


THquotient3mmmm = [mmmm([1,0,0,1]), mmmm([1,2+2*i,0,1]),
```

```
mmmm([1,0,2+2*i,1]), mmmm([1,2+2*i,2+2*i,1])]

THsinglecosets = [c*d for c in THquotient1mmmm for d in THquotient3mmmm]
```

Therefore, $|G/TH| = 16$. We must now determine a set of representatives for $TH\backslash G/TH$. Let g_1 and g_2 belong to 'THsinglecosets'. Then the double coset THg_1TH is equal to the double coset THg_2TH if and only if there is an element $h \in TH$ such that $hg_1TH = g_2TH$; indeed, this happens if and only if $(hg_1)^{-1}g_2 \in TH$.

We claim that a set of representatives for $TH\backslash G/TH$ is given by:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi^3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi \\ \pi^3 & 1 \end{pmatrix} \right\}$$

The following Sage check:

```
def leftrep(g):
    ginverse = g.adjoint()
    for x in THsinglecosets:
        if ginverse*x in TH:
            return x
```

returned, for example, that

$$TH \begin{pmatrix} 1 & \pi^3 \\ 0 & 1 \end{pmatrix} TH = TH \begin{pmatrix} 1 & 0 \\ \pi^3 & 1 \end{pmatrix} TH \text{ and}$$
$$TH \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix} TH = TH \begin{pmatrix} 1 & -\pi \\ 0 & 1 \end{pmatrix} TH = TH \begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix} TH.$$

Finally, we can check that we have the correct number of double cosets in the space $TH\backslash G/TH$ by checking the degree of each double coset. By Lemma 3.1.1, $\deg(THgTH) = [TH : TH \cap g^{-1}THg]$. The following algorithm:

```
x = mmmm([1,1+i,0,1])
y = mmmm([1,2+2*i,0,1])
u = mmmm([1,1+i,2+2*i,1])
THdoublecosets = [mmmm([1,0,0,1]), x,y,u]
```

```
for g in THdoublecosets:
    ginverse = g.adjoint()
    c = 0
    for h in TH:
        test = ginverse*h*g
        if test in TH:
            c = c+1
            degree = (4*48)/c
    print([g, degree])
```

gave the output:

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, 1\right]; \left[\begin{pmatrix}1 & \pi\\ 0 & 1\end{pmatrix}, 6\right]; \left[\begin{pmatrix}1 & \pi^3\\ 0 & 1\end{pmatrix}, 3\right]; \left[\begin{pmatrix}1 & \pi\\ \pi^3 & 1\end{pmatrix}, 6\right]$$

which verifies the claim. Hence we have shown that $\mathcal{H}(G, TH)$ is a 4-dimensional vector space with basis $\hat{1}, \hat{x}, \hat{y}, \hat{u}$.

By the proof of Lemma 3.1.1, we know that a double coset can be written as a disjoint union of single right cosets

$$THgTH = \bigcup_i THgh_i \text{ where } h_i \in TH/(TH \cap gTHg^{-1})$$

so to multiply the double cosets, we must find specific representatives for $TH/(TH \cap gTHg^{-1})$ for each representative g in $TH\backslash G/TH$. Again using Sage, we found the following data:

$$THxTH = \bigcup_{i=1}^{6} THxh_i \text{ for}$$
$$h_i \in \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix}i & i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}i & 0\\ i & -i\end{pmatrix}\right\};$$

$$THyTH = \bigcup_{j=1}^{3} THyh_j \text{ for}$$
$$h_j \in \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right\};$$

and

$$THuTH = \bigcup_{i=1}^{6} THuh_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ i & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

Recall the definition of multiplication in the Hecke algebra (3.1). The following code multiplies the double cosets:

```
righthforx = [mmmm([1,0,0,1]),mmmm([1,0,1,1]),mmmm([i,0,0,-i]),
mmmm([i,i,i,0]),mmmm([1,-1,1,0]),mmmm([i,0,i,-i])]
righthfory = [mmmm([1,0,0,1]),mmmm([1,0,1,1]),mmmm([1,-1,1,0])]
righthforu = [mmmm([1,0,0,1]),mmmm([i,0,0,-i]),mmmm([0,i,i,0]),
mmmm([0,-1,1,0]),mmmm([i,0,i,-i]),mmmm([1,0,1,1])]


for h1 in righthforx:
    for h2 in righthfory:
        answer = x*h1*y*h2
        for b in TH:
            for jj in range(4):
                if mmmm(b*answer) == THdoublecosets[jj]:
                    print(jj)
```

For example, the above code, as written, gave the output $[1, 3, 3]$. That is,

$$\hat{x} * \hat{y} = \hat{x} + 2\hat{u}$$

as claimed. The other relations were calculated in the same fashion. □

Observe that inside $\mathcal{H}(G, TH)$ we have the ring $\mathbb{C}[\hat{y}]/(\hat{y}^2 - 2\hat{y} - 3) = \mathbb{C}[\hat{y}]/(\hat{y}+1) \oplus \mathbb{C}[\hat{y}]/(\hat{y} - 3)$. It follows that the eigenvalues of $\hat{y}$ on any representation of $\mathcal{H}(G, TH)$ are -1 and 3. Moreover, since $\mathcal{H}(G, TH)$ is commutative, all of its irreducible (complex) representations are 1-dimensional. In fact, its representation theory can be described by the following character table.

	x	y	u
χ_1	2	-1	-2
χ_2	-2	-1	2
χ_3	6	3	6
χ_4	-2	3	-2

Proposition 3.2.3, as well as the character table above, completely describes $\mathcal{H}(G, TH)$. We turn now to the Hecke algebra $\mathcal{H}(G, ZH)$.

Define new matrices:

$$tx = t * x,$$

$$xt = x * t,$$

$$txt = t * x * t,$$

$$ty = t * y,$$

$$tu = t * u.$$

Proposition 3.2.4. *The Hecke algebra* $\mathcal{H}(G, ZH)$ *is* 10*-dimensional as a vector space over* $\mathbb{C}$*, with basis*

$$\{\hat{1}, \hat{t}, \hat{x}, \hat{tx}, \hat{xt}, \hat{txt}, \hat{y}, \hat{ty}, \hat{u}, \hat{tu}\}$$

The algebra $\mathcal{H}(G, ZH)$ *is non-commutative. Its multiplication table is:*

	1	t	x	tx	xt	txt	y	yt	u	ut
1	1	t	x	tx	xt	txt	y	yt	u	ut
t	t	1	tx	x	txt	xt	yt	y	ut	u
x	x	xt	$ut+y$	$3t+2x$	$u+yt$	$3+2xt$	$txt+u$	$tx+ut$	$2y+2tx+ut$	$2yt+2txt+u$
tx	tx	txt	$u+yt$	$3+2tx$	$ut+y$	$3t+2txt$	$xt+ut$	$x+u$	$2yt+2x+u$	$2y+2xt+ut$
xt	xt	x	$3t+2x$	$ut+y$	$3+2xt$	$u+yt$	$tx+ut$	$txt+u$	$2yt+u+2txt$	$2y+ut+2tx$
txt	txt	tx	$3+2tx$	$u+yt$	$3t+2txt$	$ut+y$	$x+u$	$xt+ut$	$2y+ut+2xt$	$2yt+u+2x$
y	y	yt	$txt+u$	$xt+ut$	$tx+ut$	$x+u$	$2y+3$	$3t+2yt$	$u+2x+2txt$	$ut+2xt+2tx$
yt	yt	y	$xt+ut$	$txt+u$	$x+u$	$tx+ut$	$2yt+3t$	$3+2y$	$ut+2tx+2xt$	$u+2txt+2x$
u	u	ut	$2y+ut+2xt$	$2yt+2txt+u$	$2yt+u+2x$	$2y+2tx+ut$	$u+2txt+2x$	$ut+2tx+2xt$	$6+2y+2xt+2tx+2ut$	$6t+2yt+2x+2txt+2u$
ut	ut	u	$2yt+u+2txt$	$2y+2xt+ut$	$2y+ut+2tx$	$2yt+2x+u$	$ut+2xt+2tx$	$u+2x+2txt$	$6t+2yt+2x+2txt+2u$	$6+2y+2xt+2tx+2ut$

Proof of Proposition 3.2.4. As mentioned above, the vector space structure of $\mathcal{H}(G, ZH)$ can be computed using the structure of $\mathcal{H}(G, TH)$. Recall (3.6) that each TH-double coset can be written as a union of ZH-double cosets: if $g \in G$, then

$$THgTH = ZHgZH \cup ZHtgZH \cup ZHgtZH \cup ZHtgtZH$$

We must determine such decompositions for all the double cosets in $\mathcal{H}(G, TH)$. It is straightforward to show that there are *disjoint* unions:

$$TH = ZH \cup ZHt \tag{3.8}$$

$$THyTH = HZyHZ \cup HZytHZ \tag{3.9}$$

$$THxTH = HZxHZ \cup HZxtHZ \cup HZtxHZ \cup HZtxtHZ \tag{3.10}$$

$$THuTH = HZuHZ \cup HZtuHZ \tag{3.11}$$

For example, we know a priori, that $THyTH = ZHyZH \cup ZHtyZH \cup ZHytZH \cup ZHtytZH$. However, in G, $yt = ty$ and so $tyt = y$. Thus $THyTH = ZHyZH \cup ZHtyZH$. Suppose that $ZHyZH = ZHtyZH$. Then there must be $h_1, h_2 \in H$ such that $y = h_1 ztyh_2$. However, if we reduce the matrices in Z modulo π^3, we are left with $\{\pm\text{Id}\}$; reducing the matrices in H modulo π^3 does nothing. On the other hand,

$$ty = \begin{pmatrix} 2+i & 2+2i \\ 0 & 2-i \end{pmatrix} \equiv \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \pmod{\pi^3} \text{ and}$$
$$y = \begin{pmatrix} 1 & 2+2i \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\pi^3},$$

and there are no elements $h_1, h_2 \in H$ such that $h_1 tyh_2 = y$. Thus our original assumption was incorrect: $ZHyZH$ is not equal to $ZHtyZH$.

Decompositions (3.8)-(3.11) yield the 10-dimensional basis of $\mathcal{H}(G, ZH)$. The algebra structure can be computed using minor modifications of the code given in the proof of Proposition 3.2.3 (see Appendix B.1). The non-commutativity is clear from the multiplication table.

□

Finally, we shall calculate $\mathcal{H} = \mathcal{H}(G, H)$. Define

$$tz = t * z,$$

$$yz = y * z,$$

$$ytz = y * tz.$$

Theorem 3.2.5. *The Hecke algebra $\mathcal{H}$ is non-commutative. It is 14-dimensional as a vector space, with basis the set*

$$\{\hat{1}, \hat{z}, \hat{t}, \hat{tz}, \hat{x}, \hat{xt}, \hat{tx}, \hat{txt}, \hat{y}, \hat{yz}, \hat{yt}, \hat{ytz}, \hat{u}, \hat{ut}\}.$$

Overleaf is its multiplication table.

	1	z	t	tz	x	xt	tx	txt	y	yz	yt	ytz	u	ut
1	1	z	t	tz	x	xt	tx	txt	y	yz	yt	ytz	u	ut
z	z	1	tz	t	x	xt	tx	txt	yz	y	ytz	yt	u	ut
t	t	tz	1	z	tx	txt	x	xt	yt	ytz	y	yz	ut	u
tz	tz	t	z	1	tx	txt	x	xt	ytz	yt	yz	y	ut	u
x	x	x	xt	xt	2y+2yz+2ut	2yt+2ytz+2u	6t+6tz+4x	6+6z+4xt	txt+u	txt+u	tx+ut	tx+ut	4y+4yz+4tx+2ut	4yt+4ytz+4txt+2u
xt	xt	xt	x	x	6t+6tz+4x	6+6z+4xt	2y+2yz+2ut	2yt+2ytz+2u	tx+ut	tx+ut	txt+u	txt+u	4yt+4ytz+4txt+2u	4y+4yz+4tx+2ut
tx	tx	tz	txt	txt	2yt+2ytz+2u	2y+2yz+2ut	6+6z+4tx	6t+6tz+4txt	xt+ut	xt+ut	x+u	x+u	4yt+4ytz+4x+2u	4y+4yz+4xt+2ut
txt	txt	txt	tx	tx	6+6z+4tx	6t+6tz+4txt	2yt+2ytz+2u	2y+2yz+2ut	x+u	x+u	xt+ut	xt+ut	4y+4yz+4xt+2ut	4yt+4ytz+4x+2u
y	y	yz	yt	ytz	txt+u	tx+ut	xt+ut	x+u	3+2yz	3z+2y	3t+2ytz	3tz+2yt	2txt+2x+u	2tx+2xt+ut
yz	yz	y	ytz	yt	txt+u	tx+ut	xt+ut	x+u	3z+2y	3+2yz	3tz+2yt	3t+2ytz	2txt+2x+u	2tx+2xt+ut
yt	yt	ytz	y	yz	xt+ut	xt+u	txt+u	tx+ut	3t+2ytz	3tz+2yt	3+2yz	3z+2y	2xt+2tx+ut	2txt+2x+u
ytz	ytz	yt	yz	y	xt+ut	x+u	txt+u	tx+ut	3tz+2yt	3t+2ytz	3z+2y	3+2yz	2tx+2xt+ut	2txt+2x+u
u	u	u	ut	ut	4yz+4y+4xt+2ut	4ytz+4yt+4x+2u	4ytz+4yt+4txt+2u	4yz+4y+4tx+2ut	2txt+2x+u	2txt+2x+u	2tx+2xt+ut	2tz+2xt+ut	12+12z+4xt+4tx +4y+4yz+4ut	12t+12tz+4x+4txt +4yt+4ytz+4u
ut	ut	ut	u	u	4ytz+4yt+4txt+2u	4yz+4y+4tx+2ut	4yz+4y+4xt+2ut	4ytz+4yt+4x+2u	2xt+2tx+ut	2xt+2tx+ut	2x+2txt+u	2x+2txt+u	12t+12tz+4x+4txt +4yt+4ytz+4u	12+12z+4xt+4tx +4y+4yz+4ut

Proof of Theorem 3.2.5. Note that, since $HZ = H \cup Hz$, if $\hat{g} \in \mathcal{H}(G, ZH)$, then either

$$\begin{aligned} HZgHZ =& HgH, \text{ or there is a disjoint union,} \\ HZgHZ =& HgH \cup HgzH. \end{aligned}$$

One can show that if $g = u, ut, txt, x, tx, xt$, then $HZgHZ = HgH$, whereas for $g = 1, y, yt, t$ we have a disjoint union $HZgHZ = HgH \ \cup \ HgzH$. In other words, $\mathcal{H}$ is 14-dimensional as a $\mathbb{C}$-vector space. To see this, first observe that

$$\begin{aligned} HZgHZ = HgH &\Rightarrow HZgtHZ = HgtH, \qquad (3.12) \\ HZgHZ = HgH &\Rightarrow HZtgHZ = HtgH \\ HZgHZ = HgH &\Rightarrow HZtgtHZ = HtgtH. \end{aligned}$$

Indeed, suppose that $HZgHZ = HgH$. Then, $HZgtHZ = HZgHtZ = HZgHZt = HgHt = HgtH$. This proves (3.12). The other two relations follow from the same arguments. Hence, to prove that $HZgHZ = HgH$ for $g = u, ut, txt, x, tx, xt$, it is sufficient to show $HZgHZ = HgH$ for $g = u, x$. Observe, moreover, that

$$HZgHZ = HgH \Leftrightarrow \exists\, h_1, h_2 \in H \text{ such that } g = h_1 z g h_2.$$

Suppose that $g = u$. Then

$$zu = \begin{pmatrix} 1+2i & -1-i \\ 2+2i & 1+2i \end{pmatrix}.$$

Put $h_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right), h_2 = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We have

$$h_1 z u h_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+2i & -1-i \\ 2+2i & 1+2i \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2+2i \\ 1+i & 1 \end{pmatrix} = u.$$

Now let $g = x$.

$$zx = \begin{pmatrix} 1+2i & -1-i \\ 0 & 1+2i \end{pmatrix}.$$

Put $h_1 = \left(\begin{smallmatrix} -1 & 0 \\ -2 & -1 \end{smallmatrix}\right), h_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. We have

$$h_1 z x h_2 = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1+2i & -1-i \\ 0 & 1+2i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix} = x.$$

On the other hand, suppose that $g = y$, and suppose that there exist $h_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $h_2^{-1} = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$, so that $y = h_1 z y h_2$. Then

$$h_1 \begin{pmatrix} 1+2i & 2+2i \\ 0 & 1+2i \end{pmatrix} = \begin{pmatrix} 1 & 2+2i \\ 0 & 1 \end{pmatrix} h_2^{-1}.$$

That is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1+2i & 2+2i \\ 0 & 1+2i \end{pmatrix} = \begin{pmatrix} 1 & 2+2i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a(1+2i) & a(2+2i)+b(1+2i) \\ c(1+2i) & c(2+2i)+d(1+2i) \end{pmatrix} = \begin{pmatrix} e+g(2+2i) & f+h(2+2i) \\ g & h \end{pmatrix}$$

$$\Rightarrow c(1+2i) = g \Rightarrow \ c = g = 0 \text{ or } 2,$$

$$\Rightarrow e = a(1+2i) \Rightarrow \ a = e = 0 \text{ or } 2.$$

Hence, we get a contradiction, so $HZyHZ \neq HyH$.

In Sage, we define the basis of double cosets by

```
Hdoublecosets = [mmmm([1,0,0,1]),z,t,t*z,x,x*t,t*x,t*x*t,y,y*z,
y*t,y*t*z,u,u*t]
```

To determine the structure of $\mathcal{H}$ as an algebra, we must write each H-double coset as a disjoint union of single H-cosets. There are two cases. If $ZHgZH = HgH$, and $ZHgZH = \cup_{i=1}^{n} ZHgh_i$, then since $ZH = H \cup Hz$, we have

$$\begin{aligned} HgH &= \bigcup_{i=1}^{n} ZHgh_i \\ &= \bigcup_{i=1}^{n} (H \cup Hz)gh_i \\ &= \bigcup_{i=1}^{n} Hgh_i \cup \bigcup_{i=1}^{n} Hgzh_i. \end{aligned}$$

For example, $ZHxZH = \cup_{i=1}^{3} ZHxh_i$, so $HxH = \cup_{i=1}^{3} Hxh_i \cup \cup_{i=1}^{3} Hxzh_i$. If, on the other hand, $ZHgZH \neq HgH$, and $ZHgZH = \cup_{i=1}^{n} ZHgh_i$, then

$$HgH = \bigcup_{i=1}^{n} Hgh_i.$$

Thus the job of finding representatives for H-single cosets is done: the most we need to do is to multiply the matrices in "littlehdashforg" by z on the left, to form "righthforg":

```
righthforx1 = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1]),
z,z*mmmm([1,1,1,2]),z*mmmm([1,0,1,1])]
righthforxt1 = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1]),
z,z*mmmm([1,1,1,2]),z*mmmm([1,0,1,1])]
righthfortx1 = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1]),
z,z*mmmm([1,1,1,2]),z*mmmm([1,0,1,1])]
righthfortxt1 = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1]),
z,z*mmmm([1,1,1,2]),z*mmmm([1,0,1,1])]
righthfory1 = [mmmm([1,0,0,1]),mmmm([1,0,1,1]),mmmm([1,-1,1,0])]
righthforyz1 = righthfory1
righthforyt1 = righthfory1
righthforytz1 = righthfory1
righthforu1 = [mmmm([1,0,0,1]),mmmm([1,1,0,1]),mmmm([1,-1,1,0]),
mmmm([0,-1,1,0]),mmmm([1,0,1,1]),mmmm([0,-1,1,1]),z,z*mmmm([1,1,0,1]),
z*mmmm([1,-1,1,0]),z*mmmm([0,-1,1,0]),z*mmmm([1,0,1,1]),
z*mmmm([0,-1,1,1])]
righthfortu1 =[mmmm([1,0,0,1]),mmmm([1,1,0,1]),mmmm([1,-1,1,0]),
mmmm([0,-1,1,0]),mmmm([1,0,1,1]),mmmm([0,-1,1,1]),
z,z*mmmm([1,1,0,1]),z*mmmm([1,-1,1,0]),z*mmmm([0,-1,1,0]),
z*mmmm([1,0,1,1]),z*mmmm([0,-1,1,1])]
```

Note that to check we have the correct number of single H-cosets inside each double H-coset, we can use a “degree” argument as exemplified in Appendix B.1.

The following multiplies the double cosets:

```
for h1 in righthforx1:
    for h2 in righthfory1:
        answer = x*h1*y*h2
        for b in H:
            for jj in range(14):
                if mmmm(b*answer) == Hdoublecosets[jj]:
                    print(jj)
```

The above example algorithm gave the output [7, 12]. That is, in $\mathcal{H}$,

$$\hat{x} * \hat{y} = t\hat{x}t + \hat{u}.$$

□

3.3 Calculation of the genuine quotient of $\mathcal{H}(\overline{SL_2}(\mathcal{O}_\pi), \widehat{K}_\pi(4))$

In Section 3.2, we determined the Hecke algebra $\mathcal{H}(SL_2(\mathcal{O}_\pi), K_\pi(4))$. In this section, we shall modify the method to calculate the metaplectic counterpart of this: $\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4))$. As in the previous section, we shall show that this Hecke algebra is isomorphic to the Hecke algebra of a pair of finite groups.

We shall define a central extension $\overline{G}$ of $G = SL_2(\mathcal{O}_\pi/(\pi^4))$ and a lift $\widehat{H}$ of the subgroup $H = SL_2(\mathbb{Z}/4\mathbb{Z})$ of G to $\overline{G}$. With this notation, we will have an isomorphism

$$\mathcal{H}(\overline{SL}_2(\mathcal{O}_\pi), \widehat{K}_\pi(4)) \cong \mathcal{H}(\overline{G}, \widehat{H}). \tag{3.13}$$

3.3.1 An explicit 2-cocycle defined modulo 4

To construct $\overline{G}$, we use the following:

Lemma 3.3.1. *Let $\sigma_\pi \in H^2(SL_2(F_\pi), \mu_2)$ be the cohomology class corresponding to the metaplectic cover of $SL_2(F_\pi)$. Then there is an element $\Sigma \in H^2(G, \mu_2)$ such that the inflation of Σ to $SL_2(\mathcal{O}_\pi)$ is equal to the restriction of σ_π to $SL_2(\mathcal{O}_\pi)$.*

The proof of Lemma 3.3.1 is the realm of K-theory: it follows from the fact that $K_2(\mathcal{O}_\pi) \cong K_2(F_\pi)$, as well as the result that the map

$$K_2(\mathcal{O}_\pi) \longrightarrow K_2(\mathcal{O}_\pi/(\pi^4))$$

is surjective. This can be gleaned from [14], although we shall give a new proof of Lemma 3.3.1 below in Proposition 3.3.2 since we require an explicit formula for the cocycle Σ.

Consider the extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{SL}_2(\mathcal{O}_\pi) \xrightarrow{p} SL_2(\mathcal{O}_\pi) \longrightarrow 1, \tag{3.14}$$

which is the restriction to $SL_2(\mathcal{O}_\pi)$ of the extension (1.12) we saw in Chapter 1. Recall that β_π is the associated normalised cocycle. Suppose that β_π corresponds to the section τ. That is, suppose that τ is a map $\tau : SL_2(\mathcal{O}_\pi) \to \overline{SL_2}(\mathcal{O}_\pi)$ and, for all $g, g' \in SL_2(\mathcal{O}_\pi)$,

$$\beta_\pi(g, g') = \tau(g)\tau(g')\tau(gg')^{-1} \tag{3.15}$$

By Proposition 1.1.4, τ restricted to $K_\pi(4)$ is a homomorphism.

Let $SL_2(\mathcal{O}_\pi, 4)$ denote the subgroup of $SL_2(\mathcal{O}_\pi)$ whose elements are congruent to the identity modulo (π^4). For any $g \in SL_2(\mathcal{O}_\pi)$, consider the map

$$\begin{aligned} X_g : SL_2(\mathcal{O}_\pi, 4) &\longrightarrow \mu_2 \\ \alpha &\longmapsto \tau(g^{-1}\alpha g)(\tau(g)^{-1}\tau(\alpha)\tau(g))^{-1} \end{aligned}$$

This makes sense because $\tau(g^{-1}\alpha g)$ and $\tau(g)^{-1}\tau(\alpha)\tau(g)$ have the same image under p. Using (3.15), we can re-write X_g as

$$X_g(\alpha) = \frac{\beta_\pi(g^{-1}, \alpha g)\beta_\pi(\alpha, g)}{\beta_\pi(g^{-1}, g)}$$

Let R be a set of (left) representatives for the group $SL_2(\mathcal{O}_\pi, 4)$ in $SL_2(\mathcal{O}_\pi)$. If $g \in SL_2(\mathcal{O}_\pi)$, decompose it as $g = rh$ where $r \in R$, and $h \in SL_2(\mathcal{O}_\pi, 4)$.

Definition 3.3.1. *Let S be the map $S : SL_2(\mathcal{O}_\pi) \to \overline{SL_2}(\mathcal{O}_\pi)$ given by*

$$S(g) = \{g, \beta_\pi(r, h)\}$$

where $g = rh$ for $r \in R$ and $h \in SL_2(\mathcal{O}_\pi, 4)$. Define a cocycle Σ by

$$\begin{aligned} \Sigma : SL_2(\mathcal{O}_\pi) \times SL_2(\mathcal{O}_\pi) &\longrightarrow \mu_2 \\ \Sigma(g_1, g_2) &= S(g_1)S(g_2)S(g_1g_2)^{-1} \text{ for all } g_1, g_2 \in SL_2(\mathcal{O}_\pi). \end{aligned}$$

Since the cocycles β_π and Σ determine the same extension (3.14), they differ by a 2-coboundary which we shall call ∂S_κ:

$$\Sigma = \beta_\pi \cdot \partial S_\kappa \text{ where } \partial S_\kappa(g_1, g_2) = S_\kappa(g_1)S_\kappa(g_2)S_\kappa(g_1g_2)^{-1} \text{ for all } g_1, g_2 \in SL_2(\mathcal{O}_\pi),$$
$$\text{where } S_\kappa(g) = \beta_\pi(r, h) \text{ when } g = rh \text{ for } r \in R, h \in SL_2(\mathcal{O}_\pi, 4).$$

Proposition 3.3.2. *Suppose that $X_g(\alpha) = 1$ for all $g \in SL_2(\mathcal{O}_\pi)$ and for all $\alpha \in SL_2(\mathcal{O}_\pi, 4)$. Then, the cocycle Σ is defined on the group $SL_2(\mathcal{O}_\pi)/SL_2(\mathcal{O}_\pi, 4) \cong SL_2(\mathcal{O}/4)$. That is,*

$$\Sigma(g_1, g_2) = \Sigma(g_1', g_2') \text{ if } g_1 \equiv g_1' \ (\text{mod } 4) \text{ and } g_2 \equiv g_2' \ (\text{mod } 4).$$

Thus Σ can be regarded as a 2-cocycle on the finite group G.

Proof. Observe that the function X_g can be re-written:

$$X_g(\alpha) = \{\alpha, 1\}^g \{\alpha^g, 1\}^{-1} \text{ where } \alpha^g = g^{-1}\alpha g.$$

If $X_g(\alpha) = 1$ for all $g \in SL_2(\mathcal{O}_\pi)$ and for all $\alpha \in SL_2(\mathcal{O}_\pi, 4)$, then

$$\{\alpha^g, 1\} = \{\alpha, 1\}^g. \tag{3.16}$$

If $g \in SL_2(\mathcal{O}_\pi)$, write $g = rh$ for $r \in R$ and $h \in SL_2(\mathcal{O}_\pi, 4)$, and recall that $S(g) = \{g, \beta_\pi(r, h)\}$. If $\gamma \in SL_2(\mathcal{O}_\pi, 4)$, then

$$\begin{aligned}
S(g\gamma) =& \{g\gamma, \beta_\pi(r, h\gamma)\} \text{ since } g\gamma = r(h\gamma), \\
=& \{g, \beta_\pi(r, h\gamma)\}\{\gamma, 1\}\{1, \beta_\pi(g, \gamma)^{-1}\} \\
=& \{g, \beta_\pi(rh, \gamma)\beta_\pi(r, h)\beta_\pi(h, \gamma)^{-1}\}\{\gamma, 1\}\{1, \beta_\pi(g, \gamma)^{-1}\} \text{ by the cocycle identity (1.1)} \\
=& \{g, \beta_\pi(g, \gamma)\beta_\pi(r, h)\}\{\gamma, 1\}\{1, \beta_\pi(g, \gamma)^{-1}\} \text{ since } \beta_\pi(h, \gamma) = 1 \\
=& \{g, \beta_\pi(g, \gamma)\beta_\pi(r, h)\}\{1, \beta_\pi(g, \gamma)^{-1}\}\{\gamma, 1\} \\
=& \{g, \beta_\pi(r, h)\}\{\gamma, 1\} \\
=& S(g)S(\gamma).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
S(\gamma g) =& S(g\gamma^g) = S(g)\{\gamma^g, 1\} \\
=& S(g)\{\gamma, 1\}^g \text{ by (3.16)} \\
=& S(g)S(g)^{-1}\{\gamma, 1\}S(g) \\
=& \{\gamma, 1\}S(g) \\
=& S(\gamma)S(g).
\end{aligned}$$

Observe that if $g \equiv g' \pmod 4$ then $g' = g\gamma$ for some $\gamma \in SL_2(\mathcal{O}_\pi, 4)$, and $g' = \gamma' g$ for some $\gamma' \in SL_2(\mathcal{O}_\pi, 4)$. Thus, if $g_1, g_2 \in SL_2(\mathcal{O}_\pi)$ and $\gamma \in SL_2(\mathcal{O}_\pi, 4)$, then

$$\begin{aligned}\Sigma(\gamma g_1, g_2) =& S(\gamma g_1)S(g_2)S(\gamma g_1 g_2)^{-1} \\ =& \{\gamma, 1\}S(g_1)S(g_2)(\{\gamma, 1\}S(g_1 g_2))^{-1} \\ =& \{\gamma, 1\}S(g_1)S(g_2)S(g_1 g_2)^{-1}\{\gamma, 1\}^{-1} \\ =& \{\gamma, 1\}\Sigma(g_1, g_2)\{\gamma, 1\}^{-1} \\ =& \Sigma(g_1, g_2).\end{aligned}$$

$$\begin{aligned}\Sigma(g_1, g_2\gamma) =& S(g_1)S(g_2\gamma)S(g_1 g_2 \gamma)^{-1} \\ =& S(g_1)S(g_2)\{\gamma, 1\}(S(g_1 g_2)\{\gamma, 1\})^{-1} \\ =& S(g_1)S(g_2)\{\gamma, 1\}\{\gamma, 1\}^{-1}S(g_1 g_2)^{-1} \\ =& \Sigma(g_1, g_2).\end{aligned}$$

To finish the proof of Proposition 3.3.2, we must show that the function $X_g(\alpha)$ is identically 1 for all $g \in SL_2(\mathcal{O}_\pi)$ and all $\alpha \in SL_2(\mathcal{O}_\pi, 4)$. To prove this, we need a further result:

Theorem 3.3.3. *Let σ be a 1-cocycle on a group L generated by $l_1, \cdots, l_n$. If $\sigma(l_i) = 1$ for all i, then $\sigma = 1$.*

One can prove Theorem 3.3.3 by induction.

Lemma 3.3.4. *The function $X_g(\alpha)$ takes the value 1 at all $\alpha \in SL_2(\mathcal{O}_\pi, 4)$ and all $g \in SL_2(\mathcal{O}_\pi)$.*

Proof. We will use the following properties of the function $X_g(\alpha)$:

1. As a function of α, $X_g \in \mathrm{Hom}(SL_2(\mathcal{O}_\pi, 4), \mu_2)$. Indeed, if $\alpha_1, \alpha_2 \in SL_2(\mathcal{O}_\pi, 4)$,

a simple proof gives:

$$\begin{aligned}
X_g(\alpha_1\alpha_2) =&\tau((\alpha_1\alpha_2)^g)\{\tau(\alpha_1\alpha_2)^{\tau(g)}\}^{-1}\\
=&\tau(\alpha_1^g\alpha_2^g)\{\tau(\alpha_1\alpha_2)^{\tau(g)}\}^{-1}\\
=&\tau(\alpha_1^g)\tau(\alpha_2^g)\{\tau(\alpha_1\alpha_2)^{\tau(g)}\}^{-1}\\
=&\tau(\alpha_1^g)\tau(\alpha_2^g)\{\tau(\alpha_1)^{\tau(g)}\tau(\alpha_2)^{\tau(g)}\}^{-1}\\
=&\tau(\alpha_1^g)X_g(\alpha_2)\{\tau(\alpha_1)^{\tau(g)}\}^{-1}\\
=&X_g(\alpha_1)X_g(\alpha_2)
\end{aligned}$$

2. As a function of g, $X_g \in Z^1(SL_2(\mathcal{O}_\pi), \mathrm{Hom}(SL_2(\mathcal{O}_\pi, 4), \mu_2))$ is a 1-cocycle; explicitly,

$$X_{g_1g_2}(\alpha) = X_{g_1}(\alpha)X_{g_2}(\alpha^{g_1}).$$

If $\alpha_1, \alpha_2 \in SL_2(\mathcal{O}_\pi, 4)$, then by property (2),

$$X_{g\alpha_1}(\alpha_2) = X_g(\alpha_2)X_{\alpha_1}(g^{-1}\alpha_2 g) = X_g(\alpha_2). \tag{3.17}$$

Recall that in Section 1.1, B_π was used to denote the Borel subgroup of $GL_2(F_\pi)$. Suppose that $b \in B_\pi \cap SL_2(\mathcal{O}_\pi)$. Then $X_b(\alpha) = 1$ for all $\alpha \in SL_2(\mathcal{O}_\pi, 4)$. Hence by property (2),

$$X_{gb}(\alpha) = X_g(\alpha). \tag{3.18}$$

Furthermore, if $\alpha_1 \equiv \alpha_2 \pmod{\pi^6}$: that is, if $\alpha_1\alpha_2^{-1} \in SL_2(\mathcal{O}_\pi, \pi^6)$, then

$$X_g(\alpha_1) = X_g(\alpha_2) \text{ by part (4) of Proposition 1.1.1.} \tag{3.19}$$

Equation (3.17) shows that is it sufficient to prove that $X_g(\alpha)$ is trivial for g in the coset space $G = SL_2(\mathcal{O}_\pi)/SL_2(\mathcal{O}_\pi, 4)$. Observe that X_g is still a 1-cocycle on G: if $g_1, g_2 \in SL_2(\mathcal{O}_\pi)$ and $\alpha_1, \alpha_2, \beta \in SL_2(\mathcal{O}_\pi, 4)$,

$$\begin{aligned}
X_{g_1\alpha_1g_2\alpha_2}(\beta) =&X_{g_1g_2^{(\alpha^{-1})}\alpha_1\alpha_2}(\beta)\\
=&X_{g_1g_2^{(\alpha^{-1})}}(\beta)\\
=&X_{g_1}(\beta)X_{g_2^{(\alpha^{-1})}}(\beta^{g_1})\\
=&X_{g_1}(\beta)X_{g_2}((\beta^{g_1})^{\alpha_1})\\
=&X_{g_1}(\beta)X_{g_2}(\beta^{g_1\alpha_1})\\
=&X_{g_1\alpha_1}(\beta)X_{g_2\alpha_2}(\beta^{g_1\alpha_1}).
\end{aligned}$$

Let $B(\mathcal{O}/4)$ be the subgroup of G defined by:

$$B(\mathcal{O}/4) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in (\mathcal{O}/4)^\times, b \in \mathcal{O}/4 \right\}$$

The group G is generated by $B(\mathcal{O}/4)$ and the matrix $g_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, by Theorem 3.3.3 and equations (3.17) - (3.19), it is sufficient to show that $X_g(\alpha)$ is trivial for $g = g_0$, and for all α_i in a set of generators for the group $SL_2(\mathcal{O}_\pi, \pi^4)/SL_2(\mathcal{O}_\pi, \pi^6)$.

A set of generators $\alpha_1, \cdots, \alpha_6$ for the abelian group $SL_2(\mathcal{O}_\pi, \pi^4)/SL_2(\mathcal{O}_\pi, \pi^6)$ is given by:

$$\alpha_1 = \begin{pmatrix} 1+\pi^4+i\pi^6 & \pi^6 \\ -\pi^6 & 1-\pi^4+i\pi^6 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1+i\pi^4-i\pi^6 & \pi^6 \\ -\pi^6 & 1-i\pi^4-i\pi^6 \end{pmatrix},$$
$$\alpha_3 = \begin{pmatrix} 1 & \pi^4 \\ 0 & 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 & i\pi^4 \\ 0 & 1 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 1 & 0 \\ \pi^4 & 1 \end{pmatrix}, \alpha_6 = \begin{pmatrix} 1 & 0 \\ i\pi^4 & 1 \end{pmatrix}.$$

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general element of $SL_2(\mathcal{O}_\pi, \pi^4)/SL_2(\mathcal{O}_\pi, \pi^6)$.

$$\begin{aligned} X_{g_0}(\alpha) &= \frac{1}{(a,b)_\pi^{v(b)}(b,d)_\pi(c,d)_\pi^{v(c)+1}} \text{ if } c \neq 0, b \neq 0, \text{ and} \\ X_{g_0}(\alpha) &= \frac{1}{(a,d)_\pi(c,d)_\pi^{1+v(c)}} \text{ when } c \neq 0, b = 0, \text{ and} \\ X_{g_0}(\alpha) &= \frac{1}{(b,d)_\pi(a,b)_\pi^{v(b)}} \text{ when } c = 0. \end{aligned}$$

However, if both $c \neq 0$ and $b \neq 0$, then both b and c are squares; if $c \neq 0$ but $b = 0$ then both a and d equal 1; if $c = 0$ then again, $a = d = 1$. Thus $X_{g_0}(\alpha_i) = 1$ for all $1 \leq i \leq 6$.

Therefore, Lemma 3.3.4, and hence Proposition 3.3.2, are proved.

□

Observe that we have constructed an extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1, \tag{3.20}$$

with an associated 2-cocycle Σ.

3.3.2 Calculation of the genuine quotient

Consider the double cosets $\widehat{H}\{1,1\}\widehat{H}$ and $\widehat{H}\{1,-1\}\widehat{H}$. The latter generates the group ring $\mathbb{C}[\widehat{H}\{1,-1\}\widehat{H}, \widehat{H}\{1,1\}\widehat{H}]$ which is easily seen to be contained in the centre of $\mathcal{H}(\overline{G},\widehat{H})$. In fact, there is an isomorphism

$$\begin{aligned}\mathbb{C}[\widehat{H}\{1,-1\}\widehat{H}, \widehat{H}\{1,1\}\widehat{H}] &\longrightarrow \mathbb{C}\oplus\mathbb{C}\\ \widehat{H}\{1,-1\}\widehat{H} &\longmapsto (1,-1)\end{aligned}$$

Schur's Lemma [8, p. 430] implies that on any irreducible (complex) representation of $\mathcal{H}(\overline{G},\widehat{H})$, the double coset $\widehat{H}\{1,-1\}\widehat{H}$ acts as 1 or -1. We shall define the *genuine* Hecke algebra $\mathcal{H}(\overline{G},\widehat{H})_{\mathrm{gen}}$ to be the quotient of $\mathcal{H}(\overline{G},\widehat{H})$ in which $\widehat{H}\{1,-1\}\widehat{H}$ is identified with the scalar $-1\in\mathbb{C}$. That is,

$$\mathcal{H}(\overline{G},\widehat{H})_{\mathrm{gen}} = \mathcal{H}(\overline{G},\widehat{H}) / \left(\widehat{H}\{g,-1\}\widehat{H} + \widehat{H}\{g,1\}\widehat{H}\right).$$

Thus in $\mathcal{H}(\overline{G},\widehat{H})_{\mathrm{gen}}$, $\widehat{H}\{g,-1\}\widehat{H} = -\widehat{H}\{g,1\}\widehat{H}$. If we write $\hat{g}$ for the double coset $\widehat{H}\{g,1\}\widehat{H}$, then $-\hat{g}$ shall mean $\widehat{H}\{g,-1\}\widehat{H}$.

Similarly, put

$$\overline{\mathcal{H}} = \mathcal{H}(\overline{SL_2}(\mathcal{O}_\pi), \widehat{K}_\pi(4)) / \left(\widehat{K}_\pi(4)\{g,-1\}\widehat{K}_\pi(4) + \widehat{K}_\pi(4)\{g,1\}\widehat{K}_\pi(4)\right).$$

The right-hand side is the quotient of $\mathcal{H}(\overline{SL_2}(\mathcal{O}_\pi), \widehat{K}_\pi(4))$ in which $\widehat{K}_\pi(4)\{1,-1\}\widehat{K}_\pi(4)$ is identified with -1. In analogy with Lemma 3.2.1, we have an isomorphism of rings

$$\overline{\mathcal{H}} \cong \mathcal{H}(\overline{G},\widehat{H})_{\mathrm{gen}}.$$

In this section, we shall calculate $\overline{\mathcal{H}}$.

Recall the extension (3.20). Let p be the map $\overline{G}\to G$. The pre-image of a double coset $HgH\subset G$ is a union

$$p^{-1}(HgH) = \widehat{H}\{g,1\}\widehat{H} \cup \widehat{H}\{g,-1\}\widehat{H}. \tag{3.21}$$

Note that, a priori, the union may or may not be disjoint. However,

Lemma 3.3.5. *For each g in a set of representatives for $H\backslash G/H$, the union* (3.21) *is disjoint.*

Proof. For each $g \in G$, there are two cases to consider. Either,

$$\widehat{H}\{g,-1\}\widehat{H} = \widehat{H}\{g,1\}\widehat{H}, \text{ or}$$
$$\widehat{H}\{g,-1\}\widehat{H} \cap \widehat{H}\{g,1\}\widehat{H} = \emptyset$$

Now, $\widehat{H}\{g,-1\}\widehat{H} = \widehat{H}\{g,1\}\widehat{H}$ if and only if $\{g,-1\} \in \widehat{H}\{g,1\}\widehat{H}$. We can write:

$$\begin{aligned}\widehat{H}\{g,1\}\widehat{H} &= \{\{h,1\}\{g,1\}\{h',1\} \ : h,h' \in H\} \\ &= \{\{hg,\Sigma(h,g)\}\{h',1\} \ : h,h' \in H\} \\ &= \{\{hgh',\Sigma(h,g)\Sigma(hg,h')\} \ : h,h' \in H\}\end{aligned}$$

Hence $\{g,-1\} \in \widehat{H}\{g,1\}\widehat{H}$ if:

$$\text{there is an } h \in H \cap gHg^{-1} \text{ such that } \Sigma(h,g)\Sigma(hg,g^{-1}h^{-1}g) = -1. \tag{3.22}$$

Recall the environment:

```
F.<i> = NumberField(x^2+1)
R = F.ring_of_integers()
pi = F.ideal(1+i)
k = R.residue_field(pi,'b')
kk = R.quotient_ring(2,'b')
kkk = R.quotient_ring(2*pi,'b')
kkkk = R.quotient_ring(4,'b')
kkkkk = R.quotient_ring(4*pi,'b')
M = MatrixSpace(F,2)
m = MatrixSpace(k,2)
mm = MatrixSpace(kk,2)
mmm = MatrixSpace(kkk,2)
mmmm = MatrixSpace(kkkk,2)
```

The following four algorithms calculate the Hilbert symbol $(x,y)_\pi$ for two integral elements $x,y \in \mathcal{O}_\pi$. The first, "wild hilbert symbol pi", takes an argument $x \in F^\times$, and produces the symbol $(\pi,x)_\pi$. The second, "wild hilbert symbol i", takes an argument $x \in F^\times$, and gives the output $(i,x)_\pi$. The third, "wild hilbert symbol odd", is defined for two odd elements $x,y \in F^\times$, and the fourth, "wild hilbert symbol", simply assumes that x and y are in $F^\times$.

```
def wild_hilbert_symbol_pi(x):
    val = x.valuation(pi)
    newx = x*(1+i)^-val
    test = kkk(newx)
    while test != 1:
        newx = newx *i
        test = kkk(newx)
    a = newx.trace()/2
    b = (newx*-i).trace()/2
    return (-1)^((a-b-b^2-1)/4)

def wild_hilbert_symbol_i(x):
    val = x.valuation(pi)
    newx = x*(1+i)^-val
    return (-1)^((newx.norm()-1)/4)

def wild_hilbert_symbol_odd(x,y):
    if kk(x) == kk(i):
        temp1 = wild_hilbert_symbol_i(y)
    else:
        temp1 = 1
    if kk(y) == kk(i):
        temp2 = wild_hilbert_symbol_i(x)
    else:
        temp2 = 1
    return temp1*temp2

def wild_hilbert_symbol(x,y):
    valx = x.valuation(pi)
    valy = y.valuation(pi)
    oddx = x*(1+i)^-valx
    oddy = y*(1+i)^-valy
```

```
temp1 = wild_hilbert_symbol_pi(oddx)^valy
temp2 = wild_hilbert_symbol_pi(oddy)^valx
temp3 = wild_hilbert_symbol_odd(oddx, oddy)
return temp1*temp2*temp3
```

Recall that if $g \in SL_2(\mathcal{O}_\pi)$, we can decompose it as $g = rh$ where r belongs to a set of representatives for $SL_2(\mathcal{O}_\pi)/SL_2(\mathcal{O}_\pi, 4)$ and $h \in SL_2(\mathcal{O}_\pi, 4)$. Since the section S corresponding to the cocycle Σ is defined by

$$S(g) = \{g, \beta_\pi(r, h)\},$$

it is necessary to implement both the decomposition of a general element g and the cocycle β_π. This we do below. For $g \in SL_2(\mathcal{O}_\pi)$, "S decomposition($g$)" returns a decomposition $g = rep * \gamma$ where "rep" is a representative of the congruency class of g modulo 4, and γ is in $SL_2(\mathcal{O}_\pi, 4)$.

```
congruencequotient0m = [m([1,0,0,1]),m([0,1,1,0]),m([1,1,0,1]),
m([1,0,1,1]),m([1,1,1,0]),m([0,1,1,1])]

congruencequotient1mm = [mm([1,0,0,1]),mm([1,(1+i),0,1]),
mm([1,0,(1+i),1]),mm([1,(1+i),(1+i),1]),
mm([1+(1+i),(1+i),(1+i),1+(1+i)]),mm([1+(1+i),0,0,1+(1+i)]),
mm([1+(1+i),0,(1+i),1+(1+i)]),mm([1+(1+i),(1+i),0,1+(1+i)])]

congruencequotient2mmm = [mmm([1,0,0,1]),mmm([1,(1+i)^2,0,1]),
mmm([1,0,(1+i)^2,1]),mmm([1,(1+i)^2,(1+i)^2,1]),
mmm([1+(1+i)^2,(1+i)^2,(1+i)^2,1+(1+i)^2]),
mmm([1+(1+i)^2,0,0,1+(1+i)^2]),
mmm([1+(1+i)^2,0,(1+i)^2,1+(1+i)^2]),
mmm([1+(1+i)^2,(1+i)^2,0,1+(1+i)^2])]

congruencequotient3mmmm = [mmmm([1,0,0,1]),mmmm([1,(1+i)^3,0,1]),
mmmm([1,0,(1+i)^3,1]),mmmm([1,(1+i)^3,(1+i)^3,1]),
mmmm([1+(1+i)^3,(1+i)^3,(1+i)^3,1+(1+i)^3]),
```

```
mmmm([1+(1+i)^3,0,0,1+(1+i)^3]),
mmmm([1+(1+i)^3,0,(1+i)^3,1+(1+i)^3]),
mmmm([1+(1+i)^3,(1+i)^3,0,1+(1+i)^3])]



lifts0 = [M([1,0,0,1]),M([0,1,-1,0]),M([1,1,0,1]),M([1,0,1,1]),
M([1,-1,1,0]), M([0,-1,1,1])]

lifts1 = [M([1,0,0,1]),M([1,(1+i),0,1]),M([1,0,(1+i),1]),
M([1+(1+i)^2,1+i,1+i,1]),M([1+(1+i),-(1+i),(1+i),1-(1+i)]),
M([1+(1+i)+(1+i)^2,-3*(1+i)^2,(1+i)^2,1+(1+i)+i*(1+i)^4]),
M([1+(1+i) +(1+i)^2 - 3*(1+i)^3,-3*(1+i)^2,
-i*(1+i)^4 + (1+i) + i*(1+i)^5,1+(1+i)+i*(1+i)^4]),
M([1+(1+i)+(1+i)^2,-3*(1+i)^2,(1+i)^2,1+(1+i)+i*(1+i)^4])
*M([1,1+i,0,1])]

lifts2 = [M([1,0,0,1]),M([1,(1+i)^2,0,1]),M([1,0,(1+i)^2,1]),
M([1+(1+i)^4,(1+i)^2,(1+i)^2,1]),
M([1+(1+i)^2,-(1+i)^2,(1+i)^2, 1-(1+i)^2]),
M([1+(1+i)^2 +i*(1+i)^3,-(1+i)^4 - i*(1+i)^5,
-i*(1+i)^3 + (1+i)^4,1+(1+i)^2 - i*(1+i)^3 + i*(1+i)^5 - (1+i)^6]),
M([1+(1+i)^2+ i*(1+i)^3,-i*(1+i)^3 + (1+i)^4,(1+i)^2,
1+(1+i)^2 - i*(1+i)^3]),
M([1+(1+i)^2+ i*(1+i)^3,(1+i)^2,-i*(1+i)^3 + (1+i)^4,
1+(1+i)^2 - i*(1+i)^3])]

lifts3 = [M([1,0,0,1]),M([1,(1+i)^3,0,1]),M([1,0,(1+i)^3,1]),
M([1+(1+i)^6,(1+i)^3,(1+i)^3,1]),
M([1+(1+i)^3,-(1+i)^3,(1+i)^3,1-(1+i)^3]),
M([1+(1+i)^3 + (1+i)^6,i*(1+i)^5 - (1+i)^6 - (1+i)^9,(1+i)^6,
1-(1+i)^3 - (1+i)^9]),
M([1+(1+i)^3 + (1+i)^6,i*(1+i)^5 - (1+i)^6 - (1+i)^9,(1+i)^6,
```

```
1-(1+i)^3 - (1+i)^9])*M([1,0,(1+i)^3,1]),
M([1+(1+i)^3 + (1+i)^6,i*(1+i)^5 - (1+i)^6 - (1+i)^9,(1+i)^6,
1-(1+i)^3 - (1+i)^9])*M([1,(1+i)^3,0,1])]

def S_decomposition(g):
    r = congruencequotient0m.index(m(g))
    rep0 = lifts0[r]
    g2 = rep0.inverse()*g
    s = congruencequotient1mm.index(mm(g2))
    rep1 = lifts1[s]
    g3 = rep1.adjoint()*g2
    j = congruencequotient2mmm.index(mmm(g3))
    rep2 = lifts2[j]
    g4 = rep2.adjoint()*g3
    n = congruencequotient3mmmm.index(mmmm(g4))
    rep3 = lifts3[n]
    gamma = rep3.adjoint()*g4
    rep = rep0*rep1*rep2*rep3
    return([rep,gamma])
```

The following code is used to calculate the cocycle Σ. This is done in a number of steps: "Sigma1" is a cocycle which is not defined modulo 4, and is not normalised, in the sense that when $k, k' \in SL_2(\mathcal{O}_\pi, 4)$, Sigma1$(k, k')$ is not necessarily 1.

```
def X(g):
    if g[1,0] == 0:
        return(g[1,1])
    else:
        return(g[1,0])

def Sigma1(g,h):
    Xgh = X(g*h)
    Xg = X(g)
```

```
    Xh = X(h)
    return wild_hilbert_symbol(Xgh*Xg,Xgh*Xh)
```

The function "kappa pi" is the splitting $SL_2(\mathcal{O}_\pi, 4) \to \mu_2$, and "sigma2" is the cocycle β_π: that is, it takes the value 1 on $SL_2(\mathcal{O}_\pi, 4) \times SL_2(\mathcal{O}_\pi, 4)$ but it is not defined on $SL_2(\mathcal{O}/4)$.

```
def kappa_pi(g):
    c = g[1,0]
    d= g[1,1]
    if c*d ==0:
        return 1
    elif c.valuation(pi)%2:
        return wild_hilbert_symbol(c,d)
    else:
        return 1


def Sigma2(g,h, gh):
    Xgh = X(gh)
    Xg = X(g)
    Xh = X(h)
    return wild_hilbert_symbol(Xgh*Xg, Xgh*Xh) *
                kappa_pi(g)*kappa_pi(h)*kappa_pi(gh)
```

Below, the function "S(g)" is the section $S : SL_2(\mathcal{O}_\pi) \to \overline{SL_2}(\mathcal{O}_\pi)$ defining Σ, and "Sigma3" is the cocycle Σ itself.

```
def S(g):
    sdecomp = S_decomposition(g)
    rep = sdecomp[0]
    gamma = sdecomp[1]
    Xg = X(g)
    Xgamma = X(gamma)
    Xrep = X(rep)
```

```
    return wild_hilbert_symbol(Xg*Xgamma, Xg*Xrep)
                   *kappa_pi(gamma)*kappa_pi(rep)*kappa_pi(g)


def Skappa(g):
    sdecomp = S_decomposition(g)
    rep = sdecomp[0]
    gamma = sdecomp[1]
    Xg = X(g)
    Xgamma = X(gamma)
    Xrep = X(rep)
    return wild_hilbert_symbol(Xg*Xgamma, Xg*Xrep)
                   *kappa_pi(gamma)*kappa_pi(rep)


def Sigma3(g,h,gh):
    Xgh = X(gh)
    Xg = X(g)
    Xh = X(h)
    return wild_hilbert_symbol(Xgh*Xg, Xgh*Xh)
                   *Skappa(g)*Skappa(h)*Skappa(gh)
```

Next, we define the group H. Note that it is necessary to lift all the elements of H to $SL_2(\mathbb{Z})$ in order to use the cocycle Σ. We call this set of lifts "Hlift".

```
H1 = [mmmm([1,0,0,1]), mmmm([1,1,0,1]),mmmm([1,-1,1,0]),
mmmm([0,-1,1,0]), mmmm([1,0,1,1]), mmmm([0,-1,1,1])]
H2 = [mmmm([1,0,0,1]), mmmm([-1,0,0,-1])]
H3 = [mmmm([1,0,0,1]), mmmm([1,2,0,1])]
H4 = [mmmm([1,0,0,1]), mmmm([1,0,2,1])]

H = [a*b*c*d for a in H1 for b in H2 for c in H3 for d in H4]

H1lift = [M([1,0,0,1]), M([1,1,0,1]),M([1,-1,1,0]),
M([0,-1,1,0]), M([1,0,1,1]), M([0,-1,1,1])]
```

```
H2lift = [M([1,0,0,1]), M([-1,0,0,-1])]
H3lift = [M([1,0,0,1]), M([1,2,0,1])]
H4lift = [M([1,0,0,1]), M([1,0,2,1])]

Hlift = [a*b*c*d for a in H1lift for b in H2lift
for c in H3lift for d in H4lift]
```

We define the representatives for the double cosets $H\backslash G/H$:

```
x = mmmm([1,1+i,0,1])
y = mmmm([1,(1+i)^3,0,1])
u = mmmm([1,(1+i),(1+i)^3,1])
t = mmmm([2+i,0,0,2-i])
z = mmmm([1+2*i,0,0,1+2*i])
xt = x*t
tx = t*x
txt = t*x*t
yt = y*t
yz = y*z
ytz = y*t*z
ut = u*t
tz = t*z

xlift = M([1,1+i,0,1])
ylift = M([1,(1+i)^3,0,1])
ulift = M([-3,1+i,2*i-2,1])
tlift = M([6+i,4,-4+4*i,-2+3*i])
zlift = M([1+2*i,4,-4,-3+6*i])
xtlift = xlift*tlift
txlift = tlift*xlift
txtlift = tlift*xlift*tlift
ytlift = ylift*tlift
```

```
yzlift = ylift*zlift
ytzlift = ylift*tlift*zlift
utlift = ulift*tlift
tzlift = tlift*zlift

Hdoublecosets = [mmmm([1,0,0,1]),z,t,tz,x,xt,tx,txt,y,yz,yt,ytz,u,ut]

Hdoublecosets_inverses = [g.adjoint() for g in Hdoublecosets]

Hdoublecosetlifts =  [M([1,0,0,1]),zlift,tlift,tzlift,xlift,
xtlift,txlift,txtlift,ylift,yzlift,ytlift,ytzlift,ulift,utlift]
```

Finally, we test statement (3.22) with the following programme:

```
for h in Hlift:
    hinverse = h.adjoint()
    for g in Hdoublecosets:
        glift = Hdoublecosetlifts[Hdoublecosets.index(g)]
        gliftinverse = glift.adjoint()
        temp = glift*h*gliftinverse
        if temp in Hlift:
            print Sigma3(temp, glift, temp*glift)
*Sigma3(temp*glift,hinverse,temp*glift*hinverse)
```

Since this returned the value 1 in all cases, Lemma 3.3.5 is proved.

We have shown that the dimension of $\overline{\mathcal{H}}$, as a $\mathbb{C}$-vector space, is 14. To find its structure as an algebra, we must multiply elements. Observe that

$$\begin{aligned}\widehat{H}\{g,1\}\widehat{H} &= \bigcup_{h\in H} \widehat{H}\{g,1\}\{h,1\}\\ &= \bigcup_{h\in H/(H^g\cap H)} \widehat{H}\{gh,\Sigma(g,h)\}\end{aligned}$$

The following programme computes $\Sigma(g,h)$ for each g defined by the basis $\{\widehat{H}\{g,1\}\widehat{H}\}$ of $\overline{\mathcal{H}}$, and for each $h\in H/(H^g\cap H)$. For each g, 'hforg' is a list of h such that Hgh is a single coset inside HgH.

```
hfor1 = [mmmm([1,0,0,1])]
hforz = hfor1
hfort = hfor1
hfortz = hfor1
hforx = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1]),
mmmm([1,0,2,1]),mmmm([0,-1,1,0]),mmmm([1,0,-1,1])]
hforxt = hforx
hfortx = hforx
hfortxt = hforx
hfory = [mmmm([1,0,0,1]),mmmm([1,0,1,1]),mmmm([1,-1,1,0])]
hforyz = hfory
hforyt = hfory
hforytz = hfory
hforu =  [mmmm([1,0,0,1]),mmmm([1,1,0,1]),mmmm([1,-1,1,0]),
mmmm([0,-1,1,0]),mmmm([1,0,1,1]),mmmm([0,-1,1,1]),mmmm([1,0,2,1]),
mmmm([-1,-1,2,1]),mmmm([0,-1,1,2]),mmmm([1,1,1,2]),
mmmm([0,-1,1,-1]),mmmm([-1,0,1,-1])]
hfortu = hforu

h_for = [hfor1,hfort,hforz,hfortz,hforx,hforxt,hfortx,hfortxt,
hfory,hforyz, hforyt,hforytz,hforu,hfortu]
```

We must find lifts to characteristic zero.

```
hfor1lift = [M([1,0,0,1])]
hforzlift = hfor1lift
hfortlift = hfor1lift
hfortzlift = hfor1lift
hforxlift = [M([1,0,0,1]),M([1,1,1,2]),M([1,0,1,1]),M([1,0,2,1]),
M([0,-1,1,0]),M([1,0,-1,1])]
hforxtlift = hforxlift
hfortxlift = hforxlift
hfortxtlift = hforxlift
```

```
hforylift = [M([1,0,0,1]),M([1,0,1,1]),M([1,-1,1,0])]
hforyzlift = hforylift
hforytlift = hforylift
hforytzlift = hforylift
hforulift = [M([1,0,0,1]),M([1,1,0,1]),M([1,-1,1,0]),M([0,-1,1,0]),
M([1,0,1,1]),M([0,-1,1,1]),M([1,0,2,1]),M([-1,-1,2,1]),
M([0,-1,1,2]),M([1,1,1,2]),M([0,-1,1,-1]),M([-1,0,1,-1])]
hfortulift = hforulift

h_for_lifts = [hfor1lift,hfortlift,hforzlift,hfortzlift,hforxlift,
hforxtlift,hfortxlift,hfortxtlift,hforylift,hforyzlift,hforytlift,
hforytzlift,hforulift,hfortulift]
```

'HgH' is a list of the single cosets Hgh inside the double coset HgH, and 'HgHmet' expresses each double coset $\widehat{H}\{g,1\}\widehat{H}$ as a list of single cosets $\widehat{H}\{gh,\Sigma(g,h)\}$.

```
HgH = [[Hdoublecosets[index]*h for h in h_for[index]]
for index in range(14)]

HgHmet = [[[Hdoublecosetlifts[index]*hlift,
Sigma3(Hdoublecosetlifts[index],hlift,Hdoublecosetlifts[index]*hlift)]
for hlift in h_for_lifts[index]] for index in range(14)]
```

'Hecke multiply 2" multiplies all the double cosets in $\overline{\mathcal{H}}$.

```
Hecke_algebra = ZZ^14
var('HZ,HT,HTZ,HXH,HXTH,HTXH,HTXTH,HYH,HYZH,HYTH,HYTZH,HUH,HUTH')
evaluation = matrix(14,
[1,HZ,HT,HTZ,HXH,HXTH,HTXH,HTXTH,HYH,HYZH,HYTH,HYTZH,HUH,HUTH])

def Hecke_multiply2(r,s):
     sum_of_answers = Hecke_algebra(0)
     Hg1H = HgH[r]
     Hg2H = HgH[s]
```

```
        Hg1Hmet = HgHmet[r]
        Hg2Hmet = HgHmet[s]
        for a in Hg1H:
            for b in Hg2H:
                answer = a*b
                for rep_inverse in Hdoublecosets_inverses:
                    test = answer*rep_inverse
                    if test in H:
                        alift = Hg1Hmet[Hg1H.index(a)]
                        blift = Hg2Hmet[Hg2H.index(b)]
                        ablift = alift[0]*blift[0]
                        ind = Hdoublecosets_inverses.index(rep_inverse)
                        replift = Hdoublecosetlifts[ind]
                        hlift = replift*ablift.inverse()
                        twist1 = Sigma3(hlift,ablift,replift)
                        twist2 = Sigma3(alift[0],blift[0],ablift)
                        twist = twist1 * twist2 * alift[1] * blift[1]
                        sum_of_answers[ind] = sum_of_answers[ind] +twist
        return((sum_of_answers*evaluation)[0])


    for r in range(14):
        for s in range(14):
            ans = Hecke_multiply2(r,s)
            print(evaluation[r],evaluation[s],ans)
```

This yields the following multiplication table for the genuine Hecke algebra $\overline{\mathcal{H}}$:

```
((1), (1), 1)
((1), (HZ), HZ)
((1), (HT), HT)
((1), (HTZ), HTZ)
((1), (HXH), HXH)
((1), (HXTH), HXTH)
```

```
((1), (HTXH), HTXH)
((1), (HTXTH), HTXTH)
((1), (HYH), HYH)
((1), (HYZH), HYZH)
((1), (HYTH), HYTH)
((1), (HYTZH), HYTZH)
((1), (HUH), HUH)
((1), (HUTH), HUTH)
((HZ), (1), HZ)
((HZ), (HZ), 1)
((HZ), (HT), -HTZ)
((HZ), (HTZ), -HT)
((HZ), (HXH), -HXH)
((HZ), (HXTH), -HXTH)
((HZ), (HTXH), -HTXH)
((HZ), (HTXTH), -HTXTH)
((HZ), (HYH), HYZH)
((HZ), (HYZH), HYH)
((HZ), (HYTH), -HYTZH)
((HZ), (HYTZH), -HYTH)
((HZ), (HUH), -HUH)
((HZ), (HUTH), -HUTH)
((HT), (1), HT)
((HT), (HZ), -HTZ)
((HT), (HT), 1)
((HT), (HTZ), -HZ)
((HT), (HXH), HTXH)
((HT), (HXTH), HTXTH)
((HT), (HTXH), HXH)
((HT), (HTXTH), HXTH)
((HT), (HYH), HYTH)
((HT), (HYZH), -HYTZH)
```

```
((HT), (HYTH), HYH)
((HT), (HYTZH), -HYZH)
((HT), (HUH), -HUTH)
((HT), (HUTH), -HUH)
((HTZ), (1), HTZ)
((HTZ), (HZ), -HT)
((HTZ), (HT), -HZ)
((HTZ), (HTZ), 1)
((HTZ), (HXH), HTXH)
((HTZ), (HXTH), HTXTH)
((HTZ), (HTXH), HXH)
((HTZ), (HTXTH), HXTH)
((HTZ), (HYH), HYTZH)
((HTZ), (HYZH), -HYTH)
((HTZ), (HYTH), -HYZH)
((HTZ), (HYTZH), HYH)
((HTZ), (HUH), -HUTH)
((HTZ), (HUTH), -HUH)
((HXH), (1), HXH)
((HXH), (HZ), -HXH)
((HXH), (HT), HXTH)
((HXH), (HTZ), HXTH)
((HXH), (HXH), 2*HUTH + 2*HYH - 2*HYZH)
((HXH), (HXTH), 2*HUH + 2*HYTH + 2*HYTZH)
((HXH), (HTXH), 6*HT + 6*HTZ + 4*HXH)
((HXH), (HTXTH), 4*HXTH - 6*HZ + 6)
((HXH), (HYH), HTXTH + HUH)
((HXH), (HYZH), -HTXTH - HUH)
((HXH), (HYTH), HTXH + HUTH)
((HXH), (HYTZH), HTXH + HUTH)
((HXH), (HUH), -4*HTXH - 2*HUTH - 4*HYH + 4*HYZH)
((HXH), (HUTH), -4*HTXTH - 2*HUH - 4*HYTH - 4*HYTZH)
```

```
((HXTH), (1), HXTH)
((HXTH), (HZ), -HXTH)
((HXTH), (HT), HXH)
((HXTH), (HTZ), HXH)
((HXTH), (HXH), 6*HT + 6*HTZ + 4*HXH)
((HXTH), (HXTH), 4*HXTH - 6*HZ + 6)
((HXTH), (HTXH), 2*HUTH + 2*HYH - 2*HYZH)
((HXTH), (HTXTH), 2*HUH + 2*HYTH + 2*HYTZH)
((HXTH), (HYH), HTXH + HUTH)
((HXTH), (HYZH), -HTXH - HUTH)
((HXTH), (HYTH), HTXTH + HUH)
((HXTH), (HYTZH), HTXTH + HUH)
((HXTH), (HUH), 4*HTXTH + 2*HUH + 4*HYTH + 4*HYTZH)
((HXTH), (HUTH), 4*HTXH + 2*HUTH + 4*HYH - 4*HYZH)
((HTXH), (1), HTXH)
((HTXH), (HZ), -HTXH)
((HTXH), (HT), HTXTH)
((HTXH), (HTZ), HTXTH)
((HTXH), (HXH), -2*HUH + 2*HYTH + 2*HYTZH)
((HTXH), (HXTH), -2*HUTH + 2*HYH - 2*HYZH)
((HTXH), (HTXH), 4*HTXH - 6*HZ + 6)
((HTXH), (HTXTH), 6*HT + 4*HTXTH + 6*HTZ)
((HTXH), (HYH), -HUTH + HXTH)
((HTXH), (HYZH), HUTH - HXTH)
((HTXH), (HYTH), -HUH + HXH)
((HTXH), (HYTZH), -HUH + HXH)
((HTXH), (HUH), 2*HUH - 4*HXH - 4*HYTH - 4*HYTZH)
((HTXH), (HUTH), 2*HUTH - 4*HXTH - 4*HYH + 4*HYZH)
((HTXTH), (1), HTXTH)
((HTXTH), (HZ), -HTXTH)
((HTXTH), (HT), HTXH)
((HTXTH), (HTZ), HTXH)
```

```
((HTXTH), (HXH), 4*HTXH - 6*HZ + 6)
((HTXTH), (HXTH), 6*HT + 4*HTXTH + 6*HTZ)
((HTXTH), (HTXH), -2*HUH + 2*HYTH + 2*HYTZH)
((HTXTH), (HTXTH), -2*HUTH + 2*HYH - 2*HYZH)
((HTXTH), (HYH), -HUH + HXH)
((HTXTH), (HYZH), HUH - HXH)
((HTXTH), (HYTH), -HUTH + HXTH)
((HTXTH), (HYTZH), -HUTH + HXTH)
((HTXTH), (HUH), -2*HUTH + 4*HXTH + 4*HYH - 4*HYZH)
((HTXTH), (HUTH), -2*HUH + 4*HXH + 4*HYTH + 4*HYTZH)
((HYH), (1), HYH)
((HYH), (HZ), HYZH)
((HYH), (HT), HYTH)
((HYH), (HTZ), HYTZH)
((HYH), (HXH), HTXTH - HUH)
((HYH), (HXTH), HTXH - HUTH)
((HYH), (HTXH), HUTH + HXTH)
((HYH), (HTXTH), HUH + HXH)
((HYH), (HYH), 2*HYZH + 3)
((HYH), (HYZH), 2*HYH + 3*HZ)
((HYH), (HYTH), 3*HT - 2*HYTZH)
((HYH), (HYTZH), 3*HTZ - 2*HYTH)
((HYH), (HUH), 2*HTXTH - HUH - 2*HXH)
((HYH), (HUTH), 2*HTXH - HUTH - 2*HXTH)
((HYZH), (1), HYZH)
((HYZH), (HZ), HYH)
((HYZH), (HT), -HYTZH)
((HYZH), (HTZ), -HYTH)
((HYZH), (HXH), -HTXTH + HUH)
((HYZH), (HXTH), -HTXH + HUTH)
((HYZH), (HTXH), -HUTH - HXTH)
((HYZH), (HTXTH), -HUH - HXH)
```

```
((HYZH), (HYH), 2*HYH + 3*HZ)
((HYZH), (HYZH), 2*HYZH + 3)
((HYZH), (HYTH), -3*HTZ + 2*HYTH)
((HYZH), (HYTZH), -3*HT + 2*HYTZH)
((HYZH), (HUH), -2*HTXTH + HUH + 2*HXH)
((HYZH), (HUTH), -2*HTXH + HUTH + 2*HXTH)
((HYTH), (1), HYTH)
((HYTH), (HZ), -HYTZH)
((HYTH), (HT), HYH)
((HYTH), (HTZ), -HYZH)
((HYTH), (HXH), HUTH + HXTH)
((HYTH), (HXTH), HUH + HXH)
((HYTH), (HTXH), HTXTH - HUH)
((HYTH), (HTXTH), HTXH - HUTH)
((HYTH), (HYH), 3*HT - 2*HYTZH)
((HYTH), (HYZH), -3*HTZ + 2*HYTH)
((HYTH), (HYTH), 2*HYZH + 3)
((HYTH), (HYTZH), -2*HYH - 3*HZ)
((HYTH), (HUH), -2*HTXH + HUTH + 2*HXTH)
((HYTH), (HUTH), -2*HTXTH + HUH + 2*HXH)
((HYTZH), (1), HYTZH)
((HYTZH), (HZ), -HYTH)
((HYTZH), (HT), -HYZH)
((HYTZH), (HTZ), HYH)
((HYTZH), (HXH), HUTH + HXTH)
((HYTZH), (HXTH), HUH + HXH)
((HYTZH), (HTXH), HTXTH - HUH)
((HYTZH), (HTXTH), HTXH - HUTH)
((HYTZH), (HYH), 3*HTZ - 2*HYTH)
((HYTZH), (HYZH), -3*HT + 2*HYTZH)
((HYTZH), (HYTH), -2*HYH - 3*HZ)
((HYTZH), (HYTZH), 2*HYZH + 3)
```

```
((HYTZH), (HUH), -2*HTXH + HUTH + 2*HXTH)
((HYTZH), (HUTH), -2*HTXTH + HUH + 2*HXH)
((HUH), (1), HUH)
((HUH), (HZ), -HUH)
((HUH), (HT), HUTH)
((HUH), (HTZ), HUTH)
((HUH), (HXH), 2*HUTH + 4*HXTH + 4*HYH - 4*HYZH)
((HUH), (HXTH), 2*HUH + 4*HXH + 4*HYTH + 4*HYTZH)
((HUH), (HTXH), -4*HTXTH + 2*HUH - 4*HYTH - 4*HYTZH)
((HUH), (HTXTH), -4*HTXH + 2*HUTH - 4*HYH + 4*HYZH)
((HUH), (HYH), -2*HTXTH - HUH + 2*HXH)
((HUH), (HYZH), 2*HTXTH + HUH - 2*HXH)
((HUH), (HYTH), -2*HTXH - HUTH + 2*HXTH)
((HUH), (HYTZH), -2*HTXH - HUTH + 2*HXTH)
((HUH), (HUH), 4*HTXH + 4*HXTH - 4*HYH + 4*HYZH - 12*HZ + 12)
((HUH), (HUTH), 12*HT + 4*HTXTH + 12*HTZ + 4*HXH - 4*HYTH - 4*HYTZH)
((HUTH), (1), HUTH)
((HUTH), (HZ), -HUTH)
((HUTH), (HT), HUH)
((HUTH), (HTZ), HUH)
((HUTH), (HXH), -4*HTXTH + 2*HUH - 4*HYTH - 4*HYTZH)
((HUTH), (HXTH), -4*HTXH + 2*HUTH - 4*HYH + 4*HYZH)
((HUTH), (HTXH), 2*HUTH + 4*HXTH + 4*HYH - 4*HYZH)
((HUTH), (HTXTH), 2*HUH + 4*HXH + 4*HYTH + 4*HYTZH)
((HUTH), (HYH), -2*HTXH - HUTH + 2*HXTH)
((HUTH), (HYZH), 2*HTXH + HUTH - 2*HXTH)
((HUTH), (HYTH), -2*HTXTH - HUH + 2*HXH)
((HUTH), (HYTZH), -2*HTXTH - HUH + 2*HXH)
((HUTH), (HUH), -12*HT - 4*HTXTH - 12*HTZ - 4*HXH + 4*HYTH + 4*HYTZH)
((HUTH), (HUTH), -4*HTXH - 4*HXTH + 4*HYH - 4*HYZH + 12*HZ - 12)
```

Write $\bar{g}$ for $\{g, 1\} \in \overline{G}$.

Proposition 3.3.6. *$\widehat{H}\bar{z}\widehat{H}$ belongs to the centre of $\overline{\mathcal{H}}$. We have an isomorphism of rings*

$$\overline{\mathcal{H}} \cong \overline{\mathcal{H}}_{z=1} \oplus \overline{\mathcal{H}}_{z=-1}$$

where $\overline{\mathcal{H}}_{z=1}$ is the summand of $\overline{\mathcal{H}}$ on which $\widehat{H}\bar{z}\widehat{H}$ acts as 1, and $\overline{\mathcal{H}}_{z=-1}$ is the summand on which $\widehat{H}\bar{z}\widehat{H}$ acts as -1.

This is clear since $\bar{z}$ is in the centre of $\overline{G}$ and $(\widehat{H}\bar{z}\widehat{H})^2 = 1$ in $\overline{\mathcal{H}}$.

Theorem 3.3.7. *$\overline{\mathcal{H}}_{z=1}$ is commutative and 4-dimensional as a $\mathbb{C}$-vector space, with basis*

$$\{\hat{1}, \hat{y}, \hat{t}, \hat{yt}\}.$$

It has 4 irreducible 1-dimensional representations, given by

$\hat{t} \longmapsto 1$	$\hat{t} \longmapsto 1$	$\hat{t} \longmapsto -1$	$\hat{t} \longmapsto -1$
$\hat{y} \longmapsto 3$	$\hat{y} \longmapsto -1$	$\hat{y} \longmapsto 3$	$\hat{y} \longmapsto -1$
$\hat{yt} \longmapsto 3$	$\hat{yt} \longmapsto -1$	$\hat{yt} \longmapsto -3$	$\hat{yt} \longmapsto 1$

Note, from the multiplication table above, that setting $\widehat{H}\bar{z}\widehat{H} = 1$ forces $\widehat{H}\bar{x}\widehat{H} = \widehat{H}\bar{u}\widehat{H} = 0$.

Theorem 3.3.8. *The algebra $\overline{\mathcal{H}}_{z=-1}$ is non-commutative and 10-dimensional as a vector space, with basis*

$$\{\hat{1}, \hat{t}, \hat{x}, \hat{xt}, \hat{tx}, \hat{txt}, \hat{y}, \hat{yt}, \hat{u}, \hat{ut}\}.$$

Its centre, $\mathcal{Z}(\overline{\mathcal{H}}_{z=-1})$, is 4-dimensional, generated by $\{\hat{1}, \hat{z}_1 = \hat{t} + \hat{yt}, \hat{z}_2 = \hat{x} + \hat{txt} + 4\hat{yt}, \hat{z}_3 = \hat{tx} + \hat{xt} - 2\hat{y}\}$. There are 4 central characters:

$\chi_1(\hat{z}_1) = 2$	$\chi_2(\hat{z}_1) = -2$	$\chi_3(\hat{z}_1) = 2$	$\chi_4(\hat{z}_1) = -2$
$\chi_1(\hat{z}_2) = 0$	$\chi_2(\hat{z}_2) = 0$	$\chi_3(\hat{z}_2) = 12$	$\chi_4(\hat{z}_2) = -12$
$\chi_1(\hat{z}_3) = -6$	$\chi_2(\hat{z}_3) = -6$	$\chi_3(\hat{z}_3) = 6$	$\chi_4(\hat{z}_3) = 6$

The irreducible representations of $\overline{\mathcal{H}}_{z=-1}$ are:

1.

$$\overline{\mathcal{H}}_{z=-1}/(\hat{z}_1 = \chi_1(\hat{z}_1), \hat{z}_2 = \chi_1(\hat{z}_2), \hat{z}_3 = \chi_1(\hat{z}_3)) \longrightarrow \quad \mathbb{C}$$

2.

$$\overline{\mathcal{H}}_{z=-1}/(\hat{z}_1 = \chi_2(\hat{z}_1), \hat{z}_2 = \chi_2(\hat{z}_2), \hat{z}_3 = \chi_2(\hat{z}_3)) \longrightarrow \quad \mathbb{C}$$

3.

$$\begin{aligned} \overline{\mathcal{H}}_{z=-1}/(\hat{z}_1 = \chi_3(\hat{z}_1), \hat{z}_2 = \chi_3(\hat{z}_2), \hat{z}_3 = \chi_3(\hat{z}_3)) \longrightarrow& \quad M_2(\mathbb{C}) \\ \hat{u} \longmapsto& \begin{pmatrix} 0 & 48 \\ 1 & 0 \end{pmatrix} \\ \hat{t} \longmapsto& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

4.

$$\begin{aligned} \overline{\mathcal{H}}_{z=-1}/(\hat{z}_1 = \chi_4(\hat{z}_1), \hat{z}_2 = \chi_4(\hat{z}_2), \hat{z}_3 = \chi_4(\hat{z}_3)) \longrightarrow& \quad M_2(\mathbb{C}) \\ \hat{u} \longmapsto& \begin{pmatrix} 0 & 48 \\ 1 & 0 \end{pmatrix} \\ \hat{t} \longmapsto& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Each of these representations is an isomorphism, hence

$$\overline{\mathcal{H}}_{z=-1} \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

Proof. Setting $\widehat{H}\bar{z}\widehat{H} = -1$, we have (from the table above)

$$\hat{yt} * \hat{z} = -y\hat{t}z$$
$$\Rightarrow \hat{yt} = y\hat{t}z;$$

$$\hat{yz} * \hat{z} = \hat{y}$$
$$\Rightarrow \hat{yz} = -\hat{y};$$

$$\hat{tz} * \hat{z} = -\hat{t}$$
$$\Rightarrow \hat{tz} = \hat{t}.$$

Thus, $\overline{\mathcal{H}}_{z=-1}$ is 10-dimensional, with basis

$$\{\hat{1}, \hat{t}, \hat{x}, \hat{xt}, \hat{tx}, \hat{txt}, \hat{y}, \hat{yt}, \hat{u}, \hat{ut}\}.$$

A straightforward (but long) check shows that $\mathcal{Z}(\overline{\mathcal{H}}_{z=-1})$, is 4-dimensional, generated by $\{\hat{1}, \hat{z}_1, \hat{z}_2, \hat{z}_3\}$. The following is the multiplication table for $\mathcal{Z}(\overline{\mathcal{H}}_{z=-1})$:

	1	z_1	z_2	z_3
1	1	z_1	z_2	z_3
z_1	z_1	4	$2z_3 + 12$	$2z_2 - 6z_1$
z_2	z_2	$2z_3 + 12$	$12z_3 + 72$	$6z_2$
z_3	z_3	$2z_2 - 6z_1$	$6z_2$	36

Indeed,

$$\begin{aligned}\hat{z}_1^2 &= (\hat{t} + \hat{yt})(\hat{t} + \hat{yt}) \\ &= 1 + 2\hat{y} - 2\hat{y} + 3 \\ &= 4.\end{aligned}$$

$$\begin{aligned}\hat{z}_2^2 &= (\hat{x} + \hat{txt} + 4\hat{yt})(\hat{x} + \hat{txt} + 4\hat{yt}) \\ &= 2\hat{ut} + 4\hat{y} + 4\hat{xt} + 12 + 4(\hat{tx} + \hat{ut}) + 4\hat{tx} + 12 - 2\hat{ut} + 4\hat{y} + 4(-\hat{ut} + \hat{xt}) \\ &+ 4(\hat{ut} + \hat{xt}) + 4(\hat{tx} - \hat{ut}) + 16(-2\hat{y} + 3) \\ &= -24\hat{y} + 12\hat{xt} + 12\hat{tx} + 72 \\ &= 12(\hat{tx} + \hat{xt} - 2\hat{y}) + 72 \\ &= 12\hat{z}_3 + 72.\end{aligned}$$

$$\begin{aligned}\hat{z}_3^2 &= (\hat{tx} + \hat{xt} - 2\hat{y})(\hat{tx} + \hat{xt} - 2\hat{y}) \\ &= 4\hat{tx} + 12 - 2\hat{ut} + 4\hat{y} - 2(-\hat{ut} + \hat{xt}) + 2\hat{ut} + 4\hat{y} + 4\hat{xt} + 12 - 2(\hat{tx} + \hat{ut}) - 2(\hat{ut} + \hat{xt}) \\ &- 2(\hat{tx} - \hat{ut}) + 4(-2\hat{y} + 3) \\ &= 36.\end{aligned}$$

$$\begin{aligned}\hat{z}_1\hat{z}_2 &= (\hat{x} + \hat{txt} + 4\hat{yt})(\hat{t} + \hat{yt}) \\ &= \hat{xt} + \hat{tx} + 4\hat{y} + \hat{tx} + \hat{ut} - \hat{ut} + \hat{xt} + 4(-2\hat{y} + 3) \\ &= 2\hat{xt} + 2\hat{tx} - 4\hat{y} + 12 \\ &= 12 + 2\hat{z}_3.\end{aligned}$$

$$\begin{aligned}\hat{z}_2\hat{z}_3 &= (\hat{x} + \hat{txt} + 4\hat{yt})(\hat{tx} + \hat{xt} - 2\hat{y}) \\ &= 12\hat{t} + 4\hat{x} + 2\hat{u} + 4\hat{yt} - 2\hat{txt} - 2\hat{u} - 2\hat{u} + 2\hat{yt} + 2\hat{yt} + 12\hat{t} + 4\hat{txt} + 2\hat{u} + 4\hat{txt} \\ &\quad - 4\hat{u} + 4\hat{u} + 4\hat{x} - 24\hat{t} + 16\hat{yt} \\ &= 6\hat{x} + 24\hat{yt} + 6\hat{txt} \\ &= 6\hat{z}_2.\end{aligned}$$

$$\begin{aligned}\hat{z}_1\hat{z}_3 &= (\hat{t} + \hat{yt})(\hat{tx} + \hat{xt} - 2\hat{y}) \\ &= 2\hat{x} + 2\hat{txt} + 2\hat{yt} - 6\hat{t} \\ &= 2(\hat{x} + \hat{txt} + 4\hat{yt}) - 6\hat{yt} - 6\hat{t} \\ &= 2\hat{z}_2 - 6\hat{z}_1.\end{aligned}$$

Clearly, if $\chi : \mathcal{Z}(\overline{\mathcal{H}}_{z=-1}) \to \mathbb{C}$ is a ring homomorphism, then

$$\chi(\hat{z}_1) = \pm 2 \text{ and } \chi(\hat{z}_3) = \pm 6.$$

If $\chi(\hat{z}_1) = 2$ and $\chi(\hat{z}_3) = 6$, the equation

$$\chi(\hat{z}_1\hat{z}_2) = 2\chi(\hat{z}_3) + 12 \tag{3.23}$$

implies $\chi(\hat{z}_2) = 12$. This gives χ_3. If $\chi(\hat{z}_1) = 2$ and $\chi(\hat{z}_3) = -6$, then by (3.23), $\chi(\hat{z}_2) = 0$. This gives χ_1. If $\chi(\hat{z}_1) = -2$, then again by (3.23), $\chi(\hat{z}_3) = 6$ gives $\chi(\hat{z}_2) = -12$, and $\chi(\hat{z}_3) = -6$ gives $\chi(\hat{z}_2) = 0$. This gives the values of the two remaining characters.

For ease of notation, define, for $1 \leq i \leq 4$,

$$\overline{\mathcal{H}}^{\chi_i}_{z=-1} = \overline{\mathcal{H}}_{z=-1}/\langle \hat{\zeta} - \chi_i(\hat{\zeta}) \text{ for all } \hat{\zeta} \in \mathcal{Z}(\overline{\mathcal{H}}_{z=-1})\rangle.$$

Consider $\overline{\mathcal{H}}^{\chi_1}_{z=-1}$. The relation $\hat{z}_3 = \chi_1(\hat{z}_3)$: that is, $\hat{tx} + \hat{xt} - 2\hat{y} = -6$, gives, on multiplication by $\hat{t}$ on the left,

$$\hat{x} + \hat{txt} - 2\hat{yt} = -6\hat{t}. \tag{3.24}$$

The relation $\hat{z}_1 = \chi_1(\hat{z}_1)$ implies that $\hat{yt} = 2 - \hat{t}$, and $\hat{z}_2 = \chi_1(\hat{z}_2)$ implies

$$\begin{aligned} \hat{x} + \hat{txt} + 8 - 4\hat{t} = 0 \Leftrightarrow \\ -8 - \hat{txt} + 4\hat{t} = \hat{x}. \end{aligned} \tag{3.25}$$

Substituting (3.25) into (3.24) gives

$$\begin{aligned} \hat{txt} - 8 - \hat{txt} + 4\hat{t} - 2\hat{yt} = -6\hat{t} \Leftrightarrow \\ -8 + 10\hat{t} = 2\hat{yt} \Leftrightarrow \\ -4 + 5\hat{t} = \hat{yt}. \end{aligned} \tag{3.26}$$

On the other hand, since $\hat{yt} = 2 - \hat{t}$, (3.26) gives

$$\begin{aligned} -4 + 5\hat{t} = 2 - \hat{t} \Leftrightarrow \\ -6 + 6\hat{t} = 0 \Leftrightarrow \\ 6\hat{t} = 6 \Leftrightarrow \\ \hat{t} = 1. \end{aligned}$$

If $\phi : \overline{\mathcal{H}}^{\chi_1}_{z=-1} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^m)$ is a ring homomorphism (for some integer $m \geq 1$), then

$$\phi(\hat{xt}) = \phi(\hat{tx}) = \phi(\hat{txt}) = \phi(\hat{x});$$

$$\phi(\hat{yt}) = \phi(\hat{y})$$

and,

$$\phi(\hat{ut}) = \phi(\hat{u}).$$

The relation $\hat{yt} = 2 - \hat{t}$ gives $\hat{y} = 1$. Using the relation $\hat{tx} + \hat{xt} - 2\hat{y} = -6$, we get that

$$\begin{aligned} \hat{x} + \hat{x} - 2 = -6 \Leftrightarrow \\ 2\hat{x} = -4 \Leftrightarrow \\ \hat{x} = -2. \end{aligned}$$

Observe that in $\overline{\mathcal{H}}_{z=-1}$,

$$\hat{x} * \hat{y} = \hat{txt} + \hat{u}.$$

Multiplying by $\hat{t}$ on the left of the relation $\hat{yt} = 2 - \hat{t}$ shows that $\hat{y} = 2\hat{t} - 1$. Using this, and (3.25), we have

$$\begin{aligned}
(-8 - \hat{txt} + 4\hat{t}) * (2\hat{t} - 1) = \hat{txt} + \hat{u} &\Leftrightarrow \\
-20\hat{t} + 16 + \hat{txt} - 2\hat{tx} = \hat{txt} + \hat{u} &\Leftrightarrow \\
\hat{u} = 16 - 20\hat{t} - 2\hat{tx} &\Leftrightarrow \\
\hat{u} = 16 - 20 + 4 \text{ using } \hat{t} = 1, \hat{tx} = \hat{x} = -2 &\Leftrightarrow \\
\hat{u} = 0.
\end{aligned}$$

We have shown that there is a homomorphism

$$\begin{aligned}
\overline{\mathcal{H}}^{\chi_1}_{z=-1} &\longrightarrow \mathbb{C} \\
\hat{t} &\longmapsto 1, \\
\hat{x} &\longmapsto -2, \\
\hat{tx} &\longmapsto -2, \\
\hat{xt} &\longmapsto -2, \\
\hat{y} &\longmapsto 1, \\
\hat{yt} &\longmapsto 1, \\
\hat{ut} &\longmapsto 0, \\
\hat{u} &\longmapsto 0.
\end{aligned}$$

The 1-dimensional representation of $\overline{\mathcal{H}}^{\chi_2}_{z=-1}$ is calculated in much the same way.

Consider $\overline{\mathcal{H}}^{\chi_3}_{z=-1}$. We shall show that $\{\hat{1}, \hat{t}, \hat{u}, \hat{ut}\}$ is a basis for $\overline{\mathcal{H}}^{\chi_3}_{z=-1}$. By the relation $\hat{z}_1 = \chi_3(\hat{z}_1)$, we have $\hat{yt} = 2 - \hat{t}$ and therefore $\hat{y} = 2\hat{t} - 1$. In $\overline{\mathcal{H}}_{z=-1}$,

$$\begin{aligned}
\hat{txt} * \hat{txt} &= -2\hat{ut} + 4\hat{y} \\
&= -2\hat{ut} + 4(2\hat{t} - 1) \\
&= -2\hat{ut} + 8\hat{t} - 4. \qquad (3.27)
\end{aligned}$$

On the other hand, by associativity, $\widehat{txt} * \widehat{txt} = \widehat{tx} * \hat{t}^2 * \widehat{xt} = \widehat{tx} * \widehat{xt} = \hat{t} * \hat{x}^2 * \hat{t}$, and

$$\begin{aligned} \hat{t} * \hat{x}^2 * \hat{t} &= \hat{t}(4\hat{x} - 12)\hat{t} \\ &= 4\widehat{txt} - 12. \end{aligned} \tag{3.28}$$

Putting (3.27) and (3.28) together gives

$$\begin{aligned} 4\widehat{txt} - 12 &= -2\widehat{ut} + 8\hat{t} - 4 \Leftrightarrow \\ 4\widehat{txt} &= -2\widehat{ut} + 8\hat{t} + 8 \Leftrightarrow \\ \widehat{txt} &= -\frac{1}{2}\widehat{ut} + 2\hat{t} + 2. \end{aligned} \tag{3.29}$$

By the relation $\hat{z}_2 = \chi_3(\hat{z}_2)$, we have

$$\begin{aligned} \hat{x} &= 12 - 4\widehat{yt} - \widehat{txt} \\ &= 12 - 4(2 - \hat{t}) - \widehat{txt} \\ &= 4 + 4\hat{t} - \widehat{txt} \\ &= 2 + 2\hat{t} + \frac{1}{2}\widehat{ut} \text{ using (3.29)} \end{aligned} \tag{3.30}$$

Multiplying by $\hat{t}$ on the left gives

$$\widehat{tx} = 2 + 2\hat{t} - \frac{1}{2}\hat{u},$$

and on the right gives

$$\widehat{xt} = 2 + 2\hat{t} + \frac{1}{2}\hat{u}.$$

That is, $\{\hat{1}, \hat{t}, \hat{u}, \widehat{ut}\}$ is a basis for $\overline{\mathcal{H}}^{\chi_3}_{z=-1}$.

In $\overline{\mathcal{H}}^{\chi_3}_{z=-1}$,

$$\begin{aligned} \hat{u}^2 &= 4\widehat{tx} + 4\widehat{xt} - 8\hat{y} + 24 \\ &= 4\hat{z}_3 + 24 \\ &= 48. \end{aligned}$$

Also, $(\widehat{ut})^2 = -\hat{u}^2 = -48$, and $\widehat{ut} = -\widehat{tu}$. Thus $\overline{\mathcal{H}}^{\chi_3}_{z=-1}$ has a basis $\{\hat{1}, \hat{u}, \hat{t}, \widehat{ut}\}$ subject to the relations $\hat{t}^2 = 1, \hat{u}^2 = 48, (\widehat{ut})^2 = -48, \widehat{tu} = -\widehat{ut}$. Hence there is a homomorphism

$$\begin{aligned}\overline{\mathcal{H}}^{\chi_3}_{z=-1} &\longrightarrow M_2(\mathbb{C})\\ \hat{u} &\longmapsto \begin{pmatrix}0 & 48\\ 1 & 0\end{pmatrix}\\ \hat{t} &\longmapsto \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}$$

which is also an isomorphism.

Finally, consider $\overline{\mathcal{H}}^{\chi_4}_{z=-1}$. Calculations as above show that $\{\hat{1}, \hat{t}, \hat{u}, \hat{ut}\}$ is a basis for $\overline{\mathcal{H}}^{\chi_4}_{z=-1}$; the relations are $\hat{u}^2 = 48, (\hat{ut})^2 = -48$ and $\hat{ut} = -\hat{tu}$. Thus there is a representation

$$\begin{aligned}\overline{\mathcal{H}}^{\chi_4}_{z=-1} &\longrightarrow M_2(\mathbb{C})\\ \hat{u} &\longmapsto \begin{pmatrix}0 & 48\\ 1 & 0\end{pmatrix}\\ \hat{t} &\longmapsto \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}$$

as above, which is an isomorphism.

□

3.4 Representations of $\overline{SL_2}(F_\pi)$ containing non-zero $\widehat{K}_\pi(4)$-fixed vectors

Let $\varpi_\pi(\chi_1, \chi_2)$ be the component at π of a level one cuspidal automorphic representation of $SL_2(\mathbb{A})$. Thus, $\varpi_\pi(\chi_1, \chi_2)$ is an unramified principal series representation, which is an irreducible admissible representation of $SL_2(F_\pi)$, and whose space of $K_\pi := SL_2(\mathcal{O}_\pi)$-fixed vectors is not zero. In fact,

$$\dim(\varpi_\pi(\chi_1, \chi_2)^{K_\pi}) = 1,$$

and

$$\chi_1, \chi_2 : F_\pi^\times/\mathcal{O}_\pi^\times \longrightarrow \mathbb{C}^\times \text{ are unramified.}$$

Define $\chi : F_\pi^\times/\mathcal{O}_\pi^\times \to \mathbb{C}^\times$ by $\chi = \chi_1\chi_2$; χ is the central character of $\varpi_\pi(\chi_1, \chi_2)$. Since both χ_1 and χ_2 are even, $\varpi_\pi(\chi_1, \chi_2)$ is in the image of $\widetilde{S}_\pi$, the modified local Flicker correspondence (Theorem 1.4.13). We shall denote its pre-image by $\overline{\varpi}_\pi(\bar{\chi})$. The representation $\overline{\varpi}_\pi(\bar{\chi})$ is of the principal series; its (genuine) central character $\bar{\chi}$ is defined by:

$$\bar{\chi}\left(\left\{\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2}\end{pmatrix}, \xi\right\}\right) = \xi\,\chi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right) \quad \text{for } a \in F_\pi^\times, \xi \in \mu_2,$$

and $\overline{\varpi}_\pi(\bar{\chi})$ is level one, since

$$\dim(\overline{\varpi}_\pi(\bar{\chi})^{\widehat{K}_\pi(4)}) > 0.$$

In this section, we shall calculate the dimension of $\overline{\varpi}_\pi(\bar{\chi})^{\widehat{K}_\pi(4)}$ and we shall determine this space as a representation of the genuine Hecke algebra $\overline{\mathcal{H}}$.

3.4.1 The extension of the central character

Recall the subgroups $T_v, N_v, B_v \subset GL_2(F_v)$ we defined in Section 1.1. We want to think of these instead as subgroups of $SL_2(F_\pi)$ or $SL_2(\mathcal{O}_v)$. To this end, define, for $R = F_\pi$ or $\mathcal{O}_\pi$,

$$T(R) = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \,\middle|\, a \in R^\times\right\};$$

$$N(R) = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \,\middle|\, b \in R\right\};$$

$$B(R) = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \,\middle|\, a \in R^\times,\ b \in R\right\} = N(R)T(R) = T(R)N(R),$$

and let $\overline{T}(R), \overline{N}(R), \overline{B}(R)$ be their respective pre-images in $\overline{SL_2}(R)$.

In the notation of Subsection 1.4.1, the representation $\overline{\varpi}_\pi(\bar{\chi})$ of $\overline{SL}_2(F_\pi)$ is of the form $\mathrm{Ind}\,_{\overline{B}(F_\pi)}^{\overline{SL}_2(F_\pi)}(\bar{\chi}_0)$, for some genuine character $\bar{\chi}_0$ of $\overline{B}(F_\pi)$ which is trivial on $\widehat{N}(F_\pi)$ and which agrees with $\bar{\chi}$ on $\overline{T}(F_\pi)^2$. Note that, in the case of GL_2, the principal series representations are induced from a representation of $\overline{B}(F_\pi)$ which is itself induced from $\overline{T}^0(F_\pi)\widehat{N}(F_\pi)$ where $\overline{T}^0(F_\pi)$ is a maximal abelian subgroup of

$\overline{T}(F_\pi)$. In the case of SL_2, the group $\overline{T}(F_\pi)$ is itself abelian, so we can simply induce from $\overline{B}(F_\pi) = \overline{T}(F_\pi)\widehat{N}(F_\pi)$.

To find such a character $\bar{\chi}_0$, we must extend $\bar{\chi}$ to the whole of $\overline{T}(F_\pi)$.

Since the Hecke algebra acting on $\mathrm{Ind}\,{}^{\overline{SL}_2(F_\pi)}_{\overline{B}(F_\pi)}(\bar{\chi}_0)$ is $\overline{\mathcal{H}} = \mathcal{H}(\overline{K}_\pi, \widehat{K}_\pi(4))$, it will be sufficient to calculate $\mathrm{Ind}\,{}^{\overline{SL}_2(F_\pi)}_{\overline{B}(F_\pi)}(\bar{\chi}_0)$ as a $\overline{K}_\pi$-module. Observe that

$$\overline{SL}_2(F_\pi)/\overline{B}(F_\pi) = SL_2(F_\pi)/B(F_\pi) = K_\pi/B(\mathcal{O}_\pi) = \overline{K_\pi}/\overline{B}(\mathcal{O}_\pi).$$

It follows that there is a $\overline{K}_\pi$-isomorphism

$$\mathrm{Ind}\,{}^{\overline{SL}_2(F_\pi)}_{\overline{B}(F_\pi)}(\bar{\chi}_0) \cong \mathrm{Ind}\,{}^{\overline{K}_\pi}_{\overline{B}(\mathcal{O}_\pi)}(\bar{\chi}_0).$$

Our task now is to find a character $\bar{\chi}_0$ of $\overline{T}(\mathcal{O}_\pi)$ which is trivial on $\widehat{T}(\mathcal{O}_\pi)^2$ (since χ is unramified) and genuine. Since this character will have the property that

$$\bar{\chi}_0(\{g, -1\}) = -1 \text{ for all } g \in K_\pi,$$

it will suffice to determine $\bar{\chi}_0$ on the subgroup $\widehat{T}(\mathcal{O}_\pi) \cong T(\mathcal{O}_\pi)$.

We shall use the isomorphism $T(\mathcal{O}_\pi) \cong \mathcal{O}_\pi^\times$ to abuse notation: we'll write $\bar{\chi}_0(a)$ for $\bar{\chi}_0\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)\right)$. Consider the extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{T}(\mathcal{O}_\pi) \longrightarrow T(\mathcal{O}_\pi) \longrightarrow 1;$$

it corresponds to the restriction of the cocycle ω_π to $T(\mathcal{O}_\pi)$ which we gave in (1.6). The character $\bar{\chi}_0$ shall be a section $T(\mathcal{O}_\pi) \to \mu_2$ corresponding to the 2-cocycle ω_π.

Lemma 3.4.1. *Define* $\bar{\chi}_0 : \overline{T}(\mathcal{O}_\pi) \to \mathbb{C}^\times$ *by*

$$\bar{\chi}_0(\{i^a 3^b (1+2i)^c, \xi\}) = \xi\,(-1)^{ac} \text{ for } a, b, c \in \{0,1\} \text{ and } \xi \in \mu_2.$$

Then $\bar{\chi}_0$ *is a genuine character of* $\overline{T}(\mathcal{O}_\pi)$ *which is trivial on* $\widehat{T}(\mathcal{O}_\pi)^2$.

Proof. Note that $\bar{\chi}_0$ is well-defined since

$$\begin{aligned} T(\mathcal{O}_\pi)/T(\mathcal{O}_\pi)^2 \cong \mathcal{O}_\pi^\times/\mathcal{O}_\pi^{\times 2} \cong \left(\mathcal{O}/\pi^5\right)^\times/\{\pm 1\} &= < i > \times < 3 > \times < 1+2i > \\ &= C_2 \times C_2 \times C_2. \end{aligned}$$

This means that every element of $\mathcal{O}_\pi^\times/\mathcal{O}_\pi^{\times 2}$ has a unique representation in the form $i^a 3^b (1+2i)^c$ for $a, b, c \in \{0,1\}$.

Clearly, $\bar{\chi}_0$ is genuine. To see that $\bar{\chi}_0$ is a character, suppose that $x = \{i^a 3^b (1+2i)^c, 1\}$ and $y = \{i^{a'} 3^{b'} (1+2i)^{c'}, 1\}$ are two elements of $\mathcal{O}_\pi^\times/\mathcal{O}_\pi^{\times 2} \times \{1\}$. Of course, we shall think of x and y as elements of $\widehat{T}(\mathcal{O}_\pi) = T(\mathcal{O}_\pi) \times \{1\}$. Now,

$$\begin{aligned} x * y =& \{i^a 3^b (1+2i)^c, 1\} * \{i^{a'} 3^{b'} (1+2i)^{c'}, 1\} \\ =& \{i^a 3^b (1+2i)^c i^{a'} 3^{b'} (1+2i)^{c'}, \omega_\pi(i^a 3^b (1+2i)^c, i^{a'} 3^{b'} (1+2i)^{c'})\} \\ =& \{i^{a+a'} 3^{b+b'} (1+2i)^{c+c'}, (i^a 3^b (1+2i)^c, i^{a'} 3^{b'} (1+2i)^{c'})_\pi\} \text{ by Lemma 1.1.5} \\ =& \{i^{a+a'} 3^{b+b'} (1+2i)^{c+c'}, (i,3)_\pi^{ab'+a'b} (i,1+2i)_\pi^{ac'+a'c} (3,1+2i)_\pi^{bc'+b'c}\} \\ =& \{i^{a+a'} 3^{b+b'} (1+2i)^{c+c'}, (i,1+2i)_\pi^{ac'+a'c}\} \text{ since } (i,3)_\pi = 1 \text{ and } (3,1+2i)_\pi = 1 \\ =& \{i^{a+a'} 3^{b+b'} (1+2i)^{c+c'}, (-1)^{ac'+a'c}\} \text{ since } (i,1+2i)_\pi = -1. \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{\chi}_0(x * y) =& (-1)^{(a+a')(c+c')} (-1)^{ac'+a'c} \\ =& (-1)^{ac+a'c'}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \bar{\chi}_0(x)\bar{\chi}_0(y) =& (-1)^{ac} (-1)^{a'c'} \\ =& (-1)^{ac+a'c'}. \end{aligned}$$

Hence $\bar{\chi}_0$ is a character. Finally, $\bar{\chi}_0$ is obviously trivial on $\widehat{T}(\mathcal{O}_\pi)^2$.

□

Remark 3.4.1. The centre of $\overline{K}_\pi$ is

$$\left\{ \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \pm 1 \right\} \right\} \cong C_2 \times C_2.$$

The central character of $\operatorname{Ind}_{B(\mathcal{O}_\pi)}^{\overline{K}_\pi}(\bar\chi_0)$ is given by:

$$\bar\chi_0\left(\left\{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, 1\right\}\right) = 1$$
$$\bar\chi_0\left(\left\{\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, -1\right\}\right) = -1$$
$$\bar\chi_0\left(\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, -1\right\}\right) = -1$$
$$\bar\chi_0\left(\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, 1\right\}\right) = 1.$$

This can easily be seen by the fact that $-1 = i^2 \in \mathcal{O}_\pi^{\times 2}$. In fact, this is the only choice of central character whose restriction to $\mathcal{O}_\pi^{\times 2}$ is trivial.

3.4.2 The action of $\overline{\mathcal{H}}$

To save chalk, put $W = \operatorname{Ind}_{B(\mathcal{O}_\pi)}^{\overline{K}_\pi}(\bar\chi_0)$. By definition (see (1.25)),

$$W = \left\{f : \overline{K}_\pi \to \mathbb{C} \mid f(gb) = \bar\chi_0(b)f(g) \text{ for all } b \in \overline{B}(\mathcal{O}_\pi), g \in \overline{K}_\pi;\right.$$
$$\left. f \text{ is locally constant}\right\}.$$

Note that in this case, the normalising factor $|\cdot|_\pi^{\frac{1}{2}}$ is trivial because $b \in \overline{B}(\mathcal{O}_\pi)$ (if $b = \left(\begin{smallmatrix} a & 0\\ 0 & a^{-1}\end{smallmatrix}\right)$, then $|a|_\pi = 1$). We shall find a basis for the space of $\widehat{K}_\pi(4)$-fixed vectors in W. Again, by definition,

$$W^{\widehat{K}_\pi(4)} = \Big\{f : \overline{K}_\pi \to \mathbb{C} \mid f(kgb) = \bar\chi_0(b)f(g) \text{ for all } k \in \widehat{K}_\pi(4), b \in \overline{B}(\mathcal{O}_\pi), g \in \overline{K}_\pi;$$
$$f \text{ is locally constant.}\}$$

Let $B(\mathcal{O}/4)$ be $B(R)$ as above, with $R = \mathcal{O}/4$. There are bijections

$$K_\pi(4)\backslash K_\pi/B(\mathcal{O}_\pi) \cong SL_2(\mathbb{Z}/4)\backslash SL_2(\mathcal{O}/4)/B(\mathcal{O}/4) \cong SL_2(\mathbb{Z}/4)\backslash \mathbb{P}^1(\mathcal{O}/4), \quad (3.31)$$

and, further, that $|SL_2(\mathbb{Z}/4)\backslash \mathbb{P}^1(\mathcal{O}/4)| = 4$; a set of double coset representatives for $K_\pi(4)\backslash K_\pi/B(\mathcal{O}_\pi)$ is given by:

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}i & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}-i & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right\}. \quad (3.32)$$

In fact, since

$$\widehat{K}_\pi(4)\backslash\overline{K}_\pi/\overline{B}(\mathcal{O}_\pi) = K_\pi(4)\backslash K_\pi/B(\mathcal{O}_\pi),$$

a set of coset representatives for $\widehat{K}_\pi(4)\backslash\overline{K}_\pi/\overline{B}(\mathcal{O}_\pi)$ is:

$$\left\{\hat{h} = \{h, 1\} \mid h \in K_\pi(4)\backslash K_\pi/B(\mathcal{O}_\pi)\right\}$$

Each element of $W^{\widehat{K}_\pi(4)}$ is determined by its values on the representatives (3.32). If $\hat{h}$ is such a representative, and there is an element which has value 1 on $\hat{h}$, and 0 on all other representatives, then we call this element $\mathbb{1}_{\hat{h}}$. The elements of the form $\mathbb{1}_{\hat{h}}$ form a basis for $W^{\widehat{K}_\pi(4)}$.

Suppose that $k \in \widehat{K}_\pi(4)$ and $b \in \overline{B}(\mathcal{O}_\pi)$.

$$\begin{aligned}\mathbb{1}_{\hat{h}}(k\hat{h}b) =& \bar{\chi}_0(b)\mathbb{1}_{\hat{h}}(\hat{h})\\ =& \bar{\chi}_0(b),\end{aligned}$$

It follows that if $k\hat{h}b = k'\hat{h}b'$ then we must have $\bar{\chi}_0(b) = \bar{\chi}_0(b')$. This means that for each $\hat{h}$, $\bar{\chi}_0$ must be trivial on $\overline{B}(\mathcal{O}_\pi) \cap (\hat{h}^{-1}\widehat{K}_\pi(4)\hat{h})$.

We shall write h for $\hat{h} = \{h, 1\}$ and we shall think of the representatives (3.32) as their images under (3.31): that is, as the elements

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}i\\1\end{bmatrix}, \begin{bmatrix}-i\\1\end{bmatrix}, \begin{bmatrix}2i\\1\end{bmatrix}$$

of $\mathbb{P}^1(\mathcal{O}/4)$. Then the basis for $W^{\widehat{K}_\pi(4)}$ can be re-written as:

$$\left\{\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}, \mathbb{1}_{\left[\begin{smallmatrix}i\\1\end{smallmatrix}\right]}, \mathbb{1}_{\left[\begin{smallmatrix}-i\\1\end{smallmatrix}\right]}, \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\right\}$$

Lemma 3.4.2. *The representation $W^{\widehat{K}_\pi(4)}$ is 2-dimensional as a complex vector space. A basis is given by*

$$\left\{\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}, \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\right\}.$$

Proof. We shall show that $\bar{\chi}_0$ is not trivial on $\overline{B}(\mathcal{O}_\pi) \cap (\hat{h}^{-1}\widehat{K}_\pi(4)\hat{h})$ when $\hat{h} = \{\left(\begin{smallmatrix}i & -1\\1 & 0\end{smallmatrix}\right), 1\}$ and $\{\left(\begin{smallmatrix}-i & -1\\1 & 0\end{smallmatrix}\right), 1\}$. Using (3.31), this is equivalent to showing that $\bar{\chi}_0$ is not trivial on $B(\mathcal{O}/4) \cap h^{-1}SL_2(\mathbb{Z}/4)h$ when $h = \left(\begin{smallmatrix}i & -1\\1 & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}-i & -1\\1 & 0\end{smallmatrix}\right)$.

Let $h_1 = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix}$. Observe that

$$h_1^{-1} SL_2(\mathbb{Z}/4) h_1 = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} \,\middle|\, a, b, c, d, \in \mathbb{Z}/4, ad - bc \equiv 1 \pmod 4 \right\}$$
$$= \left\{ \begin{pmatrix} d + ic & -c \\ -b - c + i(d-a) & a - ic \end{pmatrix} \,\middle|\, a, b, c, d, \in \mathbb{Z}/4, ad - bc \equiv 1 \pmod 4 \right\}.$$

Then,

$$B(\mathcal{O}/4) \cap h_1^{-1} SL_2(\mathbb{Z}/4) h_1 = \left\{ \begin{pmatrix} a + ic & -c \\ 0 & a - ic \end{pmatrix} \,\middle|\, a, c \in \mathbb{Z}/4 \right\}$$

In particular, setting $a = 2, c = -1$, we find that

$$\bar{\chi}_0 \left(\begin{pmatrix} 2 - i & 1 \\ 0 & (2-i)^{-1} \end{pmatrix} \right) = -1 \text{ since } 2 - i = i^3(1 + 2i).$$

Let $h_2 = \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix}$. Observe that

$$h_2^{-1} SL_2(\mathbb{Z}/4) h_2 = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix} \,\middle|\, \right.$$
$$\left. a, b, c, d, \in \mathbb{Z}/4, ad - bc \equiv 1 \pmod 4 \right\}$$
$$= \left\{ \begin{pmatrix} d - ic & -c \\ -b - c - i(d-a) & a + ic \end{pmatrix} \,\middle|\, \right.$$
$$\left. a, b, c, d, \in \mathbb{Z}/4, ad - bc \equiv 1 \pmod 4 \right\}.$$

Then,

$$B(\mathcal{O}/4) \cap h_2^{-1} SL_2(\mathbb{Z}/4) h_2 = \left\{ \begin{pmatrix} a - ic & -c \\ 0 & a + ic \end{pmatrix} \,\middle|\, a, c \in \mathbb{Z}/4 \right\}$$

In particular, setting $a = 2, c = 1$, we find that

$$\bar{\chi}_0 \left(\begin{pmatrix} 2 - i & 1 \\ 0 & (2-i)^{-1} \end{pmatrix} \right) = -1 \text{ since } 2 - i = i^3(1 + 2i).$$

This implies that the functions $\mathbb{1}_{\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]}, \mathbb{1}_{\left[\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right]}$ do not exist.

One can prove in the same way that $\bar{\chi}_0$ *is* trivial on $B(\mathcal{O}/4) \cap h^{-1}SL_2(\mathbb{Z}/4)h$ when $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2i & -1 \\ 1 & 0 \end{smallmatrix}\right)$. □

The Hecke algebra $\overline{\mathcal{H}}$ acts on $W^{\widehat{K}_\pi(4)}$ in the following way. Let $\tilde{g} \in \overline{K}_\pi$ and let $\mathcal{T}$ be the double coset $\widehat{K}_\pi(4)\tilde{g}\widehat{K}_\pi(4)$:

$$\begin{aligned}\mathcal{T} =& \widehat{K}_\pi(4)\tilde{g}\widehat{K}_\pi(4) \\ =& \coprod_m \widehat{K}_\pi(4)\tilde{g}_m.\end{aligned}$$

We saw in Subsection 1.4.1.4 that $\overline{K}_\pi$ acts on the right of W by left translation; explicitly:

$$(f\tilde{g})(x) = f(\tilde{g}x) \text{ for } f \in W, \tilde{g} \in \overline{K}_\pi.$$

The action of $\mathcal{T}$ is

$$\begin{aligned}(f\mathcal{T})(x) &= \sum_m (f\tilde{g}_m)(x) \\ &= \sum_m f(\tilde{g}_m x).\end{aligned}$$

In particular, if x, y belong to $\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2i \\ 1 \end{smallmatrix}\right]\}$, then

$$\begin{aligned}(\mathbb{1}_x\mathcal{T})(y) =& \sum_m \mathbb{1}_x(\tilde{g}_m y) \\ =& \sum_{m:\, \widehat{K}_\pi(4)x\overline{B}(\mathcal{O}_\pi) = \widehat{K}_\pi(4)\tilde{g}_m y\overline{B}(\mathcal{O}_\pi)} \mathbb{1}_x(\tilde{g}_m y) \\ =& \sum_{\widehat{K}_\pi(4)xb = \widehat{K}_\pi(4)\tilde{g}_m y} \mathbb{1}_x(xb) \\ =& \sum_{\widehat{K}_\pi(4)xb = \widehat{K}_\pi(4)\tilde{g}_m y} \bar{\chi}_0(b).\end{aligned}$$

In practice, we shall work "mod 4" using the isomorphism $\overline{\mathcal{H}} = \mathcal{H}(\overline{K}_\pi, \widehat{K}_\pi(4)) \cong \mathcal{H}(\overline{SL}_2(\mathcal{O}/4), \widehat{SL}_2(\mathbb{Z}/4)) = \mathcal{H}(\overline{G}, \widehat{H})$. Recall (Proposition 3.3.6) that there is a decomposition of $\overline{\mathcal{H}}$:

$$\overline{\mathcal{H}} \cong \overline{\mathcal{H}}_{z=1} \oplus \overline{\mathcal{H}}_{z=-1};$$

we have an action of either summand, $\overline{\mathcal{H}}_{z=1}$ or $\overline{\mathcal{H}}_{z=-1}$, according to whether the double coset $\widehat{H}\bar{z}\widehat{H}$ acts as 1 or -1 respectively.

Proposition 3.4.3. *The representation $W^{\widehat{K}_\pi(4)}$ is a representation of $\overline{\mathcal{H}}_{z=1}$.*

Proof. We are required to show that $\widehat{H}\bar{z}\widehat{H}$ acts trivially on $W^{\widehat{K}_\pi(4)}$. First recall that

$$\begin{aligned}\widehat{H}\bar{z}\widehat{H} =&\widehat{H}\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}, 1\right\}\widehat{H}\\ =&\widehat{H}\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}, 1\right\}.\end{aligned}$$

Now,

$$\begin{aligned}\left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{z}\widehat{H}\right)\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, 1\right\} =&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}, 1\right\}\right)\\ =&\bar{\chi}_0\left(\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}, 1\right\}\right)\\ =&1.\end{aligned}$$

$$\begin{aligned}\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{z}\widehat{H}\right)\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}, 1\right\} =&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}, 1\right\}\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}, 1\right\}\right)\\ =&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}, 1\right\}\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}, 1\right\}\right)\\ =&\bar{\chi}_0\left(\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}, 1\right\}\right)\\ =&1.\end{aligned}$$

Hence $\widehat{H}\bar{z}\widehat{H}$ acts trivially as claimed. □

Since $\overline{\mathcal{H}}_{z=1}$ is commutative (Theorem 3.3.7), it follows that $W^{\widehat{K}_\pi(4)}$ must be reducible: it must be the sum of two irreducible subspaces. Furthermore, since $\overline{\mathcal{H}}_{z=1}$ is generated as an algebra by $\hat{y}$ and $\hat{t}$, where

$$\begin{aligned}\hat{y}^2 =&2\hat{y}+3\\ \hat{t}^2 =&1,\end{aligned}$$

each of these irreducible subspaces must be an eigenspace for the action of $\hat{y}$ and $\hat{t}$.

Theorem 3.4.4. *As a representation of $\overline{K}_\pi$,*

$$W^{\widehat{K}_\pi(4)} \cong W(3) \oplus W(-1),$$

where $W(3)$ (resp. $W(-1)$) is the 3 (resp. -1)-eigenspace of $\hat{y}$.

Proof. We shall calculate the action of $\hat{t}$.

$$\begin{aligned} \widehat{H}\bar{t}\widehat{H} =& \widehat{H}\left\{\begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}, 1\right\}\widehat{H} \\ =& \widehat{H}\left\{\begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}, 1\right\}. \end{aligned}$$

Now,

$$\begin{aligned} \left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{t}\widehat{H}\right)\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right\} =& \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}, 1\right\}\right) \\ =& \bar{\chi}_0\left(\left\{\begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}, 1\right\}\right) \\ =& -1. \end{aligned}$$

That is,

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{t}\widehat{H}\right) = -\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}.$$

Further,

$$\begin{aligned}\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{t}\widehat{H}\right)\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\} =& \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2+i & 0\\0 & 2-i\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\\
=& \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2 & 2-i\\2-i & 0\end{pmatrix},\right.\right.\\
& \left.\left.\Sigma\left(\begin{pmatrix}2+i & 0\\0 & 2-i\end{pmatrix},\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\right)\right\}\right)\\
=& \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2-i & 0\\0 & 2+i\end{pmatrix},\right.\right.\\
& \Sigma\left(\begin{pmatrix}2+i & 0\\0 & 2-i\end{pmatrix},\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\right)\\
& \left.\left.\Sigma\left(\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},\begin{pmatrix}2-i & 0\\0 & 2+i\end{pmatrix}\right)\right\}\right)\\
=& \bar{\chi}_0\left(\left\{\begin{pmatrix}2-i & 0\\0 & 2+i\end{pmatrix},1\right\}\right)\\
=& -1.\end{aligned}$$

Hence,

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{t}\widehat{H}\right) = -\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}.$$

We have shown that $\hat{t}$ acts as -1 on $W^{\widehat{K}_\pi(4)}$. We shall now calculate the action of $\hat{y}$. Recall that

$$\begin{aligned}\widehat{H}\bar{y}\widehat{H} =& \widehat{H}\left\{\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},1\right\}\widehat{H}\\
=& \bigcup_{i=1}^{3}\widehat{H}\left\{y_i,1\right\} \text{ where}\\
& y_1 = \begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix}, y_2 = \begin{pmatrix}-1+2i & -2+2i\\1 & 1\end{pmatrix}, y_3 = \begin{pmatrix}-1+2i & -1\\1 & 0\end{pmatrix}.\end{aligned}$$

Then,

$$
\begin{aligned}
\left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\} =&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1&2+2i\\0&1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\}\right)\\
&+\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i&-2+2i\\1&1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\}\right)\\
&+\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i&-1\\1&0\end{pmatrix},1\right\}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\}\right)
\end{aligned}
$$

We shall consider each summand on the right-hand side in turn.

$$
\begin{aligned}
\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1&2+2i\\0&1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\}\right) =&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1&2+2i\\0&1\end{pmatrix},1\right\}\right)\\
=&\bar{\chi}_0\left(\left\{\begin{pmatrix}1&2+2i\\0&1\end{pmatrix},1\right\}\right)\\
=&1.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i&-2+2i\\1&1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\}\right)\\
=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i&-1+2i\\1&1\end{pmatrix},\Sigma\left(\begin{pmatrix}2i&-1+2i\\1&1\end{pmatrix},\begin{pmatrix}1&-1\\0&1\end{pmatrix}\right)\right\}\right)\\
=&0,
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i&-1\\1&0\end{pmatrix},1\right\}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right\}\right)\\
&=\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i&-1\\1&0\end{pmatrix},\Sigma\left(\begin{pmatrix}2i&-1\\1&0\end{pmatrix},\begin{pmatrix}1&-1\\0&1\end{pmatrix},\right)\right\}\right)\\
&=0.
\end{aligned}
$$

These calculations show that

$$
\left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)=\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}+c\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}
$$

for some complex number c.

Next, we evalulate $\left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)$ on $\left(\begin{smallmatrix}2i & -1\\1 & 0\end{smallmatrix}\right)$. We have

$$\begin{aligned}\left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\} =&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\\&+\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -2+2i\\1 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\\&+\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -1\\1 & 0\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\end{aligned}$$

We shall consider each summand on the right-hand side in turn.

$$\begin{aligned}&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\\=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2 & -1\\1 & 0\end{pmatrix},\Sigma\left(\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\right)\Sigma\left(\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\right)\right\}\right)\\=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2 & -1\\1 & 0\end{pmatrix},1\right\}\right)\\=&1.\end{aligned}$$

The second summand:

$$
\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -2+2i\\ 1 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix},1\right\}\right)\\
=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-2 & 1-2i\\ 1+2i & -1\end{pmatrix},\Sigma\left(\begin{pmatrix}-1+2i & -2+2i\\ 1 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\right\}\right)\\
=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1-2i\\ 1+2i & -1\end{pmatrix},\Sigma\left(\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & -1-2i\\ 1+2i & -1\end{pmatrix}\right)\right\}\right)\\
=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1+2i & -1\\ 0 & 1+2i\end{pmatrix},\Sigma\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}1+2i & -1\\ -4 & 1+2i\end{pmatrix}\right)\right\}\right)\\
=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & -1-2i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix},\Sigma\left(\begin{pmatrix}1 & -1-2i\\ 0 & 1\end{pmatrix},\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}\right)\right\}\right)\\
=&\bar{\chi}_0\left(\left\{\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix},1\right\}\right)\\
=&1.
\end{aligned}
$$

The third summand:

$$
\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix},1\right\}\right)\\
=&\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1-2i & 1-2i\\ 2i & -1\end{pmatrix},\Sigma\left(\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\right\}\right)\\
=&\ 0.
\end{aligned}
$$

Thus,

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)=\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}+2\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}.$$

The next step is to calculate $\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)$ on $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. We have

$$
\begin{aligned}
\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},1\right\}=&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 2+2i\\ 0 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},1\right\}\right)\\
&+\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -2+2i\\ 1 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},1\right\}\right)\\
&+\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},1\right\}\right)
\end{aligned}
$$

We shall consider each summand on the right-hand side in turn. The first summand:

$$\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},1\right\}\right)\\
&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},1\right\}\right)\\
&0.
\end{aligned}$$

The second summand:

$$\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -2+2i\\1 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},1\right\}\right)\\
&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}2i & -1+2i\\1 & 1\end{pmatrix},\Sigma\left(\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix},\begin{pmatrix}2i & -1+2i\\1 & 1\end{pmatrix}\right)\right\}\right)\\
&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix},\Sigma\left(\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right)\right\}\right)\\
&=\bar{\chi}_0\left(\left\{\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix},1\right\}\right)\\
&=1.
\end{aligned}$$

The third summand:

$$\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -1\\1 & 0\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},1\right\}\right)\\
&=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},\Sigma\left(\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\right)\right\}\right)\\
&=1.
\end{aligned}$$

So far,

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},1\right\}=2\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}+c\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\text{ for some complex number } c.$$

We must calculate the action of $\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)$ on $\left(\begin{smallmatrix}2i & -1\\1 & 0\end{smallmatrix}\right)$. We have

$$\begin{aligned}\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\bar{y}\widehat{H}\right)\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\} =& \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\\ &+\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -2+2i\\1 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\\ &+\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -1\\1 & 0\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\end{aligned}$$

We shall consider each summand on the right-hand side in turn. The first summand:

$$\begin{aligned}&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix},1\right\}\right)\\ &=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2+4i & -1\\1 & 0\end{pmatrix},\Sigma\left(\begin{pmatrix}1 & 2+2i\\0 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\1 & 0\end{pmatrix}\right)\right\}\right)\\ &=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\-2-4i & 1\end{pmatrix},\Sigma\left(\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix},\begin{pmatrix}1 & 0\\-2-4i & 1\end{pmatrix}\right)\right\}\right)\\ &=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-4i & 1\end{pmatrix},\Sigma\left(\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\-4i & 1\end{pmatrix}\right)\right\}\right)\\ &=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}1 & 0\\-4i & 1\end{pmatrix},1\right\}\right)\\ &=0.\end{aligned}$$

The second summand:

$$\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\left(\begin{pmatrix}-1+2i & -2+2i\\ 1 & 1\end{pmatrix},1\right\}\left\{\left(\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix},1\right\}\right)$$

$$= \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\left(\begin{pmatrix}-2 & 1-2i\\ 1+2i & -1\end{pmatrix},\Sigma\left(\left(\begin{pmatrix}-1+2i & -2+2i\\ 1 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\right)\right\}\right)$$

$$= \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\left(\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1+2i & -1\\ 0 & 1+2i\end{pmatrix},\Sigma\left(\left(\begin{pmatrix}-1+2i & -2+2i\\ 1 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\right)\right\}\right)$$

$$= \mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\left(\begin{pmatrix}1+2i & -1\\ 0 & 1+2i\end{pmatrix},\Sigma\left(\left(\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}1+2i & -1\\ 0 & 1+2i\end{pmatrix}\right)\right)\right.\right.$$

$$\left.\left.\Sigma\left(\left(\begin{pmatrix}-1+2i & -2+2i\\ 1 & 1\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\right)\right\}\right)$$

$$= 0.$$

The third summand:

$$\begin{aligned}
&\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},1\right\}\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix},1\right\}\right)\\
&=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1+2i & 1-2i\\ 2i & -1\end{pmatrix},\Sigma\left(\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\right\}\right)\\
&=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}-1 & 1\\ 2i & -1-2i\end{pmatrix},\Sigma\left(\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\right.\right.\\
&\left.\left.\Sigma\left(\begin{pmatrix}-1 & 1\\ 2i & -1-2i\end{pmatrix},\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}\right)\right\}\right)\\
&=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i & -1-2i\\ 1 & -1\end{pmatrix},\Sigma\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\begin{pmatrix}-1 & 1\\ 2i & -1-2i\end{pmatrix}\right)\right.\right.\\
&\Sigma\left(\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\\
&\left.\left.\Sigma\left(\begin{pmatrix}-1 & 1\\ 2i & -1-2i\end{pmatrix},\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}\right)\right\}\right)\\
&=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix},\Sigma\left(\begin{pmatrix}2i & -1-2i\\ 1 & -1\end{pmatrix},\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\right.\right.\\
&\Sigma\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\begin{pmatrix}-1 & 1\\ 2i & -1-2i\end{pmatrix}\right)\Sigma\left(\begin{pmatrix}-1+2i & -1\\ 1 & 0\end{pmatrix},\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix}\right)\\
&\left.\left.\Sigma\left(\begin{pmatrix}-1 & 1\\ 2i & -1-2i\end{pmatrix},\begin{pmatrix}1+2i & 0\\ 0 & 1+2i\end{pmatrix}\right)\right\}\right)\\
&=\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\left(\left\{\begin{pmatrix}2i & -1\\ 1 & 0\end{pmatrix},1\right\}\right)\\
&=1.
\end{aligned}$$

Finally, we have

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}\widehat{H}\overline{y}\widehat{H}\right)=2\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}+\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}.$$

Choose another basis

$$\left\{\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}+\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]},\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]}-\mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right\}$$

of $W^{\widehat{K}_\pi(4)}$. Let $W(3)$ be the line spanned by $\left\{\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} + \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right\}$. Observe that

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} + \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right)\widehat{H}\bar{y}\widehat{H} = 3\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} + \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right);$$

and

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} + \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right)\widehat{H}\bar{t}\widehat{H} = -1\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} + \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right).$$

Thus $W(3)$ is the 3-eigenspace of $\hat{y}$ and the -1-eigenspace of $\hat{t}$.

Define $W(-1)$ to be the complement of $W(3)$ in $W^{\widehat{K}_\pi(4)}$: that is, the line spanned by $\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} - \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}$. We have

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} - \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right)\widehat{H}\bar{y}\widehat{H} = -1\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} - \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right);$$

and

$$\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} - \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right)\widehat{H}\bar{t}\widehat{H} = -1\left(\mathbb{1}_{\left[\begin{smallmatrix}2i\\1\end{smallmatrix}\right]} - \mathbb{1}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\right).$$

Thus $W(-1)$ is the -1-eigenspace of $\hat{y}$ and the -1-eigenspace of $\hat{t}$. □

Appendix A

Notes on Chapter 2

A.1 Sage code

The following Sage code is used to calculate $H^2(\sim \backslash D', \kappa_{\mathbb{Z}}) \cong \mathbb{Z}^{(5)}$.

Recall the code used to determine the action of the elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ on $V := \mathrm{Ind}_{\Gamma'}^{\Gamma}(\kappa_{\mathbb{Q}})$: that is, the list ‘gammaactionK’.

```
F.<i> = NumberField(x^2+1)
R = F.ring_of_integers()
pi = F.ideal(1+i)
k = R.residue_field(pi,'b')
kk = R.quotient_ring(2,'b')
kkk = R.quotient_ring(2*pi,'b')
kkkk = R.quotient_ring(4,'b')
kkkkk = R.quotient_ring(4*pi,'b')
M = MatrixSpace(F,2)
m = MatrixSpace(k,2)
mm = MatrixSpace(kk,2)
mmm = MatrixSpace(kkk,2)
mmmm = MatrixSpace(kkkk,2)

quotient0 = [M([1,0,0,1]),M([0,1,-1,0]),M([1,1,0,1]),M([1,0,1,1]),
M([1,-1,1,0]), M([0,-1,1,1])]
```

```
quotient1 = [M([1,0,0,1]),M([i,0,0,-i]), M([1,1+i,0,1]),
M([1,0,1+i,1]), M([i,1+i,0,-i]), M([i,0,1+i,-i]),
M([1,1+i,1+i,1+2*i]), M([i,1+i,1+i,2-i])]

quotient2 = [M([1,0,0,1]),M([1,2,0,1]), M([1,0,2,1]),
M([-1,0,0,-1]), M([-1,2,0,-1]), M([-1,0,2,-1]), M([1,2,2,5]),
M([-5,2,2,-1])]

quotient3 = [M([1,0,0,1]),M([3-6*i,4,-4,-1-2*i]), M([1,2+2*i,0,1]),
M([1,0,2+2*i,1]), M([-5+2*i,2+2*i,4,-1-2*i]),
M([-5+2*i,4,2+2*i,-1-2*i]), M([1,2+2*i,2+2*i,1+8*i]),
M([-1+2*i,2+2*i,2+2*i,3-2*i])]

quotient0m = [m([1,0,0,1]),m([0,1,1,0]),m([1,1,0,1]),m([1,0,1,1]),
m([1,1,1,0]),m([0,1,1,1])]

quotient1mm = [mm([1,0,0,1]), mm([i,0,0,-i]), mm([1,1+i,0,1]),
mm([1,0,1+i,1]), mm([i,1+i,0,-i]), mm([i,0,1+i,-i]),
mm([1,1+i,1+i,1+2*i]), mm([i,1+i,1+i,2-i])]

quotient2mmm = [mmm([1,0,0,1]), mmm([1,2,0,1]),
mmm([1,0,2,1]), mmm([-1,0,0,-1]),
 mmm([-1,2,0,-1]), mmm([-1,0,2,-1]),
mmm([1,2,2,5]), mmm([-5,2,2,-1])]

quotient3mmmm = [mmmm([1,0,0,1]), mmmm([3-6*i,4,-4,-1-2*i]),
mmmm([1,2+2*i,0,1]), mmmm([1,0,2+2*i,1]),
mmmm([-5+2*i,2+2*i,4,-1-2*i]), mmmm([-5+2*i,4,2+2*i,-1-2*i]),
mmmm([1,2+2*i,2+2*i,1+8*i]), mmmm([-1+2*i,2+2*i,2+2*i,3-2*i])]

representatives = [a*b for a in quotient1 for b in quotient3]
```

```
def Decomposition(gamma):
    r = quotient0m.index(m(gamma))
    rep0 = quotient0[r]
    gamma2 = gamma*rep0.inverse()
    a = quotient1mm.index(mm(gamma2))
    rep1 = quotient1[a]
    gamma3 = rep1^-1*gamma2
    u = quotient2mmm.index(mmm(gamma3))
    rep2 = quotient2[u]
    gamma4 = gamma3*rep2^-1
    b = quotient3mmmm.index(mmmm(gamma4))
    rep3 = quotient3[b]
    gamma5 = rep3^-1*gamma4
    return([rep1,rep3,gamma5,rep2*rep0])

gammainverselist = [M([0,-i,-i,0]),M([0,1,-1,0]),M([1,-i,-i,0]),
 M([1,-1,1,0])]

def residuesymbol(x,y):
    K = R.residue_field(y)
    xbar = K(x)
    answer = xbar^((norm(y)-1)/2)
    if answer == K(1):
        return 1
    elif answer == K(-1):
        return -1
    elif answer == K(0):
        return 0

def legendresymbol(x,y):
    factors = F.factor(y)
```

```
        answer = prod([residuesymbol(x,p[0]) for p in factors if p[1]%2])
        return answer

    def kappa(A):
        c = A[1][0]
        d = A[1][1]
        return legendresymbol(c,d)

    matrixlist = []
    for kk in range(4):
        gamma = gammainverselist[kk]
        for ii in range(64):
            r = representatives[ii]
            answer = Decomposition(gamma*r)
            newrep = answer[0]*answer[1]
            jj = representatives.index(newrep)
            kappavalue = kappa(answer[2])
            matrixlist.append([ii,jj,kk,kappavalue])

    def func(ii,jj,kk):
        for entry in matrixlist:
            if entry[0]==ii and entry[1]==jj and entry[2]==kk:
                return entry[3]
        return 0
```

Finally,

```
gammaactionK = [Matrix([[func(ii,jj,kk) for ii in range(64)]
for jj in range(64)]) for kk in range(4)]
```

Then we put

```
kernels = [(1-gammaactionK[kk]).right_kernel() for kk in range(4)]
```

```
generators = []
for W in kernels:
    generators=generators+W.basis()


L = ZZ^64
subspace_Z=L.span(generators)
```

The group 'H2Z' is $H^2(\sim \backslash D', \kappa_{\mathbb{Z}})$.

```
H2_Z=L.quotient(subspace_Z)
```

The output is that 'H2Z' has rank 5.

A.2 Boundary cohomology

The purpose of this appendix is to show that

$$H^2(\Gamma_\infty, E_{2,2}(F)) \cong F, \text{ and}$$
$$H^2(\Gamma_\infty, \mathrm{Ind}\,{}^{\Gamma}_{\Gamma'}(\kappa_F) \bigotimes_F E_{2,2}(F)) \cong F^{(5)}.$$

Recall that if 2 is invertible in M, then:

$$H^2(\Gamma_\infty, M) \cong M^{\{\pm 1\}}/((1-a')M + (1-d')M + (1-e')M). \tag{A.1}$$

We define, as in Appendix A.1,

```
F.<i> = NumberField(x^2+1)
R = F.ring_of_integers()
pi = F.ideal(1+i)
k = R.residue_field(pi,'b')
kk = R.quotient_ring(2,'b')
kkk = R.quotient_ring(2*pi,'b')
kkkk = R.quotient_ring(4,'b')
kkkkk = R.quotient_ring(4*pi,'b')
M = MatrixSpace(F,2)
m = MatrixSpace(k,2)
```

```
mm = MatrixSpace(kk,2)
mmm = MatrixSpace(kkk,2)
mmmm = MatrixSpace(kkkk,2)

quotient0 = [M([1,0,0,1]),M([0,1,-1,0]),M([1,1,0,1]),M([1,0,1,1]),
M([1,-1,1,0]), M([0,-1,1,1])]

quotient1 = [M([1,0,0,1]),M([i,0,0,-i]), M([1,1+i,0,1]),
M([1,0,1+i,1]), M([i,1+i,0,-i]), M([i,0,1+i,-i]),
M([1,1+i,1+i,1+2*i]), M([i,1+i,1+i,2-i])]

quotient2 = [M([1,0,0,1]),M([1,2,0,1]), M([1,0,2,1]),
M([-1,0,0,-1]), M([-1,2,0,-1]), M([-1,0,2,-1]), M([1,2,2,5]),
M([-5,2,2,-1])]

quotient3 = [M([1,0,0,1]),M([3-6*i,4,-4,-1-2*i]), M([1,2+2*i,0,1]),
M([1,0,2+2*i,1]), M([-5+2*i,2+2*i,4,-1-2*i]),
M([-5+2*i,4,2+2*i,-1-2*i]), M([1,2+2*i,2+2*i,1+8*i]),
M([-1+2*i,2+2*i,2+2*i,3-2*i])]

quotient0m = [m([1,0,0,1]),m([0,1,1,0]),m([1,1,0,1]),m([1,0,1,1]),
m([1,1,1,0]),m([0,1,1,1])]

quotient1mm = [mm([1,0,0,1]), mm([i,0,0,-i]), mm([1,1+i,0,1]),
mm([1,0,1+i,1]), mm([i,1+i,0,-i]), mm([i,0,1+i,-i]),
mm([1,1+i,1+i,1+2*i]), mm([i,1+i,1+i,2-i])]

quotient2mmm = [mmm([1,0,0,1]), mmm([1,2,0,1]),
mmm([1,0,2,1]), mmm([-1,0,0,-1]),
 mmm([-1,2,0,-1]), mmm([-1,0,2,-1]),
mmm([1,2,2,5]), mmm([-5,2,2,-1])]
```

```
quotient3mmmm = [mmmm([1,0,0,1]), mmmm([3-6*i,4,-4,-1-2*i]),
mmmm([1,2+2*i,0,1]), mmmm([1,0,2+2*i,1]),
mmmm([-5+2*i,2+2*i,4,-1-2*i]), mmmm([-5+2*i,4,2+2*i,-1-2*i]),
mmmm([1,2+2*i,2+2*i,1+8*i]), mmmm([-1+2*i,2+2*i,2+2*i,3-2*i])]

representatives = [a*b for a in quotient1 for b in quotient3]

def Decomposition(gamma):
    r = quotient0m.index(m(gamma))
    rep0 = quotient0[r]
    gamma2 = gamma*rep0.inverse()
    a = quotient1mm.index(mm(gamma2))
    rep1 = quotient1[a]
    gamma3 = rep1^-1*gamma2
    u = quotient2mmm.index(mmm(gamma3))
    rep2 = quotient2[u]
    gamma4 = gamma3*rep2^-1
    b = quotient3mmmm.index(mmmm(gamma4))
    rep3 = quotient3[b]
    gamma5 = rep3^-1*gamma4
    return([rep1,rep3,gamma5,rep2*rep0])
```

We define the function “kappa” as in Appendix A.1. Put

```
gammaboundarylist = [M([-i,0,0,i]),M([-i,-1,0,i]), M([1,-1,0,1])]
```

“gammaboundarylist” is the list of inverses of $\{a', d', e'\}$. Proceeding:

```
matrixlist = []
for ll in range(3):
    gamma = gammaboundarylist[ll]
    for ii in range(64):
        r = representatives[ii]
        answer = Decomposition(gamma*r)
```

```
        newrep = answer[0]*answer[1]
        jj = representatives.index(newrep)
        kappavalue = kappa(answer[2])
        matrixlist.append([ii,jj,ll,kappavalue])

def func(ii,jj,kk):
    for entry in matrixlist:
        if entry[0] == ii and entry[1] == jj and entry[2] == kk:
            return entry[3]
    return 0
```

Finally, we calculate the quotient (A.1) using the following:

```
E = MatrixSpace(F,3)
gammaboundaryactionS = [E([-1,0,0,0,1,0,0,0,-1]),
E([-1,i,1,0,1,-2*i,0,0,-1]), E([1,1,1,0,1,2,0,0,1])];
gammaboundaryactionSC = [E([-1,0,0,0,1,0,0,0,-1]),
E([-1,-i,1,0,1,2*i,0,0,-1]),E([1,1,1,0,1,2,0,0,1])]

gammaboundaryactionSSC = []
for r in range(3):
    temp1 = gammaboundaryactionS[r]
    for s in range(3):
        temp2 = gammaboundaryactionSC[s]
        if r == s:
            answer = temp1.tensor_product(temp2)
            gammaboundaryactionSSC.append(answer)

Space1_SSC = MatrixSpace(F,9)
Space2_SSC = F^9
images_SSC = [Space1_SSC(1-gammaboundaryactionSSC[p]).image()
for p in range(3)]
```

```
generators_SSC = []
for W in images_SSC:
    generators_SSC=generators_SSC+W.basis()

subspace_SSC = Space2_SSC.span(generators_SSC);
H2_boundarySSC = Space2_SSC.quotient(subspace_SSC)
```

The space "H2 boundarySSC" is of dimension 1. It is $H^2(\Gamma_\infty, E_{2,2}(F))$.
Next we calculate $H^2(\Gamma_\infty, \mathrm{Ind}\,{}_{\Gamma'}^{\Gamma}(\kappa_F) \underset{F}{\otimes} E_{2,2}(F))$.

```
gammaboundaryactionK = [Matrix([[func(ii,jj,kk) for ii in range(64)]
for jj in range(64)]) for kk in range(3)]

gammaboundaryactionKSSC = []
for r in range(3):
    temp1 = gammaboundaryactionSSC[r]
    for s in range(3):
        temp2 = gammaboundaryactionK[s]
        if r == s:
            answer = temp1.tensor_product(temp2)
            gammaboundaryactionKSSC.append(answer)

V = MatrixSpace(F,576)
VV = F^576
images_KSSC = [V(1-gammaboundaryactionKSSC[p]).image() for p in range(3)]

generators_KSSC = []
for W in images_KSSC:
    generators_KSSC=generators_KSSC+W.basis()

subspace_KSSC = VV.span(generators_KSSC)

H2_boundaryKSSC = VV.quotient(subspace_KSSC)
```

The space "H2 boundaryKSSC" is of dimension 5.

A.3 The definition of cusp cohomology

Suppose that Υ is a finite index subgroup of $SL_2(\mathcal{O})$, and M is a finite-dimensional complex representation of Υ. The purpose of Appendix A.3 is to show that the definition of the cusp cohomology $H^q(\Upsilon, M)$ as given in the introduction, is equivalent to the definition we gave in Section 2.4. More precisely, we must show that the image of the map (see (0.5))

$$j : H^q_{\text{cts}}(SL_2(\mathbb{C}), L^2_0(\Upsilon\backslash SL_2(\mathbb{C}))^\infty \otimes M) \longrightarrow H^q(\Upsilon, M)$$

which we shall still denote by $H^q_{\text{cusp}}(\Upsilon, M)$, is the same as the kernel of the restriction map $H^q(\Upsilon, M) \to H^q(U(\Upsilon), M)$ for $q = 1, 2$ (see Section 2.4.1), which we shall denote by $H^q_!(\Upsilon, M)$.

Recall, from the introduction, that $L^2_d(\Upsilon\backslash SL_2(\mathbb{C}))$ is the discrete spectrum of $L^2(\Upsilon\backslash SL_2(\mathbb{C}))$. The inclusion of the space of smooth vectors $L^2_d(\Upsilon\backslash SL_2(\mathbb{C}))^\infty$ into $C^\infty(\Upsilon\backslash SL_2(\mathbb{C}))$ induces a map

$$H^q_{\text{cts}}(SL_2(\mathbb{C}), L^2_d(\Upsilon\backslash SL_2(\mathbb{C}))^\infty \otimes M) \longrightarrow H^q(\Upsilon, M)$$

whose image we denote by $H^q_{(2)}(\Upsilon, M)$, and

$$H^q_!(\Upsilon, M) \subset H^q_{(2)}(\Upsilon, M). \tag{A.2}$$

On the other hand, Borel [4] has shown that

$$H^q_{\text{cusp}}(\Upsilon, M) \subset H^q_!(\Upsilon, M). \tag{A.3}$$

We observed in Section 1.4.1.2 that the only irreducible unitary representation (ϖ_∞, H) of $SL_2(\mathbb{C})$ which occurs in the discrete spectrum $L^2_d(\Upsilon\backslash SL_2(\mathbb{C}))$ and which satisfies $H^q_{\text{cts}}(SL_2(\mathbb{C}), H \otimes M) \neq 0$ is the continuous series representation we denoted by $(\varpi_\infty(\nu_1, \nu_2), B(\nu_1, \nu_2))$. In fact,

$$H^q_{\text{cts}}(G_{\mathbb{C}}, B(\nu_1, \nu_2) \otimes M) = \begin{cases} \mathbb{C} & \text{if } q = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

One can show that $(\varpi_\infty(\nu_1, \nu_2), B(\nu_1, \nu_2))$ is tempered, and by a result of Wallach and Harish-Chandra [37, Theorem 4.3], it occurs with its full multiplicity in the cuspidal spectrum $L_0^2(\Upsilon\backslash SL_2(\mathbb{C}))$. This means that

$$H^q_{\text{cusp}}(\Upsilon, M) \cong H^q_{(2)}(\Upsilon, M). \tag{A.4}$$

It follows, from (A.2), (A.3) and (A.4), that

$$H^q_{\text{cusp}}(\Upsilon, M) \cong H^q_!(\Upsilon, M).$$

Appendix B

Notes on Chapter 3

B.1 Multiplying double cosets in $\mathcal{H}(G, ZH)$

Recall that the decompositions:

$$\begin{aligned} TH &= ZH \cup ZHt \\ THyTH &= ZHyZH \cup ZHytZH \\ THxTH &= ZHxZH \cup ZHxtZH \cup ZHtxZH \cup ZHtxtZH \\ THuTH &= ZHuZH \cup ZHtuZH \end{aligned} \tag{B.1}$$

(where all the unions are disjoint) gave us a basis for $\mathcal{H}(G, ZH)$ as a vector space. To find its structure as an algebra, we must multiply double cosets, and to do this we must write each double ZH-coset as a disjoint union of single ZH-cosets:

$$ZHgZH = \bigcup_i ZHgh_i \text{ where } h_i \in ZH/(ZH \cap gZHg^{-1}).$$

That is, we must find specific representatives for $ZH/(ZH \cap gZHg^{-1})$ for each representative g in $ZH\backslash G/ZH$. Sage gives the following data:

$$ZHxZH = \bigcup_{i=1}^{3} ZHxh_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\};$$

$$ZHxtZH = \bigcup_{i=1}^{3} ZHxth_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\};$$

$$ZHtxZH = \bigcup_{i=1}^{3} ZHtxh_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\};$$

$$ZHtxtZH = \bigcup_{i=1}^{3} ZHtxth_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\};$$

$$ZHyZH = \bigcup_{i=1}^{3} ZHyh_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\};$$

$$ZHytZH = \bigcup_{i=1}^{3} ZHyth_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\};$$

$$ZHuZH = \bigcup_{i=1}^{6} ZHuh_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\};$$

and

$$ZHutZH = \bigcup_{i=1}^{6} ZHuth_i \text{ for}$$
$$h_i \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\}.$$

A "degree" argument can help us check the above decompositions. For example, by (B.1),

$$\deg_{ZH}(THyTH) = \deg_{ZH}(ZHyZH \cup ZHytZH)$$

Since $\deg_{ZH}(TH) = 2$, it follows that $\deg_{ZH}(THyTH) = 2 * \deg_{TH}(THyTH) = 2 * 3 = 6$. Since

$$\left|ZH/(ZH \cap yZHy^{-1})\right| = \left|ZH/(ZH \cap ytZH(yt)^{-1})\right|,$$

we must have $\deg_{ZH}(ZHyZH) = \deg_{ZH}(ZHytZH) = 3$.

The following code multiplies the double cosets:

```
HZdoublecosets = [mmmm([1,0,0,1]),z,z*t,z*y,z*y*t,x,x*t,t*x,t*x*t,
u,u*t]


littlehdashforx = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1])]
littlehdashforxt = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1])]
littlehdashforyt = [mmmm([1,0,0,1]),mmmm([1,0,1,1]),mmmm([1,-1,1,0])]
littlehdashfory = [mmmm([1,0,0,1]),mmmm([1,0,1,1]),mmmm([1,-1,1,0])]
littlehdashforu = [mmmm([1,0,0,1]),mmmm([1,1,0,1]),mmmm([1,-1,1,0]),
mmmm([0,-1,1,0]),mmmm([1,0,1,1]),mmmm([0,-1,1,1])]
littlehdashfortu = [mmmm([1,0,0,1]),mmmm([1,1,0,1]),mmmm([1,-1,1,0]),
mmmm([0,-1,1,0]),mmmm([1,0,1,1]),mmmm([0,-1,1,1])]
littlehdashfortx = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1])]
littlehdashfortxt = [mmmm([1,0,0,1]),mmmm([1,1,1,2]),mmmm([1,0,1,1])]
for h1 in littlehdashforx:
    for h2 in littlehdashfory:
        answer = x*h1*y*h2
        for b in ZH:
            for jj in range(11):
                if mmmm(b*answer) == HZdoublecosets[jj]:
                    print (jj)
```

This code as written, returned, for example, $[8, 9]$. That is,

$$\hat{x} * \hat{y} = \hat{txt} + \hat{u}.$$

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