The reduced modified Ostrovsky equation: integrability and breaking

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Abstract

The reduced modified Ostrovsky equation is a reduction of the modified Korteweg-de Vries equation, in which the usual linear dispersive term with a third-order derivative is replaced by a linear non-local integral term, representing the effect of background rotation. Here we study the case when the cubic nonlinear term has the same polarity as the rotation term. This equation is integrable provided certain slope constraints are satisfied. We demonstrate, through theoretical analysis and numerical simulations, that when this constraint is not satisfied at the initial time, then wave breaking inevitably occurs.

I. INTRODUCTION

It is well-known that the extended Korteweg-de Vries, or Gardner, equation can be used to model internal solitary waves in the atmosphere and ocean, see for instance the reviews by Grimshaw [1, 2] and Helfrich and Melville [3],

$$u_t + \mu u u_x + \nu u^2 u_x + \lambda u_{xxx} = 0 \,. \tag{1}$$

Here $u(x, t)$ is the amplitude of an appropriate linear long wave mode, with linear long wave speed c_0, and (1) is expressed in a frame moving with that speed. The coefficients μ, ν, λ are found from certain internal expressions involving the modal function and the background density stratification. When the coefficient $\mu = 0$, (1) becomes the modified KdV equation. This case can be realised in practice, for instance, when the background stratification is that for a two-layer fluid, with equal layer depths. However, when the effects of background rotation through the Coriolis parameter f need to be taken into account, an extra term is needed, and (1) is replaced by

$$(u_t + \mu u u_x + \nu u^2 u_x + \lambda u_{xxx})_x = \gamma u \,, \tag{2}$$

where $\gamma = f^2/2c_0 \neq 0$. When (1) is just the KdV equation, that is the coefficient $\nu = 0$, then (2) becomes the Ostrovsky equation, see Ostrovsky [4], Grimshaw [5], or the review by Grimshaw *et al.* [6].

Our concern here is with the reduced modified Ostrovsky equation which is obtained by setting $\lambda = 0$ and $\mu = 0$ in (2),

$$u_t + \nu u^2 u_x = \gamma u \,. \tag{3}$$

Importantly, we note that when $\gamma = 0$, equation (3) reduces to a modified Hopf equation. It is then easily demonstrated that all localized solutions, or all periodic solutions, will break. That is the solution will develop an infinite slope in finite time. The issue then is how this breaking is affected when $\gamma \neq 0$. This is to be contrasted with the regularisation by the mKdV equation when breaking is replaced by the emergence of modulated periodic waves, see Kamchatnov *et al.* [7]. This present study is motivated by the recent article by Grimshaw *et al.* [8] who examined the reduced Ostrovsky equation, that is (2) with $\lambda, \nu = 0$. They showed that then the equation is either integrable and solutions exist for all time, or wave breaking occurs, depending on a certain criterion on the curvature of the initial condition.

Since (3) can be regarded as the reduced Ostrovsky equation with cubic nonlinearity rather than quadratic nonlinearity, we expect that a similar type of analysis can be applied here, and that is indeed the case, with the curvature criterion there being replaced here with slope criteria.

A canonical form of the reduced modified Ostrovsky equation is obtained by setting $u(\gamma/2|\nu|)^{1/2}\tilde{u}$, $t = \tilde{t}/\gamma$, $x = \tilde{x}$, so that

$$(u_t \pm \frac{u^2 u_x}{2})_x = u\,. \tag{4}$$

Here the ˜ has been removed. There are two equations, called respectively MROI, MROII according to the sign $\pm$ in the coefficient in the cubic term, corresponding to the sign of $\gamma\nu$ in (2). MROII is also known as the short pulse equation arising in nonlinear optics, see Sakovich and Sakovich [9, 10], Liu *et al.* [11], Pelinovsky and Sakovich [12] and the references therein. Note that equation (4) has the symmetry that if u is a solution, so also is $-u$. Also, the equation is invariant under the transformation $u(x,t) = D\tilde{u}(\tilde{x},\tilde{t})$ where $\tilde{x} = x/D, \tilde{t} = Dt$ for any $D > 0$, and in particular, the slope $u_x = \tilde{u}_{\tilde{x}}$ is invariant. The equation has a zero mass constraint for periodic or localised solutions,

$$\int_{\mathcal{D}} u\,dx = 0\,, \tag{5}$$

where $\mathcal{D}$ is the periodic, or infinite, domain. There are also conservation laws for "momentum" and "energy", similar to those for the Ostrovsky equation, see Grimshaw and Helfrich [13]. Further, multiplying (4) by u_x yields

$$\{\frac{\partial}{\partial t} \pm \frac{u^2}{2}\frac{\partial}{\partial x}\}u_x^2 + 2uu_x(\pm u_x^2 - 1) = 0\,,$$

and hence there is another conservation law,

$$F_t \pm (\frac{u^2 F}{2})_x = 0\,, \quad F = |1 \mp u_x^2|^{1/2}\,. \tag{6}$$

This conservation law is crucial for the issue of integrability and breaking.

II. INTEGRABILITY

To establish integrability, we follow the same procedure used for the reduced Ostrovsky equation, see Grimshaw *et al.* [8]. Thus we transform to characteristic coordinates,

$$x = \theta(X,T) = X \pm \int_0^T \frac{U^2(X,T')}{2}\,dT'\,, \quad t = T\,, \quad u(x,t) = U(X,T)\,. \tag{7}$$

Equivalently these are defined by

$$\frac{dx}{dt} = \pm \frac{u^2}{2}, \quad \text{where} \quad x = X \quad \text{at} \quad t = 0. \tag{8}$$

It then follows that

$$U_T = u_t \pm \frac{u^2 u_x}{2}, \quad U_X = \phi u_x,$$

$$\phi = \theta_X = 1 \pm \int_0^T U(X, T') U_X(X, T') \, dT', \quad \phi_T = \pm U U_X. \tag{9}$$

The Jacobian of the transformation is

$$J = \frac{\partial(x, t)}{\partial(X, T)} = \phi.$$

Thus equation (4) becomes

$$U_{XT} = \phi U, \quad \text{and so} \quad U U_{XTT} - U_T U_{XT} = \pm U^3 U_X. \tag{10}$$

This is more conveniently written as the system

$$\phi_T = \pm W U, \quad W_T = \phi U, \quad W = U_X. \tag{11}$$

Eliminating U yields the conservation law

$$(\phi^2 \mp W^2)_T = 0, \tag{12}$$

showing that $\phi^2 \mp W^2$ is conserved along the characteristics $X = $ constant. Importantly, this is just the conservation law (6) since

$$\phi^2 \mp W^2 = \phi^2(1 \mp u_x^2) = \phi^2 F^2. \tag{13}$$

Next consider the evolution from an initial profile $u(x, 0) = u_0(x)$ for some smooth $u_0(x)$. Since $X(x, 0) = x, \phi = 1, T = 0$ when $t = 0$,

$$\phi^2 \mp W^2 = 1 \mp W_0^2, \quad \phi F = F_0, \quad T \geq 0, \tag{14}$$

where $\quad W_0 = W(X, 0) = U_X(X, 0) = u_{0x}(x), \quad F_0(X) = |1 \mp u_{0x}^2|^{1/2}.$

We next introduce the transformations associating a unique $Z(X, T)$ with each pair (ϕ, W),

$$\phi = F_0(X) \cosh Z, \quad W = F_0(X) \sinh Z, \quad \text{for} \quad \text{MROI}, \quad u_{0x}^2 < 1, \tag{15}$$

$$\phi = F_0(X) \sinh Z, \quad W = F_0(X) \cosh Z, \quad \text{for} \quad \text{MROI}, \quad u_{0x}^2 > 1, \tag{16}$$

$$\phi = F_0(X) \cos Z, \quad W = F_0(X) \sin Z, \quad \text{for} \quad \text{MROII}. \tag{17}$$

These imply in turn

$$Z_T = U \,, \quad Z_{XT} = F_0(X) \sinh Z \,, \quad \text{for} \quad \text{MROI} \,, \quad u_{0x}^2 < 1 \,, \tag{18}$$

$$Z_T = U \,, \quad Z_{XT} = F_0(X) \cosh Z \,, \quad \text{for} \quad \text{MROI} \,, \quad u_{0x}^2 > 1 \,, \tag{19}$$

$$Z_T = U \,, \quad Z_{XT} = F_0(X) \sin Z \,, \quad \text{for} \quad \text{MROII} \,. \tag{20}$$

Thus Z satisfies the sinh-Gordon, cosh-Gordon or sine-Gordon equation respectively. All are integrable equations, but only (18, 20) have soliton solutions. For further progress we examine each of MROI, MROII separately.

III.   ANALYSIS AND NUMERICAL RESULTS

A.   The reduced modified Ostrovsky equation MROI

1. $|u_{0x}| < 1$ everywhere

First we examine the case of a $+$ sign in (4), which is the main case of interest here. If the initial slope $|u_{0x}| < 1$ for all x, then $0 < F_0(X) \le 1$ for all X. It follows from (14) that then $W_0^2 < 1$ for all X, and so $\phi > |W| \ge 0$ for all X, T. Breaking cannot occur and since the slope $u_x = U_X/\phi = W/\phi$, $|u_x| < 1$ for all x, t. The equation remains integrable for all t.

2. $|u_{0x}| > 1$ somewhere

Next suppose that there is a set of intervals $X_1 < x < X_2$ where $|u_{0x}| > 1$ with equality only at the end points, so that $F_0(X_{1,2}) = 0$. Since F is conserved, see (14), $F = 0$ is conserved on characteristics, that is, $F(X_{1,2}, T) = 0$ for all $T \ge 0$. When the initial value of $1 - u_{0x}^2$, takes both positive and negative values, then as long as the solution exists, that is $0 < \phi < \infty$, the arguments above show that the X, T domain is divided into regions where $|W/\phi| = |u_x| > 1$, namely the region between the characteristic boundaries $X = X_{1,2}$ and the remaining regions where $|u_x| < 1$ with $|u_x| = 1$ on $X = X_{1,2}$. Moreover, the wave cannot break in the regions of the (X, T) domain where $|u_x| < 1$.

In the region $X_1 < X < X_2$,

$$F_0(X) = [u_{0x}^2 - 1]^{1/2} = [W_0^2 - 1]^{1/2} > 0 \,, \tag{21}$$

and it follows that $|u_x| > 1$, throughout the region. In this case, we can use (16) to obtain the cosh-Gordon equation (19). This can be written in terms of ϕ or F alone, as follows

$$\{\frac{\psi_X}{[1 + \psi^2]^{1/2}}\}_T = \{\frac{\psi_T}{[1 + \psi^2]^{1/2}}\}_X = F_0(1 + \psi^2)^{1/2}, \quad \psi = \frac{\phi}{F_0} = \frac{1}{F}. \tag{22}$$

Integrating (22) yields

$$\psi_X = (1 + \psi^2)^{1/2}\{-\frac{F_{0X}}{F_0(1 + F_0^2)^{1/2}} + \int_0^T (1 + \psi^2(X, T'))^{1/2} \, dT'\}. \tag{23}$$

Hence, in any region where $F_{0X} \leq 0, \psi_X > 0$ and hence ψ cannot take its minimum value in any such region. Instead a minimum can only be reached where $F_{0X} > 0$, that is where $|u_{0x}| > 1$ and is increasing.

The system (11) subject to the initial conditions that $\phi = 1, W = u_{0x}$ at $T = 0$ is solved to spectral accuracy following Esler *et al.* [14] and Grimshaw *et al.* [8]l. As noted by Esler *et al.* [14] solving in characteristics space is particularly useful for investigating wave breaking as the wave breaks when an order unity quantity passes smoothly through zero. Esler *et al.* [14] show that the characteristic integrations can be carried smoothly past breaking where they agree closely with finite volume integrations which fit "equal area" shocks to the waves after breaking.

The system (11) was integrated numerically with spectral accuracy by performing the integration in Fourier space and then normalising in real space using the result that the trapezium rule is spectrally accurate for periodic functions. Integrations up to $T = 100$ with 2048 nodes showed F was conserved with an accuracy of 10^{-6}. Figure 1 shows the wave profile $u(X, T_b)$ and the Jacobian $\phi(X, T_b)$ at the instant of breaking, $T = T_b = 108.25$ for the initial profile

$$u_{0x}(x) = b \cos x, \tag{24}$$

with $b = 1.00005$ so that $\max[u_{0x}^2 - 1]^{1/2} = 10^{-2}$ For this initial condition F_0 has period π and thus so too does ϕ, while u_{0x} has period 2π and hence so too do W and U. The wave first breaks in small intervals surrounding $X = \pi/2$ and $X = 3\pi/2$ where $|u_{0x} > 1$. In particular, the wave first breaks just below the points of maximum $|u_{0x}|$ at $X = \pi/2$ and $X = 3\pi/2$. Let the minimum of the Jacobian ϕ in $(0, \pi)$ at time T be $\phi_m(T)$ and lie at $X_m(T)$. Figure 2a shows $\phi_m T)$ for the same evolution as Figure 1, with $\phi_m(T)$ first vanishing at $T_b = 108.25$ and Figure 2b shows that $X_m(T)$ lies below, but close to $X = \pi/2$ as expected. Figure 3 shows the variation in the time to breaking, T_b with F_m, the value of
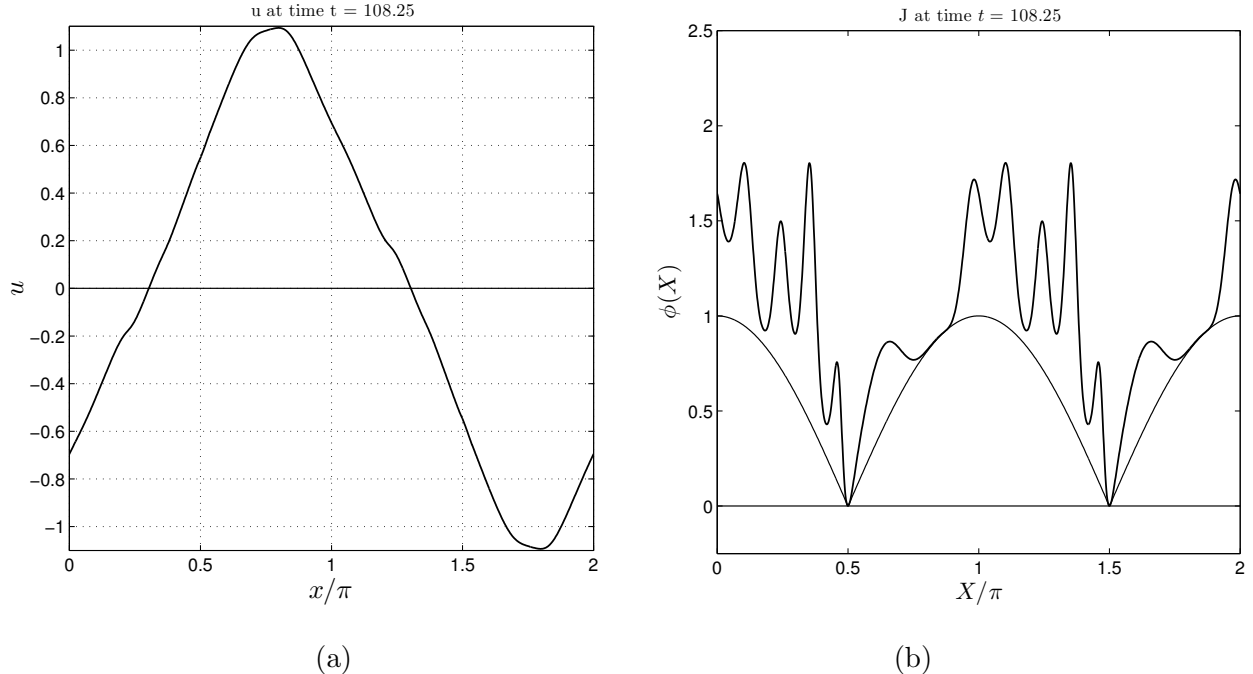
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FIG. 1. (a) The wave profile $u(x, t_b)$ at the instant of breaking, $T = t = t_b = 108.25$, for the initial profile (24) with $b = 1.00005$ so that $\max[u_{0x}^2 - 1)^{1/2}] = 10^{-2}$ computed with $N = 1024$ nodes. (b) The Jacobian $\phi(X, t_b)$ at the instant of breaking. The thinner curve shows $F_0(X) = |1 - u_{0x}^2(X))|^{1/2}$. The initial profile slope satisfies $|u_{0x}| > 1$ in small intervals surrounding $X = \pi/2$ and $X = 3\pi/2$. As expected from (14), the Jacobian exceeds $F_0(X)$ outside these intervals and so breaking first occurs inside these intervals..

F_0 at the point of initial maximum gradient for the initial profile (24) with varying b. The behaviour with decreasing F_m is consistent with the relation

$$T_b = A F_m^{-4/3} \tag{25}$$

for some constant A.

B. The modified reduced Ostrovsky equation MROII

Although this case is not our main interest here, we record here some known results for completeness and comparison with the case MROI, see Liu *et al.* [11] and Pelinovsky and Sakovich [12] for instance. Now the initial condition is such that $F_0(X) \geq 1$ is defined for all X as a bounded smooth function of X. Hence $MROI$ is integrable provided that $\phi \neq 0$,
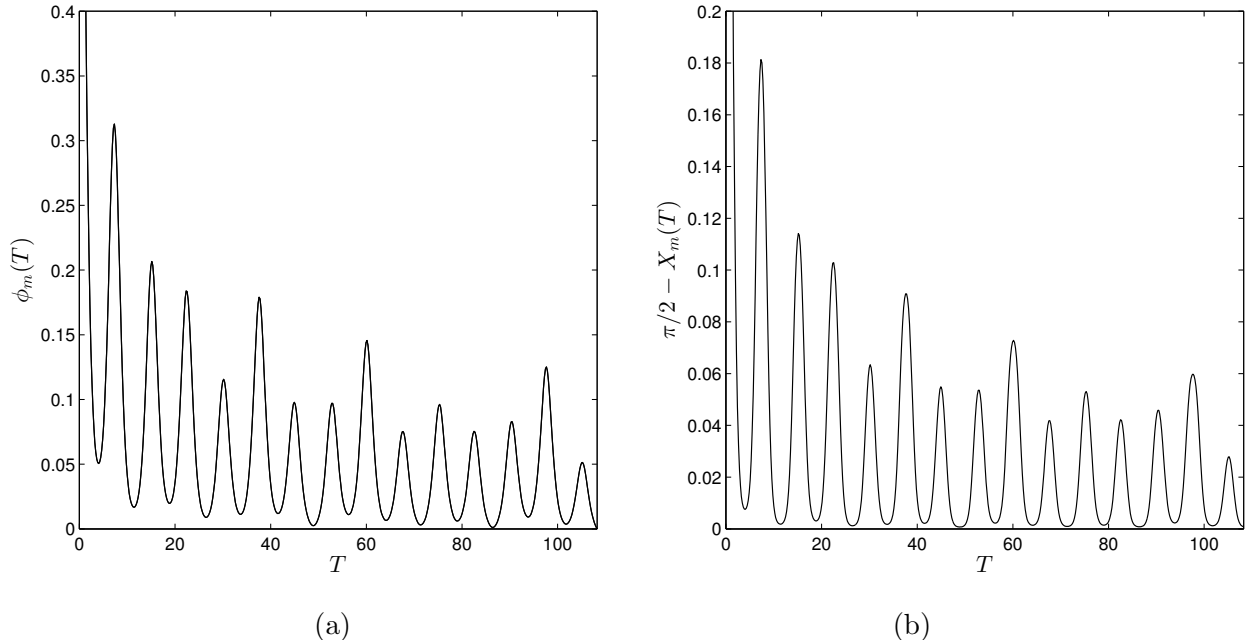
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FIG. 2. (a) $\phi_m(T)$, the minimum of the Jacobian in $(0, \pi)$ at time T for the evolution of figure 1. $\phi_m(T)$ first vanishes at $t_b = 108.25$. (b) The distance $X_m(T)$, the position of the minimum of the Jacobian, lies below $\pi/2$.

which is the case unless $u_x \to \infty$, that is unless breaking occurs. Also $W = U_X$ is bounded for all X, T, and so the case $\phi \to \infty$ cannot occur. At $T = 0, \phi = 1, |Z| < \pi/2$, see (17). and so $\phi > 0$ until breaking occurs, when $\phi = 0, Z = \pm\pi/2$. Hence the issue of breaking/not breaking in MROI (4) from an initial condition $u(x, 0) = u_0(x)$ can be restated in terms of the sine-Gordon (SG) equation (20), which can be written as

$$Z_{YT} = \sin Z, \quad Y = \int^X F_0(X') \, dX'. \tag{26}$$

That is $u_0(x)$ generates a $Z(X, 0) = Z_0(X)$ initial condition for the SG equation (20) such that $-\pi/2 < Z_0(X) < \pi/2$. Then if the evolving solution is such that $-\pi/2 < Z(Y, T) < \pi/2$ for all $T \geq 0$, then there is no breaking. In particular, note that the kink solution of the SG equation violates this condition, and hence any initial condition for the SG equation which generates a kink, corresponds to a breaking solution for the MROII equation. Equation (26) has an infinite set of conservation laws inherited from the SG equation, and especially we
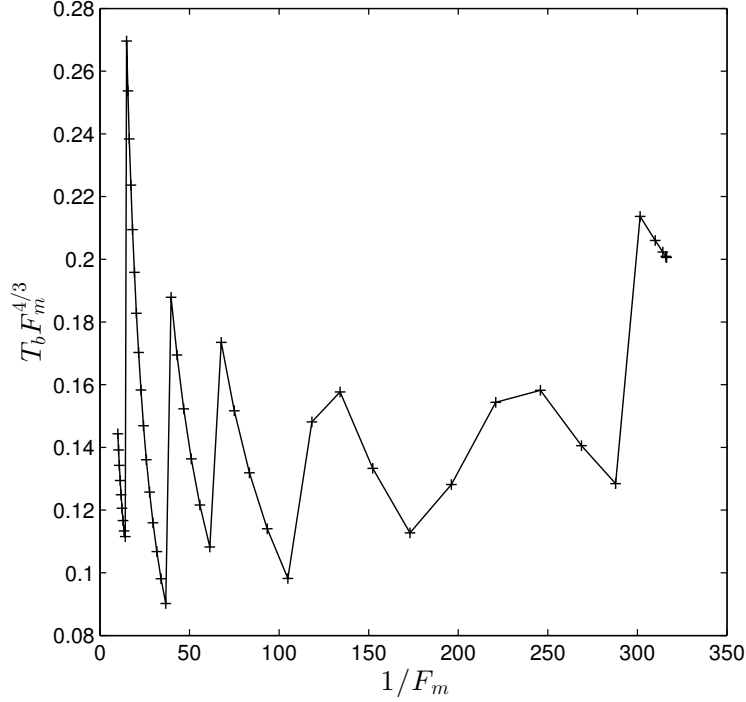
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FIG. 3. The scaled time to breaking, $F_m^{4/3} t_b$, as a function of $1/F_m$, where F_m is the value of F_0 at the point of initial maximum gradient for the initial profile (24) with varying b.

note

$$
\begin{aligned}
&\left(\frac{Z_Y^2}{2}\right)_T - (1 - \cos Z)_Y = 0\,, \\
&(1 - \cos Z)_T - \left(\frac{Z_T^2}{2}\right)_Y = 0\,, \\
&\left(-Z_{YY}^2 + \frac{Z_Y^4}{4}\right)_T + (Z_Y^2 \cos Z)_Y = 0\,, \\
&(Z_T^2 \cos Z)_T + \left(-Z_{TT}^2 + \frac{Z_T^4}{4}\right)_Y = 0\,.
\end{aligned}
\tag{27}
$$

Using these and other conservation laws, it can be shown that, on the one hand, there is now breaking and solutions exist for all time if the initial condition has sufficiently small amplitude and slope, while on the other hand, an initial condition with a large amplitude and slope will break, see Liu *et al.* [11] and Pelinovsky and Sakovich [12]. However, a more precise breaking condition analogous to that for MROI in section 3.1 is elusive.

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IV. CONCLUSION

For the MROI equation (4), we have shown that when the initial condition $u(x,0) = u_0(x)$ is such that $|u_{0x}| > 1$ somewhere, then wave breading inevitably occurs, and does so in a location close to the left-hand end point of the region where $|u_{0x}| > 1$. On the other hand, if $u_{0x}| < 1$ everywhere, then the equation is integrable, since it can be transformed to the sinh-Gordon equation (18). The precision of this result is analogous to that found for the reduced Ostrovsky equation, see Grimshaw *et al.* [8], where the slope condition is replaced by a curvature condition. On the other hand, although the MROII equation (4) can be transformed to the integrable sine-Gordon equation (20), we have not found any similar precise wave-breaking conditions, although the available numerical and analytical evidence suggests that initial conditions with sufficiently steep slopes lead to breaking, but solutions exist for all time when the initial slopes are sufficiently small, see Liu *et al.* [11] and Pelinovsky and Sakovich [12].

Finally, we recall that the MROI, MROII equations are obtained from extended Ostrovsky equation (2) by omitting the quadratic nonlinear term, and the third-order linear dispersive term. When wave-breaking occurs, then this latter term needs to be restored, with the outcome that the potential discontinuity with an infinite slope is, in practice, resolved by the formation of an undular bore, see Kamchatnov *et al.* [7]. Hence, we infer that the breaking condition, now expressed in terms of the original variables, $|u_{0x}| > |\gamma/2\nu|^{1/2}$ implies the formation of an undular bore in the full equation (2) (with $\mu = 0$). Thus, as expected strong rotation, (large γ) inhibits the formation of undular bores.

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