Shadow Boundaries of Convex Bodies

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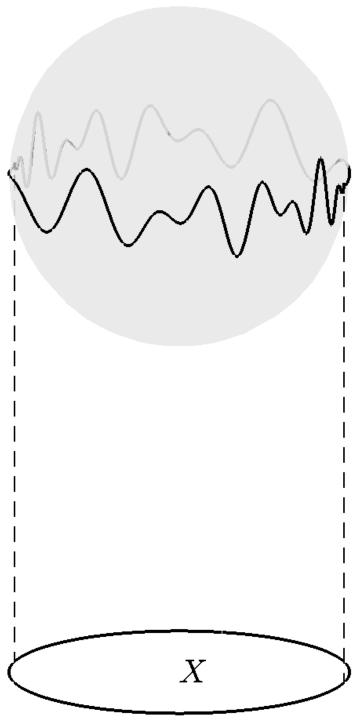
I, Louise Jottrand confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Chapter 1 is joint work with David Larman and Peter Mani.

Abstract

If C is a convex body in $\mathbb{R}^n$ and X a k-dimensional linear subspace of $\mathbb{R}^n$, we denote by $\mathbf{S}(C, X)$ the shadow boundary of C over X which is defined as the collection of all points which belong to C and to one of its tangent $(n-k)$-flats orthogonal to X. For almost all directions in $\mathbb{R}^3$, the shadow boundary is a curve encompassing the body C. It has been established long ago by G. Ewald, D.G. Larman and C.A. Rogers [11] that, for every given C, $\mathbf{S}(C, X)$ is almost always a topological $(k-1)$-sphere. As a follow on from this result, in 1974 Peter McMullen asked whether most of these shadow boundaries would have finite "length" [15]. This is already shown to be true for polytopes and also true for general convex bodies when the dimension of the subspace X is 1 or $n-1$. Here we show that almost all shadow boundaries have finite "length" whatever the dimension k, $1 \leq k < n$, of the subspace X.



The set of shadow boundaries of infinite "length" has also been considered in the context of Baire category. In 1989,

P. Gruber and H. Sorger proved that, in the Baire category sense, most pairs (C, X), where C is a convex body in $\mathbb{R}^n$ and X an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, produce shadow boundaries $\mathbf{S}(C, X)$ of infinite length. Here we show that this result also holds for pairs (C, X) where X is a k-dimensional subspace, $1 \leq k < n$. We also consider the length of increasing paths in the 1-skeleton of a convex body.

We conclude with observations and open questions arising from the work on shadow boundaries of the first two chapters.

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Contents

Chapter 1

Almost all shadow boundaries have finite length

1.1 Introduction

1.1.1 Background / Premise

Following up on a research problem of 1957, Klee asked if the boundary of a d-dimensional convex body could contain line segments in all directions [10]. In 1960 McMinn [12] answered this question by showing that:

The set D of directions of line segments lying on the boundary of a 3-dimensional convex body is contained in the union of the ranges of a countable family of Lipschitz functions each on the 1-dimensional closed unit ball B_1 to the surface of the 2-dimensional unit sphere S_2. By virtue of the Lipschitz nature of these functions, they possess total differentials (Lebesgue measure) almost everywhere and their ranges are compact and have finite one dimensional measure.

Besicovitch followed with a simplification of his proof in 1963 [13]. Finally Ewald, Larman and Rogers generalised the result to n dimensions in their 1970 publication [11]. Specifically, they proved:

Theorem (Ewald, Larman and Rogers [11])

If $1 \leq r \leq n-1$ and K a convex body in $\mathbb{E}^n$, the points $\pm G(F)$, corresponding to the r-flats F in $\mathbb{E}^n$ meeting the boundary of K in a set of linear dimension r, form a set on I_r^n of σ-finite $r(n-r-1)$-dimensional Hausdorff measure.

The shadow boundary of a convex body over a subspace X of $\mathbb{R}^n$ is the set of points of its boundary which project onto the boundary of its shadow on X. We call a shadow boundary sharp if its projection is injective.

From Ewald, Larman and Rogers' result we know that almost all shadow boundaries are sharp.

The aim of this chapter is to prove a further property of these sharp shadow boundaries which was first suggested at a workshop in Oberwolfach in 1974 by Peter McMullen. He asked whether *sharp shadow boundaries also have finite length*.

This was answered in the affirmative by Peter Steenaerts in 1985 [7] for the cases where X is an l-dimensional subspace of $\mathbb{E}^n$ and $l = 1$ or $n-1$. See Appendix A for a translation of [7] to English.

The work in this chapter generalises this result to shadow boundaries over subspaces of any dimension l, where $1 \leq l \leq n$.

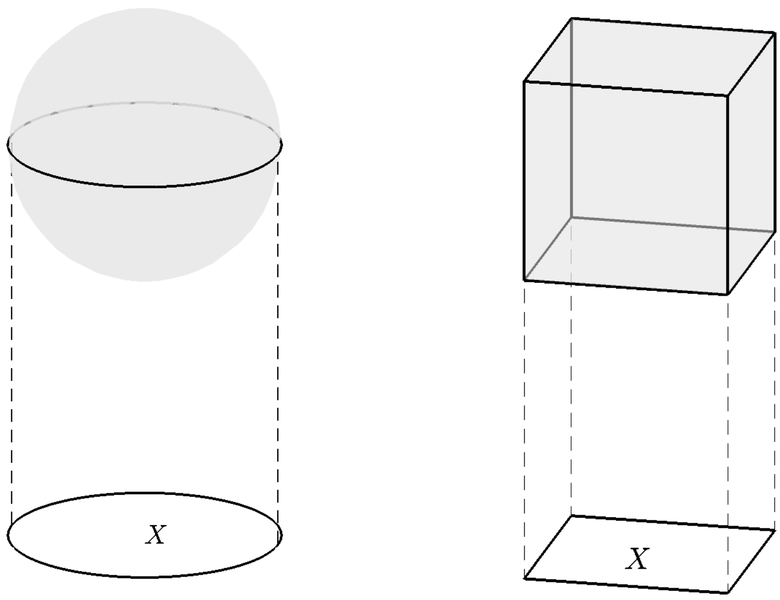
1.1.2 Basic Construction

Consider a convex body C in $\mathbb{R}^3$. We will use the cube and the sphere as examples to illustrate certain properties. Choose a two-dimensional subspace X of $\mathbb{R}^3$, say the horizontal plane $\mathbb{R}^2$.

The first step in constructing the shadow boundary of C over X is to orthogonally project C onto X. In the case of a sphere, the projection would be a disc and in the case of a cube with one of its faces parallel to X, we would get a square.

Now take the relative boundary of this projection and apply the inverse projection map to it. For the sphere, the relative boundary of the projection is a circle, which will map to an infinite hollow cylinder. In the case of the cube, we are mapping the outline of a square to an infinite box with empty interior. Both of these are oriented with their axis of symmetry being vertical. See Figure 1.1.

Figure 1.1: Shadow boundary of sphere and cube



The shadow boundary of our convex body with respect to the subspace given is defined as the intersection of the inverse projection we have just described and the original body C. The sphere's shadow boundary will be a circle on its boundary (equator) and the shadow boundary of the cube will be the union of its four vertical faces.

Looking at our two examples, there is an obvious difference between the end results. A circle is a 1-dimensional curve in $\mathbb{R}^2$, whereas the shadow boundary of the cube is a 2-dimensional compact surface in $\mathbb{R}^3$. Another way of looking at it is that corresponding to each point of the relative boundary of the projection, there is a unique point on the shadow boundary of the sphere, but for the cube there is an entire line segment. This illustrates the notion that a shadow boundary can be sharp.

Formally: A shadow boundary of a convex body is sharp if the projection map restricted to the shadow boundary is injective.

Here, the sphere has a sharp shadow boundary whereas the cube does not. However, if you consider the shadow boundary of the cube in other directions then it will almost always be sharp (i.e. always except for a negligible number of directions. Namely the directions contained in the three planes parallel to its faces).

1.1.3 Notation

- $\mathbb{R}^k = \text{lin}\{e_1, e_2, ..., e_k\} \subset \mathbb{R}^n$.

- $\mathcal{O}_n$ denotes the normalised Haar measure on the orthogonal group $O(n)$.

  Definition 1. *If G is a locally compact group then a left invariant Haar measure on G is a Borel measure μ satisfying the following conditions:*

  i) $\mu(xE) = \mu(E)$ for every $x \in G$ and every measurable $E \subseteq G$.

  ii) $\mu(U) > 0$ for every nonempty open set $U \subseteq G$.

  iii) $\mu(K) < \infty$ for every compact set $K \subseteq G$.

  This measure is right invariant if property i) is replaced by

  i) $\mu(Ex) = \mu(E)$ for every $x \in G$ and every measurable $E \subseteq G$.

  If this measure is both right and left invariant it is known as the Haar measure.

- $\Gamma(k)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$.

- $\Gamma(k, l)$ is the set $\{(X, Y) \in \Gamma(k) \times \Gamma(l) : X \subset Y\}$, for $0 \le k \le l \le n$.

- The measures $\gamma(k)$ on $\Gamma(k)$ and $\gamma(k, l)$ on $\Gamma(k, l)$ are given by:

$$\begin{aligned} \gamma(k)[M] &= \mathcal{O}_n\{r \in O(n) : r[\mathbb{R}^k] \in M\} \text{ where } M \subset \Gamma(k), \\ \gamma(k,l)[N] &= \mathcal{O}_n\{r \in O(n) : (r[\mathbb{R}^k], r[\mathbb{R}^l]) \in N\} \text{ where } N \subset \Gamma(k,l). \end{aligned}$$

- $\mathscr{H}^n$ is the n-dimensional Hausdorff measure.
- $\mathscr{L}^n$ is the n-dimensional Lebesgue measure.
- $\text{Lip}(f)$ is the Lipschitz constant of a mapping f.
- π_X denotes the orthogonal projection of $\mathbb{R}^n$ onto a subspace $X \in \mathbb{R}^n$.
- $X^\perp$ denotes the kernel of the map π_X.
- $W_i(C)$ denotes the i^{th} Minkowski Quermass integral of the convex body C in $\mathbb{R}^n$. Quermass integrals are related to the concept of mixed volumes. See Chapter 4 in R. Schneider [22].

Using this notation, we may now formally define shadow boundaries, intermediate shadow boundaries and the sets of subspaces producing sharp shadow boundaries.

Definition 2. $\mathbf{S}(C,X) = \{p \in C : (p + X^\perp) \cap \text{int}(C) = \emptyset\} = C \cap \pi_X^{-1}[\text{rel bd}(\pi_X C)]$ *is the shadow boundary of* C *over* X.

Definition 3. $\mathbf{P}(C,k) = \{X \in \Gamma(k) : \pi_X|_{\mathbf{S}(C,X)}$ *is injective*$\}$ *is the set of subsets* X *in* $\Gamma(k)$ *for which the shadow boundary* $\mathbf{S}(C,X)$ *is sharp.*

Definition 4. $\sigma(C,X) = (\pi_X|_{\mathbf{S}(C,X)})^{-1}$*,* $\sigma(C,X) : \text{relbd}(\pi_X C) \to \mathbf{S}(C,X)$ *for* $X \in \mathbf{P}(C,k)$*, is the lifting map onto the shadow boundary.*

Definition 5. $\mathbf{S}(C,X,Y) = (\pi_Y C) \cap \pi_X^{-1}(\text{relbd}\, \pi_X C)$ *for* $(X,Y) \in \Gamma(k,l)$ *is the intermediate shadow boundary of* C *over* (X,Y)*.*

Definition 6. $\mathbf{P}(C,k,l) = \Gamma(k,l) \cap [\mathbf{P}(C,k) \times \mathbf{P}(C,l)]$*, for* $(X,Y) \in \Gamma(k,l)$*.* $\mathbf{P}(C,k,l)$ *is the set of pairs* (X,Y) *for which the intermediate shadow boundary* $\mathbf{S}(C,X,Y)$ *is sharp.* $\pi_X|_{\mathbf{S}(C,X,Y)}$ *carries* $\mathbf{S}(C,X,Y)$ *homeomorphically onto* relbd $\pi_X C$*.*

$\sigma(C,X,Y)$ is the inverse of $\pi_X|_{\mathbf{S}(C,X,Y)}$ and $\sigma(C,X) = \sigma(C,Y) \circ \sigma(C,X,Y)$ holds for all $(X,Y) \in \mathbf{P}(C,k,l)$.

1.1.4 Outline

We shall prove:

Theorem 1. Let C be a convex body in $\mathbb{R}^n$ and define the maps $\varphi(C,l) : \Gamma(l) \to [0,\infty]$, for $l \in \{1,\dots,n-1\}$, by $\varphi(C,l)[X] = \mathscr{H}^{l-1}\big(\mathbf{S}(C,X)\big)$. Then each $\varphi(C,l)$ is $\gamma(l)$-measurable, and satisfies $\int \varphi(C,l)\,\mathrm{d}\gamma(l) \leq a(l,n)W_{n-l+1}(C)$.

$\varphi(C,l)[X]$ represents the $(l-1)$-dimensional Hausdorff measure of the shadow boundary of C over X and $a(l,n)$ is a constant independent of C. This theorem says that, averaged over all l-dimensional subspaces X, the $(l-1)$-dimensional Hausdorff measure of the shadow boundary of a given convex body C is less than or equal to $a(l,n)W_{n-l+1}(C)$, a constant multiple of C's $(n-l+1)$-dimensional Quermass integral.

Corollary 1. *Almost all shadow boundaries have finite "length".*

To prove Theorem 1, we shall use the fact that the average length of the shadow boundary of a convex polytope in direction X, over all subspaces $X \in \Gamma(l)$, is finite and can be expressed in terms of Quermass integrals (Lemma 1.2.2).

We will then relate the length of the shadow boundary of a convex body to the length of the relative boundary of its projection by proving that the lifting maps $\sigma(C,X)$ are rectifiable for all C and almost all $X \in \mathbf{P}(C,l)$ (Theorem 2).

This will require us first to show that the lifting maps $\sigma(C,X,Y)$ are rectifiable for all $(X,Y) \in \mathbf{P}(C,l,l+1)$ (Proposition 1) using the concept of intermediate shadow boundary (introduced above) as well as the Lebesgue area (which we shall prove is equivalent to the Hausdorff measure in our case) and its lower semi-continuity property.

This will then allow us to prove (Proposition 2) that given $\varepsilon > 0$ there exist a constant b and a compact subset $\mathbf{M}[C|(X,Y)]$ of the relative boundary of the projection of C which satisfy the following conditions:

- $\sigma(C,X,Y)[\mathbf{M}[C|(X,Y)]]$ covers all but ε of the intermediate shadow boundary $\mathbf{S}(C,X,Y)$ of C and

- $\sigma(C, X, Y)$ restricted to $\mathbf{M}[C|(X, Y)]$ has Lipschitz constant less than or equal to b.

From here we will show that this is also true for shadow boundaries (Proposition 3) and, using induction on the dimension l, we will deduce the rectifiability of the lifting maps $\sigma(C, X)$ (Theorem 2).

It will then remain to show that the average measure of the l-dimensional shadow boundary of C is bounded. By approximating convex bodies by polytopes we will be able to use our estimate of Lemma 1.2.2, combined with the rectifiability of $\sigma(C, X)$ and the lower semi-continuity of the Lebesgue area, to obtain the desired result (Theorem 1).

1.2 Preliminary Results

Almost all shadow boundaries are sharp (Ewald, Larman and Rogers [11] and Zalgaller [8]):

Lemma 1.2.1. *For every convex body C and every pair of integers (k, l) with $0 \leq k \leq l \leq n$, we have:*

$$\gamma(k)[\mathbf{P}(C, k)] = 1, \tag{1.1}$$

$$\gamma(k, l)[\mathbf{P}(C, k, l)] = 1. \tag{1.2}$$

Proof. is quoted in [8]

For $Y \in \Gamma(l)$, set $\Delta(Y) = \{X \in \Gamma(k) : X \subset Y\}$ and notice that the orthogonal group $O(n)$ acts naturally on $\Delta(Y)$.

If $A \subset \Gamma(k, l)$ is a Borel set, it follows from Fubini's Theorem that

$$\int_{Y \in \Gamma(l)} \big(\delta(Y)[A_Y]\big) \, \mathrm{d}\gamma(l)[Y] = \gamma(k, l)[A], \tag{1.3}$$

where we have written $A_Y = \{X \in \Delta(Y) : (X, Y) \in A\}$ and $\delta(Y)$ stands for the $O(Y)$-invariant Radon measure on $\Delta(Y)$. Let us set $\mathbf{P}(Y) = \{X \in \Delta(Y) :$

$\pi_X|_{\mathbf{S}(C,X,Y)}$ is injective$\}$ and use (1.2) and (1.3) in order to establish

$$\gamma(k,l)[\mathbf{P}(C,k,l)] = \int_{\mathbf{P}(C,l)} \varphi \, \mathrm{d}\gamma(l) = \gamma(l)[\mathbf{P}(C,l)] = 1,$$

where φ stands for $\delta(Y)[\mathbf{P}(Y)]$. This proves the claim in (1.2). $\square$

As with many questions relating to convex bodies, the case of polytopes is covered separately. The following lemma shows that almost all shadow boundaries of convex polytopes have finite length and gives a precise value for their average measure given a convex polytope P.

Lemma 1.2.2. *For every pair of integers k, l, with $0 \le k < l \le n$, there is a number $\alpha(k,l) > 0$ such that the equation*

$$\int \mathscr{H}^{k-1}\big(\mathbf{S}(P,X,Y)\big) \, \mathrm{d}\gamma(k,l)[X,Y] = \alpha(k,l) W_{n-k+1}(P)$$

holds for each n-polytope $P \subset \mathbb{R}^n$.

Proof. For $Y \in \Gamma(l)$ write $\Delta(Y) = \{X \in \Gamma(k) : X \subset Y\}$ and $O(Y) = \{\rho \in O(n) : \rho(Y) = Y\}$.

Denote by $\delta(Y)$ the normalised regular $O(Y)$-invariant outer Borel measure on $\Delta(Y)$. The function $(X,Y) \mapsto \mathscr{H}^{k-1}\big(\mathbf{S}(P,X,Y)\big)$ is continuous on $\mathbf{P}(P,k,l)$ and therefore is $\gamma(k,l)$-measurable.

Consider the set $F(Y)$ of all $(k-1)$-dimensional faces of the polytope $\pi_Y(P)$, $Y \in \Gamma(l)$. Let $\alpha(Y,G)$ be the exterior angle of $\pi_Y(P)$ at $G \in F(Y)$ in Y. The incidence function $\varepsilon_Y : \Delta(Y) \times F(Y) \to \{0,1\}$ is given by

$$\begin{aligned} \varepsilon_Y(X,G) &= 1, && \text{if } [\operatorname{aff} G + X] \cap \operatorname{relint}\big(\pi_Y(P)\big) = \emptyset \text{ and} \\ \varepsilon_Y(X,G) &= 0, && \text{otherwise.} \end{aligned}$$

Let $\Delta_0(Y) = \{X \in \Delta(Y) : (X,Y) \in \mathbf{P}(P,k,l)\}$. Using Fubini's Theorem, the definition and properties of $\varepsilon_Y(X,G)$ and the relation between the exterior angle

$\alpha(Y, G)$ and the Quermass integral $W_{n-k+1}(P)$ [22], we establish

$$\begin{aligned}
\int_{\Gamma(k,l)} \mathscr{H}^{k-1} & [\mathbf{S}(P, X, Y)] \,\mathrm{d}\gamma(k,l)(X,Y) \\
&= \int_{\Gamma(l)} \Big[\int_{\Delta_0(Y)} \mathscr{H}^{k-1}[\mathbf{S}(P, X, Y)] \,\mathrm{d}\delta(Y)(X) \Big] \,\mathrm{d}\gamma(l)(Y) \\
&= \int_{\Gamma(l)} \Big[\sum_{G \in F(Y)} \mathscr{H}^{k-1}(G) \int_{\Delta(Y)} \varepsilon_Y(X, G) \,\mathrm{d}\delta(Y)[X] \Big] \,\mathrm{d}\gamma(l)(Y) \\
&= \int_{\Gamma(l)} \Big[\sum_{G \in F(Y)} \mathscr{H}^{k-1}(G) \alpha(Y, G) \Big] \,\mathrm{d}\gamma(l)[Y] \\
&= b(k,l) \int_{\Gamma(l)} W_{l-k+1}(\pi_Y(P)) \,\mathrm{d}\gamma(l)(Y) \\
&= a(k,l) W_{n-k+1}(P),
\end{aligned}$$

where $b(k, l)$ and $a(k, l) > 0$ do not depend on the polytope P. □

1.3 Lebesgue Area and Rectifiability

Definition 7. *A continuous mapping $f : \mathbb{R}^k \to \mathbb{R}^n$ is said to be polyhedral if and only if the domain of f can be triangulated so that f maps each simplex of the triangulation affinely onto a rectilinear simplex of n-space.*

Definition 8. *The Lebesgue area $\mathbf{L}_P(f)$ of a continuous map $f : P \to \mathbb{R}^n$, where $P \subset \mathbb{R}^k$ for some k, is defined as the lower limit of the areas of polyhedral maps approximating f.(Where any sensible definition of area gives the same result for the areas of polyhedral maps).*

Definition 9. *A function $f : \mathbb{R}^k \to \mathbb{R}^n$ is said to be lower semi-continuous if at each point $a \in \mathbb{R}^k$ and for each $h < f(a)$ there is a neighbourhood V of a such that $h < f(x)$ for each $x \in V$.*

By definition, *the Lebesgue area is lower semi-continuous.*

We now introduce the notion of rectifiability which arises in geometric measure theory. Most references to rectifiability found seem to be in the works of Federer. He gives various definitions of rectifiability for sets. We shall only state the one we require and shall introduce the concept of a rectifiable map.

Definition 10. *A set $F \subset \mathbb{R}^n$ is called $(\mathscr{H}^k, k)$-rectifiable if there exist Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$, $i = 1, 2, ...$ such that:*

$$\mathscr{H}^k\big(F \setminus \cup_{i=1}^{\infty} f_i(\mathbb{R}^k)\big) = 0.$$

Definition 11. *Let $A \subset \mathbb{R}^n$ be a Borel set and $f : A \to \mathbb{R}^n$ a continuous map. f is* $\mathscr{H}^k$-rectifiable *if there exists a sequence $(M_i)_{i \in \mathbb{N}}$ of compact sets $M_i \subset A$, such that*

$$\mathscr{H}^k\big(\mathrm{Im}(f) \setminus \cup\{f[M_i] : i \in \mathbb{N}\}\big) = 0,$$

$$f|_{M_i} \text{ is Lipschitzian for all } i.$$

Definition 12. *A* surface *above some set $A \subset \mathbb{R}^k$ is a continuous map $f : A \to \mathbb{R}^{k+1}$, which satisfies*

$$[f(x) - x] \in \mathrm{lin}\{e_{k+1}\}, \textit{ for every } x \in A.$$

Notice that if the surface f is $\mathscr{H}^k$-rectifiable in the sense of Definition 11 and also satisfies $\mathscr{H}^k\big(\mathrm{Im}(f)\big) < \infty$, then $\mathrm{Im}(f)$ is a $(\mathscr{H}^k, k)$-rectifiable set.

The converse however does not hold. Here is an example to illustrate this fact:

Example 1. *Choose a compact set $F \subset [0, 1]$ with positive Lebesgue measure such that F contains no intervals. For $t \in [0, 1]$, consider the distance $d(t, F) = \inf\{|t - y| : y \in F\}$ and the number $f(t) = 1 - d(t, F)$. The graph G of the function g, given by $g(x) = \int_0^x f(t)\,dt$, is a C^1-curve of finite arclength in $\mathbb{R}^2$. Now, let $\rho \in SO(2)$ be a rotation carrying e_1 to $(1/\sqrt{2})(e_1 - e_2)$, and $\pi : \mathbb{R}^2 \to \mathbb{R}^1$ the orthogonal projection onto the x-axis. $P = \pi(\rho G)$ has finite length and for every $x \in P$ we find exactly one point which satisfies $\pi\sigma(x) = x$, where σ is the lifting map onto ρG. Note that $\rho G = \mathrm{Im}(\sigma)$ is $(\mathscr{H}^1, 1)$-rectifiable, whereas $\sigma : P \to \mathbb{R}^2$ is not $\mathscr{H}^1$-rectifiable, due to our choice of F. Similar examples exist in higher dimensions.*

Lemma 1.3.1. *If $P \subset \mathbb{R}^k$ is a k-polyhedron and $f : P \to \mathbb{R}^{k+1}$ is a surface above P with finite Lebesgue area $\mathbf{L}_P(f)$, then $\mathbf{L}_P(f) = \mathscr{H}^k\big(\mathrm{Im}(f)\big)$ and furthermore, $\mathrm{Im}(f)$ is $(\mathscr{H}^k, k)$-rectifiable.*

We shall not include a proof of this result as [4] is devoted entirely to this task.

Definition 13. *Given a surface $f : P \to \mathbb{R}^{k+1}$ and a rectilinear triangulation $\mathscr{F}$* [9] *of the k-polyhedron $P \subset \mathbb{R}^k$, we say that f is a piecewise linear surface over P, relative to $\mathscr{F}$, if $f|_s$ is the restriction of some affine map for every k-simplex $s \in \mathscr{F}$.*

Definition 14. *Consider a piecewise linear surface $f : P \to \mathbb{R}^{k+1}$, relative to some triangulation $\mathscr{F}$ of the k-polyhedron $P \subset \mathbb{R}^k$. Given any set M of real numbers, we define the set:*

$$\Lambda(f, M) = \sum\{\mathscr{H}^k(f[s]) : s \in \mathscr{F}, \mathrm{Lip}(f|_s) \in M\}.$$

A subdivision argument shows that $\Lambda(f, M)$ does not depend on the specific triangulation $\mathscr{F}$.

Lemma 1.3.2. *Consider a k-polyhedron $P \subset \mathbb{R}^k$ and a surface $f : P \to \mathbb{R}^{k+1}$ over P. Assume that f is not $\mathscr{H}^k$-rectifiable whereas $F = \mathrm{Im}(f)$ is $(\mathscr{H}^k, k)$-rectifiable.*

Then there exist numbers $\alpha > 0$ and $\varepsilon(c) > 0$, for every $c \in [1, \infty[$, such that each piecewise linear surface $g : P \to \mathbb{R}^{k+1}$ with $||f(x) - g(x)|| \leq \varepsilon(c)$ for all $x \in P$ satisfies

$$\Lambda(g, [c, \infty[) \geq \alpha.$$

To prove this lemma we require the following results by Federer [1].

Lemma (3.2.18 in [1])

If W is a $(\mathscr{H}^m, m)$-rectifible and $\mathscr{H}^m$-measurable subset of $\mathbb{R}^n$, and if $1 < \lambda < \infty$, then there exist compact subsets $K_1, K_2, ...$ of $\mathbb{R}^m$ and Lipschitzian maps $\psi_1, \psi_2, ...$ of $\mathbb{R}^m$ into $\mathbb{R}^n$ such that $\psi_1(K_1), \psi_2(K_2), ...$ are disjoint subsets of W with

$$\mathscr{H}^m[W \setminus \cup_{i=1}^{\infty} \psi_i(K_i)] = 0,$$

and, for each positive integer i,

$$\mathrm{Lip}\,(\psi_i) \leq \lambda, \qquad \psi_i|K_i \text{ is univalent}, \qquad \mathrm{Lip}[(\psi_i|K_i)^{-1}] \leq \lambda,$$

$$\lambda^{-1}|v| \leq |\langle v, D\psi_i(a)\rangle| \leq \lambda|v| \quad \text{for} \quad a \in K_i, v \in \mathbb{R}^m.$$

In this context, *univalent* is equivalent to bijective.

Theorem (3.1.16 in [1])

If $A \subset \mathbb{R}^m$, $f : A \to \mathbb{R}^n$ and

$$\text{ap lim sup}_{x \to a} |f(x) - f(a)|/|x - a| < \infty,$$

for $\mathscr{L}^m$ almost all x in A, then for each $\varepsilon > 0$ there exists a map $g : \mathbb{R}^m \to \mathbb{R}^n$ of class 1 such that

$$\mathscr{L}^m(A \setminus \{x : f(x) = g(x)\}) < \varepsilon.$$

Here $\mathscr{L}^m$ stands for the m-dimensional Lebesgue measure. We refer the reader to page 159 in [1] for a definition of the approximate lim sup used above.

Proof. of Lemma 3.2.

Since F is $(\mathscr{H}^k, k)$-rectifiable we may use Lemma 3.2.18 and Theorem 3.1.16 in [1] to find sequences $(U_i)_{i \in \mathbb{N}}$, $(\varphi_i)_{i \in \mathbb{N}}$ and $(K_i)_{i \in \mathbb{N}}$ such that U_i is open in $\mathbb{R}^k$, $\varphi_i : U_i \to \mathbb{R}^{k+1}$ is a C^1-embedding and the sets $K_i \subset U_i$ are compact and satisfy $\varphi_i[K_i] \subset F$ as well as $\mathscr{H}^k(F \setminus \cup\{\varphi_i[K_i] : i \in \mathbb{N}\}) = 0$.

Denote by $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ the orthogonal projection and write $A_i = \{p \in K_i :$ the derivative $D(\pi \circ \varphi_i)(p)$ of $\pi \circ \varphi_i$ is not bijective$\}$. A_i is compact and the inverse mapping theorem produces an open set $V_i \subset U_i$ such that $K_i \setminus A_i$ lies in V_i, and $(\pi \circ \varphi_i)|_{V_i}$ is a C^1-immersion.

Let $W_i = (\pi \circ \varphi_i)[V_i]$. For each $p \in W_i$, denote by $g_i(p)$ the unique point in $\pi^{-1}(p) \cap \varphi_i[V_i]$. For all k-dimensional polyhedra $Q \subset W_i$, $g_i|_Q$ is Lipschitz. Hence, if f fails to be $\mathscr{H}^k$-rectifiable, we obtain an index $i \in \mathbb{N}$ and a positive number α which satisfy

$$\mathscr{H}^k(\varphi_i[A_i]) = 2\alpha.$$

Since the Jacobian of $\pi \circ \varphi_i$ vanishes everywhere on A_i, it follows that $\mathscr{H}^k(\pi \circ \varphi_i[A_i]) = 0$.

Thus we can associate to every number $c \in [1, \infty[$ an open set $O(c) \subset W_i$ with $(\pi \circ \varphi_i[A_i]) \subset O(c)$ and

$$c^k \mathscr{H}^k[O(c)] < \alpha/2. \tag{1.4}$$

Since $\mathscr{H}^k$ is a radon measure in $\mathbb{R}^{k+1}$, there exists a k-polyhedron $Q(c)$ in $O(c)$ such that

$$\mathscr{H}^k\big(\varphi_i[A_i] \cap f[Q(c)]\big) > 3\alpha/2. \tag{1.5}$$

If Lemma 1.3.2 did not hold, then there would be a sequence $(g_i)_{i\in\mathbb{N}}$ of piecewise linear surfaces over P with $\lim_{i\to\infty} g_i = f$ uniformly and $\Lambda(g_i, [c, \infty[) < \alpha$, for all i. Here c stands for the number introduced in (1.4) above. We may now write

$$\begin{aligned} \mathscr{H}^k(g_i[Q]) &= \Lambda\big(g_i|_Q, [1, \infty[\big) = \Lambda\big(g_i|_Q, [1, c[\big) + \Lambda\big(g_i|_Q, [c, \infty[\big) \\ &< c^k \mathscr{H}^k(Q) + \alpha < 3\alpha/2. \end{aligned}$$

Lemma 1.3.1 implies that $3\alpha/2 \geq \mathbf{L}_Q(f|_Q) = \mathscr{H}^k\big(f[Q]\big)$ which contradicts (1.5). Therefore Lemma 1.3.2 must hold. □

1.4 Intermediate Shadow Boundaries

1.4.1 Polytopes

Definition 15. *For $0 < \rho < \tau < \infty$, denote by $\mathscr{P}(\rho, \tau)$ the collection of all polytopes $P \subset \mathbb{R}^n$ with $\rho\,\mathbb{B}^n < P < \tau\,\mathbb{B}^n$.*

Definition 16. *Consider the spaces*

$$\begin{aligned} \mathbf{B}(l) &= \{(p, X) \in \mathbb{S}^{n-1} \times \Gamma(l) : p \in X\}, \text{ where } 1 \leq l \leq n, \\ \mathbf{B}(k, l) &= \{(p, X, Y) \in \mathbb{S}^{n-1} \times \Gamma(k, l) : p \in X\}, \text{ where } 1 \leq k \leq l \leq n. \end{aligned}$$

Let $C \subset \mathbb{R}^n$ be a convex body with $o \in \text{int}(C)$, then for $\mathbf{M} \subset \mathbf{B}(l)$ and $X \in \Gamma(l)$, we write

$$\mathbf{M}[C|X] = \{p \in \text{relbd}[\pi_X(C)] : (p/\|p\|, X) \in \mathbf{M}\}.$$

Similarly, for $\mathbf{N} \subset \mathbf{B}(k,l)$ *and* $(X,Y) \in \Gamma(k,l)$*, we write*

$$\mathbf{N}[C|(X,Y)] = \{p \in \text{relbd}[\pi_X(C)] : [p/\|p\|, (X,Y)] \in \mathbf{N}\}.$$

The next Lemma asserts that the shadow boundary $\mathbf{S}(P,X,Y)$ of a polytope $P \in \mathscr{P}(\rho,\tau)$ cannot be steep in too many places.

Lemma 1.4.1. *Given integers* k,l *with* $1 \leq k < l \leq n$ *and positive numbers* ε, ρ, τ *with* $\rho < \tau$*, we find a number* $c = c(k,l,\varepsilon,\rho,\tau) \geq 1$ *and for each polytope* $P \in \mathscr{P}(\rho,\tau)$ *a compact set* $\mathbf{N} = \mathbf{N}(P,k,l,\varepsilon) \subset \mathbf{B}(k,l)$ *which satisfy:*

(1) $\mathbf{N}[P|(X,Y)] \neq \emptyset \quad \Rightarrow \quad (X,Y) \in \mathbf{P}(P,k,l)$.

(2) $\mathbf{N}[P|(X,Y)]$ *is the union of those faces* F *of* $\pi_X(P)$ *for which* $\text{Lip}(\sigma(P,X,Y)|F) \leq c$.

(3) $A = \{(X,Y) \in \mathbf{P}(P,k,l) : \mathscr{H}^{k-1}\big(\mathbf{S}(P,X,Y)\setminus\sigma(P,X,Y)(\mathbf{N}[P|(X,Y)])\big) \geq \varepsilon\}$ *is* $\gamma(k,l)$*-measurable, with* $\gamma(k,l)[A] < \varepsilon$.

Proof. For $Y \in \Gamma(l)$ and $h \in [0,l]$, let us write:

$$\Delta(Y,h) = \{X \in \Gamma(h) : X \subset Y\} \text{ and } O(Y,h) = \{\rho \in O(h) : \rho Y = Y\}.$$

Consider the normalised regular $O(Y,h)$-invariant outer Borel measure $\delta(Y,h)$ on $\Delta(Y,h)$ and the spaces:

$$\begin{aligned}\Delta_0 &= \big\{Y \in \Delta(\mathbb{R}^l, l-1) : \{e_{l-k+1}, \dots, e_l\} \subset Y\big\},\\ \Delta_1 &= \big\{X \in \Delta(\mathbb{R}^l, k) : X^\perp \cap \text{lin}\{e_{l-k+1}, \dots, e_l\} = \{0\}\big\}.\end{aligned}$$

If $X \in (\Delta(\mathbb{R}^l,k) \setminus \Delta_1)$, then the map $\pi_X|_{\text{lin}\{e_{l-k+1},\cdots,e_l\}}$ is injective and has an inverse σ_X defined on a subspace of X. Associate to $M \subset \Delta_0$ and $b \in [1,\infty[$, the collections:

$$\begin{aligned}S(M) &= \{X \in \Delta(\mathbb{R}^l,k) : \exists \text{ a flat } Y \in M \text{ with } X \subset Y\} \text{ and}\\ S_b(M) &= \{X \in S(M) \setminus \Delta_1 : \text{Lip}(\sigma_X) \leq b\}.\end{aligned}$$

Using the notation from Lemma 1.2.2, choose a positive number a such that $a(k,l)W_{n-k+1}(P) \leq a$ for all polytopes $P \in \mathscr{P}(\rho,\tau)$. A simple calculation shows that there exists a number $c \in [1,\infty[$ for which

$$\delta(\mathbb{R}^l,k)[S_c(M)] > (1-\varepsilon^2/a)\,\delta(\mathbb{R}^l,k)[S(M)], \tag{1.6}$$

whenever $M \subset \Delta_0$ is a closed subset.

Now, given a polytope $P \in \mathscr{P}(\rho,\tau)$ and a pair (X,Y) of spaces in $\mathbf{P}(P,k,l)$, write

$$\begin{aligned}
F(P,Y) &= \{Z : Z \text{ is a } (k-1)\text{-dimensional face of } \pi_Y(P)\},\\
F(P;X,Y) &= \{U \in F(P,Y) : [\mathrm{aff}(U)+X] \cap [\mathrm{relint}\,\pi_Y(P)] = \emptyset\} \text{ and}\\
F_c(P;X,Y) &= \{U \in F(P;X,Y) : \mathrm{Lip}[\sigma(P,X,Y)|_{\pi_X(U)}] \leq c\}.
\end{aligned}$$

Let us also associate to every face $U \in F(P,Y)$ the spaces:

$$\begin{aligned}
G(P;U,Y) &= \{X \in \Delta(Y,k) : [\mathrm{aff}(U)+X] \cap [\mathrm{relint}\,\pi_Y(P)] = \emptyset\} \text{ and}\\
G_c(P;U,Y) &= \{X \in \Delta(Y,k) : \mathrm{Lip}[\sigma(P,X,Y)|_{\pi_X(U)}] \leq c\},
\end{aligned}$$

and denote by $\alpha(P;U,Y)$ the exterior angle of $\pi_Y(P)$ at U, measured in the space Y. With $\mathbf{B}(k,l)$ according to Definition 16, consider the space $\mathbf{N}' \subset \mathbf{B}(k,l)$ defined by

$$\mathbf{N}'[P|(X,Y)] = \begin{cases} \cup\{\pi_X(U) : U \in F_c(P;X,Y)\} & \text{if } (X,Y) \in \mathbf{P}(P,k,l)\\ \emptyset & \text{otherwise}\end{cases}$$

and the space

$$A'(P) = \big\{\xi \in \mathbf{P}(P,k,l) : \mathscr{H}^{k-1}\big[\mathbf{S}(P,\xi) \setminus \sigma(P,\xi)(\mathbf{N}'[P|\xi])\big] \geq \varepsilon\big\},$$

which is $\gamma(k,l)$-measurable. We now proceed to show that

$$\gamma(k,l)[A'(P)] < \varepsilon \quad \forall P \in \mathscr{P}(\rho,\tau). \tag{1.7}$$

Indeed, assume that some polytope $P \in \mathscr{P}(\rho,\tau)$ does not satisfy (1.7) and write for $\xi \in \mathbf{P}(P,k,l)$,

$$\begin{aligned}
H(\xi) &= F(P,\xi) \setminus F_c(P,\xi) \quad \text{as well as}\\
\varphi(\xi) &= \Sigma\{\mathscr{H}^{k-1}(S) : S \in H(\xi)\}.
\end{aligned}$$

Our assumption about P implies:

$$\int_{\mathbf{P}(P,k,l)} \varphi(X,Y)\,\mathrm{d}\gamma(k,l) \geq \varepsilon\,\gamma(k,l)[A'(P)] \geq \varepsilon^2. \tag{1.8}$$

However, statement (1.6) leads to

$$\begin{aligned}
\int_{\mathbf{P}(P,k,l)} \varphi & \quad (X,Y)\,\mathrm{d}\gamma(k,l) \\
&= \int_{\Gamma(l)} \big[\int_{\Delta(Y,k)} \Sigma\{\mathscr{H}^{k-1}(U)\,\mathrm{d}\delta(Y,k) : U \in H(X,Y)\}\big]\,\mathrm{d}\gamma(l)[Y] \\
&= \int_{\Gamma(l)} \big[\Sigma\{\mathscr{H}^{k-1}(U)\delta(Y,k)[G(P;U,Y) \setminus G_c(P;U,Y)] : U \in F(P,Y)\}\big]\,\mathrm{d}\gamma(l)[Y] \\
&< (\varepsilon^2/a) \int_{\Gamma(l)} \big[\Sigma\{\mathscr{H}^{k-1}(U)\alpha(P;U,Y) : U \in F(P,Y)\}\big]\,\mathrm{d}\gamma(l)[Y] \\
&= (\varepsilon^2/a)a(k,l)W_{n-k+1}(P) \leq \varepsilon^2,
\end{aligned}$$

which contradicts (1.8), so (1.7) is established.

Hence we can associate to each polytope $P \in \mathscr{P}(\rho,\tau)$ a compact set $\mathbf{N}(P) \subset \mathbf{N}'(P)$ which satisfies, for all $\xi \in \mathbf{P}(P,k,l)$, either $\mathbf{N}[P|\xi] = \emptyset$ or $\mathbf{N}[P|\xi] = \mathbf{N}'[P|\xi]$ and furthermore, $\gamma(k,l)[A(P)] < \varepsilon$. Here we have written $A(P) = \{\xi \in \mathbf{P}(P,k,l) : \mathscr{H}^{k-1}(\mathbf{S}(P,\xi) \setminus \sigma(P,\xi)[\mathbf{N}[P|\xi]]) \geq \varepsilon\}$. Lemma 1.4.1 follows, with the above choice of c and $\mathbf{N}(P)$, $P \in \mathscr{P}(\rho,\tau)$. □

1.4.2 Smooth Convex Bodies

1.4.2.1 Rectifiability of $\sigma(C,\xi)$

Definition 17. *If C and D are compact convex sets in $\mathbb{R}^n$ with* aff(C) = aff(D)*, $o \in$* int(C)*, denote by $\rho\,(C,D)$: relbd$\,(C) \to$ relbd$\,(D)$ the* radial projection*, for which $\rho\,(C,D)[x] \in$* pos$\{x\}$*, whenever x belongs to* relbd$\,(C)$.

Definition 18. *A convex body C is called* smooth*, if every point $p \in$* relbd$\,(C)$ *lies in a unique supporting hyperplane of C.*

Lemma 1.4.2. *Let $C \subset \mathbb{R}^n$ be a smooth convex body with $o \in$* int(C)*. Given $\lambda \in]1,\infty[$ and $l \in \{1,\dots,n\}$, we can find a number $\varepsilon(C,\lambda,l) > 0$ such that*

$$\mathrm{Lip}[\rho\,(\pi_X C, D)] \leq \lambda \quad \text{and} \quad \mathrm{Lip}[\rho\,(D, \pi_X C)] \leq \lambda, \tag{1.9}$$

whenever D is a convex body in some space $X \in \Gamma(l)$ with $o \in \operatorname{relint}(D)$ and $\operatorname{aff}(D) = X$, which satisfies $d(D, \pi_X(C)) \leq \varepsilon(C, \lambda, l)$. Here, $d(\cdot, \cdot)$ stands for the Hausdorff distance between compact subsets of X.

We are interested in the measurability properties of the function $\xi \mapsto \mathscr{H}^{l-1}[\mathbf{S}(C,\xi)]$, $\xi \in \mathbf{P}(C, l, l+1)$, associated with a convex body $C \in \mathbb{R}^n$. To this end, we study the Lebesgue area $\mathbf{L}(C,\xi)$ of the lifting map $\sigma(C,\xi)$ and remember its lower semicontinuity.

Definition 19. *Consider a convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int}(C)$, and associate to every $l \in \{1, \dots, n-1\}$ and every $\xi = (X, Y) \in \mathbf{P}(C, l, l+1)$ the space $\mathscr{D}(C,\xi)$ of all polytopes $D \subset X$ with $o \in \operatorname{relint}(D)$.*

Remembering Definition 17*, for each $D \in \mathscr{D}(C,\xi)$, we define the maps $\mathfrak{C}(C,D)$ and $\tilde{\mathfrak{C}}(C,D)$ by*

$$\begin{aligned} \mathfrak{C}(C,D) &= \sigma(C,\xi) \circ \rho\,(D, \pi_X C), \\ \tilde{\mathfrak{C}}(C,D)[p] &= p + [\sigma(C,\xi)[q] - q], \quad p \in \operatorname{relbd}(D), \quad q = \rho\,(D, \pi_X C)[p]. \end{aligned}$$

Notice that $\tilde{\mathfrak{C}}(C,D)$ is a surface above $\operatorname{relbd}(D)$.

Definition 20. *For a convex body C and a polytope $D \in \mathscr{D}(C,\xi)$ define*

$$\begin{aligned} \mathbf{L}(C,D) &= \Sigma\{\mathbf{L}_S\big(\mathfrak{C}(C,D)|_S\big) : S \in F(D)\}, \\ \mathbf{L}(C,\xi) &= \limsup\{\mathbf{L}(C,D) : D \in \mathscr{D}(C,\xi)\}, \end{aligned}$$

where $F(D)$ stands for the collection of all $(l-1)$-dimensional faces of D, and $\mathbf{L}_S$ denotes the Lebesgue area above S.

It has been established that $\mathbf{L}(C,D) = \mathbf{L}(C,\xi)$, whenever C is smooth and D belongs to $\mathscr{D}(C,\xi)$. We shall only need the following weaker statement:

Lemma 1.4.3. *Given a smooth convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int}(C)$ and numbers $l \in \{1, \dots, n-1\}$ and $\mu > 1$, we find a number $\delta > 0$ such that the inequality*

$$(1/\mu)\, \mathscr{H}^{l-1}[\mathbf{S}(C,\xi)] \leq \mathbf{L}(C,D) \leq \mu\, \mathscr{H}^{l-1}[\mathbf{S}(C,\xi)], \tag{1.10}$$

holds for every $\xi = (X, Y) \in \mathbf{P}(C, l, l+1)$ *and every body* $D \in \mathscr{D}(C, \xi)$ *with* $d(D, \pi_X C) \leq \delta$.

Proof. Choose $\lambda > 1$ with $\lambda^{2(l-1)} < \mu$ and let $\delta = \varepsilon(C, \lambda, l)$ from Lemma 1.4.2. It is enough to assume $\xi = (\mathbb{R}^l, \mathbb{R}^{l+1})$. Given some $D \in \mathscr{D}(C, \xi)$ with $d(D, \pi_X C) \leq \delta$, write $\bar{C} = \pi^{-1}(\text{relbd}\, \pi_X C)$, $\bar{D} = \pi^{-1}(\text{relbd}\, D)$ where $\pi : \mathbb{R}^{l+1} \to \mathbb{R}^l$ is the orthogonal projection.

Lemma 1.4.2 implies that there exists an open neighbourhood N of 1 in $\mathbb{R}$ such that:

$$U = \{\nu p + \alpha e_{l+1} : \nu \in N, p \in \text{relbd}(D), \alpha \in \mathbb{R}\}$$

is a neighbourhood of $\bar{D}$ in $\mathbb{R}^{l+1}$ and the map φ, defined by

$$\varphi\left(\nu p + \alpha e_{l+1}\right) = \nu \rho(D, \pi_X C)[p] + \alpha e_{l+1}$$

is a Lipschitz homeomorphism between U and some neighbourhood V of $\bar{C}$, with $\text{Lip}(\varphi) < \lambda$ and $\text{Lip}(\varphi^{-1}) < \lambda$.

It follows from Lemma 1.3.1 that $\mathbf{L}_S\big[\tilde{\mathfrak{C}}(C, D)|_S\big] = \mathscr{H}^{l-1}\big[\tilde{\mathfrak{C}}(C, D)[S]\big]$ for each face S in $F(D)$. Recall that $\tilde{\mathfrak{C}}(C, D) = \varphi^{-1} \circ \mathfrak{C}(C, D)$ and using transformation formulae for the Hausdorff measure and the Lebesgue area under Lipschitz maps we see that:

$$(1/\lambda)^{l-1} \mathbf{L}_S\big(\mathfrak{C}(C, D)|_S\big) \leq \mathbf{L}_S\big(\tilde{\mathfrak{C}}(C, D)|_S\big) \leq \lambda^{l-1} \mathbf{L}_S\big(\mathfrak{C}(C, D)|_S\big) \qquad \text{and}$$
$$(1/\lambda)^{l-1} \mathscr{H}^{l-1}\big(\mathfrak{C}(C, D)[S]\big) \leq \mathscr{H}^{l-1}\big(\tilde{\mathfrak{C}}(C, D)[S]\big) \leq \lambda^{l-1} \mathscr{H}^{l-1}\big(\mathfrak{C}(C, D)[S]\big)$$

from which we can establish that the following inequality

$$(1/\mu)\, \mathscr{H}^{l-1}\big(\mathfrak{C}(C, D)[S]\big) \leq \mathbf{L}_S\big(\mathfrak{C}(C, D)|_S\big) \leq \mu\, \mathscr{H}^{l-1}\big(\mathfrak{C}(C, D)[S]\big), \qquad (1.11)$$

holds for every face $S \in F(D)$. Fubini's Theorem for Lebesgue measure implies that $\mathscr{H}^{k+1}(\text{Im}(f)) = 0$, whenever $P \subset \mathbb{R}^k$ is a polytope and $f : P \to \mathbb{R}^{k+1}$ is a surface. Consequently, we obtian that $\mathscr{H}^{l-1}(\tilde{\mathfrak{C}}(C, D)[S \cap T]) = 0$ for each pair S, T of $(l-1)$-faces, with $T \neq S$. Since φ is Lipschitzian, we also have $\mathscr{H}^{l-1}(\mathfrak{C}(C, D)[S \cap T]) =$

0. This leads to $\mathscr{H}^{l-1}(\mathbf{S}(C, \mathbb{R}^l, \mathbb{R}^{l+1})) = \Sigma\{\mathscr{H}^{l-1}(\mathfrak{C}(C,D)[S]) : S \in F(D)\}$ and together with (1.11), to a proof of Lemma 1.4.3. □

Notice that Lemma 1.4.3 implies that:

$$\mathscr{H}^{l-1}(\mathbf{S}(C,\xi)) = \mathbf{L}(C,\xi), \text{ for every } \xi \in \mathbf{P}(C,l,l+1).$$

Proposition 1. *The lifting map $\sigma(C,\xi)$ is $\mathscr{H}^{l-1}$-rectifiable for every smooth convex body C in $\mathbb{R}^n$, every number $l \in \{1, \dots, n-1\}$ and $\gamma(l, l+1)$-almost every $\xi \in \mathbf{P}(C, l, l+1)$.*

Proof. Following Definition 20, associate to each convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$, the map $\mathbf{L}(C) : \mathbf{P}(C,l,l+1) \to]o, \infty]$, given by $\mathbf{L}(C)[\xi] = \mathbf{L}(C,\xi)$.

Claim:

$$\mathbf{L}(C) \text{ is lower semicontinuous.} \tag{1.12}$$

Suppose this is false. We may choose a sequence $(\xi_k)_{k\in\mathbb{N}}$ in $\mathbf{P}(C,l,l+1)$, converging to $\xi \in \mathbf{P}(C,l,l+1)$, such that

$$\left[\mathbf{L}(C,\xi) / \lim_{k\to\infty} \mathbf{L}(C,\xi_k)\right] = \mu > 1.$$

With δ as in Lemma 1.4.3, consider a sequence $(r_k)_{k\in\mathbb{N}}$ in $O(n)$ which satisfies

$$\lim_{k\to\infty} r_k = \mathrm{Id}(\mathbb{R}^n) \text{ and } r_k(\xi_k) = \xi, \text{ for every k.}$$

Pick an element $D \in \mathscr{D}(C,\xi)$ with $d(D, \pi_X C) < \delta$ and $d(D, \pi_{X(k)}(C)) < \delta$ for all k, then $[\mathfrak{C}(r_k(C), D)]_{k\in\mathbb{N}}$ converges to $\mathfrak{C}(C,D)$ uniformly. The lower semicontinuity of the Lebesgue area leads to

$$\mathbf{L}_S\left[\mathfrak{C}(C,D)|_S\right] \le \liminf\{\mathbf{L}_S\left[\mathfrak{C}(r_k(C), D)|_S\right] : k \to \infty\},$$

and with Lemma 1.4.3, to

$$\begin{aligned} \mu\, \mathbf{L}(C,\xi) &\le (1/\mu) \lim\{\mathbf{L}(r_k C, \xi) : k \to \infty\} \\ &= (1/\mu) \lim\{\mathbf{L}(C,\xi_k) : k \to \infty\}, \end{aligned}$$

which contradicts the definition of μ. Which proves our claim, statement (1.12).

Next, consider a sequence $(P_i)_{i\in\mathbb{N}}$ of n-polytopes in $\mathbb{R}^n$, converging to C. Lemma 1.2.1 tells us that

$$R_0 = \mathbf{P}(C,l,l+1) \cap \ \cap\{\mathbf{P}(P_i,l,l+1) : i \in \mathbb{N}\}$$

is a Borel set with $\gamma(l,l+1)[R_0] = 1$. Given the equations

$$\mathscr{H}^{l-1}[S(P_i,\xi)] = \mathbf{L}(P_i,\xi) = \mathbf{L}(P_i,D) \text{ for all } i,\xi \text{ and } D \in \mathscr{D}(P_i,\xi),$$

we have:

$$\mathbf{L}(C,\xi) \leq \liminf\{\mathbf{L}(P_i,\xi) : i \to \infty\}, \text{ when } \xi \in R_0. \tag{1.13}$$

Now, Fatou's Lemma, together with Lemma 1.2.2, produce the inequality

$$\begin{aligned}\int_{R_0} \mathbf{L}(C)\,\mathrm{d}\gamma(l,l+1) &\leq \liminf\Big\{\int_{R_0} \mathbf{L}(P_i)\,\mathrm{d}\gamma(l,l+1) : i \to \infty\Big\} \\ &= a(l,l+1)W_{n-k+1}(C) < \infty,\end{aligned}$$

and therefore

$$\gamma(l,l+1)\big[\{\xi \in R_0 : \mathbf{L}(C)(\xi) < \infty\}\big] = 1. \tag{1.14}$$

Choose numbers ρ,τ such that $P_i \in \mathscr{P}(\rho,\tau)$ for every $i \in \mathbb{N}$. Lemma 1.4.1 provides numbers $c_j = c(l,l+1,(1/j),\rho,\tau)$ and compact sets $\mathbf{N}_{ij} = \mathbf{N}(P_i,l,l+1,(1/j))$ which satisfy its conditions (1) - (3). Define the following abbreviations:

$$\begin{aligned}
\mathbf{N}_j &= \limsup\{\mathbf{N}_{ij} : i \to \infty\}, \\
\mathbf{N} &= \limsup\{\mathbf{N}_j : j \to \infty\}, \\
A_{ij} &= \mathbf{P}(P_i,l,l+1) \setminus A[\mathbf{N}(P_i,l,l+1,1/j)], \\
A_j &= \limsup\{A_{ij} : i \to \infty\}, \\
A &= \limsup\{A_j : j \to \infty\}, \\
R_1 &= \{\xi \in R_0 : \mathbf{L}(C)(\xi) < \infty\}, \\
R_2 &= R_1 \cap A,
\end{aligned}$$

where $A[\mathbf{N}(P_i, l, l+1, 1/j)]$ is defined as in Lemma 1.4.1. Combine (1.14) with Lemma 1.4.1 to establish the equation

$$\gamma(l, l+1)[R_2] = \gamma(l, l+1)[A] = 1. \tag{1.15}$$

It remains to show that

$$\sigma(C, \xi) \text{ is } \mathscr{H}^{l-1}\text{-rectifiable for every } \xi = (X, Y) \in A. \tag{1.16}$$

Choose a polytope $D \in \mathscr{D}(C, \xi)$ and remember that $\rho(D, \pi_X C)|_S$ is a C^1-diffeomorphism for each facet $S \in F(D)$, so it is enough for us to establish that

$$\tilde{\mathfrak{C}}(C, D)|_S \text{ is always } \mathscr{H}^{l-1}\text{-rectifiable.} \tag{1.17}$$

Since $\xi \in R_1$, it follows that $\mathbf{L}_S(\tilde{\mathfrak{C}}(C, D)|_S) < \infty$ and, with Lemma 1.3.1, that $\tilde{\mathfrak{C}}(C, D)[S]$ is $(\mathscr{H}^{l-1}, l-1)$-rectifiable. So by Lemma 1.3.2, if (1.16) does not hold, we can find a number $\alpha > 0$ such that $\Lambda(f, [c, \infty[) \geq \alpha$, for each $c \in [1, \infty[$ and every piecewise linear surface $f : S \to [Y \cap \pi_X^{-1}(S)]$ close enough to $\tilde{\mathfrak{C}}(C, D)|_S$.

Since $\rho(\pi_X C, D)$ is a Lipschitz map, there exists a number $\beta > 0$ for which

$$\Lambda\big[\sigma(P, \xi), [b, \infty[\big] \geq \beta, \tag{1.18}$$

whenever $b \in [1, \infty[$, $P \subset \mathbb{R}^n$ is a polytope close to C and ξ lies in the space $\mathbf{P}(P, l, l+1)$.

Notice that $\xi \in A_j$ for some $j > 1/\beta$ and choose a strictly increasing sequence of numbers i_k, $k \in \mathbb{N}$ such that $\xi \in A_{i_k, j}$ for all k. This leads to $\lim_{k\to\infty} P_{i_k} = C$, $\xi \in \mathbf{P}(P_i, l, l+1)$, $\Lambda\big[\sigma(P_{i_k}, \xi), [c, \infty[\big] \leq 1/j$ for every integer k, contrary to (1.18). And thus, statement (1.16) and hence Proposition 1 are established. $\square$

1.4.2.2 Approximation of Shadow Boundaries by Rectifiable Sets

Now that we have shown that the lifting map $\sigma(C, \xi)$ is rectifiable, we wish to find the sets $\mathbf{M}$ (as defined in Definition 16) which will approximate our projection well and on which the lifting map will have bounded Lipschitz constant.

Definition 21. *Given a smooth convex body $C \subset \mathbb{R}^n$ and integers i, j, l, write:*

$$\mathbf{T}(C,l,i,j) = \{\xi = (X,Y) \in \mathbf{P}(C,l,l+1) : \exists \text{ a compact set } M \subset \operatorname{relbd} \pi_X C$$
$$\text{such that } \operatorname{Lip}[\sigma(C,\xi)|_M] \le i \text{ and } \mathscr{H}^{l-1}\big(\mathbf{S}(C,\xi) \setminus \sigma(C,\xi)[M]\big) \le 1/j\}$$

Lemma 1.4.4. *Each of the sets $\mathbf{T}(C,l,i,j)$, defined above, are closed in $\mathbf{P}(C,l,l+1)$.*

Proof. Consider a sequence $(\xi_k)_{k\in\mathbb{N}}$ in $\mathbf{T}(C,l,i,j)$, converging to some element $\xi \in \mathbf{P}(C,l,l+1)$. If we write $\xi_k = (X_k, Y_k)$ and $\xi = (X,Y)$, we can choose a compact set $M_k \subset \operatorname{relbd} \pi_{X_k} C$ which satisfies

$$\operatorname{Lip}\big[\sigma(C,\xi_k)|_{M_k}\big] \le i \text{ and } \mathscr{H}^{l-1}\big(\mathbf{S}(C,\xi_k) \setminus \sigma(C,\xi_k)[M_k]\big) \le 1/j. \tag{1.19}$$

By taking a subsequence, if necessary, we may assume that $(M_k)_{k\in\mathbb{N}}$ converges to a compact set $M \subset \operatorname{relbd} \pi_X C$. Clearly, $\operatorname{Lip}[\sigma(C,\xi)|_M] \le i$.

If Lemma 1.4.4 did not hold, it would follow that

$$\mathscr{H}^{l-1}\big(\mathbf{S}(C,\xi) \setminus \sigma(C,\xi)[M]\big) > \lambda^2/j, \text{ for some number } \lambda > 1. \tag{1.20}$$

As in the proof of Proposition 1, consider a sequence $(r_k)_{k\in\mathbb{N}}$ in the group $O(n)$, such that $\lim\{r_k : k \to \infty\} = \operatorname{Id}(\mathbb{R}^n)$ and $r_k(\xi_k) = \xi$, for all k.

Using the notation from Lemma 1.4.3, choose an element $D \in \mathscr{D}(C,\xi)$ which satisfies: $d\big(D, \pi_X(r_k C)\big) \le \delta(C,\lambda,l)$ for all large enough numbers k and also: $\mathscr{H}^{l-1}\big(\cup\{\mathfrak{C}(C,D)[S] : S \in F_0(D)\}\big) \ge \lambda^2/j$, where $F_0(D) = \{S \in F(D) : M \cap \rho\,(D, \pi_X C)[S] = \emptyset\}$.

Remember that $\operatorname{relbd} \pi_X C \setminus M$ is open in $\operatorname{relbd} \pi_X C$ and that for all $S \in F_0(D)$ and for all large enough k,

$$\rho\,\big(D, \pi_X(r_k C)[S]\big) \cap r_k(M_k) = \emptyset. \tag{1.21}$$

Hence, we can use (1.11) from the proof of Lemma 1.4.3, together with the lower semicontinuity of Lebesgue area to get:

$$\begin{aligned}\mathscr{H}^{l-1} &\quad \big(\cup\{\mathfrak{C}\big(r_k(C),D\big)[S] : S \in F_0(D)\}\big) \\ \geq &\quad 1/\lambda \sum_{S\in F_0(D)} L_S(\mathfrak{C}\big(r_k(C),D\big)|_S) \\ \geq &\quad 1/\lambda \sum_{S\in F_0(D)} L_S(\mathfrak{C}(C,D)|_S) \\ \geq &\quad (1/\lambda)^2\mathscr{H}^{l-1}\big(\cup\{\mathfrak{C}(C,D)[S] : S \in F_0(D)\}\big) > 1/j,\end{aligned}$$

when k is large enough. Remembering (1.21), this would imply that:

$$1/j < \mathscr{H}^{l-1}\big(\cup\{\mathfrak{C}\big(r_k(C),D\big)[S] : S \in F_0(D)\}\big) < \mathscr{H}^{l-1}\big(\mathbf{S}(C,\xi_k)\setminus\sigma(C,\xi_k)[M_k]\big)$$

which contradicts (1.19). Thus (1.20) does not hold and Lemma 1.4.4 follows. □

We now proceed to select the sets $\mathbf{M} \subset \operatorname{relbd}(\pi_X C)$ in a measurable way.

Lemma 1.4.5. *Let $\pi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^r$ be the orthogonal projection. Consider a Borel set $A \subset \mathbb{R}^r$ and associate to every $x \in A$ a family $\mathfrak{K}(x)$ of non-empty compact sets in $\mathbb{R}^s$, such that:*

$$\cup\{\mathfrak{K}(x) : x \in A\} \text{ is bounded,} \tag{1.22}$$

and whenever a sequence $(x_i)_{i\in\mathbb{N}}$ in A converges to $x \in A$, and the sequence $(B_i)_{i\in\mathbb{N}}$, with $B_i \in \mathfrak{K}(x_i)$ converges to some compact set $B \subset \mathbb{R}^s$, then

$$B \in \mathfrak{K}(x). \tag{1.23}$$

Then there exists a Borel set $\mathbf{M} \subset \mathbb{R}^r \times \mathbb{R}^s$ which satisfies:

$$\mathbf{M} \cap \pi^{-1}(x) \in \mathfrak{K}(x) \quad \textit{for all } x \in A.$$

Proof. Set $W = \{x \in \mathbb{R}^s : 0 \leq \langle x, e_i\rangle \leq 1, \forall\, i\}$ and $\mathscr{M}_p = \{(1/2)^p(W+t) : t \in \mathbb{Z}^s\}$, $p \in \mathbb{N}$. Let Φ_p be the collection of all finite subsets of $\mathscr{M}_p$. For $C \in \Phi_p$, write:

$$t(C) = \{W \in \mathscr{M}_{p-1} : \exists \text{ a cube } V \in C \text{ such that } V \subset W\}$$

and choose a linear ordering $\leq_p$ on each Φ_p such that:

$$\cup C \subset \cup D \quad \Rightarrow \quad C \leq_p D, \quad \text{for } C, D \text{ in } \Phi_p, \quad and \tag{1.24}$$

$$C \leq_p D \quad \text{always implies} \quad t(C) \leq_{(p-1)} t(D). \tag{1.25}$$

Given a compact set $K \subset \mathbb{R}^s$ and a number $p \in \mathbb{N}$, consider the collection $\varphi(K,p) = \{W \in \mathscr{M}_p : K \cap W \neq \emptyset\}$ and observe that $\varphi(K,p)$ always lies in the collection Φ_p and, by equation (1.22), so does $\psi(x,p) = \max\{\varphi(K,p) : K \in \mathfrak{K}(x)\}$.

With $R_p(C) = \{x \in A : \psi(x,p) = C\}$ we derive from (1.23) and (1.24) that $R_p(C)$ is a closed set in $\cup\{R_p(D) : D \in \Phi_p, D \geq C\}$ for all $C \in \Phi_p$. Hence, we have $\mathbf{M}_p = \cup\{R_p(C) \times [\cup C] : C \in \Phi_p\}$ as well as $\mathbf{M} = \cap\{\mathbf{M}_p : p \in \mathbb{N}\}$ are Borel sets.

This, together with (1.23) leads to $\mathbf{M} \cap \pi^{-1}(x) \in \mathfrak{K}(x)$, whenever $x \in A$, and Lemma 1.4.5 follows. □

Proposition 2. *Consider a smooth convex body* $C \in \mathbb{R}^n$ *with* $o \in \text{int}(C)$*, an integer* $l \in \{1, \dots, n-1\}$ *and a number* $\varepsilon > 0$*. There exists an element* $b \in [1, \infty[$ *and a compact set* $\mathbf{M} \subset \mathbf{B}(l, l+1)$ *such that*

$$\text{Lip}\big(\sigma(C,\xi)|\mathbf{M}[C|\xi]\big) \leq b, \quad \forall\, \xi \in \mathbf{P}(C,l,l+1), \quad and \tag{1.26}$$

$$\gamma(l,l+1)\{\xi \in \mathbf{P}(C,l,l+1) : \mathscr{H}^{l-1}\big(\mathbf{S}(C,\xi) \setminus \sigma(C,\xi)\big[\mathbf{M}[C|\xi]\big]\big) \geq \varepsilon\} \leq \varepsilon. \tag{1.27}$$

Proof. Choose an integer $j > 0$ with $(1/j) < \varepsilon$. Lemma 1.4.4 and Proposition 1 establish the equation

$$\gamma(l,l+1)\big[\cup\{\mathbf{T}(C,l,i,j) : i \in \mathbb{N}\}\big] = 1,$$

which implies $\gamma(l,l+1)\big[\mathbf{T}(C,l,i,j)\big] > 1 - \varepsilon$, for some $i \in \mathbb{N}$. Set $\tau(p,X,Y) = (X,Y)$ which determines a fibration $\tau : \mathbf{B}(l,l+1) \to \Gamma(l,l+1)$. Notice that none of the spaces

$$\begin{aligned}\mathfrak{K}'(\xi) = \; & \{\mathbf{K} \subset \tau^{-1}(\xi) : \mathbf{K} \text{ is compact, Lip}(\sigma(C,\xi)|_{\mathbf{K}[C|\xi]}) \leq i, \\ & \mathscr{H}^{l-1}(\mathbf{S}(C,\xi) \setminus \sigma(C,\xi)\big[\mathbf{K}[C|\xi]\big]) \leq 1/j\}\end{aligned}$$

are empty, for any $\xi \in \mathbf{T}(C, l, i, j)$. With $r = \dim \Gamma(l, l+1)$, we find a finite covering $(U_i)_{i=1}^t$ of $\mathbf{T}(C, l, i, j)$ by pairwise disjoint Borel sets and sequences of homeomorphisms: $\varphi_i : U_i \to V_i \subset \mathbb{R}^r$, $\psi_i : [\tau^{-1}(U_i)] \to W_i \subset \mathbb{R}^{r+s}$, which satisfy $\pi \circ \psi_i(x) = \varphi_i \circ \tau(x)$, whenever $x \in \tau^{-1}(U_i)$. Here $\pi : \mathbb{R}^{r+s} \to \mathbb{R}^r$ denotes the orthogonal projection. Lemma 1.4.5 produces a Borel set $\mathbf{M}' \subset \mathbf{B}(l, l+1)$, such that

$$\operatorname{Lip}(\sigma(C,\xi)|_{\mathbf{M}'[C|\xi]}) \leq i, \quad \text{if } \xi \in \mathbf{P}(C, l, l+1), and \tag{1.28}$$

$$\gamma(l,l{+}1)\{\xi \in \mathbf{P}(C,l,l{+}1) : \mathscr{H}^{l-1}\big(\mathbf{S}(C,\xi) \backslash \sigma(C,\xi)(\mathbf{M}'[C|\xi])\big) > 1/j\} < \varepsilon. \tag{1.29}$$

If we write

$$d[(p,X,Y),(p',X',Y')] = d(p,p') + d(X \cap \mathbb{B}^n, X' \cap \mathbb{B}^n) + d(Y \cap \mathbb{B}^n, Y' \cap \mathbb{B}^n),$$

we obtain an $O(n)$-invariant metric d on the space $\mathbf{B}(l, l+1)$. Let $\beta(l, l+1)$ be the corresponding $[\dim \mathbf{B}(l, l+1)]$-dimensional normalised Hausdorff measure. A Fubini type theorem leads to

$$\beta(l, l+1)[\mathbf{N}] = \int \mathscr{H}^{l-1}[\mathbf{N} \cap \tau^{-1}(\xi)] \mathrm{d}\,\gamma(l, l+1)(\xi).$$

Since $\beta(l, l+1)$ is regular, there exists a compact set $\mathbf{M} \subset \mathbf{M}'$ for which the properties (1.28) and (1.29) also hold. Proposition 4.2 follows. $\square$

So we have shown that for any smooth convex body C in $\mathbb{R}^n$ and a given integer $1 \leq l < n$, we can find a set $\mathbf{M} \subset \mathbf{B}(l, l+1)$ which defines subsets $\mathbf{M}[C|\xi]$ of the relative boundary of the projection of C onto X such that:

- For almost every $\xi \in \Gamma(l, l+1)$, $\sigma(C,\xi)(\mathbf{M}[C|\xi])$ covers almost all of the intermediate shadow boundary of C, and
- The lifting map $\sigma(C,\xi)$ restricted to the sets $\mathbf{M}[C|\xi]$ is Lipschitz for every ξ such that the intermediate shadow boundary is sharp.

This result only relates to *intermediate shadow boundaries* and as such leads to the induction process used in the following section to prove our result in the case of shadow boundaries.

1.5 Bodies of Positive Span

Definition 22. *The convex body C has* positive span $\eta > 0$ *if there exists a convex body $D \subset \mathbb{R}^n$ such that $C = D + \eta\,\mathbb{B}^n$. Notice that in particular that C is smooth.*

Definition 23. *Let $C \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int}(C)$. For $1 \leq k < l \leq n$, $\varepsilon > 0$, $\mathbf{P} \subset \mathbf{B}(k)$, $\mathbf{Q} \subset \mathbf{B}(k,l)$ and $\mathbf{R} \subset \mathbf{B}(l)$ write:*

$$\begin{aligned}
\mathbf{F}(C,\mathbf{P},\varepsilon) &= \{X \in \mathbf{P}(C,k) : \mathscr{H}^{k-1}\big(\mathbf{S}(C,X) \setminus \sigma(C,X)\big[\mathbf{P}[C|X]\big]\big) > \varepsilon\} \\
\mathbf{G}(C,\mathbf{Q},\varepsilon) &= \{(X,Y) \in \mathbf{P}(C,k,l) : \\
&\qquad \mathscr{H}^{k-1}\big(\mathbf{S}(C,X,Y) \setminus \sigma(C,X,Y)\big[\mathbf{Q}[C|(X,Y)]\big]\big) > \varepsilon\} \\
\mathbf{G}(C,\pi_X\mathbf{R},\varepsilon) &= \{(X,Y) \in \mathbf{P}(C,k,l) : \\
&\qquad \mathscr{H}^{k-1}\big(\mathbf{S}(C,X,Y) \setminus \sigma(C,X)\big[\pi_X\mathbf{R}[C|Y]\big]\big) > \varepsilon\}
\end{aligned}$$

Lemma 1.5.1. *Let $C \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int}(C)$ and positive span η. Given $\varepsilon \in]0,1[$ and $l \in \{1,\dots,n-1\}$, we find a number $\mathbf{x} = \mathbf{x}(C,l,\varepsilon)$ such that whenever $\mathbf{O} \subset \mathbf{B}(l+1)$ is an open set with $\gamma(l+1)\big[\mathbf{F}(C,\mathbf{O},\mathbf{x})\big] < \mathbf{x}$, it will satisfy the inequality*

$$\gamma(l,l+1)\big[\mathbf{G}\big(C,\pi_X\mathbf{O},\varepsilon\big)\big] < \varepsilon.$$

Proof. For $Y \in \Gamma(l+1)$, define:

$$\Delta(Y) = \{X \in \Gamma(l) : X \subset Y\} \quad \text{and} \quad O(Y) = \{\rho \in O(n) : \rho(Y) = Y\},$$

and let $\delta(Y)$ denote the normalised $O(Y)$-invarient outer Borel measure on $\Delta(Y)$. Define also: $\Delta_0(Y) = \{X \in \Delta(Y) : (X,Y) \in \mathbf{P}(C,l,l+1)\}$.

Letting

$$\Delta(Y,\mathbf{O},\beta) = \{X \in \Delta_0(Y) : \mathscr{H}^{l-1}\big(\mathbf{S}(C,X) \setminus \sigma(C,X)\big[\pi_X(\mathbf{O}[C|Y])\big]\big) > \beta\},$$

we will establish that there exists a number $k \geq 1$ such that the inequality:

$$\delta(Y)[\Delta(Y,\mathbf{O},\beta)] \leq (k/\beta)\mathscr{H}^{l}\big(\mathbf{S}(C,Y) \setminus \sigma(C,Y)\big[\mathbf{O}[C|Y]\big]\big), \tag{1.30}$$

holds for all $Y \in \mathbf{P}(C, l+1)$, every open set $\mathbf{O} \subset \mathbf{B}(l+1)$ and every $\beta > 0$.

With $a(k) = (\mathscr{L}^k(\mathbb{B}^k)/2^k) \in]0, \infty[$ where $\mathbb{B}^k$ is the k-dimensional unit ball, we define the outer measures $\mathscr{H}^k_\mu$, $\mu > 0$, $k \in \{0, 1, \ldots, n\}$ over $\mathbb{R}^n$ by setting

$$\mathscr{H}^k_\mu(M) = a(k)\ \inf\{\sum_{u \in \nu}[\tau(u)]^k : \nu \text{ is a countable covering of } M,$$

$$\text{such that } \tau(u) = \operatorname{diam}(u) \leq \mu, \forall u \in \nu\}.$$

Take $\alpha = \mathscr{H}^l\big(\mathbf{S}(C, Y) \setminus \sigma(C, Y)\big[\mathbf{O}[C|Y]\big]\big)$, and let us choose a decreasing sequence $(\mu_i)_{i \in \mathbb{N}}$ and an increasing sequence $(\alpha_i)_{i \in \mathbb{N}}$ of real numbers which satisfy

$$\lim_{i \to \infty} \mu_i = 0,\ \lim_{i \to \infty} \alpha_i = \alpha,\ \text{as well as } \alpha_i > \mathscr{H}^l_{\mu_i}\big(\mathbf{S}(C, Y) \setminus \sigma(C, Y)\big[\mathbf{O}[C|Y]\big]\big).$$

Remember that $\lim_{\mu \to 0} \mathscr{H}^k_\mu(M) = \mathscr{H}^k(M)$. Let ν_i be a countable covering of $\mathbf{S}(C, Y) \setminus \sigma(C, Y)\big[\mathbf{O}[C|Y]\big]$ by open convex sets, such that $\tau(u) \leq \mu_i$ for all $u \in \nu_i$ and $a(l)\Sigma\{\big(\tau(u)\big)^l : u \in \nu_i\} \leq \alpha_i$. Define the real valued functions ψ_i on the space $\Delta_0(Y)$ by

$$\psi_i(X) = \sum\{\tau\big[u \cap \big(\mathbf{S}(C, X) \setminus \sigma(C, X)\big[\pi_X(\mathbf{O}[C|Y])\big]\big)\big]^{l-1} : u \in \nu_i\}$$

and notice that ψ_i is always $\delta(Y)$-measureable. Define

$$\begin{aligned} &\xi(u, X) = 1 \quad \text{if } u \cap \big(\mathbf{S}(C, X) \setminus \sigma(C, X)\big[\pi_X(\mathbf{O}[C|Y])\big]\big) \neq \emptyset \\ &\xi(u, X) = 0 \qquad\qquad\qquad \text{otherwise,} \end{aligned}$$

from which we obtain the inequality

$$\int_{\Delta_0(Y)} \psi_i \, \mathrm{d}(\delta Y) \leq \sum\{\int_{\Delta_0(Y)} [\tau(u)]^{l-1} \xi(u, X) \, \mathrm{d}(\delta(Y)[X]) : u \in \nu_i\},$$

and therefore

$$\int_{\Delta_0(Y)} \psi_i \, \mathrm{d}(\delta Y) \leq \sum\{[\tau(u)]^{l-1} \delta(Y)[E(u)] : u \in \nu_i\},$$

where $E(u) = \{X \in \Delta_0(Y) : \xi(u, X) = 1\}$.

Since C has positive span η, the outer normal map [24] $n : \mathrm{bd}(C) \to \mathbb{S}^{n-1}$ of C satisfies $\mathrm{Lip}(n) \leq 1/\eta$. If the point

$$p \in u \cap \big(\mathbf{S}(C,X) \setminus \sigma(C,X)\big[\pi_X(\mathbf{O}[C|Y])\big]\big),$$

then the outer normal vector $n[p]$ lies in $\pi_Y^{-1}[X]$, hence we obtain

$$E(u) \subset F(u) = \{X \in \Delta_0(Y) : \pi_Y^{-1}[X] \cap n[u] \neq \emptyset\}.$$

In view of $\mathrm{Lip}(n) \leq 1/\eta$, we find a spherical ball of diameter $2\tau(u)/\eta$, containing $n[u]$ and a constant $\lambda > 0$, independent of Y, $\mathbf{O}$ and β such that

$$\delta(Y)[F(u)] \leq \lambda\tau(u), \quad \text{for all } u \in \nu_i \text{ and for all } i \in \mathbb{N}. \tag{1.31}$$

Since the inequality $\psi_i(X) > \beta/a(l-1)$ holds for every element X in the Borel set $D_i = \{X \in \Delta_0(Y) : \mathscr{H}^{l-1}\big(\mathbf{S}(C,X) \setminus \sigma(C,X)\big[\pi_X(\mathbf{O}[C|Y])\big]\big) > \beta\}$, we derive from (1.31) the fact that

$$\beta\,\delta(Y)[D_i] \leq a(l-1)\int_{\Delta_0(Y)} \psi_i \,\mathrm{d}\delta(Y) \leq \lambda \sum_{u\in\nu_i} [\tau(u)]^l \leq \lambda\, a(l-1)\alpha_i.$$

Remembering that $(D_i)_{i\in\mathbb{N}}$ is an ascending sequence with $\cup\{D_i : i \in \mathbb{N}\} = \Delta(Y,\mathbf{O},\beta)$, we obtain (1.30) with $k = \lambda\, a(l-1)$. We write $\mathbf{x} = \mathbf{x}(C,l,\varepsilon) = \varepsilon^2/2k$ and choose an open set $\mathbf{O} \subset \mathbf{B}(l+1)$, which satisfies $\delta(Y)[A] < \mathbf{x}$, where $A = \{Y \in \mathbf{P}(C,l+1) : \mathscr{H}^l\big(\mathbf{S}(C,Y) \setminus \sigma(C,Y)[\mathbf{O}[C|Y]]\big) > \mathbf{x}\}$.

We now derive from (1.30), for the space $B = \{(X,Y) \in \mathbf{P}(C,l,l+1) : \mathscr{H}^{l-1}\big(\mathbf{S}(C,X) \setminus \sigma(C,X)\big[\pi_X(\mathbf{O}[C|Y])\big]\big) > \varepsilon\}$ the inequality:

$$\begin{aligned}\gamma(l,l+1)[B] &= \gamma(l+1)[A] + \int \delta(Y)[\Delta(Y,\mathbf{O},\varepsilon)]\mathrm{d}\,\gamma(l+1)(Y)\\ &< \mathbf{x} + (k/\varepsilon)\mathbf{x} < \mathbf{x} + \varepsilon/2 < \varepsilon.\end{aligned}$$

where integration extends over the space $\mathbf{P}(C,l+1) \setminus A$. Lemma 5.1 follows. □

Proposition 3. *Consider a convex body $C \subset \mathbb{R}^n$ with $o \in \mathrm{int}(C)$ and positive span η. Given elements $l \in \{1,\ldots,n-1\}$ and $\varepsilon \in]0,\infty[$, we find a number $e = e(C,l,\varepsilon) > 1$*

and a compact set $\mathbf{P} = \mathbf{P}(C, l, \varepsilon) \subset \mathbf{B}(l)$ *such that*

$$
\begin{aligned}
\gamma(l)[\mathbf{F}(C, \mathbf{P}, \varepsilon)] &< \varepsilon \quad \textit{and} & (1.32)\\
\operatorname{Lip}\big(\sigma(C, X)|\mathbf{P}[C|X]\big) &\leq e \quad \textit{whenever } X \in \mathbf{P}(C, l). & (1.33)
\end{aligned}
$$

Proof. By induction on $n - l$*, where the case* $n - l = 1$ *is covered by Proposition 2.*

For the inductive step with $(n - l) \geq 2$, we pick a number ε_1 from $]0, \varepsilon/2[$ and determine $\mathbf{x} = \mathbf{x}(C, l, \varepsilon_1)$ according to Lemma 1.5.1. We may assume that $\mathbf{x} \leq \varepsilon_1$ and obtain, by the inductive assumption, an element $\bar{e} \in]1, \infty[$, together with a compact set $\bar{\mathbf{P}} \subset \mathbf{B}(l + 1)$, which satisfy $\gamma(l + 1)\big[\mathbf{F}(C, \bar{\mathbf{P}}, \mathbf{x})\big] < \mathbf{x}$ as well as $\operatorname{Lip}\big(\sigma(C, Y)|\bar{\mathbf{P}}[C|Y]\big) \leq \bar{e}$ for all $Y \in \mathbf{P}(C, l + 1)$.

Next, we choose a number ε_2 such that $\varepsilon_2 \bar{e}^{\,(l-1)} < \varepsilon_1$ and determine $b = b(C, l, \varepsilon_2)$ and $\mathbf{M} = \mathbf{M}(C, l, \varepsilon_2)$ according to Proposition 2. Consider the Borel sets:

$$
\begin{aligned}
A_1 &= \{(X, Y) \in \mathbf{P}(C, l, l + 1) : \mathscr{H}^{l-1}\big(\mathbf{S}(C, X) \setminus \sigma(C, X)\big[\pi_X \bar{\mathbf{P}}[C|Y]\big]\big) \leq \varepsilon_1\} \text{ and}\\
A_2 &= \{(X, Y) \in \mathbf{P}(C, l, l + 1) :\\
&\qquad \mathscr{H}^{l-1}\big(\mathbf{S}(C, X, Y) \setminus \sigma(C, X, Y)\big[\mathbf{M}[C|(X, Y)]\big]\big) < \varepsilon_2\}.
\end{aligned}
$$

They satisfy the inequalities: $\gamma(l, l + 1)[A_i] \geq 1 - \varepsilon_i$ since, by Lemma 1.5.1:

$$
\begin{aligned}
\gamma(l, l + 1)\big[\mathbf{G}\big(C, \pi_X \bar{\mathbf{P}}, \varepsilon_1\big)\big] &= \{(X, Y) \in \mathbf{P}(C, l, l + 1) :\\
&\qquad \mathscr{H}^{l-1}\big(\mathbf{S}(C, X) \setminus \sigma(C, X)\big[\pi_X \bar{\mathbf{P}}[C|Y]\big]\big) > \varepsilon_1\}\\
&< \varepsilon_1
\end{aligned}
$$

and by Proposition 2:

$$
\begin{aligned}
&\{(X, Y) \in \mathbf{P}(C, l, l + 1) :\\
&\quad \mathscr{H}^{l-1}\big(\mathbf{S}(C, X, Y) \setminus \sigma(C, X, Y)\big[\mathbf{M}[C|(X, Y)]\big]\big) \geq \varepsilon_2\} \leq \varepsilon_2.
\end{aligned}
$$

Due to the regularity of $\gamma(l, l + 1)$, we obtain a compact set A_0 in $A_1 \cap A_2$ with

$$\gamma(l, l + 1)[A_0] \geq 1 - \varepsilon_1 - \varepsilon_2 > 1 - \varepsilon.$$

Let $\tau : \Gamma(l, l+1) \to \Gamma(l)$ be the projection defined by $\tau(X, Y) = X$. A Fubini type theorem for the fiber bundle $\big(\Gamma(l, l+1), \tau, \Gamma(l)\big)$ leads to:

$$\gamma(l)[B_0] > 1 - \varepsilon, \qquad \text{where } B_0 = \tau[A_0]. \tag{1.34}$$

Now let us establish that

$$\exists \text{ a Borel measurable map } \varphi : B_0 \to A_0 \text{ for which } \tau \circ \varphi = \mathrm{Id}_{B_0}. \tag{1.35}$$

Similarly to the proof of Proposition 2, we choose a decomposition $(U_i)_{1\leq i\leq t}$ of B_0 by Borel sets and sequences of homeomorphisms $\varphi_i : U_i \to V_i \subset \mathbb{R}^r$ and $\psi_i : (\tau^{-1}U_i) \to W_i \subset \mathbb{R}^{r+s}$, such that $\pi \circ \psi_i = \varphi_i \circ \tau$, always. Here, $r = \dim(\Gamma(l))$, $r + s = \dim(\Gamma(l, l+1))$ and $\pi : \mathbb{R}^{r+s} \to \mathbb{R}^r$ denotes the orthogonal projection.

By applying Lemma 1.4.5 to the sets V_i and the systems $\mathfrak{K}(\varphi_i x) = \{\psi_i(\xi) : \xi \in A_0 \cap \tau^{-1}x\}$, $x \in U_i$, we obtain a Borel set $L \subset A_o$ such that $L \cap \tau^{-1}(x)$ is a singleton, for every $x \in B_o$. A map φ which satisfies (1.35) can be constructed by setting $L \cap \tau^{-1}(x) = \{\varphi x\}$.

Look at the Borel set $\mathbf{D} = \{(p, X) \in \mathbf{B}(l) : (p, X, Y) \in \mathbf{M} \text{ and } (p, Y) \in \bar{\mathbf{P}},\ \text{with } \varphi(X) = (X, Y)\}$. We want to establish that

$$\mathbf{F}(C, \mathbf{D}, \varepsilon) \cap B_0 = \emptyset. \tag{1.36}$$

Suppose not, then we can choose $X \in \big[\mathbf{F}\big(C, \mathbf{D}, \varepsilon\big) \cap B_0\big]$ and write $\varphi(X) = (X, Y)$. Since $X \in B_0$ and $\mathrm{Im}(\varphi) \in A_0 \subset A_1 \cap A_2$, it follows that $\varphi(X) \in A_0$ and hence $\varphi(X) = (X, Y) \in A_1 \cap A_2$. Thus, the following inequalities hold for X:

$\mathscr{H}^{l-1}\big(\mathbf{S}(C, X) \setminus \sigma(C, X)[\pi_X \bar{\mathbf{P}}[C|Y]]\big) \leq \varepsilon_1$ and

$\mathscr{H}^{l-1}\big(\mathbf{S}(C, X, Y) \setminus \sigma(C, X, Y)[\mathbf{M}[C|(X, Y)]]\big) \leq \varepsilon_2 < \varepsilon_2 \bar{e}^{(l-1)} < \varepsilon_1$

We also have $\mathbf{D}[C|X] \subset \mathbf{M}[C|(X, Y)] \cap \pi_X \big[\bar{\mathbf{P}}[C|Y]\big]$ by definition of $\mathbf{D}$. Therefore:

$$\mathscr{H}^{l-1}\big(\mathbf{S}(C, X) \setminus \sigma(C, X)[\mathbf{D}[C|X]]\big) \leq \varepsilon_1 < \varepsilon$$

contradicting our choice of X, and therefore (1.36) is established.

Next, we shall derive from the definitions of $\mathbf{D}, \mathbf{M}$ and $\bar{\mathbf{P}}$ that

$$\text{Lip}\big(\sigma(C,X)|\mathbf{D}[C|X]\big) \le \bar{e}\, b, \qquad \forall\, X \in \mathbf{P}(C,l). \tag{1.37}$$

We know that $\text{Lip}\big(\sigma(C,Y)|\bar{\mathbf{P}}[C|Y]\big) \le \bar{e}$, where $\bar{e} \in]1,\infty[$ and $\text{Lip}\big(\sigma(C,X,Y)|\mathbf{M}[C|(X,Y)]\big) \le b$ where $b \in [1,\infty[$. Hence

$$\begin{aligned} \text{Lip}\big(\sigma(C,X)|\mathbf{D}[C|X]\big) &= \text{Lip}\big(\sigma(C,Y) \circ \sigma(C,X,Y)|\mathbf{D}[C|X]\big) \\ &\le \bar{e}b, \end{aligned}$$

since $\mathbf{D}[C|X] \subset \mathbf{M}[C|(X,Y)] \cap \pi_X\big[\bar{\mathbf{P}}[C|Y]\big]$.

Pick a real number $e = e(C,l,\varepsilon) \in]\bar{e}b,\infty[$ and a compact set $\mathbf{P} = \mathbf{P}(C,l,\varepsilon) \subset \mathbf{D}$. In view of (1.34) and (1.36), we know that $\gamma(l)[\mathbf{F}(C,\mathbf{D},\varepsilon)] < \varepsilon$ and therefore, $\gamma(l)[\mathbf{F}(C,\mathbf{P},\varepsilon)] < \varepsilon$ also. In addition, (1.37) implies that $\text{Lip}\big(\sigma(C,X)|\mathbf{P}[C|X]\big) \le e$, for all $X \in \mathbf{P}(C,l)$. Thus, we may conclude that $e(C,l,\varepsilon)$ and $\mathbf{P}(C,l,\varepsilon)$ satisfy the requirements of Proposition 3.

□

The relative boundary of the projection of C onto X has finite $\mathscr{H}^{l-1}$ measure. Hence, since the lifting map $\sigma(C,X)$ is Lipschitz on $\mathbf{M}[C|X]$ which covers the entire relative boundary of $\pi_X C$ bar a set of measure zero (Proposition 3) we can now conclude that almost all shadow boundaries of convex bodies with positive span have finite length.

1.6 Principal Results

Definition 24. *Associate to every convex body $C \in \mathbb{R}^n$ and every number $l \in \{1,\dots,n-1\}$ the space*

$$\mathbf{P}_0(C,l) = \{X \in \mathbf{P}(C,l) : \sigma(C,X) \text{ is rectifiable}\}.$$

Theorem 2. $\gamma(l)[\mathbf{P}_0(C,l)] = 1$ *for every convex body C in $\mathbb{R}^n$ and every integer $l \in \{1,2,\dots,n-1\}$.*

Proof. We may assume that $o \in \text{int}\,(C)$ and begin with the case where:

(1) C has positive span.

Given a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of positive numbers converging to zero, we construct the elements $e_i = e_i(C, l, \varepsilon_i) \in]1, \infty[$ and the compact sets $\mathbf{P}_i = \mathbf{P}_i(C, l, \varepsilon_i)$ according to Proposition 3. By this same Proposition, we obtain the inequality $\gamma(l)[\mathbf{N}] \leq \liminf_{i\to\infty} \gamma(l)[\mathbf{N}_i] = 0$, where we have written $\mathbf{N}_i = \mathbf{F}(C, \mathbf{P}_i, \varepsilon_i)$ and $\mathbf{N} = \liminf_{i\to\infty} \mathbf{N}_i$. Proposition 3 also implies that $\mathbf{P}_0(C, l) \supset (\mathbf{P}(C, l) \setminus \mathbf{N})$, and Theorem 2 is established under assumption (1).

If we now drop this assumption for C, we observe that $C_1 = C + \mathbb{B}^n$ has positive span. Therefore it is enough to prove that:

(2) $\mathbf{P}_0(C_1, l) \subset \mathbf{P}_0(C, l)$.

For $X \in \mathbf{P}_0(C_1, l)$, set $\bar{C} = \pi_X(C), \bar{C}_1 = \pi_X C_1$, and let $\nu : \mathbb{R}^n \to C$ be the nearest point map of C, $\bar{\nu}$ the nearest point map of $\bar{C}$. Each of the collections $A_i = \{p \in \text{relbd}\,\bar{C}$: there exists an l-dimensional Euclidean ball $B \subset \bar{C}$ of radius $1/i$, such that $p \in \text{relbd}(B)\}$ is compact. A classical statement in convex geometry [17] establishes the equality

$$\mathscr{H}^{l-1}(\text{relbd}(\bar{C}) \setminus \cup\{A_i : i \in \mathbb{N}\}) = 0.$$

Furthermore, $\bar{\nu}_i = \bar{\nu}|_{B_i}$ is a homeomorphism between $B_i = \text{relbd}(\bar{C}_1) \cap \bar{\nu}^{-1}[A_i]$ and A_i, with $\text{Lip}(\bar{\nu}_i^{-1}) \leq i$. Since X belongs to $\mathbf{P}_0(C_1, l)$, we obtain a sequence $(\mathbf{M}_j)_{j\in\mathbb{N}}$ of compact sets in $\text{relbd}(\bar{C}_1)$, such that $\mathscr{H}^{l-1}(\mathbf{S}(C_1, X) \setminus \cup\{\sigma(C_1, X)[\mathbf{M}_j] : j \in \mathbb{N}\}) = 0$, as well as $\lambda_j := \text{Lip}\,(\sigma(C_1, X)[\mathbf{M}_j]) < \infty$, always.

With $\mathbf{N}_{ij} = A_i \cap \bar{\nu}[\mathbf{M}_j]$, we notice that $\sigma(C, X)|_{\mathbf{N}_{ij}} = (\nu \circ \sigma(C_1, X) \circ \bar{\nu}_i^{-1})|_{\mathbf{N}_{ij}}$ and consequently, $\text{Lip}(\sigma(C, X)|_{\mathbf{N}_{ij}}) \leq i\lambda_j$. In addition,

$$\begin{aligned}&\mathscr{H}^{l-1}(\mathbf{S}(C, X) \setminus \cup\{\sigma(C, X)[\mathbf{N}_{ij}] : i, j \in \mathbb{N}\}) \\ &\qquad \leq \mathscr{H}^{l-1}(\mathbf{S}(C_1, X) \setminus \cup\{\sigma(C_1, X)[\mathbf{M}_j] : j \in \mathbb{N}\}) = 0.\end{aligned}$$

This implies that $X \in \mathbf{P}_0(C, l)$.

Therefore (2) holds and Theorem 2 follows. $\square$

Proof. of Theorem 1.

We assume that $o \in \text{int}(C)$, and choose a sequence $(Q_j)_{j\in\mathbb{N}}$ of polytopes in $\mathbb{R}^n$, converging to C. With the notation of Definition 24, for a given l, set $\mathbf{P} = \mathbf{P}_0(C,l) \cap \cap\{\mathbf{P}_0(Q_j,l) : j \in \mathbb{N}\}$, and by Theorem 2, Fatou's Lemma and Lemma 1.2.2, notice that it is sufficient to establish that:

$$\varphi(C,l)|_{\mathbf{P}} \text{ is lower semicontinuous and} \tag{1.38}$$

$$\varphi(C,l)[X] \leq \liminf_{j\to\infty} \varphi(Q_j,l)[X], \text{ for every } X \in \mathbf{P}. \tag{1.39}$$

Given $X \in \mathbf{P}$ and a sequence $(X_i)_{i\in\mathbb{N}}$ in $\mathbf{P}$, converging to X, we pick elements $r_i \in O(n)$, $i \in \mathbb{N}$, such that $\lim_{i\to\infty} r_i = \text{Id}(\mathbb{R}^n)$ and $r_i(X_i) = X$, always. Now we consider an l-simplex $S \subset X$ with $o \in \text{relint}(S)$. Denote by $F(S)$ the collection of all $(l-1)$-dimensional faces of S. For every $P \in F(S)$, choose an affine map $\varphi_P : \mathbb{R}^{l-1} \to X$, which carries the standard $(l-1)$-simplex $T = \text{conv}\{0, e_1, e_2, \dots, e_{l-1}\}$ onto P. Associate to every element P of $F(S)$ the maps $\alpha(P)$, $\beta(i,P)$ and $\gamma(j,P)$, $i, j \in \mathbb{N}$, defined by

$$\begin{aligned}
\alpha(P) &= \sigma(C,X) \circ \rho(S, \pi_X C) \circ \varphi_P|_T \\
\beta(i,P) &= \sigma(r_i C, X) \circ \rho(S, \pi_X[r_i C]) \circ \varphi_P|_T \\
\gamma(j,P) &= \sigma(Q_j, X) \circ \rho(S, \pi_X Q_j) \circ \varphi_P|_T.
\end{aligned}$$

where $\rho(S, \pi_X \cdot)$ is the radial projection introduced in Definition 17. Let $\mathscr{I}^{l-1}$ be the $(l-1)$-dimensional integral geometric measure in $\mathbb{R}^n$ [5] and denote by $\gamma_X(\psi)$ the stable integral geometric area [5] of the continuous map $\psi : T \to \mathbb{R}^n$ over $X \subset T$. It follows from the definition of $\mathbf{P}$ that $\alpha(P)$, $\beta(i,P)$ and $\gamma(j,P)$ are $\mathscr{H}^{l-1}$-rectifiable, for every i, j and P. Hence, (8) on p312 in [5] and the fact that the equations $\mathscr{I}^{l-1}(\psi[X]) = \gamma_X(\psi)$ hold whenever ψ is a continuous map and $\psi|_X$ is Lipshchitz [see Note following proof], lead to:

$$\mathscr{H}^{l-1}\big(\text{Im}(\tau)\big) = \mathscr{I}^{l-1}\big(\text{Im}(\tau)\big) = \gamma_T(\tau), \tag{1.40}$$

for all $\tau \in \{\alpha(P)\} \cup \{\beta(i,P) : i \in \mathbb{N}, P \in F(S)\} \cup \{\gamma(j,P) : j \in \mathbb{N}, P \in F(S)\}$.

The lower semicontinuity of the integral geometric area [5] and equation (1.40) imply that

$$\begin{aligned}
\mathscr{H}^{l-1}\big(S(C,X)\big) &= \sum_{P\in F(S)} \mathscr{H}^{l-1}\big(\mathrm{Im}(\alpha(P)\big) \\
&= \sum_{P\in F(S)} \gamma_T\big(\alpha(P)\big) \\
&= \sum_{P\in F(S)} \liminf_{i\to\infty} \gamma_T\big(\beta(i,P)\big) \\
&\le \liminf_{i\to\infty} \sum_{P\in F(S)} \mathscr{H}^{l-1}\big(\mathrm{Im}(\beta(i,P))\big) \\
&= \liminf_{i\to\infty} \mathscr{H}^{l-1}\big(S(C,X_i)\big).
\end{aligned}$$

Hence (1.38) follows.

At the same time, we obtain:

$$\begin{aligned}
\mathscr{H}^{l-1}\big(S(C,X)\big) &= \sum_{P\in F(S)} \gamma_T\big(\alpha(P)\big) \\
&\le \sum_{P\in F(S)} \liminf_{j\to\infty} \gamma_T\big(\gamma(j,P)\big) \\
&\le \liminf_{j\to\infty} \mathscr{H}^{l-1}\big(S(Q_j,X)\big),
\end{aligned}$$

and (1.39) is established too.

The proof of Theorem 1 is thus completed. □

Note

The purpose of this note is to justify the statement made prior to equation (1.40).

In [5] Federer defines the integralgeometric measure and the stable integralgeometric area. Our statement relies on a combination of properties also present in his paper. Here we state the relevant properties and explain how they are used. Define the following:

$\mathcal{L}_k$	is the k-dimensional Lebesgue measure on $\mathbb{E}^n$;
G_n	is the group of orthogonal transformations of $\mathbb{E}^n$;
R	is an element of G_n;
ϕ_n	is the Haar measure over G_n;
p_n^k	is the projection of $\mathbb{E}^n$ onto $\mathbb{E}^k$ s.t. $p_n^k(x) = (x_1, \ldots, x_k) \in \mathbb{E}^k$ where $x = (x_1, \ldots, x_n) \in \mathbb{E}^n$;
$N(f, X, y)$	is the number of elements in the set $X \cap \{x \mid f(x) = y\},$ for a function $f,$ a set X and a point y;
$\beta(n, k)$	$= \dfrac{\alpha(k) \cdot \alpha(n-k)}{\alpha(n) \cdot \binom{n}{k}},$ where $\alpha(k)$ is the volume of the k-dimensional unit ball;
$S(f, X, y)$	is the stable multiplicity. $S(f, X, y) = \liminf_{g \to f} N(g, X, y)$.

The k-dimensional integralgeometric stable area on $C_n(X)$ is defined as

$$\beta(n,k)^{-1} \cdot \int_{G_n} \int_{\mathbb{E}_k} S(p_n^k \circ R \circ f, X, z) d\mathcal{L}_k z d\phi_n R.$$

The k-dimensional integralgeometric Favard measure is defined as

$$\mathscr{F}_n^k(X) = \beta(n,k)^{-1} \cdot \int_{G_n} \int_{\mathbb{E}_k} N(p_n^k \circ R, X, y) \mathrm{d}\mathcal{L}_k y \mathrm{d}\phi_n R. \tag{1.41}$$

This implies that the integralgeometric measure of $f(X)$ is equal to

$$\mathscr{F}_n^k(f(X)) = \beta(n,k)^{-1} \cdot \int_{G_n} \int_{\mathbb{E}_k} N(p_n^k \circ R \circ f, X, y) \mathrm{d}\mathcal{L}_k y \mathrm{d}\phi_n R. \tag{1.42}$$

We also know that

$$S(h, X, z) = N(h, X, z),$$

for $\mathcal{L}_k$ almost all z in $\mathbb{E}_k$ whenever h is a Lipschitz map in $C_k(X)$.

Hence, when f is a Lipschitz map in $C_n(X)$:

$$\beta(n,k)^{-1} \cdot \int_{G_n} \int_{\mathbb{E}_k} S(p_n^k \circ R \circ f, X, z) d\mathcal{L}_k z d\phi_n R$$
$$= \beta(n,k)^{-1} \cdot \int_{G_n} \int_{\mathbb{E}_k} N(p_n^k \circ R \circ f, X, y) d\mathcal{L}_k y d\phi_n R$$

Hence the k-dimension integralgoemetric area of the continuous function ψ over X is equal to the integralgeometric measure of the image of ψ.

Using the notation of our theorem this is equivalent to $\mathscr{I}^{l-1}(\psi[X]) = \gamma_X(\psi)$ when ψ is a continuous map and $\psi|_X$ is Lipshchitz.

Chapter 2

Baire category and shadow boundaries of infinite length

2.1 Introduction

In this chapter we introduce the notion of Baire category and expand some previous results relating to shadow boundaries.

2.1.1 Baire category

René Baire introduced the notion of a Baire space and Baire categories at the end of the 19th century. Here are some fundamental definitions.

Definition 25. *A* nowhere dense set *is a set whose closure has empty interior.*

Definition 26. *A set is* of first Baire category *if it is the countable union of nowhere dense sets. A set that is not of first Baire category is* of second Baire category.

Definition 27. *The complement of a set of first Baire category is called a* residual *set.*

Definition 28. *A space is* Baire *if every set of first Baire category has a complement of second Baire category.*

Definition 29. *If a property holds for all elements of a Baire space except for those of a set of first Baire category we say that the property holds for* most *elements and that the elements having the property are* typical.

The Baire Category Theorem gives sufficient conditions for a space to be Baire. We will use the statement below.

Baire Category Theorem: Any complete metric space is a Baire space.

2.1.2 Shadow Boundaries of Typical Convex Bodies. Measure Properties [18]

In [18], Gruber and Sorger consider shadow boundaries of convex bodies in $\mathbb{E}^d$ produced by $(d-1)$-dimensional subspaces. Looking at pairs (C, X) where C is a convex body in $\mathbb{E}^d$ and X is a $(d-1)$-dimensional linear subspace, they show that for most pairs (in the Baire category sense cf. Definition 29) the shadow boundary $\mathbf{S}(C, X)$ is sharp and has infinite $(d-2)$-dimensional Hausdorff measure.

In the next two sections we look at extensions of this result to other sets of subspaces.

2.2 Shadow Boundaries over Directions Within a Hyperplane

The motivation behind this extension is to apply it to increasing paths in the one skeleton of a convex body.

2.2.1 Notation

S^{d-1} denotes the $(d-1)$-dimensional unit sphere in $\mathbb{E}^d$

$\Gamma(d-1)$ is the set of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$. If $X \in \Gamma(d-1)$ then there is $\bar{x} \in S^{d-1}$ orthogonal to X.

$\mathbf{S}(C, X)$ is the shadow boundary of C over X.

$\mathscr{C}$ is the set of all convex bodies in $\mathbb{E}^d$.

$\delta^{\mathscr{H}}$ is the Hausdorff metric on the space of all non empty compact sets in $\mathbb{E}^d$, in particular over $\mathscr{C}$.

$\mathscr{H}^d$ denotes the d-dimensional Hausdorff measure.

Given a $(d-1)$-dimensional subspace L in $\mathbb{E}^d$ let S_L^{d-2} be the $(d-2)$-dimensional unit sphere in $\mathbb{E}^d$ such that $S_L^{d-2} \subset L$.

$\Gamma_L(d-1)$ is the set of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$ orthogonal to L. That is the $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$ which contain the complement of L.

Let $\mathscr{S}$ be the set of all pairs $(C, X) \in \mathscr{C} \times \Gamma(d-1)$ for which $\mathbf{S}(C, X)$ is sharp and $\mathscr{S}_L$ be the set of all pairs $(C, X) \in \mathscr{C} \times \Gamma_L(d-1)$ for which $\mathbf{S}(C, X)$ is sharp.

Definition 30. *We say that a class $\mathcal{G}$ of open sets forms a basis for the open sets of a topological space Ω if each of these sets of Ω is a union of sets from $\mathcal{G}$.*

2.2.2 Basic Properties

Property 1. *Let X and Y be Baire spaces such that Y has a countable basis, and let $R \subset X \times Y$ be residual. Then for most $x \in X$, the set $\{y \in Y : (x, y) \in R\}$ is residual in Y.*

Property 2. *Let X, Y be Baire spaces and let $R \subset X$ be residual. Then $R \times Y$ is residual in the Baire space $X \times Y$.*

Property 3. *If R is residual in a Baire space X and S is residual in R then S is residual in X.*

Property 4. *The set of strictly convex bodies is residual in $\mathscr{C}$* [19] *and since $\mathscr{S}_L$ contains the set $\{C \in \mathscr{C} : C$ is strictly convex$\} \times \Gamma_L(d-1)$ by* Property 2*:*

$$\mathscr{S}_L \text{ is residual in } \mathscr{C} \times \Gamma_L(d-1). \tag{2.1}$$

Property 5. *Let $C, C_i \in \mathscr{C}, i = 1, 2, ...,$ be such that $C_i \to C$. If $z_i \in$ bd C_i, $i = 1, 2, \ldots,$ and $z_i \to z$, say, then $z \in$ bd C.*

Definition 31. *When $(C, X) \in \mathscr{S}_L$, the* natural parametrization *of $\mathbf{S}(C, X)$ is defined by the function $f = f_{C,X} : \pi_X(\mathbf{S}(C, X)) \to \mathbf{S}(C, X)$ such that for $x \in \pi_X(\mathbf{S}(C, X))$ we have $\pi_X(f(x)) = x$. f is bijective and continuous.*

Definition 32. *A* parametrization *of* $\mathbf{S}(C, X)$ *is defined by any function mapping a* $(d-2)$*-dimensional topological sphere onto* $\mathbf{S}(C, X)$.

Definition 33. *We say two parametrizations* $f : S \to \mathbf{S}(C, X)$, $g : T \to \mathbf{S}(C, X)$ *of* $\mathbf{S}(C, X)$ *are* equivalent *if there is a homeomorphism* $h : S \to T$ *such that* $f = g \circ h$.

Proposition 4. *Let* (C, X), $(C_i, X_i) \in \mathscr{S}_L, i = 1, 2, ...$ *be such that* $(C_i, X_i) \to (C, X)$. *Then there are continuous bijective parametrizations* g *of* $\mathbf{S}(C, X)$ *(equivalent to* $f_{C,X}$*) and* g_i *of* $\mathbf{S}(C_i, X_i)$ *(equivalent to* f_{C_i,X_i}*) respectively, all defined on the same* $(d-2)$*-dimensional sphere and such that* $g_i \to g$ *uniformly.*

Proof. Proposition (5) in [18] shows that this result holds for pairs (C, X), $(C_i, X_i) \in \mathscr{S}, i = 1, 2, \ldots$. Since any pair belonging to $\mathscr{S}_L$ also belongs to $\mathscr{S}$, Proposition (5) in [18] implies Proposition 4. □

Due to certain limitations of Hausdorff measure we need to introduce the $(d-2)$-*dimensional integral geometric stable area* γ_{d-2} and some of its properties (for a precise definition we refer the reader to [5]). The properties below are drawn from the work of H. Federer. For Properties 6 and 8 see [5] p. 325; for Property 7 see [33] p. 182; for Property 9 see [5] p. 319.

Let S be a $(d-2)$-dimensional topological sphere in $\mathbb{E}^d$. Then the following properties hold:

Property 6. *If* S *is a triangulated polytopal sphere and* $g : S \to \mathbb{E}^d$ *is continuous, injective and affine on each simplex of* S*, then*

$$\mathscr{H}^{d-2}(g(S)) = \gamma_{d-2}(g).$$

Property 7. *If* $g : S \to \mathbb{E}^d$ *is continuous and injective, then*

$$\mathscr{H}^{d-2}(g(S)) \geq \gamma_{d-2}(g).$$

Property 8. *If $g : S \to \mathbb{E}^d$ and $g_i : S \to \mathbb{E}^d, i = 1, 2, ...$, are continuous and $g_i \to g$ uniformly on S, then*

$$\gamma_{d-2}(g) \leq \liminf_{i \to \infty} \gamma_{d-2}(g_i).$$

Property 9. *If T is another $(d-2)$-dimensional topological sphere in $\mathbb{E}^d$ and $h : T \to S$ a homeomorphism, then*

$$\gamma_{d-2}(g) = \gamma_{d-2}(g \circ h).$$

2.2.3 Main Result

Theorem 3. *Given a $(d-1)$-dimensional subspace L in $\mathbb{E}^d$, let S^{d-2} be contained in L. Then for most pairs $(C, X) \in \mathscr{C} \times \Gamma_L(d-1)$ the shadow boundary $\mathbf{S}(C, X)$ is sharp and has infinite $(d-2)$-dimensional Hausdorff measure.*

Here 'most' is used in the sense of definition 29. Our proof requires the following result:

Lemma 2.2.1. *For $n = 1, 2, ...$, the set of pairs $(Q, X) \in \mathscr{S}_L$ where Q is a convex polytope and for which $\mathscr{H}^{d-2}(\mathbf{S}(Q, X)) > n$ holds is dense in $\mathscr{S}_L$.*

We shall prove the following, which implies Lemma 2.2.1.

Lemma 2.2.2. *Let $(P, X) \in \mathscr{S}_L$, P a polytope. Then there are polytopes P_i, $i = 1, 2, ...$, with $(P_i, X) \in \mathscr{S}_L$, $P_i \to P$, $\mathscr{H}^{d-2}(\mathbf{S}(P_i, X)) \to +\infty$.*

Proof. Without loss of generality we may assume that the origin o is contained in the interior of $\pi_X P$. Let ρ denote the radial projection of $X \setminus \{o\}$ onto the the relative boundary $S = \pi_X \mathbf{S}(P, X)$ of $\pi_X P$. Clearly, the following holds:

Property 10. *There exists a constant $\alpha > 1$ such that:*

If T is a (closed) convex surface in X containing $\pi_X P$ and contained in $2\pi_X P$, then

$$(1/\alpha)\|x - y\| \leq \|\rho(x) - \rho(y)\| \leq \alpha \|x - y\| \quad \textit{for } x, y \in T \tag{2.2}$$

For $i = 1, 2, ...$ let S_i be a simplicial convex surface in X with the following properties:

i) each facet F of S_i has diameter $< 1/i^2$,

ii) S_i contains $\pi_X P$ and is contained in $(3/2)\pi_X P$,

iii) $\|x - \rho(x)\| < 1/i$ for each $x \in S_i$.

Choose a point a in the relative interior of each F and to each a assign a point b outside S_i such that:

Property 11. *b is close enough to a for the line segments joining different b's to intersect* rel int(conv(S_i)).

The simplicial convex surface T_i constructed from S_i by replacing each F by the $(d-1)$-dimensional simplices obtained by connecting the boundary simplices of F with b, has the following properties:

i) each facet of T_i has diameter $< 1/i^2$,

ii) T_i contains $\pi_X P$ and is contained in $2\pi_X P$,

iii) $\|x - \rho(x)\| < 1/i$ for each $x \in T_i$.

Define a function $h_i : T_i \to \mathbb{R}$ as follows $h_i(x) = 0$ for x on the boundary of any facet F of S_i, $h_i(b) = 1/i$ and interpolate linearly in between. Clearly,

$$\max\{|h_i(x)| : x \in T_i\} = 1/i \tag{2.3}$$

and with Property 11 *i)* this shows that:

Property 12. *Each facet of the polytopal surface $\{x + h_i(x)\bar{x} : x \in T_i\}$ has slope (gradient) at least i with respect to X (where $\bar{x}$ is the unit vector perpendicular to X).*

To the natural parametization $f_{P,X}$ of $\mathbf{S}(P, X)$ there corresponds a function $e : S = \pi_X(\mathbf{S}(P, X)) \to \mathbb{R}$ such that

$$f_{P,X}(x) = x + e(x)\bar{x} \text{ for } x \in S.$$

The slopes of facets of $\mathbf{S}(P, X)$ are bounded by β say. Then (2.2) and Property 11 *ii)* imply that each facet of the polytopal surface $\{x + e(r(x))\bar{x} : x \in T_i\}$ has slope $\leq \alpha\beta$ with respect to X. With Property 12 this shows that each facet of the polytopal surface

$$U_i = \{x + (e(r(x)) + h_i(x))\bar{x} : x \in T_i\} \text{ has slope } \geq i - \alpha\beta,$$

and therefore

$$\begin{aligned} \mathscr{H}^{d-2}(U_i) &\geq (1 + (i - \alpha\beta)^2)^{1/2} \mathscr{H}^{d-2}(T_i) \\ &\geq (1 + (i - \alpha\beta)^2)^{1/2} \mathscr{H}^{d-2}(S) \to +\infty \quad \text{as } i \to \infty \end{aligned}$$

From Property 11 *iii)* and (2.3) we have that

$$\delta^H(\mathbf{S}(P, X), U_i) < 2/i. \tag{2.4}$$

Define

$$P_i = \text{conv}(\{v \in \text{vert } P, v \notin \mathbf{S}(P, X)\} \cup U_i).$$

Then by Property 11 *ii)* $\mathbf{S}(P_i, X) = U_i$ and therefore $(P_i, X) \in \mathscr{S}_L$. The definition of P_i together with (2.4) imply $\delta^{\mathscr{H}}(P, P_i) \to 0$ as $i \to \infty$. Since $\mathbf{S}(P_i, X) = U_i$, (2.4) shows that $\mathscr{H}^{d-2}(\mathbf{S}(P_i, X)) \to +\infty$ as $i \to \infty$. Thus Lemma 2.2.2 holds.

□

Definition 34. *For $n = 1, 2, ...$, let*

$$\mathscr{S}_{L,n} = \{(C, X) \in \mathscr{S}_L : \gamma_{d-2}(f_{C,X}) \leq n\}.$$

Proof. of Thm 3

Statement 1. *$\mathscr{S}_{L,n}$ is closed in $\mathscr{S}_L$.*

Pick a sequence of pairs $(C_i, X_i) \in \mathscr{S}_{L,n}, i = 1, 2, ...$ converging to $(C, X) \in \mathscr{S}_L$. We need to show that $(C, X) \in \mathscr{S}_{L,n}$. By Proposition 4 there are continuous bijective parametrizations g of $\mathbf{S}(C, X)$ equivalent to $f_{C,X}$ and g_i of $\mathscr{S}(C_i, X_i)$ equivalent to f_{C_i,X_i} all defined on the same $(d-2)$-dimensional sphere $S \subset X$ and such that $g_i \to g$ uniformly on S. By Property 8 and Property 9 we have:

$$\gamma_{d-2}(f_{C,X}) = \gamma_{d-2}(g) \leq \liminf_{i\to\infty} \gamma_{d-2}(g_i) = \liminf_{i\to\infty} \gamma_{d-2}(f_{C_i,X_i}) \leq n$$

Hence $(C, X) \in \mathscr{S}_{L,n}$, which proves Statement 1.

Next we show:

Statement 2. *$\mathscr{S}_{L,n}$ has empty interior in $\mathscr{S}_L$.*

Suppose the opposite. By Lemma 2.2.1 we may choose a pair $(Q, X) \in \mathscr{S}_{L,n}$ with $\mathscr{H}^{d-2}(\mathbf{S}(Q, X)) > n$ where Q is a simplicial polytope. As $\pi_X(\mathbf{S}(Q, X))$ can be considered as a triangulated $(d-2)$-dimensional sphere in X, Property 6 yields:

$$\gamma_{d-2}(f_{Q,X}) = \mathscr{H}^{d-2}(\mathbf{S}(Q, X)) > n.$$

Hence $(Q, X) \notin \mathscr{S}_{L,n}$ which is a contradiction. Hence, Statement 2 is established.

From Statements 1 and 2, we get that $\mathscr{S}_{L,n}$ is nowhere dense for $n = 1, 2, ...$ thus:

$$\mathscr{S}_L \setminus \bigcup_{n=1}^{\infty} \mathscr{S}_{L,n} = \{(C, X) \in \mathscr{S}_L : \gamma_{d-2}(f_{C,X}) \leq n\} \text{ is residual in } \mathscr{S}_L$$

Applying Property 7 we see that $\{(C, X) \in \mathscr{S}_L : \mathscr{H}^{d-2}(\mathbf{S}(C, X)) = \infty\}$ is residual in $\mathscr{S}_L$ and therefore residual in $\mathscr{C} \times \Gamma_L(d-1)$ by Property 3 and equation (2.1).

This concludes the proof of Theorem 3. □

2.2.4 Applications

In [16], Larman and Rogers show the existence of increasing paths on the 1-skeleton of convex bodies. A corollary to Theorem 1 will show the existence of such paths with infinite length.

First, an introduction to increasing paths:

Let l be a linear function on $\mathbb{E}^d$ and let L be the $(d-1)$-dimensional linear subspace such that $l(x) = a$ for all $x \in L$, where $a \in \mathbb{R}$ is a constant.

Definition 35. *The* 1-skeleton *of a convex body C in $\mathbb{E}^d$ is the set of all points of C that are not the centre of any 2-dimensional spherical ball contained in C.*

Definition 36. *The* exposed 1-skeleton *of C is the set of points of C which belong to a tangent plane to C whose total intersection with C is of linear dimension 0 or 1.*

Theorem 4. *(Larman and Rogers [16])*
Let L be a non-constant linear function on $\mathbb{E}^d$ and let K be a convex body in $\mathbb{E}^d$. Then there are continuous maps s_1, s_2 of the closed interval $[0,1]$ to the exposed one-skeleton of K with

$$\begin{aligned} L(s_i(0)) &= \inf_{k \in K} L(k) \\ L(s_i(t_1)) &< L(s_i(t_2)), \text{ when } 0 \leq t_1 < t_2 \leq 1, \\ L(s_i(1)) &= \sup_{k \in K} L(k) \quad \text{for } i = 1, 2. \end{aligned}$$

Further, the paths can be separated by a $(d-1)$-dimensional plane, in that a plane π can be chosen such that the sets

$$s_i(t),\ 0 < t < 1 \quad \text{for } i = 1, 2$$

lie in opposite open half-spaces determined by π.

Using Theorem 3 we show that for most pairs (C, L) there are increasing paths on the 1-skeleton of a convex body which have infinite length.

Corollary 2. *For most pairs $(C, L) \in \mathscr{C} \times \Gamma(d-1)$ there exists an increasing path in the 1-skeleton of C from $m = \{c \in C : l(c) = \min_{k \in C} l(k)\}$ to $M = \{c \in C : l(c) = \max_{k \in C} l(k)\}$ of infinite length.*

Proof. In fact there are at least two! Theorem 3 tells us that given L, most pairs $(C, X) \in \mathscr{C} \times \Gamma_L(d-1)$ produce sharp shadow boundaries with infinite length.

A sharp shadow boundary is simply a closed path on the 1-skeleton of C. The sets m and M are contained in hyperplanes L_1 and L_2, respectively, both parallel to L. A shadow boundary over a subspace orthogonal to L will intersect m and M. Thus if we decompose our shadow boundary into four distinct paths such that:

$$\begin{aligned} p_1 &= \mathbf{S}(C, X) \cap L_1 \\ p_2 &= \mathbf{S}(C, X) \cap L_2 \\ p_3, p_4 &= \text{remaining sections,} \end{aligned}$$

then p_3 and p_4 are increasing paths on the 1-skeleton of C. □

2.3 Shadow Boundaries over Subspaces of all Dimensions

We now look at another generalisation of Gruber and Sorger's result on shadow boundaries.

It has been shown in [18] that: For most pairs $(C, X) \in \mathscr{C} \times \Gamma(d-1)$ the shadow boundary $S(C, X)$ is sharp and has infinite $(d-2)$-dimensional Hausdorff measure. We wish to extend this result to shadow boundaries over subspaces of any dimension $2 \leq l < d$. Our aim is to prove that:

Theorem 5. *For most pairs $(C, X) \in \mathscr{C} \times \Gamma(l)$, where $2 \leq l < d$, the shadow boundary $S(C, X)$ is sharp and has infinite $(l-1)$-dimensional Hausdorff measure.*

2.3.1 Notation

Definition 37. *Let $\mathscr{S}(l)$ denote the set of all pairs $(C, X) \in \mathscr{C} \times \Gamma(l)$ such that $\mathbf{S}(C, X)$ is sharp.*

Definition 38. *The* natural parametrisation *of* $\mathbf{S}(C, X)$ *for* $(C, X) \in \mathscr{S}(l)$ *is defined by the function* $f = f_{C,X} : \pi_X(\mathbf{S}(C, X)) \to \mathbf{S}(C, X)$ *such that for* $x \in \pi_X(\mathbf{S}(C, X)))$ *we have* $\pi_X(f(x)) = x$. f *is bijective and continuous.*

Definition 39. *A* parametrisation *of* $\mathbf{S}(C, X)$ *is defined by a function mapping an* $(l-1)$*-dimensional topological sphere onto* $\mathbf{S}(C, X)$.

2.3.2 Preliminary Results

Proposition 5. *Given a constant* $2 \leq l < d$*, let* $(C, X), (C_i, X_i) \in \mathscr{S}(l)$*,* $i = 1, 2, ...$ *with* $(C_i, X_i) \to (C, X)$*. Then there are continuous one-to-one parametrisations* g *of* $\mathbf{S}(C, X)$ *(equivalent to* $f_{C,X}$*) and* g_i *of* $\mathbf{S}(C_i, X_i)$ *(equivalent to* f_{C_i,X_i}*) respectively, all defined on the same* $(l-1)$*-dimensional sphere and such that* $g_i \to g$ *uniformly.*

Proof. The proof of this result is based on the proof of (5) in [18].

Assume the origin o lies in the interior of C and C_i for all $i = 1, 2, ...$ This is definitely achievable by applying the same translation to all bodies and, if need be, deleting finitely many pairs (C_i, X_i). Then there exists an $(l-1)$-dimensional sphere S in X, centered at o and contained in C and all C_i.

We may now assume that no X_i is parallel to X (deleting finitely many pairs if necessary).

Define the parametrisation $g : S \to \mathbf{S}(C, X)$, $g(x) = z$, as follows: if y is the intersection point of the half ray originating at o in direction x and $\pi_X(\mathbf{S}(C, X))$, then z is the unique point in $\mathbf{S}(C, X)$ such that $\pi_X z = y$. Define $g_i : S \to \mathbf{S}(C_i, X_i)$ similarly.

Clearly, $g, g_i, i = 1, 2, ...$ are equivalent to $f_{C,X}, f_{C_i,X_i}$ respectively. Since $\mathbf{S}(C, X), \mathbf{S}(C_i, X_i)$ are sharp, g, g_i are one-to-one. Need to show:

$$x, x_i \in S(i = 1, 2, ...)\, x_i \to x \text{ implies } g_i(x_i) \to g(x). \tag{2.5}$$

As $C_i \to C$, the sequence $(g_i(x_i))$, $i = 1, 2, ...$ is bounded. Hence it suffices to show that any convergent subsequence of $(g_i(x_i))$ has limit $g(x)$. Let $(g_{i_k}(x_{i_k}))$ converge to z, say. By Property 5, $z \in \text{bd}\, C$.

Now, the support planes of C_{i_k} through $g_{i_k}(x_{i_k})$ parallel to $\bar{x}_{i_k}$ converge to the support plane of C through z parallel to $\bar{x}$. This implies

$$y_{i_k} = \pi_{X_{i_k}}(g_{i_k}(x_{i_k})) \to y = \pi_X(z) \in \pi_X(\mathbf{S}(C, X)).$$

Therefore, the radial projection centered at o of y_{i_k} onto S converges to the radial projection of y. Now, the radial projection of y_{i_k} is x_{i_k} and $x_{i_k} \to x$ therefore the radial projection of y onto S is x. The definition of g shows that $z = g(x)$, thus proving (2.5).

A similar argument yields that

$$g, g_i, i = 1, 2, \ldots \text{ are continuous on } S.$$

It remains to show that

$$g_i \to g \text{ uniformly on } S. \tag{2.6}$$

If this were false there would exist

$$\varepsilon > 0 \text{ such that for some } x_{i_k} \in S, (k = 1, 2, \ldots)\ |g_{i_k}(x_{i_k}) - g(x_{i_k})| \geq \varepsilon. \tag{2.7}$$

The compactness of S implies that there exists a convergent subsequence of (x_{i_k}). After renumbering, let us assume (x_{i_k}) converges to x, say. Then $g_{i_k}(x_{i_k}) \to g(x)$ by (2.5) and the continuity of g implies $g(x_{i_k}) \to g(x)$. The last two statements contradict (2.7) therefore (2.6) holds.

The proof of Proposition 5 is thus complete. □

Direct consequences of this are:

Proposition 6. *For any $(C, X) \in \mathscr{S}(l)$ the natural parametrisation $f_{C,X}$ of $S(C, X)$ is continuous and one-to-one.*

Proposition 7. *If $(C, X), (C_i, X_i) \in \mathscr{S}(l)$, $i = 1, 2, \ldots$, and $(C_i, X_i) \to (C, X)$, then $\mathbf{S}(C_i, X_i) \to \mathbf{S}(C, X)$.*

We now introduce a few properties of the $(l-1)$-dimensional integral geometric stable area γ_{l-1}. For Properties 6 and 8 see [5] p. 325; for Property 7 see [33] p. 182;

for Property 9 see [5] p. 319.

Let S be a $(l-1)$-dimensional topological sphere in $\mathbb{E}^d$. Then the following properties hold:

Property 13. *If S is a triangulated polytopal sphere and $g : S \to \mathbb{E}^d$ is continuous, injective and affine on each simplex of S, then*

$$\mathscr{H}^{l-1}(g(S)) = \gamma_{l-1}(g).$$

Property 14. *If $g : S \to \mathbb{E}^d$ is continuous and injective, then*

$$\mathscr{H}^{l-1}(g(S)) \geq \gamma_{l-1}(g).$$

Property 15. *If $g : S \to \mathbb{E}^d$ and $g_i : S \to \mathbb{E}^d, i = 1, 2, ...$, are continuous and $g_i \to g$ uniformly on S, then*

$$\gamma_{l-1}(g) \leq \liminf_{i\to\infty} \gamma_{l-1}(g_i).$$

Property 16. *If T is another $(l-1)$-dimensional topological sphere in $\mathbb{E}^d$ and $h : T \to S$ a homeomorphism, then*

$$\gamma_{l-1}(g) = \gamma_{l-1}(g \circ h).$$

2.3.3 Main Result

The proof of Theorem 5 requires the following lemma

Lemma 2.3.1. *The set of pairs $(Q, X) \in \mathscr{S}(l)$ where Q is a convex polytope and for which $\mathscr{H}^{l-1}(\mathbf{S}(Q, X)) > n$ holds is dense in $\mathscr{S}(l)$, $(n = 1, 2, ...)$.*

We shall prove the following which implies Lemma 2.3.1

Lemma 2.3.2. *Let $(P, X) \in \mathscr{S}(l)$, P a polytope. Then there are polytopes P_i, $i = 1, 2, ...$, with $(P_i, X) \in \mathscr{S}(l)$, $P_i \to P$ and $\mathscr{H}^{l-1}(\mathbf{S}(P, X)) \to +\infty$.*

Proof. Without loss of generality, let $o \in \pi_X P$. Let ρ be the radial projection of $X\setminus\{o\}$ onto the relative bound $S = \pi_X(\mathbf{S}(P, X))$ of $\pi_X P$. There exists a constant $\alpha > 1$ such that the following holds:

Let T be a closed convex surface in X containing $\pi_X P$ and contained in $2\pi_X P$. Then:

$$(1/\alpha)\|x - y\| \leq \|\rho(x) - \rho(y)\| \leq \alpha\|x - y\| \text{ for } x, y \in T. \tag{2.8}$$

($\|\ \|$ stands for the euclidean norm).

For $i = 1, 2, ...$, define simplicial convex surfaces S_i in X with the following properties:

i) Each facet F of S_i has diameter $< 1/i^2$,

ii) S_i contains $\pi_X P$ and is contained in $(3/2)\pi_X P$,

iii) $\|x - \rho(x)\| < 1/i$ for each $x \in S_i$.

For each F of S_i pick a point a in its relative interior. To each a assign a point b outside of S_i close enough for the line segments joining different bs to intersect the relative interior of conv(S_i) and such that the following holds:

Property 17. *We can construct a surface T_i by replacing each facet F of S_i by the $(l - 1)$-dimensional simplex formed by joining the boundary simplices of F with b. T_i has properties*

i) Each facet of T_i has diameter $< 1/i^2$,

ii) T_i contains $\pi_X P$ and is contained in $2\pi_X P$,

iii) $\|x - \rho(x)\| < 1/i$ for each $x \in T_i$.

Define the function $h_i : T_i \to \mathbb{R}$ by $h_i(x) = 0$ if $x \in$ bd$F$ for some F of S_i and $h_i(b) = 1/i$, interpolate linearly in between. Clearly,

$$\max\{|h_i(x)| : x \in T_i\} = 1/i. \tag{2.9}$$

Consider the polytopal surface $\{x + h_i(x)\bar{x} : x \in T_i\}$ (where $\bar{x}$ is the unit vector perpendicular to X). By Property 17 *i)*:

$$\text{each of its facets has slope } > 1/i \text{ with respect to } X. \tag{2.10}$$

There exists a function $e : \pi_X(\mathbf{S}(P,X)) \to \mathbb{R}$ corresponding to the natural parametrisation $f_{P,X}$ of $\mathbf{S}(P,X)$ such that

$$f_{P,X}(x) = x + e(x)\bar{x} \ \text{ for } \ x \in \pi_X(\mathbf{S}(P,X)). \tag{2.11}$$

As P is a polytope, the slopes of the facets of $\mathbf{S}(P,X)$ are bounded by a constant, β say. By (2.8) and Property 17 *ii)*, each facet of the polytopal surface $\{x + e(\rho(x))\bar{x} : x \in T_i\}$ has slope $\leq \alpha\beta$ with respect to X. Define

$$U_i = \{x + (e(\rho(x)) + h_i(x))\bar{x} : x \in T_i\}. \tag{2.12}$$

The previous inequality along with (2.10) show that each facet of U_i has slope $\geq i - \alpha\beta$ with respect to X. This implies:

$$\begin{aligned}
\mathscr{H}^{l-1}(U_i) &\geq (1 + (i - \alpha\beta)^2)^{1/2}\mathscr{H}^{l-1}(T_i) \\
&\geq (1 + (i - \alpha\beta)^2)^{1/2}\mathscr{H}^{l-1}(\pi_X(\mathbf{S}(P,X))) \\
&\to +\infty \ \text{ as } \ i \to \infty.
\end{aligned}$$

(Note: the $(l-1)$ Hausdorff measure of the sets U_i, T_i and $\pi_X(\mathbf{S}(P,X))$ is equivalent to their $(l-1)$-dimensional surface area.) Property 17 *iii)* and equation (2.9) imply

$$\delta^{\mathscr{H}}(\mathbf{S}(P,X), U_i) < 2/i. \tag{2.13}$$

Define the sets

$$P_i = \text{ conv}(\{v \in \text{ vert } P, v \notin \mathbf{S}(P,X)\} \cup U_i).$$

Then by Property 17 *ii)*, $\mathbf{S}(P_i, X) = U_i$ which implies $(P_i, X) \in \mathscr{S}(l)$. The definition of P_i and equation (2.13) yield $\delta^{\mathscr{H}}(P, P_i) \to 0$ as $i \to \infty$. Since $\mathbf{S}(P_i, X) = U_i$, equation (2.13) implies $\mathscr{H}^{l-1}(\mathbf{S}(P_i, X)) = U_i \to +\infty$ as $i \to \infty$. Thus Proposition 2.3.2 holds. $\square$

Proof. of Theorem 5

Define, for $n = 1, 2, ...$, the sets

$$\mathscr{S}_n(l) = \{(C, X) \in \mathscr{S}(l) : \gamma^{l-1}(f_{C,X}) \leq n\}.$$

We first show

$$\mathscr{S}_n(l) \text{ is closed in } \mathscr{S}(l). \tag{2.14}$$

Let $(C_i, X_i) \in \mathscr{S}_n(l)$, $i = 1, 2, ...$, converge to $(C, X) \in \mathscr{S}(l)$, then it suffices to show $(C, X) \in \mathscr{S}_n(l)$. Proposition 5 implies there exist one-to-one continuous parametrisations g of $\mathbf{S}(C, X)$ equivalent to $f_{C,X}$ and g_i of $\mathbf{S}(C_i, X_i)$ equivalent to f_{C_i,X_i} all defined on the same $(l-1)$-dimensional sphere $S \subset X$ and such that $g_i \to g$ uniformly on S.

Properties 15 and 16 imply

$$\gamma^{l-1}(f_{C,X}) = \gamma^{l-1}(g) \leq \liminf_{i\to\infty} \gamma^{l-1}(g_i) = \liminf_{i\to\infty} \gamma^{l-1}(f_{C_i,X_i}) \leq n.$$

Therefore $(C, X) \in \mathscr{S}_n(l)$ which completes the proof of (2.14).

We now show

$$\mathscr{S}_n(l) \text{ has empty interior in } \mathscr{S}(l). \tag{2.15}$$

Assume the converse. By Proposition 2.3.1 we can choose a pair $(Q, X) \in \mathscr{S}_n(l)$ with $\mathscr{H}^{l-1}(\mathbf{S}(Q, X)) > n$ where Q is a simplicial polytope. We may consider $\pi_X(\mathbf{S}(Q, X))$ as a triangulated $(l-1)$-dimensional sphere in X and by Property 13 $\gamma^{l-1}(f_{Q,X}) = \mathscr{H}^{l-1}(\mathbf{S}(Q, X)) > n$. Therefore $(Q, X) \notin \mathscr{S}_n(l)$ which contradicts our assumption. The proof of (2.15) is thus complete.

Statements (2.14) and (2.15) imply

$$\mathscr{S}(l) \setminus \cup_{n=1}^{\infty} \mathscr{S}_n(l) = \{(C, X) \in \mathscr{S}(l) : \gamma^{l-1}(f_{C,X}) = \infty\} \text{ is residual in } \mathscr{S}(l).$$

Then Property 14 implies that the set $\{(C, X) \in \mathscr{S}(l) : \mathscr{H}^{l-1}(\mathbf{S}(C, X)) = \infty\}$ is residual in $\mathscr{S}(l)$ and therefore also residual in $\mathscr{C} \times \Gamma(l-1)$.

$\square$

2.4 Connections

Intuitively, these results may seem to contradict the results of the previous chapter. Here we shall reconcile this apparent contradiction.

Our measure theory results state that given a convex body C and a dimension $l \leq d$ *almost all* shadow boundaries $\mathbf{S}(C, X)$, where $X \in \Gamma(l)$, have finite $(l-1)$-dimensional Hausdorff measure. Here *almost all* means the property holds except on a set of measure zero.

Our Baire category results state that for *most* pairs (C, X), where C is a convex body in $\mathbb{E}^d$ and X is an l-dimensional linear subspace, the shadow boundary $\mathbf{S}(C, X)$ has infinite length. Here *most* means the property holds for all elements except those of a set of first Baire category.

The most obvious way to reconcile these two perspectives would be the existence of a set of first Baire category of measure 1. It is possible to construct nowhere dense sets with positive measure and as a set of first Baire category is the countable union of nowhere dense sets it is possible to find a set of first Baire category with measure 1.

An example of such a nowhere dense set is the Smith-Volterra-Cantor set. It is constructed similarly to the typical Cantor set by removing successively smaller open intervals from the unit interval. Starting with the interval $[0, 1]$ the first step consists in removing an interval of length a quarter centered at $1/2$. The remaining set is

$$\left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right].$$

Next we remove a sixteenth from each of the remaining intervals leaving

$$\left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right].$$

The general step is to remove an interval of length $1/2^{2n}$ from the center of each of the 2^{n-1} intervals created at the previous step.

The resulting set is nowhere dense as it contains no intervals. The total measure of intervals removed from $[0, 1]$ is

$$\sum_{n=1}^{\infty} 2^{n-1}(1/2^{2n}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \frac{1}{2}.$$

Thus the measure of the set remaining is $1/2$.

This construction can be adapted to remove intervals of length $\alpha 1/2^{2n}$, where α is a constant, which will in turn vary the measure of the remaining set to reach any value less than 1.

Thus it is not a contradiction for almost all shadow boundaries to have finite length and most of them to have infinite length.

Chapter 3

Further work

In this chapter I have collected all other problems considered throughout my PhD. Most arise from work presented previously, mostly from our work on finite shadow boundaries. As it has been said by a contributor to our main results one has a duty to pose the questions prompted by one's work in order to enable successors to know where to start.

3.1 Increasing paths of finite length on the 1-skeleton of a convex body

3.1.1 Existence of increasing paths on the 1-skeleton of a convex body

Having shown that almost all shadow boundaries have finite length we questioned whether our result or method could be used to show that other structures on the boundary of a convex body have finite length.

The first of these was proving that increasing paths in the 1-skeleton of a convex body have finite length.

Definition 40. *The* 1*-skeleton of a convex body* C *in* $\mathbb{E}^d$ *is the set of points of* C *that are not the centre of any* 2*-dimensional spherical ball contained in* C.

In 1971, Larman and Rogers proved the existence of strictly increasing paths in the 1-skeleton of a convex body. They conjectured:

Conjecture 1. [16]

Let L be a non-constant linear function on $\mathbb{E}^d$. Let e_0 be an extreme point of a convex body K in $\mathbb{E}^d$. Then there is a continuous map s of the closed interval $[0,1]$ to the 1-skeleton of K with

$$\begin{aligned} s(0) &= e_0, \\ L(s(t_1)) &< L(s(t_2)), \quad \textit{when } 0 \leq t_1 < t_2 \leq 1, \\ L(s(1)) &= \sup_{k \in K} L(k). \end{aligned}$$

They were unable to reach a firm opinion as to the truth or falsehood of this statement. (A counterexample to this conjecture was later found by S. Gallivan [21] who constructed a three dimensional convex body with an extreme point from which there is no strictly increasing path).

They instead proved a refinement of this result concerning the *exposed* 1-skeleton of a convex body.

Definition 41. *A point e of a convex body C belongs to the exposed 1-skeleton of C if it lies in a plane tangent to C, whose total intersection with C is of linear dimension 0 or 1.*

Theorem 6. [16]

Let L be a non-constant linear function on $\mathbb{E}^d$ and let K be a convex body in $\mathbb{E}^d$. Then there are continuous maps s_1, s_2 of the closed interval $[0,1]$ to the exposed one-skeleton of K with

$$\begin{aligned} L(s_i(0)) &= \inf_{k \in K} L(k) \\ L(s_i(t_1)) &< L(s_i(t_2)), \ \textit{when } 0 \leq t_1 < t_2 \leq 1, \\ L(s_i(1)) &= \sup_{k \in K} L(k) \quad \textit{for } i = 1, 2. \end{aligned}$$

Further, the paths can be separated by a $(d-1)$-dimensional plane, in that a plane π can be chosen such that the sets

$$s_i(t),\ 0 < t < 1 \quad \textit{for } i = 1, 2,$$

lie in opposite open half-spaces determined by π.

Both of these statements are generalisations of the paths produced by the simplex algorithm for convex polytopes.

Simplex Algorithm

If L is a linear function on $\mathbb{E}^d$ and if v_0 is a vertex of a convex polytope P, then there is a finite sequence of vertices $v_0, v_1, v_2, \ldots, v_m$ of P, the line segments

$$v_0v_1, v_1v_2, \ldots, v_{m-1}v_m$$

being edges of P, with

$$L(v_0) < L(v_1) < \ldots < L(v_m) = \sup_{p \in P} L(p).$$

3.1.2 Can we always find an increasing path of finite length in the 1-skeleton of a convex body?

Increasing paths in the 1-skeleton of a convex polytope are composed of vertices and edges like the one described above. Vertices do not contribute to the total measure of a path as they are of dimension 0. Convex polytopes are bounded and hence all of their edges have measure less than or equal to the diameter of the polytope.

Thus increasing paths on the 1-skeleton of a convex polytope, being composed of a finite number of finite length edges, have finite length.

Question 1. *Can this be extended to general convex bodies?*

Sharp shadow boundaries of a convex body over 2-dimensional subspaces are contained in its 1-skeleton. Given an appropriate linear function on the space, these sharp shadow boundaries can be decomposed into four sections, two of which will be increasing paths as defined above.

As we are only interested in sharp shadow boundaries over 2-dimensional subspaces, from now on the term *shadow boundary* will strictly refer to sharp shadow boundaries over 2-dimensional subspaces.

Clearly, there are many more paths in the 1-skeleton of a convex body than there are sharp shadow boundaries. This is true as sharp shadow boundaries are paths in the 1-skeleton and paths are not restricted to points belonging to the intersection of C with tangent planes orthogonal to the subspace corresponding to the shadow boundary.

We would like to show

Conjecture 2. *Given a linear function on the space $\mathbb{E}^n$ and a convex body C, there is a continuous map s of the closed interval $[0,1]$ to the 1-skeleton of C with*

$$\begin{aligned} L(s(0)) &= \inf_{k \in C} L(k), \\ L(s(t_1)) &= L(s(t_2)), \quad \textit{when } 0 \leq t_1 < t_2 \leq 1, \\ L(s(1)) &= \sup_{k \in C} L(k) \quad \textit{and} \\ \mathscr{H}^1(L(s[0,1])) &< \infty. \end{aligned}$$

3.1.3 Applying Theorem 1

This could seem like a rather straightforward objective as we are simply looking to show the existence of increasing paths of finite length rather than say anything about the proportion of them which have finite length. As mentioned before, sharp shadow boundaries are paths in the 1-skeleton and almost all of them have finite length thus it should be straightforward to prove our conjecture.

However, our proof that almost all shadow boundaries have finite length relies on an upper bound on the average measure of the shadow boundaries of a convex body over all subspaces of a particular dimension.

Given a convex body C and a linear function L on the space $\mathbb{E}^n$, the shadow boundaries of C corresponding to increasing paths are limited to those over subspaces orthogonal to the plane $\pi : L(x) = 0$.

Unfortunately, this set of directions has measure zero within $\mathbb{E}^n$. Hence it could be contained in the set of measure zero of 'bad' directions. This would mean that no shadow boundaries corresponding to directions within this hyperplane have finite length. Hence we can not directly guarantee the existence of an increasing path of finite length.

3.1.4 Adapting the proof of Theorem 1

Since simply applying our result is not possible, the next approach is to prove an analogous result for the average length of shadow boundaries over the directions within a hyperplane.

The proof of Theorem 1 in Chapter 1 relies on approximating our given convex body by a sequence of polytopes. Then we use the fact that the average measure of the shadow boundary of a convex polytope is equal to a constant multiple of its Quermass integral (Lemma 1.2.2).

Unfortunately, this result again relies on an average over all directions within $\mathbb{R}^n$ and there is no straightforward way to adapt the proof to directions within a hyperplane.

To summarise:

Although we know increasing paths on polytopes have finite length, we can not infer a bound on this length. This means that although we can approximate a convex body by polytopes with finite increasing paths there is nothing to say this sequence will tend to a finite path on the boundary of the convex body.

3.2 Is there a convex body with a plane of bad directions?

The idea here is to find out if there is a way of bounding the length of shadow boundaries corresponding to increasing paths on the 1-skeleton of polytopes approximating a convex body.

If yes, then there is an equivalent of Lemma 1.2.2 for shadow boundaries over subspaces orthogonal to a fixed hyperplane. Thus the method used in Chapter 1 can be used.

If no, the method used in Chapter 1 can not be applied and we need a different approach.

Formally, we wish to answer the following:

Question 2. *Given a non constant linear function L on $\mathbb{E}^n$, can we find a sequence of polytopes $(P_i)_{i=1}^{\infty}$ converging to a convex body C such that the shadow boundaries of P_i over subspaces orthogonal to π are unbounded as $i \to \infty$?*

Here π denotes the plane where L is identically 0, that is $\pi = \{x : L(x) = 0\}$. A subsidiary question would be:

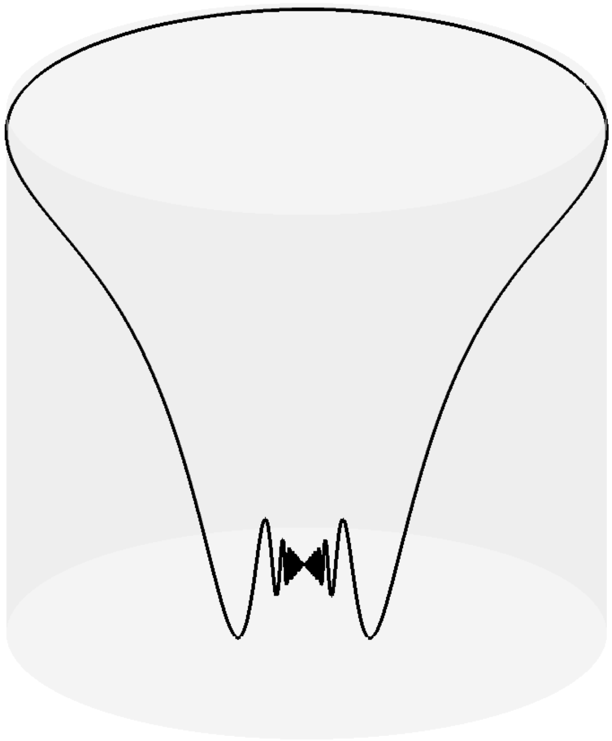
Question 3. *Given a non constant linear function L, two tac planes t_1 and t_2 parallel to π and a number M, can we construct a polytope such that all increasing paths from t_1 to t_2 are longer than M?*

Let us look at the problem in three dimensions. We need to find structures which induce long paths. An example of a shadow boundary with infinite measure is that constructed from the graph of $x \sin(1/x)$.

Example 1. *See* Figure 3.1.

Take a cylinder and map the graph of $x \sin(1/x)$ onto its curved face such that the x-axis is parallel to the circular faces. Now remove the cylinder to leave the curved graph of $x \sin(1/x)$. Taking the convex hull of the graph we obtain a 3-dimensional

Figure 3.1: Infinite shadow boundary



convex body. Taking the linear 2-dimensional subspace X to be parallel to the x-axis, we construct the shadow boundary over X. The resulting shadow boundary is the graph of $x\sin(1/x)$ near the origin which is known to have infinite 1-dimensional measure.

In general, oscillating paths (as constructed above) as well as spirals can lead to long (or infinite) paths. See Figure 3.2.

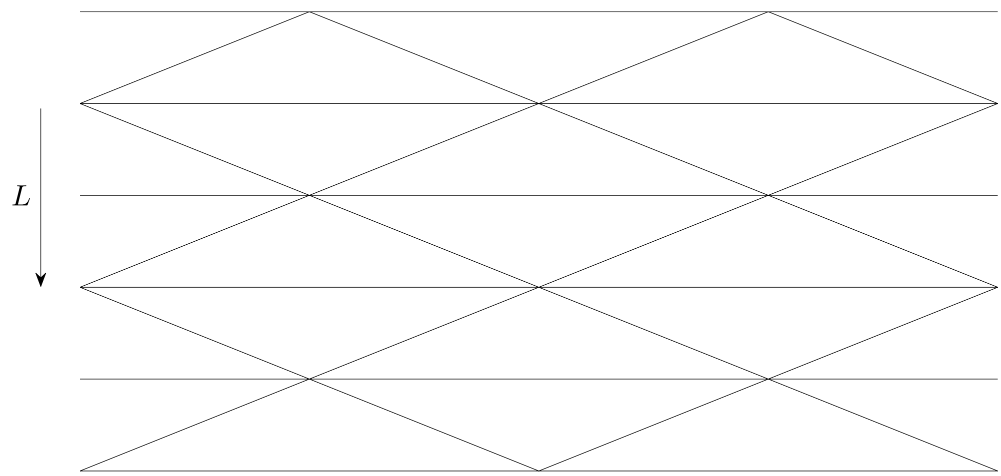
3.2.1 Triangle structure - initial examples

We consider the following facial structure (Fig 3.3):

The idea behind this structure is that any increasing path will have to travel a long distance sideways in order to cover a short vertical distance. By altering the number of points and the "flatness" of the triangles we can increase the distance travelled to reach

Figure 3.2: Oscillating and spiral paths

Figure 3.3: Uniform mesh



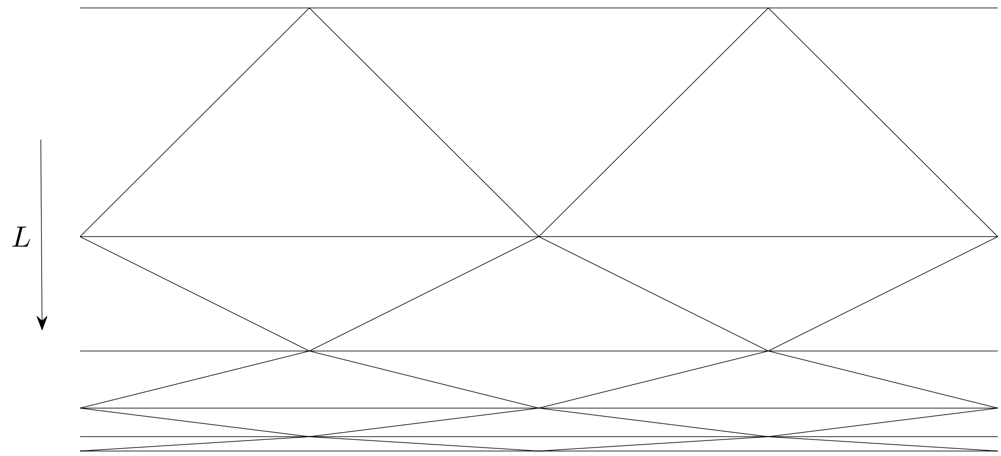
a certain height. In particular, by increasing the flatness as we tend to $\max_{x \in C} L(x)$ we will get infinite paths (Fig 3.4).

Question 4. *Can we construct polytopes with this facial structure which approximate a convex body?*

This would imply that each vertex of the structure in Figures 3.3 or 3.4 lies on the boundary of the convex body being approximated and that no extra edges which would 'short circuit' our long path structure are created.

Applying this structure to the cylinder while respecting the two conditions above

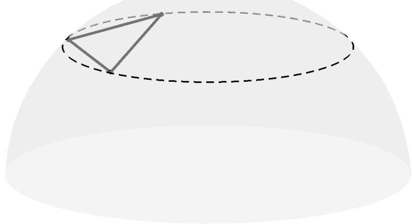
Figure 3.4: Scaled mesh



leads to the construction of a non convex polytope. Hence this is not a viable approximation.

An attempt on the sphere using a slightly modified structure (in which the points on each level become closer to account for the fact that the radii of the circles corresponding to each level are decreasing) is equally unsuccessful due to convexity breaking down. This can easily be seen by looking at a triangle from our structure placed in the sphere with its vertices on the boundary.

Figure 3.5:

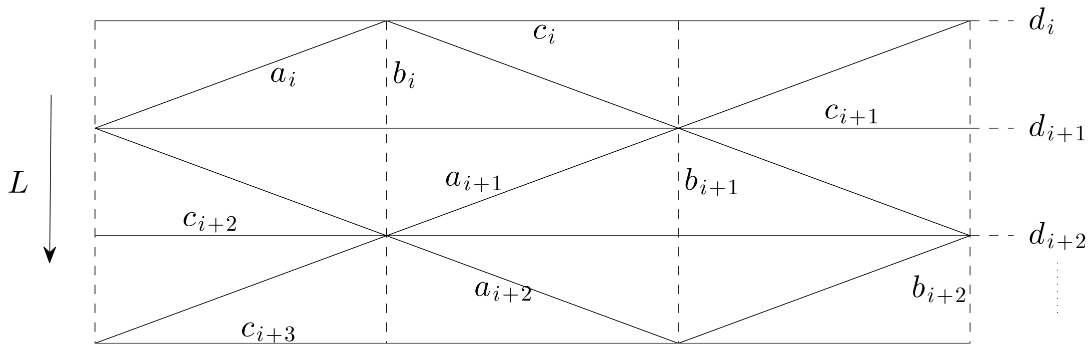


The plane containing this triangle intersects the sphere in a disc (see Fig 3.5). In order to preserve convexity, the triangle neighbouring the first one along its long edge must not lie within the cap cut by the plane containing the first triangle. As the vertices of all the triangles in our structure must lie on the boundary of the sphere, we can not maintain the proportions of the triangles.

This observation about the shape of the intersection cut by the plane containing the initial triangle means the body required must have ellipsoidal section when cut by a plane close to the boundary. Examples of such bodies can be obtained by rotating a 2-dimensional parabola around its y-axis to obtain a 'bowl' shaped body. The linear function on the space would increase as y tends to zero. As the main length contribution to the infinite paths considered is determined by the behaviour close to the maximal value of L, adapting our structure to these bowls would be sufficient.

3.2.2 Essential Conditions

These initial examples lead us to consider which conditions need to be satisfied and which convex body might allow us to construct approximating polytopes which satisfy them.



Let us define the sequences a_1, b_i, c_i, d_i and r_i, where

d_i is the vertical distance from o (i.e. the bottom of the bowl);

r_i is the radius of the bowl at level d_i;

c_i is the distance between two vertices on level d_i;

b_i is the distance between a vertex on level d_i and the corresponding edge on level d_{i+1};

a_i is the diagonal distance between a vertex on level d_i and one on level d_{i+1}.

What needs to be satisfied?

1. Convexity

2. $\sum_i a_i = \infty$ and $\sum_i b_i < \infty$, where a_i is the diagonal distance between levels i and $i+1$ and b_i is the vertical distance between levels i and $i+1$

3. $a_i^2 = b_i^2 + c_i^2$ and $c_i = r_i \sin(\pi/c)$, where c is the number of points on each level, c_i is the distance between two points on level i and r_i is the 'radius' at each level.

Each of these conditions leads to different ways of approaching the problem.

Convexity is assured if the gradient of b_i is greater than that of b_{i+1}. There is also a distinction to be made between b_is being part of a 'downward' triangle and those being part of an 'upward' triangle. The latter having a longer length than the former. The equations bellow refer to the case of a 'bow-tie' ($^{\triangledown}_{\vartriangle}$) and of a lozenge ($\lozenge$) respectively:

$$f(r_i)[r_{i+1} - r_{i+2}h] + f(r_{i+1})[r_{i+2}h - r_ih] + f(r_{i+2})[r_ih - r_{i+1}] > 0, \quad (3.1)$$

$$f(r_i)[r_{i+1}h - r_{i+2}] + f(r_{i+1})[r_{i+2} - r_i] + f(r_{i+2})[r_i - r_{i+1}h] > 0; \quad (3.2)$$

where r_i is the radius at level d_i and $h = \cos\frac{\pi}{c}$ where c is the number of points on each level.

It has been suggested that the a_is should behave similarly to $1/i$ whereas the b_is should behave like $1/i^2$. This may be a bit ambitious and so it is suggested $a_i = 1/i^\alpha$ and $b_i = 1/i^\beta$, where $0 < \alpha < 1, \beta > 1$ are constants.

Setting a_i and b_i as above and using the conditions listed above to determine values for r_i or α or β was attempted by solving a system of equations using the software package Mathematica, but this didn't lead to anything conclusive.

After all these observations, the approach with the most chance of success seems to be defining the r_is recursively.

This means picking a function which might work and values for r_1 and r_2. From these, the gradient of the face between r_1 and r_2 and where it will intersect the curve of the function can be calculated. Thus the next r_i can be picked to be below this second point of intersection and therefore guarantee convexity.

This process has been executed with x^2 and x^3 for various values of r_1 and r_2, using Mathematica to calculate the values by recursion. Mathematica sets the value r_{i+2} to be the 'smaller' intersection point of the line determined by r_i and r_{i+1} and the curve $f(x) = x^2$.

Unfortunately, these trials have not lead to anything conclusive as all trials have lead to finite paths. This reinforces our initial impression that given a convex body and a non constant linear function on the space there are increasing paths in the 1-skeleton of finite length.

3.3 Vanishing Line Segments

Continuing to focus on structures on the boundary of convex bodies, we look back at Ewald, Larman and Rogers paper [11] in which they show:

Theorem 7. Thm 2 in [11]

If K is a convex body in $\mathbb{E}^n$, the set S, of end points of the vectors drawn from the origin in the directions of line segments on the surface of K, is a set of σ-finite $(n-2)$-dimensional Hausdorff measure on the $(n-1)$-dimensional surface of the unit ball.

Similar results on the measure of the set of r-dimensional balls on the boundary of a convex body were shown in the same paper. These results led to the conclusion that almost all shadow boundaries are sharp. An extension of the set of directions of line segments on the boundary of a convex body is to consider the set of directions of vanishing line segments.

Definition 42. *Let $\underset{\sim}{k}$ be a boundary point of a convex body K in $\mathbb{E}^d$ and $H(\underset{\sim}{k})$ be a support plane of K at $\underset{\sim}{k}$ which does not contain a line segment of K. For each $\varepsilon > 0$ let $H(\varepsilon, \underset{\sim}{k})$ denote the hyperplane parallel to $H(\underset{\sim}{k})$ and at distance ε from $H(\underset{\sim}{k})$ which lies on the same side of $H(\underset{\sim}{k})$ as K.*

Let $W(H(\varepsilon,\underset{\sim}{k})\cap K)$ denote the minimal width of $H(\varepsilon,\underset{\sim}{k})\cap K$ in $H(\varepsilon,\underset{\sim}{k})$. $H^*(\underset{\sim}{k})$ denotes the $(n-1)$-dimensional subspace parallel to $H(\underset{\sim}{k})$.

If $\underset{\sim}{u}\in H^*(\underset{\sim}{k})$ let $W(u,H(\varepsilon,\underset{\sim}{k})\cap K)$ denote the width of $H(\varepsilon,\underset{\sim}{k})\cap K$ in direction $\underset{\sim}{u}$. If

$$\lim_{\varepsilon\to 0}\frac{W(H(\varepsilon,\underset{\sim}{k})\cap K)}{W(\underset{\sim}{u},H(\varepsilon,\underset{\sim}{k})\cap K)}=0,$$

we say that $\underset{\sim}{u}$ is the direction of a vanishing line segment of K.

The questions we'd like to answer are:

Question 5. *Is it true that the set of directions of vanishing line segments on the surface of a convex body has σ-finite $(d-2)$-dimensional Hausdorff measure / zero $(d-1)$-dimensional Hausdorff measure?*

Question 6. *If we look at a particular hyperplane, is it true that the set of directions of vanishing line segments orthogonal to this plane have zero $(d-2)$-dimensional measure?*

We start by outlining at the proof of the result for line segments.

3.3.1 Outline of proof of Theorem 7

Let $\mathscr{P}$ be the collection of all pairs of parallel planes such that each plane of the pair is parallel to one of the coordinate planes, meets the perpendicular coordinate axis in a rational point and meets K in an interior point. Clearly $\mathscr{P}$ is countable.

Each line segment on the boundary will meet both planes of some pair in $\mathscr{P}$, say π_0 and π_1. Then this line segment will lie on the boundary of the least convex cover of $(\pi_0\cap K)\cup(\pi_1\cap K)$ and meet π_0 and π_1.

Some subtleties later, we can now reduce the problem to a purely $(n-1)$-dimensional one by projecting the configuration orthogonally onto π_0. If K_0, K_1 are $(n-1)$-dimensional convex bodies in π_0,π_1 write

$$K_0^*=K_0, K_1^*=K_1-e_1,\ K_1^* \text{ is the projection of } K_1 \text{ onto } \pi_0 \text{ and}$$

$$L^* = \{x_1 - x_0 : x_0, x_1 \text{ are points of contact of parallel tac planes to } K_0^*, K_1^*\}.$$

Then it suffices to show $\mathscr{H}^{n-2}(L^*)$ is finite.

Apply Lemma 5 in [11] to the vector sum $K^* = K_0^* + K_1^*$. For sufficiently small $\varepsilon > 0$ we can cover K^* by a sequence of caps $C_1, \ldots, C_m$ each with minimal width lying between 2ε and $36(n-1)\varepsilon$ and with

$$\sum_{i=1}^{m} V_{n-1}(C_i) < \varepsilon\Delta, \quad \text{where } \Delta \text{ is independent of } \varepsilon.$$

Show $L^* \subset \cup_{i=1}^{m}(C_i^{(1)} - C_i^{(0)})$ where $C_i^{(0)}, C_i^{(1)}$ are caps cuts from K_0^*, K_1^*, respectively, by the closed half space $H_i(t_i)$ which cuts C_i from K^*.

Show $C_i^{(1)} - C_i^{(0)}$ is contained in the set D_i

By Lemma 7 in [11] D_i can be covered by a system of N_i spherical balls of diameter d_i. Then $\cup_{i=1}^{m} D_i$ and hence L^* will be covered by a system of N_1 balls of diameter d_1, N_2 balls of diameter d_2,..., N_m balls of diameter d_m. This covering depends on $\varepsilon > 0$ but is valid for ε arbitrarily small thus

$$\mathscr{H}^{n-2}(L^*) \text{ is finite.}$$

3.3.2 Approach

Is there a way of picking out the directions of vanishing line segments and covering them by a cap covering as in the proof of Theorem 7?

We would potentially need to cover a slightly bigger region but economically enough to get finite measure. The current covering may well be suitable to cover vanishing line segments as well as line segments on the boundary.

The proof of Theorem 7 relies on 'catching' line segments between a pair of parallel planes. Here, the only representative of a vanishing line segment on the boundary is a point. This does not give us any information about whether a vanishing line segment occurs at that point or not and if one does, there is no indication as to its direction.

The information we need is contained in the caps of vanishing line segments. Is there a way of 'catching' these caps in the same way line segments are caught in the proof of Theorem 7? Is there a way of making the vanishing line segments 'appear' somehow?

3.3.3 Representing a vanishing line segment by a line segment on the bounding hyperplane of its cap

Associate to a cap cut from K by a plane $H(\varepsilon, \underset{\sim}{k})$ corresponding to a vanishing line segment in direction $\underset{\sim}{u}$ the maximal line segment in direction $\underset{\sim}{u}$ contained in $H(\varepsilon, \underset{\sim}{k}) \cap K$.

This line segment would certainly indicate the direction of the vanishing line segment but it would not be on the surface of K. The covering used in the proof of Theorem 7 covers more than just the boundary of the convex body. For a given $\varepsilon > 0$, it actually covers $K \setminus K_\varepsilon$, where K_ε is the inner parallel body of K at distance ε. So potentially, the line segment would be contained in the existing covering. However, the line segment would depend on the value of ε chosen to determine the cap. As the proof relies on the covering working for all values of ε, and in particular as $\varepsilon \to 0$, the two limiting processes may interfere with each other and it is not obvious that the covering would be valid.

3.3.4 Adding tangent line segments in the direction of vanishing line segments at the points of the boundary at which they occur

The idea here is to make the vanishing line segments 'appear' on the boundary in order to adapt the existing proof more easily.

Essentially, a new body is created by adding tangent line segments in the direction of vanishing line segments at the points of the boundary at which they occur. The new convex body is constructed by taking the convex hull of K and this set of additional line segments. Applying the proof of Theorem 7 to this new body should yield the result required.

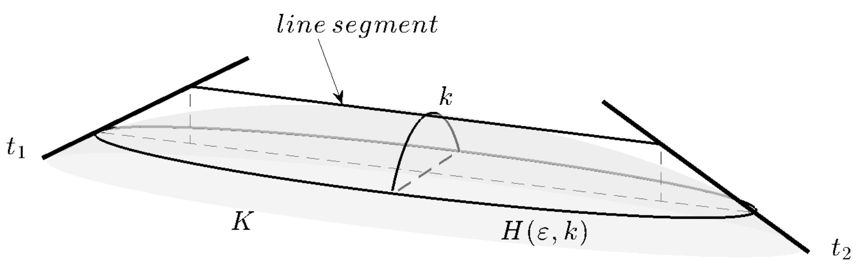
The problem with this method is determining whether these new line segments will interfere with any existing line segments or vanishing line segments. If this is the case our measure would be invalid as it would not be taking into account all the line segments and vanishing line segments present on the original body.

Suggestions to solve this problem are to:

- Determine a ratio between the minimum width and width in direction $\underset{\sim}{u}$ of a vanishing line segment cap which is small enough to guarantee no other vanishing line segments or line segments intersect the cap. If $\delta > 0$ is our ratio, then since
$$\lim_{\varepsilon \to 0} \frac{W(H(\varepsilon, \underset{\sim}{k}) \cap K)}{W(\underset{\sim}{u}, H(\varepsilon, \underset{\sim}{k}) \cap K)} = 0,$$
for $\delta > 0$ there exists $\varepsilon > 0$ such that
$$\frac{W(H(\varepsilon, \underset{\sim}{k}) \cap K)}{W(\underset{\sim}{u}, H(\varepsilon, \underset{\sim}{k}) \cap K)} < \delta.$$

- Once caps are fixed (by the ratio described above or otherwise), look at a cap cut by $H(\varepsilon, \underset{\sim}{k})$. Take the planes t_1, t_2 tangent to K at either end of the maximal line segment in direction $\underset{\sim}{u}$ contained in $K \cap H(\varepsilon, \underset{\sim}{k})$. Then the end points of the tangent line segment at $\underset{\sim}{k}$ in direction $\underset{\sim}{u}$ are determined by the tangent planes t_1, t_2.

Figure 3.6: Adding a tangent line segment



It is however not clear how one would determine a suitable ratio.

3.3.5 Classifying caps by size

As stated above, given $\delta > 0$ for each vanishing line segment there exists $\varepsilon > 0$ such that

$$\frac{W(H(\varepsilon, \underset{\sim}{k}) \cap K)}{W(\underset{\sim}{u}, H(\varepsilon, \underset{\sim}{k}) \cap K)} < \delta. \tag{3.3}$$

Then each vanishing line segment has a unique cap assigned to it. These caps have the characteristic of being 'long' in direction $\underset{\sim}{u}$ (the direction of the vanishing line segment they represent). They may therefore be classified into classes V_m defined as follows,

$$V_m = \{\underset{\sim}{u} : W(\underset{\sim}{u}, H(\varepsilon, \underset{\sim}{k}) \cap K) \geq 1/m \text{ for } \varepsilon \text{ s.t.(3.3) holds}\},\ m = 1, 2, 3, \ldots.$$

The set of planes dividing the space can then be considered in pairs at distance $1/m$ of each other. Proving each set V_m has finite measure will lead to the set of all direction having finite measure.

3.3.6 Cone of directions

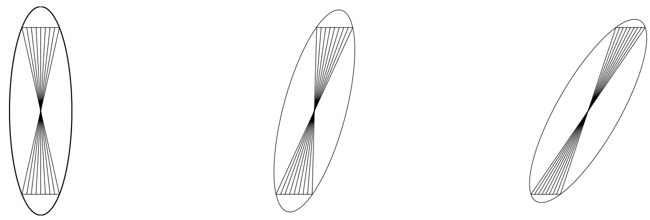
The distinguishing feature of the caps corresponding to vanishing line segments is that they are 'slim'.

Assume a given cap is caught by a pair of parallel planes p_1 and p_2, say. Then the set $H(\varepsilon, \underset{\sim}{k}) \cap K$ is also intersected by the planes p_1 and p_2. The line segments $p_1 \cap H(\varepsilon, \underset{\sim}{k}) \cap K$ and $p_2 \cap H(\varepsilon, \underset{\sim}{k}) \cap K$ define a trapezium in $H(\varepsilon, \underset{\sim}{k}) \cap K$. Looking at the directions of the line segments with an end point in $p_1 \cap H(\varepsilon, \underset{\sim}{k}) \cap K$ and one in $p_2 \cap H(\varepsilon, \underset{\sim}{k}) \cap K$ we obtain Figure 3.7. This is defined as the *cone of directions* of a cap.

In the case of a vanishing line segment cap, this cone should have a rather small principal angle. The line segments $p_1 \cap H(\varepsilon, \underset{\sim}{k}) \cap K$ and $p_2 \cap H(\varepsilon, \underset{\sim}{k}) \cap K$ should also have length much smaller than $1/m$. Could something be said about the caps caught by looking at the line segments described above? Could the cone of directions be used in some way?

One thing that needs to be taken into account is that the pair of planes will not necessarily intersect the cap orthogonally to $\underset{\sim}{u}$. In which case we would get something like Figure 3.7. The principal angle of the cone of directions is largest when the planes intersect the cap at right angles to $\underset{\sim}{u}$. As the principal angle gets smaller the thiner the cap is, considering values of the principal angle smaller than or equal to that when the planes the cap orthogonally to $\underset{\sim}{u}$ would be sufficient.

Figure 3.7: Cone of directions



3.4 Conclusion/Future outlook

To conclude:

One could ask whether increasing paths on the 1-skeleton of a convex body (as described in [16]) have finite length? This is clearly true in the case of convex polytopes given that these paths are composed of a finite number of edges of finite length (the simplex algorithm implies the finite number of edges and the length of each edge is bounded by the diameter of the body and is therefore finite).

However, extending this to general convex bodies using the results we have just shown is not straightforward as averaging takes place over a hyperplane rather than over the entire space.

A possible extension would be to prove that the lifting maps $\sigma(C, X)$ are Lipschitz rather than rectifiable. This would obviously not be true for all shadow boundaries given the limitations we encountered proving rectifiability.

Another area of interest is vanishing line segments. These are defined by:

Definition: *Given a point k on the boundary of a convex body C, define $H(k)$ to be the support hyperplane to C at k. Looking at caps cut from C by hyperplanes $H(\varepsilon, k)$ parallel to and at distance ε from $H(k)$, if there exists a direction u such that*

$$\lim_{\varepsilon \to 0} \frac{\text{min width of } (H(\varepsilon, k) \cap C)}{\text{width in direction } u \text{ of } (H(\varepsilon, k) \cap C)} = 0,$$

then we say u is the direction of a vanishing line segment of C.

We know that the set of directions of line segments on the boundary of a convex body has σ-finite $(n-2)$-dimensional measure. The question is whether the set of directions of vanishing line segments of a convex body have zero $(n-1)$-dimensional measure.

The study of vanishing line segments is closely linked to showing that the lifting maps of shadow boundaries are Lipschitz. In fact, the Lipschitz property would imply the vanishing line segment result.

Appendix A

Average measure of the shadow boundaries of convex bodies.

Mittlere Schattengrenzenlänge konvexer Körper

P. Steenaerts

Translation by L. Jottrand.

A.1 Introduction and problem statement

We place ourselves in n-dimensional Euclidean space $\mathbb{E}^n$. We take a point $o \in \mathbb{E}^n$ to be the origin and points $x \in \mathbb{E}^n$ are referred to by their position vector. Our basis is an orthonormal system of vectors $\{e_i\}_{i=1}^n$, $\langle e_j, e_i \rangle = \delta_{ij}$. A convex body $K \in \mathbb{E}^n$ is taken to be a compact convex set of points with non empty interior.

For a fixed direction, assign to each convex body the measure of its shadow boundary in that direction. From this we obtain a functional over the class of all convex bodies. In "Lichtgrenzen und Leichtsinn" [23], Blaschke remarked that this functional varies discontinuously in the transition from polytopes to smooth convex bodies. The starting point of this work is a problem posed by P. McMullen [15] in 1974 concerning the average measure of shadow boundaries of convex bodies in 3-dimensional Euclidean space.

In the following we will answer this question more generally by studying two n-dimensional variables. A key fact is taken from the result that the average measure of shadow boundaries (as well as the measures of shadow boundaries in a given direction) in the transition from polytopes to smooth convex bodies is lower semi-continuous. Along with this comes a new substantial additivity property of the Lebesgue area.

Take $K \subset \mathbb{E}^n$ a convex body with $o \in \text{int } K$ and let $n \geq 3$. Denote by Γ_1^n the compact $(n-1)$-dimensional Grassmann manifold of all lines through the origin o of $\mathbb{E}^n$. For $G \in \Gamma_1^N$, we define

$$\begin{aligned} \mathbf{\Sigma}(K,G) &:= \{x \mid x \in \text{bd}(K), (x+G) \cap \text{int}(K) = \emptyset\} \quad \text{and} \\ \mathbf{\Pi}(K,G) &:= \text{relbd}\,(\pi_G(K)), \end{aligned}$$

where $\pi_G : \mathbb{E}^n \to F$ is the normal projection in direction G onto the subspace F of $\mathbb{E}^n$, where F is the orthogonal complement of G.

$\mathbf{\Sigma}(K,G)$ is called the *shadow boundary of K in direction G*. Clearly

$$\pi_G(\mathbf{\Sigma}(K,G)) = \mathbf{\Pi}(K,G).$$

We say $\mathbf{\Sigma}(K,G)$ is *sharp* when π_G, restricted to $\mathbf{\Sigma}(K,G)$, is a homomorphism between $\mathbf{\Sigma}(K,G)$ and $\mathbf{\Pi}(K,G)$. Furthermore, γ_1^n is the *normalised Haar measure* on Γ_1^n and so $\gamma_1^n(\Gamma_1^n) = 1$

Set $\Gamma_0(K) := \{G \mid G \in \Gamma_1^n, \mathbf{\Sigma}(K,G) \text{ is sharp}\}$. λ_{n-2}^n is the $(n-2)$-dimensional Hausdorff measure in $\mathbb{E}^n$.

Definition 43.

$$\begin{aligned} \alpha(K) &:= \int_{\Gamma_1^n} \lambda_{n-2}^n[\mathbf{\Sigma}(K,G)]d\gamma_1^n. \\ \beta(K) &:= \int_{\Gamma_1^n} \lambda_{n-2}^n[\mathbf{\Pi}(K,G)]d\gamma_1^n. \end{aligned}$$

$\alpha(K)$ *is the* average measure of the shadow boundary of K. $\beta(K)$ *is the average measure of the relative boundary of the normal projection of K onto an $(n-1)$-dimensional plane.*

Finally, set

$$q(K) := \alpha(K)/\beta(K),$$

and further, let B be a ball and P an n-polytope. Then McMullen's problem can be stated thus for all convex bodies $K \subset \mathbb{E}^n$:

$$1 = q(B) \leq q(K) \leq q(P)?$$

Often, the proof of these two inequalities is sufficient. We will prove them in the case of smooth convex bodies K.

A.2 Calculating $q(P)$ for n-polytopes P

First, we will show that the functional q takes a constant value over the class of all n-polytopes.

A.2.1 $\beta(P)$

We use Cauchy's projection formula (see H. Hadwiger [24]):

$$\omega_n \int_{\Gamma_1^n} W_1'(\pi_G(P)) \mathrm{d}\gamma_1^n = \omega_{n-1} W_2(P).$$

ω_k denotes the volume of the k-dimensional unit ball, i.e. $\omega_k = \pi^{k/2}/\Gamma(1 + k/2)$. W_v is the vth Minkowski Quermassintegral relative to $\mathbb{E}^{n-1}$. Applying the equality $(n-1)W_1'(\pi_G(P)) = \lambda_{n-2}^n[\mathbf{\Pi}(P, G)]$ the projection formula yields

$$\beta(P) = \frac{(n-1)\,\omega_{n-1}}{\omega_n} W_2(P) = \frac{(n-1)\,\omega_{n-1}}{n\omega_n} M(P),$$

where M represents the integral of the average curvatures.

A.2.2 $\alpha(P)$

$\Delta^i(P)$ is the set of i-dimensional faces of P, $0 \leq i \leq n-1$; $r \in \Delta^{n-2}(P)$, $r = f_1 \cap f_2$, where $f_1, f_2 \in \Delta^{n-1}(P)$; n_1, n_2 are the outer normal vectors on f_1 and f_2, ϑ_r is the exterior angle of the face r.

$$\alpha(P) = \frac{1}{n\,\omega_n} \sum_{r \in \Delta^{n-2}(P)} g(r) \lambda_{n-2}^n(r), \text{ with } g(r) = \frac{n\,\omega_n}{\pi} \vartheta_r.$$

It follows that:

$$\alpha(P) = \frac{1}{\pi} \sum_{r \in \Delta^{n-2}(P)} \vartheta_r \lambda_{n-2}^n(r).$$

From Steiner's formula we have that the volume of the outer parallel body A_ρ of A at distance ρ, $(0 \leq \rho < \infty)$:

$$V(A_\rho) = \sum_{\nu=0}^{n} \binom{n}{\nu} W_\nu(A) \rho^\nu.$$

A direct calculation gives:

$$V(P_\rho) = \ldots + \sum_{r \in \Delta^{n-2}(P)} \frac{\vartheta_r \rho^2}{2} \lambda_{n-2}^n(r) + \ldots$$

Through comparison it follows: $\binom{n}{2} W_2(P) = \frac{1}{2} \sum_{r \in \Delta^{n-2}(P)} \vartheta_r \lambda_{n-2}^n(r)$ and from this:

$$\alpha(P) = (n(n-1)/\pi) W_2(P) = ((n-1)/\pi) M(P).$$

A.2.3 From 2.1 and 2.2:

Lemma A.2.1. *For all n-polytope $P \subset \mathbb{E}^n$ we have: $q(P) = n\,\omega_n / \pi \omega_{n-1}$.*

Remark:

For $n \geq 3$, we know from J. Rätz [25] that $n\,\omega_n/\omega_{n-1} \geq 4$; whereby it effectively follows that $q(P) > 1$.

A.3 Measure theoretic foundations

In this section we will establish several measure theoretic facts which will help us with our problem.

A.3.1 The measurability of $\Gamma_0(K)$ with respect to γ_1^n

Lemma A.3.1. $\Gamma_0(K)$ *is a Borel set in Γ_1^n. Further, it is a G_δ set.*

Proof. For $i \in \mathbb{N}$, set:

$$X_i = \{G \in \Gamma_1^n : \exists \text{ a line segment } S(G) \subset \text{bd}K, \text{with length } \ell(S(G)) \geq 1/i \text{ and aff}(S(G)) - \text{aff}(S(G)) = G\}$$

Here, aff$A$ stands for the affine hull of the set A. Now we may easily conclude:

1. X_i is compact for all i,

2. $\cup_{i=1}^{\infty} X_i = \Gamma_1^n \setminus \Gamma_0(K)$.

$\square$

Lemma A.3.2. *We have* $\gamma_1^n(\Gamma_0(K)) = 1$ *and thereby* $\gamma_1^n(\Gamma_1^n \setminus \Gamma_0(K)) = 0$.

Proof. This follow directly from a work by Ewald, Larman and Rogers [11]. $\square$

A.3.2 The measure space $(\Gamma_0(K), \mathfrak{B}, \gamma_0)$

$\mathfrak{B}$ is the family of Borel sets in the space $\Gamma_0(K)$.
$\mathfrak{B}' := \{\Gamma_0(K) \cap Y | Y \subset \Gamma_1^n, Y$ is a Borel set in $\Gamma_1^n\}$.

Lemma A.3.3. $\mathfrak{B} = \mathfrak{B}'$

Proof. Due to general measure theory. $\square$

From Lemmas A.3.1 and A.3.3, it follows that any $X \in \mathfrak{B}$ is a Borel set in Γ_1^n; set $\gamma_0(X) := \gamma_1^n(X)$; from this arises the measure space $(\Gamma_0(K), \mathfrak{B}, \gamma_0)$ which we will simply denote by $\Gamma_0(K)$ from now on.

A.3.3 The Hausdorff measure λ_k^n

For the definition and elementary properties of the k-dimensional Hausdorff measures λ_k^n in $\mathbb{E}^n$ $(k \in \mathbb{N}, 1 \leq k \leq n)$ we refer to H. Federer [1].

It is common knowledge that all Borel sets in $\mathbb{E}^n$ are λ_k^n-measurable and so, in particular $\mathbf{\Pi}(K,G)$ and $\mathbf{\Sigma}(K,G)$ are λ_k^n-measurable for $g \in \Gamma_0(K)$ and $k = n-2$. If $\mathbf{\Pi}(K,G)$ lies in $\mathbb{E}^{n-1}$, it follows from the Hausdorff measure's immersion invariance that $\lambda_{n-2}^n(\mathbf{\Pi}(K,G)) = \lambda_{n-2}^{n-1}(\mathbf{\Pi}(K,G))$; compare with C. A. Rogers [20], pp. 50 and 53.

A.4 Main conjecture and Consequences

A.4.1 Main Conjecture

$K \subset \mathbb{E}^n$ is a convex body. The functionals defined in the introduction can now be defined thus:

$$\begin{aligned} \alpha(K) &:= \int_{\Gamma_0(K)} \lambda^n_{n-2}(\mathbf{\Sigma}(K,G))\mathrm{d}\gamma_0 \quad \text{and} \\ \beta(K) &:= \int_{\Gamma_0(K)} \lambda^n_{n-2}(\mathbf{\Pi}(K,G))\mathrm{d}\gamma_0. \end{aligned}$$

For $G \in \Gamma_0(K)$ we define $g(K,G) := \lambda^n_{n-2}(\mathbf{\Pi}(K,G)) = (n-1)W_1'(\pi_G(K))$; g is continuous in both arguments (relative to the Blaschke-Hausdorff metric and the topology on $\Gamma_0(K)$ respectively). As a constant multiple of the Minkowski Quermassintegral $W_2(K)$, $\beta(K) = \int_{\Gamma_0(K)} g(K,G)\mathrm{d}\gamma_0$ is a continuous function of K (see the calculation of $\beta(K)$ in section 2.1). For $G \in \Gamma_0(K)$ we define $f(K,G) := \lambda^n_{n-2}(\mathbf{\Sigma}(K,G))$.

Theorem 8. Main Conjecture:
$f(K,G)$ is polyhedrally lower semi-continuous in the first argument and lower semi-continuous in the second. This means:

a) P_i n-polytopes, $P_i \to K (i \to \infty)$, $G \in \cap_{i=1}^{\infty}\Gamma_0(P_i) \cap \Gamma_0(K)$
$\Rightarrow f(K,G) \leq \liminf_{i\to\infty} f(P_i,G)$.

b) G_i, $G \in \Gamma_0(K)$, $G_i \to G (i \to \infty) \Rightarrow f(K,G) \leq \liminf_{i\to\infty} f(K,G_i)$.

A.4.2 Implications

We henceforth assume $G \in \Gamma_0(K)$.

1. $0 < g(K,G) \leq f(K,G) \leq \infty$. The second inequality follows roughly from Corollary 11 in H. Federer [1], p. 176, if we consider that $\mathbf{\Pi}(K,G) = \pi_G(\mathbf{\Sigma}(K,G))$ and Lip $\pi_G \leq 1$, where Lip π_G denotes the Lipschitz constant of the projection mapping π_G.

2. As a lower semi-continuous function in the second argument, $f(K, G)$ is $(\Gamma_0(K), \mathfrak{B}, \gamma_0)$-measurable and thus $\alpha(K) = \int_{\Gamma_0(K)} f(K, G)\mathrm{d}\gamma_0$ exists.

3. $0 < \beta(K) \leq \alpha(K) \leq \infty$. This follows from the monotonicity of the integral.

4. $\alpha(K)$ is a polygonally lower semi continuous function of K.

   Proof. P_i n-polytopes, $P_i \to K$ $(i \to \infty)$, $G \in \Gamma_0 := \cap_{i=1}^{\infty}\Gamma_0(P_i) \cap \Gamma_0(K)$. Γ_0 is a G_δ set relative to $\Gamma_0(K)$ and $\gamma_0(\Gamma_0) = 1$.

$$\begin{aligned}\alpha(K) &= \int_{\Gamma_0(K)} f(K, G)\mathrm{d}\gamma_0 = \int_{\Gamma_0} f(K, G)\mathrm{d}\gamma_0 \\ &\leq \int_{\Gamma_0)} \liminf_{i\to\infty} f(P_i, G)\mathrm{d}\gamma_0 \leq \liminf_{i\to\infty} \int_{\Gamma_0} f(P_i, G)\mathrm{d}\gamma_0 \\ &= \liminf_{i\to\infty} \int_{\Gamma_0(P_i)\cap\Gamma_0(K)} f(P_i, G)\mathrm{d}\gamma_0 = \liminf_{i\to\infty} \alpha(P_i).\end{aligned}$$

   Each step is due to the following respectively: $\gamma_0(\Gamma_0(K) \setminus \Gamma_0) = 0$, the polyhedral lower semi-continuity of f in the first argument and the monotonicity of the integral, Fatou's Lemma and finally to $\gamma_0[(\Gamma_0(P_i) \cap \Gamma_0(K)) \setminus \Gamma_0] = 0$. □

5. $q(K) = \alpha(K)/\beta(K)$ is a polyhedral lower semi continuous function of K since $q(K)$ is the quotient of a polyhedral lower semi-continuous and a continuous function.

From these observations we immediately get the solution to McMullen's Problem: Choose a sequence of n-polytopes P_i, with $P_i \to K$ $(i \to \infty) \Rightarrow$

$$q(K) \leq \liminf_{i\to\infty} q(P_i) = \frac{n\,\omega_n}{\pi\,\omega_{n-1}}.$$

A.5 Shadow boundary convergence

A.5.1 Convergence of shadow boundaries approximations

Let $K \subset \mathbb{E}^n$ be a convex body, P_i are n-polytopes with $P_i \to K$ $(i \to \infty)$, furthermore $o \in \text{int}\, K \cap (\cap_{i=1}^{\infty}\text{int}\, P_i)$; $G \in \Gamma_0$. In this section we will show that the shadow boundaries on P_i in direction G converge to the shadow boundaries on K in direction G.

We may take $G = \text{lin}\{e_n\}$ and $F := \text{lin}\{e_1, \dots, e_{n-1}\}$ (by the notation $\text{lin}\, A$ we understand the linear hull of the set A). Choose an $(n-1)$-simplex $\Delta \subset F$, with $o \in \text{relint}\, \Delta$. Without loss of generality, we may assume $\Delta \subset \cap_{i=1}^{\infty} P_i \cap K$ and let $S := \text{relbd}\, \Delta$.

For $x \in S$ and $i \in \mathbb{N}$, set:

$$p(x) := \text{pos}\{x\} \cap \mathbf{\Pi}(K, G) \quad \text{and} \quad p_i(x) := \text{pos}\{x\} \cap \mathbf{\Pi}(P_i, G).$$

where $\text{pos}A$ denotes the positive hull of the set A. $\pi_G|_{\mathbf{\Sigma}(K,G)}$ is the restriction of π_G to $\mathbf{\Sigma}(K, G)$. $\pi_G|_{\mathbf{\Sigma}(K,G)}$ is a homeomorphism between $\mathbf{\Sigma}(K, G)$ and $\mathbf{\Pi}(K, G)$. $\sigma(K, G) : \mathbf{\Pi}(K, G) \to \mathbf{\Sigma}(K, G)$, $\sigma(K, G) := (\pi_G|_{\mathbf{\Sigma}(K,G)})^{-1}$; $\sigma(P_i, G)$ is defined similarly.

Define a mapping $\varphi : S \to \mathbb{E}^n$ and for each $i \in \mathbb{N}$ a mapping $\varphi_i : S \to \mathbb{E}^n$ as follows:

$$\varphi(x) := \sigma(K, G)[p(x)], \quad \varphi_i(x) := \sigma(P_i, G)[p_i(x)]$$

φ, φ_i are homeomorphisms between S and S and $\mathbf{\Sigma}(P_i, G)$ respectively. Using some elementary convergence properties we can easily establish:

Lemma A.5.1. $\varphi_i \to \varphi\, (i \to \infty)$ *uniformly on* S.

A.5.2 Convergence of shadow boundaries on linear approximations

Take $G, G_i \in \Gamma_0(K), G_i \to G\, (i \to \infty)$. In this case we prove that the shadow boundaries of K in direction G_i converge to the shadow boundary of K in direction G.

G, F, S and the mappings p, φ and $\sigma(K, G)$ are as in Section A.5.1. F_i is the hyperplane orthogonal to G_i through o. $(\delta_i)_{i \in \mathbb{N}}$ is a sequence of proper rotations of $\mathbb{E}^n$ about o, with $\delta_i(F) = F_i$ for all $i \in \mathbb{N}$ which converge to the identity on $\mathbb{E}^n$. (For example take δ_i to be a rotation around o such that in the 2-dimensional plane defined by G and G_i, the angle of rotation is equal to the angle between between G and G_i). $S_i := \delta_i(S)$.

For $x \in S_i$ and all $i \in \mathbb{N}$, we define $q_i(x) := \text{pos}\{x\} \cap \mathbf{\Pi}(K, G_i)$ and $\psi_i : S \to \mathbb{E}^n$ by $\psi_i(x) := \sigma(K, G_i)[q_i(\delta_i(x))]$. The definition of $\sigma(K, G_i)$ is clear from the definition of $\sigma(K, G)$. ψ_i is a homeomorphism between S and $\mathbf{\Sigma}(K, G_i)$.

Lemma A.5.2. $\psi_i \to \varphi\,(i \to \infty)$ *uniformly on* S.

Proof. Instead of a complete analysis of convergence considerations, we will provide an outline of a proof:

Take $x \in S$ at random; without loss of generality, we may assume $\psi_i(x) \to y\,(i \to \infty)$, $y \in \text{bd}K$; as $G_i \to G\,(i \to \infty)$ and because $(\psi_i(x) + G_i) \cap \text{int}K = \emptyset$, it follows that $(y + G) \cap \text{int}K = \emptyset \Rightarrow y \in \mathbf{\Sigma}(K, G)$. Finally, show simultaneously $y = \varphi(x)$. Uniform convergence is achieved indirectly. □

A.6 Lebesgue area and Hausdorff measure

A.6.1 The concept of Lebesgue area

$k, n \in \mathbb{N}, k \leq n$; $I \subset \mathbb{E}^n$ denotes a fixed compact, polyhedral k-cell (not necessarily convex).

$$C_{k,n}(I) := \{f : I \to \mathbb{E}^n \mid f \text{ continuous}\}$$

Definition 44. *$p \in C_{k,n}(I)$ is called polyhedral if I can be triangulated such that p maps each simplex of the triangulation barycentrically (affinely) onto a 'linearly equivalent' simplex in $\mathbb{E}^n$.*

Each 'linearly equivalent' simplex in $\mathbb{E}^n$ has an elementary k-dimensional measure. That is its k-dimensional Hausdorff measure can be calculated from the coordinates of its vertices using appropriate determinants. Applying this to polyhedral functions, all sensible definitions of the area give the same result: every simplex of a given triangulation of I will be assigned the elementary k-dimensional measure of its image in $\mathbb{E}^n$, then these values will be added over all simplices of the triangulation. This results in $L_k^n(p;I)$.

In the following we shall lay out the relevant definitions and lemmas. Additionally, we refer the reader to H. Federer [2], pp. 90-94 and S. Saks [6], pp. 164-165.

The class of polyhedral functions lies densely in $C_{k,n}(I)$ relative to the topology of uniform convergence. Now let $f \in C_{k,n}(I)$ be arbitrary.

Definition 45. $L_k^n(f;I) := \inf\{\liminf_{i\to\infty} L_k^n(p_i;I) \,|\, (p_i)_{i\in\mathbb{N}}$ *a sequence of polyhedral functions in* $C_{k,n}(I)$*, with* $p_i \to f\ (i \to \infty)$ *uniformly on* $I\}$. $L_k^n(f;I)$ *is the* k*-dimensional Lebesgue area of* f *over* I.

Lemma A.6.1. $f, f_i \in C_{k,n}(I), i \in \mathbb{N}, f_i \to f\ (i \to \infty)$ *uniformly on* I

$\Rightarrow L_k^n(f;I) \le \liminf_{i\to\infty} L_k^n(f_i;I)$.

This means that the Lebesgue area of uniform approximations in $C_{k,n}(I)$ is lower semicontinuous.

Proof. See extract from S. Saks [6] mentioned above. □

A.6.2 Connection with the Hausdorff measure

$f \in C_{k,n}(I); y \in \mathbb{E}^n; N(f;y)$ denotes the number of elements (may be ∞) of the set $\{x \in I | f(x) = y\}$; $N(f;y)$ is the multiplicity function of y in f.

Definition 46.

$$N_k^{*n}(f) := \int_{\mathbb{E}^n} N(f;y)\, d\lambda_k^n(y).$$

$N_k^{*n}(f)$ *is the* k*-dimensional Hausdorff area of* f.

Lemma A.6.2. *Each of the following statements imply*

$$N_k^{*n}(f) = L_k^n(f).$$

i) f *is a Lipschitz mapping,*

ii) $k = 2$.

Proof. See the reference to H. Federer [2] mentioned above. □

A.6.3 Non parametric surfaces

$f : \mathbb{E}^n \to \mathbb{R}$ is fexed; define a function $\bar{f} : \mathbb{E}^n \to \mathbb{E}^{n+1}$, by $\bar{f}(x) = (x_1, \ldots, x_n, f(x))$ for $x = (x_1, \ldots, x_n) \in \mathbb{E}^n$. $W \subset \mathbb{E}^n$ is finitely rectilinearly triangulable. If W is an n-complex, $\bar{f}$ defines an n-dimensional non parametric surface over W and it follows that:

Lemma A.6.3.

$$L_n^{n+1}(\bar{f}; W) = \lambda_n^{n+1}(\bar{f}(W))$$

Proof. This is the main result in one of H. Federer's works [4]. □

A.6.4 Core measure theoretic problem

From the preliminaries above it is possible to reduce the main conjecture (see Section A.4) to a purely measure theoretic problem.

$K \subset \mathbb{E}^n$ is a convex body with $o \in \text{int}K$; $G \in \Gamma_0(K)$. In addition, the assumptions and notation from Section A.5.1 apply.

$$\text{KEY QUESTION: Does it follow that } \lambda_{n-2}^n(\varphi(S)) = L_{n-2}^n(\varphi; S)? \tag{A.1}$$

The accuracy of (A.1), and Lemma A.6.1 would validate part $a)$ of the main conjecture. For part $b)$ we will argue using the notation from section A.5.2 and, amongst others, the application of (A.1), Lemma A.6.1 and the "movement invariance" of the Lebesgue area as follows:

$$\begin{aligned}
\lambda_{n-2}^n(\varphi(S)) &= L_{n-2}^n(\varphi; s) \leq \liminf_{i\to\infty} L_{n-2}^n(\psi_i; S) \\
&= \liminf_{i\to\infty} L_{n-2}^n(\psi_i \circ (\delta_i|_s)^{-1}; \delta_i(S)) \\
&= \liminf_{i\to\infty} L_{n-2}^n(\sigma(K, G_i) \circ q_i; \delta_i(S)) \\
&= \liminf_{i\to\infty} \lambda_{n-2}^n(\psi_i(S))
\end{aligned}$$

with which part $b)$ of the main conjecture is established.

A.6.5 Two helpful statements

The same notation as in section A.6.1 holds. $S_j, j \in \{1, \ldots, n\}$ stands for an $(n-2)$-face of Δ. Then:

Lemma A.6.4.

$$\lambda_{n-2}^n(\varphi(S)) = \sum_{j=1}^n \lambda_{n-2}^n(\varphi(S_j)).$$

Proof.

$$\begin{aligned}\lambda_{n-2}^n(\varphi(S)) &= \lambda_{n-2}^n(\varphi(S_1 \cup \cdots \cup S_n)) \\ &= \lambda_{n-2}^n(\varphi(S_1) \cup \cdots \cup \varphi(S_n)) \\ &= \sum_{j=1}^n \lambda_{n-2}^n(\varphi(S_j)).\end{aligned}$$

The last inequality is based on the fact that, for $j \neq k$, we have $\lambda_{n-2}^n(\varphi(S_j) \cap \varphi(S_k)) = 0$. Set $U := S_j \cap S_k$; U is an $(n-3)$-face of Δ. With the injectivity of φ and the embedding invariance of the Hausdorff measure, it follows:

$$\lambda_{n-2}^n(\varphi(S_j) \cap \varphi(S_k)) = \lambda_{n-2}^n(\varphi(S_j \cap S_k)) = \lambda_{n-2}^n(\varphi(U)) = \lambda_{n-2}^{n-1}(\varphi(U)).$$

Intermediate observations: For $x \in U$ we decompose $\varphi(x)$ into two components, namely $\varphi(x) := (p(x), h(x))$. $p(x)$ is the component in the $(n-2)$-plane which goes through U and o; $h(x)$ the component in the direction $\text{lin}\{e_n\}$. Define $\delta : U \to \mathbb{E}^N$ as follows $\delta(x) := (x, h(x))$ and $\tau : \delta(U) \to \varphi(U)$ as $\tau(\delta(x)) := \varphi(x)$. τ is a Lipschitz mapping (since p naturally has the Lipschitz property). Using Corollary 11 in H. Federer [1], p176, the embedding invariance of the Hausdorff measure and induction from Statement 2 in I. P. Natanson [26], p332, we get:

$$\begin{aligned}\lambda_{n-2}^{n-1}(\varphi(U)) &= \lambda_{n-2}^{n-1}(\tau(\delta(U)))) \leq (\text{Lip } \tau)^{n-2}\lambda_{n-2}^{n-1}(\delta(U)) \\ &= (\text{Lip } \tau)^{n-2}\lambda_{n-2}^{n-2}(\delta(U)) = (\text{Lip } \tau)^{n-2}L^{n-2}(\delta(U)) = 0\end{aligned}$$

with which the proof of Lemma A.6.4 is complete. L^{n-2} denotes the $(n-2)$-dimensional Lebesgue measure. □

Observe now the analog construction to the intermediate study with T instead of U, where T is an $(n-2)$-face of Δ. It follows:

Lemma A.6.5.

$$L^{n-1}_{n-2}(\delta;T) = L^{n}_{n-2}(\delta;T).$$

Proof. Set $\mathbb{E}^{n-1} :=$ aff $T+$ lin $\{e_n\}$, where $+$ denotes Minkowski addition; $\mathbb{E}^{n-1}$ is an $(n-1)$-dimensional surface in $\mathbb{E}^n$. T is an $(n-2)$-simplex, $\delta : T \to \mathbb{E}^{n-1}$. Finally, π denotes the orthogonal projection of $\mathbb{E}^n$ onto $\mathbb{E}^{n-1}$. $L^{n}_{n-2}(\delta;T) \leq L^{n-1}_{n-2}(\delta;T)$ is trivial. $L^{n-1}_{n-2}(\delta;T) \geq L^{n}_{n-2}(\delta;T)$ comes from the fundamental fact that the orthogonal projection will at most reduce elementary volumes. □

A.7 An additivity property of Lebesgue area

A.7.1 Smoothness of continuous functions

P is an n-polytope in $\mathbb{E}^n$; $f : P \to \mathbb{R}$ is continuous; for $x \in P$ we define $\bar{f} : P \to \mathbb{E}^{n+1}$ as $\bar{f}(x) := (x, f(x))$. $g : \mathbb{E}^n \to \mathbb{R}$ is an extension of f; $\bar{g}(x) := (x, g(x))$ for $x \in \mathbb{E}^n$; $\bar{g} : \mathbb{E}^n \to \mathbb{E}^{n+1}$. For the natural number i write $B^n_{1/i} := \{y \in \mathbb{E}^n \,|\, ||y|| < 1/i\}$. $g_i : \mathbb{E}^n \to \mathbb{R}$ is given by $g_i(x) := (i^n/\omega_n) \int_{B^n_{1/i}} g(x+z)\, \mathrm{d}L^n(z)$, where L^n is the Lebesgue measure over $\mathbb{E}^n$. For $x \in \mathbb{E}^n$, set $\bar{g}_i(x) := (x, g_i(x))$; $\bar{g}_i : \mathbb{E}^n \to \mathbb{E}^{n+1}$.

Lemma A.7.1. *For all $i \in \mathbb{N}$, g_i is a Lipschitz mapping over all compact subsets $B \subset \mathbb{E}^n$, in particular over P.*

Proof. We give a proof only for P, as we are only studying g_i over the compact subset $B \supset P_1$, where P_1 is the outer parallel body of P at distance 1. $x, y \in P$ are chosen at random; set $A(x,y) := |g_i(x) - g_i(y)|/||x-y||$; it suffices to show that $A(x,y) \leq c$, where c is a constant independent of x and y. We may assume that x and y are close to each other, say $||x-y|| < 2/i$. Using an elementary approximation, it follows easily that:

$$A(x,y) \leq (2i \max |g_{|B}|\, \omega_{n-1}/\omega_n) := c$$

□

Lemma A.7.2. *a) $g_i \to g\,(i \to \infty)$ uniformly on P and also $g_i \to f\,(i \to \infty)$ uniformly on P.*

b) $L_n^{n+1}(\bar{g}_i; P) \to L_n^{n+1}(\bar{g}; P) = L_n^{n+1}(\bar{f}; P)\,(i \to \infty)$.

Proof. a) comes from the uniform continuity of g on P_1.
b) is Theorem 3.8 in R. N. Tompson [27] p398. □

A further application of the method in Lemma A.7.1 on a Lipschitz mapping gives, as we will see in what follows, a C^1 function.

For $p \in \mathbb{E}^n$, $\rho > 0$ and an L^n-integrable function $f : \mathbb{E}^n \to \mathbb{R}$ we set $B_\rho^n(p) := \{q \in \mathbb{E}^n |\, ||q - p|| \leq \rho\}$ and $f_\rho : \mathbb{E}^n \to \mathbb{R}$ with $f_\rho(p) := \int_{B_\rho^n(p)} f(x)\, \mathrm{d}L^n(x)$.

Lemma A.7.3. *$f : \mathbb{E}^n \to \mathbb{R}$ is a Lipschitz function $\Rightarrow$ f_ρ is continuously differentiable for all $\rho > 0$.*

Proof. This is a well known property of the mean function f_ρ (compare definition 6.15 and Remark 6.16 in H. Federer [3]). □

A.7.2 Helpful fact from Alpert-Toralballa

$P \subset \mathbb{E}^n$ is an n-polytope; $f : P \to \mathbb{R}, f \in C^1; \bar{f} : P \to \mathbb{E}^{n+1}$ with $\bar{f}(x) := (x, f(x))$ for all $x \in P$.

Lemma A.7.4. *Under the conditions stated above there is a unique sequence of piecewise linear simplicial non-parametric functions $\bar{h}_i$ inscribed in $\bar{f}(P)$ with*

$$L_n^{n+1}(\bar{h}_i; P) \to L_n^{n+1}(\bar{f}; P)\,(i \to \infty)$$

Proof. Follows directly from Theorems 1) and 2) in L. I. Alpert- L. V. Toralballa [29], where the idea of "unique sequence" is defined. □

For our purposes, we still need to assert that the $\bar{h}_i$s converge uniformly to $\bar{f}\,(i \to \infty)$ over P. To construct $\bar{h}_i$, L. I. Alpert and L. V. Toralballa did as follows: $(\mathscr{C}_i)_{i\in\mathbb{N}}$ is a sequence of finite simplicial components of P with mesh size $1/i$; the simplices involved will then be called “right-angled” (definition in [29]). The existence of such components for $n > 3$ is, however, an open question. Lemma 6.12 in [28] with some slight changes from [29] can help towards this point. $h_i : P \to \mathbb{R}$ is the piecewise linear function belonging to $\mathscr{C}_i$, that is: $h_i(p) = f(p)$ for $p \in \Delta^0(\mathscr{C}_i)$ and is affine in between. $\Delta^0(\mathscr{C}_i)$ denotes the vertex set of $\mathscr{C}_i$. For $x \in P$, we define $\bar{h}_i : P \to \mathbb{E}^{n+1}$ by $\bar{h}_i(x) := (x, h_i(x))$. First, it follows easily:

Lemma A.7.5. $h_i \to f, \bar{h}_i \to \bar{f}\,(i \to \infty)$ *uniformly on* P *and the* $\bar{h}_i$*s are piecewise linear and non-parametric*

Let us summarise the results of the current section: $P \subset \mathbb{E}^n$ is an n-polytope, $f : P \to \mathbb{R}$ is continuous; for any function $r : P \to \mathbb{R}$, we define $\bar{r} : P \to \mathbb{E}^{n+1}$ by $\bar{r} := (x, r(x))$ for all $x \in P$. Then we have

Lemma A.7.6. *For the definition of the Lebesgue area of non-parametric functions* $\bar{f}$, *one can limit oneself to non-parametric approximating polyhedric functions* $\bar{p}_v$, *that is: there exists a sequence of piecewise linear functions* $p_v : P \to \mathbb{R}, v \in \mathbb{N}$, *with the following properties:*

1. $\bar{p}_v \to \bar{f}\,(v \to \infty)$ *uniformly on* P,
2. $L_n^{n+1}(\bar{p}_v; P) \to L_n^{n+1}(\bar{f}; P)\,(v \to \infty)$.

Proof. Use Lemmas A.7.1 to A.7.5. □

A.7.3 Additivity statement

In the following we will show that under subdivision of the domain of definition, the Lebesgue area behaves additively. Similar results have until now only been known for a few special cases (see J. Serrin [30], p440). In the context of our work, this additivity property has great significance.

$P \subset \mathbb{E}^n$ is an n-polytope; for $X \subset \text{bd}\, P$, $\mathscr{F}(X)$ is the space of all continuous functions $f : X \to \mathbb{R}$ with the supremum norm $|| \cdot ||_X$. If X is a polyhedron, we denote by $\mathscr{F}_0(X) \subset \mathscr{F}(X)$ the subspace of all existing piecewise linear functions. For $f \in \mathscr{F}_0(X)$, denote by $I(F)$ the $(n-1)$-dimensional elementary geometric content of the graphs $\Gamma(f) \subset \mathbb{E}^{n+1}$ of f. Now, $f \in \mathscr{F}(X)$ and $\varphi : \mathbb{N} \to \mathscr{F}_0(X)$ is a sequence in $\mathscr{F}_0(X)$, with $\text{lin}_{i\to\infty}||\varphi(i) - f|| = 0$. Set $\lambda(\varphi, f) := \liminf_{i\to\infty} I(\varphi(i))$; then we define

$$\mathscr{L}_X(f) := \inf\{\lambda(\varphi, f)|\, \varphi \text{ is a sequence in } \mathscr{F}_0(X), \text{ with } \lim_{i\to\infty} ||\varphi(i) - f|| = 0\}$$

Statement 3. *X_1 and X_2 are proper polyhedra in* bd$P$*, with* $\text{int}(X_1 \cap X_2) = \emptyset$*;*
$f : X_1 \cup X_2 \to \mathbb{R}$ is continuous. Then

$$\mathscr{L}_{X_1}(f|_{X_1}) + \mathscr{L}_{X_2}(f|_{X_2}) = \mathscr{L}_{X_1\cup X_2}(f).$$

Here, int$Z$ means the interior of $Z \subset \text{bd}P$ relative to the topology on bd$P$. The statement above, in conjunction with Lemmas A.7.6 and A.6.5, directly produces the following:

Corollary 3. *If X_1 and X_2 are facets of P (i.e. $(n-1)$-faces) then we also have:*

$$L^{n+1}_{n-1}(\bar{f}; X_1) + L^{n+1}_{n-1}(\bar{f}; X_2) = L^{n+1}_{n-1}(\bar{f}; X_1 \cup X_2)$$

Remark 1. *A straightforward inductive argument shows that statement 7.1 as well as the corollary are valid for any finite number of polyhedric subsets in* bd$P$*, respectively, facets of P.*

Proof. of Statement 1.
a) Trivially, we have : $\mathscr{L}_{X_1}(f|_{X_1}) + \mathscr{L}_{X_2}(f|_{X_2}) \leq \mathscr{L}_{X_1\cup X_2}(f)$.
b) Conversely: If one of the values on the left side is infinite we are done. So from now on we can assume that $\mathscr{L}_{X_1}(f|_{X_1})$ and $\mathscr{L}_{X_2}(f|_{X_2})$ are finite.

If our statement is false, there is $f \in \mathscr{F}(X_1 \cup X_2)$ and $\varepsilon > 0$ in $\mathbb{R}$ such that:

$$\mathscr{L}_{X_1}(f|_{X_1}) + \mathscr{L}_{X_2}(f|_{X_2}) + 3\varepsilon < \mathscr{L}_{X_1\cup X_2}(f). \tag{A.2}$$

Let $\delta > 0$ is a real number with the property:

$$2\delta\, \lambda^n_{n-2}(X_1 \cap X_2) < \varepsilon. \tag{A.3}$$

For $i \in \{1, 2\}$, we set $f_i := f|_{X_i}$; $g_i \in \mathscr{F}_0(X_i)$ is chosen such that $\|g_i - f_i\|_{X_I} \leq \delta/4$ and $I(g_i) \leq \mathscr{L}_{X_I}(f_i) + \varepsilon$. We choose $\alpha > 0$ so that all $h \in \{f, g_1, g_2\}$ and all p, q in the area of definition of h, with $\|p - q\| \leq \alpha$, give: $|h(p) - h(q)| \leq \delta/4$. $\|p - q\|$ is the euclidian distance between p and q.

Choose a simplicial decomposition $\mathscr{C}$ of $X_1 \cup X_2$ so that X_i is triangulated by a subcomplex $\mathscr{C}_i$ of $\mathscr{C}$, g_i is affine over each simplex of $\mathscr{C}_i$ and the diameter of each simplex of $\mathscr{C}$ is at most equal to α. $\mathscr{C}_0 := \mathscr{C}_1 \cap \mathscr{C}_2$ is then a triangulation of $X_0 := X_1 \cap X_2$.

Here $<$ is a linear ordering of the vertex set $\Delta^0(\mathscr{C})$ of $\mathscr{C}$. $\mathfrak{D}$ denotes the set of all $(n-1)$-simplices $x \in \Delta^{n-1}(\mathscr{C})$, for which $x \cap X_0 \neq \phi$. $\Delta^{n-1}(\mathscr{C})$ stands for the set of all $(n-1)$-simplices of $\mathscr{C}$. Eventually after further subdivisions of $\mathscr{C}$, one can assume that for all $x \in \mathfrak{D}, x \cap X_0$ is an edge $k(x)$ of x. At first, $x \cap X_0$ could be the union of multiple edges of various dimensions in x.

For $x \in \mathfrak{D}, \ell(x) :=$ conv$(\Delta^0(x) \backslash k(x))$, where conv$A$ denotes the convex hull of the set A. For $0 < \beta < 1$ we set

$$U(x, \beta) := U\{p + \beta(x - p) | p \in k(x)\}.$$

$m(x)$ is the convex hull of $\ell(x) \cup \{n(x)\}$, where $n(x)$ is the centre of gravity of $k(x)$.

Here $<_1$ is the ordering of the vertex set $\Delta^0(m(x))$ of $k(x)$ induced by $<$; $<_2$ it the linear ordering of $\Delta^0(m(x))$ in which $n(x)$ precedes any other vertex and the ordering of $\Delta^0(\ell(x))$ is induced by $<$.

$\mathscr{C}'(x)$ is the natural simplicial decomposition of $k(x) \times m(x)$ with respect to $<_1$ and $<_2$ (compare S. Eilenberg - N. Steenrod [31], p.66-67). $\varphi(x, \beta)$ is the combinatorial isomorphism from $k(x) \times m(x)$ onto $U(x, \beta)$, by which $(p, n(x))$ is mapped onto p $(p \in \Delta^0(k(x)))$ and (p, q) is mapped onto the vertex of $U(x, \beta)$ on the segment $[p, q]$, where $p \in \Delta^0(k(x))$ and $q \in \Delta^0(l(x))$.

$\mathscr{C}'(x,\beta)$ is the simplical decomposition of $U(x,\beta)$ isomorphic to $\mathscr{C}'(x)$ conveyed by $\varphi(x,\beta)$. Set $U(\beta) := \cup\{U(x,\beta)|x \in \mathfrak{D}\}$ and $\mathscr{C}'(\beta) := \cup\{\mathscr{C}'(x,\beta)|x \in \mathfrak{D}\}$. Now we immediately have:

$$\begin{aligned}
&U(\beta) \text{ is a neighbourhood of } X_0 \text{ in } X_1 \cup X_2, \\
&\mathscr{C}'(\beta) \text{ is a simplical decomposition of } U(\beta). \qquad \text{(A.4)}
\end{aligned}$$

We define a piecewise linear mapping $g(\beta) : X_1 \cup X_2 \to \mathbb{R}$: for $p \in \text{cl}[(X_1 \cup X_2)\backslash U(\beta)]$ we set $g(\beta)[p] := (g_1 \cup g_2)[p]$; for $p \in \Delta^0(\mathscr{C}_0)$, we have $g(\beta)[p] := \frac{1}{2}(g_1(p) + g_2(p))$ and $g(\beta)$ is defined by linear interpolation over the simplices of $\mathscr{C}'(\beta)$. We want to show, there exists $\beta > 0$ s.t.

$$I(g(\beta)) \quad \leq \mathscr{L}_{X_1}(f|_{X_1}) + \mathscr{L}_{X_2}(f|_{X_2}) + 3\varepsilon \qquad \text{(A.5)}$$

$$\text{and} \qquad \|g(\beta) - f\|_{X_1 \cup X_2} \leq \delta \quad \text{hold.} \qquad \text{(A.6)}$$

If this holds for all $\delta > 0$ which satisfies equation (A.3), then since (A.5) and (A.6) contradict (A.2), our statement is proved.

First, we take care of the proof of (A.6). p is a point from $X_1 \cup X_2$, say $p \in X_1$. To begin with, take p not belonging to $U(\beta)$ then, from our assumptions, $g(\beta)[p] = g_1(p)$ and $|g(\beta)[p] - f(p)| = |g_1(p) - f(p)| \leq \delta/4$. No p is a vertex of a simplex of $\mathscr{C}'(\beta)$. In the case $p \notin \mathscr{C}_0 \subset \mathscr{C}'(\beta)$ we again have $g(\beta)[p] = \frac{1}{2}(g_1(p) + g_2(p))$ hence

$$|g(\beta)[p] - f(p)| \leq \frac{1}{2}[|g_1(p) - f(p)| + |g_2(p) - f(p)] \leq \delta/4.$$

Now p is a random point in a simplex x of $\mathscr{C}'(\beta)$. As the diameter of x is at most α, for each vertex q from x, $|f(p) - f(q)| \leq \delta/4$ holds. From the observations above, $|f(q) - g(\beta)[p]| \leq \delta/4$ also holds for all $q \in \Delta^0(x)$. $q_1, \ldots, q_k$ are the vertices of x. There are non negative numbers α_i $(i = 1, \ldots, k)$ with $\sum_{i=1}^k \alpha_i = 1$ and $p = \sum_{i=1}^k \alpha_i q_i$. From this:

$$
\begin{aligned}
|g(\beta)[p] - f(p)| &= \Big|\sum_{i=1}^{k} \alpha_i \,[g(\beta)[q_i] - f(q_i)] + \sum_{i=1}^{k} \alpha_i \,[f(q_i) - f(p)]\Big| \\
&\leq \sum_{i=1}^{k} \alpha_i \,\delta/4 + \sum_{i=1}^{k} \alpha_i \,\delta/4 = \delta/2,
\end{aligned}
$$

and (A.6) follows.

It remains to show (A.5). We have

$$
I(g(\beta)) = I(g_1) + I(g_2) + \sum_1 (I(g(\beta)|_y) - I(g_1|_y) + \sum_2 (I(g(\beta)|_y) - I(g_2|_y)
$$

where $\sum_i$ for $i = \{1, 2\}$ denotes summation over $y \in \Delta^{n-1}\mathscr{C}'(\beta)$, with $y \subset X_i$. To prove (A.5), it suffices to show that for each $x \in \mathfrak{D}, x \subset X_i$ and $y \in \Delta^{n-1}\mathscr{C}'(x)$, there exists a function $h(y) \,:]0, 1[\to \mathbb{R}$, with $\lim_{\beta\to 0} h(y)[\beta] = 0$ such that the following holds:

$$
I(g(\beta)|_{y(\beta)}) - I(g_i|_{y(\beta)}) \leq \frac{\delta}{n-1}\,\lambda^n_{n-2}(k(x)) + h(y)[\beta]. \tag{A.7}
$$

Here $y(\beta) \in \mathscr{C}'(\beta)$ is the simplex from which$\varphi(x, \beta)$ is produced. $I(g(\beta)|_{y(\beta)})$ is simply the elementary geometric content of the $(n-1)$-simplex which lies above $y(\beta)$ in the graph of $g(\beta)$.

To show (A.7), we first look at the case where $k(x)$ is $(n-2)$-dimensional. We assume $i = 1$. $D(\beta)$ is the graph of $g(\beta)|_{y(\beta)}, D_1(\beta)$ is the graph of $g_1|_{y(\beta)}$. $D(\beta)$ and $D_1(\beta)$ are $(n-1)$-simplices. $D_1 := \lim_{\beta\to 0} D_1(\beta)$ is an $(n-2)$-simplex equivalent to the section of the graph of g_1 above $k(x)$. Here, $\lim_{\beta\to 0} I(D_1(\beta)) = I(\lim_{\beta\to 0} D_1(\beta)) = 0$ and to prove (A.7), it suffices to show that:

$$
\lim_{\beta\to 0} I(D(\beta)) \leq \frac{\delta}{n-1}\lambda^n_{n-2}(k(x)), \tag{A.8}
$$

holds.

We can assume that $D := \lim_{\beta\to 0} D(\beta)$ is an $(n-1)$-simplex. $(q_1, \ldots, q_{n-1})$ is an ordering of the vertices of $k(x)$ for which D has the following vertices: $(q_1, g_1(q_1)), \ldots, (q_i, g_1(q_i)), (q_i, \frac{1}{2}(g_1(q_i)+g_2(q_i))), \ldots, (q_{n-1}, \frac{1}{2}(g_1(q_{n-1})+g_2(q_{n-1})))$

for some suitable $i \in \{1, \ldots, n-1\}$. D_0 is the simplex with the vertices $q_1, \ldots, q_i, (q_i, s), q_{i+1}, \ldots, q_{n-1}$ where we have set $s := \frac{1}{2}(g_1(q_i) + g_2(q_i)) - g_1(q_i)$ and $(q_j, 0)$ is identified with q_j for $j \in \{1, \ldots, n-1\}$.

Notice that $|s| = \frac{1}{2}|g_1(q_i) - g_2(q_i)| = \frac{1}{2}|g_1(q_i) - f(q_i) + f(q_i) - g_2(q_i)| \leq \delta/4$ holds. We want to show that D and D_0 have the same $(n-1)$-dimensional content. We assume $k(x)$ lies in $\mathbb{E}^{n-2}$ and regard D, as well as D_0, as a subset of $\mathbb{E}^{n-1}$. Now q is a point in $k(x)$. $\mathbb{E} \subset \mathbb{E}^{n-1}$ is a 2-dimensional plane through q, q_i and $q_i + e_{n-1}$; we assume $q \neq q_i$.

Set $s := \mathbb{E} \cap k(x)$. The endpoints of the line segment s are q_i and a point $r \in \text{conv}\{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_{n-1}\}$. Fix $\alpha \in]0,1[$ so that $q = \alpha q_i + (1-\alpha) r$ holds, assume $q \neq r$. $D \cap \mathbb{E}$ is a triangle with the vertices $(q_i, g_1(q_i)), (q_i, \frac{1}{2}(g_i(q_i) + g_2(q_i))), r'$ where $r' \in D$ is the point with $\pi(r') = r$. $\pi : \mathbb{E}^{n-1} \to \mathbb{E}^{n-2}$ denotes here the orthogonal projection.

$D \cap (q + \mathbb{R}e_{n-1})$ is a line segment of length $\alpha|s|$, where $\alpha = \|q - r\| / \|q_i - r\|$. Similarly, we find that $D_0 \cap (q + \mathbb{R}e_{n-1})$ is a line segment of length $\alpha|s|$. From Fubini's Theorem, we then have:

$$\lambda^n_{n-2}(D) = \lambda^n_{n-2}(D_0) \leq (1/n-1)\lambda^n_{n-2}(k(x))\,\delta/4$$

and (A.7) follows in this case. Notice as well that $\lambda^n_{n-2}(D) = I(D) = I(\lim_{\beta\to 0} D(\beta)) = \lim_{\beta \to 0} I(D(\beta))$ holds, from which (A.8) follows immediately. If $k(x)$ is less than $(n-2)$-dimensional, $\lim_{\beta\to 0} D(\beta)$ and $\lim_{\beta\to 0} D_0(\beta)$ are at most $(n-2)$-dimensional and we conclude:

$$\lim_{\beta\to 0} I(D(\beta)) = I(\lim_{\beta \to 0} D(\beta)) = 0.$$

So (A.8) also holds in this case. Hence Statement A.7.1 is established. □

A.7.4 Lebesgue area and Hausdorff measure on polytopes

P and R are $(n-1)$-polytopes in $\mathbb{E}^{n-1}$, with $\text{int}P \cap \text{int}R \neq \emptyset$, say $o \in \text{int}P \cap \text{int}R$. $p : \text{bd}P \to \text{bd}R$ is the radial projection from o; naturally, p is a Lipschitz mapping.

$s \subset \text{bd}P$ is polyhedric; $Q := p(s); Q \subset \text{bd}R$ is polyhedric. $f : Q \to \mathbb{R}$ is continuous; $\bar{f} : Q \to \mathbb{E}^n$ is defined by $\bar{f}(x) := (x, f(x))$ for $x \in Q$. Then the following holds:

Lemma A.7.7. $L^n_{n-2}(\bar{f}; Q) = L^n_{n-2}(\bar{f} \circ p; s)$.

We will do without the proof of this statement as it is awkward and technical.

From now on, we adopt the conditions from the beginning of section A.6.4. S stands for the relative boundary of Δ relative to F. Furthermore, P is an $(n-1)$-polytope in F with $o \in T := \text{relbd}P$. $\varphi(x) = (p(x), h(x))$, where $p : S \to \mathbf{\Pi}(K, G)$ denotes the radial projection from o and $h(x)$ is the e_n component of $\varphi(x), h : S \to \mathbb{R}$. $\kappa_P : S \to T$ denotes the radial projection from o and $\varphi_P : T \to \mathbb{E}^n$ is defined for all $y \in T$ by $\varphi_P(y) := (y, h(\kappa_P^{-1}(y)))$. We now have:

Lemma A.7.8. $L^n_{n-2}(\varphi_P \circ \kappa_P; S) = \lambda^n_{n-2}((\varphi_P \circ \kappa_P)(S))$

Proof. $P_i \, (i = 1, \ldots, m)$ are the facets ($(n-2)$-faces) of P.

$$\begin{aligned}
\lambda^n_{n-2}((\varphi_P \circ \kappa_P)(S)) &= \lambda^n_{n-2}(\varphi_P(T)) = \lambda^n_{n-2}(\varphi_P(P_1 \cup \cdots \cup P_m)) \\
&= \sum_{i=1}^{m} \lambda^n_{n-2}(\varphi_P(P_i)) = \sum_{i=1}^{m} \lambda^{n-1}_{n-2}(\varphi_P(P_i)) = \sum_{i=1}^{m} L^{n-1}_{n-2}(\varphi_P; P_i) \\
&= \sum_{i=1}^{m} L^n_{n-2}(\varphi_P; P_i) = L^n_{n-2}(\varphi_P; T) = L^n_{n-2}(\varphi_P \circ \kappa_P; S).
\end{aligned}$$

The non trivial implications come from the proof of Lemma A.6.4 with φ_P instead of δ; the imbedding invariance of the Hausdorff measure; Lemma A.6.3, with $n-2$ instead of n; Lemma 6.5 with φ_P instead of S; the Corollary and Remark to Statement A.7.1 with n instead of $n+1$ and finally, from Lemma A.7.7 with several substitutions. $\square$

A.8 Smooth polyhedral approximation

The aim of this section is, based on Lemma A.7.8, by approximating a convex body K by polyhedra, to find a positive answer to our key question (A.1). In addition though, we must assume that K is smooth which means that each point on the boundary of K belongs to a distinct supporting hyperplane.

A.8.1 Approximation statement

The following is crucial to the subsequent work.

Statement 4. *$K \subset \mathbb{E}^n$ is a smooth convex body with $o \in \text{int}K$. $(P_i)_{i\in\mathbb{N}}$ denotes a sequence of n-polytopes with $P_i \supset K$ and $P_i \to K$ $(i \to \infty)$. $\pi_i : \text{bd}K \to \text{bd}P_i$ denotes the radial projection from o; $\varepsilon > 0$. Then there exists $i_0 \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ with $i \geq i_0$, the following holds*

$$1 - \varepsilon \leq \text{ Lip } (\pi_i, \pi_i^{-1}) \leq 1 + \varepsilon.$$

Proof. The last line in the statement above says that the Lipschitz constant of π_i as well as that of π_i^{-1} lie within the given bounds. We will only sketch here the proof for the right hand side; the rest follows similarly.

Contrapositive:

There is $\varepsilon > 0$ such that for all $i_0 \in \mathbb{N}$ there exists an $i \geq i_0$ with Lip $\pi_i > 1 + \varepsilon$.

Possibly by choosing a subsequence of $(P_i)_{iin\mathbb{N}}$, we may assume Lip $\pi_i > 1 + \varepsilon$ for all $i \in \mathbb{N}$. Therefore, there exist $x_i, y_i \in \text{bd}K, x_i \neq y_i$ for all i with

$$\frac{\|\pi_i(x_i) - \pi_i(y_i)\|}{\|x_i - y_i\|} > 1 + \varepsilon. \tag{A.9}$$

$G_i := \text{aff}\{x_i, y_i\}; H_i := \text{aff}\{\pi_i(x_i), \pi_i(y_i)\}; E_i := \text{aff}\{o, G_i\} = \text{aff}\{o, H_i\}$. Without loss of generality, we can assume:

$$x_i \to p, y_i \to q \, (i \to \infty), p, q \in \text{bd}K.$$

Case 1: $p \neq q$.

Trivially, it holds that: $\pi_i(x_i) \to p$ and $\pi_i(y_i) \to q \, (i \to \infty)$ so

$$\lim_{i\to\infty} \frac{\|\pi_i(x_i) - \pi_i(y_i)\|}{\|x_i - y_i\|} = \frac{\|p - q\|}{\|p - q\|} = 1.$$

contradicting (A.9).

Case 2: $p = q$.

Possibly after choosing a subsequence, we can assume $G_i \to G, H_i \to H, E_i \to$

$E\,(i \to \infty)$, where G, H are straight lines and $E := \text{aff}\{o, G\}$.

Properties of G and H

1) $p \in G \cap H$, trivially.

2) $G \subset T_pK$, where T_pK is the $(n-1)$-dimensional plane tangent to K at p. Otherwise, G cuts a chord from K of length $g\,(g > 0)$ which implies one can find $i_0 \in \mathbb{N}$ such that G_i, for $i \geq i_0$ cuts a chord from K of length $g_i \geq g/2$. Now, $g_i = \|x_i - y_i\|$ which contradicts the fact that $(\|x_i - y_i\|)_{i \in \mathbb{N}}$ converges to 0.

3) $H \subset T_pK$. Otherwise, H cuts a chord $\text{conv}\{p, q\}$ from $K, q \in \text{bd}K$, of positive length ($H \cap$ int$K \neq \emptyset$). For all $\delta > 0$ there exists $i_0(\delta) \in \mathbb{N}$ so that for $i \geq i_0, i \in \mathbb{N}$ we have $H_i \cap$ bd$K = \{p_i, q_i\}$, where $p_i \in U_\delta(p), q_i \in U_\delta(q)$. Here $U_\delta(p)$ denotes the open δ-ball at p. Look at all the chords of K with an endpoint in $U_\delta(p)$ and $U_\delta(q)$. ℓ_δ is the infimum of their length. As δ decreases ℓ_δ grows and for δ small enough $\ell_\delta > 0$. If $\pi_i(x_i) \to p, \pi_i(y_i) \to p\,(i \to \infty)$ there exists $j_0(\delta) \in \mathbb{N}$ such that for $i \geq j_0, \|\pi_i(x_i) - \pi_i(y_i)\| < \ell_\delta$ holds. This implies that for $i \geq \max\{i_0, j_0\}$ we have $H_i \cap$ bdP_i has at least 3 elements which is a contradiction.

4) $H = G$; since $H \subset E \cap T_pK = G$.

   Pick a linear isometry $\varphi_i : E_i \to E$ with $\varphi_i(o) = o, \varphi_i(x_i) \in \text{pos}\{p\}$. In addition, set $\varphi_i(G_i) := \bar{G}_i, \varphi_i(H_i) := \bar{H}_i$. Without loss of generality, we may assume $\bar{G}_i \to \bar{G}, \bar{H}_i \to \bar{H}\,(i \to \infty)$ where $\bar{G}$ and $\bar{H}$ are straight lines in E through p. From $G = H$ it follows directly that $\bar{G} = \bar{H}$. From this we conclude immediately that $o \notin \bar{G}$. As one can easily see and laboriously show using analytic geometry on the 2-dimensional plane E, it holds that:

$$\frac{\|\varphi_i(\pi_i(x_i)) - \varphi_i(\pi_i(y_i))\|}{\|\varphi_i(x_i) - \varphi_i(y_i)\|} \to 1 \quad (i \to \infty).$$

By considering the isometry property of φ_i, this contradicts equation (A.9) and statement A.8.1 holds.

$\square$

A.8.2 Conclusions

$K \subset \mathbb{E}^N$ is a smooth convex body with $o \in \text{int}K; \varepsilon > 0$. According to Statement A.8.1, there exists an n-polytope $P \subset \mathbb{E}^n$ with $o \in \text{int}P$ and $1 - \varepsilon \leq \text{Lip}\ (\pi, \pi^{-1}) \leq 1 + \varepsilon$, where $\pi : \text{bd}P \to \text{bd}K$ is the radial projection from o. For appropriate P this is equivalent to:

$$\begin{aligned}(1-\varepsilon)\|x-y\| \leq \quad & \|\pi(x)-\pi(y)\| && \leq (1+\varepsilon)\|x-y\| \quad \text{and} \\ (1-\varepsilon)\|\pi(x)-\pi(y)\| \leq \quad & \|x-y\| && \leq (1+\varepsilon)\|\pi(x)-\pi(y)\|\end{aligned}$$

for all x, y in bd$P$. Pick an n-simplex $\Delta \subset \mathbb{E}^n$ with $o \in \text{int}\Delta$ and $\Delta \subset \text{int}K \cap$ int$P$; $S := \text{bd}\Delta$. Z_P denotes the cylinder in $\mathbb{E}^{n+1}$ over the boundary of P, $\alpha : Z_P \to \text{bd}P$ is the normal projection. Z_K is the cylinder in $\mathbb{E}^{n+1}$ over the boundary of K. $\bar{\pi} : Z_P \to Z_K$ induces an extension of π on Z_P by $\bar{\pi}(z) = \bar{\pi}(\alpha(z) + \lambda_z e_{n+1}) := \pi(\alpha(z)) + \lambda_z e_{n+1}$ for $z \in Z_P$. $\bar{\pi}$ is one to one and naturally:

$$\begin{aligned}(1-\varepsilon)\|z-w\| \leq \quad & \|\bar{\pi}(z)-\bar{\pi}(w)\| && \leq (1+\varepsilon)\|z-w\| \quad \text{and} \\ (1-\varepsilon)\|\bar{\pi}(z)-\bar{\pi}(w)\| \leq \quad & \|z-w\| && \leq (1+\varepsilon)\|\bar{\pi}(z)-\bar{\pi}(w)\|\end{aligned}$$

hold for all $z, w \in Z_P$. In short: $1 - \varepsilon \leq \text{Lip}\ (\pi, \pi^{-1}) \leq 1 + \varepsilon$.

$f : S \to Z_P$ is continuous. By $L_{n-1}^{n+1}(f; S)$ we understand the $(n-1)$-dimensional Lebesgue area of f over S, which we can also define using approximating Lipschitz mappings instead of polyhedric functions. $L_{n-1}^{*n+1}(f; S)$ is defined similarly, although the approximating Lipschitz function l_j used must also satisfy the condition that Im$(l_j) \subset Z_P$, where Im(l_j) is the image of S in $\mathbb{E}^{n+1}$ under the mapping l_j.

Lemma A.8.1. $L_{n-1}^{*n+1}(f; S) = L_{n-1}^{n+1}(f; S)$.

Proof. a) $L^{*n+1}_{n-1}(f;S) \geq L^{n+1}_{n-1}(f;S)$ is trivial.

b) Pick a sequence of polyhedric functions $p_i : S \to \mathbb{E}^{n+1}$, with $p_i \to f\ (i \to \infty)$ uniformly on S and $L^{n+1}_{n-1}(p_i;S) \to L^{n+1}_{n-1}(f;S)\ (i \to \infty)$. Without loss of generality we may assume that for all $i \in \mathbb{N}$ Im(p_i) lies outside Z_P. $\nu : \mathbb{E}^{n+1} \to Z_P$ is the "nearest point map" relative to Z_P. Based on the Busemann-Feller Lemmas, ν is a contraction (see Lemma 3 on page 35 in P. McMullen - G. C. Shepard [32] where on p. 31 there is also the definition of the nearest point map). Naturally, $\nu \circ p_i$ is a Lipschitz map for all $i \in \mathbb{N}$ and $\nu \circ p_i \to f\ (i \to \infty)$ uniformly on S.

$$\begin{aligned} \Rightarrow L^{*n+1}_{n-1}(f;S) &\leq \liminf_{i\to\infty} L^{n+1}_{n-1}(\nu \circ p_i; S) \\ &= \liminf_{i\to\infty} \int_{\mathbb{E}^{n+1}} N(\nu \circ p_i; y)\, d\lambda^{n+1}_{n-1}(y). \end{aligned}$$

The last equality is a generalisation of Lemma A.6.2 *i)* (see Theorem 6.18 in H. Federer [3]).

C_i is a simplicial decomposition of S into $(n-1)$-simplices $S_j\ (j = 1, \ldots, n_i)$ so that p_i is affine and one to one over each simplex S_j. So it is true that:

$$\begin{aligned} L^{*n+1}_{n-1}(f;S) &\leq \liminf_{i\to\infty} \int_{\mathbb{E}^{n+1}} \Big[\sum_{j=1}^{n_i} N(\nu \circ p_i|S_j; y)\Big] \mathrm{d}\lambda^{n+1}_{n-1}(y) \\ &= \liminf_{i\to\infty} \sum_{j=1}^{n_i} \Big[\int_{\mathbb{E}^{n+1}} N(\nu \circ p_i|S_j; y)\, \mathrm{d}\lambda^{n+1}_{n-1}(y)\Big] \\ &= \liminf_{i\to\infty} \sum_{j=1}^{n_i} \Big[\int_{p_i(S_j)} J_{n-1}(\nu)\, \mathrm{d}L^{n-1}(y)\Big], \end{aligned}$$

where L^{n-1} denotes the $(n-1)$-dimensional Lebesgue measure. Here $J_{n-1}(\nu)$ is the Jacobian area distortion factor for ν; which means $J_{n-1}(\nu)|_{w\in p_i(S_j)}$ gives the ratio of the $(n-1)$-dimensional Hausdorff measure of the image and pre-image of any cube under the linear derivative $\nu'|_w$: aff $p_i(S_j) \to \mathbb{E}^{n+1}$ of ν at the point w (since ν is a Lipschitz mapping, ν' exists almost everywhere). This last observation follows directly from Theorem 5.9 and Definitions 2.7 and 2.8 in H. Federer [33]. Since ν is a contraction, it obviously holds that $0 \leq J_{n-1}(\nu) \leq 1$. Along with this, we get

$$\begin{aligned} L_{n-1}^{*n+1}(f;S) &\leq \liminf_{i\to\infty} \sum_{j=1}^{n_i} \lambda_{n-1}^{n+1}(p_i(S_j)) = \liminf_{i\to\infty} L_{n-1}^{n+1}(p_i;S) \\ &= \lim_{i\to\infty} L_{n-1}^{n+1}(p_i;S) = L_{n-1}^{n+1}(f;S). \end{aligned}$$

□

Remark 2. *Lemma A.8.1 gives the corresponding result in the case where f is a continuous mapping from S to Z_K.*

Lemma A.8.2.

$$\frac{1}{(1+\varepsilon)^{n-1}} L_{n-1}^{n+1}(f;S) \leq L_{n-1}^{n+1}(\bar{\pi}\circ f;S) \leq (1+\varepsilon)^{n-1} L_{n-1}^{n+1}(f;S).$$

Proof. By Lemma 8.1, there exist Lipschitz functions $f_i : S \to Z_P$, with $f_i \to F$ $(i \to \infty)$ uniformly on S and $L_{n-1}^{n+1}(f_i;S) \to L_{n-1}^{n+1}(f;S)$ $(i \to \infty)$. The mappings $\bar{\pi}\circ f_i : S \to Z_K$ are Lipschitz for all $i \in \mathbb{N}$ and we have $\bar{\pi}\circ f_i \to \bar{\pi}\circ f$ $(i \to \infty)$ uniformly on S. With the generalisation of Lemma A.6.2 *i)* already mentionned in the proof of Lemma A.8.1, the one to one ness of $\bar{\pi}$ and the fact that Lip $\bar{\pi} \leq 1+\varepsilon$, we easily obtain:

$$\begin{aligned} L_{n-1}^{n+1}(\bar{\pi}\circ f;S) &\leq \liminf_{i\to\infty} L_{n-1}^{n+1}(\bar{\pi}\circ f_i;S) \\ &= \liminf_{i\to\infty} \int_{\mathbb{E}^{n+1}} N(\bar{\pi}\circ f_i;y)\,\mathrm{d}\lambda_{n-1}^{n+1}(y) \\ &= \liminf_{i\to\infty} \int_{\mathbb{E}^{n+1}} N(f_i;\bar{\pi}^{-1}(y))\,\mathrm{d}\lambda_{n-1}^{n+1}(y) \\ &\leq (1+\varepsilon)^{n-1} \liminf_{i\to\infty} \int_{\mathbb{E}^{n+1}} N(f_i;\bar{\pi}^{-1}(y))\,\mathrm{d}\lambda_{n-1}^{n+1}(\bar{\pi}^{-1}(y)) \\ &= (1+\varepsilon)^{n-1} \liminf_{i\to\infty} L_{n-1}^{n+1}(f_i;S) \\ &= (1+\varepsilon)^{n-1} \lim_{i\to\infty} L_{n-1}^{n+1}(f_i;S) \\ &= (1+\varepsilon)^{n-1} L_{n-1}^{n+1}(f;S), \end{aligned}$$

with this the right hand side is established. The left hand side follows analogously. □

The corresponding facts, as shown above for the Lebesgue area, can now be shown for the Hausdorff measure.

Lemma A.8.3.

$$\frac{1}{(1+\varepsilon)^{n-1}}\lambda_{n-1}^{n+1}(f(S)) \leq \lambda_{n-1}^{n+1}((\bar{\pi}\circ f)(S)) \leq (1+\varepsilon)^{n-1}\lambda_{n-1}^{n+1}(f(S)).$$

Proof. Follows directly from Lip $(\bar{\pi}, \bar{\pi}^{-1}) \leq 1+\varepsilon$. □

A.9 Main statement and further questions

A.9.1 Main statement

Under certain conditions on the convex bodies taken into consideration, we succeed in proving McMullen's conjecture. For the exact problem statement, in particular the definition of the function q, refer to the introduction.

Theorem 9. Main Theorem

For all smooth convex bodies $K \subset \mathbb{E}^n$ we have:

$$1 \leq q(K) \leq \frac{n\,\omega_n}{\pi\,\omega_{n-1}}$$

Proof. By considering the main conjecture and the measure theoretic Key question (A.1), it suffices to show

$$L_{n-1}^{n}(\varphi; S) = \lambda_{n-2}^{n}(\varphi(S))$$

Therefore, we adopt the conditions in section A.6.4.

1) If both sides of the inequality are infinite there is nothing to show.

2) Both sides are finite:

   Contrapositive: $|L_{n-2}^{n}(\varphi; S) - \lambda_{n-2}^{n}(\varphi(S))| = \delta > 0$. $F := \text{lin}\{e_1, \ldots, e_{n-1}\}$. Statement A.8.1 with $n-1$ instead of n and $\pi_G(K)$ for K (see introduction) guarantee that for each $\varepsilon > 0$ there exists an $(n-1)$-polytope $P_\varepsilon \subset F$, with $o \in \text{int}P_\varepsilon$ and $1-\varepsilon \leq$ Lip $(P_\varepsilon, P_\varepsilon^{-1}) \leq 1+\varepsilon$, where P_ε : relbd$P_\varepsilon \to \mathbf{\Pi}(K, G)$ denotes the radial projection from o; note that as well as K, $\pi_G(K)$ is smooth. Based on Lemmas A.8.2 and A.8.3, with n instead of $n+1$, $\varphi_{P_\varepsilon} \circ \kappa_{P_\varepsilon}$ instead of f

anf φ instead of $\bar{\pi} \circ f$, we can pick $\varepsilon > 0$ such that the following two statements hold:

$$|L^n_{n-2}(\varphi; S) - L^n_{n-2}(\varphi_{P_\varepsilon} \circ \kappa_{P_\varepsilon}; S)| < \delta/2 \quad \text{and}$$
$$|\lambda^n_{n-2}(\varphi(S)) - \lambda^n_{n-2}((\varphi_{P_\varepsilon} \circ \kappa_{P_\varepsilon})(S))| < \delta/2,$$

where φ_{P_ε} and κ_{P_ε} are defined similarly to φ_P and κ_P from section A.7.4. From this we can conclude directly:

$$L^n_{n-2}(\varphi_{P_\varepsilon} \circ \kappa_{P_\varepsilon}; S) \neq \lambda^n_{n-2}((\varphi_{P_\varepsilon} \circ \kappa_{P_\varepsilon})(S)),$$

which contradicts Lemma A.7.8.

3) One side is finite and the other is infinite is impossible by Lemmas A.8.2, A.8.3 (with the same switches as in 2) above) and Lemma A.7.8 (with P_ε instead of P).

□

A.9.2 Conclusions and further questions

1) "Long shadow boundaries are rare". $K \subset \mathbb{E}^n$ is as always a smooth convex body. As $q(K) = \alpha(K)/\beta(K)$ from our main statement and $\beta(K)$ (as a constant multiple of the Minkowski Quermass integral $W_2(K)$) are both finite, it follows immediately that $\alpha(K) < \infty$.

   Set $\Lambda := \{G | \, G \in \Gamma_o(K); f(K,G) = \infty\}$. $F(K,G)$ was defined in section A.4.1. Since $f(K,G)$ is lower semi continuous in the second argument due to $b)$ of the main conjecture, for a fixed K we have:

   Corollary 4. *Λ is γ_0-measurable and $\gamma_0(\Lambda) = 0$.*

   For the definition of γ_0 refer back to section A.3.2.

2) The question corresponding to McMullen's problem for $(n-2)$-dimensional projections and hence 1-dimensional shadow boundaries (topological circumference) can be solved similarly. It is in fact easier and can be done without the

smoothness of K. Instead of λ^n_{n-2} and L^n_{n-2} take λ^n_1 and L^n_1. The corresponding key question for the proof of the main conjecture results from a direct generalisation of Lemma A.6.2 *ii)* or the main conjecture follows directly, without using Lebesgue area, from the fact that λ^n_1 agrees with the natural arc length, which is lower semi continuous over approximations. Compare with Remark β), page 164 in F. Hausdorff [34] and Result 5.8, page 20 in H. Busemann [35].

For polytopes, take the analog function q and the same constant values as in the problem solved above (see H. Hadwiger [36]).

3) The case of k-dimensional projections and $(n - k - 1)$-dimensional shadow boundaries, for $1 < k < n - 2$ is harder and is being worked on sucessfully by D. G. Larman and P. Mani.

4) *Open questions:* Out of the numerous problems, we pick two.

   a) Can the smoothness property of the convex body K in the approximation statement and in the main statement be left out?

   b) When do we have equality in the main statement, ie: which convex body aside from the ball satisfy $q(K) = 1$ and which along with polytopes are characterised by $q(K) = \pi\omega_{n-1}$?

Bibliography

[1] H. Federer 'Geometric Measure Theory' Die Grundlagen der mathematischen Wissenschaften, Springer-Verlag, 1969.

[2] H. Federer 'Hausdorff Measure and Lebesgue Area' Proc. Natl. Acad. Sci, 90-94 (1950).

[3] H. Federer 'On Lebesgue area', Ann. of Math. vol.61 (1955) pp. 289-353.

[4] H. Federer 'The Area of a Nonparametric Surface', Proc. Amer. Math. Soc, vol. 11, No. 3 (Jun 1960), pp. 436-439.

[5] H. Federer 'Measure and area' Bull. Amer. Math. Soc. 58, 302-378 (1952)

[6] S. Saks 'Theory of the Integral' Monografie Matematyczne. 7 (2nd ed.). Warszawa-Lwów: G.E. Stechert & Co.. pp. VI+347 (1937)

[7] P. Steenaerts 'Mittlere Schattengrenzenlänge konvexer Körper' Results in Math. 8 (1985), 54- 77.

[8] V. A. Zalgaller 'k-dimensional Directions Singular for a Convex Body F in $\mathbb{R}^n$' Instituta im. V. A. Steklova Akademii Nauk SSSR, Vol. 27, pp. 67-72, 1972.

[9] G. Ziegler 'Lectures on Polytopes' Grad.Texts in Math. 152, Springer-Verlag 1998

[10] V. L. Klee 'Can the boundary of a d-dimensional convex body contain segments in all directions?' Amer. Math. Monthly, 76 (1969), 408-410.

[11] G. Ewald, D. G. Larman and C. A. Rogers 'The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space', MATHEMATIKA 17 (1970)

[12] T. J. McMinn 'On the line segments of a convex surface in $\mathbb{E}^3$' Pacific J. MATH. 10 (1960) 943-46.

[13] A. S. Besicovitch 'On the set of directions of linear segments on a convex surface' Proc. Amer. Math. Soc. Symp. Pure Math. (convexity), 7 (1963) 24-25

[14] L. C. Evans and R. F. Gariepy 'Measure Theory and Fine Properties of Functions' CRC Press (1992)

[15] P. McMullen, List of problems, Problem 3). Oberwolfach, May 1974.

[16] D. G. Larman and C. A. Rogers 'Increasing Paths on the One-Skeleton of a Convex Body and the Directions of Line Segments on the Boundary of a Convex Body' Proceedings of the London Mathematical Society Dec. 1971.

[17] P. McMullen 'On the Inner Parallel Body of a Convex Body' Israel Journal of Mathematics, Volume 19, Number 3 (1974), 217-219

[18] P. Gruber and H. Sorger: 'Shadow Boundaries of Typical Convex Bodies. Measure Properties' MATHEMATIKA, 36 (1989), 142-152.

[19] V. Klee: 'Some new results on smoothness and rotundity in normed linear spaces'. Math. Ann., 139 (1959), 51-63.

[20] C. A. Rogers: 'Hausdorff Measures' Cambridge Univ. Press (1970)

[21] S. Gallivan 'Properties of the one-skeleton of a convex body', PhD Thesis, University College London, June 1974.

[22] R. Schneider 'Convex Bodies: The Brunn-Minkowski Theory' Encyclopedia of Math., Cam. Uni. Press, 1993.

[23] W. Blaschke 'Vorlesungen über Integralgeometrie' VEB Deutscher Verlag der Wissenschaften, Berlin, 1955, 3. Auflage.

[24] H. Hadwiger 'Vorlesungen über Inhalt, Oberfläche und Isoperimetrie' Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[25] J.Rätz 'Über Inhalt und Oberfläche von Kugel und Zylinder' Math-Phys. Semesterber. 15 (1968) 88-93.

[26] I. P. Natanson 'Theorie der Funktionen einer reellen Veränderlichen' Akademie-Verlag Berlin, 1954.

[27] R. N. Tompson 'Areas of k-dimensional nonparametric surfaces in k+1 space' Trans. Amer. Math. Soc. 77 (1954), 374-407.

[28] H. Federer 'On Lebesgue area' Ann. of Math. 61 Nr. 2 (1955), 289-353.

[29] L. I. Alpert and L. V. Toralballa 'An elementary definition of surface area in $\mathbb{E}^{N+1}$ for smooth surfaces' Pacific J. Math. 40 Nr. 2 (1972) 261-268.

[30] J. Serrin 'On the area of curved surfaces' Amer. Math. Monthly 68 (1961), 435-440.

[31] S. Eilenberg and N. Steenrod 'Foundations of algebraic topology' Princeton University Press, Princeton, 1952.

[32] P. McMullem and G. C. Shephard 'Convex Polytopes and the upper bound conjecture' Cambridge University Press, London-New York, 1971.

[33] H. Federer 'The (ϕ, k) rectifiable subsets of n space' Trans. Amer. Math. Soc. 62 (1947), 114-192.

[34] F. Hausdorff 'Dimension und äusseres' Mass. Math. Ann. 79 (1918), 157-179.

[35] H. Busemann 'The geometry of geodesics' Academic Press, New York, 1955.

[36] H. Hadwiger 'Lineare, polyedrisch-limitierbare Eikörperfunktionale' Unveröffentlichtes Manuskript, Bern, 1976.