# Pricing Fixed-Income Securities in an Information-Based Framework

Lane P. Hughston* and Andrea Macrina†‡

* Department of Mathematics, Imperial College London
London SW7 2AZ, United Kingdom
† Department of Mathematics, King's College London
London WC2R 2LS, United Kingdom
‡ Institute of Economic Research, Kyoto University
Kyoto 606-8501, Japan

**Abstract**

In this paper we introduce a class of information-based models for the pricing of fixed-income securities. We consider a set of continuous-time processes that describe the flow of information concerning market factors in a monetary economy. The nominal pricing kernel is at any given time assumed to be given by a function of the values of information processes at that time. By use of a change-of-measure technique we derive explicit expressions for the price processes of nominal discount bonds, and deduce the associated dynamics of the short rate of interest and the market price of risk. The interest rate positivity condition is expressed as a differential inequality. We proceed to model the price level, which at any given time is also taken to be a function of the values of the information processes at that time. A simple model for a stochastic monetary economy is introduced in which the prices of nominal discount bonds and inflation-linked notes can be expressed in terms of aggregate consumption and the liquidity benefit generated by the money supply.

Key words: Fixed-income securities, interest rate theory, inflation, inflation-linked securities, non-linear filtering, incomplete information.
This version: April 27, 2010.
Email: lane.hughston@imperial.ac.uk, andrea.macrina@kcl.ac.uk

# 1 Introduction

The idea of information-based asset pricing (Macrina 2006, Brody *et al.* 2007, 2008a,b, Hughston & Macrina 2008) is that the market filtration should be modeled in such a way that it is generated by a set of processes that carry information about the future cash flows generated by tradable securities. One can regard each such cash flow as a random variable that is in turn given by a function of one or more independent random variables called "market factors" or, more succinctly, "$X$-factors". The information processes that generate the market filtration are associated with the various $X$-factors in such a way that the value of each $X$-factor is revealed at some designated time by the associated information process. The simplest examples of information processes are those based on Brownian bridges (Brody *et al.* 2007, 2008a, Rutkowski & Yu 2007), and gamma bridges (Brody *et al.* 2008b), which lead to highly tractable asset pricing models; more general information processes can be constructed based on Lévy random bridges (Hoyle *et al.* 2009).

The purpose of the present paper is to present a simple class of information-based models for interest rates, foreign exchange, and inflation. The point of view is the following. We retain the premise that the $X$-factors represent the fundamental factors, the values of which are revealed from time to time, that determine the cash flows generated by primary securities. We also accept the view that the market filtration is generated collectively by the information processes associated with these factors. In a macroeconomic setting with a dynamic equilibrium, it is appropriate to assume the existence of a universal pricing kernel associated with the choice of a suitable base currency. We shall call this the nominal pricing kernel associated with the given base currency. The pricing kernel is necessarily adapted to the market filtration, and is therefore given by a functional of the trajectories of the information processes up to the time at which the value of the pricing kernel is to be determined. A similar property holds for the pricing kernel associated with any other currency or unit of exchange. The models for interest rates and foreign exchange that we develop are characterized by the following additional assumptions: (a) that the information processes collectively have the Markov property with respect to the market filtration, and (b) that the pricing kernels associated with each currency under consideration can at any given time be expressed as a function of the values taken by the information processes at that time. In the case of inflation, we take a similar point of view, adapting the "foreign exchange analogy" (Hughston 1998, Jarrow &

Yildirim 2003, Mercurio 2005, Brody *et al.* 2008, Hinnerich 2008). In this scheme the price level is given by the ratio of the real and the nominal pricing kernels. These are given, in the models developed in the present paper, by functions of the current levels of the relevant information processes.

# 2 One-factor models

For simplicity we consider first the case of a single $X$-factor and a single information process. The resulting theory can be worked out rather explicitly, and from this example one can see how the general case can be approached when there are several currencies and many $X$-factors. In the single-factor case we proceed as follows.

The market will be modelled as usual by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We assume that $\mathbb{P}$ is the "real" probability measure, and that $\{\mathcal{F}_t\}$ is the market filtration. The filtration will be modelled in the following manner. Let time 0 denote the present, and fix a time $U > 0$. We introduce a continuous random variable $X_U$ taking values in $\mathbb{R}$, with probability density $p(x)$. The restriction to a continuous random variable is for convenience. With this "$X$-factor" we associate an information process $\{\xi_{tU}\}_{0\leq t\leq U}$ defined by

$$\xi_{tU} = \sigma t X_U + \beta_{tU}. \tag{2.1}$$

Here $\sigma$ is an information flow-rate parameter, and the Brownian bridge process $\{\beta_{tU}\}_{0\leq t\leq U}$ is taken to be independent of the market factor $X_U$. As remarked in Brody *et al.* 2007 (see also Rutkowski & Yu 2007) it is a straightforward exercise making use of well-known properties of the Brownian bridge to show that $\{\xi_{tU}\}$ has the Markov property with respect to its own filtration. We shall assume that $\{\xi_{tU}\}$ generates the market filtration. Hence for each $t \in [0, U]$ the sigma-algebra $\mathcal{F}_t$ is defined by

$$\mathcal{F}_t = \sigma\left(\{\xi_{sU}\}_{0\leq s\leq t}\right). \tag{2.2}$$

It should be evident that $U$ acts as a kind of "sunset" for the economy, that there is only one piece of information to be revealed, and once it has been revealed then that is the end of the story. This is of course an artifact of the simplicity of our assumptions, and in a more realistic model we can expect the revelation of $X$-factors to proceed indefinitely into the future, the more distant ones being generally less important than the nearer.

The pricing kernel $\{\pi_t\}$ will be assumed to be given by a positive function of time $t$ and the value of the information process at $t$. Thus, we have

$$\pi_t = F(t, \xi_{tU}). \tag{2.3}$$

Given the pricing kernel, we can work out the value processes of various assets. In the simple economy under consideration, the "primary" assets are those that deliver a single cash flow at time $U$ given by a suitably integrable function $H(X_U)$ that depends on the outcome $X_U$. The value of such a security at time $t \leq U$ is given by

$$H_t = \frac{1}{\pi_t}\mathbb{E}^{\mathbb{P}}\left[\pi_U H(X_U) \,|\, \mathcal{F}_t\right]. \tag{2.4}$$

For each choice of $H(X_U)$ we obtain a tradable security. We also consider the discount-bond system associated with the given pricing kernel. Let us write $P_{tT}$ for the price at time $t$ of a bond that pays one unit of currency at time $T$ for $t \leq T \leq U$. Then for each $T \in [0, U]$ we have:

$$P_{tT} = \frac{1}{\pi_t}\mathbb{E}^{\mathbb{P}}\left[\pi_T \,|\, \mathcal{F}_t\right]. \tag{2.5}$$

Finally, we consider various "derivative" assets. These deliver prescribed cash flows at one or more times in the interval $(0, U)$ determined by the values of the basic assets and the discount bonds at various times. More generally we can consider "information derivatives" for which the cash flows can depend in an essentially arbitrary way on the information available up to the time of the cash flow. For example, let the payoff of a security at time $T$ be given by $G(T, \xi_{TU})$ where $G(t, \xi)$ is a function of two variables. Then the value of this asset at $t \leq T$ is

$$G_t = \frac{1}{\pi_t}\mathbb{E}^{\mathbb{P}}\left[\pi_T \, G(T, \xi_{TU}) \,|\, \mathcal{F}_t\right]. \tag{2.6}$$

We observe that the value at time $t$ of a $T$-maturity option on a primary security takes this form.

# 3 Interest rates in a one-factor model

Let us consider in more detail the properties of discount bonds. Recalling that the information process has the Markov property, we see that (2.5) reduces to the following expression:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[F(T,\xi_{TU})\,|\,\xi_{tU}\right]}{F(t,\xi_{tU})}. \tag{3.1}$$

We proceed to work out the conditional expectation. To this end we recall one further property of the information process. This is the existence of the so-called "bridge measure" $\mathbb{B}$. Under the bridge measure (Brody *et al.* 2007) the information process $\{\xi_{tU}\}$ is a Brownian bridge over the interval $[0, U)$. The change-of-measure density martingale for the transformation from $\mathbb{P}$ to $\mathbb{B}$ is given by a process $\{M_t\}_{0\leq t<U}$ defined by

$$M_t = \left(\int_{-\infty}^{\infty} p(x)\exp\left[\frac{U}{U-t}\left(\sigma x\,\xi_{tU} - \tfrac{1}{2}\,\sigma^2x^2t\right)\right]\mathrm{d}x\right)^{-1}. \tag{3.2}$$

Applying Ito's formula, one can show that

$$\frac{\mathrm{d}M_t}{M_t} = -\frac{\sigma U}{U-t}\,\mathbb{E}^{\mathbb{P}}\left[X_U\,|\,\xi_{tU}\right]\mathrm{d}W_t, \tag{3.3}$$

where the process $\{W_t\}$ defined by

$$W_t = \xi_{tU} + \int_0^t \frac{1}{U-s}\,\xi_{sU}\,\mathrm{d}s - \sigma U\int_0^t \frac{1}{U-s}\,\mathbb{E}^{\mathbb{P}}\left[X_U\,|\,\xi_{sU}\right]\mathrm{d}s. \tag{3.4}$$

is an $(\{\mathcal{F}_t\},\mathbb{P})$ Brownian motion on $[0, U)$. Thus, in the information-based approach the Brownian motions that drive asset prices always arise as "secondary" objects—innovation processes—rather than as primary drivers. For further details of the properties of the change-of-measure martingale $\{M_t\}$ and related processes appearing in its definition, see Macrina 2006, chapter 3. Bearing in mind that the random variable $M_t$ can be expressed as a function of $t$ and $\xi_{tU}$, as given by (3.2), we can without loss of generality introduce a positive function $f(t,\xi_{tU})$ such that

$$\pi_t = M_t\,f(t,\xi_{tU}), \tag{3.5}$$

and as a consequence we obtain

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[M_T\,f(T,\xi_{TU})\,|\,\xi_{tU}\right]}{M_t\,f(t,\xi_{tU})}. \tag{3.6}$$

The appearance of the change-of-measure density in this formula enables us to use the conditional version of Bayes formula to re-express $\{P_{tT}\}$ in terms of an expectation with respect to the bridge measure:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[f(T, \xi_{TU}) \,|\, \xi_{tU}\right]}{f(t, \xi_{tU})}. \tag{3.7}$$

Since the information process is a $\mathbb{B}$-Brownian bridge we know that at each time $t$ the random variable $\xi_{tU}$ is $\mathbb{B}$-Gaussian. Armed with this fact, we proceed as follows. We introduce a random variable $Y_{tT}$ defined by

$$Y_{tT} = \xi_{TU} - \frac{U-T}{U-t}\, \xi_{tU}. \tag{3.8}$$

It is evident that $Y_{tT}$ is $\mathbb{B}$-Gaussian, and a short calculation making use of properties of the Brownian bridge shows that $Y_{tT}$ has mean zero and variance

$$\mathrm{Var}^{\mathbb{B}}\left[Y_{tT}\right] = \frac{(T-t)(U-T)}{U-t}. \tag{3.9}$$

We observe that $Y_{tT}$ and $\xi_{tU}$ are $\mathbb{B}$-independent. This can be checked by calculating the relevant covariance. Next we express $Y_{tT}$ in terms of a standard Gaussian variable $Y$, with mean zero and variance unity. Thus we write

$$Y_{tT} = \nu_{tT} Y, \tag{3.10}$$

where

$$\nu_{tT} = \sqrt{\frac{(T-t)(U-T)}{U-t}}. \tag{3.11}$$

Then we rewrite (3.7) in terms of $Y$ to obtain

$$P_{tT} = \frac{1}{f(t, \xi_{tU})}\, \mathbb{E}^{\mathbb{B}}\left[ f\left(T, \nu_{tT} Y + \frac{U-T}{U-t}\, \xi_{tU}\right) \middle|\, \xi_{tU}\right]. \tag{3.12}$$

Since $Y$ and $\xi_{tU}$ are $\mathbb{B}$-independent, the conditional expectation in (3.12) reduces to a Gaussian integral over the range of $Y$, and thus:

$$P_{tT} = \frac{1}{\sqrt{2\pi}\, f(t, \xi_{tU})} \int_{-\infty}^{\infty} f\left(T, \nu_{tT}\, y + \frac{U-T}{U-t}\, \xi_{tU}\right) \exp\left(-\tfrac{1}{2}\, y^2\right) \mathrm{d}y. \tag{3.13}$$

We can write this equation more compactly by introducing a function of three variables $\tilde{f}(t,T,\xi)$ as follows:

$$\tilde{f}(t,T,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(T, \nu_{tT}\, y + \frac{U-T}{U-t}\,\xi\right) \exp\left(-\tfrac{1}{2}\,y^2\right) \mathrm{d}y. \tag{3.14}$$

Then, the bond price is given by

$$P_{tT} = \frac{\tilde{f}(t,T,\xi_{tU})}{f(t,\xi_{tU})}. \tag{3.15}$$

For any particular choice of $f$ it is straightforward to simulate the dynamics of the bond price since, conditional on the outcome of the underlying factor $X_U$, the information process is $\mathbb{P}$-Gaussian.

# 4 Pricing kernel, nominal interest rate, and market price of risk

Let us proceed to derive the dynamics of the pricing kernel (3.5). We apply the product rule to obtain

$$\mathrm{d}\pi_t = f_t\,\mathrm{d}M_t + M_t\,\mathrm{d}f_t + \mathrm{d}M_t\,\mathrm{d}f_t, \tag{4.1}$$

where $f_t = f(t,\xi_{tU})$. The dynamical equation for the change-of-measure density martingale is (3.3). We shall assume that $f(t,\xi)$ has a continuous first derivative with respect to $t$, denoted $\dot{f}(t,\xi)$, and a continuous second derivative with respect to $\xi$, denoted $f''(t,\xi)$. Hence

$$\mathrm{d}f_t = \dot{f}_t\,\mathrm{d}t + f'_t\,\mathrm{d}\xi_{tU} + \tfrac{1}{2}\,f''_t\,(\mathrm{d}\xi_{tU})^2. \tag{4.2}$$

In terms of the innovation process defined by (3.4), the dynamical equation for $\{\xi_{tU}\}$ is

$$\mathrm{d}\xi_{tU} = \frac{1}{U-t}\left(\sigma\,U\,\mathbb{E}\left[X_U \,|\, \xi_{tU}\right] - \xi_{tU}\right)\mathrm{d}t + \mathrm{d}W_t. \tag{4.3}$$

Thus $(\mathrm{d}\xi_{tU})^2 = \mathrm{d}t$, and with (3.2) and (4.3) at hand a calculation shows that

$$\mathrm{d}\pi_t = M_t\left(\dot{f}_t - \frac{\xi_{tU}}{U-t} f'_t + \tfrac{1}{2} f''_t\right)\mathrm{d}t \,+\, M_t\left(f'_t - \frac{\sigma U}{U-t}\,\mathbb{E}\left[X_U \,|\, \xi_{tU}\right]\, f_t\right)\mathrm{d}W_t. \tag{4.4}$$

or equivalently,

$$\frac{\mathrm{d}\pi_t}{\pi_t} = \frac{1}{f_t}\left(\dot{f}_t - \frac{\xi_{tU}}{U-t}\, f'_t + \tfrac{1}{2} f''_t\right)\mathrm{d}t \\ + \frac{1}{f_t}\left(f'_t - \frac{\sigma U}{U-t}\,\mathbb{E}\left[X_U \,|\, \xi_{tU}\right]\, f_t\right)\mathrm{d}W_t. \tag{4.5}$$

The drift of the pricing kernel is minus the short rate of interest, and the volatility is minus the market price of risk:

$$\frac{\mathrm{d}\pi_t}{\pi_t} = -r_t\,\mathrm{d}t - \lambda_t\,\mathrm{d}W_t. \tag{4.6}$$

Comparing coefficients, we deduce that the nominal short rate and market price of risk are given respectively by

$$r_t = \frac{1}{f_t}\left(\frac{\xi_{tU}}{U-t}\, f'_t - \tfrac{1}{2}\, f''_t - \dot{f}_t\right), \tag{4.7}$$

and

$$\lambda_t = \frac{\sigma U}{U-t}\,\mathbb{E}\left[X_U \,|\, \xi_{tU}\right] - \frac{f'_t}{f_t}. \tag{4.8}$$

It is natural in the context of some applications to impose the condition that the short rate should be positive. This condition is evidently given by

$$\frac{\xi_{tU}}{U-t}\, f'_t - \tfrac{1}{2}\, f''_t - \dot{f}_t > 0. \tag{4.9}$$

The interest-rate positivity condition is equivalent to the following differential inequality:

$$\frac{x}{U-t}\, f'(t,x) - \tfrac{1}{2} f''(t,x) - \dot{f}(t,x) > 0. \tag{4.10}$$

# 5 Pricing in a multi-factor setting

We introduce a set of $X$-factors $\{X_{T_1}, \ldots, X_{T_n}\}$, labeled by a series of dates $T_k$ $(k = 1, \ldots, n)$ such that $0 < T_1 < \ldots < T_n$. With each $X$-factor we associate an information process $\{\xi_{tT_k}\}$ defined by

$$\xi_{tT_k} = \sigma_k\, t\, X_{T_k} + \beta_{tT_k}. \tag{5.1}$$

The information processes associated with different $X$-factors are taken to be independent. The market filtration $\{\mathcal{F}_t\}$ is assumed to be generated by collection of information processes:

$$\mathcal{F}_t = \sigma\left(\{\xi_{sT_1}\}_{0\le s\le t}, \ldots, \{\xi_{sT_n}\}_{0\le s\le t}\right). \tag{5.2}$$

As a generalisation of the Markov model introduced in the previous section, we consider the following multi-factor model for $\pi_t$:

$$\pi_t = M_t^{(1)} \cdots M_t^{(n)}\, f(t, \xi_{tT_1}, \ldots, \xi_{sT_n}). \tag{5.3}$$

Here $f(t, \xi_1, \xi_2, \ldots, \xi_n)$ is a function of $n+1$ variables. The unit-initialized $\mathbb{P}$-martingales $\{M_t^{(k)}\}_{k=1,\ldots,n}$ are defined by

$$\frac{\mathrm{d}M_t^{(k)}}{M_t^{(k)}} = -\frac{\sigma_k\, T_k}{T_k - t}\, \mathbb{E}\left[X_{T_k} \,|\, \xi_{tT_k}\right] \mathrm{d}W_t^{(k)}, \tag{5.4}$$

where for each $k$ the $\mathbb{P}$-Brownian motion $\{W_t^{(k)}\}$ is defined by

$$W_t^{(k)} = \xi_{tT_k} + \int_0^t \frac{1}{T_k - s}\, \xi_{sT_k} \mathrm{d}s - \sigma_k T_k \int_0^t \frac{1}{T_k - s}\, \mathbb{E}\left[X_{T_k} \,|\, \xi_{sT_k}\right] \mathrm{d}s. \tag{5.5}$$

Since the information processes are independent, it follows that

$$\mathrm{d}W^{(j)} \mathrm{d}W^{(k)} = \delta^{jk}\, \mathrm{d}t. \tag{5.6}$$

Let us focus on the pricing of a nominal discount bond with maturity $T < T_1$. The price of the bond is given by:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[M_T^{(1)} \cdots M_T^{(n)}\, f\left(T, \xi_{TT_1}, \ldots, \xi_{TT_n}\right) \,\Big|\, \xi_{tT_1}, \ldots, \xi_{tT_n}\right]}{M_t^{(1)} \cdots M_t^{(n)}\, f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}. \tag{5.7}$$

Here we have used the fact that the information processes are Markovian. Since the information processes are independent, the product of $\mathbb{P}$-martingales given by $M_t^{(1)} \cdots M_t^{(n)}$ for $t$ in the interval $[0, T_1)$ is itself a $\mathbb{P}$-martingale. This martingale induces a bridge measure that has the effect of turning the information processes into $\mathbb{B}$-Brownian bridges. More precisely, under the bridge measure each information process has, over the interval $[0, T_1)$, the distribution of a standard Brownian bridge on the interval from 0 to the termination time of the information process. Thus we have

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[f\left(T, \xi_{TT_1}, \ldots, \xi_{TT_n}\right) \,\Big|\, \xi_{tT_1}, \ldots, \xi_{tT_n}\right]}{f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}, \tag{5.8}$$

where all of the variables appearing are $\mathbb{B}$-Gaussian. Next we introduce a set of random variables $Y^{(1)}_{tT}, Y^{(2)}_{tT}, \ldots, Y^{(n)}_{tT}$ defined by

$$Y^{(k)}_{tT} = \xi_{TT_k} - \frac{T_k - T}{T_k - t}\,\xi_{tT_k}. \tag{5.9}$$

Since $\{\xi_{tT_k}\}$ is a $\mathbb{B}$-Brownian bridge, it follows that $Y^{(k)}_{tT}$ is Gaussian with mean zero and variance

$$\mathrm{Var}^{\mathbb{B}}\left[Y^{(k)}_{tT}\right] = \frac{(T-t)(T_k-T)}{T_k-t}. \tag{5.10}$$

We introduce therefore an $n$-dimensional set of standard Gaussian variables $(Y_1, \ldots, Y_n)$. The variable $Y_k$ stands in relation to $Y^{(k)}_{tT}$ via the formula

$$Y^{(k)}_{tT} = \nu^{(k)}_{tT}\, Y_k, \tag{5.11}$$

where

$$\nu^{(k)}_{tT} = \sqrt{(T-t)(T_k-T)/(T_k-t)}. \tag{5.12}$$

In terms of the standard Gaussian random variables, the bond price at time $t$ can thus be written in the form

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[f\left(T,\, \nu^{(1)}_{tT}Y_1 + \frac{T_1-T}{T_1-t}\xi_{tT_1}\,, \ldots\,, \nu^{(n)}_{tT}Y_n + \frac{T_n-T}{T_n-t}\xi_{tT_n}\right) \mid \xi_{tT_1}\cdots\xi_{tT_n}\right]}{f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}. \tag{5.13}$$

We observe that the random variables $Y^{(k)}_{tT}$ and $\xi_{tT_k}$ are $\mathbb{B}$-independent. The expression for the bond price thus reduces to a definite integral:

$$P_{tT} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{f(T,\, \nu^{(1)}_{tT}\,y_1 + \frac{T_1-T}{T_1-t}\,\xi_{tT_1}\,, \ldots\,, \nu^{(n)}_{tT}\,y_n + \frac{T_n-T}{T_n-t}\,\xi_{tT_n})}{f(t, \xi_{tT_1}, \ldots, \xi_{tT_n})} \times \frac{1}{\left(\sqrt{2\pi}\right)^n} \exp\left[-\tfrac{1}{2}\left(y_n^2 + \ldots + y_1^2\right)\right] \mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{5.14}$$

That is to say, we obtain an expression of the form

$$P_{tT} = \frac{\tilde{f}\left(t, T, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}{f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}, \tag{5.15}$$

where

$$
\begin{aligned}
&\tilde{f}(t, T, \xi_1, \xi_2, \dots, \xi_n) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(T,\, \nu_{tT}^{(1)}\, y_1 + \frac{T_1 - T}{T - t}\, \xi_1\,,\, \dots\,,\, \nu_{tT}^{(n)}\, y_n + \frac{T_n - T}{T - t}\, \xi_n\right) \\
&\qquad\qquad \times \frac{1}{\left(\sqrt{2\pi}\right)^n} \exp\left[-\tfrac{1}{2}\left(y_n^2 + \ldots + y_1^n\right)\right] \mathrm{d}y_n \cdots y_1. \qquad (5.16)
\end{aligned}
$$

A multi-currency environment, with an interest rate system for each currency, can be handled similarly. We consider $N+1$ currencies, writing $\{\pi_t\}$ for the pricing kernel of the "domestic" or "base" currency, and $\{\pi_t^i\}$, $i = 1, \dots, N$, for the pricing kernels of the foreign currencies. We introduce $n$ information processes, and assume that each of the pricing kernels is a function of the current levels of the information processes. The prices associated with the $N$ foreign currencies, expressed in units of the domestic currency, are the ratios of the various foreign pricing kernels to the domestic pricing kernel. For a realistic model, we expect to have $n \geq 2N + 1$.

# 6 Positivity condition

Let us consider the class of functions $f(t, \xi_1, \dots, \xi_n)$ for which the pricing kernel (5.3) is a supermartingale. We need thus to derive the dynamics of the pricing kernel and to work out its drift. Let us assume that $f(t, \xi_1, \dots, \xi_n)$ is in $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n)$. We let $\dot{f}$ denote the derivative with respect to $t$, $\partial_k f$ the derivative with respect to the $k$-th coordinate, and $\partial_{kk} f$ the second derivative with respect to the $k$-th coordinate. Then the dynamical equation of the pricing kernel is:

$$
\begin{aligned}
\frac{\mathrm{d}\pi_t}{\pi_t} &= \frac{1}{f_t}\left[\dot{f} + \sum_{k=1}^{n}\left(\tfrac{1}{2}\,\partial_{kk} f_t - \frac{\xi_{tT_k}}{T_k - t}\,\partial_k f_t\right)\right] \mathrm{d}t \\
&\quad + \frac{1}{f_t}\sum_{k=1}^{n}\left(\partial_k f_t - \frac{\sigma_k T_k}{T_k - t}\, X_{tT_k}\, f_t\right) \mathrm{d}W_t^k, \qquad (6.1)
\end{aligned}
$$

where $X_{tT_k} = \mathbb{E}[X_{T_k} \,|\, \xi_{tT_k}]$. The multi-factor short rate process is thus

$$
r_t = \frac{1}{f_t}\left[\sum_{k=1}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k f_t - \frac{1}{2}\,\partial_{kk} f_t\right) - \dot{f}_t\right], \qquad (6.2)
$$

and for the $k$-th component of the market price of risk vector we obtain

$$\lambda_t^k = \frac{1}{f_t}\left(\frac{\sigma_k T_k}{T_k - t}\, X_{tT_k} f_t - \partial_k f_t\right). \tag{6.3}$$

If we impose the condition of a positive short rate process, then the following condition needs to be satisfied:

$$\sum_{k=1}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k f_t - \tfrac{1}{2}\,\partial_{kk} f_t\right) - \dot{f}_t > 0. \tag{6.4}$$

A sufficient condition for (6.4) is that the function $f(t, \xi_1, \ldots, \xi_n)$ satisfies

$$\sum_{k=1}^{n}\left[\frac{x_k}{T_k - t}\,\partial_k f(t, \xi_1, \ldots, \xi_n) - \tfrac{1}{2}\,\partial_{kk} f(t, \xi_1, \ldots\, \xi_n)\right] - \dot{f}(t, \xi_1, \ldots, \xi_n) > 0. \tag{6.5}$$

# 7 Inflation-linked products

The technique used for nominal discount bonds can be adapted to the pricing of inflation-linked assets. In what follows we focus on the pricing of inflation-linked discount bonds. We denote the price level (e.g., the consumer price index) by $\{C_t\}$, and note that the relation between the nominal pricing kernel $\{\pi_t\}$, the real pricing kernel $\{\pi_t^R\}$, and the price level is

$$C_t = \frac{\pi_t^R}{\pi_t}. \tag{7.1}$$

We take the view that the dynamics of the price level should be derived from the dynamics of the pricing kernels. We return to this point shortly, when we introduce the elements of a stochastic monetary economy. Once models for the nominal and the real pricing kernels have been constructed, then the dynamical equation of the price level follows as a result of (7.1).

It will be convenient to define an inflation-linked discount bond as a security that at its maturity $T$ generates a single cash flow equal to the price level $C_T$ prevailing at that time. Thus, the price $\{Q_{tT}\}_{0\leq t\leq T}$ of an inflation-linked discount bond is given by

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[\pi_T\, C_T \,|\, \mathcal{F}_t\right]}{\pi_t}. \tag{7.2}$$

Using the relationship (7.1) we can write this alternatively as

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[\pi_T^R \,|\, \mathcal{F}_t\right]}{\pi_t}. \tag{7.3}$$

We shall construct models for the nominal and real pricing kernels following the approach presented in the earlier sections. For simplicity we introduce a pair of independent market factors $X_{T_1}$ and $X_{T_2}$, where $0 \le t \le T < T_1 < T_2$, along with the associated Brownian-bridge information processes $\{\xi_{tT_1}\}$ and $\{\xi_{tT_2}\}$ that generate the filtration, and set

$$\pi_t = M_t^{(1)} M_t^{(2)}\, f\left(t, \xi_{tT_1}, \xi_{tT_2}\right), \tag{7.4}$$

and

$$\pi_t^R = M_t^{(1)} M_t^{(2)}\, g\left(t, \xi_{tT_1}, \xi_{tT_2}\right), \tag{7.5}$$

where the $\mathbb{P}$-martingales $\{M_t^{(1)}\}$ and $\{M_t^{(2)}\}$ are given by expressions analogous to (3.2). The price of an inflation-linked bond is thus

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[M_T^{(1)} M_T^{(2)}\, g(T, \xi_{TT_1}, \xi_{TT_2}) \,\middle|\, \xi_{tT_2}, \xi_{tT_2}\right]}{M_t^{(1)} M_t^{(2)}\, f\left(t, \xi_{tT_1}, \xi_{tT_2}\right)}. \tag{7.6}$$

Here we have made use of the Markov property of the information processes. Then we change measure from $\mathbb{P}$ to $\mathbb{B}$ to obtain

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[g(T, \xi_{TT_1}, \xi_{TT_2}) \,\middle|\, \xi_{tT_2}, \xi_{tT_1}\right]}{f(t, \xi_{tT_1}, \xi_{tT_2})}. \tag{7.7}$$

The conditional expectation reduces to a Gaussian integral and the bond price process can be expressed as follows:

$$Q_{tT} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(T, \nu_{tT}^{(1)} y_1 + \frac{T_1 - T}{T_1 - t}\, \xi_{tT_1}, \nu_{tT}^{(2)} y_2 + \frac{T_2 - T}{T_2 - t}\, \xi_{tT_2})}{f(t, \xi_{tT_1}, \xi_{tT_2})} \times \exp\left[-\tfrac{1}{2}\left(y_1^2 + y_2^2\right)\right] \mathrm{d}y_1 \mathrm{d}y_2. \tag{7.8}$$

In such a setting the nominal discount bond has the following price:

$$P_{tT} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(T, \nu_{tT}^{(1)} y_1 + \frac{T_1 - T}{T_1 - t}\, \xi_{tT_1}, \nu_{tT}^{(2)} y_2 + \frac{T_2 - T}{T_2 - t}\, \xi_{tT_2})}{f(t, \xi_{tT_1}, \xi_{tT_2})} \times \exp\left[-\tfrac{1}{2}\left(y_1^2 + y_2^2\right)\right] \mathrm{d}y_1 \mathrm{d}y_2. \tag{7.9}$$

The dynamics of the real pricing kernel can be computed analogously to that of the nominal pricing kernel. Since the real interest rate may be positive or negative, there is no supermartingale condition on $g$. Inserting the expressions for the nominal and the real pricing kernels in (7.1), one obtains the dynamics of the price level. In the case of a general $n$-factor model the pricing kernels are given by expressions of the following form:

$$\pi_t = M_t^{(1)} \cdots M_t^{(n)} \, f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right), \tag{7.10}$$

and

$$\pi_t^R = M_t^{(1)} \cdots M_t^{(n)} \, g\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right). \tag{7.11}$$

The price level is then given by

$$C_t = \frac{g\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}{f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}, \tag{7.12}$$

and for the dynamics of $\{C_t\}$ we obtain:

$$\begin{aligned}\frac{\mathrm{d}C_t}{C_t} = &\left\{ \frac{1}{f_t}\left[\sum_{k=1}^{n} \frac{\xi_{tT_k}}{T_k - t}\, \partial_k f_t - \tfrac{1}{2}\sum_{k=1}^{n} \partial_k \partial_k f_t - \dot{f}\right] \right. \\ &\quad - \frac{1}{g_t}\left[\sum_{k=1}^{n} \frac{\xi_{tT_k}}{T_k - t}\, \partial_k g_t - \tfrac{1}{2}\sum_{k=1}^{n} \partial_k \partial_k g_t - \dot{g}\right] \\ &\quad + \sum_{k=1}^{n} \frac{\sigma_k T_k}{T_k - t}\, X_{tT_k} \left(\frac{1}{g_t}\partial_k g_t - \frac{1}{f_t}\partial_k f_t\right) \\ &\quad \left. - \frac{1}{g_t f_t}\sum_{k=1}^{n} \partial_k g_t \, \partial_k f_t + \frac{1}{f_t^2}\sum_{k=1}^{n} \left(\partial_k f_t\right)^2 \right\} \mathrm{d}t \\ &+ \sum_{k=1}^{n} \left(\frac{1}{g_t}\, \partial_k g_t - \frac{1}{f_t}\, \partial_k f_t\right) \mathrm{d}W_t^{(k)},\end{aligned} \tag{7.13}$$

where $X_{tT_k} = \mathbb{E}\left[X_{T_k} \,|\, \xi_{tT_k}\right]$. In this calculation we have used the relationship

$$\mathrm{d}\xi_{tT_k} = \frac{1}{T_k - t}\left(\sigma_k \, T_k \, X_{tT_k} - \xi_{tT_k}\right) \mathrm{d}t + \mathrm{d}W_t^{(k)}. \tag{7.14}$$

The drift of the price level is the instantaneous rate of inflation:

$$I_t = r_t - r_t^R + \lambda_t \left(\lambda_t - \lambda_t^R\right). \tag{7.15}$$

The volatility of the price level on the other hand is given by the difference between the nominal and the real market prices of risk. Verification of these results is achieved by calculating the dynamics of the nominal and the real pricing kernels. In particular, we have (4.6) together with

$$\frac{\mathrm{d}\pi_t^R}{\pi_t^R} = -r_t^R\,\mathrm{d}t - \lambda_t^R\,\mathrm{d}W_t. \tag{7.16}$$

A calculation shows that

$$\begin{aligned}\frac{\mathrm{d}\pi_t}{\pi_t} = \frac{1}{f_t}\left[\dot{f}_t - \sum_{k=1}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k\, f_t - \tfrac{1}{2}\partial_k\partial_k\, f_t\right)\right]\mathrm{d}t \\ + \frac{1}{f_t}\sum_{k=1}^{n}\left(\partial_k f_t - \frac{\sigma_k T_k}{T_k - t}\,X_{tT_k}\, f_t\right)\mathrm{d}W_t^{(k)}\end{aligned} \tag{7.17}$$

and

$$\begin{aligned}\frac{\mathrm{d}\pi_t^R}{\pi_t^R} = \frac{1}{g_t}\left[\dot{g}_t - \sum_{k=m}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k\, g_t - \tfrac{1}{2}\partial_k\partial_k\, g_t\right)\right]\mathrm{d}t \\ + \frac{1}{g_t}\sum_{k=m}^{n}\left(\partial_k g_t - \frac{\sigma_k T_k}{T_k - t}\,X_{tT_k}\, g_t\right)\mathrm{d}W_t^{(k)}.\end{aligned} \tag{7.18}$$

The resulting dynamics of the price level can then be written in the form

$$\frac{\mathrm{d}C_t}{C_t} = \left[r_t - r_t^R + \lambda_t(\lambda_t - \lambda_t^R)\right]\mathrm{d}t + (\lambda_t - \lambda_t^R)\,\mathrm{d}W_t. \tag{7.19}$$

# 8 Stochastic monetary economy

So far we have indicated how the pricing of fixed-income securities, in particular the nominal and inflation-linked discount bond systems, can be modelled in an information-based framework. We have shown how the nominal and real pricing kernels, and hence the price level, can be modelled in terms of information processes. It is our goal now to consider the relationship between the two pricing kernels, and to develop a simple macroeconomic model based on (a) the liquidity benefit of the money supply, and (b) the rate of consumption of goods and services. This will be carried out in the context of the pricing theory developed in the previous sections. A macroeconomic pricing model that suits the present investigation is that presented in Hughston & Macrina (2008). In this finite-time-horizon model, agents maximize the expected utility derived from the consumption of goods and services and from the liquidity benefit of money supply.

There are three exogenously-specified processes that form the basis of such an economy: (1) the real per-capita rate of consumption of goods and services $\{k_t\}$, (2) the per-capita money supply $\{m_t\}$, and (3) the rate of liquidity benefit $\{\eta_t\}$ provided per unit of money supply. The product $\eta_t m_t$ is the instantaneous benefit rate in nominal units provided by the money supply at time $t$. The goal of the representative agent is to find the optimal strategy for consuming goods and services and for taking advantage of the benefit of the supply of money. Since the liquidity benefit is measured in nominal units, we use the price level to convert its units to those of good and services. The "real" liquidity benefit rate $\{j_t\}$ is thus given by

$$j_t = \frac{\eta_t m_t}{C_t}. \tag{8.1}$$

The agent's rate of utility derived from real consumption and real liquidity benefit is modelled by a bivariate utility function $U(x, y) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ of the Sidrauski (1967) type satisfying $U_x > 0$, $U_{xx} < 0$, $U_y > 0$, $U_{yy} < 0$ and $U_{xx}U_{yy} > (U_{xy})^2$. The strategy that delivers the agent the highest level of total expected utility over the period $[0, T]$ is found by maximising

$$I = \mathbb{E}\left[\int_0^T \Gamma_t U(j_t, k_t)\mathrm{d}t\right], \tag{8.2}$$

when the agent has a budget $H_0$ given by

$$H_0 = \mathbb{E}\left[\int_0^T \pi_t C_t \left(j_t + k_t\right) \mathrm{d}t\right]. \tag{8.3}$$

Here $\Gamma_t$ is a psychological discount factor which we take to be deterministic. In equilibrium, a relation is thus determined between $C_t$, $j_t$, and $k_t$.

An exact solution can be found in the case where the utility function is of a separable bivariate logarithmic type:

$$U(x, y) = a \ln(x) + b \ln(y), \tag{8.4}$$

where $a$ and $b$ are constants. We obtain the following expressions for the pricing kernels:

$$\pi_t = \frac{a_t}{\eta_t\, m_t} \quad \text{and} \quad \pi_t^R = \frac{b_t}{k_t}, \tag{8.5}$$

where $a_t = a\,\Gamma_t/\lambda$ and $b_t = b\,\Gamma_t/\lambda$, and $\lambda$ is determined by the budget constraint. The price level is then given by

$$C_t = \frac{b}{a}\,\frac{\eta_t m_t}{k_t}. \tag{8.6}$$

Next we establish a link between this model and the information-based approach presented in the previous sections. We revert to a low-dimensional example in which the economy is driven by a pair of macroeconomic factors $X_{T_1}$ and $X_{T_2}$ with associated information processes $\{\xi_{tT_k}\}_{k=1,2}$. The nominal and real pricing kernels are taken to be

$$\pi_t = M_t^{(1)} M_t^{(2)}\, f(t, \xi_{tT_1}, \xi_{tT_2}) \quad \text{and} \quad \pi_t^R = M_t^{(1)} M_t^{(2)}\, g(t, \xi_{tT_1}, \xi_{tT_2}), \tag{8.7}$$

where $\{M_t^{(1)}\}$ and $\{M_t^{(2)}\}$ are the $\mathbb{P}$-martingales defined by (5.4). We assume that the real rate of consumption, the money supply and the nominal rate of specific liquidity benefit are given by functions of the form

$$k_t = k(t, \xi_{tT_1}, \xi_{tT_2}), \quad m_t = m(t, \xi_{tT_1}, \xi_{tT_2}), \quad \eta_t = \eta(t, \xi_{tT_1}, \xi_{tT_2}). \tag{8.8}$$

By comparison with (8.5) we thus obtain the following relationships:

$$f(t, \xi_{tT_1}, \xi_{tT_2}) = \frac{a_t}{M_t^{(1)} M_t^{(2)}\, \eta(t, \xi_{tT_1}, \xi_{tT_2})\, m(t, \xi_{tT_1}, \xi_{tT_2})}, \tag{8.9}$$

and

$$g(t, \xi_{tT_1}, \xi_{tT_2}) = \frac{b_t}{M_t^{(1)} M_t^{(2)}\, k(t, \xi_{tT_1}, \xi_{tT_2})}. \tag{8.10}$$

Applying (7.8) and (7.9), we are then able to work out prices of bonds in a monetary economy in which asset prices fluctuate in line with emerging information about macroeconomic factors influencing the economy. The price of an inflation-linked bond is:

$$\begin{aligned} Q_{tT} = {} & \frac{M_t^{(1)} M_t^{(2)}\, \eta(t, \xi_{tT_1}, \xi_{tT_2})\, m(t, \xi_{tT_1}, \xi_{tT_2})}{a_t} \\ & \times \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{b_T\, \exp\left[-\frac{1}{2}\left(y_1^2 + y_2^2\right)\right]\, \mathrm{d}y_1\, \mathrm{d}y_2}{M_T^{(1)}\left(z(y_1)\right) M_T^{(2)}\left(z(y_2)\right) k\left(T, z(y_1), z(y_2)\right)}, \end{aligned} \tag{8.11}$$

where

$$z(y_k) = \nu_{tT}^{(k)}\, y_k + \frac{T_k - T}{T_k - t}\, \xi_{tT_k}, \qquad k = 1, 2. \tag{8.12}$$

The corresponding nominal discount bond price is

$$P_{tT} = \frac{M_t^{(1)} M_t^{(2)}\, \eta(t, \xi_{tT_1}, \xi_{tT_2})\, m(t, \xi_{tT_1}, \xi_{tT_2})}{a_t}$$
$$\times \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{a_T \, \exp\left[-\frac{1}{2}\,(y_1^2 + y_2^2)\right] \, \mathrm{d}y_1 \, \mathrm{d}y_2}{M_T^{(1)}\,(z(y_1))\, M_T^{(2)}\,(z(y_2))\, \eta\,(T, z(y_1), z(y_2))\, m(T, z(y_1), z(y_2))}. \tag{8.13}$$

Similar formulae can be derived in the case of a separable power-utility function. In such a situation, the nominal pricing kernel also depends on the real rate of consumption.

**Acknowledgments.** The authors are grateful to J. Akahori, D. C. Brody, H. Gretarsson, M. Hinnerich, T. Honda, E. Hoyle, E. Mackie, R. Miura, K. Ohashi, P. A. Parbhoo, G. Sarais, J. Sekine, and W. T. Shaw for useful discussions. We are grateful to seminar participants at Ritsumeikan University, Kusatsu, Japan, the KIER-TMU International Workshop on Financial Engineering 2009, Tokyo, and ICS Hitotsubashi University, Tokyo, for helpful comments. LPH acknowledges support from Lloyds TSB, Shell International, and the Aspen Center for Physics.

**References**.

1. D. C. Brody, J. Crosby & H. Li (2008) Convexity adjustments in inflation-linked derivatives. Risk Magazine, September 2008 issue, 124-129.

2. D. C. Brody, L. P. Hughston & A. Macrina (2007) Beyond hazard rates: a new framework for credit-risk modelling. In *Advances in Mathematical Finance, Festschrift Volume in Honour of Dilip Madan*, edited by R. Elliott, M. Fu, R. Jarrow & J.-Y. Yen. Birkhäuser, Basel and Springer, Berlin.

3. D. C. Brody, L. P. Hughston & A. Macrina (2008a) Information-based asset pricing. International Journal of Theoretical and Applied Finance Vol. 11, 107-142.

4. D. C. Brody, L. P. Hughston & A. Macrina (2008b) Dam rain and cumulative gain. Proceedings of the Royal Society London A, Vol. 464, No. 2095, 1801-1822.

5. M. Hinnerich (2008) Inflation-indexed swaps and swaptions. Journal of Banking and Finance, Vol. 32, No. 11, 2293-2306.

6. L. P. Hughston (1998) Inflation derivatives. Merrill Lynch and King's College London report, with added note (2004).

7. L. P. Hughston & A. Macrina (2008) Information, inflation, and interest. Advances in Mathematics of Finance. Banach Center Publications, Vol. 83, 117-138. Institute of Mathematics, Polish Academy of Science, Warsaw.

8. R. Jarrow & Y. Yildirim (2003) Pricing treasury inflation protected securities and related derivative securities using an HJM model. Journal of Financial and Quantitative Analysis, No. 38, 409.

9. A. Macrina (2006) An information-based framework for asset pricing: $X$-factor theory and its applications. PhD thesis, King's College London.

10. F. Mercurio (2005) Pricing inflation-indexed derivatives. Quantitative Finance, Vol. 5, No. 3, 289-302.

11. M. Rutkowski & N. Yu (2007) An extension of the Brody-Hughston-Macrina approach to modeling of defaultable bonds. International Journal of Theoretical and Applied Finance Vol. 10, 557-589.

12. M. Sidrauski (1967) Rational choice and patterns of growth in a monetary economy. American Economic Review, Vol. 57, No. 2, 534-544.