Composition Operators on Weighted Bergman Spaces

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Abstract

In the late 1960's, E.A. Nordgren and J.V. Ryff studied composition operators on the Hardy space H^2. They provided upper and lower bounds on the norms of general composition operators and gave the exact norm in the case where the symbol map is an inner function.

Composition operators themselves, on various other spaces, have been studied by many authors since and much deep work has been done concerning them. Recently, however B.D. MacCluer and T. Kriete have developed the study of composition operators on very general weighted Bergman spaces of the unit disk in the complex plane. My starting point is this work.

Composition operators serve well to link the two areas of analysis, operator theory and complex function theory. The products of this link lie deep in complex analysis and are diverse indeed. These include a thorough study of the Schröeder functional equation:

$$\sigma \circ \varphi = \lambda \sigma$$

and its solutions, see [16] and the references therein, in fact some of the well known conjectures can be linked to composition operators. Nordgren, [12],

has shown that the Invariant Subspace Problem can be solved by classifying the minimal invariant subspaces of a certain composition operator on H^2, and de Branges used composition operators to prove the Bieberbach conjecture.

In this thesis, I use various methods from complex function theory to prove results concerning composition operators on weighted Bergman spaces of the unit disk, the main result is the confirmation of two conjectures of T. Kriete, which appeared in [7]. I also construct, in the final chapter, inner functions which map one arbitrary weighted Bergman space into another.

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Chapter 1

Preliminaries

1.1 Some basic definitions

In this thesis I study composition operators on weighted Bergman spaces. These spaces consist of functions which are analytic in the unit disk

$$\mathbb{D} = \{z\colon |z| < 1\},$$

and satisfy the growth condition

$$\|f\| = \left\{ \iint_{\mathbb{D}} |f(z)|^2 G(|z|) dA(z) \right\}^{1/2} < \infty. \tag{1.1}$$

Here dA denotes normalised 2-dimensional Lebesgue measure on the unit disk. That is

$$dA = \frac{dx\, dy}{\pi} = \frac{r dr\, d\theta}{\pi}.$$

The constant is chosen so that $A(\mathbb{D}) = 1$. $G(r)$ will always denote a continuous, non-increasing function on $[0, 1)$ which satisfies

$$G(r) \to 0 \qquad \text{as } r \to 1.$$

This space, we will denote A_G^2, and call the *weighted Bergman space with weight G*. It is a Hilbert space with inner product

$$\langle f, g \rangle = \iint_{\mathbb{D}} f(z)\overline{g(z)}G(|z|)dA(z).$$

Now, suppose that φ is a function analytic in $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, then by the *composition operator* C_φ we mean the linear operator acting on A_G^2 as

$$C_\varphi(f) = f \circ \varphi.$$

We will mainly be concerned with C_φ as an operator mapping A_G^2 into itself.

The spaces A_G^2 were introduced in [8] where a systematic study of composition operators acting on them is undertaken.

The adjoint operator C_φ^* is the unique operator acting on A_G^2 as

$$\langle C_\varphi(f), g \rangle = \langle f, C_\varphi^*(g) \rangle.$$

We define the operator norm of C_φ as

$$\|C_\varphi\| = \sup_{\|f\|=1} \|C_\varphi(f)\| = \sup_{f \in A_G^2} \frac{\|C_\varphi(f)\|}{\|f\|},$$

then the following relation is true

$$\|C_\varphi\| = \|C_\varphi^*\| = \sqrt{\|C_\varphi C_\varphi^*\|}.$$

The study of composition operators was pioneered in the late 1960s by Nordgren, Ryff and Schwartz, see [11, 14, 15]. They studied C_φ acting on H^2, the Hardy space. In the past 15 years, this area has received much attention once more. Composition operators have been linked to some very deep areas of analysis and operator theory, see for example [16]. Recently Nordgren, Rosenthal and Wintrobe have shown that the solution of the *Invariant subspace problem* for Hilbert space is equivalent to studying the minimal invariant subspaces of a certain composition operator on H^2, see [12].

The question of what conditions on the symbol φ determine when the composition operator C_φ maps A^2_G boundedly into itself has been much studied, see [4] for an introduction. A complete solution to this was given in [8] and [7] by Kriete and MacCluer. It was shown that C_φ maps A^2_G boundedly into itself if, and only if the following condition is satisfied:

$$\limsup_{r\to 1^-} \frac{G(r)}{G(M(r,\varphi))} < \infty, \qquad \text{where } M(r,\varphi) = \max_{|z|=r} |\varphi(z)|. \tag{1.2}$$

As an example, we consider the weighted Bergman spaces with weight function

$$G(r) = (1-r)^\alpha, \qquad \alpha > 0.$$

These are the so-called *standard weighted Bergman spaces* which we will henceforth write as A^2_α. In this case the above criterion is just

$$\limsup_{r\to 1} \left(\frac{1-r}{1-M(r,\varphi)} \right)^\alpha < \infty.$$

This is equivalent to requiring that the *angular derivative* be positive at each point in the unit circle $\partial\mathbb{D}$. This is always true however, as we will now see.

1.2 Geometric Function Theory

In this section we introduce some definitions which will be important throughout the rest of this thesis.

1.2.1 The angular derivative

As with most introductions to geometric function theory in the unit disk, we must first state the following.

Theorem 1 (Schwarz-Pick Theorem). *If φ is an analytic self-map of the unit disk into itself, then*

$$\left|\frac{\varphi(w)-\varphi(z)}{1-\overline{\varphi(w)}\varphi(z)}\right| \leq \left|\frac{w-z}{1-\overline{w}z}\right|.$$

If equality holds in the above for **any** *$z \neq w$, then φ is an automorphism of the disk.*

This theorem has a different interpretation see [1]. If we define the *pseudohyperbolic metric* in the disk as

$$d(z,w) = \left|\frac{z-w}{1-\overline{z}w}\right|,$$

then the pseudohyperbolic metric is clearly invariant under conformal maps. Since the shortest path from 0 to $z \in \mathbb{D}$ is along the radius, and conformal maps map straight lines onto circles; we must have that the geodesics of this metric are either arcs of circles orthogonal to the unit circle, or radii.

Theorem 1 can be restated by saying that self-maps of $\mathbb{D}$ are contractions in the pseudohyperbolic metric, and the automorphisms of $\mathbb{D}$ are the only isometries. As a simple corollary we have the following.

Corollary 1. *If φ is an analytic self-map of the unit disk, then*

$$|\varphi(z)| \leq \frac{|z| + |\varphi(0)|}{1 + |z|\,|\varphi(0)|}.$$

In particular we have that

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1-|\varphi(0)|}{1+|\varphi(0)|},$$

and hence

$$\sup_{r\in[0,1)} \left(\frac{1-r}{1-M(r,\varphi)}\right)^{\alpha} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha} < \infty$$

for any function φ, which proves the claim asserted earlier, that all φ give bounded composition operators on A^2_α.

1.2.2 Julia's Lemma

Let us define the following quantity. Given a self-map of the unit disk φ, let

$$d_\varphi(\zeta) = \liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|}, \tag{1.3}$$

where the limit is taken as $z \to \zeta$ along **any** path in $\mathbb{D}$. An important observation is that the limit is finite only if

$$\lim_{z\to\zeta} |\varphi(z)| = 1.$$

This quantity plays an important role in the determination of the growth of φ at the point ζ. More precisely we have the following

Lemma 1 (Julia's lemma). *Suppose ζ is in the unit circle, and $d_\varphi(\zeta)$, as defined in* (1.3)*, is finite. Let $\{a_n\}$ be a sequence along which this lower limit is achieved and for which $\varphi(a_n)$ converges to $\eta \in \partial\mathbb{D}$. Then for every $z \in \mathbb{D}$*

$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2} \leq d_\varphi(\zeta)\frac{|\zeta-z|^2}{1-|z|^2}.$$

Furthermore, if equality holds at **any** *point $z \in \mathbb{D}$, then φ is an automorphism of $\mathbb{D}$.*

For a proof of this, see [4]. An interesting application of Julia's lemma to Alexandrov measures exists. The *Alexandrov measure* of φ at $\alpha \in \partial\mathbb{D}$ is the positive Borel measure σ_α defined on the unit circle by

$$Re\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\sigma_\alpha(\zeta).$$

The fact that σ_α exists follows from the Herglotz representation and the fact that

$$|\varphi(z)| < 1, \qquad z \in \mathbb{D}.$$

If $\varphi(\alpha) = \beta$ then Julia's lemma implies that

$$\sigma_\beta = \frac{1}{d_\varphi(\alpha)}\delta_\alpha + \mu_\beta,$$

where μ_β is another such positive measure. Here δ_α denotes the Dirac point mass at α.

Proof.

$$\text{Re }\frac{\beta + \varphi(z)}{\beta - \varphi(z)} - \frac{1}{d_\varphi(\alpha)}\frac{\alpha + z}{\alpha - z} = \frac{1 - |\varphi(z)|^2}{|\beta - \varphi(z)|^2} - \frac{1}{d_\varphi(\alpha)}\frac{1 - |z|^2}{|\alpha - z|^2} \geq 0.$$

Hence, Herglotz representation theorem implies there is a positive measure, μ_β on $\partial\mathbb{D}$ such that the above is equal to

$$\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_\beta(\zeta).$$

In fact, it is easy to see that if $\alpha_n : n = 1, \ldots, N$ are points on the unit circle, such that $\varphi(\alpha_n) = \beta$ then there is a positive measure ν_β with $\nu_\beta(\partial\mathbb{D}) = O(1/N)$ such that

$$\sigma_\beta = \nu_\beta + \frac{1}{N}\sum_{n=1}^{N} \frac{1}{d_\varphi(\alpha_n)}\delta_{\alpha_n}.$$

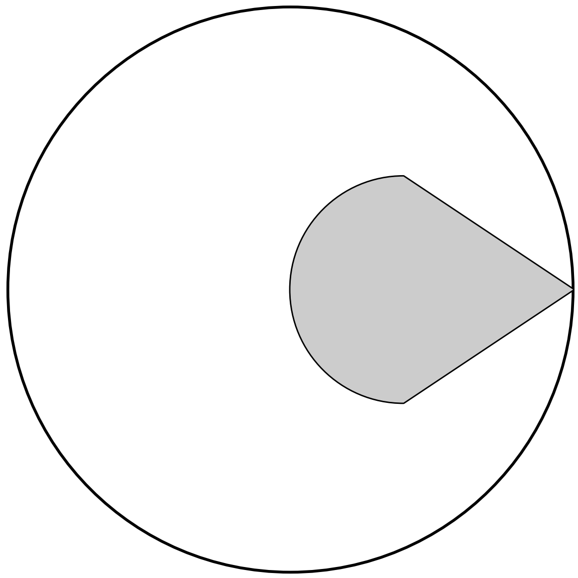

Figure 1.1: A typical Stolz angle

1.2.3 Julia-Caratheodory's Theorem and angular derivatives

A function, analytic in the unit disk, is said to have a *non-tangential limit* at a point $\zeta \in \partial\mathbb{D}$ if the limit of $f(z)$ as $z \to \zeta$ exists in each *Stolz angle*. A Stolz angle is a region similar to the one shown in figure 1.1, the only important aspect of this region is the angle at ζ. In fact it can be replaced by any region which has this angle at ζ.

We say that φ has a *finite angular derivative* at ζ if

$$\lim_{z \to \zeta} \frac{\varphi(z) - \eta}{z - \zeta}$$

exists and is finite in each Stolz angle. Here η is the image of $\varphi(\zeta)$ on the unit circle. If this limit exists, we denote it $\varphi'(\zeta)$. Notice it is not the same as the derivative since we need $\eta \in \partial\mathbb{D}$. They are related as follows however, (see [4] for a proof).

Theorem 2 (Julia-Caratheodory's Theorem). *If φ is an analytic self map of the unit disk $\mathbb{D}$, and $\zeta \in \partial\mathbb{D}$ then the following are equivalent:*

1. $d_\varphi(\zeta) < \infty$.

2. *φ has finite angular derivative at ζ.*

3. *Both φ and φ' have finite nontangential limit at ζ, with $|\eta| = 1$, where $\eta = \lim_{z\to\zeta} \varphi(z)$.*

Moreover, when these conditions hold, we have $\lim_{r\to 1} \varphi'(r\zeta) = \varphi'(\zeta) = d_\varphi(\zeta)\overline{\zeta}\eta$

This theorem concludes the section.

1.3 Boundedness of C_φ.

In this section we briefly look at the boundedness of C_φ on the standard weighted Bergman spaces, A^2_α for $\alpha > 0$. Although we have already seen, from the characterisation of bounded composition operators given by Kriete and MacCluer, that all composition operators are bounded on these spaces, we will now study this a little more. The aim of this section is to highlight some important aspects concerning the boundedness of composition operators. We begin with the following theorem of Littlewood:

Theorem 3 (Littlewood's subordination theorem). *Let φ be an analytic self-map of the unit disk $\mathbb{D}$ such that $\varphi(0) = 0$. Then if f is analytic in $\mathbb{D}$,*

$$\int_0^{2\pi} |f \circ \varphi(re^{i\theta})|^2 d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$

for any $0 < r < 1$.

A proof of this theorem is given in [5, page 10]. So, given any weight function G we have that C_φ is bounded as an operator on A^2_G and

$$\|C_\varphi\| \leq 1$$

whenever $\varphi(0) = 0$.

Now given any self-map φ of $\mathbb{D}$, we have the following decomposition

$$\varphi = \varphi_1 \circ \psi \circ \varphi_2,$$

where $\psi(0) = 0$ and φ_1 and φ_2 are automorphisms of $\mathbb{D}$, and hence each composition operator has the decomposition

$$C_\varphi = C_{\varphi_2} C_\psi C_{\varphi_1}.$$

Since bounded operators on any Hilbert space form a Banach algebra we must have that C_φ is bounded if, and only if C_{φ_1} and C_{φ_2} are. Hence we must show that all automorphisms of the disk give bounded composition operators.

This is easy however. If we look at the equivalent weights $(1 - r^2)^\alpha$ then we find that, using a simple change of variables formula, if

$$\psi(z) = \frac{a - z}{1 - \overline{a}z}$$

is an automorphism of the disk and f is analytic in the disk then

$$\begin{aligned}\iint_{\mathbb{D}} |f\circ\psi(z)|^2(1-|z|^2)^\alpha dA &= \iint_{\mathbb{D}} |f(z)|^2(1-|\psi(z)|^2)^\alpha \frac{1}{|\psi'(z)|} dA \\ &= \iint_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\alpha \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha-1} dA.\end{aligned}$$

It doesn't take much calculation to show that

$$\left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha-1}$$

is bounded in $\mathbb{D}$.

Thus unboundedness of composition operators is essentially determined by the behaviour of the automorphism induced composition operators. This is an important fact to notice. For some weight functions, notably those for which

$$\lim_{r\to 1}\frac{G(r)}{(1-r)^p} = 0, \qquad \text{for every } p > 0,$$

there are unbounded composition operators. This, then, must mean that automorphisms never give bounded composition operators on these spaces, which is true.

1.4 Inner functions

Used by Beurling to characterise the invariant subspaces of the shift operator on H^2, inner functions are probably the most important class of self-maps of the unit disk.

Definition 1. *A self-map, φ, of $\mathbb{D}$ is said to be an inner function if*

$$\lim_{r\to 1}|\varphi(r\zeta)| = 1$$

for almost all $\zeta \in \partial\mathbb{D}$. Note that this limit exists a.e. by Fatou's theorem.

There are essentially only two types of inner functions, Blaschke products and Singular inner functions.

1.4.1 Blaschke Products

Given a finite or infinite sequence $\{z_n\}$ in $\mathbb{D}$, define the *Blaschke product*

$$B(z) = \prod \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}.$$

Now a simple calculation shows that if $|z| \leq R < 1$

$$\left| 1 - \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z} \right| = \left| \frac{(z_n + |z_n| z)(1 - |z_n|)}{z_n(1 - \bar{z}_n z)} \right| \leq \frac{2(1 - |z_n|)}{1 - R}.$$

Hence, if $\{z_n\}$ satisfies the *Blaschke condition*

$$\sum_n (1 - |z_n|) < \infty,$$

then we see that $B(z)$ coverges on compact subsets of $\mathbb{D}$ and is not identically 0 there. Hence it represents an analytic function on $\mathbb{D}$. It can also be shown that $|B(e^{i\theta})| = 1$ for almost all $e^{i\theta} \in \partial\mathbb{D}$, so that B is an inner function.

Note that B has zeros at the points $\{z_n\}$ and only at these points.

1.4.2 Singular inner functions

Whereas Blaschke products represent functions with prescribed zeros in the disk, Singular inner functions represent functions with prescribed zeros on the unit circle. Recall that a measure μ on $\partial\mathbb{D}$ is singular with respect to Lebesgue arc-length measure on $\partial\mathbb{D}$ if the derivative of the function $\mu([0, t])$

is zero almost everywhere with respect to Lebesgue arc-length measure. For example the dirac point-mass, δ_ζ which assigns to an arbitrary set E, the following measure.

$$\delta_\zeta(E) = \begin{cases} 1 & \text{if } \zeta \in E, \\ 0 & \text{otherwise.} \end{cases}$$

is singular. We will henceforth assume that *singular* means *singular with respect to Lebesgue arc-length measure*.

Definition 2. *Let μ be a singular measure on $\partial\mathbb{D}$, then we call S a singular inner function if it has the representation:*

$$S(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)\right\}.$$

Note that if we define the Poisson kernel by

$$P_z(e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2},$$

then

$$|S(z)| = \exp\left\{-\int_0^{2\pi} P_z(e^{it}) d\mu(t)\right\}.$$

It can be shown that there are no other inner functions, and by that we mean that every inner function, I, has the form

$$I = e^{i\lambda} z^n B(z) S(z)$$

where n is some non-negative integer, B is a Blaschke product, S is a singular inner function and λ is real.

1.5 Results

In the following chapters I continue the work on composition operators on large weighted Bergman spaces initiated by Kriete and MacCluer in [8, 9, 7]. In the second chapter I look at unbounded composition operators, the main results are that the sets $E_T = \{f \in A^2_G : Tf \in A^2_G\}$ are of Baire category 1, where T is either C_φ or C^*_φ. I also give a construction of functions in A^2_G whose image is not in A^2_G. These constructions rely heavily on properties of the reproducing kernels of A^2_G.

In the third chapter, I look at bounded composition operators. In [7] Tom Kriete asked the following question of a particular range of weight functions G:

Question: Suppose $G_i = e^{-h_i}$, $i = 1, 2$ are two weight functions such that

$$\frac{h_1'(r)}{h_2'(r)} \to \infty \qquad \text{as } r \to 1,$$

does there exist a C_φ bounded on $A^2_{G_2}$ but not on $A^2_{G_1}$?

I answer this question in the affirmative. The main construction is a function analytic in the disk $\mathbb{D}$ such that

$$\limsup_{r\to 1}(M(r,\varphi) - r)h_1'(r) = \infty,$$

$$\text{but} \quad \limsup_{r\to 1}(M(r,\varphi) - r)h_2'(r) < \infty.$$

This construction has interesting implications concerning a theorem of Burns and Krantz in [3]. We give some consequences of this theorem including the answer to a second question asked in [7].

In the final chapter, I discuss the problem of finding an inner function such that its composition operator maps arbitrarily large Bergman spaces into arbitrarily small Bergman spaces. This is shown to be equivalent to the construction of an inner function with certain slow growth. I construct both a Blaschke product and a singular inner function which solve this problem.

Chapter 2

Unbounded Composition Operators

In this chapter we use *Baire's category theorem* to prove that the pre-image of C_φ and C_φ^* intersected with A_G^2 are of Baire category 1 if C_φ is unbounded. As consequences we construct functions $f \in A_G^2$ for which $C_\varphi f \notin A_G^2$ and similarly for C_φ^*. A subset $\mathcal{S}$ of a topological space $\mathcal{X}$ is said to be *nowhere dense* if, for every point $s \in \bar{\mathcal{S}}$, the closure of $\mathcal{S}$, and every open set $\mathcal{U}$ containing s, there is a point in $\mathcal{U}$ and an open set containing that point which is not in $\mathcal{S}$. Then a topological space is said to be of the *first category* if it is the *countable union of nowhere dense subsets.* We thus have the following famous theorem.

Theorem 4 (Baire's category theorem). *If $\mathcal{X}$ is a complete metric space; then $\mathcal{X}$ is not of the first category in itself.*

We write K_z as the reproducing kernel of A_G^2, i.e. the unique function in A_G^2

with the property

$$f(w) = \langle f, K_w \rangle \qquad \forall f \in A^2_G$$

K_z also has the useful property, $\|K_z\| = \|K_{|z|}\|$.

We have the following theorem of T. Kriete, which relates the growth of $\|K_r\|$, to the decay of $G(r)$, see [7].

Theorem 5. *Suppose $G = e^{-h}$ is an admissible weight function as defined below. Then there exists an increasing function $\beta(r)$, $(0 < r < 1)$ such that $\beta(r) \to \infty$ and*

$$\frac{\|K_r\|^2 G(r)}{\beta(r)} \to 1.$$

as $r \to 1$.

The definition of an admissible weight is very technical, however it can be seen that most 'well-behaved' weight functions are admissible. The definition of admissible is given in [7] where an admissible weight is called a quick admissible weight.

Definition 3. *If v is defined by the equation*

$$G(r) = \exp -v\left(\log\left(\frac{1}{r}\right)\right),$$

then G is an admissible weight function if there is a $t_0 > 0$ such that v is of class C^4 on $(0, t_0)$ and the following 9 conditions hold:

1. *$v'(t) < 0$, $v''(t) > 0$, and $v'''(t) < 0$ on $(0, t_0)$.*

2. *The function*

$$\frac{1}{v''(t)}\left(2+\frac{d^2}{dt^2}\frac{1}{v''(t)}\right)$$

is positive and increasing on $(0, t_0)$.

3. *The positive function* $-v'''(t)/v''(t)$ *is decreasing on* $(0, t_0)$.

4. $$\frac{v'''(t)\sqrt{v(t)}}{v''(t)v'(t)} \to 0 \qquad \text{as } t \to 0.$$

5. $v'''(t)/v''(t)^{\frac{3}{2}} \to 0$ *as* $t \to 0$.

6. *The positive function* $-v'''(t)/v''(t)^2$*, which must tend to* 0 *with* t *by the above condition, does so in an almost monotone sense as follows: there exists a positive function* $g(t)$ *defined and increasing for small* $t > 0$*, so that*

$$-\frac{v'''(t)/v''(t)^2}{g(t)}$$

is bounded above and away from 0 *for* t *near* 0.

7. $v'(t)v''(t)e^{-v(t)}$ *remains bounded as* $t \to 0$.

8. $-tv'(t) \to \infty$ *as* $t \to 0$.

9. $-tv'(t)/v(t)$ *remains bounded as* $t \to 0$.

We will always assume that our weight function G is admissible as we will use Theorem 5 extensively.

2.1 The adjoint composition operator

Now if f is an arbitrary function in A^2_G, we have

$$\begin{aligned} \langle f, C_\varphi^* K_z \rangle = \langle C_\varphi f, K_z \rangle &= f \circ \varphi(z) \\ &= \langle f, K_{\varphi(z)} \rangle. \end{aligned}$$

Hence we see that

$$C_\varphi^* K_z = K_{\varphi(z)}.$$

This is the crux of the following proof.

Theorem 6. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic with maximum modulus function $M(r)$ satisfying*

$$\limsup_{r \to 1^-} \frac{G(r)}{G(M(r))} = \infty.$$

Then the set

$$E = \{ f \in A^2_G \ : \ C_\varphi^* f \in A^2_G \}$$

is of Baire category 1.

Proof. Now by hypothesis there exists a sequence $r_n \to 1^-$ such that

$$\lim_{n \to \infty} \frac{G(r_n)}{G(M(r_n))} = \infty.$$

Thus for n large enough $M(r_n) > r_n$; hence from Theorem 5 we see that since β is increasing:

$$\frac{\|K_{M(r_n)}\|}{\|K_{r_n}\|} \sim \frac{G(r_n)}{G(M(r_n))} \frac{\beta(M(r_n))}{\beta(r_n)} \geq \frac{G(r_n)}{G(M(r_n))} \to \infty$$

and so there is a sequence $(z_n) \in \mathbb{D}$ such that

$$\lim_{n\to\infty} \frac{\|K_{\varphi(z_n)}\|}{\|K_{z_n}\|} = \infty$$

We can now split E up into a countable union of nowhere dense subsets, as follows:

Set

$$E_m = \{f \in A^2_G \ : \ \|C^*_\varphi f\| \leq m\} \qquad m = 0, 1, \ldots$$

then clearly we have

$$E = \bigcup_{m=0}^{\infty} E_m.$$

Thus it suffices to show that E_m is nowhere dense. To do this we need to show that for any $f \in E_m$ and $\epsilon > 0$ there is a function inside the ϵ-ball around f that isn't in the set E_m.

To procceed, we take any $\epsilon > 0$ and any $f \in E_m$, then choose an n so that if $p_n(z) = \left(1 - \frac{1}{n}\right) f(z)$, we have $\|f - p_n\| = \frac{1}{n}\|f\| < \epsilon/2$.

Now by the calculation at the beginning of the proof we can choose an n large enough so that

$$\frac{\|K_{\varphi(z_n)}\|}{\|K_{z_n}\|} > \frac{4m}{\epsilon}. \tag{2.1}$$

Then if we let

$$g(z) = p_n(z) + \frac{\epsilon}{2}\frac{K_{z_n}(z)}{\|K_{z_n}\|},$$

we get

$$\|f - g\| \leq \|f - p_n\| + \frac{\epsilon}{2}\left\|\frac{K_{z_n}}{\|K_{z_n}\|}\right\| < \epsilon.$$

But

$$\begin{aligned} \|C_\varphi^* g\| &= \left\| C_\varphi^* p_n + \frac{\epsilon}{2} C_\varphi^* \frac{K_{z_n}}{\|K_{z_n}\|} \right\| \\ &\geq \left| \frac{\epsilon}{2} \frac{\|K_{\varphi(z_n)}(z)\|}{\|K_{z_n}\|} - \|C_\varphi^* p_n\| \right| \\ &> \frac{\epsilon}{2} \frac{\|K_{\varphi(z_n)}(z)\|}{\|K_{z_n}\|} - m \\ &> m, \end{aligned}$$

by (2.1). So g is not in E_m. Hence E_m is nowhere dense and E is of Baire category 1.

We note that this theorem provides some insight into exactly why a function which fails to satisfy (1.2) gives an unbounded composition operator. A reason why the reproducing kernel function is important in this respect is due to the inequality: $|f(z)| \leq \|f\| \, \|K_z\|$; which holds for all functions f in A_G^2 and more importantly, noticing that the inequality becomes an equality when $f = K_z$. So in this sense, the reproducing kernel, K_z, provides us with a function which has the worst growth at the point z.

We can, in fact, use the kernel functions to construct functions, f, in A_G^2 for which $\|C_\varphi^* f\| = \infty$ as follows. Firstly, we choose a sequence of complex numbers (ϵ_n) with the following properties:

$$\sum_{n=0}^{\infty} |\epsilon_n| < \infty, \tag{2.2}$$

$$n^2|\epsilon_n| \to \infty \;\; n \to \infty. \tag{2.3}$$

Secondly, we can choose a sequence $(z_n) \subset \mathbb{D}$ with the following:

$$\frac{\|K_{\varphi(z_n)}\|}{\|K_{z_n}\|} \geq n^2, \tag{2.4}$$

$$\sum_{n=0}^{\infty}(1 - |\varphi(z_n)|) < \infty, \tag{2.5}$$

$$\prod_{j \neq k} \left| \frac{\varphi(z_k) - \varphi(z_j)}{1 - \overline{\varphi(z_j)}\varphi(z_k)} \right| \geq \delta > 0. \tag{2.6}$$

Remarks: a) The sequence (z_n) can be constructed by simply choosing a subsequence of the sequence used in the proof of Theorem 6. b) We will also see that we may replace the n^2 in (2.3) and (2.4) by any suitable increasing function of n tending to infinity rapidly enough.

Now with these sequences, we define a function

$$f = \sum_{n=0}^{\infty} \epsilon_n \frac{K_{z_n}}{\|K_{z_n}\|}$$

Then this function converges uniformly in compact subsets of the unit disk since for $|z| \leq R < 1$ we have $|K_{z_n}(z)| \leq \|K_{z_n}\| \, \|K_z\| \leq \|K_{z_n}\| \, \|K_R\|$ so that

$$|f(z)| \leq \|K_R\| \sum |\epsilon_n| < \infty,$$

by (2.2). Hence, f is analytic in $\mathbb{D}$ and since

$$\|f\| \leq \sum |\epsilon_n| < \infty$$

it is in A^2_G.

We claim that this function is such that $C_\varphi^*(f)$ is not in A_G^2. To prove this we use the Riesz Representation Theorem and show that the linear functional defined by $C_\varphi^*(f)$ is unbounded. To do this we must find a family of functions $(f_k) \subset A_G^2$ such that

$$\frac{|\langle f_k, C_\varphi^*(f)\rangle|}{\|f_k\|} \to \infty \qquad k \to \infty.$$

We let

$$B_k(z) = \prod_{j \neq k} \left(\frac{z - \varphi(z_j)}{1 - \overline{\varphi(z_j)} z} \right).$$

By condition (2.5), this Blaschke product converges. Now we let $f_k = B_k K_{\varphi(z_k)}$ so that we have

$$\begin{aligned} |\langle f_k, C_\varphi^* f\rangle| &= \left| \sum \frac{\overline{\epsilon_n}}{\|K_{z_n}\|} \langle B_k K_{\varphi(z_k)}, K_{\varphi(z_n)}\rangle \right| \\ &= |\epsilon_k||B_k(\varphi(z_k))| \frac{\|K_{\varphi(z_k)}\|^2}{\|K_{z_k}\|} \\ &\geq \delta k^2 |\epsilon_k| \, \|K_{\varphi(z_k)}\|. \end{aligned}$$

Hence, since $\|f_k\| \leq \|K_{\varphi(z_k)}\|$,

$$\frac{|\langle f_k, C_\varphi^* f\rangle|}{\|f_k\|} \geq \delta k^2 |\epsilon_k| \to \infty. \tag{2.7}$$

$C_\varphi^*(f)$ must, then, give an unbounded linear functional on A_G^2 and therefore it cannot be a member of A_G^2, which is what we were required to prove.

2.2 The composition operator

The above construction can be used to prove our second category theorem.

Theorem 7. *If φ is analytice in $\mathbb{D}$ and G is an admissible weight such that*

$$\limsup_{r\to 1} \frac{G(r)}{G(M(r,\varphi))} = \infty,$$

then the set

$$F = \{g \in A_G^2 \ : \ C_\varphi(g) \in A_G^2\}$$

is of Baire category 1 in A_G^2.

Proof. We prove this in the same way we proved Theorem 6, but first we need a substitute for the functions $\dfrac{K_{z_n}}{\|K_{z_n}\|}$ used in that proof.
Let us define the linear functionals T_k as

$$T_k(g) = \left\langle g, \frac{C_\varphi(f_k)}{\|f_k\|} \right\rangle,$$

where f_k are the functions constructed in the proof of theorem 6. Now suppose $\|T_k\| \le m$ for all k, then for fixed g

$$|T_k(g)| \le m\|g\|$$

for $k = 0, 1, \ldots$. But by (2.7) we see that

$$\frac{|\langle f, C_\varphi(f_k)\rangle|}{\|f\|\|f_k\|} \to \infty$$

That is

$$\frac{|T_k(f)|}{\|f\|} \to \infty.$$

We must, then, have

$$\limsup_{k\to\infty} \|T_k\| = \infty.$$

But by Riesz's representation theorem

$$\|T_k\| = \frac{\|C_\varphi f_k\|}{\|f_k\|}.$$

So

$$\limsup_{k\to\infty} \frac{\|C_\varphi f_k\|}{\|f_k\|} = \infty.$$

Hence by a suitable choice of subsequence, (n_k), if we let

$$g_k = \frac{f_{n_k}}{\|f_{n_k}\|}$$

we have our family of functions with the properties: $\|g_k\| = 1$ and $\|C_\varphi(g_k)\| \to \infty$ as $k \to \infty$. The rest of the proof is as the proof of Theorem 6 and is therefore omitted.

We can now construct a function $f \in A^2_G$ such that $C_\varphi f \notin A^2_G$ as follows using the above functions g_k. We will assume that the conditions (2.2) to (2.6) hold for the sequences (ϵ_n) and (z_n). Then as before, we see that the function

$$h(z) = \sum_{n=0}^{\infty} \epsilon_k g_k$$

is in A^2_G, but that

$$\frac{|\langle K_{z_{n_j}}, C_\varphi h\rangle|}{\|K_{z_{n_j}}\|} \to \infty.$$

So $C_\varphi h \notin A^2_G$. The details are as above and are therefore omitted.

Chapter 3

Bounded Composition Operators

3.1 Introduction

We define

$$\mathcal{C}(G) = \{\varphi \,:\, C_\varphi \text{ is bounded as an operator mapping } A^2_G \text{ into } A^2_G\}.$$

On one hand, we have shown in Section 1.3 that for $G(r) = (1-r)^\alpha$, $\alpha > 0$, $\mathcal{C}(G)$ consists of all analytic self-maps of $\mathbb{D}$; this is a consequence of the fact that all such maps of $\mathbb{D}$ can be decomposed as $\varphi = \varphi_1 \circ \psi \circ \varphi_2$, where $\psi(0) = 0$ and φ_1 and φ_2 are automorphisms of the disk.

On the other hand, it has been shown in [9] that if we write $G = e^{-h}$ and assume that $\limsup_{r\to 1}(1-r)^3 h'(r) = \infty$, then

$$\mathcal{C}(G) = \{\varphi \,:\, \varphi(z) = e^{i\lambda}z \text{ or } |\varphi'(\zeta)| > 1\ \forall\, \zeta \in \partial\mathbb{D}\}, \tag{3.1}$$

where $|\varphi'(\zeta)|$ denotes the angular derivative of φ at $\zeta \in \partial\mathbb{D}$, as defined in

Chapter 1. Thus, in these ranges, $\mathcal{C}(G)$ is independent of the choice of G.

We say that G is a *fast weight* if

$$\lim_{r\to 1}\frac{G(r)}{(1-r)^p}=0 \qquad \forall p>0.$$

These weights should be seen as the opposite of the standard weights. The Bergman spaces they define contain functions with very fast growth, and more importantly, functions with essential singularities on the boundary. It is not surprising that automorphisms of the unit disk do not give bounded composition operators on these spaces.

The question was raised in [7] of what monotonicity, if any, does $\mathcal{C}(G)$ display in the range where G is a fast weight, but $(1-r)^3h'(r)$ remains bounded as $r\to 1$?

It is readily seen from the condition (1.2) that if $G_i=e^{-h_i}$, for $i=1,2$, are fast weights such that

$$\frac{h_2'(r)}{h_1'(r)}\leq C,$$

then we have

$$\begin{aligned}
\frac{G_2(r)}{G_2(M(r,\varphi))} &= \exp\left(h(M(r,\varphi))-h(r)\right)\\
&= \exp\int_r^{M(r,\varphi)} h_2'(s)ds\\
&\leq \exp C\int_r^{M(r,\varphi)} h_1'(s)ds\\
&= \left(\frac{G_1(r)}{G_1(M(r,\varphi))}\right)^C.
\end{aligned}$$

Hence,

$$\mathcal{C}(G_1) \subseteq \mathcal{C}(G_2).$$

See [7, Theorem 4]. It can also be shown that if

$$\lim_{r\to 1} \frac{h_2'(r)}{h_1'(r)} = 0,$$

then

$$\mathcal{C}(G_1) = \mathcal{C}(G^*).$$

Here $G^* = G_1/G_2$. Put another way, given a weight G, one can find another weight $\hat{G}$ such that $G/\hat{G} \to \infty$ or 0 but $\mathcal{C}(G) = \mathcal{C}(\hat{G})$. This can also be seen by noticing that $\mathcal{C}(G) = \mathcal{C}(G^p)$ for any $p > 0$. Hence monotonicity of $\mathcal{C}(G)$ does not depend on the weight function G in itself, but rather, on h'.

3.2 Main Results

We prove the following theorem which was conjectured in [7].

Theorem 8. *Let $G_i = e^{-h_i}$ be such that $(1-r)^3 h_i'(r)$ remains bounded as $r \to 1$, for each $i = 1, 2$. Suppose that*

$$\lim_{r\to 1} \frac{h_1'(r)}{h_2'(r)} = \infty.$$

Then

$$\mathcal{C}(G_1) \subsetneqq \mathcal{C}(G_2).$$

We note that if there are constants $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{h_1'(r)}{h_2'(r)} \leq c_2$$

then $\mathcal{C}(G_1) = \mathcal{C}(G_2)$, and so the theorem can be considered as exploring the converse to this statement.

We will say that an increasing function $\omega(r)$ belongs to the class π_0 if whenever the two quantities $a(r)$ and $b(r)$ are such that

$$\lim_{r\to 1} \frac{1-a(r)}{1-b(r)} = 1,$$

then there are constants $c_1, c_2 > 0$, depending only on $a(r)$ and $b(r)$, such that

$$c_1 \leq \frac{\omega(a(r))}{\omega(b(r))} \leq c_2.$$

We will say that a weight function $G = e^{-h} \in \Pi$ if $h' \in \pi_0$. Notice firstly, that π_0 contains all functions such that

$$\omega(r) = (1-r)^{-\gamma} p\left(\log \frac{1}{1-r}\right),$$

where p is a polynomial, and secondly, that $\omega \in \pi_0$ is actually a condition on the growth of $\omega'(r)$.

From now on, we will assume $G_2 \in \Pi$, which we may do by replacing h_2 by another *smoother* function h_2^* for which $h_2(r)/h_2^*(r)$ remains bounded as $r \to 1$.

To prove Theorem 8 we will construct explicitly a function φ such that C_φ is bounded on $A^2_{G_2}$ but not bounded on $A^2_{G_1}$.

Let $0 \leq r_1 < r_2 < \cdots < r_n \to 1$ as $n \to \infty$. Then for a sequence $\Lambda = (\lambda_n)_{n=1}^{\infty}$ define

$$F(r) = F(r;\Lambda) = \sum_{n=1}^{\infty} \chi_{[r_n, r_{n+1})}(r) \left(\frac{1}{1-r}\right)^{\lambda_n},$$

where $\chi_{[r_n,r_{n+1})}(r)$ is the *characteristic function* for the interval $[r_n, r_{n+1})$.

Lemma 2. *Let* $\rho : [0,1) \to [0,\infty)$ *be an increasing continuous function with*

$$\lim_{r\to 1} \rho(r) = \infty, \tag{3.2}$$

$$\forall\, \delta > 0 \quad \lim_{r\to 1} (1-r)^{\delta} \rho(r) \to 0. \tag{3.3}$$

Then there is a sequence (r_n) *with*

$$\limsup_{n\to\infty} \frac{n}{\log \frac{1}{1-r_n}} > 0. \tag{3.4}$$

Moreover $1 - r_n = \mathcal{O}(1 - r_{n+1})$, *and there exists a sequence,* $\Lambda = (\lambda_n)$ *with* $\lambda_n \to 0$, *such that*

$$\rho(r) \sim F(r;\Lambda) \qquad \text{as } r \to 1. \tag{3.5}$$

We will postpone the proof of this lemma until later.

We now define

$$\mathcal{L}(r;\rho) = (1-r)\frac{\rho'(r)}{\rho(r)}, \text{ and } \qquad \mathcal{F}(r;\rho) = \frac{\log \rho(r)}{\log \dfrac{1}{1-r}}.$$

The following corollary shows the importance of $\mathcal{F}$ and $\mathcal{L}$.

Corollary 2. *If* $\rho(r) \in C^1$ *is such that*

$$\mathcal{L}(r;\rho) < \mathcal{F}(r;\rho) \tag{3.6}$$

then the sequence (λ_n) *of Lemma 2 associated with* ρ *is decreasing.*

Since the proof of this corollary depends on the proof of lemma 2, we postpone it until we have proved the lemma.

It is clear that the condition (3.6) is sufficient for ρ to be in π_0. Hence membership of ρ in π_0 can be seen as a growth condition on

$$(\log\log\rho)'.$$

We also need the following lemma which was proved with help from J. Clunie, to whom I am greatly indebted. We write $H_- = \{z : \text{Im } z < 0\}$ for the lower half-plane.

Lemma 3. *If* $1 \leq b \leq 3$ *then*

$$Re\,\frac{(1-z)^b}{z} \leq \frac{(1-|z|)^b}{|z|}.$$

Proof. We will show that the above real part is decreasing for $z = re^{i\theta}$ and $\theta \in [0, \pi]$. Since the function $(1-z)^b/z$ is clearly symmetric about the real axis, this will suffice.

We have

$$\begin{aligned}\frac{\partial}{\partial\theta}\text{Re}\,\frac{(1-z)^b}{z} &= \text{Re}\,iz\left(\frac{-bz(1-z)^{b-1}-(1-z)^b}{z^2}\right)\\ &= \text{Re}\,-\frac{i}{z}(1-z)^{b-1}(1+(b-1)z)\\ &= \text{Im}\,\frac{(1-z)^{b-1}}{z}(1+(b-1)z) = v(z).\end{aligned}$$

Now let $c = b-1$, so that $c \in [0, 2]$. We will deal only with the domain $D_\delta = \{z : \text{Im } z > 0, \delta < |z| < 1\}$. This will suffice since, as we have already mentioned, $\dfrac{(1-z)^b}{z}$ is symmetric about the real axis and is defined everywhere except 0. Since we want $v(z) \le 0$, we will write

$$v(z) = \text{Im } \frac{(1-z)^c}{z} + c(1-z)^c = \text{Im } f_1(z) + f_2(z).$$

Firstly, it is clear that since $Arg(1-z) \in [-\pi/2, 0]$ and $c \le 2$, we must have that Im $f_2(z) \le 0$, and so we only need to show that Im $f_1(z) \le 0$.

But since $0 < c \le 2$

$$\frac{1}{f_1(z)} = \frac{z}{(1-z)^c}$$

is the univalent, conformal map which takes $\mathbb{D}$ onto the domain

$$L = \{z \,:\, \left|Arg(z + 2^{-c})\right| < c\pi/2\}.$$

Clearly $1/f_1$ maps the upper half of the unit disk into the upper half plane since it is univalent, and since the map $\dfrac{1}{w}$ maps the upper half plane into the lower and vice versa, we must have

$$f_1(D_\delta) \subset H_-,$$

as required. The result now follows since δ is arbitrary. □

The crux of the proof of Theorem 8 is the following construction.

Proposition 1. *a) Suppose that*

$$\lim_{r \to 1} \frac{\rho_1(r)}{\rho_2(r)} = \infty.$$

Then there is a function g, analytic in $\mathbb{D}$ such that

$$\limsup_{r \to 1} g(r)\rho_1(r) = \infty,$$
$$\text{but} \quad \sup_{0<r<1} g(r)\rho_2(r) < \infty.$$

Moreover if

$$\varphi(z) = z + t(1-z)^a g(z),$$

where t is chosen small enough, then

$$M(r, \varphi) = \varphi(r), \qquad \textit{for } r \textit{ sufficiently close to } 1.$$

b) Suppose, on the other hand, that

$$\lim_{r \to 1} \frac{\rho_2(r)}{\rho_1(r)} = \infty.$$

Then there is a function h, analytic in $\mathbb{D}$ such that

$$\liminf_{r \to 1} h(r)\, \rho_1(r) = 0,$$
$$\text{but} \quad \liminf_{r \to 1} h(r)\, \rho_2(r) > 0.$$

Moreover we have that $h(z) \neq 0$ in $\mathbb{D}$ and again, if

$$\psi(z) = z + t\frac{(1-z)^a}{h(z)},$$

where t is chosen small enough, then

$$M(r, \psi) = \psi(r), \qquad \textit{for } r \textit{ sufficiently close to } 1.$$

Proof. Part a). The function g, which we will construct, is of the form

$$g(z) = \sum_{i=1}^{\infty} a_i (1-z)^{\lambda_i},$$

where the a_i will be 0 infinitely often.

Now we can find a function $\tau_0(r) \uparrow \infty$ such that

$$\frac{\rho_1(r)}{\rho_2(r)} \frac{1}{\tau_0(r)} \to \infty \qquad \text{as } r \to 1, \tag{3.7}$$

and

$$\frac{\rho_2(r)}{\tau_0(r)} \to \infty \qquad \text{as } r \to 1. \tag{3.8}$$

Now, let $M = (\mu_n)$ and $\Lambda = (\lambda_n)$ be the sequences of Lemma 2 associated with ρ_2 and τ_0 respectively. We can assume that these sequences are decreasing since, by the corollary to lemma 2, this depends only on the growth of $\mathcal{L}(r; \rho_2)$ and $\mathcal{L}(r; \tau_0)$, and these can be replaced by suitable functions which satisfy this criterion.

Given a subsequence r_{n_k} of the r_n, which we will choose later, let

$$a_i = \begin{cases} \dfrac{1}{\rho_2(r_{n_k})} - \dfrac{1}{\rho_2(r_{n_{k+1}})} & \text{if } i = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that

$$\begin{aligned} \rho_1(r_{n_k}) g(r_{n_k}) &\geq \rho_1(r_{n_k}) \sum_{i \geq n_k} a_i (1 - r_{n_k})^{\lambda_i} \\ &\geq \frac{\rho_1(r_{n_k})}{\rho_2(r_{n_k})} (1 - r_{n_k})^{\lambda_{n_k}} \\ &\sim \frac{\rho_1(r_{n_k})}{\rho_2(r_{n_k})} \frac{1}{\tau_0(r_{n_k})} \to \infty \qquad \text{as } k \to \infty. \end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned}\sup_{r_n\le r\le r_{n+1}} \rho_2(r)g(r) &\sim \sup_{r_n\le r\le r_{n+1}} \sum_{i=1}^{\infty} a_i(1-r)^{\lambda_i-\mu_n} \\ &\le \sum_{i=1}^{m(n)} a_i(1-r_n)^{\lambda_i-\mu_n} + \sum_{i>m(n)} a_i(1-r_{n+1})^{\lambda_i-\mu_n},\end{aligned}$$

with $m(n)$ defined as the largest integer, m, such that $\lambda_m \ge \mu_n$. The above does not exceed

$$\begin{aligned}&\sum_{i=1}^{\infty} a_i + c\sum_{i>m(n)} a_i(1-r_n)^{-\mu_n} \\ \le\ & 1/\rho_2(r_{n_1}) + c\rho_2(r_n)\sum_{i>m(n)} a_i.\end{aligned}$$

Let us mention two important features of the function $m(n)$. Firstly, $m(n)$ is increasing since the sequences (λ_n) and (μ_n) are decreasing, and secondly, we see that by (3.8), $m(n) < n$ for all n large enough.

We can therefore define a sequence n_k such that

$$n_k < m(n_{k+1}) < n_{k+1}.$$

Now if we assume that $m(n_{k+1})$ is always much larger than n_k, then there is a largest integer $n_{\alpha(k)}$ such that $m(n) \le n_k$ for $n = n_k, \ldots, n_{\alpha(k)}$ and $m(n) > n_k$ for $n = n_{\alpha(k)} + 1, \ldots, n_{k+1}$.

For the first case above we have

$$\rho_2(r_n)\sum_{i>m(n)} a_i \le \rho_2(r_n)\sum_{i\ge n_k} a_i \le \frac{\rho_2(r_{n_{\alpha(k)}})}{\rho_2(r_{n_k})},$$

and for the second,

$$\rho_2(r_n) \sum_{i>m(n)} a_i \leq \rho_2(r_n) \sum_{i\geq n_{k+1}} a_i \leq 1.$$

Thus we need $\dfrac{\rho_2(r_{n_{\alpha(k)}})}{\rho_2(r_{n_k})}$ to be bounded. This is clearly not true in general. However since all we required of τ_0 is that

$$\lim_{k\to\infty} \frac{\rho_1(r_{n_k})}{\rho_2(r_{n_k})} \frac{1}{\tau_0(r_{n_k})} = \infty,$$

we can simply construct a function $\tau(r)$, equal to $\tau_0(r)$ at each r_{n_k} such that, for $m(n)$ and $n_{\alpha(k)}$ defined as above with τ_0 replaced by τ, we have

1. $n_k < m(n_{k+1}) < n_{k+1}$,
2. $\dfrac{\rho_2(r_{n_{\alpha(k)}})}{\rho_2(r_{n_k})} \leq C.$

Of course, in general, (3.7) and (3.8) will not hold for such τ, but this will not matter.

To construct τ from τ_0 we let $n_{\alpha(k)}$ be the sequence constructed above from τ_0. Given some n_k let $n_{\beta(k)} < n_{\alpha(k)}$ be such that

$$\frac{\rho_2(r_{n_{\beta(k)}})}{\rho_2(r_{n_k})} \leq K,$$

where $K > 1$ is some pre-chosen constant. Clearly we can assume that the sequence (n_k) has been chosen so that

$$\tau_0(r_{n_{k+1}-1}) > \rho_2(r_{n_{\alpha(k)}}).$$

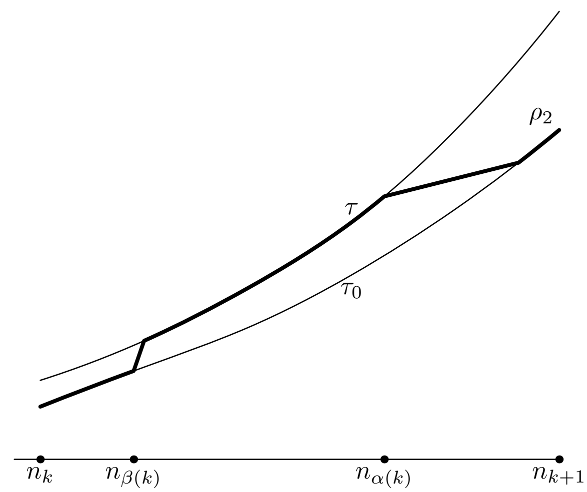

Figure 3.1: Construction of τ from ρ_2 and τ_0

Suppose, now, that τ has been defined for $r < r_{n_k}$. Then we define $\tau(r)$ for $r_{n_k} \le r \le r_{n_{k+1}}$ as follows:

$$
\begin{array}{ll}
r_{n_k} \le r \le r_{n_{\beta(k)}} & \tau(r) = \tau_0(r), \\
r_{n_{\beta(k)}} \le r \le r_{n_{\beta(k)}+1} & \tau(r) \text{ is linear, increasing from } \tau_0(r_{n_{\beta(k)}}) \text{ to } \rho_2(r_{n_{\beta(k)}+1}), \\
r_{n_{\beta(k)}+1} \le r \le r_{n_{\alpha(k)}} & \tau(r) = \rho_2(r), \\
r_{n_{\alpha(k)}} \le r \le r_{n_{k+1}-1} & \tau(r) \text{ is linear, increasing from } \rho_2(r_{n_{\alpha(k)}}) \text{ to } \tau_0(r_{n_{k+1}-1}), \\
r_{n_{k+1}-1} \le r \le r_{n_{k+1}} & \tau(r) = \tau_0(r).
\end{array}
$$

Thus $\tau(r)$ is *jumping* between $\tau_0(r)$ and $\rho_2(r)$, but is always increasing (See the above figure 3.1). We show that this function τ satisfies all that is required of it. We will use the obvious notation $m_\tau(n)$ and $m_{\tau_0}(n)$ for the functions $m(n)$ defined above for τ and τ_0 respectively.

For $n_k \le n \le n_{\beta(k)}$, τ is the same as τ_0 and so $m_\tau(n) = m_{\tau_0}(n) < n_k$.

For $n = n_{\beta(k)} + 1$, $\tau(r_n) = \rho_2(r_n)$ and so $m_\tau(n) \geq n > n_k$.

For the other values of n we see that since $\tau(r)$ is increasing from $r_{n_{\beta(k)}+1}$ to $r_{n_{k+1}}$ and is equal to ρ_2 on the interval $[r_{n_{\beta(k)}+1}, r_{n_{\alpha(k)}}]$, m_τ is increasing and so $m_\tau(n) \geq m_\tau(n_{\beta(k)} + 1) > n_k$. This concludes our construction of g.

To show that $M(r, \varphi) = \varphi(r)$ for $r \geq r_0$ when $\varphi(z) = z + t(1-z)^a g(z)$ we look at

$$|\varphi(z)|^2 = |z|^2 + 2t\text{Re}\,\overline{z}\sum_{j=1}^{\infty} a_j(1-z)^{a+\lambda_j} + t^2\left|\sum_{j=1}^{\infty} a_j(1-z)^{a+\lambda_j}\right|^2.$$

Setting $z = re^{i\theta}$ and $se^{ix} = 1 - z$, we have

$$|\varphi(z)|^2 = r^2 + 2t\text{Re}\,\sum_j a_j r e^{-i\theta} s^{a+\lambda_j} e^{i(a+\lambda_j)x} + t^2\left|\sum_j a_j s^{a+\lambda_j} e^{i(a+\lambda_j)x}\right|^2.$$

Let us, then, define

$$I_n = \left|\sum_{j=n}^{\infty} a_j s^{a+\lambda_j} e^{i(a+\lambda_j)x}\right|^2.$$

An easy calculation shows that we have the relation

$$I_n = a_n^2 s^{2(a+\lambda_n)} + 2a_n s^{a+\lambda_n}\sum_{j=n+1}^{\infty} a_j s^{a+\lambda_j}\cos(\lambda_n - \lambda_j)x + I_{n+1}.$$

Hence,

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty} I_n - I_{n+1} \\ &= \sum_{n=1}^{\infty} a_n^2 s^{2(a+\lambda_n)} + 2\sum_{n=1}^{\infty} a_n s^{a+\lambda_n}\sum_{j=n+1}^{\infty} a_j s^{a+\lambda_j}\cos(\lambda_n - \lambda_j)x. \end{aligned}$$

It follows that

$$\begin{aligned}
|\varphi(r)|^2 &- \left|\varphi(re^{i\theta})\right|^2 = 2t\sum_{j=1}^{\infty} a_j\left\{r(1-r)^{a+\lambda_j} - rs^{a+\lambda_j}\cos[(a+\lambda_j)x-\theta]\right\} \\
&+ t^2\sum_{n=1}^{\infty} a_n^2\left\{(1-r)^{2(a+\lambda_n)} - s^{2(a+\lambda_n)}\right\} \\
&+ 2t^2\sum_{n=1}^{\infty} a_n\Bigg\{(1-r)^{a+\lambda_n}\sum_{j=n+1}^{\infty} a_j(1-r)^{a+\lambda_j} \\
&\qquad -s^{a+\lambda_n}\sum_{j=n+1}^{\infty} a_j s^{a+\lambda_j}\cos(\lambda_n-\lambda_j)x\Bigg\} \\
&= 2t\mathcal{S}_1 + t^2\mathcal{S}_2 + 2t^2\mathcal{S}_3.
\end{aligned}$$

Clearly, we need this quantity to be non-negative for all r large enough. But $\mathcal{S}_1 > 0$ by Lemma 3, since

$$\text{Re }(1-r)^b - (1-re^{i\theta})^b e^{-i\theta} > 0$$

when $b \leq 3$, this is precisely where this condition is used. It is easy to see that $\mathcal{S}_2 < 0$ and $\mathcal{S}_3 < 0$ if we choose $\lambda_n - \lambda_j < \lambda_n$ small enough, which we can do.

Hence we can choose t small enough so that the above is in fact non-negative for all r close to 1.

The second part of the proposition is proved similarly.

Part b) Suppose now that

$$\frac{\rho_2(r)}{\rho_1(r)} \to \infty \qquad r \to 1.$$

Then we need a function h such that

$$\begin{aligned} \liminf_{r\to 1} \rho_2(r)h(r) &> 0, \\ \text{and } \liminf_{r\to 1} \rho_1(r)h(r) &= 0. \end{aligned} \tag{3.10}$$

However, without loss of generality, we will construct a function h so that (3.10) holds and a sequence n_k so that

$$\begin{aligned} &\lim_{k\to\infty} \rho_1(r_{n_k})h(r_{n_k}) \le C, \\ \text{and } &\lim_{k\to\infty} \rho_2(r_{n_k})h(r_{n_k}) = \infty. \end{aligned} \tag{3.11}$$

Choose $\tau_0(r) \uparrow \infty$ such that

$$\lim_{r\to 1} \frac{\rho_2(r)}{\rho_1(r)\tau_0(r)} = \infty.$$

Let (μ_n), (ν_n) and (λ_n) be the decreasing sequences associated with $\rho_2(r), \rho_1(r)$ and $\tau_0(r)$. Then we have, as before, the functions $m_1(n)$ and $m_2(n)$. Now let us define the subsequence n_k so that

$$m_i(n_{k+1}) > n_k \qquad i = 1, 2,$$

and suppose that

$$a_i = \begin{cases} \dfrac{1}{\rho_1(r_{n_k})} - \dfrac{1}{\rho_1(r_{n_{k+1}})} & i = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then as before, let

$$h_0(z) = \sum_{i=1}^{\infty} a_i(1-z)^{\lambda_i},$$

so that

$$\begin{aligned} \rho_1(r_{n_k})h_0(r_{n_k}) &\sim \sum_{i=1}^{\infty} a_i(1-r_{n_k})^{\lambda_i-\nu_{n_k}} \\ &= \sum_{i\le n_{k-1}} a_i(1-r_{n_k})^{\lambda_i-\nu_{n_k}} + \sum_{i\ge n_k} a_i(1-r_{n_k})^{\lambda_i-\nu_{n_k}} \\ &\le C + c\rho_1(r_{n_k})\sum_{i\ge n_k} a_i = C. \end{aligned}$$

Now for ρ_2 we have

$$\begin{aligned} \inf_{r_n\le r\le r_{n+1}} \frac{h_0(r)}{(1-r)^{\mu_n}} &\ge c\sum_{i\le m_2(n)} a_i(1-r_n)^{\lambda_i-\mu_n} + \sum_{i>m_2(n)} a_i(1-r_n)^{\lambda_i-\mu_n} \\ &= o(1) + \sum_{i>m_2(n)} a_i(1-r_n)^{\lambda_i-\mu_n}. \end{aligned}$$

Let $n_{\alpha(k)}$ be defined as above so that

$$m_2(n) \le n_k \qquad \text{whenever } n_{\alpha(k-1)} < n \le n_{\alpha(k)},$$

then for n in the above range, we have

$$\sum_{i>m_2(n)} a_i(1-r_n)^{\lambda_i-\mu_n} = \sum_{i\ge n_k} a_i(1-r_n)^{\lambda_i-\mu_n} \ge \frac{(1-r_n)^{\lambda_{n_k}}\rho_2(r_n)}{\rho_1(r_{n_k})}. \tag{3.12}$$

We will assume the sequences (n_k) were defined with the following constraints, which we may clearly do:

(i) n_k is close enough to $n_{\alpha(k-1)}$ so that (3.12) is bounded below for $n = n_{\alpha(k-1)}, \dots, n_k$.

(ii) For some $n_{\beta(k)} > n_k$, (3.12) is bounded below when $n = n_k, \dots, n_{\beta(k)}$.

(iii) $\dfrac{\rho_2(r_n)}{\tau_0(r_n)\rho_1(r_n)} \to \infty$ as n tends to infinity but remains always between $n_{\beta(k)}$ and $n_{\alpha(k)}$.

Then we will construct other functions $h_n(z)$, $n = 1, 2, \ldots$, which are concentrated on the points where (3.12) is not bounded below so that the sum $h(z) = \sum_n x_n h_n(z)$ satisfies (3.10) and (3.12). Here x_n is chosen such that $x_n > 0$ and $\sum x_n = 1$.

Let $(n(\gamma_k, i))$ be a sequence such that

$$n_{\beta(k)} \leq n(\gamma_k, i) \leq n_{\alpha(k)} \qquad \text{for all } k,$$

and

$$\bigcup_{k,i=0}^{\infty} \{n(\gamma_k, i)\} = \bigcup_{k=0}^{\infty} \left\{ n_{\beta(k)}, \ldots, n_{\alpha(k)} \right\}.$$

Then let

$$A_j^i = \begin{cases} \dfrac{1}{\rho_1(r_{n(\gamma_k,i)})} - \dfrac{1}{\rho_1(r_{n(\gamma_{k+1},i)})} & \text{if } j = n(\gamma_k, i), \\ 0 & \text{otherwise.} \end{cases}$$

As before, let

$$h_i(z) = \sum_{j=0}^{\infty} A_j^i (1-z)^{\lambda_j}.$$

Then the same calculation as above shows that there is a constant C independent of i such that

$$\rho_1(r_{n_k}) h_i(r_{n_k}) \leq C.$$

Moreover, for n close to $n(\gamma_k, i)$

$$\inf_{r_n \leq r \leq r_{n+1}} \rho_2(r) h_i(r) > \frac{(1-r_n)^{\lambda_{n(\gamma_k,i)}} \rho_2(r_n)}{\rho_1(r_{n(\gamma_k,i)})}.$$

Now if we consider

$$h(z) = \sum_{n=0}^{\infty} x_n h_n(z),$$

then h is a bounded analytic function in $\mathbb{D}$ that satisfies

$$\begin{aligned} \liminf_{r\to 1} \quad & \rho_2(r)h(r) > 0, \\ \text{but} \quad & \rho_1(r_{n_k})h(r_{n_k}) \le C. \end{aligned}$$

Now $h(z)$ is of the form

$$\sum_{j=0}^{\infty} X_j(1-z)^{\ell_j},$$

where ℓ_j and X_j are positive numbers. Thus $h(z) \neq 0$ since

$$\text{Re } (1-z)^b > 0 \qquad \text{when } 0 < b \le 1.$$

The calculation showing that if $\psi(z) = z + t(1-z)^a/h(z)$ then $M(r,\psi) = \psi(r)$, is now identical to the one for part a). □

We now prove Theorem 8. The function φ which we require needs to have

$$\limsup_{r\to 1} \quad \frac{G_1(r)}{G_1(M(r,\varphi))} = \infty, \tag{3.13}$$

$$\text{but} \quad \frac{G_2(r)}{G_2(M(r,\varphi))} \le C, \qquad \text{as } r \to 1. \tag{3.14}$$

Now (3.13) means that $\limsup_{r\to 1} h_1(M(r)) - h_1(r) = \infty$. But

$$\begin{aligned} & h_1(M(r) - h_1(r) \\ = \quad & h_1'(s)(M(r) - r) \qquad \text{for some } s \in (r, M(r)) \\ > \quad & h_1'(r)(M(r) - r). \end{aligned} \tag{3.15}$$

Similarly for (3.14) we have

$$\begin{aligned} & h_2'(s)(M(r)-r) \qquad \text{for some } s \in (r, M(r)) \\ \leq\ & Ch_2'(r)(M(r)-r), \end{aligned} \tag{3.16}$$

if

$$\lim_{r\to 1} \frac{1-M(r,\varphi)}{1-r} = 1.$$

Note that $G_2 \in \Pi$ means that if $a(r)$ and $b(r)$ are such that

$$\lim_{r\to 1} \frac{1-a(r)}{1-b(r)} = 1,$$

then there are constants $c_1, c_2 > 0$ such that

$$c_1 < \frac{h_2'(a(r))}{h_2'(b(r))} < c_2.$$

In the above we have used $a(r) = M(r,\varphi)$ and $b(r) = r$.

We now write

$$\begin{aligned} h_1'(r) &= \left(\frac{1}{1-r}\right)^a \omega_1(r), \\ \text{and} \quad h_2'(r) &= \left(\frac{1}{1-r}\right)^b \omega_2(r), \end{aligned}$$

where ω_i are functions which are either constant or tend to 0 (or ∞) slower than any power of $(1-r)$ as $r \to 1^-$. Then either $a > b$ or $a = b$. In the first case we may simply choose φ to be $\varphi(z) = z + t(1-z)^c$, for some $b < c < a$. Then we have

$$M(r,\varphi) - r = t(1-r)^c,$$

for r large enough, and so (3.15) tends to infinity as $r \to 1$, but (3.16) remains bounded.

In the second case, $a = b$, we employ Proposition 1 in three subcases.

Case 1: $\omega_i(r) \to \infty$ for $i = 1, 2$. We use Proposition 1 part a) to construct a function with

$$\limsup_{r \to 1} \frac{M(r, \varphi) - r}{(1 - r)^a} \rho_1(r) = \infty,$$

$$\text{but} \quad \frac{M(r, \varphi) - r}{(1 - r)^a} \rho_2(r) \leq C.$$

where $\omega_1 = \rho_1$ and $\omega_2 = \rho_2$.

Case 2: $\rho_i(r) \to 0$ for $i = 1, 2$. We use Proposition 1 part b) to construct the function we need, this time with $\rho_1 = 1/\omega_1$ and $\rho_2 = 1/\omega_2$.

Case 3: $\rho_1(r) \to \infty$ but $\rho_2(r) \to 0$ In this case the function $z + t(1 - z)^a$ will do.

Theorem 8 is thus proved.

3.3 Proof of lemma 2 and corollary 2

We now prove lemma 2. Firstly let r_n be a sequence satisfying (3.4) such that

$$\lim_{n \to \infty} \frac{\rho(r_n)}{\rho(r_{n+1})}$$

exists and is finite. Such a sequence can easily be constructed. Let $\delta_n \uparrow \infty$ be such that

$$\rho(r_n) = e^{\delta_n},$$

and write $r_n = 1 - e^{-\phi(n)}$. We will show that the sequence, $\Lambda = (\lambda_n)$, defined as

$$\lambda_n = \frac{\delta_n}{\phi(n)} \qquad n = 1, 2, \ldots$$

satisfies (3.5). Now clearly we have

$$(1 - r_n)^{\lambda_n} \rho(r_n) = \exp(-\lambda_n \phi(n) + \delta_n) = 1.$$

We also have that

$$(1 - r_{n+1})^{\lambda_n} \rho(r_{n+1}) = \exp(-\lambda_n \phi(n+1) + \delta_{n+1}),$$

and so (3.5) will follow if we can show that

$$\lim_{n\to\infty} \delta_{n+1} - \lambda_n \phi(n+1) = 0. \tag{3.17}$$

In that case we will have that for $r_n \leq r \leq r_{n+1}$, there is an $a_n \to 1$ such that

$$a_n \left(\frac{1}{1 - r_n}\right)^{\lambda_n} \leq \rho(r_n) \leq \rho(r) \leq \rho(r_{n+1}) \leq \left(\frac{1}{1 - r_{n+1}}\right)^{\lambda_{n+1}}$$

and so

$$(1 - r)^{\lambda_n} \rho(r) \leq \frac{(1 - r_n)^{\lambda_n}}{(1 - r_{n+1})^{\lambda_{n+1}}}$$

which, as we will see later, tends to 1 as $n \to \infty$. Also

$$(1 - r)^{\lambda_n} \rho(r) \geq a_n \left(\frac{1 - r_{n+1}}{1 - r_n}\right)^{\lambda_n} \sim a_n \to 1.$$

Now (3.17) is equivalent to

$$\lim_{n\to\infty} \log \rho(r_{n+1}) - \frac{\phi(n+1)}{\phi(n)} \log \rho(r_n) = 0.$$

This is equivalent to showing that

$$\lim_{n\to\infty} \log \frac{\rho(r_{n+1})}{\rho(r_n)} + \frac{\phi(n) - \phi(n+1)}{\phi(n)} \log \rho(r_n) = 0. \tag{3.18}$$

Now by virtue of (3.3) we have that

$$\lim_{n\to\infty} \frac{\log \rho(r_n)}{\phi(n)} = 0. \tag{3.19}$$

This implies that the first term in (3.18) tends to zero.

We also need to show that the second term in (3.18) tends to zero. This will follow if we can show that

$$\lim_{n\to\infty} \frac{\rho(r_n)}{\rho(r_{n+1})} = 1.$$

Since ρ is positive and increasing, we already have

$$0 < \frac{\rho(r_n)}{\rho(r_{n+1})} < 1,$$

and so if we assume, towards a contradiction, that

$$a = \lim_{n\to\infty} \frac{\rho(r_n)}{\rho(r_{n+1})} \neq 1,$$

then we know that $0 \leq a < 1$. Hence if we choose any $0 < \epsilon < 1 - a$ we can find an N so that

$$\frac{\rho(r_n)}{\rho(r_{n+1})} \leq (a + \epsilon) < 1,$$

for any $n \geq N$. But then in particular, we have that for each positive integer m,

$$\frac{\rho(r_N)}{\rho(r_{N+m})} \leq (a + \epsilon)^m,$$

or

$$\log \rho(r_{N+m}) \geq \log \rho(r_N) + m \log \frac{1}{a+\epsilon}.$$

But this means that

$$\limsup_{m\to\infty} \frac{\log \rho(r_{N+m})}{\phi(N+m)} > 0,$$

by (3.4), which is a contradiction of (3.19). Hence, we must have $a = 1$ and (3.18) does tend to zero which proves (3.5).

We also, now, prove corollary 2

Proof of corollary 2.

From the proof of lemma 2 we see that the sequence λ_n is defined to be $\mathcal{F}(r_n)$ and so the corollary follows by differentiating $\mathcal{F}$ and using the hypothesis to show it is decreasing.

Another way to see this is to note that if $\mathcal{L}(r;\rho) < \epsilon$ for $r \geq r_0$ then $(1-r)^\epsilon \rho(r)$ is decreasing for $r \geq r_0$. Hence the hypotheses merely say that $(1-r)^{\lambda_n}\rho(r)$ is decreasing whenever $r \geq r_n$.

3.4 Consequences

A second question was asked in [7] regarding the function $\mathcal{C}(G)$.

Question 1. *It is known that for fast weights, G, C_φ bounded on A^2_G implies the angular derivative satisfies*

$$|\varphi'(\zeta)| \geq 1 \qquad \forall \zeta \in \partial\mathbb{D}.$$

Does there exist a weight for which there are no further restrictions.

We are now able to answer this question.

Proposition 2. *Let*

$$H = \{\varphi\colon |\varphi'(\zeta)| \geq 1 \quad \forall \zeta \in \partial\mathbb{D}\}.$$

There is no weight function G such that

$$\mathcal{C}(G) = H.$$

Proof. Suppose, on the contrary, that there is a function G such that $\mathcal{C}(G) = H$, and as usual write $G = e^{-h}$. Now since $\mathcal{C}(G)$ is known when

$$G(r) = (1-r)^{\alpha}$$
$$\text{and when} \quad \limsup_{r\to 1}(1-r)^3 h'(r) = \infty,$$

and $\mathcal{C}(G) \neq H$ in either of these ranges, we must conclude that G is exactly in the range we have been discussing in this chapter.

Hence we can find another weight function $G_* = e^{-h_*}$ with

$$\lim_{r\to 1} \frac{h'(r)}{h'_*(r)} = \infty$$

to deduce that

$$H = \mathcal{C}(G) \subsetneqq \mathcal{C}(G_*) \subseteq H$$

which is a contradiction.

3.5 Complex Analysis

As mentioned at the beginning of this thesis, the set $\mathcal{C}(G)$ is independent of the weight G, provided

$$\limsup_{r\to 1}(1-r)^3 h'(r) = \infty. \tag{3.20}$$

This criterion comes from a result of Burns and Krantz, [3], which states that if φ is a self-map of the unit disk, and is such that

$$\varphi(z) = z + o(1-z)^3 \qquad \text{as } z \to 1,$$

then $\varphi(z) \equiv z$.

The authors of [3] prove this using the Herglotz representation to show that

$$\operatorname{Re} \frac{1+\varphi(z)}{1-\varphi(z)} - \frac{1+z}{1-z} > 0.$$

However an application of Julia's lemma suffices since

$$\operatorname{Re} \frac{1+\varphi(z)}{1-\varphi(z)} - \frac{1+z}{1-z} = \frac{1-|\varphi(z)|^2}{|1-\varphi(z)|^2} - \frac{1-|z|^2}{|1-z|^2} > 0$$

as $\varphi(z)$ has an angular derivative of 1 at 1.

A refinement of this result was used in [9] to get the above criterion (3.20). In fact the authors improved the result of Burns and Krantz to the following: If φ has angular derivative 1 somewhere on the unit circle, and

$$\liminf_{r\to 1} \frac{M(r,\varphi)-r}{(1-r)^3} = 0,$$

then $M(r,\varphi) = r$, in other words, $\varphi(z) = e^{i\lambda} z$ for some real λ.

The main result in this chapter can now be restated as

Theorem 9. *Suppose $\omega_i(r)(1-r)^3$ remains bounded as $r \to 1$ for $i = 1, 2$, where ω_i are non-decreasing functions on $[0, 1)$, tending to infinity as $r \to 1$. Suppose also that*

$$\lim_{r\to 1} \frac{\omega_1(r)}{\omega_2(r)} = \infty. \tag{3.21}$$

Then there is an analytic self-map of the unit disk φ such that

$$\begin{aligned} \limsup_{r\to 1}(M(r,\varphi) - r)\omega_1(r) &= \infty \\ \limsup_{r\to 1}(M(r,\varphi) - r)\omega_2(r) &< \infty. \end{aligned}$$

This result, then, is of independent interest given the Burns and Krantz result. It is also, therefore, worth noting that (3.21) can be replaced with

$$\limsup_{r\to 1} \frac{\omega_1(r)}{\omega_2(r)} = \infty.$$

This can be seen by inspection of the construction in Proposition 1.

It is unlikely that one could strengthen the conclusion of Theorem 9 to

$$M(r,\varphi) - r \sim \frac{1}{\omega(r)},$$

for arbitrary increasing functions $\omega(r)$. However, it would be of interest to classify such functions $\omega(r)$. As a positive result, it can be seen that for **any** function of the form

$$\tau(r) = \sum_{n=1}^{\infty} a_n (1-r)^{\lambda_n}$$

there are functions φ and ψ such that

$$M(r,\varphi) - r = (1-r)^a \tau(r) \quad \text{or } M(r,\psi) - r = \frac{(1-r)^a}{\tau(r)}.$$

Chapter 4

Growth of Inner functions

4.1 Introduction

In this chapter we consider the question of whether or not we can construct an inner function I, such that C_I maps one weighted space into another. We will, however, not consider weighted Bergman spaces as such, but rather the following spaces: let $w(r) \uparrow \infty$, as $r \to 1$, be a continuous weight function, then we define

$$H_w = \{f\colon f \text{ is analytic in } \mathbb{D} \text{ and } M(r,f) = O(w(r))\}.$$

It is clear that for every weight G, there are two weights w_1 and w_2, such that

$$H_{w_1} \subset A^2_G \subset H_{w_2},$$

so we can, without loss of generality look at these spaces.

In [2], the authors construct an inner function I such that C_I maps H_w into the little Bloch space $\mathcal{B}_0$. It is of interest to note that one cannot construct an inner function I such that the composition operator C_I which maps H_w into the Hardy spaces H^p, since MacLane showed in [10] that there are functions in H_w with no radial limits, but functions in the Hardy spaces have radial limits almost everywhere, see [5].

If we define a norm on H_w as

$$\|f\|_{H_w} = \sup_{z\in\mathbb{D}} \frac{|f(z)|}{w(|z|)},$$

then an inner function I forms a bounded composition operator from H_{w_1} to H_{w_2} if

$$\frac{|f \circ I(z)|}{w_2(|z|)} \leq \frac{w_1(|I(z)|)}{w_2(|z|)} \leq C < \infty.$$

But if we let $\phi(r) = w_1^{-1} \circ w_2(r)$, for $r > r_0$, then the last inequality above with $C = 1$ is just

$$M(r, I) \leq \phi(r). \tag{4.1}$$

Here we consider ϕ as an arbitrary increasing function mapping $[0, 1]$ into $[0, 1]$ with $\phi(1) = 1$.

Thus, for the remainder of this chapter we will consider the problem of construction an inner function I which solves (4.1).

4.2 Blaschke products

In this section we construct a Blaschke product which satisfies (4.1).

Theorem 10. *Let r_0 be given $0 < r_0 < 1$ and let $\phi(r)$ be a continuous increasing function with $\phi(1) = 1$. Then there exists a Blashke product $B(z)$ such that*

$$M(r, B) < \phi(r)$$

for $r_0 < r < 1$.

Proof. Let $\{\epsilon_n\}_1^\infty$ be a monotonic decreasing sequence satisfying

i) $\epsilon_n \to 0$ as $n \to \infty$, $\epsilon_1 < 1/2$.

ii) $\sum_{n=1}^\infty \epsilon_n < \infty$.

and define sequences $\{r_n\}$ and $\{N_n\}$ by

a) $\phi(r_n) = 1 - \epsilon_n$.

b) $r_{n+1}^{N_n} < \epsilon_n$.

Now put

$$\psi_n(z) = \frac{z^{N_n} + (1 - 2\epsilon_n)}{1 + (1 - 2\epsilon_n) z^{N_n}}.$$

Then, for $r_n \leq r \leq r_{n+1}$,

$$\psi_n(r) \leq \psi_n(r_{n+1}),$$

by the maximum modulus principle, since $M(r, \psi_n) = \psi_n(r)$.

Thus

$$\psi_n(r) \leq (1 - 2\epsilon_n) + \epsilon_n = 1 - \epsilon_n = \phi(r_n) \leq \phi(r).$$

We now let

$$B(z) = \prod_{n=1}^\infty \psi_n(z),$$

then B is a Blaschke product, and for $r_n \leq r \leq r_{n+1}$,

$$M(r, B) = B(r) < \psi_n(r) \leq \phi(r)$$

for all $r > \phi^{-1}(1 - \epsilon_0) = r_0$.

Now, the Blaschke condition is satisfied since

$$\begin{aligned}\sum(1-|z_n|) &= \sum_n N_n(1-(1-2\epsilon_n)^{1/N_n}) \\ &\sim \sum N_n\left(1-\left(1-\frac{2\epsilon_n}{N_n}+O\left(\frac{\epsilon_n^2}{N_n^2}\right)\right)\right) \\ &= \sum 2\epsilon_n + O\left(\frac{\epsilon_n^2}{N_n}\right) < \infty.\end{aligned}$$

Finally, r_0 can be made as small as possible by considering $z^M B(z)$.

4.3 Singular Inner functions

In this final section we construct a Singular inner function which solves (4.1).

Theorem 11. *Let r_0 be given, $0 < r_0 < 1$ and let $\phi(r)$ be a continuous increasing function with $\phi(1) = 1$. Then there exists a singular inner function $S(z)$ such that*

$$M(r, S) \le \phi(r),$$

for $r_0 < r < 1$.

Proof. Let ϵ_n be a sequence with $\epsilon_n \downarrow 0$ as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \epsilon_n < \infty.$$

Define a sequence $\{r_n\}$ by

$$\phi(r_n) = e^{-\frac{1}{2}\epsilon_n}.$$

The function $w = \dfrac{1+z}{1-z}$ maps $\mathbb{D}$ onto the right half plane, $\{z\colon \mathrm{Re}\ z > 0\}$ with $w(0) = 1$. Thus if r is sufficiently small,

$$\mathrm{Re}\ \frac{1+z}{1-z} > \frac{1}{2} \qquad \text{for all } z \text{ with } |z| \le r.$$

Now define a sequence $\{N_n\}$ so that

$$\operatorname{Re} \frac{1 + r_{n+1}^{N_n}}{1 - r_{n+1}^{N_n}} > \frac{1}{2}$$

for $|z| \leq r_{n+1}$.

Set

$$\varphi_n(z) = \exp\left\{-\epsilon_n \left(\frac{1 + z^{N_n}}{1 - z^{N_n}}\right)\right\},$$

so that by the maximum modulus theorem, for $r_n \leq r \leq r_{n+1}$,

$$|\varphi_n(r)| \leq |\varphi_n(r_{n+1})| < \exp -\frac{1}{2}\epsilon_n \leq \varphi(r),$$

and so

$$|\varphi_n(z)| \leq \phi(r) \qquad \text{for } r_n \leq |z| \leq r_{n+1}.$$

Set

$$S(z) = \prod_{n=1}^{\infty} \varphi_n(z) = \exp\left\{-\sum_{n=1}^{\infty} \epsilon_n \left(\frac{1 + z^{N_n}}{1 - z^{N_n}}\right)\right\}$$

and note that, because $\sum \epsilon_n < \infty$, the series converges uniformly and absolutely on every compact subset of $\mathbb{D}$. Also

$$\operatorname{Re} \; -\sum_{n=1}^{\infty} \epsilon_n \left(\frac{1 + z^{N_n}}{1 - z^{N_n}}\right) \leq 0$$

in $\mathbb{D}$, so that $S(z)$ is either an inner function or $S(z) \equiv 0$. But

$$S(0) = \exp\left\{-\sum_{n=1}^{\infty} \epsilon_n\right\} \neq 0,$$

so S is an inner function.

Also

$$M(r, S) = S(r) \leq \phi(r)$$

for all r sufficiently large. To arrange that $S(r) < \phi(r)$ for all $r > r_0$, we just replace $S(z)$ by $S(z)^N$ for some large enough N.

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