Thermodynamical Approach to Quantifying Quantum Correlations

Jonathan Oppenheim,[1,2] Michał Horodecki,[2] Paweł Horodecki,[3] and Ryszard Horodecki[2]

[1]*Theoretical Physics Institute, University of Alberta, 412 Avadh Bhatia Physics Laboratory, Edmonton, Alberta, Canada T6G 2J1*

[2]*Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80–952 Gdańsk, Poland*

[3]*Faculty of Applied Physics and Mathematics, Technical University of Gdańsk, 80–952 Gdańsk, Poland*

(Received 11 February 2002; published 11 October 2002)

We consider the amount of work which can be extracted from a heat bath using a bipartite state ρ shared by two parties. In general it is less then the amount of work extractable when one party is in possession of the entire state. We derive bounds for this "work deficit" and calculate it explicitly for a number of different cases. In particuar, for pure states the work deficit is exactly equal to the distillable entanglement of the state. A form of complementarity exists between physical work which can be extracted and distillable entanglement. The work deficit is a good measure of the quantum correlations in a state and provides a new paradigm for understanding quantum nonlocality.

DOI: 10.1103/PhysRevLett.89.180402 PACS numbers: 03.65.Ud, 03.67.–a, 05.70.–a, 89.70.+c

Strong connections exist between information and thermodynamics. Work is required to erase a magnetic tape in an unknown state [1] and bits of information can be used to draw work from single heat bath [2,3]. The second law of thermodynamics forbids the drawing of work from a single heat bath, however, if one has an engine which contains "negentropy" (bits of information) then one can draw work from it. The process does not violate the second law because the information is depleted as entropy from the heat bath accumulates in the engine. Typically, the source of information is particles in known states, and these states can be thought of as a type of fuel or resource [4]. In particular, quantum states can be used as fuel [5], and recently, physically realizable microengines have been proposed [6].

The field of quantum information theory has also yielded tantalizing connections between entanglement and thermodynamics [7]. Bipartite states (jointly held by two parties) such as the maximally entangled state

$$\Psi_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{1}$$

exhibit mysterious nonlocalities which can be exploited to perform quantum useful logical work [8] such as teleporting qubits [9]. For many states, one can distill singlets in order to perform logical work, but there is also *bound entanglement* [10] which cannot be distilled from the state and it has been proposed that this is analogous to heat [11]. Pure bipartite states can be reversible transformed into each other in a manner which is reminiscent of a Carnot cycle [12,13]. Furthermore, the preparation of certain jointly held states appears to result in a greater loss of information when the state is prepared by two separated observers than when the entire state is prepared by a single party [14]. Connections between Landauer erasure, measurement, error correction, and distillation have also been explored [15].

In this paper we ask how much work can be drawn from a single heat bath if the information is *distributed* between two separated parties Alice and Bob. It turns out that in general their engines will be more efficient when information is localized, and that the degree to which this is the case provides a powerful new paradigm to understand and quantify nonlocality in quantum mechanics.

As with the distant labs scenario for entanglement analysis, we allow Alice and Bob to perform local operations on their states, and communicate classically with each other (LOCC). We will quantify the amount of potential work that cannot be extracted by two separated parties by introducing the concept of a *work deficit* Δ, defined to be the difference in the amount of work that can be extracted from a state under LOCC versus the amount that can be extracted by a party who holds the entire state. For pure states we find that the work deficit is exactly equal to the amount of distillable entanglement E_D of the state (i.e., the number of singlets that can be extracted from the state under LOCC). This also seems to be the case for so-called maximally correlated states. We also prove bounds for Δ and show that it is a good measure for the amount of quantum correlations present in a state jointly held by two parties. A more detailed analysis of the concepts introduced here will be presented elsewhere [16].

Before proceeding with the quantum case it may be worthwhile to review the connections between information and thermodynamical work for classical states. Consider a number of classical bits n which are all initially in the standard state 0. These bits can be used to draw work from a heat reservoir of temperature T. To visualize this, one might imagine that a bit is represented by a box divided by a wall in the center. A particle placed in the left hand side represents the 0 state, while if the particle is in the right hand side, the bit is in the 1 state.

Now imagine that we know that the bit is in the 0 state. We can draw work from the heat reservoir by replacing the central wall with a piston and then allowing the particle to reversible push it out, drawing $kT \ln 2$ of work from the reservoir [17]. We now no longer know where the particle is, so the entropy of the bit has been increased by the same amount. No more work can be drawn from the bit, since we do not know on which side to place the piston.

Although we cannot extract work from an unknown state, we can extract work if we know that classical correlations exist. Imagine, for example, that we have two classical bits in unknown states, but we know that they are in the same unknown state. We can then perform a control not (cnot) gate on the bits which flips the second bit if the first bit is a 1. After the cnot gate has been performed the control bit is still in an unknown state, but the target bit is now in the 0 state and 1 bit of work can be extracted from it. In general, for a n-bit random variable X with Shannon entropy $H(X)$ one can use the first law of thermodynamics to see that the amount of work W_C that can be extracted is just the change in entropy of the state

$$W_C = n - H(X). \tag{2}$$

The same methods can be used to extract work from quantum bits (qubits) [5]. If we have n qubits in a state ρ and entropy $S(\rho)$, then one can extract

$$W_t = n - S(\rho). \tag{3}$$

This is the amount of work that can be extracted in total by someone who has access to the entire system ρ. As with the classical case, all correlations can be exploited to extract work from the state.

We now ask how much work two individuals can extract under LOCC using a shared state ρ_{AB}. We imagine that Alice and Bob each have an engine which can be used to locally extract work from a common heat bath. Then, under LOCC they try to extract the largest amount of local work W_l possible. We then define the work deficit to be the amount of potential work which cannot be extracted under LOCC

$$\Delta \equiv W_t - W_l. \tag{4}$$

Before proving some general results, it may be useful to give a few simple examples. Consider the classically correlated state

$$\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|), \tag{5}$$

where the first bit is held by Alice, and the second by Bob. We can see that Δ is zero for this case, since Alice can measure her bit, send the result to Bob who can then extract 1 bit of work by performing a cnot. Alice can then reset the memory of her measuring apparatus by using the work extracted from the bit that she held. The total amount of work extracted under LOCC is therefore 1 bit, which from Eq. (3) is the same as the amount of work that can be extracted under all operations. In more detail, the steps involved in the process are as follows: (a) Alice uses a measuring device represented by a qubit prepared in the standard state $|0\rangle$. She performs a cnot using her original state as the control qubit, and the measuring qubit as the target. (b) The measurement qubit is now in the same state as her original bit and can be dephased (i.e., decohered) in the $|0\rangle, |1\rangle$ basis so that the information is purely "classical" (dephasing simply brings the off-diagonal elements of the density matrix to zero, destroying all quantum coherence). For this state, dephasing does not change the state since the state is already classical. (c) The measuring qubit can now be sent to Bob who (d) performs a cnot using the measuring qubit as the control. His original qubit is now in the standard state $|0\rangle$. (e) Bob sends the measuring qubit back to Alice who (f) resets the measuring device by performing a cnot using her original bit as the control. Alice's state is now in the same state as it was originally, while Bob's state is known and can be used to extract 1 bit of work.

We now consider how much work can be extracted from the maximally entangled qubits of Eq. (1). The same protocol as above can be used to find $\Delta = 1$ (later we will show that this is optimal). Alice and Bob can extract 1 bit of work by following steps (a)–(f). However, unlike the previous case, the measurement in step (b) is an irreversible process and the original state and the entanglement is destroyed by the dephasing that must occur for a measurement to be made. On the other hand, someone with access to the entire state can extract 2 bit of work since the state is pure and has zero entropy.

Basic questions now arise: How much work can be drawn from a given state ρ? For which states is $\Delta = 0$? How is Δ related to entanglement?

To deal with these questions we have to state the paradigm for drawing work from bipartite states more precisely. First, we will clarify the class of operations Alice and Bob are allowed to perform. The crucial point is that here, unlike in usual LOCC schemes, one must explicitly account for all entropy transferred to measuring devices or ancillas. So in defining the class of allowable operations one must ensure that no information loss is being hidden when operations are being carried out. One way to do this is to define elementary allowable operations as follows: (a) adding separable pure state ancillas to the system; (b) local unitary operations (i.e., $U_A \otimes I_B$ or $I_A \otimes U_B$); (c) sending qubits through a dephasing channel.

The dephasing operation can be written as $\sum_i P_i \rho_{AB} P_i$ where the P_i are orthogonal local projection operators. The class of operations (c) are equivalent to local measurements and classical communication but has the advantage that we do not need to worry about erasing the memory of measuring devices. Alice or Bob could also make measurements without sending the information, but in this scheme it is wasteful. Furthermore, it is equivalent to Alice sending qubits down the dephasing channel and having Bob send the qubits back. We therefore do not need to consider measurement and sending qubits as separate operations.

We imagine that Alice and Bob share an n qubit state ρ_{AB} composed of n_A qubits held by Alice and n_B by Bob. The part of the state which Alice holds is given by tracing out the degrees of freedom associated with Bob's state, i.e., $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ and visa versa. Alice and Bob then perform combinations of operations (a)–(c) to arrive at a final state ρ'_{AB} composed of $n' = n'_A + n'_B$ qubits. The difference $k = n' - n$ is the number of pure state ancillas which they have added to the state, and therefore we must subtract k bit of work from the amount of work they can draw using the n' bit. Since they must extract the work from ρ_{AB} locally, we find that the total amount of local work that can be extracted is

$$\begin{aligned} W_l &= n'_A - S(\rho'_A) + n'_B - S(\rho'_B) - k \\ &= n - S(\rho'_A) - S(\rho'_B). \end{aligned} \tag{6}$$

The goal is now clear. Alice and Bob perform their operations to arrive at a state which has $S(\rho'_A) + S(\rho'_B)$ as low as possible. They then draw work locally using this new state ρ'_{AB}.

Consider now the state of the form

$$\rho_{AB} = \sum_{ij} p_{ij} |i_A\rangle |j_B\rangle\langle j_B| \langle i_A|, \tag{7}$$

where $|i_A\rangle$ and $|j_B\rangle$ are a local orthonormal basis. Such states can be called *classically correlated* [18]. The natural protocol is that Alice sends her part to Bob down the dephasing channel. This will not change the entropy of the state. The final state ρ' will have $S(\rho'_A) = 0$ (strictly speaking Alice will now have no system) while $S(\rho'_B) = S(\rho)$. Thus according to Eq. (6) $W_l = n - S(\rho) = W_t$, so that $\Delta = 0$ for the above states. Note that local dephasing in the eigenbases of ρ'_A, ρ'_B does not change the optimal W_l (6) but brings the state ρ'_{AB} to the form (7). This gives another method to evaluate work: instead of minimizing $S(\rho'_A) + S(\rho'_B)$ we can minimize $S(\rho')$ over classically correlated states ρ' that can be achieved from ρ by the allowed class of operations. Now we are in position to prove a general upper bound on the amount of work that can be drawn using distributed information. Our bound holds for pure states [16], but for more general states our proof relies on the following assumption (although we conjecture that the bound holds in general).

Assumption.—Bits which are sent down the communication channel are treated as classical in the sense that they are only dephased once, and not again in a second basis (which would destroy the encoded information).

Theorem: *Under this assumption the maximum amount of work that can be extracted using LOCC operations on an n qubit state ρ_{AB} is bounded by* $W_l \le n - \max\{S(\rho_A), S(\rho_B)\}$.

The proof follows after noting that Alice (or Bob), rather than directly sending the results of measurements, can reversibly copy the measurement results by performing the cnot operation with the measurement bit as the control bit and an ancilla as the target. Alice can then send the copy to Bob who can use the information stored in the copy. At the end of the whole protocol all the copies can be sent back (this follows from our assumtion) and erased by performing a second reversible cnot. Alice and Bob's protocol will therefore not be more inefficient if they keep their original measurement bits and send copies only to each other. Consider now any optimal protocol transforming ρ_{AB} into the final state ρ'_{AB} of the form (7) with minimal entropy $S(\rho'_{AB})$. As we already know the protocol can be followed by dephasing in the local eigenbases. Hence before sending copies back and erasing them, the entire system can be considered to be in another state σ_{AB}, *still* in the form (7), so [19] $S(\sigma_{AB}) \ge \max\{S(\sigma_A), S(\sigma_B)\}$. Now, as only copies of bit measurements were sent, and the bits themselves were kept, production of σ_{AB} from ρ_{AB} could only have increased local entropies because neither unitary operation nor dephasing decreases entropy. So one has

$$S(\sigma_{AB}) \ge \max\{S(\rho_A), S(\rho_B)\}, \tag{8}$$

where $S(\rho_A)$, $S(\rho_B)$ are local entropies of the initial state ρ_{AB}. Finally, because resending and erasing copies preserves the spectrum of the whole state, one has $S(\sigma_{AB}) = S(\rho'_{AB})$ which gives

$$S(\rho'_{AB}) \ge \max\{S(\rho_A), S(\rho_B)\}. \tag{9}$$

The theorem then follows directly from the fact that $n - S(\rho'_{AB})$ is an upper bound on the amount of work that can be drawn from the state ρ'_{AB}. The corresponding work deficit obtained under our assumption will be denoted by Δ_r. For mixed states it is possible that $\Delta < \Delta_r$, but we conjecture equality. Also for one way LOCC schemes (classical communication from Alice to Bob only) Δ_r coincides with the one way deficit $\Delta_{\leftarrow}$. From the theorem we have [20]

$$\Delta_r \ge \max\{S(\rho_A), S(\rho_B)\} - S(\rho). \tag{10}$$

This allows one to calculate the extractable work for pure states by exhibiting protocols that achieve this bound. To this end write a given pure state in the Schmidt decomposition $\psi = \sum_i a_i |e_i\rangle |f_i\rangle$ where e_i, f_i are local bases. Alice then performs dephasing in her basis. The resulting state is classically correlated and has entropy equal to $S(\rho_A)$ where ρ_A is the reduction of $|\psi\rangle\langle\psi|$. Note that the latter is the entanglement measure for pure states, which is unique in the asymptotic regime and is equal to the distillable entanglement E_D and the entanglement cost (i.e., the number of singlets which are required to create the state under LOCC) [12]. Thus for pure states the work deficit is exactly equal to entanglement

$$\Delta(\psi) = E(\psi). \quad (11)$$

We are able to calculate Δ_r for a broader class of states, the so-called maximally correlated states of the form

$$\rho_{AB} = \sum_{ij} \sigma_{ij} |ii\rangle\langle jj|. \quad (12)$$

To achieve bound (10) Alice dephases her part in the basis $\{|i\rangle\}$. The resulting state is $\rho' = \sum_i \sigma_{ii} |ii\rangle\langle ii|$, and has entropy equal to $S(\rho_A)$. Thus $\Delta_r(\rho) = S(\rho_A) - S(\rho)$. One can check that for states (12) local entropies are equal and no smaller than the total entropy so that $\Delta_r \geq 0$ as it should be. Now it turns out that for the above states, we know E_D [21], and it is again equal to Δ_r. An example is the mixture of state (1) with $1/\sqrt{2}(|00\rangle - |11\rangle)$: $E_D = E_D^{\rightarrow}$ (the latter being one way distillable entanglement) is equal to $1 - S(\rho)$. The explicit distillation protocol attaining this value was shown in [22]. That one cannot do better follows from the relative entropy bound [13,21] which is equal to $1 - S(\rho)$ for those states.

The above result is rather surprising because the state (12) contains bound entanglement, i.e., the entanglement cost of the state is greater than the entanglement of distillation [23]. This result shows that work can be drawn from the bound entanglement.

Although distillable entanglement cannot be used to perform physical work, it allows us to perform *logical work* (see [8]): each bit of distillable entanglement enables Alice to teleport one qubit to Bob. For these states, the total amount of extractable work W_t gets divided between physical work W_l and logical work E_D. Entanglement can therefore be thought of as a source of nonlocal negentropy which can be used to perform logical work. Just as with physical negentropy, logical negentropy cannot increase under LOCC ($\delta E_D \leq 0$). However, if one uses the state to extract physical work the ability to perform logical work is lost. Likewise, after performing logical work, the singlets are left in a maximally entropic state and the ability to perform physical work is lost. There is thus a new form of complementarity between the logical and physical work.

It is also worth investigating the connection between our approach and the measures of classical and quantum corelations introduced in [24,25]. It would also be desirable to consider collective actions on many copies of the given state. In the examples we considered, collective actions cannot help since the parameter Δ turns out to be additive.

In conclusion, we have proposed a paradigm for quantifying quantum correlations motivated by thermodynamical and operational considerations. This approach is also fruitful in multipartite settings. The emerging function Δ is nonzero for all entangled states, but need not vanish for separable states. It quantifies the part of correlations that must be destroyed during transmission via a classical channel (this is compatible with the observation that decoherence causes a Maxwell demon to be less efficient [26]). If a quantum channel were available, all information could be localized, and the full work $n - S(\rho)$ could be drawn from local heat baths. Thus the work deficit Δ quantifies truly quantum correlations. Finally, we hope that the present approach, in particular, may help discover a "new face" of the so-called *thermodynamics of entanglement*.

This work is supported by EU grant EQUIP, Contract No. IST-1999-11053. J. O. thanks R. Laflamme and R. Cleve for interesting discussions, and NSERC and DITP for support.

[1] R. Landauer, IBM J. Res. Dev. **5**, 183 (1961).
[2] L. Szilard, Z. Phys. **53**, 840–856 (1929)
[3] C. H. Bennett, Int. J. Phys. **21**, 905 (1982).
[4] L. Levitin, in *Proceedings of the Workshop on Physics and Computation, Phys. Comp. '92* (IEEE Computer Soc. Press, Silver Spring, Maryland, 1993), pp. 223–226; pp. 275–279.
[5] J. von Neumann, *Mathematische Grundlagen der Quantenmachanik* (Springer, Berlin, 1932).
[6] M. O. Scully, Phys. Rev. Lett. **87**, 220601 (2001).
[7] See recent review in M. B. Plenio and V. Vitelli, Contemp. Phys. **42**, 25–60 (2001).
[8] R. Horodecki *et al.* Phys. Rev. A **63**, 022310 (2001).
[9] C. Bennett *et al.* Phys. Rev. Lett. **70**, 1895 (1993).
[10] M. Horodecki *et al.* Phys. Rev. Lett. **80**, 5239 (1998).
[11] P. Horodecki *et al.* Acta Phys. Slovaca **47**, 148 (1998).
[12] C. Bennett *et al.* Phys. Rev. A **53**, 2046 (1996); S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).
[13] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[14] C. H. Bennett *et al.* Phys. Rev. A **59**, 1070 (1999).
[15] V. Vedral, quant-ph/9903049.
[16] M. Horodecki *et al.*, quant-ph/0207168.
[17] Henceforth, we will set $T = k = 1$ so that $kT \ln 2$ is simply "one bit" of work. We will also describe the dimension of Hilbert spaces in terms of qubits.
[18] This is a smaller class than separable states because the latter do not need to have product eigenvectors.
[19] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978).
[20] An upper bound for Δ is given by relative entropy distance from the set of classically correlated states [16].
[21] E. Rains, Phys. Rev. A **60**, 179 (1999).
[22] C. H. Bennett *et al.*, Phys. Rev. A **54**, 2046 (1996).
[23] G. Vidal, W. Dür, and J. I. Cirac, quant-ph/0112131.
[24] L. Henderson and V. Vedral, J. Phys. A, **34**, 6899 (2001).
[25] W. H. Żurek, Ann. Phys. (Leipzig) **9**, 5 (2000); H. Ollivier and W. H. Żurek, Phys. Rev. Lett. **88**, 017901 (2002).
[26] W. H. Żurek, *Feynman and Computation*, edited by A. Hey (Perseus Books, Reading, 1999), pp. 393–410.