

Are the Laws of Entanglement Theory Thermodynamical?

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We argue that on its face, entanglement theory satisfies laws equivalent to thermodynamics if the theory can be made reversible by adding certain *bound entangled states* as a free resource during entanglement manipulation. Subject to plausible conjectures, we prove that this is not the case in general, and discuss the implications of this for the *thermodynamics of entanglement*.

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The pioneering papers in quantum information theory [1–3] revealed a potential irreversibility in entanglement processing. They suggested that by local operations and classical communication (LOCC), one needs more pure entangled states to produce a state than can be drawn from it. Some researchers expressed the intuition that this is due to some “second law” in entanglement processing. The first paper making this analogy rigorous was [4]. However the authors did not discuss irreversibility, instead building the analogy in the region of full reversibility—in the domain of pure states (see also [5]). It was sometimes argued that irreversibility is where entanglement and thermodynamics differ, as, e.g., the Carnot cycle is reversible. In [6] a different point of view was presented which attempted to account for this irreversibility. The leading idea was that entanglement is analogous to *energy* and that distillation of pure entanglement is like drawing work (the amount of pure entanglement drawn from a state is E_d while the amount needed to create a state we denote by E_c). Meanwhile, the extreme case of irreversibility was discovered: the bound entangled (BE) state. One cannot draw any pure entanglement from them, but entanglement is needed to create them.

In this paper we will investigate aspects of the latter analogy and argue that it is extremely useful for understanding the basic laws of entanglement processing. We study the consequences of our proposal, and make the theory precise so that testing is possible. We first show that the very fact of irreversibility does not cancel the analogy, rather it is a constitutive element. This enables us to state the three laws of entanglement for such a theory. However, next we show that the analogy with reversible thermodynamics does not appear to hold in general. We then speculate on possible ways the basic laws of thermodynamics may still yield insights into quantum information theory.

The main observation we will need is that what constitutes thermodynamics, and distinguishes it, e.g., from mechanics, is that there are *two forms* of energy: disordered (heat) and ordered (work). The first law does not “feel” any difference between these. It says that to create

a single heat bath of energy E we need precisely this amount of work. The second law says that this creation is *irreversible*: from a single heat bath one cannot draw work. Thus the second law accounts for the basic irreversibility of the heat bath formation: it is due to loss of information. In its ideal version, thermodynamics offers also a reversible change of work into heat and vice versa (the Carnot cycle). However, this is possible only if one has heat reservoirs from the very beginning. This is depicted in Fig. 1. To summarize: (i) there is a form of energy that cannot be used to draw work (single heat baths); (ii) it can, however, be used to store work, but (iii) work can be stored in heat only if there is some heat at the beginning (iv) in the latter case work can be stored *reversibly*.

Let us now argue that thus far, entanglement processing not only does not exclude a perfect analogy with the features of thermodynamics outlined above, but suggests strongly that there is one. (We will then provide a new result, which, up to a plausible conjecture, precludes such a *prima facie* possibility.) We will follow an idea related to the entanglement-energy analogy of [6].

First it is reasonable to assume that pure state entanglement is analogous to mechanical energy, while mixed state entanglement corresponds to energy that may be partly accumulated in the form of heat. Now, distillation [1], i.e., drawing pure entanglement from mixed states, is like running an engine that produces work out of heat. There is a question: in thermodynamics we have systems

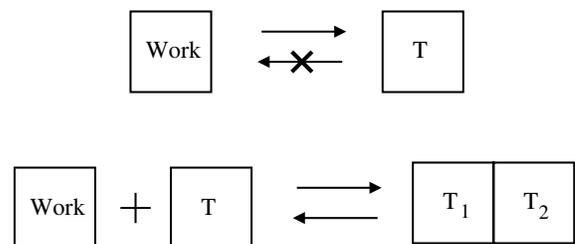


FIG. 1. Reversible and irreversible processes in thermodynamics.

that *have* energy but no work can be drawn from them—single heat baths. If we did not have a counterpart of it in entanglement processing, then a constitutive element would be missing. However, there exist so-called *bound entangled* states [7] that are entangled (have “energy”) but this energy cannot be used, i.e., changed into an ordered form. Thus BE states are the counterpart of a “single heat bath”—a completely disordered form of entanglement. Other mixed states would represent partially ordered entanglement: work stored in thermodynamical systems (e.g., pairs of reservoirs of different temperatures). The process of forming a BE state out of pure states could be thought of as analogs to Joule’s experiment establishing the equivalence between work and heat where all work is (irreversibly) changed into heat. The energy of the created heat bath is equal to the amount of work added to the system. This leads us to the conclusion: *The counterpart of total energy would be entanglement cost (amount of pure entanglement needed to produce the state).* At this moment one can ask a pointed question: does a counterpart to the first law hold in entanglement theory? Is the entanglement in a BE state still present? Or perhaps to form the state one needs entanglement but during the formation process it gets dissipated [8]. This is very much connected with the main question which we will now address: *Can we have an analog of reversible work extraction?* One might say no, because in entanglement processing there is irreversibility. This is, however, not an adequate answer. Let us argue that we can have a perfect analogy of reversible work extraction, despite the fact that generically we need more singlets to create a state than can be draw from it.

First note, that any thermodynamical system has more energy than we can draw from it (excluding the case when one of the reservoirs has zero temperature). This is due to the second law. The same is true for entanglement: for a generic mixed state $E_D < E_c$. Thus the *irreversibility* of entanglement processing could be due to a second law of entanglement that says “the disorder of entanglement can only increase.” Now, according to Fig. 1, for a given state ϱ , the following two conditions should be matched: (i) One should be able to distill E_d of pure entanglement, but in such a way that the garbage left over from the distillation procedure (denoted by ϱ_g) is a BE state with $E_c(\varrho_g) = E_c(\varrho) - E_d(\varrho) \equiv E_b(\varrho)$

$$\varrho \rightarrow \varrho_g \otimes \psi_+ \otimes n; \quad (1)$$

(ii) one should be able to form the state ϱ out of $E_d(\varrho)$ singlets plus a BE state of entanglement cost equal to $E_b(\varrho)$.

This is a consequence of *conservation of E_c* which would be analogous to the *first law*. Even though we will show that in general this will not hold, we think it is important to realize how this theory would look. Indeed perhaps reversible thermodynamics of entanglement is an important element of the total, more compli-

cated picture. Besides, the independent development [11] suggests that for some classes of mixed states this theory may be valid.

Thus, we would have the following reversible thermodynamics of entanglement: There is a unique measure of entanglement (E_c); E_d is not an autonomous measure of entanglement: it is the “work” that can be drawn in units of E_c ; processes of formation and distillation are *reversible* in the sense described above; the process of changing pure entanglement into mixed entanglement is *irreversible* [12].

Exactly as in statistical physics we would have two kinds of irreversible processes: dissipation (where E_c decreases) and pure decoherence (where E_c is constant, but E_d decreases). The first irreversibility does not exist in optimal processes (which we consider). The second could be removed if the processes start with some initial supply of BE states. Then to form a state, one would not need to create disordered entanglement out of the ordered form but only *dilute* the ordered one into the noisy entanglement of BE states.

Usually hypothetical reversibility in entanglement processing is associated with a unique measure of entanglement, in the sense that there is only one function monotonic under operations. Indeed, we can state the above in such a way that there will only be one measure of entanglement. Since bound entanglement is used only as a source of fully disordered entanglement, it is not actually a resource. Thus we can treat it as free of charge (say heat is cheap, only work costs). Our class of operations would then be as postulated in [13]: LOCC plus BE ancillas. This does not make the theory trivial, since we still cannot obtain singlets for free. Then, however, the only monotone would be E_D . Actually, it would be equal to the (regularized) relative entropy distance to the set of nondistillable states [14–16]. At this point we can even formulate a precise theorem.

Theorem 1: If under a given class of operations C that includes mixing states, one can reversibly transform ϱ into pure states, then any asymptotically continuous function E that is monotonic under the class C satisfies

$$E^\infty(\varrho) = E_R^\infty(\varrho), \quad (2)$$

where E_R^C is the relative entropy distance from the set of states that is closed under C .

The above notation indicates the regularization of a function $M(\varrho)$ given by $M^\infty(\varrho) = \lim_{n \rightarrow \infty} \frac{1}{n} M(\varrho^{\otimes n})$.

Remark: For such states we thus have a unique entanglement measure equal to the familiar regularized relative entropy of entanglement.

Proof: Uniqueness comes from reversibility and uniqueness of the entanglement measure in the pure state case [4,17] (that again comes from reversibility between pure states [3]). E_R^C is a monotone under C [18], and is also asymptotically continuous if the considered set is convex [17] (which is the case since C includes mixing).

Thus we can formulate the *second law*. Surprisingly, it has the same form as in statistical physics: it is Uhlmann monotonicity—see, e.g., [19]—saying that *under completely positive maps the relative entropy does not increase*. Indeed from this, one gets that the relative entropy distance is monotonic, from which our theorem follows. In this way the law gives the optimal entanglement that can be distilled exactly as the second law gives the Carnot efficiency. This also suggests a feedback to thermodynamics: it seems possible to express the extractable work by means of the relative entropy distance from the single heat bath states (for a fixed Hamiltonian). Indeed, Theorem 1 appears to be satisfied in thermodynamics. We hope to come back to this interesting point elsewhere. It is also rather amusing that in this theory we would have an analog to the *third law* of thermodynamics: *one cannot distill singlets with perfect fidelity*

Let us return to the question of whether we can have reversibility for all states. Recall that for this proposal: *Reversible thermodynamics of entanglement = possibility of reversible separation of bound and pure entanglement*.

Below we show a counterexample. To this end we need to be more rigorous. The first problem is that if there are NPT [20] bound entangled states, then BE cannot be for free, as such states together with some PPT [20] states are distillable, so that singlets would be for free too. Thus we should not speak about BE states, but rather about the smaller class of states that is closed under LOCC and under tensor product. Call this set the Hyper-Set (HS). Thus to produce a counterexample, we should show that there is a state for which $E_d^{\text{LOCC+HS}} < E_c^{\text{LOCC+HS}}$, where LOCC + HS is LOCC operations plus states from HS as free ancillas. One can also consider the class of hypermaps that includes the class LOCC + HS—namely the maps that do not move states outside of HS or the subclass of these consisting of the maps that are closed under tensor product.

We do not know the set HS or the class of hypermaps. However, the following result [21] adds an additional restriction: any NPT state can be distilled with the help of some PPT state. The HS would therefore not be able to include PPT states and any NPT state. The conditions on other sets are also very restrictive, and we therefore conjecture that the HS is PPT and the class is PPT maps [22] introduced in [23].

Let us therefore define PPT entanglement of formation E_f^{PPT} . Recall that the traditional entanglement of formation was defined as [2] $E_f(\varrho) = \inf \sum_i p_i S(\varrho_A^i)$, where the infimum runs over all ensembles $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ and ϱ_A^i is obtained by partial trace of ψ_i . One knows that the entanglement cost is given by the regularized entanglement of formation [24]

$$E_c = E_f^\infty. \quad (3)$$

To define the PPT version of E_f we need the notion of *phase-separated states*. A state is called phase separated

if it is either pure or PPT or a tensor product of both. Then we define

$$E_f^{\text{PPT}}(\varrho) = \inf \sum_i p_i f(\varrho_i), \quad (4)$$

where the infimum runs over decompositions of ϱ into phase-separated states, and $f(\varrho_{AB}^{\text{PPT}}) = 0$, $f(|\psi_{AB}\rangle) = S(\text{Tr}_A(|\psi\rangle_{AB}\langle\psi|)) = f(|\psi\rangle_{AB}\langle\psi| \otimes \varrho_{A'B'}^{\text{PPT}})$, i.e., we count only the pure entanglement. This quantity intuitively measures the entanglement cost under LOCC + PPT, like entanglement of formation E_f was associated with entanglement cost under LOCC. Indeed, following [24]:

Proposition 1: If E_f^{PPT} is monotonic under PPT maps, and asymptotically continuous [25], then its regularization is equal to the entanglement cost under PPT maps.

If the assumptions hold, then we have counterpart of (3) for PPT maps. Consider now the state

$$\varrho = p|\psi_+\rangle\langle\psi_+| + (1-p)|\psi_-\rangle\langle\psi_-|, \quad p \in [0, 1], \quad (5)$$

with $\psi_\pm = (1/\sqrt{2})(|00\rangle \pm |11\rangle)$. We have $E_f = E_c = H[\frac{1}{2} + \sqrt{p(1-p)}]$ [26] where $H(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy. Also $E_d^{\text{PPT}} = E_R^{\text{PPT}} = S(\varrho_A) - S(\varrho) = 1 - H(p)$ [2, 15].

We will now state the main result of the paper.

Theorem 2: For the state (5) $p = 1/2, 0, 1$ the PPT distillable entanglement is strictly smaller than the regularized PPT entanglement of formation.

Proof: We will prove that $E_f^{\text{PPT}} = E_f^{\text{PPT}\infty} = E_c$. We will actually show that it is true for all maximally correlated (MC) states [15] defined as $\varrho = \sum_{ij} a_{ij} |ii\rangle\langle jj|$. We need the following facts about MC states: (i) A state $\varrho_{AB} \otimes \sigma_{A'B'}$ is MC iff ϱ and σ are MC; (ii) any state in the support of an MC state is MC; (iii) if a MC state is PPT then it is separable. To see (i) note that (a) all pure states in the support of MC state have the same Schmidt decomposition (SD); (b) a state is MC if its eigenvectors have the same Schmidt decomposition. Thus if $\varrho_{AB} \otimes \sigma_{A'B'}$ is MC, then its eigenvectors $\psi_{AB}^i \otimes \phi_{A'B'}^i$ (where ψ_{AB}^i and $\phi_{A'B'}^i$ are eigenvectors of ϱ_{AB} and $\sigma_{A'B'}$, respectively) have the same SD. This is possible only if it is true for vectors ψ_{AB}^i and $\phi_{A'B'}^i$ separately. Thus both ϱ_{AB} and $\sigma_{A'B'}$ have to be MC. Similarly the converse holds. Fact (ii) can be found, e.g., in [15]. Proof of (iii) is straightforward.

Now we can prove the theorem. From (i) it follows that if ϱ is MC then so is $\varrho \otimes n$. Now take any ensemble $\varrho = \sum_i p_i \varrho_i$, with phase-separated ϱ_i . Because of (ii) we have that all ϱ_i are MC. Then (iii) implies that if ϱ_i is PPT, it must be separable. Otherwise it is either pure or product $|\psi\rangle\langle\psi| \otimes \sigma_{\text{PPT}}$ of a pure state and a PPT state. However, due to (i), σ_{PPT} must be MC, hence it is separable. Thus we do not have PPT states in decomposition, so that $E_f^{\text{PPT}}(\varrho^{\otimes n}) = E_f(\varrho^{\otimes n})$, hence $E_f^{\text{PPT}\infty} = E_f^\infty(\varrho)$. However, we have [26, 27] $E_f^\infty = E_f$. Of course, for states (5) with $p \neq 0, 1/2, 1$ we have $E_d^{\text{PPT}} < E_c$. This ends the proof.

Thus, under the conjectures that E_f^{PPT} is monotonic under PPT and asymptotically continuous and that HS is PPT, we obtain that there is no reversible thermodynamics of entanglement. Instead we in general have some dissipation of entanglement even in optimal processes. Since E_R^∞ is the entanglement that could be distilled in the reversible case one can be tempted to argue that the difference between E_c and E_R^∞ gives the amount of entanglement dissipated during formation, while $E_R^\infty - E_d$ is that dissipated during distillation. Then the process of formation of BE states could be nondissipative, while for maximally correlated states, the whole bound entanglement would be dissipated during formation of the state. That is, bound entanglement is needed to create the state, but is lost during creation (this is supported by the fact that one can localize the information corresponding to the bound entanglement for these states [28]).

Does it mean that thermodynamical analogies should be abandoned? First of all, let us emphasize that even if we do not have reversible thermodynamics of entanglement it is definitely instructive to know how far we are from this regime. Thus one might be able to understand the theory looking at deviations from the desired behavior. Moreover, there is still place for the analogy. It may be that one can account for the E_c that is dissipated. Perhaps we sometimes operate in the nonequilibrium regime, or perhaps when distilling entanglement we have phase transitions (cf. [29]). Then we may have irreversibility due to the release of heat. A result disproving the conjecture that E_f^{PPT} is asymptotically continuous would also reopen the possibility that these three laws may still hold as is.

Finally, in a recent development [11] a nontrivial state was exhibited, for which there is reversibility under PPT maps, even though $E_d < E_c$. If one can show that the PPT map can be realized by means of LOCC + PPT, we would obtain a nontrivial regime where reversible thermodynamics of entanglement hold.

To summarize, the features that distinguish entanglement theory from reversible thermodynamics are not due to the existence of bound entanglement (or distillation-formation irreversibility). Rather the problem is that two “phases,” bound entanglement and free (pure) entanglement cannot in general be *separated* without loss of entanglement cost. Finally, we believe that our paper provides a new, clearer picture of entanglement theory, and that further investigation into the thermodynamics of entanglement is warranted.

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