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Optimal Time Allocation for Process Improvement for Growth-Focused Entrepreneurs

Onesun Steve Yoo
University College London, School of Management, London WC1E 6BT, United Kingdom,
onesun.yoo@ucl.ac.uk

Charles J. Corbett
UCLA Anderson School of Management, 110 Westwood Plaza, Los Angeles, CA 90095, USA,
charles.corbett@anderson.ucla.edu

Guillaume Roels
UCLA Anderson School of Management, 110 Westwood Plaza, Los Angeles, CA 90095, USA,
guillaume.roels@anderson.ucla.edu

For many entrepreneurs, time is a key constraint. They need to invest time to achieve growth, but also lose time due to recurring crises. We develop a simple stochastic dynamic program to model how an entrepreneur should prioritize between improving processes to reduce crises vs. harvesting revenue or ensuring future growth. We show that it is initially optimal to prioritize process improvement: an entrepreneur should strive for high process quality early in the venture's growth process. We numerically analyze a simple heuristic derived from this optimal policy and identify the conditions under which it is (or is not) effective. It performs near-optimally except when process quality or revenue rate may deteriorate too fast, or when the cost of process improvement or revenue enhancement is too high. Our work provides a theoretical foundation for the advice found in the popular entrepreneurship and time management literature to invest time now to save time later.

Key words: entrepreneurship; process improvement; time allocation; dynamic programming
History:

1. Introduction

Consider an entrepreneur who wishes to expand her small business. Her time is severely constrained, yet some of it is spent dealing with minor but recurring crises. Most of this entrepreneur's time is devoted to the routine tasks required for running her business—but when a few hours of discretionary time become available, how should they be used? Should she harvest existing opportunities and generate cash, use that time to cultivate future growth, or use it to improve internal processes and thereby reduce the number of recurring crises? This is the question that we study here.

Our interest in studying entrepreneurs' time management was motivated by two main observations. First, many of the entrepreneurs with whom we have interacted are indeed severely time-constrained. Second, even though the popular entrepreneurship literature does sometimes argue for initially prioritizing process over revenue, it provides no theoretical underpinning for that advice.

The large number of such books testifies to the importance of time management for executives and entrepreneurs. The core premise of popular time management literature is that one must invest time today in order to save time in the future; yet absent is an explanatory theory. Drucker (1967, p. 41) emphasizes the need to prevent the "recurrent crisis" by reducing it to a routine that an unskilled worker can manage. Griessman (1994, p. 150) reminds us to "sharpen the axe", or take time to improve the process even when one is busy. Mackenzie (1997) argues that we should prevent new fires rather than spend so much time putting out old ones. Focusing on entrepreneurs, Gerber (2001) emphasizes the need for building systems (e.g., checklists, operating manuals) that can prevent entrepreneurs from dropping the ball. Hess (2012) remarks on how much time entrepreneurs spend putting out fires and argues that they should spend half a day each week thinking about big opportunities or problems. To prevent recurring crises, Ries (2011) recommends investing as much time in the process itself as the time lost when a crisis occurs. Although the call to "invest time now to save time later" has become almost a mantra, we are not aware of any theoretical foundation that supports specific recommendations concerning exactly when the entrepreneur should invest time in process improvement.

To develop such a foundation, we propose a simple model of an entrepreneur's time allocation decisions. We distinguish between four types of activity: fire-fighting (FF), which is unavoidable when a crisis occurs; and three other activities—process improvement (PI), revenue harvesting (RH), and revenue enhancement (RE)—to which the entrepreneur can devote available time if there is no crisis. We formulate a stochastic dynamic program to characterize the entrepreneur's optimal time allocation policy and then characterize when process improvement should take precedence over other activities. We show that entrepreneurs should invest more in process improvement early on, when their opportunity cost of doing so is relatively low. Because the optimal time allocation policy turns out to be too complex to fully characterize in general, we use some of its structural properties to derive a simple heuristic to better understand the structure of the optimal policy. From a numerical comparative analysis, we find that the heuristic often performs (near-)optimally, suggesting that in those cases the structure of the optimal policy is the same as that of the heuristic. The heuristic does not do well, however, when the process quality or revenue rate may drop too fast or when process improvements or revenue enhancements are too costly.

This paper's key contributions are to answer the question we posed at the start and, in so doing, to establish a theoretical foundation for the advice from the previously cited popular literature on entrepreneurship. The value of such a foundation is to make the underlying mechanisms more explicit and also to predict when those popular prescriptions might not hold, as Lévesque (2004) points out in her call for more analytical modeling in entrepreneurship research. The novelty of the current paper is to add a process improvement perspective, which is currently missing in existing work on (entrepreneurial) time allocation.

Our focus is on entrepreneurs who have an operating business that they seek to expand by investing their own time. We do not consider, for instance, entrepreneurs who have received grant or venture funding to conduct research and development into new drugs or materials—and thus whose main challenge is to develop commercially valuable intellectual property before their funding runs out.

Below, we first review relevant literature from entrepreneurship and operations management in Section 2. In Section 3 we introduce our time allocation model, and in Section 4 we discuss the optimal policy and a related heuristic. Section 5 contains numerical illustrations and experiments, and Section 6 contains concluding comments.

2. Literature Review

Our work builds on several streams of literature. The entrepreneurial time allocation literature argues that having more time available is beneficial for the venture's success, but research in this vein usually treats the cost of that time simply as lost wages or reduced leisure time. Much of the work goes back to Becker (1965), who models how individuals allocate time between work and leisure. Hakansson (1971) describes how entrepreneurs should allocate money to various investment opportunities or consumption over their lifetime, but he does not view time as a scarce resource that needs to be allocated. Lévesque and MacCrimmon (1997) examine how an entrepreneur can choose to allocate time between a wage job and a new venture, where the latter's success depends on how much time she invests in it. Several other studies explore how entrepreneurs allocate time between work and leisure via approaches that are analytical (Lévesque et al. 2002), empirical (McCarthy et al. 1990, Cooper et al. 1997), or experimental (Lévesque and Schade 2005, Burmeister-Lamp et al. 2012). Our work differs from these studies in two ways. First, rather than determining how much time overall the entrepreneur should spend on her venture, we analyze how a given amount of time should be allocated among competing priorities. Second, we allow the entrepreneur to "create" future time by investing in process improvement.

A rare empirical study on entrepreneurs' use of time is Mueller et al. (2012); their Table 1 summarizes earlier work that describes how entrepreneurs shift the focus of their attention between the venture's start-up and growth stages. These authors observe six entrepreneurs in each stage over a period of four days; they then categorize the observed activities into several types but do not (as we do) differentiate between fire-fighting and development-oriented activities.

A related body of work is the literature on managerial time allocation; this research goes back to Radner (1975) and Radner and Rothschild (1975), who model how managers should allocate time between various projects. Gifford (1992) provides a wide-ranging critique of this literature. In Seshadri and Shapira (2001), managers balance short-term maintenance activities (spending time on processes that deteriorate if left untended) and longer-term developmental activities (which aim to improve performance). These authors stipulate the proportion of time that managers should spend on both types of activity to maintain system stability; they also make numerical comparisons among various strategies for allocating attention. Our work shares some of these elements, but we allow the entrepreneur to invest in process improvement.

There is some research in entrepreneurial operations that focuses on time, but it typically addresses timing decisions and not time allocation decisions. Babich and Sobel (2004) analyze the optimal timing of an initial public offering, and others examine variations on the theme of when a venture should switch its mode from exploration to exploitation. Armstrong and Lévesque (2002) characterize the optimal time to cease product development and release the product to market, and Choi et al. (2008) describe how the optimal time for switching from exploration to exploitation depends on the nature of the opportunity.

Joglekar and Lévesque (2009) examine how allocation of funding to research and development (exploration) versus marketing (exploitation) should change over time. Lichtenstein et al. (2007) find that the timing of start-up activities affects the new firm's likelihood of emerging successfully.

This paper describes a gradual investment of time in process improvement so as to reduce the time spent fighting fires; hence it is related to the vast literature on process improvement. Two seminal works are Fine (1988) and Fine and Porteus (1989), who study gradual reductions in setup cost and gradual improvements in process quality. The key differences are that we focus on the investment of time, not money, and that we study process improvement in the context of entrepreneurship rather than production.

3. Dynamic Time Allocation Model

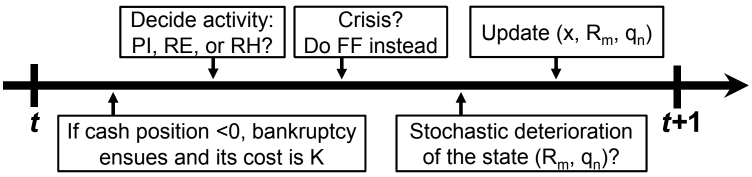
In this section we develop a model of how entrepreneurs should use their time. We use a discrete-time model, where each period t (e.g., a week) contains one block (e.g., half a day) of discretionary time available to the entrepreneur. The time horizon is assumed to be long enough (one or two years) relative to the frequency of time allocation decisions that we can assume an infinite horizon.

During each period t, the entrepreneur undertakes one of four stylized types of activity: fire-fighting (FF), process improvement (PI), revenue harvesting (RH), or revenue enhancement (RE). Examples of *fire-fighting* include dealing with suppliers about miscommunication in shipments or spending time pacifying a customer who has been kept waiting because an assistant double-booked the entrepreneur's time. *Process improvement* could amount to clarifying the written specifications for suppliers or upgrading the appointment scheduling process to prevent double-booking. *Revenue harvesting* can be selling goods delivered by a supplier or making a sale subsequent to meeting a customer. Finally, *revenue enhancement* might involve devising new products to sell or identifying new segments of customers to target. We assume that there is no multi-tasking during these discretionary time periods. (There is substantial evidence from psychology that multi-tasking is counterproductive; see e.g. Levitin 2014.)

The firm's state is characterized by a triplet (x, R, q), in which $x \in \mathbb{R}$ denotes the current cash position, R denotes the revenue that the entrepreneur would earn if the firm engaged in RH, and q denotes the current process quality (i.e., the probability that no crisis will occur). Revenue rate R and process quality q transition through ordered sets $\{R_m\}_{m=0}^{M}$ and $\{q_n\}_{n=0}^{N}$ respectively, where R_0 and q_0 and R_M and q_N are the lowest and highest achievable states for each. If the cash position falls below zero, then the firm goes bankrupt, the cost of which is K.

The sequence of events is illustrated in Figure 1. First, the firm goes bankrupt if it does not have enough cash. Second, the entrepreneur chooses how to allocate her discretionary time in that period: PI, RE, or RH. Third, a crisis erupts with a probability that depends both on the process quality and the chosen activity (PI, RE, or RH). During process improvement or revenue enhancement, a crisis erupts with probability $1 - q_n$. Actively harvesting revenue may induce more crises that the entrepreneur must resolve immediately—in this case with a higher probability $1 - \chi q_n$, where $0 < \chi \leq 1$. We assume that χ is independent of the revenue rate R_m, but this assumption can easily be generalized.

Figure 1 **Sequence of events.**



The immediate reward depends on the chosen action (PI, RE, or RH) and on whether or not there is a crisis (FF). If there is no crisis, then either PI, RE, or RH will generate a base revenue b. Revenue harvesting generates an additional one-time (net) revenue R_m. Process improvement and revenue enhancement each involve an additional cost c, which we assume, for tractability, to be equal. In the event of a crisis, the entrepreneur loses the base revenue b. Furthermore, we allow the consequences of a crisis during RH to be more severe by including an additional loss of $c^{\mathrm{FF}}(R_m) \geq 0$ which may depend on the firm's revenue rate. We assume that all earnings are re-invested in the firm, and we let δ denote the intertemporal per-period discount factor that the entrepreneur applies to money; in this way we capture the opportunity cost of money as well as various risks (e.g., regulatory, technology) beyond the entrepreneur's control.

After each action, the state (R_m, q_n) can deteriorate stochastically with a probability that depends on the action taken. Suppose there is no crisis. Following PI (or RE or RH): the entrepreneur's revenue potential R_m deteriorates by one level, if $m > 0$, with probability α^{PI} (or α^{RE} or α^{RH}); process quality q_n deteriorates by one level, if $n > 0$, with probability β^{PI} (or β^{RE} or β^{RH}); and no deterioration occurs with probability $1 - \alpha^{\mathrm{PI}} - \beta^{\mathrm{PI}}$ (or $1 - \alpha^{\mathrm{RE}} - \beta^{\mathrm{RE}}$ or $1 - \alpha^{\mathrm{RH}} - \beta^{\mathrm{RH}}$). However, if there is a crisis then the entrepreneur must engage in fire-fighting; we assume no further deterioration in such a period. (Allowing deterioration in process quality or revenue potential during fire-fighting would further strengthen the case for process improvement, so this is a conservative assumption.) The state transitions and immediate rewards are summarized in Table 1.

Table 1 Transitions and immediate net rewards from state (x, R_m, q_n).

Decision	Crisis?	Probability	Immediate net reward	Transition to State	Transition Probability
RH	No	χq_n	$b + R_m$	$(x + b + R_m, R_{m-1}, q_n)$	α^{RH}
				$(x + b + R_m, R_m, q_{n-1})$	β^{RH}
				$(x + b + R_m, R_m, q_n)$	$1 - \alpha^{\mathrm{RH}} - \beta^{\mathrm{RH}}$
	Yes	$1 - \chi q_n$	$-c^{\mathrm{FF}}(R_m)$	$(x - c^{\mathrm{FF}}(R_m), R_m, q_n)$	1
PI	No	q_n	$b - c$	$(x + b - c, R_{m-1}, q_{n+1})$	α^{PI}
				$(x + b - c, R_m, q_n)$	β^{PI}
				$(x + b - c, R_m, q_{n+1})$	$1 - \alpha^{\mathrm{PI}} - \beta^{\mathrm{PI}}$
	Yes	$1 - q_n$	0	(x, R_m, q_n)	1
RE	No	q_n	$b - c$	$(x + b - c, R_m, q_n)$	α^{RE}
				$(x + b - c, R_{m+1}, q_{n-1})$	β^{RE}
				$(x + b - c, R_{m+1}, q_n)$	$1 - \alpha^{\mathrm{RE}} - \beta^{\mathrm{RE}}$
	Yes	$1 - q_n$	0	(x, R_m, q_n)	1

The only assumption on the sequences $\{b + R_m\}$ and $\{q_n\}$ is that they are log-concave increasing in m and n, respectively. (The assumption is automatically satisfied if these sequences are concave increasing.) This is consistent with decreasing marginal returns in revenue growth or process improvement, but it also allows for convex–concave patterns similar to the S-curve commonly observed in new product diffusion (Bass 1969).

Following convention in the entrepreneurial operations management literature, we assume that the entrepreneur is risk-neutral (cf. Archibald et al. 2002, Buzacott and Zhang 2004, Tanrisever et al. 2012). As in Fine (1988) and Fine and Porteus (1989), we look for a time allocation policy π that maximizes the net present value (NPV) of expected future profit over an infinite horizon:

$$\lim_{T \to \infty} \sum_{t=0}^{T} \delta^t E_\pi \left[\Pi(x_t, R_{m(t)}, q_{n(t)} \mid a_t) \mid (x_0, R_{m(0)}, q_{n(0)}) \right].$$

Here E_π denotes the conditional expectation given policy π is employed, and $\Pi(x_t, R_{m(t)}, q_{n(t)} \mid a)$ is the expected one-period profit (or loss) associated with action $a \in \{\mathrm{RH}, \mathrm{PI}, \mathrm{RE}\}$ given the current state $(x_t, R_{m(t)}, q_{n(t)})$. Here, $m(t)$ and $n(t)$ are stochastic variables defined on $\{0, \dots, M\}$ and $\{0, \dots, N\}$ respectively, indicating the revenue rate and process quality state that applies in period t. Specifically, $\Pi(x_t, R_m, q_n \mid \mathrm{RH}) = \chi q_n(b + R_m) + (1 - \chi q_n)(-c^{\mathrm{FF}}(R_m))$ and $\Pi(x_t, R_m, q_n \mid \mathrm{PI}) = \Pi(x_t, R_m, q_n \mid \mathrm{RE}) = q_n(b - c)$. The optimal policy π^* is found by solving a dynamic program with the following value-to-go functions. Since the costs and rewards are bounded, there exists a stationary optimal policy as $T \to \infty$ (Bertsekas 2000) and so we drop the time index t. For $n = 0, \dots, N$ and $m = 0, \dots, M$, we have

$$V(x, R_m, q_n) = \begin{cases} \max\{V(x, R_m, q_n \mid \mathrm{RH}), V(x, R_m, q_n \mid \mathrm{PI}), V(x, R_m, q_n \mid \mathrm{RE})\} & \text{if } x \geq 0, \\ x - K & \text{if } x < 0. \end{cases} \tag{1}$$

In these expressions,

$$\begin{aligned} V(x, R_m, q_n \mid \mathrm{RH}) = {} & \chi q_n \big(b + R_m + \delta(1 - \alpha^{\mathrm{RH}} - \beta^{\mathrm{RH}}) V(x + b + R_m, R_m, q_n) \\ & + \delta \alpha^{\mathrm{RH}} V(x + b + R_m, R_{[m-1]^+}, q_n) + \delta \beta^{\mathrm{RH}} V(x + b + R_m, R_m, q_{[n-1]^+})\big) \\ & + (1 - \chi q_n)(-c^{\mathrm{FF}}(R_m) + \delta V(x - c^{\mathrm{FF}}(R_m), R_m, q_n)), \\ V(x, R_m, q_n \mid \mathrm{PI}) = {} & q_n \big(b - c + \delta(1 - \alpha^{\mathrm{PI}} - \beta^{\mathrm{PI}}) V(x + b - c, R_m, q_{n+1}) \\ & + \delta \alpha^{\mathrm{PI}} V(x + b - c, R_{[m-1]^+}, q_{n+1}) + \delta \beta^{\mathrm{PI}} V(x + b - c, R_m, q_n)\big) \\ & + (1 - q_n)(0 + \delta V(x, R_m, q_n)), \\ V(x, R_m, q_n \mid \mathrm{RE}) = {} & q_n \big(b - c + \delta(1 - \alpha^{\mathrm{RE}} - \beta^{\mathrm{RE}}) V(x + b - c, R_{m+1}, q_n) \\ & + \delta \alpha^{\mathrm{RE}} V(x + b - c, R_m, q_n) + \delta \beta^{\mathrm{RE}} V(x + b - c, R_{m+1}, q_{[n-1]^+})\big) \\ & + (1 - q_n)(0 + \delta V(x, R_m, q_n)), \end{aligned}$$

where $[z]^+ = \max\{0, z\}$. A stationary optimal policy tells the entrepreneur which activity to invest time in depending on the state of the firm (x, R_m, q_n). We next describe this policy in detail.

4. Time Allocation Policy and Heuristic

In Section 4.1 we examine the structure of the optimal time allocation policy, starting with the most general formulation and then successively introducing assumptions under which we attain more precise results. In Section 4.2 we introduce a simple heuristic, which will help us understand the structure of the optimal policy in those settings where their performances are similar.

4.1. Structure of the Optimal Policy

We first present a structural property of the optimal policy that holds for the most general model. (All proofs are given in the Appendix.)

PROPOSITION 1. *Suppose process improvement is optimal in states* $(x+b-c, R_{m+1}, q_n)$, $(x+b-c, R_{m+1}, q_{n-1})$, *and* $(x+b-c, R_m, q_n)$. *Then process improvement dominates revenue enhancement in state* (x, R_m, q_n).

According to this proposition, any process improvement (if done at all) has priority over revenue enhancement. More specifically, if process improvement were optimal in all states that might be reached while undertaking revenue enhancement (namely, states $(x+b-c, R_{m+1}, q_n)$, $(x+b-c, R_{m+1}, q_{n-1})$, and $(x+b-c, R_m, q_n)$), then process improvement would be preferable to revenue enhancement. In short, process improvement should normally precede revenue enhancement. Recursive application of this logic yields this structural property of the optimal policy. Process improvement has priority over revenue enhancement because it creates more time in the future by reducing the frequency of crises, which in turn creates more opportunities to improve processes, harvest revenue, or enhance revenue. Although revenue enhancement makes future revenue harvesting more profitable, it does not create additional time for other activities.

The complexity of the dynamic program in its most general form precludes further analytical characterization of the optimal policy. We shall therefore introduce a set of assumptions that can be used (successively) to characterize the optimal policy more precisely.

ASSUMPTION 1. $\alpha^{\mathrm{RE}}=0$ *and* $\beta^{\mathrm{RE}}=0$; *that is, there is no state deterioration during revenue enhancement.*

ASSUMPTION 2. *Either* $x \gg b-c$, *or* $c=b$ *and* $c^{\mathrm{FF}}(R_m)=0$ *for all* m; *that is, there is no threat of bankruptcy.*

ASSUMPTION 3. $\alpha^{\mathrm{RH}}=\beta^{\mathrm{RH}}=0$, $\alpha^{\mathrm{PI}}=\beta^{\mathrm{PI}}=0$, *and* $\delta > \bar{\delta}$ *for some* $\bar{\delta}<1$; *that is, there is no state deterioration during revenue harvesting or process improvement and the discount factor is sufficiently large.*

COROLLARY 1. *Under Assumption 1, the optimal time allocation policy will involve multiple cycles of revenue harvesting followed by process improvement followed by revenue enhancement.*

Corollary 1 states that when there is no state deterioration during RE, the optimal policy consists of multiple intervals of revenue harvesting—in between which process improvement never *immediately* follows revenue enhancement. Harvesting revenue may however be periodically necessary in order to replenish cash and avoid bankruptcy.

If costs are insignificant or if all costs for process improvement or revenue enhancement can be financed by the base revenue b (i.e., if there is no threat of bankruptcy per Assumption 2), then the optimal policy has the following structure.

Proposition 2. *Under Assumption 2, for any R_m there exists a process quality threshold $q^*(R_m)$ such that process improvement is optimal for all states (R_m, q_j) with $q_j \leq q^*(R_m)$. Moreover, $q^*(R_m) \geq q^*(R_M) \triangleq \bar{q}$.*

This proposition introduces a *minimum process quality*, $\bar{q}$, below which process improvement is optimal at all revenue rates R_m. Maintaining process quality at or above this threshold has first priority over engaging in either revenue harvesting or revenue enhancement, but this is only a necessary condition for optimality. According to Proposition 2, there is a threshold $q^*(R_m) \geq \bar{q}$ up to which the entrepreneur should improve the process. So if there are future revenue enhancement opportunities, i.e., if $R_m < R_M$, the entrepreneur may want to invest more time in process improvement than when there are no such opportunities, i.e., when $R_m = R_M$. This "overinvesting" in process improvement relative to the long-term target is analogous to building a safety stock of process quality that can be used later when revenue rates are higher.

Finally, if Assumptions 1 and 2 hold and if also there are no state deteriorations and the per-period discount factor is sufficiently large (Assumption 3), then the optimal policy can be fully characterized.

Proposition 3. *Under Assumptions 1–3, the optimal allocation of available time in state (R_m, q_n) is determined by the following decision procedure:*

1. *if $q_n < q^*(R_m)$ then do process improvement*;
2. *else if $R_m < R^*(q_n)$ then do revenue enhancement*;
3. *else do revenue harvesting.*

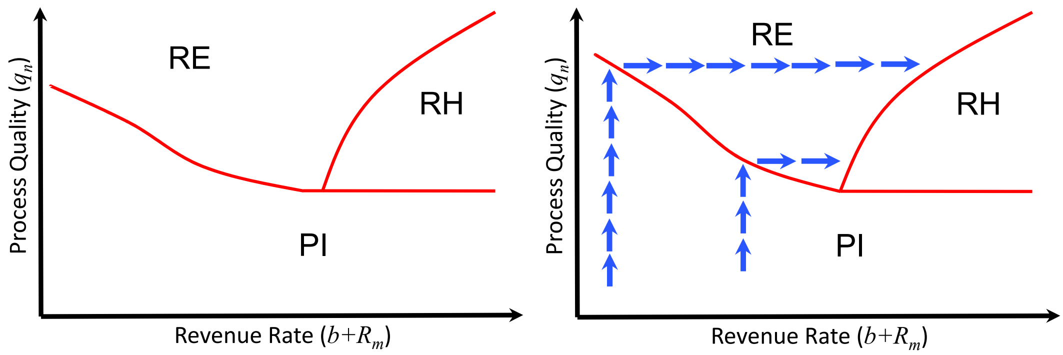
In the absence of stochastic deterioration or cash constraints, there is at most one improvement cycle. This cycle is characterized by specific thresholds: first invest time in process improvement until an *improve-up-to* level $q^*(R_m)$ has been reached; then invest time in revenue enhancement until the *enhance-up-to* level $R^*(q_n)$; then focus on harvesting revenue.

Our next proposition summarizes the comparative statics of these thresholds.

PROPOSITION 4. *Under Assumptions 1–3, the following statements hold*:
(i) *the optimal improve-up-to level* $q^*(R_m)$ *is decreasing to* $\bar{q}$ *in* R_m;
(ii) *the optimal enhance-up-to level* $R^*(q_n)$ *is increasing in* q_n.

Higher revenue rates R_m mean a higher opportunity cost of undertaking process improvements, so the entrepreneur should cease doing them sooner. In contrast, higher process quality q_n increases the value of revenue enhancement, so the entrepreneur should continue revenue enhancement for longer. The optimal improve-up-to level $q^*(R_m)$ is independent of the starting process quality q_n, and the optimal enhance-up-to level $R^*(q_n)$ is independent of the initial revenue rate R_m. Hence the (R,q) state space can be divided into three contiguous regions, as illustrated in Figure 2.

Figure 2 Optimal policy under Assumptions 1–3.



Note. Left panel: The optimal actions correspond to three contiguous regions. *Right panel:* An entrepreneur starting from a lower revenue rate spends more time on process quality, resulting in a higher final revenue rate.

The following corollary is a direct consequence of Proposition 4.

COROLLARY 2. *Under Assumptions 1–3, the entrepreneur reaches a higher revenue rate starting from* (R_a, q_a) *than from* (R_b, q_b), *where* $R_a < R_b$ *for all* q_a, q_b.

An entrepreneur with higher initial revenue rate will end up with a revenue rate lower than that of an entrepreneur who started with a lower rate. That pattern reflects an important feature of the evolution of the value of time: as harvesting revenue becomes more lucrative, time becomes more valuable—which makes it less in the interest of entrepreneurs with higher revenue streams to invest in long-term process improvement activities. This dynamic is plotted by the process improvement paths (arrows) in the right panel of Figure 2.

4.2. A Heuristic Perspective: Return on Time Invested

Because the optimal time allocation policy is typically complex, we propose a simple heuristic—based on the notion of "return on time invested" (ROTI)—whose results closely mimic those of the optimal policy in many but not all circumstances. Whenever the heuristic does well relative to the optimal policy, we presume that the two have a similar structure.

The ROTI heuristic is based on applying standard Net Present Value (NPV) analysis to value the flow of future time. Recall that δ is the intertemporal per-period discount factor applied to money by the entrepreneur and that δ captures the opportunity cost of money in addition to various risks beyond the entrepreneur's control. The net present value of \$1 per period forever is thus $\sum_{t=1}^{\infty}\delta^t = \frac{\delta}{1-\delta}$. We can similarly view the NPV of an infinite series of one-unit periods as $\frac{\delta}{1-\delta}$ periods, where we use the same discount factor as, in theory, each unit of time can be used to generate a unit of revenue. However, some future periods become unavailable due to crises, which occur with probability $1-q_n$. Therefore, the NPV of *available* future time is $q_n\sum_{t=1}^{\infty}\delta^t = \frac{\delta q_n}{1-\delta}$ periods.

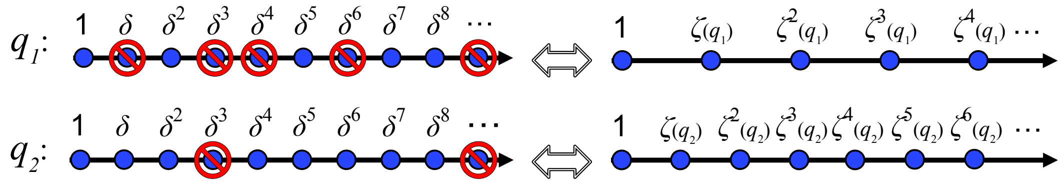
To define the crisis-adjusted discount factor $\zeta(q_n)$ for future time, which takes into account the likelihood of time being available, $\zeta(q_n)$ must satisfy $\frac{\zeta(q_n)}{1-\zeta(q_n)} = \frac{\delta q_n}{1-\delta}$; hence

$$\zeta(q_n) \triangleq \frac{\delta q_n}{1-\delta(1-q_n)}. \tag{2}$$

Our interpretation of $\zeta(q_n)$ is illustrated in Figure 3. A higher process quality q_n has the effect of increasing the present value of the supply of future available time. Hence $\zeta(q_n)$ can be viewed as a discount factor that incorporates an additional operational risk (of crises) relative to the monetary discount factor δ. The discount rate applied to time is different than that for money (but is derived from it). For brevity we will write $\zeta_n \equiv \zeta(q_n)$.

To assess whether it would be advantageous to undertake process improvement (or revenue enhancement) in a particular period, we assume—in the spirit of Fine and Porteus's (1989) *last chance policy*—that the current period is the last chance to do so and that process quality (or revenue rate) will remain unchanged in the future. Suppose the current process quality is q_n and that process improvement can increase it to $q_{n+1} > q_n$ for all

Figure 3 **Discounting of available time.**



Note. *Left-hand side:* The circles represent time periods (e.g., days), some of which are not available because of fire-fighting. With higher process quality ($q_2 > q_1$), there is less fire-fighting and the average interval between two available time periods is shorter. *Right-hand side:* The time stream can be equivalently expressed using $\zeta(q_1)$ and $\zeta(q_2)$ to denote constant time intervals corresponding to the expected delay between two available time periods.

future periods. Then the expected amount of time saved each period is $q_{n+1} - q_n$, so the total discounted time saved is $\frac{\delta(q_{n+1}-q_n)}{1-\delta}$. Therefore, one period of process improvement is worthwhile if and only if

$$\frac{\delta(q_{n+1}-q_n)}{1-\delta} > 1 \iff \frac{\zeta_n}{1-\zeta_n}\left(\frac{q_{n+1}}{q_n} - 1\right) > 1 \iff \zeta_n \frac{q_{n+1}}{q_n} > 1.$$

We define the return on time invested in process improvement as follows:

$$\mathrm{ROTI}_n^{\mathrm{PI}} \triangleq \zeta_n \frac{q_{n+1}}{q_n} = \frac{0 + \zeta_{n+1} + \zeta_{n+1}^2 + \zeta_{n+1}^3 + \cdots}{1 + \zeta_n + \zeta_n^2 + \zeta_n^3 + \cdots}. \tag{3}$$

We likewise define the return on time invested in revenue enhancement as

$$\mathrm{ROTI}_{m,n}^{\mathrm{RE}} \triangleq \zeta_n \frac{b + R_{m+1}}{b + R_m} = \frac{b + R_{m+1}}{b + R_m} \cdot \frac{0 + \zeta_n + \zeta_n^2 + \cdots}{1 + \zeta_n + \zeta_n^2 + \cdots}. \tag{4}$$

Note that $\mathrm{ROTI}_n^{\mathrm{PI}}$ is independent of revenue rate R_m whereas $\mathrm{ROTI}_{m,n}^{\mathrm{RE}}$ depends on process quality q_n. As a result, investing time in PI followed by RE leads to a return on time invested ($\mathrm{ROTI}_n^{\mathrm{PI}} \times \mathrm{ROTI}_{m,n+1}^{\mathrm{RE}}$), which is greater than the return from devoting time to RE followed by PI ($\mathrm{ROTI}_n^{\mathrm{PI}} \times \mathrm{ROTI}_{m,n}^{\mathrm{RE}}$); this statement is in line with Proposition 1. Under Assumptions 1–3, the optimal time allocation policy finds the maximum return on time invested by multiplying all ROTI in process improvements and in revenue enhancements:

$$\left(\mathrm{ROTI}_n^{\mathrm{PI}} \times \cdots \times \mathrm{ROTI}_{n^*}^{\mathrm{PI}}\right) \times \left(\mathrm{ROTI}_{m,n^*+1}^{\mathrm{RE}} \times \cdots \times \mathrm{ROTI}_{m^*,n^*+1}^{\mathrm{RE}}\right)$$

for some n^* and m^*. The thresholds $q^*(R_m)$ and $R^*(q_n)$ in Proposition 3 correspond to m^* and n^*, respectively, and can be constructed using the ROTI notions just described:

$$q^*(R_m) = \min_n \left\{ q_n \,\middle|\, \mathrm{ROTI}_n^{\mathrm{PI}} \cdot \left(\frac{\max_j \prod_{j=m}^{M} \mathrm{ROTI}_{j,n+1}^{\mathrm{RE}}}{\max_j \prod_{j=m}^{M} \mathrm{ROTI}_{j,n}^{\mathrm{RE}}} \right) < 1 \right\}, \qquad R^*(q_n) = \min_m \left\{ R_m \mid \mathrm{ROTI}_{m,n}^{\mathrm{RE}} < 1 \right\}.$$

The foregoing considerations lead to our simple heuristic, as follows.

ROTI heuristic:

1. *if* $x+b-c<0$ *then do RH*;
2. *else if* $\mathrm{ROTI}_n^{\mathrm{PI}}\left(\frac{\max_j \prod_{j=m}^{M} \mathrm{ROTI}_{j,n+1}^{\mathrm{RE}}}{\max_j \prod_{j=m}^{M} \mathrm{ROTI}_{j,n}^{\mathrm{RE}}}\right) > 1$ *then do PI*;
3. *else if* $\mathrm{ROTI}_{m,n}^{\mathrm{RE}} > 1$ *then do RE*;
4. *else do RH.*

The ROTI heuristic prescribes revenue harvesting when the alternative is imminent bankruptcy. Otherwise, the heuristic prescribes process improvement followed by revenue enhancement and then revenue harvesting—all based on the thresholds stipulated in Proposition 3. Having defined the optimal policy and a related heuristic, we can now analyze both numerically.

5. Numerical Study

Using the numerical setup described below in Section 5.1, we present (in Section 5.2) a numerical illustration of a representative sample path, after which we examine (in Section 5.3) the settings in which the ROTI heuristic does and does not perform well.

5.1. Setup of Numerical Study

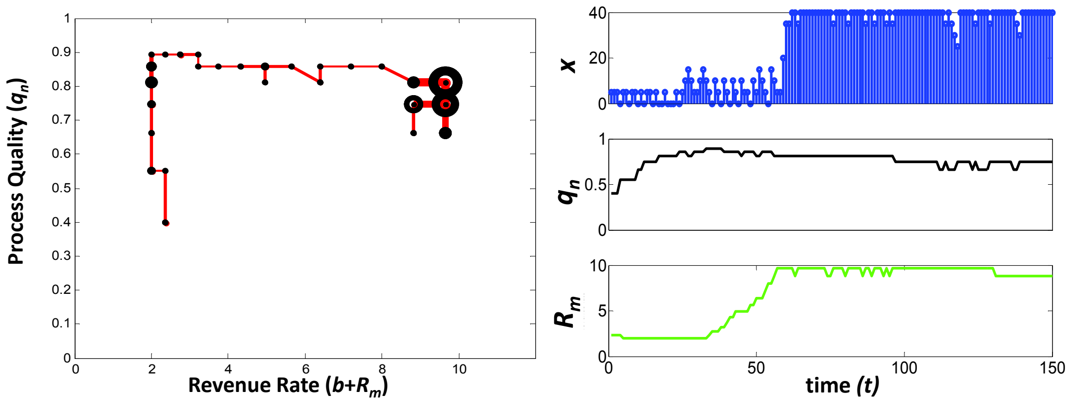
We assume the following parameters. The sequence of process quality $\{q_n\}$ is defined by $(1-q_{n+1}) = [0.75 \cdot (1-q_n)]$ for $n = \{0,\ldots,9\}$ with $q_0 = 0.2$. That is, the likelihood of firefighting declines to 75% of its previous value after every period of process improvement: from 0.8 to 0.6, then to 0.45, and so forth. The sequence of revenue rates is defined by $(b+R_{m+1}) = \left[1.2 \cdot (b+R_m) \cdot \frac{100-(b+R_m)}{100}\right]$ for $m = \{0,\ldots,15\}$ with $R_0 = -3$ and $b = 5$. This gives a convex–concave sequence of increasing revenue rates that resembles the S-curve often observed with regard to new product diffusion (Bass 1969). In each period, process quality and revenue rate can deteriorate with probability $\alpha^i = \beta^i = 0.1$ for $i \in \{\mathrm{RE}, \mathrm{RH}, \mathrm{PI}\}$. We consider continuous cash-level states $x \in [-5, 80]$, where $x = 0$ represents the bankruptcy threshold. In other words: if cash falls below zero then the firm goes bankrupt, but if cash rises above 80 then it stays at 80. We assume a large bankruptcy cost, $K = 999$, to ensure that avoiding bankruptcy is always desirable. For the discount factor we set $\delta = 0.95$.

We use a standard value iteration algorithm (Bertsekas 2000) to solve for the optimal policy and value, a mapping from $\{x_l\} \times \{R_m\} \times \{q_n\}$ to (respectively) $\{\mathrm{RH}, \mathrm{RE}, \mathrm{PI}\}$ or $\mathbb{R}$. We discretize the continuous cash state and interpolate those values that do not fall on the grid during value iteration. All code is written in MATLAB and is available from the authors upon request.

5.2. Illustration of the Optimal Policy

The left panel in Figure 4 plots a representative sample path of the optimal policy in the (R, q) state space, where the vertical and horizontal axes correspond to process quality and the revenue rate, respectively. Upward (rightward) movement in the graph signifies process improvement (revenue enhancement). The right panels illustrate the temporal dynamics of states x, R, and q.

Figure 4 **Sample path of the optimal policy.**



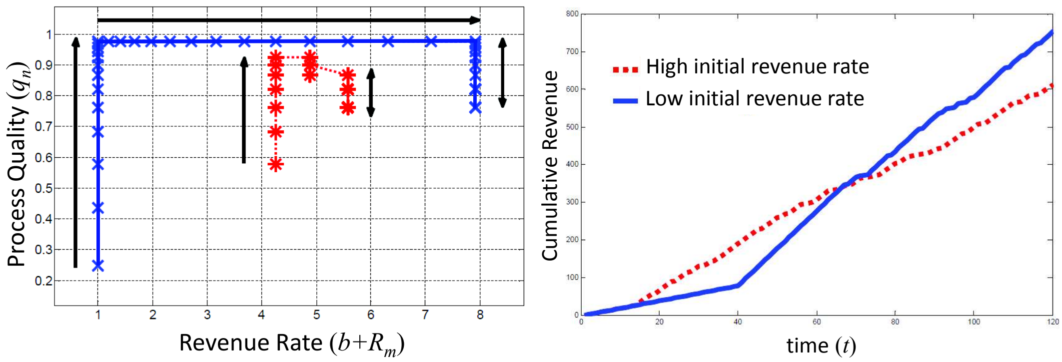
Note. In the left panel, the size of each dot is proportional to the frequency with which the corresponding state is visited. The right panel plots the cash level, process quality, and revenue rate against time. The parameters are as described in the text: $\alpha^i = \beta^i = 0.1$ for $i \in \{\mathrm{RE}, \mathrm{RH}, \mathrm{PI}\}$, and $c = c^{\mathrm{FF}}(R_m) = 1.25b$. The initial state is $(x, R_{m(0)}, q_{n(0)}) = (5, 2.76, 0.38)$.

Figure 4 illustrates several of our analytical results. The entrepreneur engages in a phase of process improvement until the improve-up-to level is reached in period 45; then engages in revenue enhancement until the enhance-up-to level is reached in period 58; and thereafter finally harvests revenue. The entrepreneur may deviate from this pattern either to harvest revenue (to avoid bankruptcy) or to do process improvement during the revenue enhancement phase (after a deterioration in process quality). This pattern is a noisy version of the one predicted by Proposition 3, which considers neither cash constraints nor stochastic deterioration. We nonetheless find that, in line with Proposition 1 and Corollary 1, revenue enhancement never immediately precedes process improvement; rightward motion (in the graph) is never followed by upward motion.

The right panel of Figure 4 illustrates "overinvesting" in process improvement ($q^*(R_m) > \bar{q}$), consistent with Proposition 2. Early on ($t = 30$), when revenue rate is lower, the entrepreneur aims for higher process quality ($q^*(R_m) > \bar{q}$) than later on ($t = 120$), when revenue rate is higher.

Although Corollary 2 assumed no stochastic deterioration, its main insights continue to hold even when process quality can deteriorate. The left panel of Figure 5 illustrates how the entrepreneur who starts at a higher revenue rate ends up harvesting revenue at a lower rate. The right panel plots the cumulative (undiscounted) revenue of the two entrepreneurs following their respective optimal policies. Note that the value function $V(R_m, q_n)$ is increasing in R_m and q_n. Hence in period $t = 0$, the entrepreneur who starts with both a higher revenue rate and a higher process quality will have a higher expected future (discounted) profits than the other entrepreneur; but in period $t = 120$, that other entrepreneur ends up at a higher revenue rate and a higher process quality and will therefore have a higher expected future (discounted) profit than the former entrepreneur. This is because the entrepreneur who starts with a higher revenue rate and a higher process quality starts harvesting revenue earlier instead of doing process improvement because his/her high opportunity cost of time, given his/her current revenue rate, inhibits process improvement. This eventually results in slower accumulation of revenue, as indicated by a less steep slope in the figure.

Figure 5 **Optimal sample paths given different initial revenue rates.**



Note. Parameters are as described in the text: $\alpha^i = 0.001$ and $\beta^i = 0.1$ for $i \in \{\text{RE}, \text{RH}, \text{PI}\}$ and with $c = c^{\text{FF}}(R_m) = b$; the initial states $(R_{m(0)}, q_{n(0)})$ are $(1, 0.25)$ and $(4.2, 0.58)$. The black arrows indicate whether the state transitions occur only in one direction or both.

5.3. Performance of the ROTI Heuristic

To better understand the structure of the optimal policy, we performed an extensive simulation comparing the performance of the ROTI heuristic to the optimal policy. We used combinations of $\alpha^i = \beta^i$ for $i \in \{\text{RH}, \text{PI}, \text{RE}\}$ within the set $\{0, 0.05, 0.1, 0.15\}$ and used $c = c^{\text{FF}}(R_m) = \lambda \cdot b$ with $\lambda \in \{0, 1, 1.25, 1.5, 2\}$. To assess the effect of cash constraints and of initial process quality and revenue rate, we used four starting points—$(\text{lo}, \text{lo}, \text{lo})$, $(\text{lo}, \text{lo}, \text{md})$, $(\text{lo}, \text{md}, \text{lo})$, and $(\text{hi}, \text{lo}, \text{lo})$—for $\{(x_0, R_0, q_0)\} \in \{5, 75\} \times \{-2.6, 0.7, 5.5\} \times \{0.4, 0.75, 0.9\} \triangleq \{\text{lo}, \text{hi}\} \times \{\text{lo}, \text{md}, \text{hi}\} \times \{\text{lo}, \text{md}, \text{hi}\}$ (where lo, md, and hi stand for low, medium, and high levels of the variables). For each case, we ran 5,000 simulations for $T = 330$ periods, which approximates an infinite horizon because $\sum_{t=T+1}^{\infty} \delta^t < 10^{-6}$. The simulation average of the sum of discounted profits (V^{sim}) and the optimal value (V^{opt}) were used to compute the suboptimality gap $\left(\frac{V^{\text{opt}} - V^{\text{sim}}}{V^{\text{opt}}}\right)$ shown in Table 2.

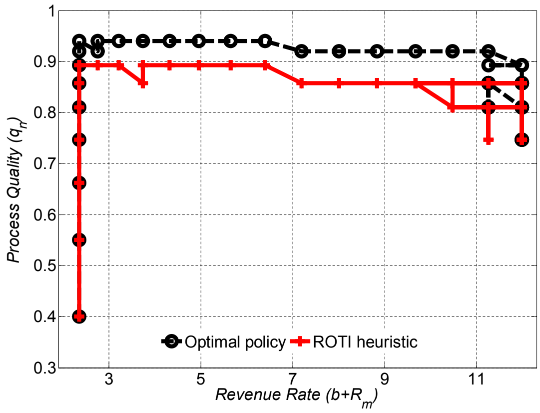
Table 2 Suboptimality gaps of the ROTI heuristic.

$\alpha = \beta$	Initial (x, R, q)	Cost of PI and RE (c) $c = 0$	$c = b$	$c = 1.25b$	$c = 1.5b$	$c = 2b$
0	(lo, lo, lo)	**2.2%**	**0.1%**	**0.0%**	**1.6%**	**3.5%**
	(lo, lo, md)	**2.1%**	**0.2%**	**0.2%**	**2.6%**	**2.6%**
	(lo, md, lo)	**2.8%**	**0.4%**	**0.3%**	**2.0%**	**2.9%**
	(hi, lo, lo)	**2.2%**	**0.0%**	**0.1%**	**0.7%**	**2.7%**
0.05	(lo, lo, lo)	**2.7%**	**1.1%**	**1.0%**	**0.3%**	10.7%
	(lo, lo, md)	**3.2%**	**0.7%**	**1.6%**	**3.8%**	7.6%
	(lo, md, lo)	**3.1%**	**0.4%**	**2.1%**	**4.7%**	6.0%
	(hi, lo, lo)	**2.7%**	**0.9%**	**0.8%**	**0.3%**	13.3%
0.1	(lo, lo, lo)	**3.5%**	**2.6%**	6.4%	**0.8%**	17.7%
	(lo, lo, md)	**4.0%**	**2.9%**	**4.5%**	**4.8%**	16.1%
	(lo, md, lo)	**3.4%**	**2.3%**	**4.4%**	**2.6%**	12.8%
	(hi, lo, lo)	**3.5%**	**3.2%**	6.9%	**0.5%**	**4.7%**
0.15	(lo, lo, lo)	**4.0%**	6.6%	18.9%	6.2%	24.9%
	(lo, lo, md)	**4.5%**	5.8%	13.0%	8.3%	17.9%
	(lo, md, lo)	**3.6%**	**3.4%**	12.2%	29.7%	35.1%
	(hi, lo, lo)	**3.9%**	7.2%	21.0%	6.8%	26.1%

Note. $b = 5$; boldface type indicates values under which the ROTI heuristic has a suboptimality gap of less than 5%.

For moderate probabilities of deterioration or cost of improvement activities (top rows or left columns), the ROTI heuristic performs near optimally for many initial states: most of the suboptimality gaps are less than 5% (boldface values). In particular, the ROTI heuristic is optimal when $c = b$, $x = \text{hi}$, and $\alpha = \beta = 0$ (Proposition 3). The first column shows that when $c < b$, the ROTI heuristic performs near optimally under all deterioration rates. In that case, fire-fighting (and losing b) is more costly than improvement activities and so the optimal policy prescribes process improvements—even when a higher rate of deterioration makes their effect more temporary. In such settings with small suboptimality gaps, the ROTI heuristic appears to mimic the optimal policy; see the representative sample paths in Figure 6.

Figure 6 **Representative sample paths: ROTI heuristic and optimal policy.**



Note. The parameters used to generate this graph are $\alpha^i = \beta^i = 0.1$ for $i \in \{\mathrm{RE}, \mathrm{RH}, \mathrm{PI}\}$, $c = c^{\mathrm{FF}}(R_m) = 1.25b$, and an initial state of $(25, -2.7, 0.4)$.

Comparing the pairs of $(\mathrm{lo}, \mathrm{lo}, \mathrm{lo})$ and $(\mathrm{hi}, \mathrm{lo}, \mathrm{lo})$ cases reveals that the ROTI heuristic's performance is relatively insensitive to the initial cash position, even though it largely ignores cash, as initial cash affects the optimal policy and the ROTI heuristic in a similar fashion. However, the heuristic appears to suffer more from the neglect of cash constraints when the cost of process improvement is high.

As expected, the heuristic does not perform well when the process quality or revenue rate has a high (i.e., 10% or more) chance of deteriorating each period or when process improvements or revenue enhancements incur high monetary costs. In those cases, a simple time allocation heuristic may not be available. In other cases, though, the fact that the heuristic performs well indicates that the simple time allocation policy it represents provides a good guideline for entrepreneurs.

6. Concluding Discussion

The popular time management literature emphasizes adages about investing time now to save time later, but without a theoretical framework. Using a stochastic dynamic program to characterize the time allocation policy of entrepreneurs, we show that they should invest more time in process improvement early on—that is, when the opportunity cost of doing so is relatively low. We derive a simple heuristic from the optimal policy and then assess its effectiveness under a wide range of parameters.

Having established the importance of process improvement, one might ask what kind of process improvements an entrepreneur could make in practice. Gerber (2001, p. 97) proclaims that entrepreneurs must "work on the business and not *in* it" and discusses the importance of developing the right processes as the firm grows. According to Hess (2012, p. 79), process improvement activities include designing "rules for mitigating financial and quality risks", writing "directions, recipes, instructing an employee how to do specific tasks or what not to do", and implementing systems that can produce "reliable, timely data, or feedback that will reveal variances or mistakes." Drucker (1967) argues that executives must systematically monitor their use of time in order to diagnose and eliminate any sources of waste. He provides practical pointers for process improvement by identifying the lack of systems or foresight, disorganization, and malfunctioning delivery of information as the main time wasters. None of these recommended actions are especially novel, yet our model helps explain why they are useful. Moreover, our analysis encourages entrepreneurs to think in terms of "return on time invested", to help decide which process improvement activities to prioritize, and when to shift their focus from process improvement to revenue enhancement and harvesting.

Our results have several implications for future research at the intersection of entrepreneurship and operations management. One could study how entrepreneurs invest time in activities that save future time and how their decisions relate to the NPV framework. Although some psychology studies compare the investment of time versus money (LeClerc et al. 1995, Soman 2001, Okada and Hoch 2004, Zauberman and Lynch 2005), we are not aware of any research that examines the present value of time streams. It is well known that process improvement efforts are difficult (Repenning and Sterman 2002) and can even result in negative feedback (Sterman et al. 1997). Future research could profitably investigate the conditions under which entrepreneurs decide to invest time in process improvement and how that depends on perceived revenue opportunities. Detailed analysis of how entrepreneurs actually use their time, perhaps building on Mueller et al. (2012), but using categories of activities as we distinguish here, would also provide important pointers for which future research would be most valuable for entrepreneurs.

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Appendix

LEMMA A-1. *$\zeta(q)$ is concave increasing in $q \in [0,1]$ with $\zeta(0)=0$ and $\zeta(1)=\delta$.*

Proof. It is clear that $\zeta(0)=0$ and $\zeta(1)=\delta$. Taking the first and second derivatives then yields $\zeta'(q)=\frac{\delta(1-\delta)}{(1-\delta(1-q))^2}>0$ and $\zeta''(q)=-\frac{2\delta^2(1-\delta)(1-\delta(1-q))}{(1-\delta(1-q))^4}<0$ for all $\delta, q\in(0,1)$. $\square$

Proof of Proposition 1. By Proposition 7.3.1 in Bertsekas (2000), the optimal stationary policy can be obtained by solving the Bellman equation corresponding to (1). In particular, if the optimal action is PI in state (x,R_m,q_n), then $V(x,R_m,q_n)=V(x,R_m,q_n\,|\,\mathrm{PI})$, where

$$\begin{aligned}
V(x,R_m,q_n\,|\,\mathrm{PI}) &= q_n\big(b-c+\delta(1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})V(x+b-c,R_m,q_{n+1})\\
&\quad+\delta\alpha^{\mathrm{PI}}V(x+b-c,R_{[m-1]^+},q_{n+1})+\delta\beta^{\mathrm{PI}}V(x+b-c,R_m,q_n)\big)+(1-q_n)(0+\delta V(x,R_m,q_n))\\
&=\frac{q_n(b-c)}{1-\delta(1-q_n)}+\zeta_n\big((1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})V(x+b-c,R_m,q_{n+1})\\
&\quad+\alpha^{\mathrm{PI}}V(x+b-c,R_{[m-1]^+},q_{n+1})+\beta^{\mathrm{PI}}V(x+b-c,R_m,q_n)\big).
\end{aligned}$$

Similarly, if the optimal action is RE in state (x,R_m,q_n), then $V(x,R_m,q_n)=V(x,R_m,q_n\,|\,\mathrm{RE})$, where

$$\begin{aligned}
V(x,R_m,q_n|RE) &= q_n\Big(b-c+\delta(1-\alpha^{RE}-\beta^{RE})V(x+b-c,R_{m+1},q_n)\\
&\quad+\delta\alpha^{RE}V(x+b-c,R_m,q_n)+\delta\beta^{RE}V(x+b-c,R_{m+1},q_{[n-1]^+})\Big)+(1-q_n)\Big(0+V_{t+1}(x,R_m,q_n)\Big)\\
&= \frac{q_n(b-c)}{1-\delta(1-q_n)}+\zeta_n\Big((1-\alpha^{RE}-\beta^{RE})V(x+b-c,R_{m+1},q_n)\\
&\quad+\alpha^{RE}V(x+b-c,R_m,q_n)+\beta^{RE}V(x+b-c,R_{m+1},q_{[n-1]^+})\Big).
\end{aligned}$$

By assumption, undertaking PI is optimal in states $(x+b-c,R_m,q_n)$, $(x+b-c,R_{m+1},q_n)$, and $(x+b-c,R_{m+1},q_{[n-1]^+})$. Then the value function in state (x,R_m,q_n) when engaging in RE is

$$\begin{aligned}
V(x,R_m,q_n\,|\,\mathrm{RE}) &= \frac{q_n(b-c)}{1-\delta(1-q_n)}+\zeta_n\big((1-\alpha^{\mathrm{RE}}-\beta^{\mathrm{RE}})V(x+b-c,R_{m+1},q_n\,|\,\mathrm{PI})\\
&\quad+\alpha^{\mathrm{RE}}V(x+b-c,R_m,q_n\,|\,\mathrm{PI})+\beta^{\mathrm{RE}}V(x+b-c,R_{m+1},q_{[n-1]^+}\,|\,\mathrm{PI})\big)\\
&=\frac{q_n(b-c)}{1-\delta(1-q_n)}+\zeta_n\left((1-\beta^{\mathrm{RE}})\frac{q_n(b-c)}{1-\delta(1-q_n)}+\beta^{\mathrm{RE}}\frac{q_{[n-1]^+}(b-c)}{1-\delta(1-q_{[n-1]^+})}\right)\\
&\quad+\zeta_n^2(1-\alpha^{\mathrm{RE}}-\beta^{\mathrm{RE}})(1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})V(x+2b-2c,R_{m+1},q_{n+1})\\
&\quad+\zeta_n^2(1-\alpha^{\mathrm{RE}}-\beta^{\mathrm{RE}})\alpha^{\mathrm{PI}}V(x+2b-2c,R_m,q_{n+1})+\zeta_n^2(1-\alpha^{\mathrm{RE}}-\beta^{\mathrm{RE}})\beta^{\mathrm{PI}}V(x+2b-2c,R_{m+1},q_n)\\
&\quad+\zeta_n^2\alpha^{\mathrm{RE}}(1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})V(x+2b-2c,R_m,q_{n+1})+\zeta_n^2\alpha^{\mathrm{RE}}\alpha^{\mathrm{PI}}V(x+2b-2c,R_{[m-1]^+},q_{n+1})\\
&\quad+\zeta_n^2\alpha^{\mathrm{RE}}\beta^{\mathrm{PI}}V(x+2b-2c,R_m,q_n)+\zeta_n\zeta_{n-1}\beta^{\mathrm{RE}}(1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})V(x+2b-2c,R_{m+1},q_n)\\
&\quad+\zeta_n\zeta_{n-1}\beta^{\mathrm{RE}}\alpha^{\mathrm{PI}}V(x+2b-2c,R_m,q_n)+\zeta_n\zeta_{n-1}\beta^{\mathrm{RE}}\beta^{\mathrm{PI}}V(x+2b-2c,R_{m+1},q_{[n-1]^+}).
\end{aligned}$$

Since RE is feasible in states $(x+b-c,R_{m-1},q_{n+1})$, $(x+b-c,R_m,q_n)$, and $(x+b-c,R_m,q_{n+1})$, we have

$$\begin{aligned}
V(x,R_m,q_n\,|\,\mathrm{PI}) &\geq \frac{q_n(b-c)}{1-\delta(1-q_n)}+\zeta_n\big((1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})V(x+b-c,R_m,q_{n+1}\,|\,\mathrm{RE})\\
&\quad+\alpha^{\mathrm{PI}}V(x+b-c,R_{[m-1]^+},q_{n+1}\,|\,\mathrm{RE})+\beta^{\mathrm{PI}}V(x+b-c,R_m,q_n\,|\,\mathrm{RE})\big)\\
&=\frac{q_n(b-c)}{1-\delta(1-q_n)}+\zeta_n\left((1-\beta^{\mathrm{PI}})\frac{q_{n+1}(b-c)}{1-\delta(1-q_{n+1})}+\beta^{\mathrm{PI}}\frac{q_n(b-c)}{1-\delta(1-q_n)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\zeta_n\zeta_{n+1}(1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})(1-\alpha^{\mathrm{RE}}-\beta^{\mathrm{RE}})V(x+2b-2c,R_{m+1},q_{n+1})\\
&+\zeta_n\zeta_{n+1}(1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})\alpha^{\mathrm{RE}}V(x+2b-2c,R_m,q_{n+1})\\
&+\zeta_n\zeta_{n+1}(1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})\beta^{\mathrm{RE}}V(x+2b-2c,R_{m+1},q_n)\\
&+\zeta_n\zeta_{n+1}\alpha^{\mathrm{PI}}(1-\alpha^{\mathrm{RE}}-\beta^{\mathrm{RE}})V(x+2b-2c,R_m,q_{n+1})+\zeta_n\zeta_{n+1}\alpha^{\mathrm{PI}}\alpha^{\mathrm{RE}}V(x+2b-2c,R_{[m-1]^+},q_{n+1})\\
&+\zeta_n\zeta_{n+1}\alpha^{\mathrm{PI}}\beta^{\mathrm{RE}}V(x+2b-2c,R_m,q_n)+\zeta_n^2\beta^{\mathrm{PI}}(1-\alpha^{\mathrm{RE}}-\beta^{\mathrm{RE}})V(x+2b-2c,R_{m+1},q_n)\\
&+\zeta_n^2\beta^{\mathrm{PI}}\alpha^{\mathrm{RE}}V(x+2b-2c,R_m,q_n)+\zeta_n^2\beta^{\mathrm{PI}}\beta^{\mathrm{RE}}V(x+2b-2c,R_{m+1},q_{[n-1]^+})\\
&\geq V(x,R_m,q_n\mid \mathrm{RE}).
\end{aligned}
$$

The last inequality follows because ζ_n is increasing by Lemma A-1 and because

$$(1-\beta^{\mathrm{PI}})\frac{q_{n+1}}{1-\delta(1-q_{n+1})}+\beta^{\mathrm{PI}}\frac{q_n}{1-\delta(1-q_n)}>(1-\beta^{\mathrm{RE}})\frac{q_n}{1-\delta(1-q_n)}+\beta^{\mathrm{RE}}\frac{q_{n-1}}{1-\delta(1-q_{n-1})}. \quad \square$$

Proof of Corollary 1. The proof is identical to the proof of Proposition 1. When $\alpha^{\mathrm{RE}}=\beta^{\mathrm{RE}}=0$, it is not necessary to assume that PI is optimal in states $(x+b-c,R_m,q_n)$ and $(x+b-c,R_{m+1},q_{n-1})$. $\square$

LEMMA A-2. *Under Assumption 2, suppose that*

$$\xi_{n+1}\zeta_n\tilde{\gamma}_n^{PI}-\xi_n+\xi_{n+1}\zeta_n\xi_n\beta^{RH}\tilde{\gamma}_n^{PI}(1-\zeta_{n-1}\tilde{\gamma}_{n-1}^{PI})+\xi_{n+1}\xi_n(1-\alpha^{RH}-\beta^{RH})(1-\zeta_n\tilde{\gamma}_n^{PI})\geq 0, \tag{A1}$$

where $\xi_n\triangleq\delta\chi q_n/(1-\delta(1-\chi q_n))$ *and* $\tilde{\gamma}_n^{PI}=(1-\alpha^{PI}-\beta^{PI})/(1-\zeta_n\beta^{PI})$*. Then, if either PI is optimal in state* (R_m,q_{n-1}) *or* $\beta^{RH}=0$*, then PI is preferred to RH in state* (R_m,q_n)*.*

Proof. Under Assumption 2, the cash state x remains constant during PI or RE or a crisis. Hence the cash state has no effect on the optimal decision. Ignoring the cash state, we have the following Bellman's equation:

$$
\begin{aligned}
V(R_m,q_n) &= \max\left\{V(R_m,q_n\mid\mathrm{RH}),V(R_m,q_n\mid\mathrm{PI}),V(R_m,q_n\mid\mathrm{RE})\right\};\ \text{here}\\
V(R_m,q_n\mid\mathrm{RH})&=\frac{\xi_n}{\delta}(R_m+b)+\xi_n\big(\alpha^{\mathrm{RH}}V(R_{[m-1]^+},q_n)+\beta^{\mathrm{RH}}V(R_m,q_{[n-1]^+})+\gamma^{\mathrm{RH}}V(R_m,q_n)\big),\\
V(R_m,q_n\mid\mathrm{PI})&=\zeta_n\big(\alpha^{\mathrm{PI}}V(R_{[m-1]^+},q_{n+1})+\beta^{\mathrm{PI}}V(R_m,q_n)+\gamma^{\mathrm{PI}}V(R_m,q_{n+1})\big),\\
V(R_m,q_n\mid\mathrm{RE})&=\zeta_n\big(\alpha^{\mathrm{RE}}V(R_m,q_n)+\beta^{\mathrm{RE}}V(R_{m+1},q_{[n-1]^+})+\gamma^{\mathrm{RE}}V(R_{m+1},q_n)\big),
\end{aligned}
$$

and $\gamma^a\triangleq 1-\alpha^a-\beta^a$ for $a\in\{\mathrm{RE},\mathrm{PI},\mathrm{RE}\}$.

Suppose that $m>0$ and $n>0$. If either $m=0$ or $n=0$, then the proof is identical once we set (respectively) $\alpha^{\mathrm{PI}}=\alpha^{\mathrm{RH}}=0$ or $\beta^{\mathrm{RH}}=\beta^{\mathrm{RE}}=0$. The proof proceeds by contradiction. We start by assuming that it is optimal to do RH in state (R_m,q_n). On the one hand, since PI is optimal in state (R_m,q_{n-1}), we have

$$
\begin{aligned}
V(R_m,q_n\mid\mathrm{RH})&=\frac{\xi_n}{\delta}(R_m+b)+\xi_n\big(\alpha^{\mathrm{RH}}V(R_{m-1},q_n)+\beta^{\mathrm{RH}}V(R_m,q_{n-1}\mid\mathrm{PI})+\gamma^{\mathrm{RH}}V(R_m,q_n\mid\mathrm{RH})\big)\\
&=\frac{\xi_n}{\delta}(R_m+b)+\xi_n\big(\alpha^{\mathrm{RH}}V(R_{m-1},q_n)+\gamma^{\mathrm{RH}}V(R_m,q_n\mid\mathrm{RH})\big)\\
&\quad+\xi_n\beta^{\mathrm{RH}}\frac{\zeta_{n-1}}{1-\beta^{\mathrm{PI}}\zeta_{n-1}}\big(\alpha^{\mathrm{PI}}V(R_{m-1},q_n)+\gamma^{\mathrm{PI}}V(R_m,q_n\mid\mathrm{RH})\big)\\
&=\frac{(\xi_n/\delta)(1-\beta^{\mathrm{PI}}\zeta_{n-1})}{(1-\gamma^{\mathrm{RH}}\xi_n)(1-\beta^{\mathrm{PI}}\zeta_{n-1})-\zeta_{n-1}\xi_n\beta^{\mathrm{RH}}\gamma^{\mathrm{PI}}}(R_m+b)\\
&\quad+\frac{\xi_n(\alpha^{\mathrm{RH}}(1-\beta^{\mathrm{PI}}\zeta_{n-1})+\beta^{\mathrm{RH}}\zeta_{n-1}\alpha^{\mathrm{PI}})}{(1-\gamma^{\mathrm{RH}}\xi_n)(1-\beta^{\mathrm{PI}}\zeta_{n-1})-\zeta_{n-1}\xi_n\beta^{\mathrm{RH}}\gamma^{\mathrm{PI}}}V(R_{m-1},q_n).
\end{aligned}
$$

On the other hand, since RH is feasible in state (R_m, q_{n+1}) and PI in (R_m, q_n), it follows that

$$\begin{aligned}
V(R_m, q_n \mid \mathrm{PI}) &\geq \zeta_n \big(\alpha^{\mathrm{PI}} V(R_{m-1}, q_{n+1}) + \beta^{\mathrm{PI}} V(R_m, q_n \mid \mathrm{PI}) + \gamma^{\mathrm{PI}} V(R_m, q_{n+1} \mid \mathrm{RH})\big) \\
&\geq \frac{\zeta_n}{1-\beta^{\mathrm{PI}}\zeta_n} \big(\alpha^{\mathrm{PI}} V(R_{m-1}, q_{n+1}) + \gamma^{\mathrm{PI}} V(R_m, q_{n+1} \mid \mathrm{RH})\big) \\
&\geq \frac{\alpha^{\mathrm{PI}}\zeta_n}{1-\beta^{\mathrm{PI}}\zeta_n} V(R_{m-1}, q_{n+1}) \\
&\quad + \frac{\gamma^{\mathrm{PI}}\zeta_n}{1-\beta^{\mathrm{PI}}\zeta_n} \frac{\xi_{n+1}}{1-\gamma^{\mathrm{RH}}\xi_{n+1}} \left(\frac{1}{\delta}(R_m + b) + \alpha^{\mathrm{RH}} V(R_{m-1}, q_{n+1}) + \beta^{\mathrm{RH}} V(R_m, q_n \mid \mathrm{PI}) \right) \\
&\geq \frac{\xi_{n+1}\gamma^{\mathrm{PI}}\zeta_n/\delta}{(1-\gamma^{\mathrm{RH}}\xi_{n+1})(1-\beta^{\mathrm{PI}}\zeta_n) - \zeta_n\xi_{n+1}\beta^{\mathrm{RH}}\gamma^{\mathrm{PI}}} (R_m + b) \\
&\quad + \frac{\zeta_n(\alpha^{\mathrm{PI}}(1-\gamma^{\mathrm{RH}}\xi_{n+1}) + \gamma^{\mathrm{PI}}\alpha^{\mathrm{RH}}\xi_{n+1})}{(1-\gamma^{\mathrm{RH}}\xi_{n+1})(1-\beta^{\mathrm{PI}}\zeta_n) - \zeta_n\xi_{n+1}\beta^{\mathrm{RH}}\gamma^{\mathrm{PI}}} V(R_{m-1}, q_{n+1}).
\end{aligned}$$

Finally, the expression

$$\begin{aligned}
\frac{(\xi_{n+1}/\delta)\gamma^{\mathrm{PI}}\zeta_n}{(1-\gamma^{\mathrm{RH}}\xi_{n+1})(1-\beta^{\mathrm{PI}}\zeta_n) - \zeta_n\xi_{n+1}\beta^{\mathrm{RH}}\gamma^{\mathrm{PI}}} &= \frac{(\xi_{n+1}/\delta)\tilde{\gamma}_n^{\mathrm{PI}}\zeta_n}{1-\gamma^{\mathrm{RH}}\xi_{n+1} - \zeta_n\xi_{n+1}\beta^{\mathrm{RH}}\tilde{\gamma}_n^{\mathrm{PI}}} \\
\geq \frac{(\xi_n/\delta)}{1-\gamma^{\mathrm{RH}}\xi_n - \zeta_{n-1}\xi_n\beta^{\mathrm{RH}}\tilde{\gamma}_{n-1}^{\mathrm{PI}}} &= \frac{(\xi_n/\delta)(1-\beta^{\mathrm{PI}}\zeta_{n-1})}{(1-\gamma^{\mathrm{RH}}\xi_n)(1-\beta^{\mathrm{PI}}\zeta_{n-1}) - \zeta_{n-1}\xi_n\beta^{\mathrm{RH}}\gamma^{\mathrm{PI}}}
\end{aligned}$$

holds if and only if (A1) holds. Moreover,

$$\begin{aligned}
&\frac{\zeta_n(\alpha^{\mathrm{PI}}(1-\gamma^{\mathrm{RH}}\xi_{n+1}) + \gamma^{\mathrm{PI}}\alpha^{\mathrm{RH}}\xi_{n+1})}{(1-\gamma^{\mathrm{RH}}\xi_{n+1})(1-\beta^{\mathrm{PI}}\zeta_n) - \zeta_n\xi_{n+1}\beta^{\mathrm{RH}}\gamma^{\mathrm{PI}}} \\
&\quad = \frac{\left(\frac{\alpha^{\mathrm{PI}}}{1-\beta^{\mathrm{PI}}\zeta_n}(1-\gamma^{\mathrm{RH}}\xi_{n+1}) + \tilde{\gamma}_n^{\mathrm{PI}}\alpha^{\mathrm{RH}}\xi_{n+1}\right)}{\tilde{\gamma}_n^{\mathrm{PI}}\xi_{n+1}} \frac{\zeta_n\tilde{\gamma}_n^{\mathrm{PI}}\xi_{n+1}}{(1-\gamma^{\mathrm{RH}}\xi_{n+1}) - \zeta_n\xi_{n+1}\beta^{\mathrm{RH}}\tilde{\gamma}_n^{\mathrm{PI}}} \\
&\quad \geq \frac{\left(\frac{\alpha^{\mathrm{PI}}}{1-\beta^{\mathrm{PI}}\zeta_n}(1-\gamma^{\mathrm{RH}}\xi_{n+1}) + \tilde{\gamma}_n^{\mathrm{PI}}\alpha^{\mathrm{RH}}\xi_{n+1}\right)}{\tilde{\gamma}_n^{\mathrm{PI}}\xi_{n+1}} \frac{\xi_n}{1-\gamma^{\mathrm{RH}}\xi_n - \zeta_{n-1}\xi_n\beta^{\mathrm{RH}}\tilde{\gamma}_{n-1}^{\mathrm{PI}}} \\
&\quad = \left(\frac{(\alpha^{\mathrm{PI}}(1-\gamma^{\mathrm{RH}}\xi_{n+1}))}{\gamma^{\mathrm{PI}}\xi_{n+1}} + \alpha^{\mathrm{RH}} \right) \frac{\xi_n}{1-\gamma^{\mathrm{RH}}\xi_n - \zeta_{n-1}\xi_n\beta^{\mathrm{RH}}\tilde{\gamma}_{n-1}^{\mathrm{PI}}} \\
&\quad > \left(\alpha^{\mathrm{PI}} \frac{\beta^{\mathrm{RH}}\zeta_{n-1}}{1-\beta^{\mathrm{PI}}\zeta_{n-1}} + \alpha^{\mathrm{RH}} \right) \frac{\xi_n}{1-\gamma^{\mathrm{RH}}\xi_n - \zeta_{n-1}\xi_n\beta^{\mathrm{RH}}\tilde{\gamma}_{n-1}^{\mathrm{PI}}},
\end{aligned}$$

where the first inequality follows from (A1) and the second inequality holds because

$$\begin{aligned}
&(1-\gamma^{\mathrm{RH}}\xi_{n+1})(1-\beta^{\mathrm{PI}}\zeta_{n-1}) - \gamma^{\mathrm{PI}}\xi_{n+1}\beta^{\mathrm{RH}}\zeta_{n-1} \\
&\quad = (1-\xi_{n+1}(1-\alpha^{\mathrm{RH}}))(1-\beta^{\mathrm{PI}}\zeta_{n-1}) + \beta^{\mathrm{RH}}\xi_{n+1}(1-\beta^{\mathrm{PI}}\zeta_{n-1} - \gamma^{\mathrm{PI}}\zeta_{n-1}) \\
&\quad = (1-\xi_{n+1}(1-\alpha^{\mathrm{RH}}))(1-\beta^{\mathrm{PI}}\zeta_{n-1}) + \beta^{\mathrm{RH}}\xi_{n+1}(1-(1-\alpha^{\mathrm{PI}})\zeta_{n-1}) > 0.
\end{aligned}$$

Therefore, $V(R_m, q_n \mid \mathrm{PI}) > V(R_m, q_n \mid \mathrm{RH})$, a contradiction. Hence PI dominates RE in state (R_m, q_n). □

Proof of Proposition 2. The proof proceeds by induction. Let

$$\theta = \min\left\{ n \,|\, \xi_{n+1}\zeta_n\tilde{\gamma}_n^{\mathrm{PI}} - \xi_n + \xi_{n+1}\zeta_n\xi_n\beta^{\mathrm{RH}}\tilde{\gamma}_n^{\mathrm{PI}}(1-\zeta_{n-1}\tilde{\gamma}_{n-1}^{\mathrm{PI}}) + \xi_{n+1}\xi_n(1-\alpha^{\mathrm{RH}}-\beta^{\mathrm{RH}})(1-\zeta_n\tilde{\gamma}_n^{\mathrm{PI}}) < 0 \right\}, \tag{A2}$$

where $\xi_n \triangleq \delta\chi q_n/(1-\delta(1-\chi q_n))$ and $\tilde{\gamma}_n^{\mathrm{PI}} \triangleq (1-\alpha^{\mathrm{PI}}-\beta^{\mathrm{PI}})/(1-\zeta_n\beta^{\mathrm{PI}})$. Define $\bar{q} = q_\theta$. Then, applying Lemma A-2 yields that $V(R_M, q_n \mid \mathrm{PI}) \geq V(R_M, q_n \mid \mathrm{RH})$ for all $q_n < \bar{q}$. Consider revenue rate R_m, and suppose that $V(R_{m+1}, q_n) = V(R_{m+1}, q_n \mid \mathrm{PI})$ for all $q_n < \bar{q}$. Then, by Proposition 1, we have $V(R_m, q_n \mid \mathrm{PI}) \geq V(R_m, q_n \mid \mathrm{RE})$ for all $q_n < \bar{q}$. Furthermore, Lemma A-2 shows that $V(R_m, q_n \mid \mathrm{PI}) \geq V(R_m, q_n \mid \mathrm{RH})$ for all $q_n < \bar{q}$. As a result, $V(R_m, q_n) = V(R_m, q_n \mid \mathrm{PI})$ for all $q_n < \bar{q}$. □

Lemma A-3. *For any $r \in \mathbb{Z}^+$ there exists a $\bar{\delta} < 1$ such that, for all $\delta \in (\bar{\delta}, 1]$, the function $\zeta_n \zeta_{n+1}^r \frac{q_{n+1}}{q_n}$ is decreasing in n for any R_m.*

Proof. Requiring the function $\zeta_n \zeta_{n+1}^r \frac{q_{n+1}}{q_n}$ to be decreasing in n is equivalent to requiring that $\zeta_n \zeta_{n+1}^r \frac{q_{n+1}}{q_n} \geq \zeta_{n+1} \zeta_{n+2}^r \frac{q_{n+2}}{q_{n+1}}$ or, equivalently, that $\frac{q_{n+1}^2}{q_n q_{n+2}} \geq \left(\frac{\zeta_{n+2}}{\zeta_{n+1}}\right)^r \left(\frac{\zeta_{n+1}}{\zeta_n}\right)$; that is,

$$\frac{q_{n+1}^2}{q_n q_{n+2}} \geq \left(\frac{q_{n+2}(1-\delta(1-q_{n+1}))}{q_{n+1}(1-\delta(1-q_{n+2}))}\right)^r \left(\frac{q_{n+1}(1-\delta(1-q_n))}{q_n(1-\delta(1-q_{n+1}))}\right).$$

Here the left-hand side (LHS) is independent of δ whereas the right-hand side (RHS) is decreasing in δ (since $\{q_n\}$ is increasing). Hence the inequality is tight for at most one $\delta \in (0,1)$. When $\delta = 1$, the inequality is satisfied because the LHS is greater than the RHS—which is equal to 1 given that $\{q_n\}$ is log-concave in q_n. As a result, there exists a $\bar{\delta} \in [0,1)$ such that the inequality is satisfied for all $\delta \in (\bar{\delta}, 1]$. $\square$

Lemma A-4. *There exists a $\bar{\delta} \geq 0$ such that, for all $\delta \in (\bar{\delta}, 1]$, the function*

$$\frac{\zeta_n (\zeta_{n+1})^{i^*(m,n+1)} q_{n+1} (R_{m+i^*(m,n+1)} + b)}{(\zeta_n)^{i^*(m,n)} q_n (R_{m+i^*(m,n)} + b)}$$

is decreasing in n, where

$$i^*(m,n) \equiv \max\left\{ i \geq 0 : \zeta_n \frac{R_{m+i} + b}{R_{m+i-1} + b} \geq 1 \right\}.$$

Proof. The proof is based on Lemmas A-1 and A-3. By definition of $i^*(m, n+1)$, we have

$$(\zeta_{n+1})^{i^*(m,n+1)-i^*(m,n)} \frac{R_{m+i^*(m,n+1)} + b}{R_{m+i^*(m,n)} + b} = \prod_{j=i^*(m,n)+1}^{i^*(m,n+1)} \zeta_{n+1} \frac{R_{m+j} + b}{R_{m+j-1} + b} \geq 1;$$

$$(\zeta_{n+1})^{i^*(m,n+2)-i^*(m,n+1)} \frac{R_{m+i^*(m,n+2)} + b}{R_{m+i^*(m,n+1)} + b} = \prod_{j=i^*(m,n+1)+1}^{i^*(m,n+2)} \zeta_{n+1} \frac{R_{m+j} + b}{R_{m+j-1} + b} < 1.$$

Applying these equalities sequentially and then using (a) Lemma A-3 (while assuming that $\delta \geq \bar{\delta}$) and (b) the fact that $\{\zeta_{n+1}/\zeta_n\}$ is decreasing (by Lemma A-1), we obtain

$$\begin{aligned}
&\frac{(\zeta_{n+1})^{i^*(m,n+1)} \zeta_n}{(\zeta_n)^{i^*(m,n)}} \frac{q_{n+1}}{q_n} \frac{R_{m+i^*(m,n+1)} + b}{R_{m+i^*(m,n)} + b} \geq \left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)} \zeta_n \frac{q_{n+1}}{q_n} \\
&\geq \left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)} \frac{q_{n+1}}{q_n} (\zeta_{n+1})^{i^*(m,n+2)-i^*(m,n+1)} \zeta_n \frac{R_{m+i^*(m,n+2)} + b}{R_{m+i^*(m,n+1)} + b} \\
&= \left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)} \frac{q_{n+1}}{q_n} (\zeta_{n+1})^{i^*(m,n+2)-i^*(m,n)} (\zeta_{n+1})^{i^*(m,n)-i^*(m,n+1)} \zeta_n \frac{R_{m+i^*(m,n+2)} + b}{R_{m+i^*(m,n+1)} + b} \\
&\geq \left(\frac{\zeta_{n+1}}{\zeta_n}\right)^{i^*(m,n)} \frac{q_{n+2}}{q_{n+1}} (\zeta_{n+2})^{i^*(m,n+2)-i^*(m,n)} (\zeta_{n+1})^{i^*(m,n)-i^*(m,n+1)} \zeta_{n+1} \frac{R_{m+i^*(m,n+2)} + b}{R_{m+i^*(m,n+1)} + b} \\
&\geq \left(\frac{\zeta_{n+2}}{\zeta_{n+1}}\right)^{i^*(m,n)} \frac{q_{n+2}}{q_{n+1}} (\zeta_{n+2})^{i^*(m,n+2)-i^*(m,n)} \zeta_{n+1} (\zeta_{n+1})^{i^*(m,n)-i^*(m,n+1)} \frac{R_{m+i^*(m,n+2)} + b}{R_{m+i^*(m,n+1)} + b} \\
&= \zeta_{n+1} \frac{(\zeta_{n+2})^{i^*(m,n+2)}}{(\zeta_{n+1})^{i^*(m,n+1)}} \frac{q_{n+2}}{q_{n+1}} \frac{R_{m+i^*(m,n+2)} + b}{R_{m+i^*(m,n+1)} + b}. \quad \square
\end{aligned}$$

Proof of Proposition 3. Under Assumptions 1–3, the cash state can be dropped from (1), and therefore the Bellman's equation simplifies to:

$$V(R_m,q_n)=\max\Bigg\{\underbrace{\frac{\chi(b+R_m)}{1-\zeta_n}}_{\text{RH}},\ \underbrace{\zeta_n V(R_m,q_{n+1})}_{\text{PI}},\ \underbrace{\zeta_n V(R_{m+1},q_n)}_{\text{RE}}\Bigg\}. \tag{A3}$$

Under Assumptions 1–3, the states for which RH is optimal are absorbing. Also, RE $\to$ PI is suboptimal by Proposition 1. Hence the only possible optimal path consists of one improvement cycle beginning with sequences of PI (if any) and followed by sequences of RE (if any) and then of RH. The optimal policy can be characterized by two thresholds: the improve-up-to level (stop PI and do RE) and the enhance-up-to level (stop RE and do RH). Starting from state (R_m,q_n), a policy that consists of j periods of PI followed by i periods of RE, with RH occurring thereafter, generates value equal to

$$(\zeta_n\zeta_{n+1}\cdots\zeta_{n+j-1})\cdot\zeta_{n+j}^{i}\frac{\chi(b+R_{m+i})}{1-\zeta_{n+j}}.$$

Thus the maximum value starting from state (R_m,q_n) can be written as

$$V(R_m,q_n)=\max_{j=0,\dots,N-n}\left(\prod_{j'=1}^{j}\zeta_{n+j'-1}\cdot\max_{i=0,\dots,M-m}\left\{\zeta_{n+j}^{i}\frac{\chi(b+R_{m+i})}{1-\zeta_{n+j}}\right\}\right). \tag{A4}$$

Note that, for all j,

$$\begin{aligned}\arg\max_{i\ge 0}\left\{\zeta_{n+j}^{i}\frac{\chi(b+R_{m+i})}{1-\zeta_{n+j}}\right\}&=\arg\max_{i\ge 0}\left\{\prod_{i'=0}^{i}\zeta_{n+j}\frac{b+R_{m+i'}}{b+R_{m+i'-1}}\frac{\chi(b+R_{m-1})}{1-\zeta_{n+j}}\right\}\\&=\max\left\{i\ge 0:\zeta_{n+j}\frac{b+R_{m+i}}{b+R_{m+i-1}}\ge 1\right\};\end{aligned}$$

where the first equality follows from the telescoping product and the second equality is by the log-concavity of $\{b+R_m\}_{m=0}^{M}$. Thus we have derived an expression for the threshold $R^*(q_n)$.

We now derive an expression for the threshold $q^*(R_m)$. Let $i^*(m,n+j)$ denote the optimal number of RE periods in state $(m,n+j)$. Then (A4) can be equivalently expressed as $V(R_m,q_n)=\max_{j=0,\dots,N-n}\prod_{j'=1}^{j}\zeta_{n+j'}\zeta_{n+j}^{i^*(m,n+j)}\frac{\chi(b+R_{m+i^*(m,n+j)})}{1-\zeta_{n+j}}$. Note that

$$\begin{aligned}&\arg\max_{j\ge 0}\left\{\prod_{j'=1}^{j}\zeta_{n+j'-1}\zeta_{n+j}^{i^*(m,n+j)}\frac{\chi(b+R_{m+i^*(m,n+j)})}{1-\zeta_{n+j}}\right\}\\&=\arg\max_{j\ge 0}\left\{\prod_{k=1}^{j}\left(\zeta_{n+k-1}\frac{\zeta_{n+k}^{i^*(m,n+k)}}{\zeta_{n+k-1}^{i^*(m,n+k-1)}}\cdot\frac{b+R_{m+i^*(m,n+k)}}{b+R_{m+i^*(m,n+k-1)}}\cdot\frac{1-\zeta_{n+k-1}}{1-\zeta_{n+k}}\right)\left(\zeta_n^{i^*(m,n)}\frac{\chi(b+R_{m+i^*(m,n)})}{1-\zeta_n}\right)\right\}\\&=\arg\max_{j\ge 0}\left\{\prod_{k=1}^{j}\left(\zeta_{n+k-1}\frac{1-\zeta_{n+k-1}}{1-\zeta_{n+k}}\frac{\zeta_{n+k}^{i^*(m,n+k)}}{\zeta_{n+k-1}^{i^*(m,n+k-1)}}\frac{b+R_{m+i^*(m,n+k)}}{b+R_{m+i^*(m,n+k-1)}}\right)\right\}\\&=\max\left\{k\ge 0:\zeta_{n+k-1}\frac{1-\zeta_{n+k-1}}{1-\zeta_{n+k}}\frac{\zeta_{n+k}^{i^*(m,n+k)}}{\zeta_{n+k-1}^{i^*(m,n+k-1)}}\frac{b+R_{m+i^*(m,n+k)}}{b+R_{m+i^*(m,n+k-1)}}\ge 1\right\};\end{aligned}$$

where the first equality is due to the telescoping product, the second equality results from simplification, and the third equality follows because the preceding line's expression (inside large parentheses) is decreasing in n when δ is large enough by Lemma A-4 and the log-concavity of $\{q_n\}$. □

Proof of Proposition 4. We first prove part (ii). According to (A3), the decision to switch from RE to RH is a stopping action because the entrepreneur will remain in state (R_m, q_n) and continue engaging in RH forever. I.e., while undertaking RE, the entrepreneur faces an optimal stopping problem (Bertsekas 2000). Doing RE one more step and then doing RH is preferable to doing RH now if and only if

$$\zeta_n V(R_{m+1}, q_n \mid RH) = \zeta_n \frac{\delta}{1-\delta} \chi q_n (R_{m+1} + b) \geq \frac{\delta}{1-\delta} \chi q_n (R_m + b) \iff \zeta_n \frac{R_{m+1} + b}{R_m + b} \geq 1. \tag{A5}$$

When $\{R_m + b\}$ is log-concave, the stopping set is absorbing and so the one-step–look-ahead policy is optimal (Bertsekas 2000, p. 176). Since the stopping set defined by (A5) is decreasing in q_n, it follows that $R^*(q)$ is nondecreasing.

We now prove part (i) of the proposition. Suppose that $V(R_{m+1}, q_n) = V(R_{m+1}, q_n \mid \mathrm{PI})$. Then, by Proposition 1, $V(R_m, q_n \mid \mathrm{PI}) \geq V(R_m, q_n \mid \mathrm{RE})$. The rest of the proof amounts to showing that $V(R_m, q_n \mid \mathrm{PI}) \geq V(R_m, q_n \mid \mathrm{RH})$. Define $\theta \geq 1$ such that $q_{n+\theta} \equiv q^*(R_{m+1}, q_n)$. Then, by (A3),

$$V(R_{m+1}, q_n) = V(R_{m+1}, q_n \mid \mathrm{PI}) = \zeta_n \cdots \zeta_{n+\theta-1} V(R_{m+1}, q_{n+\theta}).$$

It follows from our definition of θ that $V(R_{m+1}, q_{n+\theta}) > V(R_{m+1}, q_{n+\theta} \mid \mathrm{PI})$. We must therefore consider two cases: either when RH or when RE is optimal in state $(R_{m+1}, q_{n+\theta})$. Assume first that $V(R_{m+1}, q_{n+\theta}) = V(R_{m+1}, q_{n+\theta} \mid \mathrm{RH})$. Since $V(R_{m+1}, q_n \mid \mathrm{PI}) > V(R_{m+1}, q_n \mid \mathrm{RH})$, we have

$$\zeta_n \cdots \zeta_{n+\theta-1} \frac{\delta}{1-\delta} \chi q_{n+\theta} (R_{m+1} + b) > \frac{\delta}{1-\delta} \chi q_n (R_{m+1} + b), \text{ and therefore}$$

$$\begin{aligned} V(R_m, q_n \mid \mathrm{PI}) &= \zeta_n V(R_m, q_{n+1}) \geq \zeta_n \cdots \zeta_{n+\theta-1} V(R_m, q_{n+\theta}) \geq \zeta_n \cdots \zeta_{n+\theta-1} \frac{\delta}{1-\delta} \chi q_{n+\theta} (R_m + b) \\ &> \frac{\delta}{1-\delta} \chi q_n (R_m + b) = V(R_m, q_n \mid \mathrm{RH}). \end{aligned}$$

Now if $V(R_{m+1}, q_{n+\theta}) = V(R_{m+1}, q_{n+\theta} \mid \mathrm{RE})$, then by (A5) we have $\zeta_{n+\theta} \frac{R_{m+2}+b}{R_{m+1}+b} > 1$. Thus,

$$\begin{aligned} V(R_m, q_n \mid \mathrm{PI}) &= \zeta_n V(R_m, q_{n+1}) \geq \zeta_n \cdots \zeta_{n+\theta-1} \zeta_{n+\theta} V(R_{m+1}, q_{n+\theta}) \\ &= \zeta_{n+\theta} V(R_{m+1}, q_n) \geq \zeta_{n+\theta} V(R_{m+1}, q_n \mid \mathrm{RH}) = \zeta_{n+\theta} \frac{\delta}{1-\delta} \chi q_n (R_{m+1} + b). \end{aligned}$$

Because $\{R_m + b\}$ is log-concave, $\frac{R_{m+1}+b}{R_m+b} \geq \frac{R_{m+2}+b}{R_{m+1}+b}$. Hence $\zeta_{n+\theta} \frac{R_{m+1}+b}{R_m+b} > 1$; therefore, $V(R_m, q_n \mid \mathrm{PI}) > \zeta_{n+\theta} \frac{\delta}{1-\delta} \chi q_n (R_m + b) = V(R_m, q_n \mid \mathrm{RH})$.

In sum, we have found that $V(R_m, q_n \mid \mathrm{PI}) > \max\{V(R_m, q_n \mid \mathrm{RH}), V(R_m, q_n \mid \mathrm{RE})\}$; that is, $V(R_m, q_n) = V(R_m, q_n \mid \mathrm{PI})$ when $V(R_{m+1}, q_n) = V(R_{m+1}, q_n \mid \mathrm{PI})$. So if $q^*(R_{m+1}, q_n) > q_n$, then $q^*(R_m, q_n) > q_n$. □

Proof of Corollary 2. This result follows directly from Proposition 4. □