Coframes, spinors and torsion

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I, James Burnett confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

This thesis is based on five articles, four of which have been published in the Journal of Mathematical Physics, Physical Review D, Modern Physics Letters A and Journal of High Energy Physics. The fifth has been submitted to Mathematika. In these works we study several distinct problems within the broad subject area of Mathematical Physics. The common feature is that all these works deal with rotations of one form or another. In particular, we show an equivalence between the massless and massive Dirac equations and models based on the concept of rotating material points. We also solve an open problem in Einstein-Cartan theory, namely, we find a natural matter source for a non-trivial spin angular momentum tensor. Finally, we construct a complete class of non-standard (non-local) spinor field theories and examine their possible applications in Cosmology.

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Chapter 1

Introduction

1.1 Topics and themes

This thesis is based on five articles, four of which have been published in the Journal of Mathematical Physics, Physical Review D, Modern Physics Letters A and Journal of High Energy Physics. The fifth has been submitted to Mathematika. The reader may notice discrepancies in notation between the thesis and the articles. This is due to the nature of working on two main but distinct topics during the course of the PhD. Therefore, making such notational changes was unavoidable in preparing a coherent document.

The first half of the thesis details extensive work investigating an interesting link between spinors and rotating material points. The latter half deals with spinors in cosmology, specifically, with what are called non-standard spinors.

1.1.1 *Coframe*

We suggest a new geometric interpretation of both the Dirac and Weyl (massless Dirac) equations. The basic idea is to view space-time as an elastic continuum whose material points can experience no displacements, only rotations, with rotations of different material points being totally independent. The idea of rotating material points may seem exotic, however it has long been accepted in continuum mechanics within the Cosserat theory of elasticity [43]. This idea also lies at the heart of the theory of *teleparallelism* (= absolute parallelism = fernparallelismus), a subject promoted by A. Einstein and É. Cartan [37, 106, 117]. With regards to the latter it is interesting that Cartan acknowledged [35] that he drew inspiration from the 'beautiful' work of the Cosserat brothers.

An elastic continuum with no displacements, only rotations, is, of course, a limit case of Cosserat elasticity. The other limit case is classical elasticity with displacements only and no (micro)rotations.

Rotations of material points of the elastic continuum are described mathematically by attaching to each geometric point an orthonormal basis, which gives a field of orthonormal bases called the *frame* or *coframe,* depending on whether one prefers dealing with vectors or covectors. Our model will be built on the basis of exterior calculus so for us it will be more natural to use the coframe.

1.1.2 *Spinors and torsion*

General relativity is a successful theory in agreement with a vast number of observations. It is based on the Einstein-Hilbert action which yields the field equations if varied with respect to the metric. If, however, the metric and the connection (more precisely the non-Riemannian part of the connection with the connection assumed to be metric compatible) are considered as *a priori* independent variables, two field equations are obtained. The first one relates the Einstein tensor (not necessarily symmetric) to the canonical energy-momentum tensor, while the other field equation relates the skew-symmetric part of the connection, the torsion tensor, to the spin angular momentum of matter, see e.g. [67, 68, 69, 70, 66, 115]. Spin and torsion are related by algebraic equations, and torsion vanishes in the absence of sources.

The cosmological principle states that the universe is homogeneous and isotropic on very large scales. More mathematically speaking, the four dimensional spacetime (M, g) is defined by $3D$ space-like hypersurfaces of constant time which are orbits of a Lie group G action on M, with isometry group $SO(3)$. We assume all fields to be invariant under the action of G which means $\mathcal{L}_\xi g_{\mu\nu} = 0$ and $\mathcal{L}_\xi T_{\mu\nu}{}^\lambda = 0$ where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the generator of the group. This assumption reduces the cosmological metric to the well known Friedmann-Lemaître-Robertson-Walker form which is characterized by the scale factor and the geometry of the constant time hypersurfaces. If applied to the torsion of spacetime, it reduces the components compatible with the cosmological principle to a spatial axial torsion and a vector torsion part [116].

Cosmological models with torsion were pioneered by Kopczyński in [81, 82], who assumed a Weyssenhoff fluid [124] to be the source of both curvature and torsion. The cosmological principle was first extended to Einstein-Cartan theory in [116], where it was also suggested to reconsider the results in [81, 82], since the Weyssenhoff fluid turns out to be incompatible with the cosmological principle (see also [94, 14, 28]). An elaborate analysis of the most general action up to quadratic terms in curvature and torsion assuming the cosmological principle can be found in [59]. Analytical solutions of the Riemann-squared gravity have recently been discussed in a cosmological context in [83]. Non-Riemannian models of cosmology in general have been discussed in [101, 100, 102, 103].

However, nobody has so far succeeded in constructing a non-trivial spin angular momentum tensor in cosmology by minimally coupling matter fields to the geometry. We show that the minimally coupled eigenspinors of the charge conjugation operator [4, 3] yield a spin tensor compatible with the cosmological principle.

These spinors belong to a wider class of so-called flagpole spinors [44]. They are non-standard spinors according to the Wigner classification and obey the unusual property $(CPT)^2 = -\mathbb{1}$. Hence, their dominant coupling to other fields is via the Higgs mechanism or via gravity [4, 3]. The particles associated with such a field theory are naturally dark and are named Elko spinors.

1.1.3 *New class of spinors*

In recent years, our understanding of the universe has become greatly improved thanks to the high precision cosmological observations that we have available today. According to the Standard Model of Cosmology, which assumes General Relativity as the theory describing the gravitational interaction, our universe is composed by about 4% of baryons, 23% of dark matter and 73% of dark energy. Moreover, in addition to these components, we need to assume an early inflationary epoch in order to explain the current state of our universe. Although this budget enables us to successfully account for the current cosmological data, it needs to assume the existence of three unknown components from a particle physics point of view, namely dark matter, dark energy and inflation. Thus, we find that predictions based on General Relativity plus the Standard Model of particle physics are at odds with current astronomical observations, not only on cosmological scales, but also on galactic scales where dark matter plays a crucial role. This indicates failures either in particle physics or in general relativity (or both) and, in particular, it might be indicating the existence of new particles/fields as candidates for dark matter, dark energy and inflation which could arise in high energy physics [92, 5, 26, 12, 85, 34, 42, 80, 79, 88, 105, 84, 8, 122].

Spinors have played an important role in mathematics and physics throughout the last 80 years. They theoretically model particles with half integer spin, like the electron in the massive case or the neutrino (massive or massless). The spin structure of manifolds has played an important part in modern mathematics, while in mathematical physics this structure motivated the twistor program.

In the framework of particle physics all spinors used are either Dirac, Weyl (massless Dirac spinors) or Majorana spinors, ψ. Such spinors obey a field equation which is first order in the derivatives (momenta) of ψ. Cosmologically, this first order field equation implies that the average value of both $\Phi = \bar{\psi}\psi$ and the spinor energy density of a free spinor field evolves like the energy density of pressure-less dust i.e. proportional to $(1+z)^3$, where z is the redshift. Additionally, the first order nature of the field equation results in a quantum propagator, G_F, which, for large momenta p, behaves as $G_F \propto p^{-1}$. This limits the form of perturbatively renormalizable spinor self-interaction terms in the action to be no more than quadratic in ψ e.g. $\bar{\psi}\psi$ and $\bar{\psi}\gamma_\mu A^\mu\psi$. The momentum drop-off of G_F also results in ψ having a canonical mass dimension of $3/2$.

A wider range of renormalizable self-interaction terms and cosmological behavior would be allowed if one could construct a viable spinor field theory where $G_F \propto p^{-2}$, for large p, resulting in a ψ with a canonical mass dimension of unity. We refer to this entire class of spinor field theories with such properties as Non-Standard Spinors (NSS). This class of spinors is closely related to Wigner's non-standard classes [125]. Weinberg showed that, under the assumptions of Lorentz invariance (rotations and boosts) and locality (events affecting other events within their light-cones), the only spin-$1/2$ quantum field theory is that which describes standard spinors (Dirac, Weyl, Majorana). NSS will therefore violate either locality or Lorentz invariance, or possibly both. Our working assumption is that reasonable NSS models preserve Lorentz invariance, while being non-local.

Along these lines of reasoning, Ahluwalia-Khalilova and Grumiller [4, 3] constructed a NSS model using momentum space eigenspinors of the charge conjugation operator *Eigenspinoren des LadungsKonjugationsOperators* (Elko) to build a quantum field. They showed that such spinors belong to a non-standard Wigner class and exhibit non-locality [125]. They satisfy $(CPT)^2 = -\mathbb{I}$ while Dirac spinors satisfy $(CPT)^2 = \mathbb{I}$. In more mathematical terms, they belong to a wider class of spinor fields, so-called flagpole spinor fields [44]. The spinors correspond to the class 5, according to Lounesto's classification which is based on bilinear covariants, similar to Majorana spinors, see also [46, 45, 74]. Locality issues and Lorentz invariance were further investigated in [2, 1] with results along the lines of the current work. Causality has been analysed in [52, 53].

The construction of Elkos using momentum space eigenspinors, $\lambda(\mathbf{p}, h, e)$, of the charge conjugation operator leads to a spinor field with a double helicity structure. The left-handed and the right-handed spinor have opposite helicities which in turn requires a careful construction of the resulting field theory. These spinors have received quite some attention recently [22, 23, 52] and their effects in cosmology have been investigated [25, 15, 24, 61, 16, 44, 46, 45, 74, 107, 108, 17, 123].

However, as we will show in §6, Elkos spinors, defined in the way described above, are not

Lorentz invariant. We demonstrate using our construction of NSS where this Lorentz violation appears, thus confirming [2, 1]. The original analyses defined the field structure entirely in terms of momentum space basis spinors rather than, for example, starting with an action whose minimization would imply that structure. This led to the violation of Lorentz invariance being hidden in the mathematical structure of the model. In the present work, on the other hand, we start with a general action principle for NSS. When applied to the Elkos, an alternative model also based on eigenspinors of the charge conjugation operator, the violation of Lorentz invariance and other issues with their construction are explicit at the level of the action. The original Elko definition is seen to require a preferred space-like direction and is ill-defined when the momentum points along that direction. We offer a new NSS field theory which is also based on the eigenspinors of the charge conjugation operator (i.e. using the basis $\lambda(\mathbf{p}, h, e)$) which respects the rotational group $SO(3)$ but is not invariant under boosts.

We shall see that the general construction of NSS models can be seen as the choice of some operator P satisfying $P^2 = \mathbb{I}$ which acts on ψ to project out those states that would otherwise give an inconsistent Hamiltonian density. In this thesis we provide a general treatment of a class of NSS models based on an action principle and choice of operator P. We show that there is one, potentially unique, choice of P which results in a Lorentz invariant, ghost-free but non-local spinor field theory with canonical mass dimension one.

We are also interested in the cosmological behavior of general NSS models and construct the energy-momentum tensor, $T_{\mu\nu}$. For Elko spinors it appears that, at present, no one has obtained the full $T_{\mu\nu}$ as all previous works in the literature, including ours, have overlooked contributions to $T_{\mu\nu}$ from the variation of spin connection.

The remainder of this chapter provides the notation and conventions used throughout the thesis. In particular, it describes the spin connection, Pauli matrices, covariant derivative, the coframe, torsion and spinors.

Due to the work of this thesis having developed in two main parts, it is unavoidable to have separate sections for notation pertaining to Chapters 3 & 4 and Chapters 5 & 6

Chapter 2

Notation

We work, unless otherwise stated, on a 4-manifold M equipped with a Lorentzian metric g. The construction presented is local so we do not make *a priori* assumptions on the geometric structure of spacetime $\{M, g\}$. The metric g is not necessarily the Minkowski metric. Furthermore, we use the following signature $\{+, -, -, -\}$. We use local coordinates $\{x^\mu\}$ where $\mu = 0, 1, 2, 3$. We also denote $\partial_\mu = \partial/\partial x^\mu$ and assume Einstein's summation convention for repeated indices. We define the covariant derivative of a vector field V^ν as

$$\nabla_\mu V^\nu = \partial_\mu + \Gamma^\nu{}_{\mu\kappa} V^\kappa \tag{2.1}$$

where $\Gamma^\nu{}_{\mu\kappa}$ are the connection coefficients. In the case of General Relativity (curvature only, no torsion) we call our connection the Levi-Civita connection and write our connection coefficients as $\{\Gamma\}^\nu{}_{\mu\kappa}$. In the case of no curvature but non-zero torsion (Teleparallelism) we write coefficients as $|\Gamma|^\nu{}_{\mu\kappa}$. The explicit formula for the Levi-Civita connection can be derived from the metric compatibility condition

$$\nabla_\mu g_{\nu\kappa} := 0 \tag{2.2}$$

together with the condition that torsion is zero, giving

$$\{\Gamma\}^\gamma{}_{\mu\nu} = \frac{1}{2} g^{\gamma\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}). \tag{2.3}$$

Curvature is measured by the Riemann curvature tensor which is defined as

$$R_{\mu\nu\rho}{}^\sigma := 2\partial_{[\mu}\Gamma^\sigma{}_{\nu]\rho} + 2\Gamma^\sigma{}_{[\mu|\lambda}\Gamma^\lambda{}_{\rho|\nu]}, \tag{2.4}$$

where $\Gamma^{\sigma}{}_{\nu\rho}$ is the general connection and therefore can contain curvature and torsion. Two other important quantities are the Ricci tensor

$$R_{\nu\rho} := R_{\mu\nu\rho}{}^{\mu} \tag{2.5}$$

and the Ricci scalar $\mathcal{R} = R_{\nu}{}^{\nu}$.

As mentioned above, we can also encode torsion into this picture. It does not feature in the metric but appears at the level of the connection. In particular, we can write a general metric compatible ($\nabla g = 0$) connection as

$$\Gamma^{\gamma}{}_{\mu\nu} = \{\Gamma\}^{\gamma}{}_{\mu\nu} - K_{\mu\nu}{}^{\gamma} \tag{2.6}$$

where K is a tensor called contortion; it possess the anti-symmetry property $K_{\alpha\beta\gamma} = -K_{\alpha\gamma\beta}$. Torsion is defined as the anti-symmetric part of the connection,

$$T^{\gamma}{}_{\mu\nu} = (\Gamma^{\gamma}{}_{\mu\nu} - \Gamma^{\gamma}{}_{\nu\mu}) = (K_{\nu\mu}{}^{\gamma} - K_{\mu\nu}{}^{\gamma}). \tag{2.7}$$

Torsion (contortion) and the metric are independent of each other providing our universe with more degrees of freedom. The interval on our space-time is defined as

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{2.8}$$

and it does not depend on torsion (contortion).

Throughout this thesis we use Greek letters $\{\alpha, \beta, \dots\}$ for holonomic indices and Latin letters $\{j, k, \dots\}$ for anholonomic indices.

We will use Λ to represent Lorentz transformations and, in the latter part of the thesis, the cosmological constant. It will be obvious from the context as to which use of Λ is being implemented.

2.1 Coframes

Within this thesis we will use two distinct coframes, $\vartheta^{j}{}_{\alpha}$ and $e^{j}{}_{\alpha}$. They satisfy the same condition

$$\vartheta^{j}{}_{\alpha} \vartheta^{k}{}_{\beta} \eta_{jk} = g_{\alpha\beta} \tag{2.9}$$

and

$$e^j{}_\alpha e^k{}_\beta \eta_{jk} = g_{\alpha\beta} \tag{2.10}$$

where $g_{\alpha\beta}$ is the space-time metric and $\eta_{jk} = \mathrm{diag}(+1,-1,-1,-1)$ (the Minkowski metric). With both coframes we can obtain the frame versions

$$\vartheta_k{}^\beta = \eta_{jk}\vartheta^j{}_\alpha g^{\alpha\beta} \tag{2.11}$$

and

$$e_k{}^\beta = \eta_{jk} e^j{}_\alpha g^{\alpha\beta}. \tag{2.12}$$

It is important to note that the anholonomic index always comes first and the Lorentz index second. Sometimes we will suppress the Lorentz index. We will do this only when it is obvious and doesn't add any confusion.

The reader may wonder why we would introduce two coframes (frames) that, at least according to the above definitions, are the same object.

2.1.1 $e^j{}_\alpha$

The usual argument for introducing a coframe is to include spinors in curved space. Spinors require by definition to be defined clearly with respect to the Lorentz symmetry of a given space-time. Since in general a manifold in General Relativity does not necessarily respect Lorentz symmetry globally, it is necessary to introduce a local structure that defines spinor states according to Lorentz symmetry of locally flat spaces. In other words, $e^j{}_\alpha$ is a reference coframe, and all formulae are invariant under changes of this reference coframe.

2.1.2 $\vartheta^j{}_\alpha$

This second coframe is the the main feature of our alternative model in Chapters 3 & 4. Our formulae will not be invariant under changes of $\vartheta^j{}_\alpha$, so $\vartheta^j{}_\alpha$ is a true dynamical variable.

In the next section we will introduce our spinor notation and then return to the topic of including the coframes defined above.

2.2 Spinors

Spinors can be difficult to understand and often in the physics literature they are introduced without a rigorous definition. The simplest definition of a spinor is Cartan's for a spinor in 3 dimensions. Simply put, a spinor is the square root of a complex isotropic ($V_\alpha V^\alpha = 0$) vector. We can see immediately one very important feature of the spinor. We define our vector to have the components $V^\mu = (V^1, V^2, V^3)$ and being isotropic means we take the following condition into consideration

$$(V^1)^2 + (V^2)^2 + (V^3)^2 = 0. \tag{2.13}$$

Then we can define two numbers ξ_1, ξ_2 in accordance with

$$\begin{aligned} V^1 &= \xi_1^2 - \xi_2^2, \\ V^2 &= i(\xi_1^2 + \xi_2^2), \\ V^3 &= -2\xi_1\xi_2. \end{aligned} \tag{2.14}$$

These give the solutions

$$\xi_1 = \pm\sqrt{\frac{V^1 - iV_2}{2}} \text{ and } \xi_2 = \pm\sqrt{\frac{-V_1 - iV_2}{2}}. \tag{2.15}$$

If we were multiply the vector by $e^{-i\alpha}$ then according to (2.15) ξ_1 and ξ_2 will be multiplied by $e^{-i\alpha/2}$. Therefore a rotation through 2π leaves the vector unchanged but the two numbers ξ_1 and ξ_2 change sign. This pair of quantities constitutes a *spinor*. A spinor, according to Cartan [36], can be thought of as a directed or polarised isotropic vector.

Throughout this thesis we will not be discussing spinors in much detail but will be using them for various mathematical constructions or to represent something physical. We have therefore decided *not* to derive their form explicitly (which could be the topic of a book) and just introduce the properties that we need. Furthermore since there is a clear divide in the topic of this thesis, we will separate our notation section into two parts: Section 2.3 pertains to Chapters 3 & 4 and Section 2.4 to Chapters 5 & 6.

2.3 Notation for Chapters 3 & 4

Spinors, unlike "proper" vectors introduced above, do not carry Lorentz indices. Instead, they have spinor indices. We will reserve the beginning of the Latin alphabet for these $\{a, b, \dots\}$. In Chapter 3 we will be in (1+3) dimensions working with Weyl spinors (= definite helicity). They have two complex components, i.e. four real degrees of freedom. In Chapter 4 we will be in (1+2) dimensions working with a Dirac spinor. Due to the reduced dimensionality the Dirac spinor also has two complex components, i.e. four real degrees of freedom in (1+2) dimensions.

For example, when dealing with the Weyl equation (massless Dirac equation), we will be working with a Weyl spinor field ξ^b where $b = 1, 2$. The Weyl equation itself is

$$i\sigma^{\alpha}{}_{\dot{a}b}\{\nabla\}_{\alpha}\xi^{b} = 0, \tag{2.16}$$

where the Pauli matrices σ are defined below and $\{\nabla\}$ denotes the covariant derivative with respect to the Levi-Civita connection.

In Minkowski space, i.e. flat space-time with $g = \eta = \mathrm{diag}(+1, -1, -1, -1)$, Pauli matrices are defined as

$$\sigma^{\alpha}{}_{\dot{a}b} = s^{j}{}_{\dot{a}b} = \begin{pmatrix} \sigma^{0}{}_{\dot{a}b} \\ \sigma^{1}{}_{\dot{a}b} \\ \sigma^{2}{}_{\dot{a}b} \\ \sigma^{3}{}_{\dot{a}b} \end{pmatrix} := \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}. \tag{2.17}$$

For an arbitrary Lorentzian metric $g \neq \eta = \mathrm{diag}(+1, -1, -1, -1)$ Pauli matrices $s^{j}{}_{\dot{a}b}$ (note the Latin upper index!) are defined as above (see formula (2.17)) whereas Pauli matrices $\sigma^{\alpha}{}_{\dot{a}b}$ (note the Greek upper index!) are defined as Hermitian matrices $\sigma^{\alpha}{}_{\dot{a}b}$ satisfying the relation $\sigma^{\alpha}{}_{\dot{b}a}\sigma^{\beta\dot{b}c} + \sigma^{\beta}{}_{\dot{b}a}\sigma^{\alpha\dot{b}c} = 2g^{\alpha\beta}\delta_{a}{}^{c}$ where spinor indices are raised and lowered using the "metric spinor" ϵ_{ab}

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.18}$$

in accordance with the formula $\sigma_\beta{}^{\dot{a}c} := \epsilon^{\dot{a}\dot{e}}\sigma_{\beta\dot{e}d}\epsilon^{cd}$.

Of course, the matrices $\sigma^\alpha{}_{\dot{a}b}$ are expressed via $s^j{}_{\dot{a}b}$ in accordance with the formula

$$\sigma^\alpha{}_{\dot{a}b} = e_j{}^\alpha s^j{}_{\dot{a}b} \tag{2.19}$$

where $e_j{}^\alpha$ is some reference frame. This is the frame introduced in subsection 2.1.1.

We define the covariant derivatives of spinor fields as

$$\nabla_\mu \xi^a = \partial_\mu \xi^a + \Gamma^a{}_{\mu b}\xi^b, \qquad \nabla_\mu \xi_a = \partial_\mu \xi_a - \Gamma^b{}_{\mu a}\xi_b, \tag{2.20}$$

$$\nabla_\mu \eta^{\dot{a}} = \partial_\mu \eta^{\dot{a}} + \bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}}\eta^{\dot{b}}, \qquad \nabla_\mu \eta_{\dot{a}} = \partial_\mu \eta_{\dot{a}} - \bar{\Gamma}^{\dot{b}}{}_{\mu\dot{a}}\eta_{\dot{b}}, \tag{2.21}$$

where $\bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}} = \overline{\Gamma^a{}_{\mu b}}$. The explicit formula for the spinor connection coefficients $\Gamma^a{}_{\mu b}$ can be derived from the following two conditions:

$$\nabla_\mu \epsilon_{ab} = 0, \tag{2.22}$$

$$\nabla_\mu \sigma^\alpha{}_{\dot{a}b} = 0, \tag{2.23}$$

where $\nabla_\mu \sigma^\alpha{}_{\dot{a}b} = \partial_\mu \sigma^\alpha{}_{\dot{a}b} + \Gamma^\alpha{}_{\mu\beta}\sigma^\beta{}_{\dot{a}b} - \bar{\Gamma}^{\dot{c}}{}_{\mu\dot{a}}\sigma^\alpha{}_{\dot{c}b} - \Gamma^d{}_{\mu b}\sigma^\alpha{}_{\dot{a}d}$ and

$$\Gamma^\alpha{}_{\mu\beta} = \left\{ {\alpha \atop \mu\beta} \right\} \tag{2.24}$$

are the Christoffel symbols. Conditions (2.22), (2.23) give an overdetermined system of linear algebraic equations for $\operatorname{Re}\Gamma^a{}_{\mu b}$, $\operatorname{Im}\Gamma^a{}_{\mu b}$ the unique solution of which is

$$\Gamma^a{}_{\mu b} = \frac{1}{4}\sigma_\alpha{}^{\dot{c}a}\left(\partial_\mu \sigma^\alpha{}_{\dot{c}b} + \Gamma^\alpha{}_{\mu\beta}\sigma^\beta{}_{\dot{c}b}\right). \tag{2.25}$$

In Chapters 3 & 4 we will view the coframe (frame) ϑ as a dynamical variable.

We restrict our choice of local coordinates on M to those with $\det e^j{}_\alpha > 0$. This means that we work in local coordinates with specific orientation. In particular, this allows us to define the Hodge star: we define the action of $*$ on a rank r antisymmetric tensor R as

$$(*R)_{\alpha_{r+1}\ldots\alpha_4} := (r!)^{-1}\sqrt{|\det g|}\, R^{\alpha_1\ldots\alpha_r}\varepsilon_{\alpha_1\ldots\alpha_4} \tag{2.26}$$

where ε is the totally antisymmetric quantity, $\varepsilon_{0123} := +1$.

The coframe ϑ which is our dynamical variable is assumed to satisfy

$$\det \vartheta^j{}_\alpha > 0, \tag{2.27}$$

and $e^0 \cdot \vartheta^0 > 0$. These assumptions mean that we work with coframes ϑ which can be obtained from our reference coframe e by proper Lorentz transformations: $\vartheta^j = \Lambda^j{}_k\, e^k$ where the $\Lambda^j{}_k$ are real scalar functions satisfying conditions

$$\eta_{ji}\, \Lambda^j{}_k\, \Lambda^i{}_r = \eta_{kr}, \qquad \det \Lambda^j{}_k > 0, \qquad \Lambda^0{}_0 > 0.$$

We define the forward light cone (at a given point) as the set of covectors of the form $c_j\vartheta^j$ with $\eta^{jk}c_jc_k = 0$ and $c_0 > 0$. This implies, in particular, that our covector l defined by formula (3.4) lies on the forward light cone.

We define

$$\sigma_{\alpha\beta ac} := (1/2)(\sigma_{\alpha a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\beta c\dot{d}} - \sigma_{\beta a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\alpha c\dot{d}}) \tag{2.28}$$

(the first spinor index enumerates the rows and the second one the columns). These "second order" Pauli matrices are polarized, i.e. $*\sigma = \pm i\sigma$ depending on the choice of "basic" Pauli matrices $\sigma_{\alpha a\dot{b}}$. Here the explicit formula for the action of the Hodge star on second order Pauli matrices is

$$(*\sigma)_{\gamma\delta ab} := \frac{1}{2}\sqrt{|\det g|}\;\sigma^{\alpha\beta}{}_{ab}\,\varepsilon_{\alpha\beta\gamma\delta}.$$

Following from our choice of Pauli matrices we have the following polarization

$$*\,\sigma = i\sigma. \tag{2.29}$$

We can also form a complex coframe, written as

$$\begin{pmatrix} l \\ m \\ \bar{m} \\ n \end{pmatrix} \tag{2.30}$$

where

$$l := \vartheta^0 + \vartheta^3, \qquad m := \vartheta^1 + i\vartheta^2, \qquad n := \vartheta^0 - \vartheta^3. \tag{2.31}$$

(The Lorentz index has been suppressed.)

Note that formula (2.27) implies

$$* (l \wedge m) = -i(l \wedge m) \tag{2.32}$$

where the covectors l and m are defined by formulae (2.31). We chose the sign in the RHS of (2.29) so as to agree with (2.32). In other words, the meaning of condition (2.29) is that the orientation encoded in our Pauli matrices agrees with the orientation encoded in our coframe.

2.3.1 Torsion

One of the more dominant themes in this thesis is torsion. It is particularly important when we want to measure the deformation of the coframes from the reference counterparts. We define torsion for our dynamical variable $\vartheta^j{}_\alpha$ as (suppressing Lorentz indices)

$$T = \vartheta^0 \otimes d\vartheta^0 - \vartheta^1 \otimes d\vartheta^1 - \vartheta^2 \otimes d\vartheta^2 - \vartheta^3 \otimes d\vartheta^3 \tag{2.33}$$

where $(d\vartheta^j)_{\alpha\beta} = \partial_\alpha \vartheta^j{}_\beta - \partial_\beta \vartheta^j{}_\alpha$ is the exterior derivative of the coframe. We are only interested in a special irreducible part of torsion, namely the axial part, which is totally antisymmetric in all three Lorentz indices,

$$T^{\mathrm{ax}} = \frac{1}{3}\left(\vartheta^0 \wedge d\vartheta^0 - \vartheta^1 \wedge d\vartheta^1 - \vartheta^2 \wedge d\vartheta^2 - \vartheta^3 \wedge d\vartheta^3\right). \tag{2.34}$$

Here the exterior product of a covector (1-form) v and a covariant rank two antisymmetric tensor (2-form) w is defined as $(v \wedge w)_{\alpha\beta\gamma} := v_\alpha w_{\beta\gamma} + v_\gamma w_{\alpha\beta} + v_\beta w_{\gamma\alpha}$.

We identify differential forms with covariant antisymmetric tensors. Given a pair of real covariant antisymmetric tensors P and Q of rank r we define their dot product as $P \cdot Q := \frac{1}{r!} P_{\alpha_1 \ldots \alpha_r} Q_{\beta_1 \ldots \beta_r} g^{\alpha_1 \beta_1} \ldots g^{\alpha_r \beta_r}$. We also define $\|P\|^2 := P \cdot P$.

2.4 Notation for Chapters 5 & 6

In Chapter 5 we will be in 1+3 dimensions working with Elko spinors in Cosmology. They have 4 complex components, but due to the cosmological principle (= space-time is assumed to be homogeneous and isotropic), only have one real degree of freedom.

In Chapter 6 we will be in 1+3 dimensions working with generalised non-standard quantum fields. These are different mathematical objects to spinors, in some sense they are infinite-dimensional versions of the spinors from the rest of the thesis. We will introduce them rigorously in that chapter with their own notation and therefore will avoid cluttering this section with very specialised notation.

As in Chapters 5 & 6 we will be dealing with spinors with 4 complex components we will require the 4×4 analogue of the Pauli matrices, namely the Dirac matrices.

The 4×4 Dirac matrices γ^j, $j = 0,1,2,3$, in any space-time, curved or flat, are defined in terms of the 2×2 Pauli matrices σ^j as

$$\gamma^0 = \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{pmatrix}, \quad \gamma^n = \begin{pmatrix} \mathbb{O} & -s^n \\ s^n & \mathbb{O} \end{pmatrix}, \quad n = 1,2,3, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{pmatrix}, \tag{2.35}$$

where the s^n are defined in accordance with 2.17.

The matrices γ^α are then given by $\gamma^\alpha = \gamma^j {e_j}^\alpha$, and hence satisfy

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2g^{\alpha\beta}\,\mathbb{1}.$$

The covariant derivative of a 4-component complex spinor is defined as

$$\nabla_\alpha\Psi = \partial_\alpha\Psi + \Gamma_\alpha\Psi \tag{2.36}$$

where Γ_α denotes the spin connection

$$\Gamma_\alpha = \frac{i}{4}{\omega_\alpha}^{jk} f_{jk}, \qquad f^{jk} = \frac{i}{2}\left[\gamma^j, \gamma^k\right], \tag{2.37}$$

and, since we require

$$\nabla_\alpha {e^j}_\beta = \partial_\alpha {e^j}_\beta - {\Gamma^\sigma}_{\alpha\beta}{e^j}_\sigma - {\omega_\alpha}^j{}_k {e^k}_\beta = 0, \tag{2.38}$$

we have

$$\omega_\alpha{}^{jk} = e^j{}_\beta \partial_\alpha e^{k\beta} + e^j{}_\beta e^{k\sigma} \Gamma^\beta{}_{\alpha\sigma}. \tag{2.39}$$

Here $e^{k\alpha} = e^k{}_\beta g^{\alpha\beta}$.

The covariant derivative has a particular form associated with each object (vector, covector, spinor, etc) it acts on. Also, there is a distinct connection whether you have curvature, torsion or both. We will always state explicitly which form of the covariant derivative we are using throughout the thesis.

2.5 Brief introduction to teleparallelism

Given a coframe ϑ, we introduce a covariant derivative $|\nabla|$ such that $|\nabla|\vartheta = 0$. We repeat this formula giving frame and tensor indices explicitly: $|\nabla|_\alpha \vartheta^j{}_\beta = 0$. We then rewrite the formula in even more explicit form:

$$\partial_\alpha \vartheta^j{}_\beta - |\Gamma|^\gamma{}_{\alpha\beta} \vartheta^j{}_\gamma = 0 \tag{2.40}$$

where $|\Gamma|^\gamma{}_{\alpha\beta}$ are the connection coefficients. Note that formula (2.40) has three free indices j, α, β running through the values $0, 1, 2, 3$. Note also that the connection coefficient $|\Gamma|^\gamma{}_{\alpha\beta}$ has three indices α, β, γ running through the values $0, 1, 2, 3$. Hence, (2.40) can be viewed as a system of 64 inhomogeneous linear algebraic equations for the determination of the 64 unknown connection coefficients $|\Gamma|^\gamma{}_{\alpha\beta}$. It is easy to see that its unique solution is

$$|\Gamma|^\gamma{}_{\alpha\beta} = \eta_{ik} g^{\gamma\delta} \vartheta^i{}_\delta \partial_\alpha \vartheta^k{}_\beta. \tag{2.41}$$

The corresponding connection is called *teleparallel*. When writing the teleparallel covariant derivative and connection coefficients we use the "modulus" sign to distinguish these from the Levi-Civita covariant derivative and connection coefficients for which we use curly brackets.

Thus, we have two different connections: the Levi-Civita connection used primarily in the text of this thesis and the teleparallel connection used in this section. Both are metric compatible: $\{\nabla\}g = |\nabla|g = 0$. The Levi-Civita connection is uniquely determined by the metric whereas the teleparallel connection is uniquely determined by the coframe. For the

Levi-Civita connection torsion is zero whereas for the teleparallel connection curvature is zero. Thus, in a sense, the Levi-Civita and teleparallel connections are antipodes.

"Teleparallelism" stands for "distant parallelism". What is meant here is that the result of parallel transport of a vector (or a covector) does not depend on the choice of curve connecting the two points. This fact can be expressed in even simpler terms as follows. Suppose we have two covectors, u and v, of equal magnitude $||u||^2 = ||v||^2 \neq 0$, at two different points, P and Q, of our manifold (spacetime) M. We need to establish whether u and v are parallel. To do this, we use the coframe as a basis and write $u = a_j \vartheta^j$, $v = b_j \vartheta^j$. By definition, the covectors u and v are said to be parallel if $a_j = b_j$.

Formula (2.41) allows us to evaluate torsion of the teleparallel connection:

$$T^{\gamma}{}_{\alpha\beta} := |\Gamma|^{\gamma}{}_{\alpha\beta} - |\Gamma|^{\gamma}{}_{\beta\alpha} = \eta_{ik} g^{\gamma\delta} \vartheta^i{}_{\delta} (\partial_\alpha \vartheta^k{}_{\beta} - \partial_\beta \vartheta^k{}_{\alpha}) = \eta_{ik} g^{\gamma\delta} \vartheta^i{}_{\delta} (d\vartheta^k)_{\alpha\beta}$$

where d denotes the exterior derivative. Lowering the first tensor index gives a neater representation $T_{\gamma\alpha\beta} = \eta_{ik} \vartheta^i_\gamma (d\vartheta^k)_{\alpha\beta}$. Dropping Lorentz indices altogether we get

$$T = \eta_{ik} \vartheta^i \otimes d\vartheta^k. \tag{2.42}$$

It is known [70, 118, 72] that torsion decomposes into three irreducible pieces called tensor torsion, vector torsion and axial torsion. (Vector torsion is sometimes called trace torsion.) In this thesis we use only the axial piece. Axial torsion has a very simple meaning: it is the totally antisymmetric piece $T^{\mathrm{ax}}{}_{\alpha\beta\gamma} = \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha})$. Substituting (2.42) into this general formula we arrive at (2.34).

Of course, there is much more to teleparallelism than the elementary facts sketched out above. Modern reviews of the physics of teleparallelism can be found in [71, 65, 89, 48, 13, 95].

Part I

Alternative picture of particle physics.

Chapter 3

Weyl Lagrangian

Theorems 3.1 - 3.4 are the main the results of this chapter: we find an equivalence between our model based on the coframe and the standard model for a massless neutrino.

Let's first define our model in more detail.

3.1 The setup

We start by reacalling our definition of axial torsion (2.34). This 3-form is called *axial torsion of the teleparallel connection*. The geometric meaning of the latter phrase was explained in a concise fashion in the previous chapter, whereas a detailed exposition of the application of torsion in field theory and the history of the subject can be found in [58, 72]. What is important at this stage is the observation that the 3-form (2.34) is a measure of deformations generated by rotations of spacetime points.

Note that the 3-form (2.34) has the remarkable property of conformal covariance: if we rescale our metric and coframe as

$$g_{\alpha\beta} \mapsto e^{2h} g_{\alpha\beta} \tag{3.1}$$

$$\vartheta^j \mapsto e^h \vartheta^j \tag{3.2}$$

where $h : M \to \mathbb{R}$ is an arbitrary scalar function, then our 3-form is scaled as

$$T^{\mathrm{ax}} \mapsto e^{2h} T^{\mathrm{ax}} \tag{3.3}$$

without the derivatives of h appearing. The issue of conformal covariance and invariance will be examined in detail in Section 3.6.

It is tempting to use the 3-form (2.34) as our Lagrangian but the problem is that we are working in 4-space. In order to turn our 3-form into a 4-form we proceed as follows.

We recall the definition of l_α,

$$l_\alpha := \vartheta^0{}_\alpha + \vartheta^3{}_\alpha. \tag{3.4}$$

This is a nonvanishing real lightlike covector field. It will eventually (see Section 3.8) transpire that the covector field (3.4) has the geometric meaning of neutrino current.

We define our “teleparallel” Lagrangian as

$$L_{\mathrm{tele}}(\vartheta) := l \wedge T^{\mathrm{ax}}. \tag{3.5}$$

Note that formulae (2.34), (3.4), (3.5) are very simple. They do not contain spinors, Pauli matrices or covariant derivatives. The only concepts used are those of a differential form, wedge product and exterior derivative. Even the metric does not appear in formulae (2.34), (3.4), (3.5) explicitly: it is incorporated implicitly via the constraint (2.9).

3.2 Symmetries

As with any Lagrangian it is good to know what symmetries, internal or not, are available to us. We start with Lorentz transformations of the coframe:

$$\vartheta^j \overset{\Lambda}{\mapsto} \tilde{\vartheta}^j = \Lambda^j{}_k \vartheta^k \tag{3.6}$$

where the $\Lambda^j{}_k$ are real scalar functions satisfying the constraint

$$\eta_{jk} \Lambda^j{}_r \Lambda^k{}_s = \eta_{rs}. \tag{3.7}$$

Obviously, transformations (3.6), (3.7) form an infinite-dimensional Lie group. Within this group we single out an infinite-dimensional Lie subgroup H as follows.

Put

$$m_\alpha := \vartheta^1{}_\alpha + i\vartheta^2{}_\alpha. \tag{3.8}$$

The subgroup H is defined by the condition of preservation modulo $\mathrm{U}(1)$ of the complex 2-form $l \wedge m$. More precisely, a Lorentz transformation (3.6), (3.7) is included in H if and only if

$$l \wedge m \overset{\mathrm{mod}\,\mathrm{U}(1)}{=} \tilde{l} \wedge \tilde{m} \tag{3.9}$$

where $\tilde{l}_\alpha = \tilde{\vartheta}^0{}_\alpha + \tilde{\vartheta}^3{}_\alpha$ and $\tilde{m}_\alpha = \tilde{\vartheta}^1{}_\alpha + i\tilde{\vartheta}^2{}_\alpha$.

We can pause for a moment and state our first result.

Theorem 3.1. *The teleparallel Lagrangian (3.5) is invariant under the action of the group H.*

In view of Theorem 3.1 we call two coframes equivalent if they differ by a transformation from the subgroup H and gather coframes into equivalence classes according to this relation.

Let's look in more detail at our gauge group H. Consider a Lorentz transformation of the coframe (3.6) satisfying the defining condition (3.9) of our group H. (Recall that here the $\Lambda^j{}_k$ are not assumed to be constant, i.e. they are real scalar functions satisfying (3.7).) We denote this Lorentz transformation Λ.

Condition (3.9) means that Λ is a composition of two Lorentz transformations:

$$\Lambda = \Lambda''\Lambda' \tag{3.10}$$

where Λ' is a rotation by a constant angle φ in the ϑ^1, ϑ^2–plane

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \overset{\Lambda'}{\mapsto} \begin{pmatrix} l \\ e^{i\varphi} m \\ n \end{pmatrix} \tag{3.11}$$

and Λ'' is a Lorentz transformation preserving the 2-form $l \wedge m$. Our convention for writing compositions of Lorentz transformations is as follows. When looking at a Lorentz transformation (3.6) we view the real coframe as a column of height 4 with entries ϑ^k, $k = 0, 1, 2, 3$, and the Lorentz transformation itself as multiplication by a real 4×4 matrix $\Lambda^j{}_k$, so the group operation is matrix multiplication with the matrix furthest to the right acting on the coframe first. So, formula (3.10) means that Λ' acts on the coframe first.

It is known, see Section 10.122 in [11], that Lorentz transformations preserving the 2-form $l \wedge m$ admit an explicit description:

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \overset{\Lambda''}{\mapsto} \begin{pmatrix} l \\ m + fl \\ n + f\bar{m} + \bar{f}m + |f|^2 l \end{pmatrix} \tag{3.12}$$

where $f : M \to \mathbb{C}$ is an arbitrary scalar function and n is defined as

$$n_\alpha := \vartheta^0{}_\alpha - \vartheta^3{}_\alpha. \tag{3.13}$$

Substituting (3.11), (3.12) into (3.10) we arrive at the explicit formula for an element Λ of the group H:

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \overset{\Lambda}{\mapsto} \begin{pmatrix} l \\ e^{i\varphi} m + fl \\ n + fe^{-i\varphi}\bar{m} + \bar{f}e^{i\varphi}m + |f|^2 l \end{pmatrix}. \tag{3.14}$$

Let us now examine the structure of the group H.

The group of rotations in the ϑ^1, ϑ^2–plane is isomorphic to $\mathrm{U}(1)$. Hence further on we will refer to the group of Lorentz transformations of the coframe of the form (3.11) as $\mathrm{U}(1)$. Let us emphasise that the φ appearing in formula (3.11) is a constant, not a function.

Let us denote by $B^2(M)$ the group of Lorentz transformations of the coframe preserving the 2-form $l \wedge m$, see formula (3.12). In choosing the notation B^2 we follow [11] where the "M" indicates dependence on the point of the manifold M, i.e. it highlights the fact that the f appearing in formula (3.12) is a function, not a constant.

Both $\mathrm{U}(1)$ and $B^2(M)$ are abelian[1] subgroups of H. Moreover, it is easy to see that $B^2(M)$ is a normal subgroup of H, $B^2(M) \triangleleft H$, and that H is a semidirect product of $B^2(M)$ and $\mathrm{U}(1)$, $H = B^2(M) \ltimes \mathrm{U}(1)$. Here the symbol "$\ltimes$" stands for the semidirect product with the normal subgroup coming first.

[1]The group B^2 can, in fact, be characterised as the nontrivial abelian subgroup of the Lorentz group. See Appendix B in [120] for details.

The infinite-dimensional Lie group H is itself nonabelian. However, it is very close to being abelian: H contains the infinite-dimensional abelian Lie subgroup $B^2(M)$ of codimension 1.

3.3 Proof of Theorem 3.1

Let us rewrite our teleparallel Lagrangian (3.5) in terms of the complex coframe (3.4), (3.8), (3.13):

$$L_{\mathrm{tele}}(\vartheta) = (1/6)\, l \wedge (n \wedge dl - \bar{m} \wedge dm - m \wedge d\bar{m}). \tag{3.15}$$

The group H is a semidirect product of the groups $B^2(M)$ and $\mathrm{U}(1)$ so in order to check that (3.15) is invariant under the action of H it is sufficient to check that (3.15) is invariant under the actions of $B^2(M)$ and $\mathrm{U}(1)$ separately. $\mathrm{U}(1)$-invariance is obvious: just substitute (3.11) into (3.15) noting that φ is constant. Hence, it remains only to show that our teleparallel Lagrangian (3.15) is invariant under the transformation (3.12).

When substituting (3.12) into (3.15) we will get an expression which is a sum of two terms:

- a term without derivatives of the function f, and
- a term with derivatives of the function f.

Looking at our original formula (3.5) we see that the term without derivatives of the function f does not change the teleparallel Lagrangian because our transformation (3.12) preserves the covector field l and because axial torsion is an irreducible piece of torsion (i.e. the 3-form (2.34) is invariant under rigid Lorentz transformations). So it only remains to check that the term with derivatives of the function f vanishes. The term in question is

$$(1/6)\, l \wedge (-\bar{m} \wedge df \wedge l - m \wedge d\bar{f} \wedge l)$$

which is clearly zero. □

3.4 Equivalence

Before we state our second main result and prove it, we must first discuss the Weyl Lagrangian.

The accepted mathematical model for a massless neutrino field is the following complex linear partial differential equation on M know as *Weyl's equation*:

$$i\sigma^\alpha{}_{\dot{b}a}\{\nabla\}_\alpha\xi^a = 0. \tag{3.16}$$

The corresponding Lagrangian is

$$L_{\mathrm{Weyl}}(\xi) := \frac{i}{2}(\bar{\xi}^{\dot{b}}\sigma^\alpha{}_{\dot{b}a}\{\nabla\}_\alpha\xi^a - \{\nabla\}_\alpha\bar{\xi}^{\dot{b}}\sigma^\alpha{}_{\dot{b}a}\xi^a) * 1. \tag{3.17}$$

Here $*1$ is the standard volume 4-form (Hodge dual of the scalar 1), σ^α, $\alpha = 0, 1, 2, 3$, are Pauli matrices, ξ is the unknown Weyl (2-component) spinor field and $\{\nabla\}$ is the covariant derivative with respect to the Levi-Civita connection defined by formulae (2.20), (2.24).

It is well known that Weyl's Lagrangian (3.17) is $\mathrm{U}(1)$-invariant:

$$\xi \overset{\mathrm{mod\,U}(1)}{=} \tilde{\xi} \quad \Longrightarrow \quad L_{\mathrm{Weyl}}(\xi) = L_{\mathrm{Weyl}}(\tilde{\xi}).$$

In view of this we call two spinor fields equivalent if they are equal modulo $\mathrm{U}(1)$ and gather spinor fields into equivalence classes according to this relation. We call an equivalence class of spinors *nonvanishing* if its representatives do not vanish at any point.

Theorem 3.2. *The equivalence classes of coframes ϑ and nonvanishing spinor fields ξ are in a one-to-one correspondence given by the formula*

$$(l \wedge m)_{\alpha\beta} \overset{\mathrm{mod\,U}(1)}{=} \sigma_{\alpha\beta ab}\xi^a\xi^b \tag{3.18}$$

where l and m are defined by formulae (3.4) and (3.8) respectively, ϑ and ξ are arbitrary representatives of the corresponding equivalence classes and $\sigma_{\alpha\beta}$ are "second order" Pauli matrices (2.28). Furthermore, under the correspondence (3.18) we have

$$L_{\mathrm{tele}}(\vartheta) = -\frac{4}{3}\,L_{\mathrm{Weyl}}(\xi). \tag{3.19}$$

A shorter way of stating Theorem 3.2 is "the nonlinear change of variable

$$\text{coframe } \vartheta \quad \longleftrightarrow \quad \text{spinor field } \xi$$

specified by formula (3.18) shows that the two Lagrangians, $L_{\mathrm{tele}}(\vartheta)$ and $L_{\mathrm{Weyl}}(\xi)$, are the same up to a constant factor". The only problem with such a statement is that it brushes aside the important question of gauge invariance.

3.5 Proof of Theorem 3.2

The gauge group H allows us to gather coframes into equivalence classes: we call two coframes equivalent if they differ by a transformation from H. We will now establish the geometric meaning of these equivalence classes of coframes.

Let us first fix a spacetime point $x \in M$ and examine in detail the geometric meaning of the group B^2. We initially defined B^2 as the the group of Lorentz transformations preserving the 2-form $l \wedge m$. The complex nonzero antisymmetric tensor $l \wedge m$ is polarized (see (2.32)) and has the additional property $\det(l \wedge m) = 0$. It is easy to see (and this fact was extensively used in [120, 78, 118, 119, 97]) that such a tensor can be written in terms of a nonzero spinor ξ as

$$(l \wedge m)_{\alpha\beta} = -\sigma_{\alpha\beta ab}\xi^a\xi^b \tag{3.20}$$

with the spinor defined uniquely up to a sign. Thus, the group B^2 can be reinterpreted as the group of Lorentz transformations preserving a given nonzero spinor ξ and the equivalence classes of coframes are related to this spinor according to formula (3.20). Here the relationship between an equivalence class of coframes and a nonzero spinor is one-to-two because formula (3.20) allows us to change the sign of ξ.

Remark 1. *One can use the above observation to formulate an alternative definition of a spinor: a spinor is a coset of the Lorentz group with respect to the subgroup B^2. In using this definition one, however, has to decide whether to use left or right cosets as B^2 is not a normal subgroup of the Lorentz group.*

Remark 2. *In $\mathrm{SL}(2,\mathbb{C})$ notation the group B^2 is written in a particularly simple way:* $B^2 = \left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \middle| \; f \in \mathbb{C} \right\}$.

Let us now allow dependence on the spacetime point $x \in M$. Then the group $B^2(M)$ is the group of Lorentz transformations preserving a given nonzero spinor field ξ, with the equivalence classes of coframes related to the spinor field according to formula (3.20). Here the relationship between an equivalence class of coframes and a nonvanishing spinor field remains one-to-two.

Finally, let us switch from the group $B^2(M)$ to $H = B^2(M) \ltimes \mathrm{U}(1)$. This means that in our definition of equivalence classes of coframes we allow $l \wedge m$ to be multiplied by a constant complex factor of modulus 1, so formula (3.20) turns into (3.18). Here the relationship between an equivalence class of coframes and a nonvanishing spinor field becomes one-to-infinity because formula (3.18) allows us to multiply the nonvanishing spinor field ξ by a constant complex factor of modulus 1; note that this eliminates the difference between ξ and $-\xi$. It remains only to gather nonvanishing spinor fields ξ into equivalence classes as described in the beginning of Section 3.4 and we arrive at a one-to-one correspondence between equivalence classes of coframes and nonvanishing spinor fields given by the explicit formula (3.18).

In the remainder of this section we perform the nonlinear change of variable

$$\text{spinor field } \xi \quad \longrightarrow \quad \text{coframe } \vartheta$$

and show that $L_{\mathrm{Weyl}}(\xi)$ turns into $-\frac{3}{4}\,L_{\mathrm{tele}}(\vartheta)$. In order to simplify calculations we observe that we have freedom in our choice of Pauli matrices. It is sufficient to prove formula (3.19) for one particular choice of Pauli matrices, hence we will use 2.17. We are also allowed to choose $e = \vartheta$. Note that this approach is not new: it was, for example, extensively used by A. Dimakis and F. Müller-Hoissen [49, 50, 51].

We now calculate explicitly the corresponding second order Pauli matrices:

$$\sigma_{\alpha\beta ab} = \frac{1}{2}(\vartheta^j \wedge \vartheta^k)_{\alpha\beta}\, s_{jkab} \tag{3.21}$$

where

$$s_{jkab} = \begin{pmatrix} 0 & s_{01ab} & s_{02ab} & s_{03ab} \\ s_{10ab} & 0 & s_{12ab} & s_{13ab} \\ s_{20ab} & s_{21ab} & 0 & s_{23ab} \\ s_{30ab} & s_{31ab} & s_{32ab} & 0 \end{pmatrix}$$

$$:= \begin{pmatrix} \mathbb{O} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \mathbb{O} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & \mathbb{O} & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \mathbb{O} \end{pmatrix}. \qquad (3.22)$$

Substituting (3.4), (3.8) and (3.21), (3.22) into the equation (3.18) we see that this equation can be easily resolved for ξ giving

$$\xi^a \overset{\mathrm{mod}\,\mathrm{U}(1)}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad (3.23)$$

Formula (3.23) may seem strange: we are proving Theorem 3.2 for a general nonvanishing spinor field ξ but ended up with formula (3.23) which is very specific. However, there is no contradiction here because we chose Pauli matrices specially adapted to the coframe ϑ and, hence, specially adapted to the corresponding spinor field ξ.

Substituting (2.20) and (3.23) into (3.17) we get

$$\begin{aligned} &L_{\mathrm{Weyl}}(\xi) \\ &= \frac{i}{8}(\bar\xi^{\dot b}\sigma^\alpha{}_{\dot b a}\sigma_\beta{}^{\dot c a}(\partial_\alpha\sigma^\beta{}_{\dot c d} + \{\Gamma\}^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{\dot c d})\xi^d - \xi^a\sigma^\alpha{}_{\dot b a}\sigma_\beta{}^{\dot b c}(\partial_\alpha\sigma^\beta{}_{\dot d c} + \{\Gamma\}^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{\dot d c})\bar\xi^{\dot d}) * 1 \\ &\quad = \frac{i}{8}(\sigma^\alpha{}_{\dot 1 a}\sigma_\beta{}^{\dot c a}(\partial_\alpha\sigma^\beta{}_{\dot c 1} + \{\Gamma\}^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{\dot c 1}) - \sigma^\alpha{}_{\dot b 1}\sigma_\beta{}^{\dot b c}(\partial_\alpha\sigma^\beta{}_{\dot 1 c} + \{\Gamma\}^\beta{}_{\alpha\gamma}\sigma^\gamma{}_{\dot 1 c})) * 1 \\ &\qquad = \frac{i}{8}(\sigma^\alpha{}_{\dot 1 a}\sigma_\beta{}^{\dot c a}\{\nabla\}_\alpha\sigma^\beta{}_{\dot c 1} - \sigma^\alpha{}_{\dot b 1}\sigma_\beta{}^{\dot b c}\{\nabla\}_\alpha\sigma^\beta{}_{\dot 1 c}) * 1\,. \end{aligned}$$

We now write down the spinor summation indices explicitly:

$$L_{\mathrm{Weyl}}(\xi)=\frac{i}{8}(\sigma^\alpha{}_{\dot 11}\sigma_\beta{}^{\dot 21}\{\nabla\}_\alpha\sigma^\beta{}_{\dot 21}+\sigma^\alpha{}_{\dot 12}\sigma_\beta{}^{\dot 12}\{\nabla\}_\alpha\sigma^\beta{}_{\dot 11}+\sigma^\alpha{}_{\dot 11}\sigma_\beta{}^{\dot 22}\{\nabla\}_\alpha\sigma^\beta{}_{\dot 21}$$
$$-\sigma^\alpha{}_{\dot 11}\sigma_\beta{}^{\dot 12}\{\nabla\}_\alpha\sigma^\beta{}_{\dot 12}-\sigma^\alpha{}_{\dot 21}\sigma_\beta{}^{\dot 21}\{\nabla\}_\alpha\sigma^\beta{}_{\dot 11}-\sigma^\alpha{}_{\dot 21}\sigma_\beta{}^{\dot 22}\{\nabla\}_\alpha\sigma^\beta{}_{\dot 12})*1\,.$$

Note that the terms with $a=1$, $\dot c=\dot 1$ and $\dot b=\dot 1$, $c=1$ cancelled out. Finally, we substitute explicit formulae (2.19), (2.17) for our Pauli matrices which gives us

$$\begin{aligned}L_{\mathrm{Weyl}}(\xi)&=\frac{i}{8}(l^\alpha(-\bar m_\beta)\{\nabla\}_\alpha m^\beta+\bar m^\alpha(-m_\beta)\{\nabla\}_\alpha l^\beta+\bar m^\alpha l_\beta\{\nabla\}_\alpha m^\beta\\&\quad-l^\alpha(-m_\beta)\{\nabla\}_\alpha\bar m^\beta-m^\alpha(-\bar m_\beta)\{\nabla\}_\alpha l^\beta-m^\alpha l_\beta\{\nabla\}_\alpha\bar m^\beta)*1\\&=\frac{i}{8}((m\wedge\bar m)^{\alpha\beta}\{\nabla\}_\alpha l_\beta-(l\wedge\bar m)^{\alpha\beta}\{\nabla\}_\alpha m_\beta+(l\wedge m)^{\alpha\beta}\{\nabla\}_\alpha\bar m_\beta)*1\\&=\frac{i}{16}((m\wedge\bar m)^{\alpha\beta}(dl)_{\alpha\beta}-(l\wedge\bar m)^{\alpha\beta}(dm)_{\alpha\beta}+(l\wedge m)^{\alpha\beta}(d\bar m)_{\alpha\beta})*1\\&=\frac{i}{16}*((m\wedge\bar m)^{\alpha\beta}(dl)_{\alpha\beta}-(l\wedge\bar m)^{\alpha\beta}(dm)_{\alpha\beta}+(l\wedge m)^{\alpha\beta}(d\bar m)_{\alpha\beta})\\&=\frac{i}{8}([*(m\wedge\bar m)]\wedge dl-[*(l\wedge\bar m)]\wedge dm+[*(l\wedge m)]\wedge d\bar m).\end{aligned}$$

But $*(l\wedge m)=-i(l\wedge m)$ (see (2.32)) and $*(m\wedge\bar m)=+i(l\wedge n)$ so the above formula becomes

$$L_{\mathrm{Weyl}}(\xi)=-\frac{1}{8}(l\wedge n\wedge dl-l\wedge\bar m\wedge dm-l\wedge m\wedge d\bar m).$$

Comparing with (3.15) we arrive at (3.19). $\square$

3.6 Conformal invariance

Until now we have kept the metric fixed but now we shall scale the metric as (3.1) and the Pauli matrices as

$$\sigma_\alpha\mapsto e^h\sigma_\alpha. \tag{3.24}$$

Recall that here $h:M\to\mathbb{R}$ is an arbitrary scalar function. Let us also scale the spinor field as

$$\xi\mapsto e^{-(3/2)h}\xi. \tag{3.25}$$

It is well known that the Weyl Lagrangian (3.17) is invariant under the transformation (3.1), (3.24), (3.25).

Examination of formulae (3.18), (2.28), (3.42), shows that the transformation (3.1), (3.24), (3.25) (but not (3.2)) induces the following transformation of the complex coframe (3.4), (3.8), (3.13):

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \mapsto \begin{pmatrix} e^{-2h}l \\ e^{h}m \\ e^{4h}n \end{pmatrix} \tag{3.26}$$

Of course, it is easy to check directly that our teleparallel Lagrangian (3.15) is invariant under the transformation (3.26).

The transformation (3.26) is a composition of two commuting transformations: a conformal rescaling of the coframe (3.2) and a Lorentz boost

$$\begin{pmatrix} \vartheta^0 \\ \vartheta^3 \end{pmatrix} \mapsto \begin{pmatrix} \cosh 3h & -\sinh 3h \\ -\sinh 3h & \cosh 3h \end{pmatrix} \begin{pmatrix} \vartheta^0 \\ \vartheta^3 \end{pmatrix}.$$

The presence of a Lorentz boost in this argument is somewhat unnatural so we suggest below a modified version of our teleparallel Lagrangian, one for which conformal invariance is self-evident. Recall that our original teleparallel Lagrangian $L_{\mathrm{tele}}(\vartheta)$ was defined by formula (3.5) or, equivalently, in terms of the complex coframe, by formula (3.15).

Put

$$\tilde{L}_{\mathrm{tele}}(\vartheta, s) := sL_{\mathrm{tele}}(\vartheta) = sl \wedge T^{\mathrm{ax}} = (s/6)\, l \wedge (n \wedge dl - \bar{m} \wedge dm - m \wedge d\bar{m}) \tag{3.27}$$

where $s : M \to (0, +\infty)$ is a scalar function. The function s will play the role of an additional dynamical variable. In view of (3.3) the Lagrangian (3.27) does not change if we scale the coframe as (3.2), the metric as (3.1) and the scalar s as $s \mapsto e^{-3h}s$. Hence, the Lagrangian (3.27) is conformally invariant and, moreover, this conformal invariance is quite obvious.

Let us now examine the properties of the Lagrangian (3.27) for *fixed* metric. Of course, it is invariant under the action of the group H which was described implicitly in Section 3.4 and explicitly in Section 3.2 (see formula (3.14)). However, it is also invariant under the transformation

$$\begin{pmatrix} l \\ m \\ n \\ s \end{pmatrix} \mapsto \begin{pmatrix} e^{-k}l \\ m \\ e^{k}n \\ e^{k}s \end{pmatrix} \tag{3.28}$$

where $k : M \to \mathbb{R}$ is an arbitrary scalar function. The transformation (3.27) is a composition of two transformations: a Lorentz boost

$$\begin{pmatrix} \vartheta^0 \\ \vartheta^3 \end{pmatrix} \mapsto \begin{pmatrix} \cosh k & -\sinh k \\ -\sinh k & \cosh k \end{pmatrix} \begin{pmatrix} \vartheta^0 \\ \vartheta^3 \end{pmatrix}$$

and a rescaling of the scalar s, $s \mapsto e^k s$. We will denote the infinite-dimensional Lie group of transformations (3.28) by $J(M)$.

Thus, having incorporated into our original teleparallel Lagrangian (3.5) an additional dynamical variable, the positive scalar function s, we have acquired an additional gauge degree of freedom. The new (extended) gauge group is

$$\begin{aligned}\tilde{H} = H \ltimes J(M) = (B^2(M) \ltimes \mathrm{U}(1)) \ltimes J(M) \\ = (B^2(M) \ltimes J(M)) \ltimes \mathrm{U}(1) = B^2(M) \ltimes (J(M) \times \mathrm{U}(1)).\end{aligned}$$

The action of $\tilde{H}$ preserves the 2-form $l \wedge m$ modulo $\mathrm{U}(1)$ and modulo rescaling by a positive scalar function.

We have established the following analogue of Theorem 3.1.

Theorem 3.3. *The modified teleparallel Lagrangian (3.27) is invariant under the action of the group* $\tilde{H}$.

In view of Theorem 3.3 we call two sets of dynamical variables "coframe + positive scalar" equivalent if they differ by a transformation from the group $\tilde{H}$ and gather sets of dynamical variables into equivalence classes according to this relation. The following is an analogue of Theorem 3.2.

Theorem 3.4. *The equivalence classes of coframes ϑ and positive scalars s on the one hand and nonvanishing spinor fields ξ on the other are in a one-to-one correspondence given by the formula*

$$s\,(l\wedge m)_{\alpha\beta} \overset{\mathrm{mod}\,\mathrm{U}(1)}{=} \sigma_{\alpha\beta ab}\xi^a\xi^b \tag{3.29}$$

where l and m are defined by formulae (3.4) and (3.8) respectively, ϑ, s and ξ are arbitrary representatives of the corresponding equivalence classes and $\sigma_{\alpha\beta}$ are "second order" Pauli matrices (2.28). Furthermore, under the correspondence (3.29) we have

$$\tilde{L}_{\mathrm{tele}}(\vartheta,s) = -\frac{4}{3}\,L_{\mathrm{Weyl}}(\xi). \tag{3.30}$$

The proof of the first part of Theorem 3.4 (formula (3.29)) is essentially a repetition of the proof of the first part of Theorem 3.2: take the argument from the beginning of Section 3.3 and add one gauge degree of freedom.

As to the second part of Theorem 3.4 (formula (3.30)), it simply follows from the second part of Theorem 3.2 (formula (3.19)). Indeed, when we replace (3.18) by (3.29) the spinor field scales as $\xi\mapsto\sqrt{s}\,\xi$. But

$$-\frac{4}{3}\,L_{\mathrm{Weyl}}(\sqrt{s}\,\xi) = -\frac{4}{3}\,sL_{\mathrm{Weyl}}(\xi) \overset{\text{by (3.19)}}{=} sL_{\mathrm{tele}}(\vartheta) \overset{\text{by (3.27)}}{=} L_{\mathrm{tele}}(\vartheta,s)$$

giving us (3.30).

3.7 Weyl's equation in teleparallel form

Here we write down explicitly the Euler–Lagrange field equations resulting from the variation of the action

$$S_{\mathrm{tele}} := \int L_{\mathrm{tele}} = \int l\wedge T^{\mathrm{ax}} = \frac{1}{3}\,p_i\eta_{jk}\int \vartheta^i\wedge\vartheta^j\wedge d\vartheta^k \tag{3.31}$$

with respect to the coframe ϑ subject to the metric constraint (2.9). Here by p_i we denote the quartet of constants $p_i := (1\ 0\ 0\ 1)$.

The variation of the coframe is given by the formula

$$\delta\vartheta^j{}_k = F^j{}_k\vartheta^k \tag{3.32}$$

where the ${F^j}_k$ are real scalar functions satisfying the antisymmetry condition

$$F_{jk} = -F_{kj}. \tag{3.33}$$

Condition (3.33) ensures that the variation of the RHS of (2.9) is zero. Of course, the ${\Lambda^j}_k$ appearing on the RHS of (3.6) are expressed via the ${F^j}_k$ as

$${\Lambda^j}_k = \delta^j_k + {F^j}_k + \frac{1}{2}{F^j}_l {F^l}_k + \ldots$$

(an exponential series), or, in matrix notation, $\Lambda = e^F$. Hence, the matrix-function F is the linearization of the Lorentz transformation Λ about the identity.

Substituting (3.32) into (3.31) we get

$$3\delta S_{\mathrm{tele}} = p_i \eta_{jk} \int ({F^i}_l \vartheta^l \wedge \vartheta^j \wedge d\vartheta^k + {F^j}_l \vartheta^i \wedge \vartheta^l \wedge d\vartheta^k + {F^k}_l \vartheta^i \wedge \vartheta^j \wedge d\vartheta^l + \vartheta^i \wedge \vartheta^j \wedge d{F^k}_l \wedge \vartheta^l)$$

where $d{F^k}_l$ is the gradient of the scalar function ${F^k}_l$. Upon contraction with η_{jk} the second and third terms in the integrand cancel out in view of (3.33) (that this would happen was clear a priori because axial torsion is invariant under rigid Lorentz transformations) so the above formula becomes

$$3\delta S_{\mathrm{tele}} = \int (p^i \eta_{lk} F_{ij} \vartheta^j \wedge \vartheta^l \wedge d\vartheta^k + p_k {\eta_l}^i \vartheta^k \wedge \vartheta^l \wedge dF_{ij} \wedge \vartheta^j)$$

where $p^i := \eta^{ij} p_j$. Integration by parts and antisymmetrization in i, j gives

$$6\delta S_{\mathrm{tele}} = \int F_{ij} (p^i \eta_{lk} \vartheta^j \wedge \vartheta^l \wedge d\vartheta^k - p^j \eta_{lk} \vartheta^i \wedge \vartheta^l \wedge d\vartheta^k - 2p_k d(\vartheta^k \wedge \vartheta^i \wedge \vartheta^j)).$$

Thus, our field equations are

$$p^i \eta_{lk} \vartheta^j \wedge \vartheta^l \wedge d\vartheta^k - p^j \eta_{lk} \vartheta^i \wedge \vartheta^l \wedge d\vartheta^k - 2p_k d(\vartheta^k \wedge \vartheta^i \wedge \vartheta^j) = 0. \tag{3.34}$$

The field equations (3.34) are, of course, equivalent to

$$* \left[p^i \eta_{lk} \vartheta^j \wedge \vartheta^l \wedge d\vartheta^k - p^j \eta_{lk} \vartheta^i \wedge \vartheta^l \wedge d\vartheta^k - 2p_k d(\vartheta^k \wedge \vartheta^i \wedge \vartheta^j)\right] = 0. \tag{3.35}$$

The advantage of the representation (3.35) is that the left-hand sides of (3.35) are scalars and not 4-forms as in (3.34). We denote the left-hand sides of (3.35) by G^{ij}. Note the antisymmetry $G^{ij} = -G^{ji}$.

We will now rewrite our field equations (3.35) in more compact form in terms of the complex coframe (3.4), (3.8), (3.13).

We note first that $G^{12} = 4\{\nabla\}_\alpha l^\alpha$. Thus, our field equations (3.35) imply

$$\{\nabla\}_\alpha l^\alpha = 0. \tag{3.36}$$

Note that the scalar G^{03} also has a clear geometric meaning: $G^{03} = 3 * L_{\mathrm{tele}}$.

Put

$$q_j := (0\ \ 1\ \ i\ \ 0)\,, \qquad r_j := (1\ \ 0\ \ 0\ -1)\,,$$

$$A_{jk} := p_j q_k - p_k q_j, \qquad B_{jk} := p_j r_k - p_k r_j - q_j \bar{q}_k + q_k \bar{q}_j, \qquad C_{jk} := r_j \bar{q}_k - r_k \bar{q}_j.$$

The antisymmetric matrices $\operatorname{Re} A$, $\operatorname{Im} A$, $\operatorname{Re} B$, $\operatorname{Im} B$, $\operatorname{Re} C$, $\operatorname{Im} C$ are linearly independent, therefore the system of 6 real equations (3.35) is equivalent to the system of 3 complex equations

$$A_{ij}G^{ij} = 0, \qquad B_{ij}G^{ij} = 0, \qquad C_{ij}G^{ij} = 0.$$

Straightforward calculations show that $A_{ij}G^{ij}$ is zero for any coframe ϑ (this is actually a consequence of Theorem 3.1), hence our real field equations (3.35) are equivalent to the pair of complex equations

$$B_{ij}G^{ij} = 0, \qquad C_{ij}G^{ij} = 0. \tag{3.37}$$

As the systems (3.35) and (3.37) are equivalent and as equation (3.36) is a consequence of (3.35), equation (3.36) is also a consequence of (3.37). Hence we can extend the system (3.37) by adding equation (3.36): the system (3.37) is equivalent to the system (3.37), (3.36). The advantage of having (3.36) as a separate equation is that it simplifies subsequent calculations.

We now examine our system of field equations (3.37), (3.36). Straightforward calculations with account of (3.36) give

$$B_{ij}G^{ij} = -8i\bar{m}^\alpha v_\alpha, \qquad C_{ij}G^{ij} = 8in^\alpha \bar{v}_\alpha$$

where

$$v_\alpha := \{\nabla\}^\beta (l \wedge m)_{\alpha\beta} - m^\beta \{\nabla\}_\alpha l_\beta. \tag{3.38}$$

Thus, our system of field equations (3.37), (3.36) is equivalent to

$$\bar{m}^\alpha v_\alpha = 0, \qquad n^\alpha v_\alpha = 0 \tag{3.39}$$

and (3.36). But $\mathrm{Re}(\bar{m}^\alpha v_\alpha) = 2\{\nabla\}_\alpha l^\alpha$, so (3.36) is a consequence of (3.39). Hence, (3.39) is the full system of field equations. It is equivalent to the original system of field equations (3.35).

It is easy to see that for any coframe ϑ we have

$$m^\alpha v_\alpha = 0, \qquad l^\alpha v_\alpha = 0 \tag{3.40}$$

so the pair of scalar complex equations (3.39) is equivalent to the complex covector equation

$$v = 0. \tag{3.41}$$

Recall that the LHS of this equation is defined by formula (3.38).

Equation (3.41) is the compact "tetrad" representation of the Weyl equation found by Griffiths and Newing [64]. Griffiths and Newing derived (3.41) directly from Weyl's equation (2.16), without examining the Weyl Lagrangian (3.17).

Let us have a closer look at equation (3.41) so as to establish the actual number of independent "scalar" equations contained in it and the actual number of independent "scalar" unknowns. It would seem that (3.41) is a system of 4 complex "scalar" equations (4 being the number of components of the covector v) for 6 real "scalar" unknowns (6 being the dimension of the Lorentz group). But we already know that we a priori have identities (3.40) so equation (3.41) is equivalent to the pair of scalar complex equations (3.39). It is also easy to see that v is invariant under the action of the transformation (3.12), hence the set of solutions to equation (3.41) is invariant under this transformation which means that we are dealing with a pair of complex "scalar" unknowns (see argument in the beginning of Section 3.5). Thus, equation (3.41) is a system of 2 complex "scalar" equations for 2 complex "scalar" unknowns, as expected of the Weyl equation.

Note that the scalar $\bar{m}^\alpha v_\alpha$ appearing in the LHS of (3.39) is also invariant under the action of the transformation (3.12) and can be written down explicitly as $\bar{m}^\alpha v_\alpha = 2\{\nabla\}_\alpha l^\alpha - \frac{3i}{2} * L_{\mathrm{tele}}$.

3.8 Discussion of results

For Weyl's Lagrangian we found a simple teleparallel representation (3.5).

The teleparallel representation of Weyl's equation was first derived by Griffiths and Newing [64]. Our contribution is the teleparallel representation of Weyl's Lagrangian and observation that for the Lagrangian things become much simpler.

Now, formula (3.19) (as well as its generalised version (3.30)) holds for *any* Lorentzian metric so when using this formula there is really no need to assume the metric to be fixed.

Let us now examine the geometric meaning of the covector field l defined by formula (3.4). If we choose Pauli matrices in the form (2.17) and take (2.19) replacing e with ϑ, we get (3.23) which immediately implies

$$l_\alpha = \sigma_{\alpha a\dot{b}} \xi^a \bar{\xi}^{\dot{b}}. \tag{3.42}$$

Formula (3.42) remains true for any choice of Pauli matrices because its RHS has an invariant meaning. More specifically, the RHS of (3.42) is the well-known expression for the neutrino current. In light of this it is not surprising that our field equations imply that the divergence of l is zero, see formula (3.36).

The main issue with our model is that our Lagrangian (3.5) (as well as its generalised version (3.27)) is not invariant under rigid Lorentz transformations of the coframe. A possible way of overcoming this difficulty is sketched out below.

Consider the Lagrangian

$$L(\vartheta, s) := s\|T^{\mathrm{ax}}\|^2 * 1 \tag{3.43}$$

where $s : M \to (0, +\infty)$ is a scalar function which plays the role of an additional dynamical variable. This Lagrangian is Lorentz invariant and is a special case of a general quadratic Lorentz invariant Lagrangian (a general Lagrangian contains squares of all three irreducible pieces of torsion). The special feature of the Lagrangian (3.43) is that it is conformally invariant: it does not change if we rescale the coframe as (3.2) and the scalar s as $s \mapsto e^{-2h}s$.

Of course, a positive scalar s is equivalent to a positive density ρ : $\quad \rho = s\sqrt{|\det g|}$. Thus, having the scalar function s as a dynamical variable is equivalent to having the density ρ as a dynamical variable. Thinking in terms of an unknown density ρ is more natural from the physical viewpoint. However, in this chapter we will stay with the scalar s.

We vary the action $S(\vartheta, s) := \int L(\vartheta, s)$ with respect to the scalar s and with respect to the coframe ϑ subject to the metric constraint (2.9), which gives us the Euler–Lagrange field equations. The fundamental difference between our original conformally invariant Lagrangian (3.27) and the new conformally invariant Lagrangian (3.43) is that the latter is quadratic in torsion, hence the field equations for (3.43) will be second order.

Suppose now that the metric is Minkowski. It turns out that in this case one can construct an explicit solution of the field equations for (3.43). This construction proceeds as follows.

Let $l \neq 0$ be a constant real lightlike covector lying on the forward light cone and let $\boldsymbol{\vartheta}$ be a constant coframe such that $l \perp \boldsymbol{\vartheta}^1$, $l \perp \boldsymbol{\vartheta}^2$; here "constant" means "parallel with respect to the Levi-Civita connection induced by the Minkowski metric". Then, of course,

$$l = c(\boldsymbol{\vartheta}^0 + \boldsymbol{\vartheta}^3) \tag{3.44}$$

where $c > 0$ is some constant (compare with formula (3.4)). Put

$$\begin{pmatrix} \vartheta^0 \\ \vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\varphi & \pm\sin 2\varphi & 0 \\ 0 & \mp\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\vartheta}^0 \\ \boldsymbol{\vartheta}^1 \\ \boldsymbol{\vartheta}^2 \\ \boldsymbol{\vartheta}^3 \end{pmatrix}, \qquad s = \mathrm{const} > 0 \tag{3.45}$$

where $\varphi := \int l \cdot dx$ and x^α are local coordinates. Straightforward calculations show that this coframe ϑ and scalar s are indeed a solution of the field equations for (3.43). We call this solution a *plane wave* with *momentum* l. The upper sign in (3.45) corresponds to the massless neutrino and lower sign corresponds to the massless antineutrino. Note that we can distinguish the neutrino from the antineutrino without resorting to negative energies. Note also that we automatically get only one type of neutrino (left-handed) and one type of antineutrino (right-handed).

Suppose now that we are seeking solutions which are not necessarily plane waves. This can be done using perturbation theory. In the language of spinors perturbation means that we assume the spinor field to be of the form “slowly varying spinor $\times\, e^{-i\varphi}$”. We claim that application of a perturbation argument reduces the quadratic (in torsion) Lagrangian (3.43) to the linear (in torsion) Lagrangian (3.27). At the most basic level this can be explained as follows. Note that for a plane wave we have the following two identities: $T^{\mathrm{ax}}=\pm\frac{4}{3}*l$ and $l=c(\vartheta^0+\vartheta^3)$ (compare the latter with (3.44)). Thus, for a plane wave we have

$$T^{\mathrm{ax}}=\pm\frac{4}{3}c*(\vartheta^0+\vartheta^3). \tag{3.46}$$

We now linearize (in torsion) the quadratic Lagrangian (3.43) about the point (3.46). We get, up to a constant factor, the linear Lagrangian (3.27).

The bottom line is that we believe that the true Lagrangian of a massless neutrino field is the quadratic Lagrangian (3.43). The linear Lagrangian (3.27) (which is equivalent to Weyl’s Lagrangian (3.17)) arises only if one adopts the perturbative approach.

Chapter 4

Dirac Lagrangian

As we mentioned in the previous chapter, we believe that the true Lagrangian for a massless neutrino field is the quadratic Lagrangian (3.43). What about the massive Dirac equation? What happens if we add mass into (3.43)?

For our next set of results we must reduce the dimensionality of the problem. For this chapter only we will be working in (1+2)-dimensional Minkowski spacetime $\mathbb{M}^{1+2}$ with coordinates x^α, $\alpha = 0, 1, 2$, and metric $g_{\alpha\beta} = \operatorname{diag}(+1, -1, -1)$.

The Dirac equation in $\mathbb{M}^{1+2}$ is

$$[\sigma^\alpha{}_{\dot{a}b}(i\partial + A)_\alpha \pm m\sigma^3{}_{\dot{a}b}]\eta^b = 0. \tag{4.1}$$

Here m is the electron mass, σ^α are Pauli matrices (2.17) and A_α is a given external real electromagnetic field. The tensor summation index α runs through the values $0, 1, 2$, the spinor summation index b runs through the values $1, 2$ and the free spinor index $\dot{a}$ runs through the values $\dot{1}, \dot{2}$. The spinor field $\eta : \mathbb{M}^{1+2} \to \mathbb{C}^2$ is the dynamical variable (unknown quantity). The two choices of sign give two versions of the Dirac equation corresponding to spin up and down.

Equations (4.1) are, of course, a special case of the Dirac equation in dimension 1+3. The latter is a system of four complex equations for four complex unknowns and if one looks for solutions which do not depend on x^3 then this system splits into a pair of systems (4.1).

All fields are assumed to be infinitely smooth with no assumptions on their behavior at infinity. We focus on understanding the geometric meaning of equation (4.1) rather than on fitting it into the framework of operator theory.

Our model is based on the Lagrangian (3.43) introduced in the end of the previous chapter except that we will need to introduce some new ideas, concepts and notation. Though the model itself is quite simple, it is not easy to see how it generates the Dirac equation (4.1). The main difficulties are as follows.

- The dynamical variables in our model and the Dirac model are different. We will overcome this difficulty by performing a nonlinear change of dynamical variables given by the explicit formulae (4.19)–(4.21).

- We incorporate mass and electromagnetic field into our model by means of a Kaluza–Klein extension, i.e. by adding an extra spatial dimension and then separating out the extra coordinate x^3. Now, our field equation (Euler–Lagrange equation) will turn out to be nonlinear so the fact that it admits separation of variables is nontrivial. We will establish separation of variables by performing explicit calculations. We suspect that the underlying group-theoretic reason for our nonlinear field equation admitting separation of variables is the fact that our model is $\mathrm{U}(1)$-invariant, i.e. it is invariant under the multiplication of the spinor field by a complex constant of modulus 1. Hence, it is feasible that one could perform the separation of variables without writing down the explicit form of the field equation.

- Our field equation will be second order so it is unclear how it can be reduced to a first order equation (4.1). This issue will be addressed in Appendix A. Namely, in this appendix we prove an abstract lemma showing that a certain class of nonlinear second order partial differential equations reduces to pairs of linear first order equations. To our knowledge, this abstract lemma is a new result.

4.1 What changes have we made to our model?

The coframe ϑ is now a triple of orthonormal covector fields ϑ^j, $j = 0, 1, 2$, in $\mathbb{M}^{1+2}$. Each covector field ϑ^j can be written as before as $\vartheta^j{}_\alpha$ where now the tensor index $\alpha = 0, 1, 2$ enumerates only three components. Of course, orthonormality is understood as before in the Lorentzian sense: the inner product $\vartheta^j \cdot \vartheta^k = g^{\alpha\beta}\vartheta^j{}_\alpha\vartheta^k{}_\beta$ is $+1$ if $j = k = 0$, -1 if $j = k = 1$ or $j = k = 2$, and zero otherwise.

Again we have the orthonormality condition for the coframe, represented as a single tensor identity

$$g = \eta_{jk}\vartheta^j \otimes \vartheta^k \tag{4.2}$$

but where η_{jk} has changed to

$$\eta_{jk} = \eta^{jk} := \mathrm{diag}(+1, -1, -1) \tag{4.3}$$

We view the identity (2.9) as a kinematic constraint: the covector fields ϑ^j are chosen so that they satisfy (2.9), which leaves us with three real degrees of freedom at every point of $\mathbb{M}^{1+2}$. If one views $\vartheta^j{}_\alpha$ as a 3×3 real matrix-function, then condition (2.9) means that this matrix-function is pseudo-orthogonal, i.e. orthogonal with respect to the Lorentzian inner product.

We choose to work with coframes satisfying conditions

$$\det \vartheta^j{}_\alpha = +1 > 0, \qquad \vartheta^0{}_0 > 0 \tag{4.4}$$

which single out coframes that can be obtained from the trivial (aligned with coordinate lines) coframe $\vartheta^j{}_\alpha = \delta^j{}_\alpha$ by proper Lorentz transformations.

As dynamical variables in our amended model we choose the coframe ϑ and a positive density ρ. Our coframe and density are functions of coordinates x^α, $\alpha = 0, 1, 2$, in $\mathbb{M}^{1+2}$. At a physical level, making the density ρ a dynamical variable means that we view our continuum more like a fluid rather than a solid: we allow the material to redistribute itself so that it finds its equilibrium distribution. Note that the total number of real dynamical degrees of freedom contained in the coframe ϑ and positive density ρ is four, exactly as in a two-component complex-valued spinor field η.

4.1.1 Mass and electromagnetism

In order to incorporate into our model mass and electromagnetic field we perform a Kaluza–Klein extension: we extend our original (1+2)-dimensional Minkowski spacetime $\mathbb{M}^{1+2}$ to (1+3)-dimensional Minkowski spacetime $\mathbb{M}^{1+3}$ by adding the extra spatial coordinate x^3. The metric on $\mathbb{M}^{1+3}$ is $\mathbf{g}_{\boldsymbol{\alpha\beta}} = \mathrm{diag}(+1,-1,-1,-1)$. Here and further on we use **bold** type for extended quantities. Say, the use of bold type in the tensor indices of $\mathbf{g}_{\boldsymbol{\alpha\beta}}$ indicates that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ run through the values $0,1,2,3$.

We extend our coframe as

$$\boldsymbol{\vartheta}^j{}_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^j{}_\alpha \\ \\ 0 \end{pmatrix}, \qquad j=0,1,2, \qquad \boldsymbol{\vartheta}^3{}_{\boldsymbol{\alpha}} = \begin{pmatrix} 0_\alpha \\ \\ 1 \end{pmatrix} \tag{4.5}$$

where the bold tensor index $\boldsymbol{\alpha}$ runs through the values $0,1,2,3$, whereas its non-bold counterpart α runs through the values $0,1,2$. In particular, the 0_α in formula (4.5) stands for a column of three zeros.

Our original (1+2)-dimensional coframe ϑ, which was initially a function of (x^0,x^1,x^2) only, is now allowed to depend on x^3 in an arbitrary way, as long as the kinematic constraint (2.9) is maintained. Our only restriction on the choice of extended (1+3)-dimensional coframe $\boldsymbol{\vartheta}$ is the condition that the last element of the coframe is prescribed as the conormal to the original Minkowski spacetime $\mathbb{M}^{1+2}$, see formula (4.5).

We also extend our positive density ρ allowing arbitrary dependence on x^3. We retain the non-bold type for the extended ρ.

The coframe elements $\boldsymbol{\vartheta}^{\mathbf{j}}$ are different at different points $x\in\mathbb{M}^{1+3}$ and this causes deformations. As a measure of these "rotational deformations" we choose *axial torsion* which is the 3-form defined by the formula

$$\mathbf{T}^{\mathrm{ax}} := \frac{1}{3}\mathbf{o}_{\mathbf{jk}}\boldsymbol{\vartheta}^{\mathbf{j}}\wedge d\boldsymbol{\vartheta}^{\mathbf{k}} \tag{4.6}$$

where $\mathbf{o}_{\mathbf{jk}} = \mathbf{o}^{\mathbf{jk}} := \mathrm{diag}(+1,-1,-1,-1)$ and d denotes the exterior derivative on $\mathbb{M}^{1+3}$.

Here “torsion” stands for “torsion of the teleparallel connection” with “teleparallel connection” defined by the condition that the covariant derivative of each coframe element $\boldsymbol{\vartheta}^{\mathbf{j}}$ is zero; see Appendix A of [29] for a concise exposition. “Axial torsion” is the totally antisymmetric part of the torsion tensor.

4.2 The new Lagrangian

We choose the basic Lagrangian density of our mathematical model as

$$L(\boldsymbol{\vartheta},\rho) := \|\mathbf{T}^{\mathrm{ax}}\|^2\rho \tag{4.7}$$

where $\|\mathbf{T}^{\mathrm{ax}}\|^2 = \frac{1}{3!}\mathbf{T}^{\mathrm{ax}}_{\boldsymbol{\alpha\beta\gamma}}\mathbf{T}^{\mathrm{ax}}_{\boldsymbol{\kappa\lambda\mu}}\mathbf{g}^{\boldsymbol{\alpha\kappa}}\mathbf{g}^{\boldsymbol{\beta\lambda}}\mathbf{g}^{\boldsymbol{\gamma\mu}}$. The main motivation behind the choice of Lagrangian density (4.7) is the fact that it is conformally invariant: it does not change if we rescale the coframe as $\boldsymbol{\vartheta}^{\mathbf{j}} \mapsto e^{\mathbf{h}}\boldsymbol{\vartheta}^{\mathbf{j}}$, metric as $\mathbf{g}_{\boldsymbol{\alpha\beta}} \mapsto e^{2\mathbf{h}}\mathbf{g}_{\boldsymbol{\alpha\beta}}$ and density as $\rho \mapsto e^{2\mathbf{h}}\rho$ where $\mathbf{h} : \mathbb{M}^{1+3} \to \mathbb{R}$ is an arbitrary scalar function. At this point it is important to note that our Kaluza–Klein extension procedure does not actually allow for conformal rescalings because the last formula (4.5) is very specific. Thus, our logic is that we choose a Lagrangian density (4.7) which *would be* conformally invariant if not for the prescriptive nature of the Kaluza–Klein construction. This is in line with the view that mass breaks conformal invariance. The electron mass m will appear below in formulae (4.12) and (4.13).

Substituting (4.5) into (4.6) we get

$$\mathbf{T}^{\mathrm{ax}} = T^{\mathrm{ax}} - \boldsymbol{\vartheta}^3 \wedge D_3\vartheta \tag{4.8}$$

where

$$T^{\mathrm{ax}} := \frac{1}{3}\eta_{jk}\vartheta^j \wedge d\vartheta^k \tag{4.9}$$

is the axial torsion in original (1+2)-dimensional spacetime and $D_3\vartheta$ is the 2-form

$$D_3\vartheta := \frac{1}{3}\eta_{jk}\vartheta^j \wedge \partial_3\vartheta^k. \tag{4.10}$$

The 2-form $D_3\vartheta$ characterizes the rotation of the coframe ϑ as we move along the coordinate x^3 and is, in effect, an analogue of angular velocity.

Substituting (4.8) into (4.7) we rewrite our basic Lagrangian density as

$$L(\vartheta,\rho) := (\|T^{\mathrm{ax}}\|^2 + \|D_3\vartheta\|^2)\rho. \tag{4.11}$$

We now incorporate the electron mass m into our model by imposing the periodicity conditions

$$\vartheta(x^0,x^1,x^2,x^3+\pi/m) = \vartheta(x^0,x^1,x^2,x^3), \tag{4.12}$$

$$\rho(x^0,x^1,x^2,x^3+\pi/m) = \rho(x^0,x^1,x^2,x^3). \tag{4.13}$$

Conditions (4.12) and (4.13) mean that we make the coordinate x^3 cyclic with period $\frac{\pi}{m}$. In other words, we effectively roll up our third spatial dimension into a circle of radius $\frac{1}{2m}$.

Finally, we incorporate the prescribed electromagnetic (co)vector potential A into our model by formally adjusting the partial derivatives appearing in the definition of axial torsion (4.9) as

$$\partial_\alpha \mapsto \partial_\alpha + m^{-1}A_\alpha\partial_3\,, \qquad \alpha = 0,1,2\,. \tag{4.14}$$

As a result, our Lagrangian density (4.11) turns into

$$L(\vartheta,\rho) := (\|T^{\mathrm{ax}}_A\|^2 + \|D_3\vartheta\|^2)\rho, \tag{4.15}$$

where

$$T^{\mathrm{ax}}_A := T^{\mathrm{ax}} - m^{-1}A \wedge D_3\vartheta. \tag{4.16}$$

Let us summarize the above construction. The Lagrangian density that we shall be studying is given by formula (4.15) where the 3-form T^{ax}_A and 2-form $D_3\vartheta$ are defined by formulae (4.9), (4.10) and (4.16). The corresponding action (variational functional) is

$$S(\vartheta,\rho) := \int_{\mathbb{M}^{1+3}} L(\vartheta,\rho)\,dx^0dx^1dx^2dx^3\,. \tag{4.17}$$

Of course, the integral in (4.17) need not converge as we will be using it only for the purpose of deriving field equations (Euler–Lagrange equations). Our dynamical variables are the coframe ϑ and density ρ which live in the original (1+2)-dimensional spacetime but depend on the extra spatial coordinate x^3. We seek solutions which are periodic in x^3, see formulae (4.12) and (4.13).

Our field equations are obtained by varying the action (4.17) with respect to the coframe ϑ and density ρ. Varying with respect to the density ρ is easy: this gives the field equation $\|T_A^{\mathrm{ax}}\|^2 + \|D_3\vartheta\|^2 = 0$ which is equivalent to $L(\vartheta, \rho) = 0$. Varying with respect to the coframe ϑ is more difficult because we have to maintain the kinematic constraint (2.9). A technique for varying the coframe with kinematic constraint (2.9) was described in Appendix B of [29] but we do not use it in this thesis.

4.3 Switching to the language of spinors

As pointed out in the previous section, varying the coframe subject to the kinematic constraint (2.9) is not straightforward. This technical difficulty can be overcome by switching to a different dynamical variable. It is known that in dimension 1+2 a coframe ϑ and a positive density ρ are equivalent to a 2-component complex-valued spinor field $\xi = \xi^a = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$ satisfying the inequality

$$\bar{\xi}^{\dot{a}} \sigma_{3\dot{a}b} \xi^b > 0. \tag{4.18}$$

The explicit formulae establishing this equivalence are

$$\rho = \bar{\xi}^{\dot{a}} \sigma_{3\dot{a}b} \xi^b, \tag{4.19}$$

$$\vartheta^0{}_\alpha = \rho^{-1} \bar{\xi}^{\dot{a}} \sigma_{\alpha\dot{a}b} \xi^b, \tag{4.20}$$

$$(\vartheta^1 + i\vartheta^2)_\alpha = \rho^{-1} \epsilon^{\dot{c}\dot{b}} \sigma_{3\dot{b}a} \xi^a \sigma_{\alpha\dot{c}d} \xi^d. \tag{4.21}$$

Here σ are Pauli matrices and ϵ is the "metric spinor" (2.18), the free tensor index α runs through the values $0, 1, 2$, and the spinor summation indices run through the values $1, 2$ or $\dot{1}, \dot{2}$. The advantage of switching to a spinor field ξ is that there are no kinematic constraints on its components, so the derivation of field equations becomes straightforward.

Formulae (4.19)–(4.21) are a variant of those from [40]: in [40] these formulae were written for dimension 3, i.e. for 3-dimensional Euclidean space, whereas in this thesis we write them for dimension 1+2, i.e. for (1+2)-dimensional Minkowski spacetime. Both the formulae from [40] and formulae (4.19)–(4.21) are a special case of those from [64].

Remark 3. *The right-hand sides of formulae (4.19)–(4.21) are invariant under the change of sign of ξ. Hence, the correspondence between coframe and positive density on the one hand and spinor field satisfying condition (4.18) on the other is one to two. A spinor field is, effectively, a square root of a coframe and a density. The fact that the spinor field has indeterminate sign does not cause problems as long as we work on a simply connected open set, such as the whole Minkowski space $\mathbb{M}^{1+2}$. Here and further on, the notions of openness and connectedness of subsets of $\mathbb{M}^{1+2}$ are understood in the Euclidean sense, i.e. in terms of a positive 3-dimensional metric. Note that a similar issue (extraction of a single-valued "square root" of a tensor) arises in the mathematical theory of liquid crystals [9].*

We now need to express the differential forms (4.9), (4.10) and (4.16) via the spinor field ξ. This is done by direct substitution of formulae (4.19)–(4.21) giving

$$*\,T^{\mathrm{ax}}=-\frac{2i(\bar\xi^{\dot a}\sigma^\alpha{}_{\dot ab}\partial_\alpha\xi^b-\xi^b\sigma^\alpha{}_{\dot ab}\partial_\alpha\bar\xi^{\dot a})}{3\bar\xi^{\dot c}\sigma_{3\dot cd}\xi^d}\,, \tag{4.22}$$

$$(*D_3\vartheta)_\alpha=\frac{2i(\bar\xi^{\dot a}\sigma_{\alpha\dot ab}\partial_3\xi^b-\xi^b\sigma_{\alpha\dot ab}\partial_3\bar\xi^{\dot a})}{3\bar\xi^{\dot c}\sigma_{3\dot cd}\xi^d}\,, \tag{4.23}$$

$$*\,T^{\mathrm{ax}}_A=-\frac{2i(\bar\xi^{\dot a}\sigma^\alpha{}_{\dot ab}(\partial_\alpha+m^{-1}A_\alpha\partial_3)\xi^b-\xi^b\sigma^\alpha{}_{\dot ab}(\partial_\alpha+m^{-1}A_\alpha\partial_3)\bar\xi^{\dot a})}{3\bar\xi^{\dot c}\sigma_{3\dot cd}\xi^d}\,. \tag{4.24}$$

The tensor summation index α in formulae (4.22) and (4.24) and the free tensor index α in formula (4.23) run through the values $0,1,2$. Formulae (4.22) and (4.23) are, of course, a variant of those from [40]: we have simply turned 3-dimensional Euclidean space into (1+2)-dimensional Minkowski space and replaced the extra coordinate x^0 with the extra coordinate x^3.

Substituting formulae (4.24) and (4.23) into (4.15) we arrive at the following self-contained explicit spinor representation of our Lagrangian density

$$L(\xi) = -\frac{4}{9\bar\xi^{\dot c}\sigma_{3\dot cd}\xi^d} \Big(\big[i(\bar\xi^{\dot a}\sigma^\alpha{}_{\dot ab}(\partial_\alpha + m^{-1}A_\alpha\partial_3)\xi^b - \xi^b\sigma^\alpha{}_{\dot ab}(\partial_\alpha + m^{-1}A_\alpha\partial_3)\bar\xi^{\dot a})\big]^2 + \big\|i(\bar\xi^{\dot a}\sigma_{\alpha\dot ab}\partial_3\xi^b - \xi^b\sigma_{\alpha\dot ab}\partial_3\bar\xi^{\dot a})\big\|^2\Big). \quad (4.25)$$

Here and further on we write our Lagrangian density and our action as $L(\xi)$ and $S(\xi)$ rather than $L(\vartheta,\rho)$ and $S(\vartheta,\rho)$, thus indicating that we have switched to spinors. The spinor field ξ satisfying condition (4.18) is the new dynamical variable.

The field equation for our Lagrangian density (4.25) is

$$\frac{4i}{3}\Big((*T^{\mathrm{ax}}_A)\sigma^\alpha{}_{\dot ab}(\partial_\alpha + m^{-1}A_\alpha\partial_3)\xi^b + \sigma^\alpha{}_{\dot ab}(\partial_\alpha + m^{-1}A_\alpha\partial_3)((*T^{\mathrm{ax}}_A)\xi^b) - (*D_3\vartheta)_\alpha\sigma^\alpha{}_{\dot ab}\partial_3\xi^b - \sigma^\alpha{}_{\dot ab}\partial_3((*D_3\vartheta)_\alpha\xi^b)\Big) - \rho^{-1}L\sigma_{3\dot ab}\xi^b = 0 \quad (4.26)$$

where the quantities $*T^{\mathrm{ax}}_A$, $*D_3\vartheta$, ρ and L are expressed via the spinor field ξ in accordance with formulae (4.24), (4.23), (4.19) and (4.25).

We seek solutions of the field equation (4.26) which satisfy the periodicity condition

$$\xi(x^0,x^1,x^2,x^3+\pi/m) = \xi(x^0,x^1,x^2,x^3), \quad (4.27)$$

or the antiperiodicity condition

$$\xi(x^0,x^1,x^2,x^3+\pi/m) = -\xi(x^0,x^1,x^2,x^3). \quad (4.28)$$

The above periodicity/antiperiodicity conditions are our original periodicity conditions (4.12) and (4.13) rewritten in terms of the spinor field. The splitting into periodicity/antiperiodicity occurs because the spinor field corresponding to a coframe and a density is determined uniquely modulo sign, see Remark 3.

4.4 Separating out the coordinate x^3

Our field equation (4.26) is highly nonlinear and one does not expect it to admit separation of variables. Nevertheless, we seek solutions of the form

$$\xi(x^0,x^1,x^2,x^3)=\eta(x^0,x^1,x^2)\,e^{\mp imx^3}. \tag{4.29}$$

Note that such solutions automatically satisfy the antiperiodicity condition (4.28): the coframe corresponding to a spinor field of the form (4.29) experiences one full turn (clockwise or anticklockwise) in the $(\vartheta^1,\vartheta^2)$-plane as x^3 runs from 0 to $\frac{\pi}{m}$.

Substituting formula (4.29) into (4.24), (4.23), (4.19) and (4.25) we get

$$*\,T^{\mathrm{ax}}_{A\pm}=-\frac{2(\bar\eta^{\dot a}\sigma^\alpha{}_{\dot ab}(i\partial\pm A)_\alpha\eta^b-\eta^b\sigma^\alpha{}_{\dot ab}(i\partial\mp A)_\alpha\bar\eta^{\dot a})}{3\bar\eta^{\dot c}\sigma_{3\dot cd}\eta^d}\,, \tag{4.30}$$

$$(*D_3\vartheta)_\alpha=\pm\frac{4m\bar\eta^{\dot a}\sigma_{\alpha\dot ab}\eta^b}{3\bar\eta^{\dot c}\sigma_{3\dot cd}\eta^d}\,, \tag{4.31}$$

$$\rho=\bar\eta^{\dot a}\sigma_{3\dot ab}\eta^b, \tag{4.32}$$

$$L_\pm(\eta)=-\frac{16}{9\bar\eta^{\dot c}\sigma_{3\dot cd}\eta^d}\Bigl(\bigl[\tfrac12(\bar\eta^{\dot a}\sigma^\alpha{}_{\dot ab}(i\partial\pm A)_\alpha\eta^b-\eta^b\sigma^\alpha{}_{\dot ab}(i\partial\mp A)_\alpha\bar\eta^{\dot a})\bigr]^2-(m\bar\eta^{\dot a}\sigma_{3\dot ab}\eta^b)^2\Bigr) \tag{4.33}$$

where the signs agree with those in (4.29) (upper sign corresponds to upper sign and lower sign corresponds to lower sign).

Note that the quantities (4.30)–(4.33) do not depend on x^3, which simplifies the next step: substituting (4.29) into our field equation (4.26) and dividing through by the common factor $e^{\mp imx^3}$ we get

$$\frac43\Bigl((*T^{\mathrm{ax}}_{A\pm})\sigma^\alpha{}_{\dot ab}(i\partial\pm A)_\alpha\eta^b+\sigma^\alpha{}_{\dot ab}(i\partial\pm A)_\alpha((*T^{\mathrm{ax}}_{A\pm})\eta^b)\Bigr)+\frac{32m^2}{9}\sigma^3{}_{\dot ab}\eta^b-\rho^{-1}L_\pm\sigma_{3\dot ab}\eta^b=0. \tag{4.34}$$

Observe that formulae (4.30)–(4.34) do not contain x^3. Thus, we have shown that our field equation (4.26) admits separation of variables, i.e. one can seek solutions of the form (4.29).

Consider now the action

$$S_\pm(\eta) := \int_{\mathbb{M}^{1+2}} L_\pm(\eta)\, dx^0 dx^1 dx^2 \tag{4.35}$$

where $L_\pm(\eta)$ is the Lagrangian density (4.33). It is easy to see that equation (4.34) is the field equation (Euler–Lagrange equation) for the action (4.35).

In the remainder of this chapter we do not use the explicit form of the field equation (4.34), dealing only with the Lagrangian density (4.33) and action (4.35). We needed the explicit form of field equations, (4.26) and (4.34), only to justify separation of variables.

We give for reference a more compact representation of our Lagrangian density (4.33) in terms of axial torsion $T^{\mathrm{ax}}_{A\pm}$ (see formula (4.30)) and density ρ (see formula (4.32)):

$$L_\pm(\eta) = -\Big(\big(*T^{\mathrm{ax}}_{A\pm}\big)^2 - \frac{16}{9}m^2\Big)\rho\,. \tag{4.36}$$

Of course, formula (4.36) is our original formula (4.15) with x^3 separated out. The choice of dynamical variables in the Lagrangian density (4.36) is up to the user: one can either use the x^3-independent spinor field η or, equivalently, the corresponding x^3-independent coframe and x^3-independent density (the latter are related to η by formulae (4.19)–(4.21) with ξ replaced by η).

4.5 Main result

Let D_{rs} be the linear differential operator mapping undotted spinor fields into dotted spinor fields in accordance with formula

$$\eta \;\mapsto\; D_{rs}\eta = \sigma^\alpha{}_{\dot a b}(i\partial_\alpha + rA_\alpha)\eta^b + sm\sigma^3{}_{\dot a b}\eta^b \tag{4.37}$$

where the tensor summation index α runs through the values $0, 1, 2$ and the letters r and s take, independently, symbolic values $\pm$ (as in D_{rs}) or numerical values ± 1 (as in the RHS of formula (4.37)), depending on the context.

The main result of this chapter is

Theorem 4.1. *Let Ω be an open (see Remark 3) subset of $\mathbb{M}^{1+2}$ and let $\eta : \Omega \to \mathbb{C}^2$ be a spinor field satisfying the condition*

$$\bar{\eta}^{\dot{a}}\sigma_{3\dot{a}b}\eta^b > 0 \tag{4.38}$$

(compare with (4.18)). Then η is a solution of the field equation for the Lagrangian density L_+ if and only if it is a solution of the Dirac equation $D_{++}\eta = 0$ or the Dirac equation $D_{+-}\eta = 0$, and a solution of the field equation for the Lagrangian density L_- if and only if it is a solution of the Dirac equation $D_{-+}\eta = 0$ or the Dirac equation $D_{--}\eta = 0$.

Proof. Put

$$L_{rs}(\eta) := \frac{1}{2}\left[\bar{\eta}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}(i\partial_\alpha + rA_\alpha)\eta^b - \eta^b\sigma^{\alpha}{}_{\dot{a}b}(i\partial_\alpha - rA_\alpha)\bar{\eta}^{\dot{a}}\right] + sm\bar{\eta}^{\dot{a}}\sigma^3{}_{\dot{a}b}\eta^b. \tag{4.39}$$

This is the Lagrangian density for the Dirac equation $D_{rs}\eta = 0$. Formula (4.39) can be rewritten in more compact form as

$$L_{rs}(\eta) = \Big(-\frac{3}{4} * T^{\mathrm{ax}}_{Ar} + sm\Big)\rho \tag{4.40}$$

where $*T^{\mathrm{ax}}_{Ar}$, $r = \pm$, is the Hodge dual of axial torsion defined by formula (4.30) and ρ is the density defined by formula (4.32). Comparing formulae (4.36) and (4.40) we get

$$L_r(\eta) = -\frac{32m}{9}\,\frac{L_{r+}(\eta)\,L_{r-}(\eta)}{L_{r+}(\eta) - L_{r-}(\eta)}\,. \tag{4.41}$$

Note that the denominator in the above formula is nonzero because condition (4.38) can be equivalently rewritten as $L_{r+}(\eta) > L_{r-}(\eta)$.

The result now follows from formula (4.41) and Lemma 1 (see Appendix A). $\square$

4.6 The sign in the inequality (4.18)

In Section 4.3, when switching to the language of spinors, we chose to work with spinor fields ξ satisfying the inequality (4.18). It is natural to ask what happens if we choose to work with spinor fields $\tilde{\xi}$ satisfying the inequality

$$\bar{\tilde{\xi}}^{\dot{a}}\sigma_{3\dot{a}b}\tilde{\xi}^b < 0. \tag{4.42}$$

One can check that in this case all our arguments can be repeated with minor changes. Namely, in dimension 1+2 a coframe ϑ and a positive density ρ are equivalent to a 2-component complex-valued spinor field $\tilde{\xi}$ satisfying the inequality (4.42), with this equivalence described by a slightly modified version of formulae (4.19)–(4.21). In the end we get an analogue of Theorem 4.1 for such spinors.

In fact, there is no need to repeat our arguments because there is a bijection between spinor fields ξ satisfying the inequality (4.18) and spinor fields $\tilde{\xi}$ satisfying the inequality (4.42):

$$\xi \mapsto \tilde{\xi}^c = \epsilon^{cb}\sigma_{3\dot{a}b}\bar{\xi}^{\dot{a}}, \qquad \tilde{\xi} \mapsto \xi^c = \epsilon^{cb}\sigma_{3\dot{a}b}\bar{\tilde{\xi}}^{\dot{a}}. \tag{4.43}$$

We do not view the transformation (4.43) as physically significant because the primary dynamical variables in our model are the coframe and positive density, not the spinor field. We view the spinor field merely as a convenient change of dynamical variables. If two different spinor fields correspond to the same coframe and positive density we interpret them as the same particle. In group-theoretical language this means that our model is built on the basis of the pseudo-orthogonal group $\mathrm{SO}(1,2)$ rather than the spin group $\mathrm{Spin}(1,2)$.

4.7 Plane wave solutions

In this section we construct a special class of explicit solutions of the field equations for our Lagrangian density (4.15). This construction is presented, initially, in the language of spinors and under the additional assumption that the electromagnetic covector potential A is zero.

We seek solutions of the form

$$\xi(x^0, x^1, x^2, x^3) = e^{-i(p\cdot x + rmx^3)}\zeta \tag{4.44}$$

where $p = (p_0, p_1, p_2)$ is a real constant covector, r takes the values ± 1 and $\zeta \neq 0$ is a constant spinor. We shall call solutions of the type (4.44) *plane wave*. In seeking plane wave solutions what we are doing is separating out all the variables, namely, the original variables $x = (x^0, x^1, x^2)$ (coordinates on $\mathbb{M}^{1+2}$) and the extra variable x^3 (Kaluza–Klein coordinate).

As usual, our spinor field ξ is assumed to satisfy the inequality (4.18). As explained in Section 4.6, this assumption does not lead to the loss of solutions.

Our field equation (4.26) is highly nonlinear so it is not *a priori* clear that one can seek solutions in the form of plane waves. However, plane wave solutions are a special case of solutions of the type (4.29) and these have already been analyzed in preceding sections. Namely, Theorem 4.1 gives us an algorithm for the calculation of all plane wave solutions (4.44) by reducing the problem to Dirac equations

$$D_{rs}\eta = 0 \tag{4.45}$$

for the x^3-independent spinor field

$$\eta(x^0, x^1, x^2) = e^{-ip\cdot x}\zeta. \tag{4.46}$$

Here r is the same as in formula (4.44), i.e. a number taking the values ± 1, and s is another number, also taking, independently, the values ± 1. By D_{rs} we denote the differential operators (4.37).

Clearly, a Dirac equation (4.45) has a nontrivial plane wave solution η if and only if the momentum p satisfies the condition $\|p\|^2 - m^2 = 0$, so p is timelike. Our model is invariant under proper Lorentz transformations of coordinates (x^0, x^1, x^2) so without loss of generality we can assume that

$$p_1 = p_2 = 0. \tag{4.47}$$

Combining formulae (4.37), (2.17), (4.46) and (4.47) we see that the Dirac equation (4.45) takes the form

$$\begin{pmatrix} p_0 - sm & 0 \\ 0 & p_0 + sm \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = 0. \tag{4.48}$$

Equation (4.46) has a nontrivial solution satisfying the inequality (4.18) only if

$$p_0 = sm \tag{4.49}$$

with the corresponding ζ given, up to scaling by a nonzero complex factor, by the formula

$$\zeta^d = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{4.50}$$

Combining formulae (4.44), (4.47), (4.49) and (4.50) we conclude that our model admits, up to a proper Lorentz transformation of the coordinate system in $\mathbb{M}^{1+2}$ and complex scaling, four plane wave solutions and that these plane wave solutions are given by the explicit formula

$$\xi^d = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-im(sx^0+rx^3)} . \tag{4.51}$$

Here the numbers r and s can, independently, take values ± 1.

Let us now rewrite the plane wave solutions (4.51) in terms of our original dynamical variables, coframe ϑ and density ρ. Substituting formulae (2.17) and (4.51) into formulae (4.19)–(4.21) we get $\rho = 1$, $\vartheta^0{}_\alpha = \delta^0{}_\alpha$ and

$$\vartheta^1{}_\alpha = \begin{pmatrix} 0 \\ \cos 2m(sx^0 + rx^3) \\ \sin 2m(sx^0 + rx^3) \end{pmatrix}, \qquad \vartheta^2{}_\alpha = \begin{pmatrix} 0 \\ -\sin 2m(sx^0 + rx^3) \\ \cos 2m(sx^0 + rx^3) \end{pmatrix}. \tag{4.52}$$

In order to distinguish the two spins we fix x^3 and examine how the covectors ϑ^1 and ϑ^2 evolve as a function of time x^0. We say that spin is up if the rotation is counterclockwise and spin is down if the rotation is clockwise. Examination of formula (4.52) shows that we have spin up if $s = +1$ and spin down if $s = -1$.

We will now establish which of the solutions (4.52) describe the electron and which describe the positron. Let us introduce a weak constant positive electric field, $0 < A_0 < m$ and

$A_1 = A_2 = 0$. Then we can repeat the calculation leading up to formula (4.52), but now we get

$$\vartheta^1{}_\alpha = \begin{pmatrix} 0 \\ \cos 2[(sm - rA_0)x^0 + rmx^3] \\ \sin 2[(sm - rA_0)x^0 + rmx^3] \end{pmatrix},$$

$$\vartheta^2{}_\alpha = \begin{pmatrix} 0 \\ -\sin 2[(sm - rA_0)x^0 + rmx^3] \\ \cos 2[(sm - rA_0)x^0 + rmx^3] \end{pmatrix}. \quad (4.53)$$

We define quantum mechanical energy as

$$\varepsilon := |sm - rA_0| \quad (4.54)$$

which is half the angular frequency (as a function of time x^0) of the solution (4.53). Note that our energy (4.54) is by definition positive.

We say that we are dealing with an electron if $\varepsilon < m$ and with a positron if $\varepsilon > m$. Examination of formula (4.54) shows that we are looking at an electron if the signs of r and s are the same and at a positron if the signs of r and s are opposite. This means that the electron is described by a wave traveling in the negative x^3-direction whereas the positron is described by a wave traveling in the positive x^3-direction.

Our classification of plane wave solutions is summarized in Table 4.1.

Table 4.1: Classification of solutions (4.52)

	$s = +1$	$s = -1$
$r = +1$	Electron with spin up	Positron with spin down
$r = -1$	Positron with spin up	Electron with spin down

4.8 Discussion

4.8.1 Problem of vanishing density

The only technical assumption in our analysis is that the density ρ does not vanish. Rephrased in terms of the spinor field, this assumption reads as

$$\bar{\xi}^{\dot{a}}\sigma_{3\dot{a}b}\xi^b \neq 0, \tag{4.55}$$

compare with (4.18) and (4.42). We do not know how to drop the assumption (4.55).

4.8.2 Electron in curved spacetime

One of the advantages of our mathematical model is that it does not use covariant differentiation (only exterior differentiation) so the generalization to the case of a curved (1+2)-dimensional spacetime is absolutely straightforward. Covariant derivatives appear only when we switch from coframe and density to a spinor field. All our analysis, including Theorem 4.1, carries over to the case of curved spacetime. We chose our (1+2)-dimensional spacetime to be flat only to make the exposition clearer.

4.8.3 Rigid Lorentz transformations of the coframe

An interesting feature of our model is that it possesses an additional symmetry which the Dirac equation in dimension 1+2 does not possess. The symmetry in question is invariance under rigid Lorentz transformations of the coframe, i.e. transformations $\vartheta^j \mapsto \tilde{\vartheta}^j = \Lambda^j{}_k\vartheta^k$ where the $\Lambda^j{}_k$ are real constants satisfying the condition $\eta_{ji}\,\Lambda^j{}_k\,\Lambda^i{}_r = \eta_{kr}$, the η's being defined by formula (4.3). In order to see that the Dirac equation in dimension 1+2 is not invariant under rigid Lorentz transformations of the coframe we look at the Dirac Lagrangian density (4.39), switch from a spinor field η to a coframe ϑ and a density ρ which gives us (4.40) and then rewrite formula (4.40) in more explicit form as

$$L_{rs}(\eta) = \Big(-\frac{3}{4} * T^{\mathrm{ax}} + rA\cdot\vartheta^0 + sm\Big)\rho \tag{4.56}$$

where $*T^{\mathrm{ax}}$ is axial torsion in dimension 1+2, see formula (4.9). Clearly, the term $(A\cdot\vartheta^0)\rho$ in formula (4.56) is not invariant under rigid Lorentz transformations of the coframe. This non-invariance is not normally noticed because the covector field $\vartheta^0\rho$ is traditionally interpreted as the electron current, unrelated to any coframe. On the other hand, our model is invariant under rigid Lorentz transformations of the coframe even in the presence of an external electromagnetic field: this fact is established by examination of formulae (4.9), (4.10), (4.15) and (4.16).

How can the two models be mathematically equivalent? The answer is that invariance under rigid Lorentz transformations of the coframe is broken when we separate out the extra coordinate x^3. Namely, the construction described in Section 4.4 assigns a special role to the coframe element ϑ^0: it does not depend on x^3 (this follows from formulae (4.29), (4.19) and (4.20)) whereas the other two elements of the coframe rotate as functions of x^3 (this follows from formulae (4.29), (4.19) and (4.21)).

4.8.4 Our choice of Lagrangian

We chose a very particular Lagrangian density (4.7) containing only one irreducible piece of torsion (axial) whereas in teleparallelism it is traditional to choose a more general Lagrangian containing all three pieces (axial, vector and tensor) of the torsion tensor

$$\mathbf{T} := \mathbf{o_{jk}}\boldsymbol{\vartheta}^{\mathbf{j}} \otimes d\boldsymbol{\vartheta}^{\mathbf{k}}, \tag{4.57}$$

see formula (26) in [72]. Note that when Einstein introduced teleparallelism [117] he failed to identify axial torsion as a separate irreducible piece: his Lagrangian contained only two terms, the square of the full torsion tensor and the square of its vector piece.

In choosing our particular Lagrangian density (4.7) we were guided by the principles of conformal invariance, simplicity and analogy with Maxwell's theory. The analogy with Maxwell's theory is that we characterize the field strength by a differential form, replacing the electromagnetic tensor (2-form) by axial torsion (3-form). It appears that the Lagrangian density (4.7) was never examined.

4.8.5 Exclusion of gravity

We assumed the (1+2)-dimensional metric g to be prescribed (fixed) and the coframe ϑ to be chosen so as to satisfy the kinematic constraint (2.9). As explained in subsection 4.8.2, the fact that we chose the metric g to be Minkowski is irrelevant and all our analysis carries over to the case of an arbitrary Lorentzian metric in dimension 1+2. The important thing is that the metric g is not treated as a dynamical variable. This means that we chose to exclude gravity from our model.

On the other hand, in teleparallelism it is traditional to view the metric as a dynamical variable. In other words, in teleparallelism it is customary to view (2.9) not as a kinematic constraint but as a definition of the metric and, consequently, to vary the coframe ϑ without any constraints. This is not surprising as most, if not all, authors who contributed to teleparallelism came to the subject from General Relativity.

It appears that the idea of working with a coframe subject to the kinematic constraint (2.9) is new.

4.8.6 Density as a dynamical variable

We took the positive density of our continuum to be a dynamical variable whereas in teleparallelism the tradition is to prescribe it as $\rho = \sqrt{|\det g|}$. Taking ρ to be a dynamical variable is, of course, equivalent to introducing an extra real positive scalar field into our model. It appears that the idea of making the density a dynamical variable is also new.

4.8.7 Electron in dimension 1+3

The major outstanding issue is whether we can reformulate the Dirac equation in dimension 1+3 using our approach. This would mean starting from (1+3)-dimensional spacetime, performing a Kaluza–Klein extension to dimension 1+4, choosing the conformally invariant Lagrangian density (4.7) and so on, as described in Section 4.1.

It seems that the equation we get starting from (1+3)-dimensional spacetime and performing the construction described in Section 4.1 is not the Dirac equation in dimension 1+3. Our analysis is heavily dependent on dimension and, when starting from (1+3)-dimensional spacetime, we do not appear to get a factorization of the Lagrangian density of the type (4.41).

However, the equation we get in dimension 1+3, although nonlinear, seems to be very similar to the Dirac equation. The natural way of testing how close our equation is to the Dirac equation would be to calculate the energy spectrum of the electron in a given static electromagnetic field, starting with the case of the Coulomb potential (hydrogen atom).

4.8.8 Similarity with the Ashtekar–Jacobson–Smolin construction

The analysis presented in this chapter exhibits certain similarities with [6, 73] in that a 3-dimensional (or, in our case, (1+2)-dimensional) coframe ϑ is used as a dynamical variable and that a second order partial differential equation is reduced to a first order equation.

Part II

Elko spinor in cosmology.

Chapter 5

Spinors and torsion

In this second part of the thesis we will shift our focus to alternative spinors and their applications to cosmology. In particular we are interested in understanding dark matter and dark energy. Therefore, we consider spinors which are naturally dark, i.e. their interaction with the electromagnetic force is heavily suppressed. We start our investigation with a particular spinor, known as the Elko spinor.

In this chapter we will look at two applications of the Elko spinor. The first is its candidacy for dark energy, then second we will investigate its ability to source torsion which was an open problem in Einstein-Cartan theory.

5.1 A very short introduction to Elko spinors

Elko spinors [4] are similar to Majorana spinors but acquire the full four degrees of freedom of a Dirac spinor due to their helicity structure. They couple to the Higgs mechanism via $\overset{\neg}{\lambda}\lambda H^\dagger H$ and weakly to the electromagnetic field via $\overset{\neg}{\lambda}[\gamma^a, \gamma^b]\lambda F_{ab}$, however in the latter case this coupling is heavily constrained because of the masslessness of the photon, making them a candidate for dark matter. The idea of one field explaining both dark matter and dark energy has already been discussed in various approaches, see e.g. [27, 33]. One possible mass range for the Elko spinors is in the $m \simeq \mathrm{MeV}$ range.

These spinors belong to a wider class of so-called flagpole spinors [44]. They are non-standard spinors according to the Wigner classification and obey the unusual property

$(CPT)^2 = -\mathbb{1}$. Elko spinors are defined by

$$\lambda = \begin{pmatrix} \pm\sigma_2 \phi_L^* \\ \phi_L \end{pmatrix} , \tag{5.1}$$

where ϕ_L^* denotes the complex conjugate of ϕ_L and σ_2 denotes the second Pauli matrix. For a detailed treatment of the field theory of the eigenspinors of the charge conjugation operator we refer the reader to [4, 3]. Dark spinors have an imaginary bi-orthogonal norm with respect to the standard Dirac dual $\bar{\psi} = \psi^\dagger \gamma^0$, and in order for a consistent field theory to emerge the dual is defined to be

$$\overset{\neg}{\lambda}_u = i\, \varepsilon_u^v \lambda_v^\dagger \gamma^0 \,, \tag{5.2}$$

with $\varepsilon_{\{+,-\}}^{\{-,+\}} = -1 = -\varepsilon_{\{-,+\}}^{\{+,-\}}$ such that

$$\overset{\neg}{\lambda}_u(\mathbf{p}) \lambda_v(\mathbf{p}) = \pm\, 2m\, \delta_{uv} \,, \tag{5.3}$$

where $\mathbf{p}$ denotes the momentum.

Due to their formal structure Elko spinors allow for many interesting applications. For instance, in [25] it has been shown that Elko spinors naturally yield an anisotropic expansion in the context of cosmological Bianchi type I models. This allows for a suppression of the low multipole amplitude of the primordial power spectrum. The primordial power spectrum of the quantum fluctuations of Elko spinors has been investigated in [24, 61] where is was found that the small scale power spectrum essentially agrees with that of scalar field inflation while the large scale power spectrum shows new features.

5.2 Dark energy

An increasing number of independent observations indicates that we are living in an expanding universe where the expansion itself is accelerating [111, 114, 98]. It has been accepted that this requires some additional negative-pressure matter source, named dark energy. The simplest model explaining this accelerated expansion is the cosmological constant Λ which corresponds to an unusual equation of state $w = P/\rho = -1$. The Λ cold dark matter (ΛCDM) model (the standard model of cosmology) fits the present data very well. However, the numerical value of the cosmological constant is about 120 orders of magnitude smaller than the vacuum expectation value predicted by quantum field theory. This smallness problem can be addressed by considering dynamical models. The field slowly rolls down some potential, and the effective equation of state w_{eff} converges to $w_{\text{eff}} = -1$. Originally it was believed that this value should be approached from above. Recently there has been interest in phantom models where the dark energy equation of state is approached from below: $w \leq -1$, see [31, 32, 109, 91, 57, 60, 47, 41, 112, 76, 121, 75, 113, 110, 33, 90]. These models, although counter intuitive, are not excluded by current data [31, 32].

Figure 5.1, taken from [32], shows data taken from the cluster abundance, supernovae, quasar-lensing statistics and the first acoustic peak in the cosmic microwave background (CMB) radiation power spectrum. Together, they imply a convergence of the equation to one dominated by dark energy. Also, when the parameter space is expanded to include $w \leq -1$ (phantom region), the data does not rule out w converging to $w = -1$ from this region.

A universe dominated by phantom energy is very different to any we are accustomed to. The scale factor increases at a rate quicker than that of the horizon, and it is not long before gravitationally bound objects are pulled apart. Finally, the same fate is met by objects bound by the three stronger forces. Due to the success of the ΛCDM model (constant equation of state), any theory based on a dynamical equation of state would be required to reproduce the results of ΛCDM for present time. In other words, w must approach the value $w = -1$: either from $w > -1$ or $w < -1$. The majority of dynamical dark energy models are based on evolving scalar fields with a suitably chosen potential. One limitation of scalar field theories is that they are unable to cross the phantom divide without acquiring pathologies, such as negative kinetic energy.

This topic falls under the umbrella of modified gravity, which splits into two main categories: amending the geometrical (left-hand) side or the matter content (right-hand) side of

Einstein's field equations. The former requires altering gravity (changing the action), and the latter populating the universe with alternative species. Those two approaches are not entirely independent as many modified theories bring new geometrical quantities to the matter side, ultimately changing the energy content of the universe, and allowing for a new interpretation.

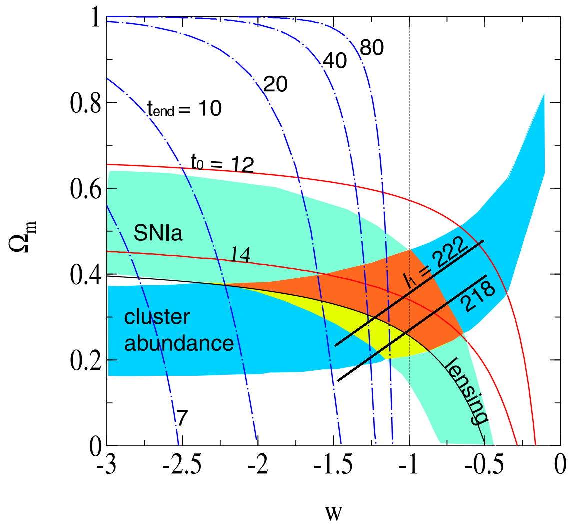

Figure 5.1: Current constraints to the $w - \Omega_m$ parameter space. The red solid curves show the age (in Gyr) of the Universe today (assuming a Hubble parameter $H_0 = 70\,\mathrm{km\,sec^{-1}\,Mpc^{-1}}$). The light shaded regions are those allowed (at 2σ confidence level) by the observed cluster abundance and by current supernova measurements of the expansion history. The dark orange shaded region shows the intersection of the cluster-abundance and supernova curves, additionally restricted (at 2σ confidence level) by the location of the first acoustic peak in the cosmic-microwave-background power spectrum and quasar-lensing statistics.

5.2.1 Cosmological Elko spinor field equations

We now introduce the standard model for cosmology, i.e. curvature and no torsion. Later in the chapter we will add torsion and define the Einstein-Cartan model. The standard model of cosmology is based upon the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \tag{5.4}$$

where $a(t)$ is the scale factor and t is cosmological time. The dynamical behavior of the universe is determined by the cosmological field equations of general relativity

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{1}{M_{\rm pl}^2}T_{\alpha\beta}, \tag{5.5}$$

where $M_{\rm pl}$ is the Planck mass which we use as the coupling constant, $1/M_{\rm pl}^2 = 8\pi G$ and $c = 1$. $T_{\alpha\beta}$ denotes the stress-energy tensor, which for a homogeneous and isotropic cosmology takes the form

$$T_{\alpha\beta} = \text{diag}(\rho, a^2P, a^2P, a^2P). \tag{5.6}$$

The cosmological field equations can be written as

$$H^2 = \frac{1}{3M_{\rm pl}^2}\rho, \tag{5.7}$$

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{5.8}$$

The dot denotes differentiation with respect to time t and the Hubble parameter H is defined by $H = \dot{a}/a$.

Let us consider a homogeneous single Elko spinor field. Following [24, 61], the effective Lagrangian density of this field can be written in terms of the *scalar* field φ as

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 + \frac{3}{8}H^2\varphi^2 - V(\varphi). \tag{5.9}$$

If the potential $V(\varphi)$ contains a standard mass term $V(\varphi) = m^2\varphi^2/2$, then we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 + \frac{3}{8}H^2\varphi^2 - \frac{1}{2}m^2\varphi^2. \tag{5.10}$$

This allows us to interpret the explicit presence of the Hubble parameter in the action as an effective mass term where the mass changes as the universe evolves, and we have

$$m_{\rm eff}^2 = m^2 - \frac{3}{4}H^2. \tag{5.11}$$

It is interesting to note that if one converts back to normal units then the second term is of the order 1×10^{-39} MeV. Therefore, the change in mass is tiny. If the universe undergoes a phase of accelerated expansion, the Hubble parameter is approximately constant. Depending on the ratio m/H, it is possible for models to attain a negative value for m_{eff}^2 without creating ghosts which have negative kinetic energy. This arises as a direct consequence of the extra coupling a spinor has, in addition to that of a scalar field, to geometry.

The energy density and the pressure of the Elko spinor field are given by

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) - \frac{3}{8}H^2\varphi^2, \tag{5.12}$$

$$P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) + \frac{1}{8}H^2\varphi^2. \tag{5.13}$$

These two equations have the important property of leaving the acceleration equation unchanged,

$$\frac{\ddot{a}}{a} = -\frac{1}{3M_{\text{pl}}}(\dot{\varphi}^2 - V(\varphi)). \tag{5.14}$$

The spinor field's potential energy may yield an accelerated expansion of the universe. It should be noted that the energy density and the pressure now explicitly depend on the Hubble parameter. These additional terms are present because the covariant derivative has more structure when acting on a spinor field. As mentioned before the 'coupling' in Eq. (5.9) can be interpreted as either the effective mass of the particle depending on the Hubble parameter [24], and therefore on the evolution of the universe, or, alternatively, regarding the gravitational coupling as time dependent [61].

The effective equation of state of the Elko spinor field is given by

$$w_{\text{eff}} = \frac{P_\varphi}{\rho_\varphi} = \frac{\frac{1}{2}\dot{\varphi}^2 - V(\varphi) + \frac{1}{8}H^2\varphi^2}{\frac{1}{2}\dot{\varphi}^2 + V(\varphi) - \frac{3}{8}H^2\varphi^2}. \tag{5.15}$$

When compared with the scalar field, (5.15) also demonstrates that crossing the phantom divide is possible without attaining a negative kinetic energy term.

We will restrict our attention to power counting renormalizable potentials. As the Elko spinor field has mass dimension one, the two allowed potentials are

$$V_1(\varphi) = \frac{1}{2}m^2\varphi^2, \tag{5.16}$$

and

$$V_2(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\alpha\varphi^4, \tag{5.17}$$

where the $V_1(\varphi)$ is the aforementioned canonical mass term, and $V_2(\varphi)$ includes the self interaction term. Finally, α is a dimensionless coupling constant.

5.3 Dark spinors as dark energy

We start by solving Eq. (5.7) for the Hubble parameter. Using Eq. (5.12) we find

$$H = \frac{1}{\sqrt{3}M_{\rm pl}} \frac{\sqrt{\dot{\varphi}^2/2 + V(\varphi)}}{\sqrt{1 + (\varphi/M_{\rm pl})^2/8}}. \tag{5.18}$$

The energy density of the Elko spinor field can be written as

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) - \frac{1}{8}\frac{\dot{\varphi}^2/2 + V(\varphi)}{1 + (\varphi/M_{\rm pl})^2/8}(\varphi/M_{\rm pl})^2 \tag{5.19}$$

$$= \Big(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\Big)\Big(1 - \frac{(\varphi/M_{\rm pl})^2/8}{1 + (\varphi/M_{\rm pl})^2/8}\Big). \tag{5.20}$$

It is precisely this latter form of the energy density which motivated [61] to interpret (5.20) as inducing a time-dependent gravitational coupling by considering $G_{tt} = 8\pi G_{\rm eff}\bar{\rho}_\varphi$ where $\bar{\rho}_\varphi$ is the standard energy density of a scalar field.

Now, we consider the conservation equation (5.8) with (5.18) and (5.20) and numerically solve the resulting equation for $\varphi(t)$ and substitute into Eqs. (5.12) and (5.13) to obtain the evolution of the effective equation of state $w_{\rm eff} = P/\rho$ and plot it as a function of the evolution parameter $a(t)$.

5.3.1 Phantom dark energy models

All of our results are in graphical form. They demonstrate that the Elko spinor has as a solution a late time convergence to that of a ΛCDM model. This would, as part of a larger class of evidence, be needed in order to qualify as a dark energy candidate.

Before we show the results we must discuss the initial conditions. We chose our initial conditions to be $w(0) = \{1/3, 0, -1/3, -2/3\}$, the first two representing radiation and dust, respectively. The initial conditions with $w(0) \leq -1/3$ correspond to an initially accelerating universe. Small changes in these initial conditions do not alter the late-time asymptotic behaviour of the solutions. We have three classes of solutions: converging, diverging and oscillating.

5.3.2 Converging models

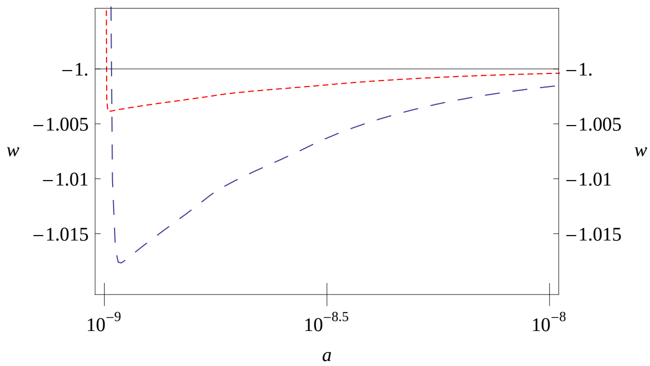

Figure 5.2: Equation of state for $V_1(\varphi)$: $M_{\rm pl} = 1$, $\dot\varphi(0) = 1$ and $w(0) = -1/3$. With $m^2 = \{0.002, 0.001\} = \{\text{red (higher)}, \text{blue (lower)}\}$

Fig. 5.2 shows the dynamical behaviour of the effective equation of state considering the potential V_1 with $M_{\rm pl} = 1$, $\dot\varphi(0) = 1$ and $w(0) = -1/3$. For the two different mass values it is possible to see that the effective equation of state almost immediately drops below the phantom divide. During the subsequent evolution, w begins to increase as further shown by Fig. 5.3 to the desired dark energy value. From Fig. 5.3 it is also evident that our model is practically indistinguishable from dark energy modelled by a cosmological constant, long before recombination when the scale factor is about $a(t) = 10^{-3}$.

We obtained very similar results for other initial values of the equation of state: $w(0) = 1/3$, $w(0) = 0$ and $w(0) = -2/3$. We have shown, for comparison, results from $w(0) = 1/3$ in Fig. 5.4 and Fig. 5.5; they qualitatively agree with the results presented in Figs. 5.2 and 5.3,

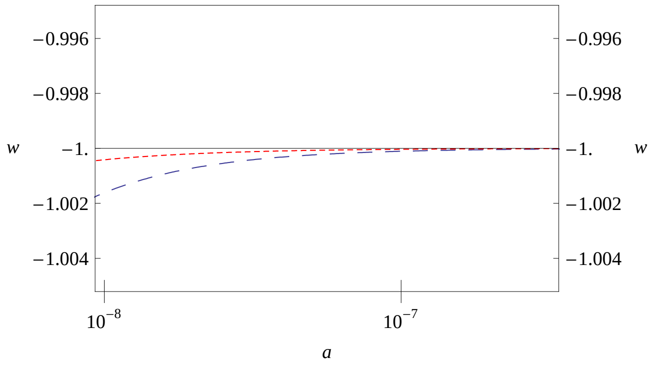

Figure 5.3: Equation of state for $V_1(\varphi)$: $M_{\rm pl} = 1$, $\dot\varphi(0) = 1$ and $w(0) = -1/3$. With $m^2 = \{0.002, 0.001\} = \{\text{red (higher)}, \text{blue (lower)}\}$.

respectively.

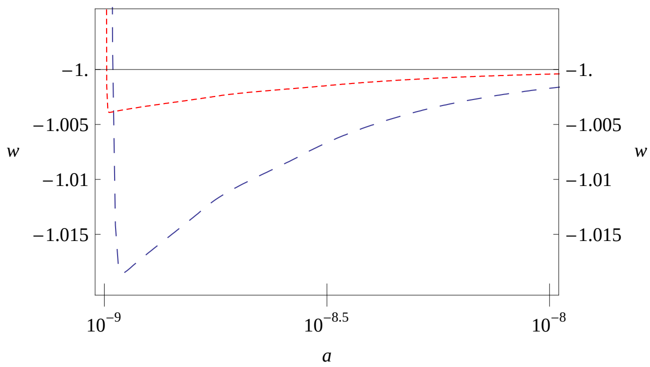

Figure 5.4: Equation of state for $V_1(\varphi)$: $M_{\rm pl} = 1$, $\dot\varphi(0) = 1$ and $w(0) = 1/3$. With $m^2 = \{0.002, 0.001\} = \{\text{red (higher)}, \text{blue (lower)}\}$.

5.3.3 Diverging models

Next, we added a self-interaction term to the potential and used $V_2(\varphi)$. Interestingly, we found, for all initial values of w and $\alpha = 1$, that the effective equation of state always diverges to $-\infty$, see Fig. (5.6). Also, we checked, although not included in here, that our numerical results for $V_2(\varphi)$ converge to results for $V_1(\varphi)$ as $\alpha \to 0$. Although it might be possible to construct models with finely tuned initial conditions such that the divergence of the equation of state would happen in the future, we believe such models are very unlikely. Hence, we are led to conclude that a dynamical dark energy model based on our Elko spinors requires their potential to be of the simplest form, namely a canonical mass term, without self interaction.

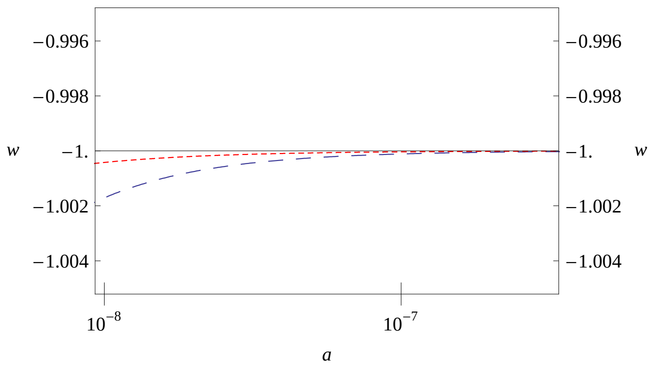

Figure 5.5: Equation of state for $V_1(\varphi)$: $M_{\text{pl}} = 1$, $\dot{\varphi}(0) = 1$ and $w(0) = 1/3$. With $m^2 = \{0.002, 0.001\} = \{\text{red (higher)}, \text{blue (lower)}\}$.

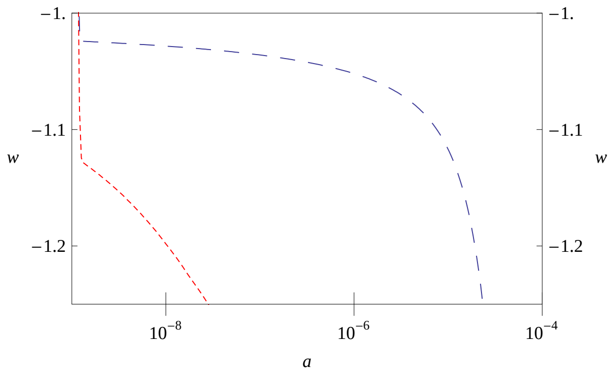

Figure 5.6: Equation of state for $V_2(\varphi)$: $M_{\text{pl}} = 1$, $w(0) = 1/3$ and $\alpha = 1$. With $m^2 = \{4, 0.02\} = \{\text{red (higher)}, \text{blue (lower)}\}$.

5.3.4 Oscillating models

Lastly, we found another set of interesting results where the equation of state oscillated between $w = 1$ and $w = -1$ for all time. The oscillation of the equation of state is very rapid, as can be seen in Fig. 5.7. This doesn't agree with current observations. Therefore these oscillating models are unphysical. This qualitative behavior does not change if we include the self-interaction term. However, if such a model could be modified it would be a prime candidate for models where the field changes its characteristic from being dark matter at early times to become dark energy at late times, see also [38, 96, 86, 127, 19] .

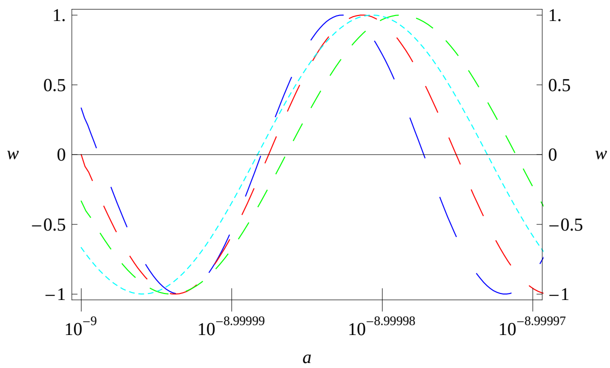

Figure 5.7: Equation of state for potential $V_1(\varphi)$: $M_{\rm pl} = 1$, $m = 0.1$ with initial conditions chosen such that $w(0) = \{1/3, 0, -1/3, -2/3\}$, respectively $\{\text{blue(long dashed), red (medium dashed), green (dashed), cyan (short dashed)}\}$

5.3.5 Discussion

An Elko spinor field is able to provide a possible model for dark matter as it couples mainly via the Higgs mechanism, but has heavily constrained interactions with the electromagnetic field. Dark spinors have a predicted $\rm MeV$ mass range and therefore experimental predictions can be formulated and possibly measured at the LHC. Our results now show that the Elko spinor field is also capable of having a dynamical equation of state which crosses the phantom divide and asymptotes to $w = -1$. This makes it a viable candidate for dark energy which cannot be ruled out experimentally.

Unlike previous phantom models, Elko spinors do not obtain negative kinetic energy on crossing the phantom divide, due to both ρ and P depending on the Hubble parameter, and therefore these models do not create ghosts. According to [32] the equation of state must not stay below the divide but converge to dark energy, therefore the Elko spinors' potential is of the simplest form, a canonical mass term $m^2\varphi^2/2$. Our Elko spinor model does not require a modification of general relativity, leaving one of the most successful models in theoretical physics untouched.

Due to the interesting nature of Elko spinors, they have been shown to give other unique properties not found with other matter sources considered in the past. For now, in a cosmological setting, Elko spinors are providing intriguing results in having the potential to be the best candidate dynamical dark energy model at hand.

5.4 Elko as a source of torsion

Due to their formal structure, Elko spinors couple differently to gravitation from scalar fields or Dirac spinors [22], eigenspinors of the parity operator. This allows for many interesting applications. For instance, in [25] it has been shown that Elko spinors naturally yield an anisotropic expansion in the context of cosmological Bianchi type I models. This allows for a suppression of the low multipole amplitude of the primordial power spectrum. The primordial power spectrum of the Elko field quantum fluctuations has been investigated in [24, 61] where it was found that the small scale power spectrum is almost in agreement with that of scalar field inflation while the large scale power spectrum shows new features.

General relativity is a successful theory in agreement with a vast number of observations. It is based on the Einstein-Hilbert action which yields the field equations if varied with respect to the metric. If, however, the metric and the connection (more precisely the non-Riemannian part of the connection) are considered as *a priori* independent variables, two field equations are obtained. The first one relates the Einstein tensor (not necessarily symmetric) to the canonical energy-momentum tensor, while the other field equation relates the skew-symmetric part of the connection, the torsion tensor, to the spin angular momentum of matter, see e.g. [67, 68, 69, 70, 66, 115]. Spin and torsion are related by algebraic equations, and torsion vanishes in the absence of sources.

The cosmological principle states that the universe is homogeneous and isotropic on very large scales. More mathematically speaking, the four dimensional spacetime (M, g) is defined by $3d$ space-like hypersurfaces of constant time which are orbits of a Lie group G action on M, with isometry group $SO(3)$. We assume all fields to be invariant under the action of G which means $\mathcal{L}_\xi g_{\mu\nu} = 0$ and $\mathcal{L}_\xi T_{\mu\nu}{}^\lambda = 0$ where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the generator of the group. This assumption reduces the cosmological metric to the well known Friedman-Lemaître-Robertson-Walker form which is characterized by the scale factor and the geometry of the constant time hypersurfaces. If applied to the torsion of spacetime, it reduces the components compatible with the cosmological principle to a spatial axial torsion and a vector torsion part [116].

Cosmological models with torsion were pioneered by Kopczyński in [81, 82], who assumed a Weyssenhoff fluid [124] to be the source of both curvature and torsion. The cosmological principle was first extended to Einstein-Cartan theory in [116], where it was also suggested to reconsider the results in [81, 82], since the Weyssenhoff fluid turns out to be incompatible with the cosmological principle (see also [94, 14, 28]). An elaborate analysis of the most general action up to quadratic terms in curvature and torsion, assuming the cosmological principle, can be found in [59]. Analytical solutions of the Riemann-squared gravity have recently been discussed in a cosmological context in [83]. Non-Riemannian models of cosmology in general have been discussed in [101, 100, 102, 103]

We will investigate the Einstein-Cartan action in the next section.

5.5 Einstein-Cartan theory with Elko spinors

The action of Einstein-Cartan gravity is

$$S = \int \Big(\frac{M_{\rm pl}^2}{2} R + \mathcal{L}_{\rm mat}\Big)\sqrt{-g}\, d^4x, \tag{5.21}$$

where R is the Ricci scalar computed from the complete connection with contortion contributions, g is the determinant of the metric, $\mathcal{L}_{mat}$ denotes the matter Lagrangian and $1/M_{\rm pl}^2 = 8\pi G$ is the coupling constant; the speed of light is set to one $(c = 1)$. The resulting field equations are

$$G_{ij} = R_{ij} - \frac{1}{2}R g_{ij} = \frac{1}{M_{\rm pl}^2}\,\Sigma_{ij}, \tag{5.22}$$

$$T^{ij}{}_k + \delta^i_k T^j{}_l{}^l - \delta^j_k T^i{}_l{}^l = M_{\rm pl}^2\,\tau^{ij}{}_k, \tag{5.23}$$

where $\tau^{ij}{}_k$ is the spin angular momentum tensor, defined by

$$\tau_k{}^{ji} = \frac{\delta \mathcal{L}_{\mathrm{mat}}}{\delta K_{ij}{}^k}, \tag{5.24}$$

and Σ_{ij} is the total energy-momentum tensor

$$\Sigma_{ij} = \sigma_{ij} + (\nabla_k - K_{lk}{}^l)(\tau_{ij}{}^k - \tau_j{}^k{}_i + \tau^k{}_{ij}), \tag{5.25}$$

where σ_{ij} is metric energy-momentum tensor

$$\sigma_{ij} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{\mathrm{mat}})}{\delta g^{ij}}. \tag{5.26}$$

The field equations (5.23) are in general 24 algebraic equations, and in the absence of spin sources torsion vanishes, torsion does not propagate.

We have not included the cosmological constant in the field equations for simplicity. It should be noted, however, that there exist models where the cosmological constant might be induced by the torsion of spacetime. Likewise, torsion could contribute to the bare cosmological constant and yield today's observed effective cosmological term, see e.g. [7, 20, 126] and also [30] for a spinorial dark energy model.

5.6 Cosmological field equations with torsion

Current observations [104, 99] suggest that the energy density of the universe is very close to the critical density, resulting in spatially flat hypersurfaces. The flat FLRW metric is

$$ds^2 = dt^2 - a(t)^2\big(dx^2 + dy^2 + dz^2\big), \tag{5.27}$$

where $a(t)$ is the scale factor. It yields the following non-vanishing holonomic Christoffel symbol components

$$\begin{aligned} \Gamma^x_{tx} = \Gamma^y_{ty} = \Gamma^z_{tz} &= \frac{\dot{a}}{a}\,, \\ \Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} &= a\dot{a}\,, \end{aligned} \tag{5.28}$$

where the dot denotes differentiation with respect to t. This then implies the following non-vanishing anholonomic Christoffel symbols Γ_n to be

$$\Gamma_n = -\frac{1}{2}\frac{\dot{a}}{a}(\gamma^0\gamma^n - \gamma^n\gamma^0) = -2\frac{\dot{a}}{a}f^{0n}, \tag{5.29}$$

$$n = 1, 2, 3. \tag{5.30}$$

When the cosmological principle is applied to the torsion tensor [116, 59] the allowed components reduce to

$$T_{110} = T_{220} = T_{330} = h(t), \tag{5.31}$$

$$T_{123} = T_{312} = T_{231} = f(t). \tag{5.32}$$

The cosmological Einstein tensor with torsion is now given by

$$G_{tt} = 3\frac{\dot{a}}{a}\Big(\frac{\dot{a}}{a} + 2h\Big) + 3h^2 - 3f^2, \tag{5.33}$$

$$G_{xx} = a^2\Big(-2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a}\big(\frac{\dot{a}}{a} + 4h\big) - 2\dot{h} - h^2 + f^2\Big), \tag{5.34}$$

$$G_{xx} = G_{yy} = G_{zz}. \tag{5.35}$$

In addition to the geometry, the matter has to be compatible with homogeneity and isotropy. This yields two classes of Elko spinors, Elko ghost spinors which satisfy $\stackrel{\neg}{\lambda}\lambda = 0$ and standard Elko spinors where $\stackrel{\neg}{\lambda}\lambda \neq 0$. The name ghost spinors refers to the fact that such spinors lead to a vanishing metric energy-momentum tensor, and hence do not affect the curvature of spacetime in general relativity, see also [62, 63, 49, 23]. A cosmological ghost spinor field can be written in the form

$$\lambda_{\{-,+\}} = \varphi(t)\,\xi, \tag{5.36}$$

$$\lambda_{\{+,-\}} = \varphi(t)\,\zeta, \tag{5.37}$$

where ξ and ζ are two linearly independent constant spinors given by

$$\xi = \begin{pmatrix} 0 \\ \pm i \\ 1 \\ 0 \end{pmatrix}, \qquad \zeta = i \begin{pmatrix} \mp i \\ 0 \\ 0 \\ -1 \end{pmatrix}, \tag{5.38}$$

with their respective dual spinors

$$\begin{aligned} \stackrel{\neg}{\xi} &= i \begin{pmatrix} 0 & i & \pm 1 & 0 \end{pmatrix}, \\ \stackrel{\neg}{\zeta} &= \begin{pmatrix} -i & 0 & 0 & \mp 1 \end{pmatrix}. \end{aligned} \tag{5.39}$$

The set of 24 algebraic equations (5.23) reduces to two independent equations relating spin and torsion if we assume homogeneity and isotropy. The torsion functions f and h can therefore be expressed [1] in terms of the matter

$$h = -\frac{\varphi^4/M_{\rm pl}^4}{4+\varphi^4/M_{\rm pl}^4}\,\frac{\dot{a}}{a}, \tag{5.40}$$

$$f = -\frac{2\varphi^2/M_{\rm pl}^2}{4+\varphi^4/M_{\rm pl}^4}\,\frac{\dot{a}}{a}, \tag{5.41}$$

which can be combined to give

$$\frac{h}{f} = \frac{1}{2}\varphi^2/M_{\rm pl}^2. \tag{5.42}$$

Therefore, an Elko ghost spinor field satisfying the cosmological principle indeed yields non-trivial contributions to the spatial axial torsion component and to the time component of the torsion vector. Hence, the spin angular momentum tensor induced by this matter source satisfies homogeneity and isotropy.

The total energy-momentum tensor Σ_{ij} for the Elko spinor matter is given by

$$\Sigma_{tt} = V_0, \tag{5.43}$$

$$\Sigma_{xx} = -a^2 V_0 + a^2\varphi^2\Big(3h - \frac{\dot{f}}{f} - 2\frac{\dot{\varphi}}{\varphi}\Big)f, \tag{5.44}$$

$$\Sigma_{xx} = \Sigma_{yy} = \Sigma_{zz}, \tag{5.45}$$

[1] These computations were performed using the software Mathematica

where $V_0 = V(0)$. This completes the formulation of the cosmological field equations. Next, we investigate the qualitative behavior of the equations of motion.

The geometrical part of the cosmological field equations (5.33)–(5.35) can, for example, be read off from [59] (cf their action L_4) which we verified. In Ref. [87], where $h = 0$ was assumed, the geometry parameter k was redefined to include the remaining torsion by $\bar{k} = k - f^2a^2/2$, see also [21].

5.7 Cosmological Elko spinor dynamics

The complete set of field equations can be reduced to a single first order differential equation in the following manner. First, all torsion functions in the field equations are written in terms of the spin tensor (5.41), thereby eliminating torsion f and h for the matter field φ. Next, we can use Eq. (5.33) and the derivative of that equation to find expressions for $\dot{a}/a$ and $\ddot{a}/a$ which are expressed entirely in terms of the matter field φ. We analyze these equations qualitatively and solve them numerically.

For the Hubble parameter $H = \dot{a}/a$ from Eq. (5.33) we find

$$H = \frac{\sqrt{V_0/M_{\rm pl}^2}}{2\sqrt{3}} \frac{4 + \varphi^4/M_{\rm pl}^4}{\sqrt{4 - \varphi^4/M_{\rm pl}^4}}. \tag{5.46}$$

Next, the terms with $\ddot{a}/a$, $\dot{a}/a$ and f and h are eliminated for φ in the spatial component of the field equation which results in

$$\frac{\dot{\varphi}}{\varphi} = -\frac{\sqrt{V_0/M_{\rm pl}^2}}{4\sqrt{3}} \frac{8 + 3\varphi^4/M_{\rm pl}^4}{12 - \varphi^4/M_{\rm pl}^4} \sqrt{4 - \varphi^4/M_{\rm pl}^4}. \tag{5.47}$$

Positivity of the square root requires $\varphi/M_{\rm pl} < \sqrt{2}$. This implies that the sign of the first derivative of the field cannot change, and hence the field value is a decreasing function of time and in fact quickly approaches zero. When this happens, the Hubble parameter asymptotes to a constant value and the universe expands according to $a \propto \exp(Ht)$.

To see this behaviour of the solutions qualitatively, let us expand Eqs. (5.46) and (5.47) about $\varphi = 0$ which leads to

$$H = \sqrt{\frac{V_0}{3M_{\rm pl}^2}} + O(\varphi/M_{\rm pl})^4, \tag{5.48}$$

$$\frac{\dot{\varphi}}{\varphi} = -\frac{1}{3}\sqrt{\frac{V_0}{3M_{\rm pl}^2}} + O(\varphi/M_{\rm pl})^4, \tag{5.49}$$

and therefore we find that a period of accelerated expansion is an attractor solution of this system of equations. Taking into account Eq. (5.41), we also find that the torsion of spacetime is quickly decreasing and approaching zero as the universe expands.

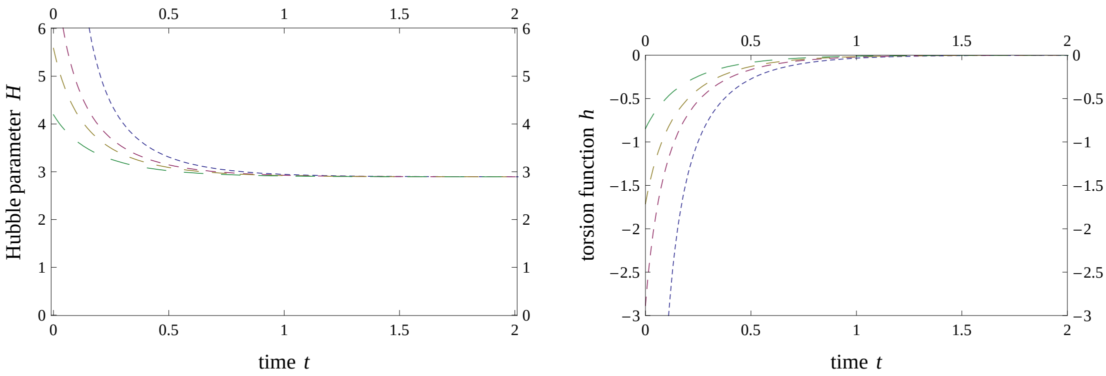

Figure 5.8: Left: Hubble parameter and right: torsion function h for $1/M_{\rm pl}^2 = 8\pi$ and $V_0 = 1$. Initial conditions of the matter field are $\varphi_i = \varphi(t = 0) = \{0.282, 0.25, 0.23, 0.20\}$, {blue (short dashed), red (dashed), (medium dashed) yellow and green (long dashed) }

.

Such a behaviour of the torsion is not unexpected, see e.g. [10]. Spinors and inflation in the context of torsion theories have received much attention in the past [56, 54, 39, 93, 77, 55, 21]. It should be pointed out, however, that matter sources considered previously violate the cosmological principle.

We numerically solve the first order differential equation (5.47) and use this solution to find the evolution of the Hubble parameter - we plot the Hubble parameter in Fig. 5.8a - which approaches a constant for different initial conditions of the field. In Fig. 5.8b the torsion function h is plotted for the same numerical solutions.

In order to give a qualitative statement about the decay rate of the torsion, in Fig. 5.9 we plot the torsion function h as a function of the number of e-foldings. We assume the total number of e-foldings to be sixty. Therefore, the torsion contribution of the spacetime becomes

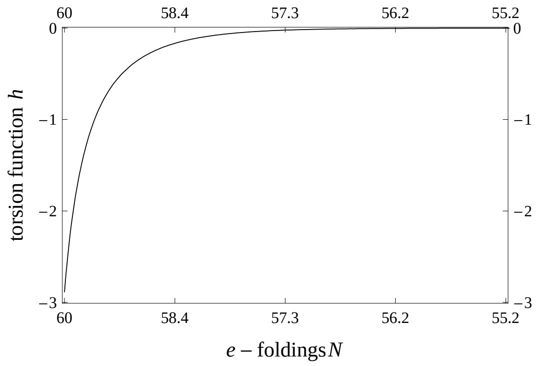

Figure 5.9: Torsion function h for $1/M_{\mathrm{pl}}^2 = 8\pi$ and $V_0 = 1$. Initial condition is $\varphi_i = \{0.25\}$.

negligible after approximately four e-foldings.

5.7.1 Discussion

We identified the Elko spinor as a matter source whose spin-angular momentum tensor is compatible with the cosmological principle. We then solved the resulting field equations of Einstein-Cartan theory. It couples to all irreducible parts of torsion and therefore leads to an interesting coupling of matter and geometry. The Elko spinor is also naturally dark in that it can only interact via the Higgs mechanism or gravity.

Our solutions of the field equations show that torsion does vanish quickly (approximately after a few e-foldings) and that the Hubble parameter has a constant value as an attractor. Both features of the model fit very well into the standard model of inflationary cosmology in that a period of accelerated expansion is an attractor solution. It is worth noting that in Einstein-Cartan theory the spins of elementary particles are thought to be the primary sources of torsion, and it is therefore expected that on large sales and over time torsion should average out or decay, respectively.

We speculate that some non-zero cosmological torsion has already been observed in the large scale anisotropies of the cosmic microwave background radiation (CMB) where torsion leaves its imprint only on the largest scales.

Chapter 6

Non Standard Spinors and Cosmology

This last chapter is much shorter than the rest and serves to summarize collaborative work on extensions to the Elko spinor. For more details we point the interested reader to the four author paper published in Journal of High Energy Physics [18].

The main theme of this work is to extend the Elko definition to include an entire class of non-standard spinors. This can be achieved by introducing a projection operator which projects out states that contribute to an ill-defined Hamiltonian operator. We begin with the criterion that a free, massive spinor free field, ψ, in flat space-time (with tetrads $e^j{}_\mu = \delta^j{}_\mu$ so $\Gamma_\mu = 0$) should obey the flat space Klein-Gordon equation,

$$\partial^2\psi = m_\psi^2\psi. \tag{6.1}$$

This suggests the following flat-space Lagrangian for ψ,

$$\mathcal{L}^{(1)}_{\mathrm{free-flat}} \quad \equiv \quad (\overset{\neg}{\psi}\overleftarrow{\not\partial})(\not\partial\psi) - m_\psi^2\overset{\neg}{\psi}\psi, \tag{6.2}$$

where $\not\partial = \gamma^\mu\partial_\mu$, and $\overset{\neg}{\psi}$ is some dual spinor to ψ defined so that $\overset{\neg}{\psi}\psi$ is a space-time scalar. We vary ψ and $\overset{\neg}{\psi}$ independently, and note that - up to a surface term - the above action ($\mathcal{L}^{(1)}_{\mathrm{free-flat}}$) is equivalent to another $\mathcal{L}^{(2)}_{\mathrm{free-flat}}$ given by,

$$\mathcal{L}^{(2)}_{\mathrm{free-flat}} \quad \equiv \quad (\partial_\mu\overset{\neg}{\psi})(\partial^\mu\psi) - m_\psi^2\overset{\neg}{\psi}\psi. \tag{6.3}$$

However, this equivalence relies on $\partial^2\psi = \not\partial^2\psi$ which is broken when the actions are promoted to curved space by taking $\partial_\mu \to \nabla_\mu$, since generally $\mathcal{R} \neq 0$ when $R_{\mu\nu\rho\sigma} \neq 0$. One must therefore choose which of the two actions to promote to curved space.

Remaining in flat space, there is a problem with both actions as they are given above. The field equation $(\partial^2 - m^2)\psi = 0$ constrains the evolution of each of the four components of ψ but does not impose any relation between the different components. We define a basis ψ_a (where $a = 1, 2, 3, 4$) on 4-spinor space, such that, $\bar{\psi}_a\psi_b = 0$ if $a \neq b$ and $\partial_\mu\psi_a = 0$. We assume that $\partial_\mu\bar{\psi}_b = 0$. However, as is well known, Lorentz invariance prevents us from defining $\bar{\psi}_a\psi_b = \delta_{ab}$, instead we can ensure that $\bar{\psi}_1\psi_1 = \bar{\psi}_2\psi_2 = 1$ and $\bar{\psi}_3\psi_3 = \bar{\psi}_4\psi_4 = -1$. Solutions of $(\partial^2 - m^2)\psi = 0$ are then given by,

$$\psi = \sum_{a,\mathbf{p}} a_a(\mathbf{p})\frac{1}{2E_p}e^{iE_pt - i\mathbf{p}\cdot\mathbf{x}}\psi_a + \sum_{a,\mathbf{p}} b_a^\dagger(\mathbf{p})\frac{1}{2E_p}e^{-iE_pt + i\mathbf{p}\cdot\mathbf{x}}\psi_a,$$

where $a_a(\mathbf{p})$ and $b_a^\dagger(\mathbf{p})$ are some functions of $\mathbf{p}$ (the 3-momentum) and $E_p = \sqrt{m^2 + \mathbf{p}^2}$. Here, $\sum_p = \int \mathrm{d}^3p$.

Let us define the Hamiltonian density $\mathcal{H} = \dot{\bar{\psi}}\bar{\pi} + \pi\dot{\psi} - \mathcal{L}^{(1)}$ where the momentum is defined as $\pi = \partial\mathcal{L}^{(1)}/\partial\dot{\psi} = \dot{\bar{\psi}}$, and $\bar{\pi} = \partial\mathcal{L}^{(1)}/\partial\dot{\bar{\psi}} = \dot{\psi}$. In flat space, the Hamiltonian density formed from $\mathcal{L}^{(2)}$ differs from that based on $\mathcal{L}^{(1)}$ only by an irrelevant total derivative which can be dropped. We then have

$$\mathcal{H} = \left[\pi\bar{\pi} + \nabla^n\bar{\psi}\nabla_n\psi + m^2\bar{\psi}\psi\right] \quad n = 1, 2, 3. \tag{6.4}$$

Taking $\epsilon_a = \bar{\psi}_a\psi_a$, one can show that

$$H = \int \mathrm{d}^3x\, \mathcal{H} = \sum_a \epsilon_a \sum_{\mathbf{p}} \frac{(E_p^2 + \mathbf{p}^2 + m^2)}{2E_p}[a_a^\dagger(\mathbf{p})a_a(\mathbf{p}) + b_a(\mathbf{p})b_a^\dagger(\mathbf{p})], \tag{6.5}$$

which then becomes

$$H = \sum_a \epsilon_a \sum_{\mathbf{p}} (E_p)[a_a^\dagger(\mathbf{p})a_a(\mathbf{p}) + b_a(\mathbf{p})b_a^\dagger(\mathbf{p})]. \tag{6.6}$$

Finally, we can assume a and b will be upgraded to operators that obey anti-commutation relations. Thus, we arrive at the following Hamiltonian

$$H = \sum_a \epsilon_a \sum_{\mathbf{p}} (E_p)[a_a^\dagger(\mathbf{p})a_a(\mathbf{p}) - b_a^\dagger(\mathbf{p})b_a(\mathbf{p})]. \tag{6.7}$$

This Hamiltonian density is ill defined, it is not positive definitive. However, we know that if we were to write the Dirac spinor in the KG equation and followed the same steps outlined above we would get a consistent Hamiltonian density. Thus, there must be a projection operation implicitly present which removes (projects out) the components of the spinor which would give an inconsistent Hamiltonian density. It is important to note that the actual energy is squared in this expression and therefore we retain the negative energy information, which is what we learned from Dirac.

Let us assume that the a_a and $a_a^\dagger$ represent annihilation and creation operators respectively, then $a_a^\dagger a_a \neq 0$ and $b_a^\dagger b_a \neq 0$. If we interpret $\stackrel{\neg}{\psi}\psi$ as the energy-density of the spinor field with $\epsilon_1 = \epsilon_2 = -\epsilon_3 = -\epsilon_4 = 1$, it follows that the spinor field can have negative energy density, unless there is some additional condition that requires $a_3 = a_4 = 0$ and $b_1 = b_2 = 0$ in the definition of ψ. Additionally, without such a requirement it would be possible to have states with both $a_a^\dagger a_a$ and $b_a^\dagger b_a \geq 0$ but with zero energy. Negative energy or ghost states lead to well known instabilities both classically and at the level of quantum field theory.

6.1 Energy-momentum tensor

The other important part of this work was to construct a full energy-momentum tensor. Thus far in our work concerning the Elko spinor the energy-momentum tensor has been calculated from the effective action. This, we find, is not the same as the energy-momentum tensor worked from the full action for a non-standard spinor. We check that in the case of the Dirac spinor the contribution from the spin connection to its energy-momentum tensor is zero. This confirms that the energy-momentum tensor for a Dirac spinor can be taken from its effective action and there are no extra terms coming from the spin connection.

Appendix A

Nonlinear second order equations which reduce to pairs of linear first order equations

Let Ω be an open subset of $\mathbb{R}^n$. We work with (infinitely) smooth vector functions $\Omega \to \mathbb{C}^m$ writing these as columns of m complex scalars. In this appendix “vector” does not carry a differential geometric meaning because we are not interested in coordinate transformations. We use Cartesian coordinates $x^1, \ldots, x^n$.

Given a pair of vector functions $u, v : \Omega \to \mathbb{C}^m$ we define their inner product in the standard Euclidean manner as $(u, v) := \int_\Omega v^* u \, dx^1 \ldots dx^n$ where the star $*$ denotes Hermitian conjugation. This integral need not converge as we will be using it only for the purpose of defining the formal adjoint of a differential operator, see next paragraph.

Let $A_\pm$ be a pair of formally self-adjoint (symmetric) first order linear partial differential operators (differential expressions) with smooth coefficients acting on smooth vector functions $\Omega \to \mathbb{C}^m$. We do not introduce any boundary conditions.

Put

$$L_\pm(u) := \mathrm{Re}(u^* A_\pm u). \tag{A.1}$$

It is easy to see that $L_\pm(u)$ is the Lagrangian density for the partial differential equation $A_\pm u = 0$. Namely, if one writes down the action (variational functional) $S_\pm(u) := \int_\Omega L_\pm(u) \, dx^1 \ldots dx^n$ then the corresponding field equation (Euler–Lagrange equation) is

$A_\pm u = 0$.

Let us now define a new Lagrangian density

$$L(u) := \frac{L_+(u)\,L_-(u)}{L_+(u) - L_-(u)} \tag{A.2}$$

and corresponding action $S(u) := \int_\Omega L(u)\,dx^1 \dots dx^n$. The field equation for the Lagrangian density (A.2) is, of course, second order and nonlinear.

Note that the notation in this appendix is self-contained and the Lagrangian densities (A.1), (A.2) should not be confused with the Lagrangian densities (4.33), (4.39) introduced in the main text (the latter have an extra subscript).

The main result of this appendix is

Lemma 1. *Let $u : \Omega \to \mathbb{C}^m$ be a vector function satisfying the condition*

$$L_+(u) \neq L_-(u). \tag{A.3}$$

Then u is a solution of the field equation for the Lagrangian density L if and only if it is a solution of the equation $A_+ u = 0$ or the equation $A_- u = 0$.

Proof. The explicit formula for the operator $A_\pm$ is

$$A_\pm = iB^\alpha_\pm \partial_\alpha + \frac{i}{2}(\partial_\alpha B^\alpha_\pm) + C_\pm \tag{A.4}$$

where $B^\alpha_\pm$ and $C_\pm$ are some smooth Hermitian $m \times m$ matrix functions and the index α runs through the values $1, \dots, n$. Substituting (A.4) into (A.1) we get

$$L_\pm(u) = \frac{i}{2}\big[u^* B^\alpha_\pm \partial_\alpha u - (\partial_\alpha u^*) B^\alpha_\pm u\big] + u^* C_\pm u. \tag{A.5}$$

Now take an arbitrary smooth function $h : \Omega \to \mathbb{R}$. Examination of formula (A.5) shows that

$$L_\pm(e^h u) = e^{2h} L_\pm(u). \tag{A.6}$$

We call the property (A.6) *scaling covariance*. Scaling covariance is a remarkable feature of the Lagrangian density of a formally self-adjoint first order linear partial differential operator.

Formulas (A.2) and (A.6) imply that the Lagrangian density L also possesses the property of scalar covariance, i.e. $L(e^h u) = e^{2h} L(u)$ for any smooth $h : \Omega \to \mathbb{R}$. Thus, all three of our Lagrangian densities, L, L_+ and L_-, have this property.

Observe now that if the vector function u is a solution of the field equation for some Lagrangian density $\mathcal{L}$ possessing the property of scaling covariance then $\mathcal{L}(u) = 0$. Indeed, let us perform a scaling variation of our vector function

$$u \mapsto u + \delta u = u + hu = e^h u + O(h^2) \tag{A.7}$$

where $h : \Omega \to \mathbb{R}$ is an arbitrary "small" smooth function with compact support, $h \in C_0^\infty(\Omega; \mathbb{R})$. Then $0 = \delta \int \mathcal{L}(u) = 2 \int h\mathcal{L}(u)$ which holds for arbitrary h only if $\mathcal{L}(u) = 0$.

In the remainder of the proof the variation $\delta u : \Omega \to \mathbb{C}^m$ of the vector function $u : \Omega \to \mathbb{C}^m$ is arbitrary and not necessarily of the scaling type (A.7). The only assumption is that $\delta u \in C_0^\infty(\Omega; \mathbb{C}^m)$.

Suppose that u is a solution of the field equation for the Lagrangian density L_+. [The case when u is a solution of the field equation for the Lagrangian density L_- is handled similarly.] Then $L_+(u) = 0$ and, in view of formula (A.3), $L_-(u) \neq 0$. Varying u we get

$$\begin{aligned} \delta \int L(u) = \int \frac{L_-(u)}{L_+(u) - L_-(u)}\, \delta L_+(u) + \int L_+(u)\, \delta \frac{L_-(u)}{L_+(u) - L_-(u)} \\ = -\int \delta L_+(u) = -\delta \int L_+(u) \end{aligned}$$

so

$$\delta \int L(u) = -\delta \int L_+(u)\,. \tag{A.8}$$

We assumed that u is a solution of the field equation for the Lagrangian density L_+ so $\delta \int L_+(u) = 0$ and formula (A.8) implies that $\delta \int L(u) = 0$. As the latter is true for an arbitrary variation of u this means that u is a solution of the field equation for the Lagrangian density L.

Suppose that u is a solution of the field equation for the Lagrangian density L. Then $L(u) = 0$ and formula (A.2) implies that either $L_+(u) = 0$ or $L_-(u) = 0$; note that in view of (A.3) we cannot have simultaneously $L_+(u) = 0$ and $L_-(u) = 0$. Assume for definiteness that $L_+(u) = 0$. [The case when $L_-(u) = 0$ is handled similarly.] Varying u and repeating the argument from the previous paragraph we arrive at (A.8). We assumed that u is a solution of the field equation for the Lagrangian density L so $\delta\int L(u) = 0$ and formula (A.8) implies that $\delta\int L_+(u) = 0$. As the latter is true for an arbitrary variation of u this means that u is a solution of the field equation for the Lagrangian density L_+. □

Remark 4. *It may seem that the variational proof presented above is "insufficiently rigorous". An alternative "completely rigorous" way of proving Lemma 1 is to write down the field equation for the Lagrangian density (A.2), (A.5) explicitly and analyze this second order nonlinear partial differential equation. The result, of course, remains the same, but the calculations become much longer.*

Remark 5. *Examination of the proof of Lemma 1 shows that the fact that the differential operators $A_\pm$ are linear and first order is not important. What is important is that their Lagrangian densities possess the scaling covariance property (A.6). As the Lagrangian density (A.2) possesses this property as well, our construction admits an obvious extension which gives a hierarchy of nonlinear partial differential equations which reduce to several separate equations.*

Example 1. *Let us give an elementary example illustrating the use of Lemma 1. Consider the pair of linear first order ordinary differential equations*

$$iu' \pm u = 0 \tag{A.9}$$

where $u : \mathbb{R} \to \mathbb{C}$ is a scalar function. Let us write down the corresponding Lagrangian densities $L_\pm(u) = \frac{i}{2}(\bar{u}u' - u\bar{u}') \pm |u|^2$ in accordance with formula (A.1) and form a new Lagrangian density $-2L(u) = \left(\frac{\bar{u}u' - u\bar{u}'}{2|u|}\right)^2 + |u|^2$ in accordance with formula (A.2). The latter gives the field equation (Euler–Lagrange equation)

$$\left(\frac{\bar{u}u' - u\bar{u}'}{2|u|^2}\,u\right)' + \frac{(\bar{u}u')^2 - (u\bar{u}')^2}{4|u|^4}\,u + u = 0. \tag{A.10}$$

Lemma 1 tells us that a smooth nonvanishing function u*, is a solution of equation (A.10) if and only if it is a solution of one of the two equations (A.9). Of course, this fact can be seen by switching to the polar representation* $u = re^{-i\varphi}$ *where* $r : \mathbb{R} \to (0, +\infty)$ *and* $\varphi : \mathbb{R} \to \mathbb{R}$.

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