Disorder and the Quantum Hall Ferromagnet.

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The distinguishing feature of the quantum Hall ferromagnet is the identity between electrical and topological charge densities of a spin distortion. In addition to the wealth of physics associated with Skyrmionic excitations of the quantum Hall ferromagnet, this identification permits a rather curious coupling of spinwaves to the disorder potential. A wavepacket of spinwaves has an associated, oscillating dipole charge distribution, due to the non-linear form of the topological density. We investigate the way in which this coupling modifies the conductivity and temperature dependence of magnetization of the quantum Hall ferromagnet.

The distinguishing feature of the quantum Hall ferromagnet (QHF) is the identity between the topological density of a spin distortion and the associated electrical charge density. This identification permits a chemical potential to stabilize topologically non-trivial groundstate spin configurations, known as Skyrmions[1]. The theoretical prediction of these states has received substantial experimental support[2] and prompted a good deal of theoretical speculation. The link between topological and electrical charge densities also produces a curious coupling of spinwaves to the disorder potential. Although a planewave spin distortion carries no charge, a wavepacket of spinwaves has an oscillating dipole charge distribution associated with it, due to the non-linear form of the topological density. Spinwaves couple to the disorder potential through this charge distribution. In this work, we investigate the way in which this coupling modifies the conductivity and temperature dependence of magnetization of the quantum Hall state.

The low energy effective action for the QHF at filling fractions $\nu = 1$ and the Laughlin filling fractions is given by[1,3]

$$\begin{aligned} S = &\int dtd^2x \left[\frac{\bar{\rho}}{2}\partial_t \mathbf{n}.\mathbf{A}[\mathbf{n}] - \frac{\rho_s}{2}|\partial_\mu \mathbf{n}|^2 + \bar{\rho} g \mathbf{B}.\mathbf{n}\right] \\ &- \int dtd^2x J_0(\mathbf{x})U(\mathbf{x}) - \int dt V\left[J_0(\mathbf{x})\right] \\ &+ \nu \int dtd^2xd^2y J_0(\mathbf{x})\epsilon_{ij}\frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|^2}J_j(\mathbf{y}), \end{aligned} \tag{1}$$

where

$$J_\mu = -\frac{e\nu}{8\pi}\epsilon_{\mu\nu\lambda}\mathbf{n}.(\partial_\nu \mathbf{n} \times \partial_\lambda \mathbf{n}). \tag{2}$$

$\mathbf{n}(\mathbf{x})$ is an O(3)-vector order parameter of unit length, describing the local polarization of the quantum Hall system. The first line of Eq.(1) is the usual low energy effective action for a ferromagnet. $\mathbf{A}[\mathbf{n}]$ is the vector potential of a unit monopole in spin space, $\bar{\rho}$ is the electron density ($\bar{\rho} = \nu/2\pi l^2$, where l is the magnetic length), ρ_s is the spin stiffness and g is the Zeeman coupling, into which we have absorbed the electron spin and the Bohr magneton for ease of notation. The second line of Eq.(1) contains terms arising due to the identity of charge and topological charge (which is embodied in Eq.(2)). The first of these terms is an interaction with the disorder potential, $U(\mathbf{x})$, and the second, $V[J_0(\mathbf{x})]$, is the Coulomb energy of the charge distribution, $J_0(\mathbf{x})$. Eq.(1) describes both the low energy spin and charge dynamics of the quantum Hall system. The quantization of Hall conductivity follows from the final term, the Hopf term[4].

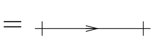
Here, we are concerned with the effect of the disorder potential upon small fluctuations, $\mathbf{l} = (l_1, l_2, 0)$, about the ferromagnetic groundstate, $\bar{\mathbf{n}} = (0, 0, 1)$; $\mathbf{n} = (l_1, l_2, \sqrt{1 - |\mathbf{l}|^2})$. The effective action and current, expanded to lowest order in these fluctuations, are

$$\begin{aligned} S = &\int d^2xdt \frac{1}{2}\bar{l}\left(\frac{\bar{\rho}}{2}\partial_t - \rho_s\nabla^2 - \bar{\rho}gB\right)l \\ &- \int d^2xdt J_0(\mathbf{x})U(\mathbf{x}), \\ J_\mu = &\, i\frac{e\nu}{8\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \bar{l}\partial_\lambda l. \end{aligned} \tag{3}$$

We use the complex notation, $l = l_1 + il_2$, $\bar{l} = l_1 - il_2$. Both the Coulomb and statistical interactions have been neglected in writing down Eq.(3). Although important in determining the size and shape of the Skyrmion excitations, the former is less relevant than the remaining terms in its effect upon spinwaves[5]. We will show later that the quantization of Hall conductivity, produced by the Hopf term, is unaffected by weak disorder. The calculations presented in this work concern the perturbative effects of weak disorder. It is worth noting that the effective action, Eq.(3), is very similar to that of electrons in a random potential, aside from the unusual form of the current density and the bosonic nature of the fields. This similarity is suggestive of the possibility of weak localization effects. These are not considered here.

We represent the bare, momentum space propagators, $\langle \bar{l}(\mathbf{q}, \tilde{\omega})l(-\mathbf{q}, -\tilde{\omega})\rangle$ and $\langle \partial_\mu \bar{l}(\mathbf{q}, \tilde{\omega})\partial_\nu l(-\mathbf{q}, -\tilde{\omega})\rangle$, by the diagrams

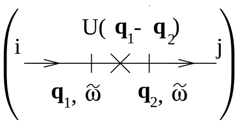
$\langle \bar{l}(\mathbf{q}, \tilde{\omega})l(-\mathbf{q}, -\tilde{\omega})\rangle =$ $= \dfrac{1}{i\bar{\rho}\tilde{\omega}/2 - E(\mathbf{q})}$

$\langle \partial_\mu \bar{l}(\mathbf{q}, \tilde{\omega})\partial_\nu l(-\mathbf{q}, -\tilde{\omega})\rangle =$  $= \dfrac{q_\mu q_\nu}{i\bar{\rho}\tilde{\omega}/2 - E(\mathbf{q})}$

where $E(\mathbf{q}) = \rho_s|\mathbf{q}|^2 + \bar{\rho}gB$ is the spin energy density.

The disorder interaction is given by

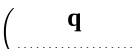
$$S_i = \int d\tilde{\omega} \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} i \left(\frac{e\nu}{8\pi}\right) \epsilon^{ij} \left(\quad \right),$$



where the frequency integral, $\int d\tilde{\omega}$, is a shorthand notation for the bosonic Matsubara frequency summation $\frac{1}{T}\sum_{n=-\infty}^{\infty} \cdots|_{\tilde{\omega}=2\pi n/T}$. Notice that the scattering off the impurity potential is entirely elastic *i.e.* the energy labels on the propagators are conserved.

In GaAs heterostructures, the disorder potential felt by the electrons in the 2DEG is due mainly to Coulomb interaction with ionized donor impurities in the n-type region[6]. This region is separated from the 2DEG by an insulating spacer layer of width d. One may obtain an expression for the correlations in the disorder potential by modeling this situation with the potential due to a random planar distribution of charge at a distance d from the 2DEG. The correlations in the disorder potential in this model are given by

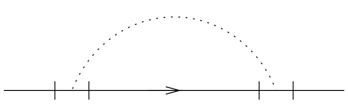
$$\begin{aligned} \langle\langle U_{\mathbf{q}} U_{\mathbf{q}'} \rangle\rangle &= (2\pi)^2 \delta(\mathbf{q}+\mathbf{q}') \left(\frac{e\sqrt{n_d}}{2\epsilon}\right)^2 \frac{e^{-2|\mathbf{q}|d}}{|\mathbf{q}|^2} \\ &= (2\pi)^2 \delta(\mathbf{q}+\mathbf{q}') \left(\quad \right), \end{aligned} \tag{4}$$



where n_d is the area density of donor impurities. This simple model of disorder somewhat overestimates the potential felt by the 2DEG. Due to Coulomb interactions between the donors, the size of the fluctuations in the disorder potential is usually much less than would be expected for a totally uncorrelated distribution of charge in the disorder plane. We follow Fogler *et al.*[7] and assume that this effect may be taken into account by interpreting n_d in Eq.(4) as a density of 'uncorrelated' donors, which is much less than the actual density of donors.

The lowest order contribution of disorder to the self-energy is

$$\Sigma(i\tilde{\omega}, \mathbf{p}) =$$



$$= K\rho_s^2 \int \frac{d^2q}{(2\pi)^2} \frac{(\mathbf{p}\times\mathbf{q})^2}{i\bar{\rho}\tilde{\omega}/2 - E(\mathbf{q}+\mathbf{p})} \frac{e^{-2d|\mathbf{q}|}}{|\mathbf{q}|^2}, \tag{5}$$

where

$$K = \frac{1}{\rho_s^2}\left(\frac{e\nu}{8\pi}\right)^2 \left(\frac{e\sqrt{n_d}}{2\epsilon}\right)^2 \tag{6}$$

is a dimensionless measure of the disorder strength. The retarded self-energy is obtained by analytic continuation to real frequencies with the substitution $i\tilde{\omega} \to \omega + i\delta$. The real and imaginary parts of the self-energy so obtained are

$$\mathcal{R}e\Sigma(\omega, \mathbf{p}) = K\rho_s^2 \int \frac{d^2q}{(2\pi)^2} \frac{(\mathbf{p}\times\mathbf{q})^2}{\bar{\rho}\omega/2 - E(\mathbf{q}+\mathbf{p})} \frac{e^{-2d|\mathbf{q}|}}{|\mathbf{q}|^2}, \tag{7}$$

$$\begin{aligned} \mathcal{I}m\Sigma(\omega, \mathbf{p}) = K\rho_s^2 \int \frac{d^2q}{(2\pi)^2} (\mathbf{p}\times\mathbf{q})^2 \frac{e^{-2d|\mathbf{q}|}}{|\mathbf{q}|^2} \\ \times \pi\delta(\bar{\rho}\omega/2 - E(\mathbf{q}+\mathbf{p})). \end{aligned} \tag{8}$$

The real part of the self-energy can be approximated from Eq.(7) in the limit $\rho_s|\mathbf{p}|^2$, $\bar{\rho}|\omega - 2gB|/2 \ll \rho_s/d^2$. The leading order contribution is proportional to $|\mathbf{p}|^2$ and provides a correction to the spinwave stiffness, $\Delta\rho_s = \mathcal{R}e\Sigma/|\mathbf{p}|^2$. For $\rho_s|\mathbf{p}|^2 > \bar{\rho}|\omega - 2gB|/2$, there is a crossover to $|\mathbf{p}|^2 \ln|\mathbf{p}|^2$ dependence. We find

$$\begin{aligned} \mathcal{R}e\Sigma(\omega, \mathbf{p}) &\simeq \frac{K\rho_s}{8\pi}|\mathbf{p}|^2 \ln\left[\frac{4|\bar{\rho}gB - \bar{\rho}\omega/2|d^2}{\rho_s}\right] \\ &\quad \text{for } \rho_s|\mathbf{p}|^2 < \bar{\rho}|\omega - 2gB|/2, \\ &\simeq \frac{K\rho_s}{8\pi}|\mathbf{p}|^2 \ln\left[4|\mathbf{p}|^2 d^2\right] \\ &\quad \text{for } \rho_s|\mathbf{p}|^2 > \bar{\rho}|\omega - 2gB|/2. \end{aligned} \tag{9}$$

The first of these expressions has been calculated by expanding Eq.(7) to lowest order in $|\mathbf{p}|^2$ and by replacing the exponential factor, $e^{-2d|\mathbf{q}|}$, with an ultra-violet cutoff, $1/2d$. The second expression is calculated exactly from Eq.(7), setting $\omega = 2gB$.

The imaginary part of the self energy may be calculated exactly when $d = 0$, with the result

$$\begin{aligned} \mathcal{I}m\Sigma(\omega, \mathbf{p}) &= -\frac{K}{8}\bar{\rho}(\omega/2 - gB)\theta\left(\omega/2 - gB\right) \\ &\quad \text{for } \rho_s|\mathbf{p}|^2 > \bar{\rho}|\omega - 2gB|/2, \\ &= -\frac{K}{8}\rho_s|\mathbf{p}|^2 \\ &\quad \text{for } \rho_s|\mathbf{p}|^2 < \bar{\rho}|\omega - 2gB|/2. \end{aligned} \tag{10}$$

The integral for finite d is much trickier and cannot be carried out analytically. For large d it is exponentially suppressed by a factor $e^{-2d|\mathbf{p}|}$.

Taken at face value, Eq.(9) implies a threshold disorder strength at which the renormalized spin-stiffness is zero at zero frequency. We interpret this as indicative of a depolarization transition to a paramagnetic state. A similar suggestion has been made by Fogler *et al.*[7] in order to explain the breakdown of spin splitting in high Landau levels. Strictly, the calculations presented here apply only for weak disorder and small $\Delta\rho_s$. That the threshold behaviour suggested here does indeed occur, may be seen in a number of ways. The most elegant of these is through a Bogomolny bound type argument[8]. The present treatment enables one to investigate the approach to this threshold.

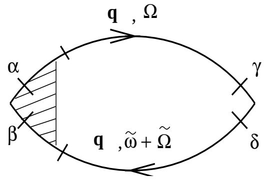
Optical conductivity. The longitudinal and transverse conductivities are given by the Kubo formula[9]:

$$\sigma_{ij}(\omega) = \frac{i}{\omega}\langle J_i(0,\tilde{\omega}) J_j(0,-\tilde{\omega})\rangle\Big|_{i\tilde{\omega}\to\omega+i\delta} \tag{11}$$

In order to determine the longitudinal conductivity, we must evaluate the following diagram:

$\langle \mathbf{J}(0,\tilde{\omega}).\mathbf{J}(0,-\tilde{\omega})\rangle$

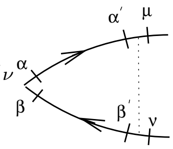
$$= -\left(\frac{e\nu}{8\pi}\right)^2 \epsilon^{i\alpha\beta}\epsilon^{i\gamma\delta}$$

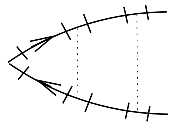
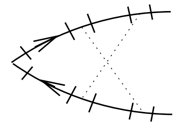


$$= -\left(\frac{e\nu}{8\pi}\right)^2 \epsilon^{i\alpha\beta}\epsilon^{i\gamma\delta}\int \frac{d^2q}{(2\pi)^2}d\tilde{\Omega}\ \ q_\mu \tilde{q}_\nu q_\gamma \tilde{q}_\delta$$
$$\times\Gamma_{\alpha\beta,\mu\nu}(\mathbf{q},i\tilde{\Omega},i\tilde{\Omega}+i\tilde{\omega})\mathcal{G}(\mathbf{q},i\tilde{\Omega}+i\tilde{\omega})\mathcal{G}(\mathbf{q},i\tilde{\Omega}), \quad (12)$$

where $q_\mu = (i\tilde{\Omega},\mathbf{q})$, $\tilde{q}_\mu = (i\tilde{\Omega}+i\tilde{\omega},\mathbf{q})$ and $\mathcal{G}(\mathbf{q},i\tilde{\Omega})$ is the full thermodynamic Green's function. The vertex function, $\Gamma_{\alpha\beta,\mu\nu}$, is given by the summation

$$\Gamma_{\alpha\beta,\mu\nu}(\mathbf{q},i\tilde{\Omega},i\tilde{\Omega}+i\tilde{\omega})$$
$$= \delta_{\alpha\mu}\delta_{\beta\nu} + \left(\frac{e\nu}{8\pi}\right)^2 \epsilon^{\alpha'\mu}\epsilon^{\beta'\nu}$$



$+ \ldots$  $+ \ldots$  $+ \ldots \quad (13)$

In fact, all contributions to the vertex function contain a factor of $q_\alpha \tilde{q}_\beta$ and there is considerable simplification in defining a new, scalar vertex function, $\gamma(\mathbf{q},i\tilde{\Omega},i\tilde{\Omega}+i\tilde{\omega})$;

$$q_\alpha \tilde{q}_\beta \gamma(\mathbf{q},i\tilde{\Omega},i\tilde{\Omega}+i\tilde{\omega}) = \Gamma_{\alpha\beta,\mu\nu}(\mathbf{q},i\tilde{\Omega},i\tilde{\Omega}+i\tilde{\omega})q_\mu \tilde{q}_\nu.$$

This definition of the vertex function is then substituted into Eqs.(11,12) to find the conductivity. After performing the summation over bosonic Matsubara frequencies and a few other standard manipulations[9], the real part of the longitudinal conductivity is given by the expression

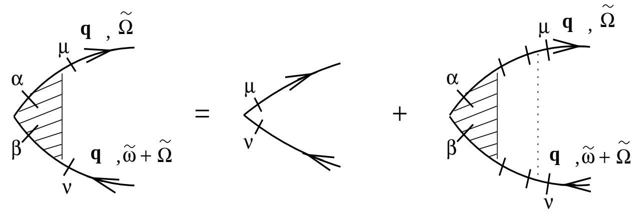
$$\sigma(\omega) = \omega\left(\frac{e\nu}{8\pi}\right)^2 \int \frac{d^2q}{(2\pi)^2}|\mathbf{q}|^2 \int_{-\infty}^{\infty}\frac{d\epsilon}{4\pi}\left[n_B(\epsilon+\omega)-n_B(\epsilon)\right]$$
$$\times\Re e\left[G^A(\mathbf{q},\epsilon)G^R(\mathbf{q},\epsilon+\omega)\gamma(\mathbf{q},\epsilon-i\delta,\epsilon+\omega+i\delta)\right.$$
$$\left.-G^R(\mathbf{q},\epsilon)G^R(\mathbf{q},\epsilon+\omega)\gamma(\mathbf{q},\epsilon+i\delta,\epsilon+\omega+i\delta)\right], \quad (14)$$

where $n_B(x)$ is the Bose occupation number. The contribution to the Hall conductivity is zero, on symmetry grounds, since the current-current correlator $\langle \mathbf{J}\times\mathbf{J}\rangle$ gives rise to a factor of $\mathbf{q}\times\mathbf{q}$ in the integrand. Compared with the analogous result for electronic conductivity[9], Eq.(14) contains an additional factor of ω^2, which ensures that the d.c. conductivity is zero. This is due to the fact that the charge fluctuations in the QHF are dipolar.

Vertex corrections. In the ladder approximation, the vertex function is given by the following Dyson's equation:

$$\Gamma_{\alpha\beta,\mu\nu}(\mathbf{q},i\tilde{\Omega},i\tilde{\omega}+i\tilde{\Omega}) = \delta_{\alpha\mu}\delta_{\beta\nu}$$
$$-\left(\frac{e\nu}{8\pi}\right)^2\left(\frac{e^2 n_d}{2\epsilon}\right)^2 \int \frac{d^2k}{(2\pi)^2}\epsilon^{b\nu}\epsilon^{d\mu}k_a k_b k_c k_d \frac{e^{-2d|\mathbf{q}-\mathbf{k}|}}{|\mathbf{q}-\mathbf{k}|^2}$$
$$\times\mathcal{G}(\mathbf{k},i\tilde{\Omega}+i\tilde{\omega})\mathcal{G}(\mathbf{k},i\tilde{\Omega})\Gamma_{\alpha\beta,ac}(\mathbf{k},i\tilde{\Omega},i\tilde{\omega}+i\tilde{\Omega}),$$

or diagrammatically,



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This Dyson's equation may be recast in terms of the scalar vertex function, $\gamma(\mathbf{q},i\tilde{\Omega},i\tilde{\omega}+i\tilde{\Omega})$:

$$\gamma(\mathbf{q},i\tilde{\Omega},i\tilde{\omega}+i\tilde{\Omega}) = 1 + \int \frac{d^2k}{(2\pi)^2}\frac{(\mathbf{q}.\mathbf{k})^2}{|\mathbf{q}|^4}W_{\mathbf{qk}}$$
$$\times\mathcal{G}(\mathbf{k},i\tilde{\Omega}+i\tilde{\omega})\mathcal{G}(\mathbf{k},i\tilde{\Omega})\gamma(\mathbf{k},i\tilde{\Omega},i\tilde{\omega}+i\tilde{\Omega}), \quad (15)$$

where

$$W_{\mathbf{qk}} = -\left(\frac{e\nu}{8\pi}\right)^2\left(\frac{e^2 n_d}{2\epsilon}\right)^2 (\mathbf{q}\times\mathbf{k})^2 \frac{e^{-2d|\mathbf{q}-\mathbf{k}|}}{|\mathbf{q}-\mathbf{k}|^2}.$$

In order to obtain Eq.(15), we have used the relation

$$\Gamma_{ab,cd}(\mathbf{k},i\tilde{\Omega},i\tilde{\omega}+i\tilde{\Omega})q_a q_b k_c k_d = \gamma(\mathbf{k},i\tilde{\Omega},i\tilde{\omega}+i\tilde{\Omega})\,(\mathbf{k}.\mathbf{q})^2\,,$$

which follows from the definition of γ and the symmetry of the disorder interaction; since the disorder potential couples only to the charge density and not to any other components of the current density, $\Gamma_{\alpha\beta,\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu}$ if either or both of μ or ν are time-like.

In order to calculate the real part of the optical conductivity we require $\gamma(\mathbf{q},\epsilon-i\delta,\epsilon+\omega+i\delta)$. Evaluating this is a very difficult task. However, several simplifying assumptions may be made. Firstly, we assume that the frequency dependence of the optical conductivity is dominated by terms in Eq.(14) other that the vertex function. Secondly, the terms $G^A G^R$ in Eq.(14) are strongly peaked within ω of $\epsilon = 2E(\mathbf{q})/\bar{\rho}$. Therefore, we have only to calculate $\gamma(\mathbf{q},2E(\mathbf{q})/\bar{\rho}) = \gamma(\mathbf{q},2E(\mathbf{q})/\bar{\rho}-i\delta,2E(\mathbf{q})/\bar{\rho}-i\delta)$. Using Eq.(15), we find

$$\gamma(\mathbf{q},2E(\mathbf{q})/\bar{\rho})$$
$$= 1 + \int \frac{d^2k}{(2\pi)^2}\frac{(\mathbf{q}.\mathbf{k})^2}{|\mathbf{q}|^4}W_{\mathbf{qk}}G^R(\mathbf{k},\epsilon)G^A(\mathbf{k},\epsilon)\gamma(\mathbf{k},\epsilon)\Big|_{\epsilon=2E(\mathbf{q})/\bar{\rho}}$$
$$= 1 + \int \frac{d^2k}{(2\pi)^2}\frac{(\mathbf{q}.\mathbf{k})^2}{|\mathbf{q}|^4}W_{\mathbf{qk}}\frac{A(\mathbf{k},\epsilon)}{2\Delta(\mathbf{k},\epsilon)},\gamma(\mathbf{k},\epsilon)\Big|_{\epsilon=2E(\mathbf{q})/\bar{\rho}}, \quad (16)$$

where $\Delta(\mathbf{k},\epsilon) = -\mathcal{I}m\Sigma(\mathbf{k},\epsilon)$ and $A(\mathbf{k},\epsilon) = -2\mathcal{I}mG^R(\mathbf{k},\epsilon)$ is the spectral function. In the limit of very weak disorder, $A(\mathbf{k},\epsilon) \approx 2\pi\delta(\bar{\rho}\epsilon/2 - E(\mathbf{k}))$. The delta function imposes the constraint $|\mathbf{k}| = |\mathbf{q}|$ and, since $\gamma(\mathbf{q},2E(\mathbf{q})/\bar{\rho}) \equiv \gamma(|\mathbf{q}|)$, Eq.(16) reduces to an algebraic equation. The solution is

$$\gamma(\mathbf{q},2E(\mathbf{q})/\bar{\rho}) = \frac{\Delta(\mathbf{k},2E(\mathbf{q})/\bar{\rho})}{\Delta_T(\mathbf{k},2E(\mathbf{q})/\bar{\rho})} \quad (17)$$

$$\Delta(\mathbf{k}, 2E(\mathbf{q})/\bar{\rho}) = \int \frac{d^2k}{(2\pi)^2} W_{\mathbf{qk}} \delta(E(\mathbf{q}) - E(\mathbf{k})) \quad (18)$$

$$\Delta_T(\mathbf{k}, 2E(\mathbf{q})/\bar{\rho}) = \int \frac{d^2k}{(2\pi)^2} W_{\mathbf{qk}} \delta(E(\mathbf{q}) - E(\mathbf{k})) \times \left(1 - \frac{(\mathbf{q}.\mathbf{k})^2}{|\mathbf{q}|^4}\right) \quad (19)$$

Eq.(18) is simply a re-writing of Eq.(8) for the imaginary part of the spinwave self-energy. The final term in the integrand of Eq.(19) is an angular weighting, $\sin^2\theta$, for scattering events, where θ is the angle between incoming and outgoing spinwave states. This should be compared with the electronic case, where the angular weighting is $1 - \cos\theta$.

Ignoring vertex corrections (substituting $\gamma = 1$), Eq.(14) reduces to

$$\sigma(\omega) = \omega \left(\frac{e\nu}{8\pi}\right)^2 \int \frac{d^2q}{(2\pi)^2} |\mathbf{q}|^2 \int_{-\infty}^{\infty} \frac{d\epsilon}{4\pi} \times [n_B(\epsilon+\omega) - n_B(\epsilon)] A(\mathbf{q},\epsilon) A(\mathbf{q},\epsilon+\omega). \quad (20)$$

A similar calculation of the finite wavevector conductivity, neglecting vertex corrections, gives

$$\sigma(\omega,\mathbf{k}) = \frac{1}{\omega} \left(\frac{e\nu}{8\pi}\right)^2 \int \frac{d^2q}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\epsilon}{4\pi} |\omega\mathbf{k} - \epsilon\mathbf{q}|^2 \times [n_B(\epsilon+\omega) - n_B(\epsilon)] A(\mathbf{q},\epsilon) A(\mathbf{q}+\mathbf{k},\epsilon+\omega).$$

In contrast to the zero wavevector conductivity, $\sigma(\omega, \mathbf{k})$ may be non-zero in the absence of disorder. Eq.(20) may now be used, in conjuction with the spinwave self-energy, Eqs.(9,10), in order to calculate the contribution of disorder scattered spinwaves to the optical conductivity. In the absence of disorder, the spectral function has a single delta-function peak, $A(\mathbf{q}, \epsilon) = 2\pi\delta(\bar{\rho}\epsilon/2 - E(\mathbf{q}))$. The effect of disorder is to broaden and shift this peak. For $T \ll g, \omega$ and weak disorder, the product $A(\mathbf{q}, \epsilon)A(\mathbf{q}, \epsilon + \omega)$, derived from Eqs.(9,10), is strongly peaked at $\bar{\rho}\epsilon/2 = E(\mathbf{q})$ and $\bar{\rho}(\epsilon + \omega)/2 = E(\mathbf{q})$ and may be approximated by

$$A(\mathbf{q},\epsilon)A(\mathbf{q},\epsilon+\omega) \approx 2\pi\delta\left(\bar{\rho}\epsilon/2 - E(\mathbf{q})\right) A(\mathbf{q},\epsilon+\omega) + 2\pi\delta\left(\bar{\rho}(\epsilon+\omega)/2 - E(\mathbf{q})\right) A(\mathbf{q},\epsilon).$$

The real part of the longitudinal optical conductivity, calculated within this approximation, is

$$\sigma(\omega) \approx \frac{K}{32\pi\rho_s^2} \left(\frac{e\nu}{8\pi}\right)^2 T^2 (1 - e^{-\omega/T}) e^{-2gB/T}. \quad (21)$$

At very small frequency, $\omega \ll KT$, the product $A(\mathbf{q}, \epsilon)A(\mathbf{q}, \epsilon + \omega)$ is no longer resolved into two peaks. The dominant frequency dependence in Eq.(20) then comes from the $n_B(\epsilon) - n_B(\epsilon + \omega)$ term. Then

$$A(\mathbf{q},\epsilon)A(\mathbf{q},\epsilon+\omega) \approx A^2(\mathbf{q},\epsilon) = \frac{2\pi\delta(\bar{\rho}\epsilon/2 - E(\mathbf{q}))}{\Im m\Sigma(\mathbf{q},\epsilon)}.$$

The energy and momentum integrals in Eq.(20) may then be carried out with the result

$$\sigma(\omega) \approx \frac{1}{\pi\rho_s^2 K} \left(\frac{e\nu}{8\pi}\right)^2 \omega^2 e^{-2gB/T} \text{ for } g \gg T. \quad (22)$$

For typical experimental systems at $\nu = 1$, an upper estimate for the disorder strength is $K \sim 0.1$ (approximating $n_d = \bar{\rho}$) and the spin stiffness $\rho_s \sim 4K$. The conductivities predicted by Eqs.(21,22) are vanishingly small and probably unmeasurable.

Magnetization The variation of magnetization with temperature, in the absence of disorder, has been calculated by Read and Sachdev[5], using a lowest order $1/N$ expansion. We extend this calculation to include the effect of disorder. Firstly, a Hopf map ($\mathbf{n} = \bar{z}_\alpha\sigma_{\alpha\beta}z_\beta$, $\sum_{\alpha=1}^{2} |z_\alpha|^2 = 1$) is used to recast the effective action, Eq.(1), into CP1 form;

$$S = \int d^2x dt \left[i\frac{\bar{\rho}}{2}\bar{z}\partial_t z + \rho_s|D_i z|^2 + \bar{\rho}gB\bar{z}\sigma^z z\right] - \int d^2x dt \left[U(\mathbf{x})J_0(\mathbf{x}) + \lambda\left(|z|^2 - 1\right)\right],$$
$$J_\nu = -\frac{i\nu e}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu\bar{z}_\alpha\partial_\lambda z_\alpha \quad (23)$$

where $D_i = \partial_i + i\theta_i$. θ_i is an auxiliary field, introduced in order to decouple quartic terms in the effective action. λ is a Lagrange multiplier that imposes the constraint. The indices on z_α have been suppressed for clarity.

To zeroth order in the $1/N$ expansion, the constraint is imposed at the mean field level in order to self-consistently determine the average value of the Lagrange multiplier, $\bar{\lambda}$[10]. The resulting gap equation is

$$\langle\langle\bar{z}z\rangle\rangle = \sum_{\sigma=\pm} \int \frac{d^2p}{(2\pi)^2} d\tilde{\Omega} \bar{\mathcal{G}}(i\tilde{\Omega}, \mathbf{p}^2, \sigma, \bar{\lambda}) = \sum_{\sigma=\pm} \int \frac{d^2p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} n_B(\epsilon) A(\epsilon, \mathbf{p}^2, \sigma, \bar{\lambda}), \quad (24)$$

where $\bar{\mathcal{G}}(i\tilde{\Omega}, \mathbf{p}^2, \sigma, \bar{\lambda})$ indicates the disorder average of the $\bar{z}z$-Green's function and $A(\epsilon, \mathbf{p}^2, \sigma, \bar{\lambda}) = -2\Im m\bar{G}_{ret}(\epsilon, \mathbf{p}^2, \sigma, \bar{\lambda})$ is the spectral function. We have carried out the frequency summation in order to obtain the final expression. The magnetization may also be calculated to this order and is given by

$$\langle\langle\bar{z}\sigma^z z\rangle\rangle = \sum_{\sigma=\pm} \int \frac{d^2p}{(2\pi)^2} d\tilde{\Omega}\sigma\bar{\mathcal{G}}(i\tilde{\Omega}, \mathbf{p}^2, \sigma, \bar{\lambda}) = \sum_{\sigma=\pm} \sigma \int \frac{d^2p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} n_B(\epsilon) A(\epsilon, \mathbf{p}^2, \sigma, \bar{\lambda}). \quad (25)$$

To O(1/N), Eq.(23), is identical to the sum of two copies of the spinwave action, Eq.(3), with the Zeeman term, $\bar{\rho}gB$, replaced with $\sigma\bar{\rho}gB + \bar{\lambda}$. The expressions for the self-energy derived above may be used directly with this replacement. In the absence of disorder, the spectral function has a single delta-function peak; $A(\epsilon, \mathbf{p}^2, \sigma, \bar{\lambda}) = 2\pi\delta(\bar{\rho}\epsilon/2 - E(\mathbf{p}^2, \sigma, \bar{\lambda}))$, where $E(\mathbf{p}^2, \sigma, \bar{\lambda}) = \rho_s \mathbf{p}^2 + \sigma\bar{\rho}gB + \bar{\lambda}$. Substitution of this into Eqs.(24,25), reproduces the result of [5]. The effect of disorder is to broaden and shift this peak. The real part of the self-energy produces a renormalization of the spin stiffness, $\rho_s \to \tilde{\rho}_s$. Upon direct substitution of Eq.(10), one finds that, to lowest order in K, the new position of the peak is at $\bar{\rho}\epsilon/2 = \tilde{E} - 4K^2\tilde{\rho}_s\mathbf{p}^2$ and so the shift due to the imaginary part of the self-energy may be incorporated as a further renormalization of the spin-stiffness. This is the dominant effect of weak disorder. The gap equation and magnetization are given by the disorder free expressions[5] with appropriately renormalized spin-stiffness[11].

The calculation of Ref.[5] shows good agreement with experiment[12] aside from at high temperatures, where the experimentally measured magnetization appears to fall below even the theoretical $\rho_s = 0$ prediction. Recent work[13] has shown that this discrepancy cannot be explained by the inclusion of higher orders in the 1/N expansion. Here, we have shown that neither can it be explained by the effects of weak disorder. In fact, to explain this observation would require spectral weight to be transfered below the Zeeman gap. This appears to be impossible so long as the groundstate remains ferromagnetic. Two possible alternative explanations lie in the effect of Skyrmions or the inclusion of the correct spinwave dispersion at high momenta. The latter approach has provided a good explanation for the dramatic reduction in magnetization with increasing temperature found at $\nu = 1/3$[14]. It is readily incorporated into the lowest order 1/N expansion in the absence of disorder, by inserting a spectral function with a delta-function peak at the correct spinwave dispersion into Eq.(24,25) and solving the resulting equations numerically.

In conclusion, we have considered the effect of weak disorder upon the quantum Hall ferromagnet. The identification of charge and topological charge of spinwave distortions allows a coupling of spins to the disorder potential. The signature of this coupling in the temperature dependence of magnetization is a reduction of the effective spin-stiffness. The effect upon conductivity is rather more interesting, although unfortunately it is probably unmeasurably small. We predict a spinwave contribution to the longitudinal optical conductivity at finite temperature.

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