Effective degrees of freedom & RFT resel counts

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Summary

In one of neuroimaging's key papers, Worsley et al. (1992) introduced the concept of a resolution element (resel) as an intuitive parametrisation of the roughness-adjusted search volume underlying random field theory's expression for corrected p-values. In another key development, Worsley and Friston (1995) showed that temporal correlation in a linear model reduced the effective degrees of freedom (eDF). Here, we illustrate a surprising connection between these quantities, which may have application to multiple comparison correction on data with elaborate statistical dependence that can make RFT over-conservative.

Theory

Resolution elements (resels)

An estimate of the (stationary) resels-per-voxel (RPV) is computed from the covariance of the spatial gradient of the normalised random fields, following Kiebel et al. (1999). In D dimensions, this can be expressed in units of full-width half-maximum (FWHM) as:

$$\mathrm{FWHM} = \mathrm{RPV}^{-1/D}. \tag{1}$$

Effective degrees of freedom (eDF or ν_e)

The sum of squares of ν independent standard normal variates defines the Chi-square distribution on ν degrees of freedom (DF); its mean and variance are ν and 2ν. Now consider n-vector $e \sim \mathcal{N}(0, V)$, where covariance V has eigenvalues d_i; $e'e$ is a d_i-weighted sum of unit Chi-square variates. The result is not Chi-square distributed, but its mean and variance are easily computed:

$$\mathrm{E}[e'e] = \sum_{i=1}^{n} d_i\,\mathrm{E}[\chi^2(1)] = \sum_{i=1}^{n} d_i = \mathrm{tr}V \tag{2}$$

$$\mathrm{V}[e'e] = \sum_{i=1}^{n} d_i^2\,\mathrm{V}[\chi^2(1)] = 2\sum_{i=1}^{n} d_i^2 = 2\,\mathrm{tr}V^2, \tag{3}$$

Matching these moments with a Chi-square on ν_e DF scaled by s, whose moments are $s\nu_e$ and $2s^2\nu_e$, yields

$$\nu_e = \frac{\mathrm{tr}^2 V}{\mathrm{tr}V^2}. \tag{4}$$

Simulations

Considering a simple 1-dimensional random process, we simulate 5000 realisations of a standard normal vector, each of which is smoothed with a 12 voxel FWHM Gaussian kernel and cropped to 100 elements to avoid edge-effects.

Results

Figure 1: Comparison of the covariance matrix estimated from smoothed data to the sampled theoretical Gaussian covariance function.

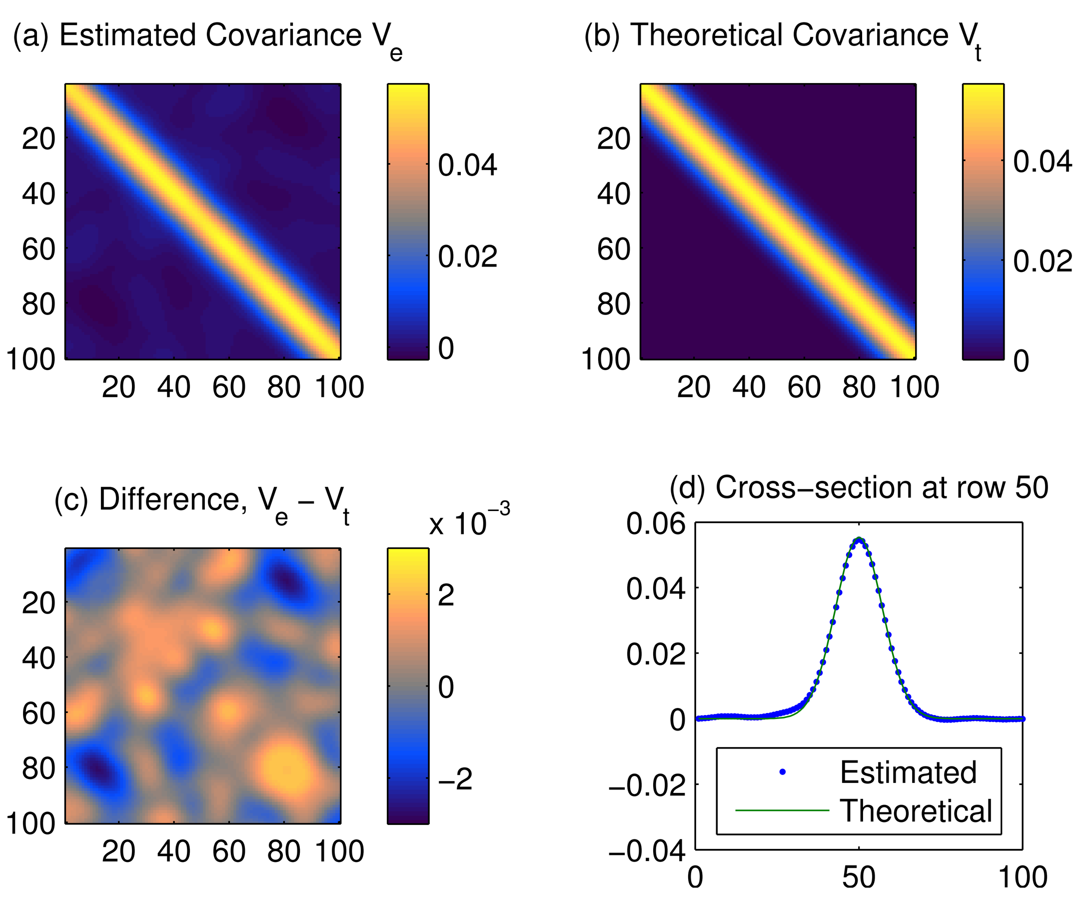


Figure 2: Eigenvectors of the theoretical covariance matrix corresponding to the four largest eigenvalues. They can be seen to resemble sinusoidal basis functions, so their eigenvalues should relate to the Fourier transform of the covariance function, which, according to the Wiener-Khinchin theorem, is the power spectral density.

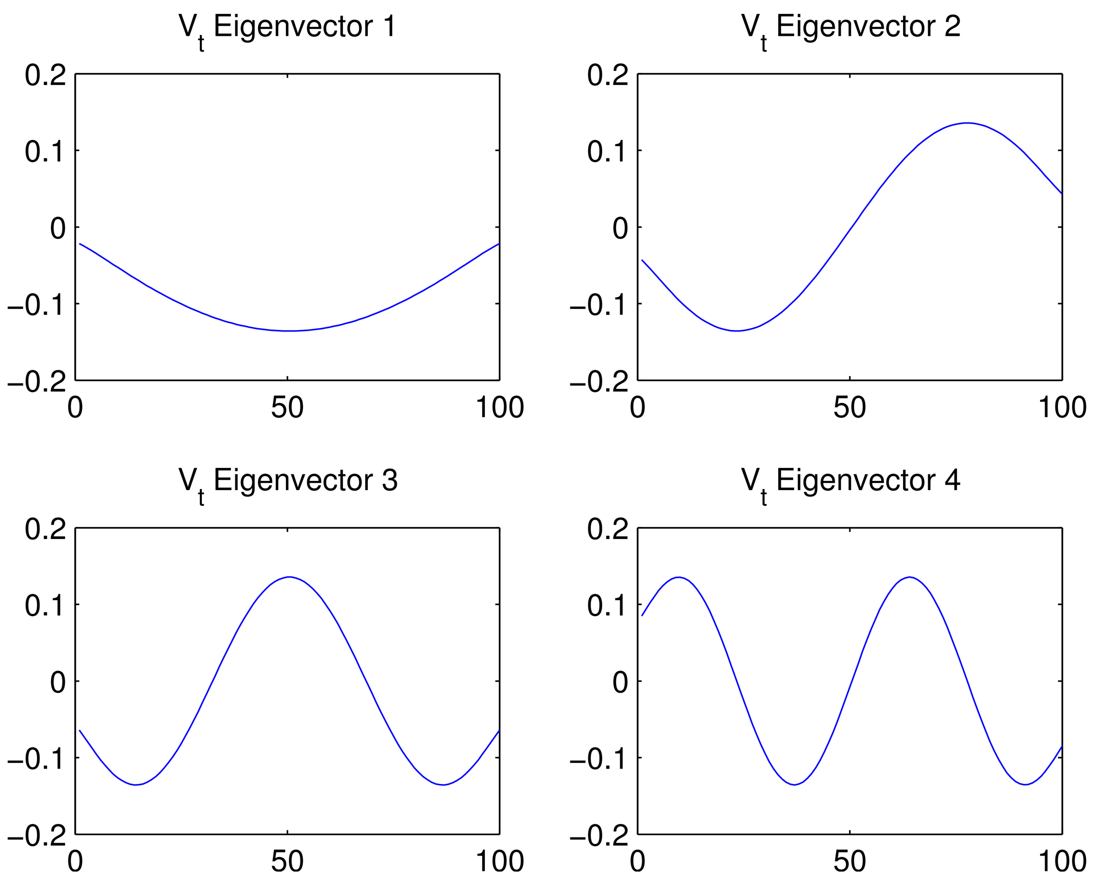
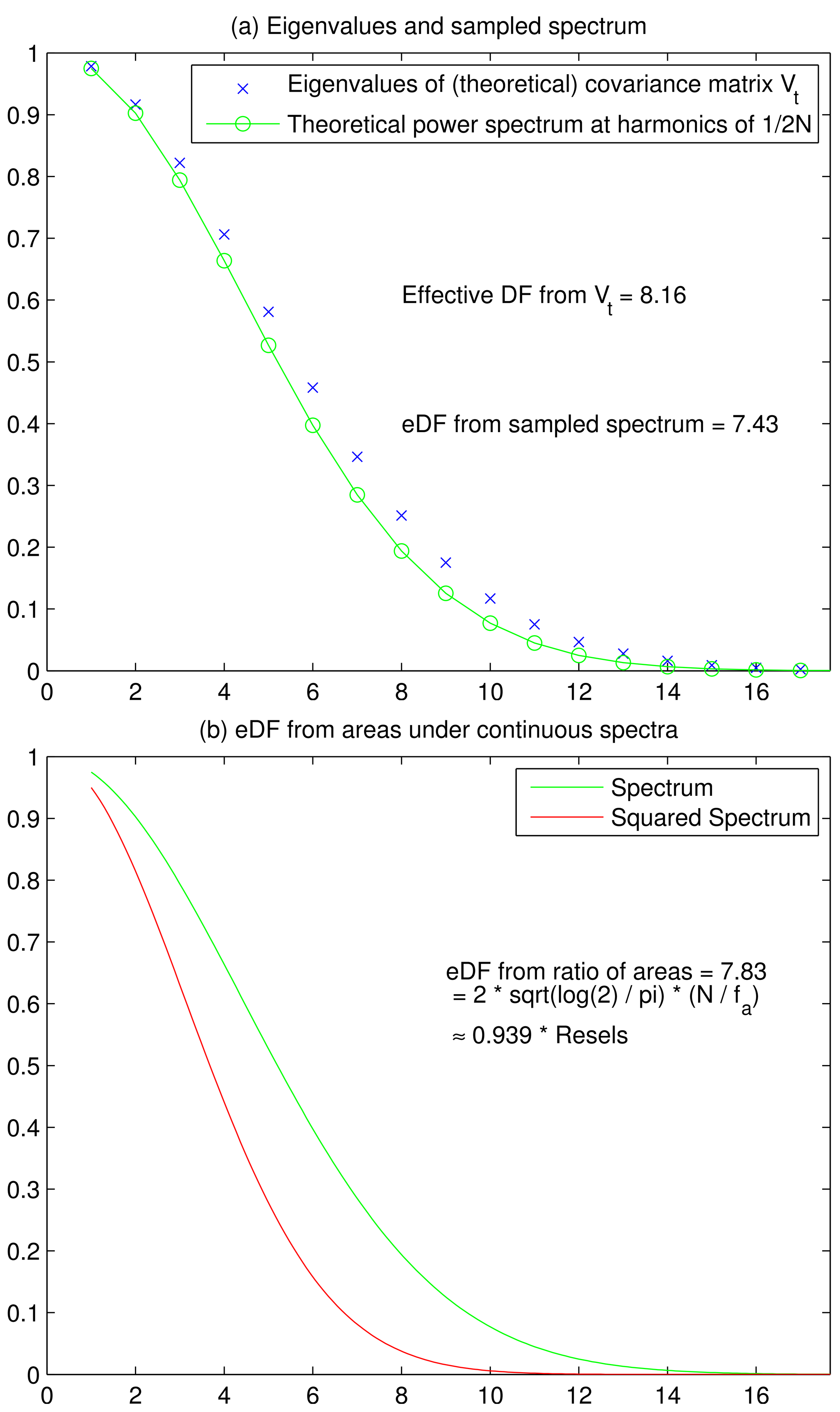


Figure 3: (a) The eigenvalues and spectral samples, showing reasonable agreement. The eDF is very close to the true resel count, and the equivalent from the spectrum is only slightly lower. (b) The continuous spectrum and its square; the ratio of the squared area to the area under the squared spectrum analytically yields eDF proportional to the expected number of resels, with scale factor near 94%.



The theoretical resel count is simply the product of the number of voxels and the RPV, or the number of voxels divided by the FWHM = 100 / 12 = 8.33; the estimated resel count (Kiebel et al., 1999) is in very close agreement: 8.29.

In the field of genomics, Gao et al. (2008) also proposed an estimator of the effective number of independent tests, defining M_{eff} such that the first M_{eff} summed eigenvalues of the correlation matrix amount to 99.5% of the total. Here, this results in an over-estimate of eDF as 14 (whether from theoretical or estimated correlation or covariance matrices).

Figure 4: Properties of the effective DF computed from the theoretical covariance matrix, over varying applied smoothness (FWHM).

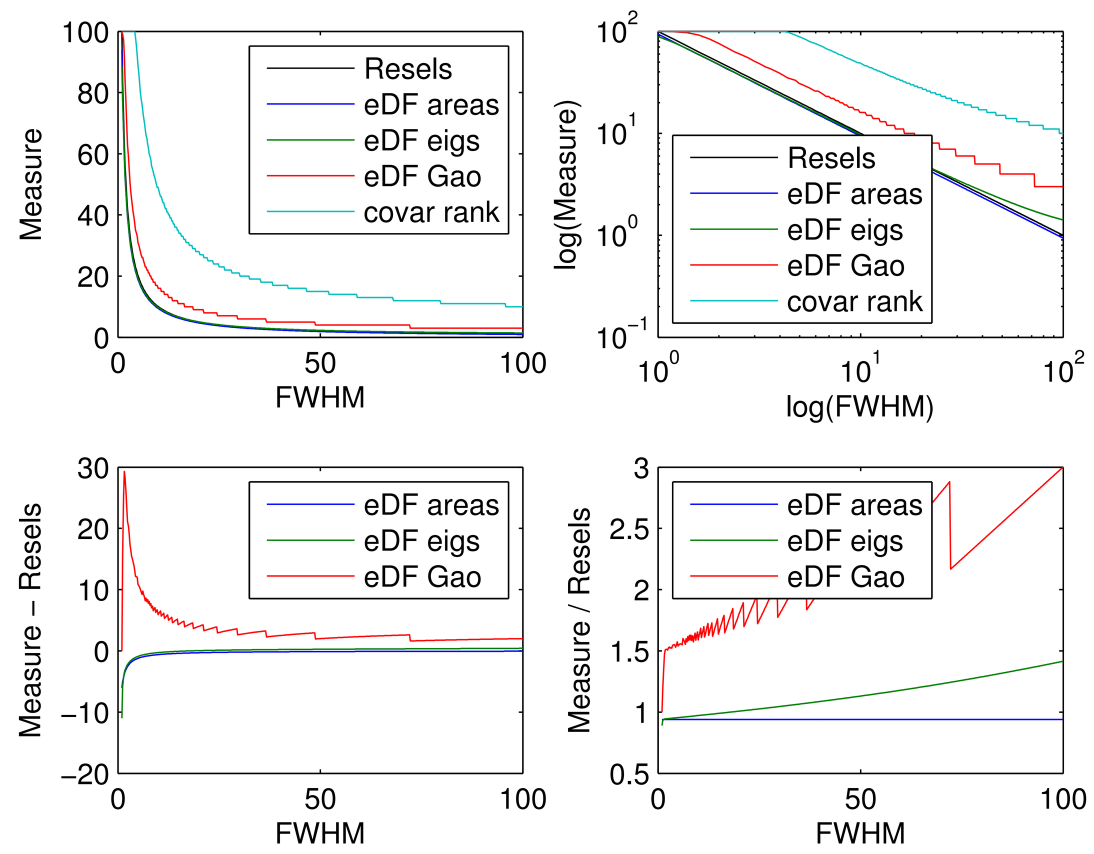
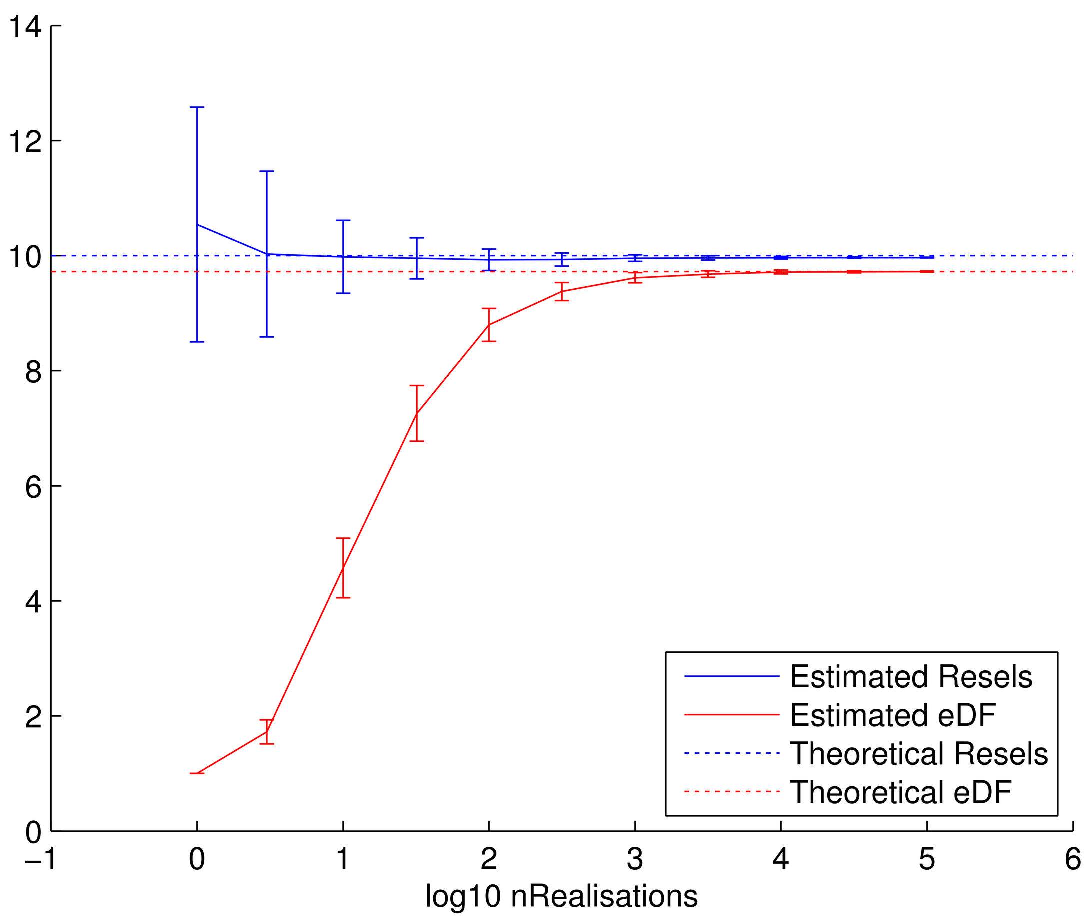


Figure 5: Properties of the estimators from empirical data over varying numbers of realisations, showing the break-down in the accuracy of the effective DF as the covariance matrix becomes less precisely estimated. Here, the applied FWHM was set to 10, the vector length remained at 100, and the number of realisations varied from 1 to 100,000 logarithmically.



Discussion

In data with complicated covariance structure, such as source reconstructed EEG or MEG, it will typically not be possible to estimate the full covariance accurately. Random field theory resolves this by focusing on the (possibly nonstationary) local smoothness, which is known to be valid but conservative in the presence of longer range correlations (Taylor and Worsley, 2007). The link between RFT and eDF shown here thus motivates considering eDF as a potentially less conservative estimate of the equivalent number of independent tests. Further research will investigate the performance of this measure in Bonferroni and related correction procedures.

References

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