Determination of Maximal Gaussian Entanglement Achievable by Feedback-Controlled Dynamics

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We determine a general upper bound for the steady-state entanglement achievable by continuous feedback for a system of any number of bosonic degrees of freedom. We apply such a bound to the specific case of parametric interactions—the most common practical way to generate entanglement in quantum optics—and single out optimal feedback strategies that achieve the maximal entanglement. We also consider the case of feedback schemes entirely restricted to local operations and compare their performance to the optimal, generally nonlocal, schemes.

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The field of quantum control is central in the current rise of quantum technologies [1,2]. In particular, the control of the coherent resources of quantum states is an issue of major interest. Most valuable and delicate among such resources is certainly quantum entanglement, whose control is a primary requisite for quantum information and communication [3–6]. This Letter addresses the question of how much entanglement can be generated by controlling the dynamics of a bosonic quantum system, and leads to the determination of optimal control schemes—achieving maximal entanglement—in relevant practical cases. In particular, we will consider systems subject to generic quadratic Hamiltonians and losses, and derive a bound on the maximal entanglement achievable, between specific bipartitions, by feedback schemes based on general continuous measurements and linear driving [7]. The class of dynamics and feedback strategies covered in our study is very important in quantum optics, and is applicable to more general continuous variable systems (ranging from atoms to nanomechanical resonators). Being crucial for the implementation of a number of quantum information protocols [8], the optimization of the generation of continuous variable entanglement has been drawing considerable attention in recent years [3–5]. Since entanglement is *not* a linear figure of merit in the quantum state's parameters, one cannot tackle this optimization with standard tools, like semidefinite programming [9], but rather requires the more detailed, specific analysis we shall present.

Notation.—We consider systems of N degrees of freedom described by pairs of canonical operators: defining a vector of operators $\hat{\mathbf{x}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^\top$, one has $[\hat{x}_j, \hat{x}_k] = i\Omega_{jk}$, where Ω is the $(2N)\times(2N)$ symplectic form: $\Omega_{jk} = \delta_{j+1,k}[1-(-1)^j]/2 - \delta_{j,k+1}[1+(-1)^j]/2$, in terms of Kronecker deltas $\delta_{j,k}$. Also, $\hat{a}_j = (\hat{q}_j + i\hat{p}_j)/\sqrt{2}$.

For a system with such a phase-space structure we can define "Gaussian states" as the states with Gaussian Wigner functions. These states are completely determined by the vector of means $\langle\hat{\mathbf{x}}\rangle$, and by the covariance matrix (CM) σ, with entries $\sigma_{jk} = (\langle\Delta\hat{x}_j\Delta\hat{x}_k\rangle + \langle\Delta\hat{x}_j\Delta\hat{x}_k\rangle)$, where $\Delta\hat{o} = (\hat{o} - \langle\hat{o}\rangle)$ for operator $\hat{o}$. The—always necessary—Robertson-Schrödinger uncertainty relation is also sufficient for Gaussian states to be physical [10]:

$$\sigma + i\Omega \geq 0. \tag{1}$$

We will consider Hamiltonians $\hat{H}$ that are at most of the second-order in $\hat{\mathbf{x}}$, so that their resulting free evolutions are affine in phase space: $\hat{H} = (1/2)\hat{\mathbf{x}}^\top H\hat{\mathbf{x}} - \hat{\mathbf{x}}^\top\Omega B\mathbf{u}(t)$, where the "Hamiltonian matrix" H is real and symmetric and B is real. The second term of $\hat{H}$ is a "linear driving" proportional to a time-dependent input $\mathbf{u}(t)$: this term will describe the control exerted over the system.

The system is considered to be open and such that each degree of freedom has its own channel to interact with the environment. Though thermal noise can also be treated along the lines we will present here, in this study we specialize for simplicity to pure losses, which are the main source of decoherence in quantum optical settings. We will thus assume a beam-splitter-like ("rotating wave") interaction between each mode and the associated modes of the bath. Under the conditions set out above, the first moments of the canonical operators evolve according to $d\langle\hat{\mathbf{x}}\rangle/dt = A\langle\hat{\mathbf{x}}\rangle + B\mathbf{u}(t)$, while the second moments obey

$$d\sigma/dt = A\sigma + \sigma A^\top + \mathbb{1}. \tag{2}$$

Here, $A = (\Omega H - \mathbb{1})/2$ is the "drift matrix", and $\mathbb{1}$ stands for the identity matrix with dimension clear from the context. We will only address stable systems, for which $(A + A^\top) < 0$. Note that, for Gaussian states, these equations describe the complete dynamics of the system.

As customary in the context of feedback control, we will now assume that the degrees of freedom of the environment can be continuously monitored on time scales which are short with respect to the system's response time [11]. The most general (efficient) measurement on the environment with outcomes continuous in time corresponds to monitoring the operators $(\hat{\mathbf{a}}^{\mathsf{T}}\mathbb{1} + \hat{\mathbf{a}}^{\dagger}\Upsilon)$, where the vector $\hat{\mathbf{a}} = (a_1, \ldots, a_N)^{\mathsf{T}}$ contains all the annihilation operators of the system, and the complex matrix Υ parametrizes the measurement. These measurements (also known as "general dyne" detections; see [1]) are very general, including heterodyne and homodyne detections as special cases, and define the broad setting of "continuous feedback" [1,7]. See [1] for a description of the POVM giving rise to such measurements. In turn, Υ defines the so-called "unraveling matrix" U, given by

$$U := \frac{1}{2}\begin{pmatrix} 1 + \mathrm{Re}[\Upsilon] & \mathrm{Im}[\Upsilon] \\ \mathrm{Im}[\Upsilon] & 1 - \mathrm{Re}[\Upsilon] \end{pmatrix}. \tag{3}$$

The only conditions on Υ are that U be symmetric and positive semidefinite. The outcome of the measurements on the environment is recorded as a "current" $\mathbf{y} = C\langle\hat{\mathbf{x}}\rangle + \frac{d\mathbf{w}}{dt}$, where $C = 2U^{1/2}\bar{C}$ and $\bar{C}_{jk} = (\delta_{2j-1,k} + \delta_{2(j-N),k})/\sqrt{2}$ for $j, k \in [1, \ldots, 2N]$. Finally, $d\mathbf{w}$ is a vector of real Wiener increments satisfying $d\mathbf{w}d\mathbf{w}^{\mathsf{T}} = \mathbb{1}dt$ [1]. Clearly this treatment, like any feedback model, applies to systems where the output channels are open to experimental scrutiny like, e.g., light modes resonating in a cavity (where leaking light can be detected). The *conditional* evolution of the moments under such continuous measurements can be derived by standard techniques (Itô calculus). It amounts to a diffusive equation with a stochastic component for the first moments $\langle\hat{\mathbf{x}}\rangle$, and to a *deterministic* Riccati equation for the second moments [9]. In our reasoning to follow, we will not make use of the details of such equations directly. We will be interested in stable systems, and will determine the maximal entanglement achievable at steady state. Hence, all we need to remark is that a CM σ is a *stabilizing solution* [12] of the Riccati equation for the second moments if and only if [9]:

$$A\sigma + \sigma A^{\mathsf{T}} + \mathbb{1} \geq 0. \tag{4}$$

Together with Ineq. (1), this relationship completely determines the set of stabilizing solutions of our conditional dynamics.

The final ingredient of the dynamics is the dependence of the linear driving $\mathbf{u}(t)$ on the history of the measurement record $\mathbf{y}(s)$ for $s < t$, which affects both first and second moments of the *unconditional*, "average", evolution (whereas the second moments of the *conditional* states are unaffected by the linear driving), and closes the control loop. We will denote the unconditional state by ϱ. Note that, for our class of dynamics, ϱ is a statistical mixture of states with the same conditional CM σ, obeying Inequality (4), and varying first moments. For Gaussian states, this implies that ϱ can be obtained from a Gaussian state ϱ_0 with CM σ and vanishing first moments by local operations and classical communication alone: $\varrho = L(\varrho_0)$, where L is some LOCC map.

The typical aim of control is to optimize the expected value of a *cost function* [1,12]. Our cost function will be the entanglement of Gaussian multimode steady states for bipartitions of 1 versus $(N-1)$ modes and "bisymmetric" bipartitions (i.e., invariant under the permutation of local modes). Such an entanglement can be quantified by the logarithmic negativity $E_{\mathcal{N}} = -\log_2\tilde{\nu}_-$, where $\tilde{\nu}_-^2$ is the smallest eigenvalue of $(-\sigma\tilde{\Omega}\sigma\tilde{\Omega}^{\mathsf{T}})$, $\tilde{\Omega}$ being the partial transposition of Ω [13,14]. Clearly, $\tilde{\nu}_-$ is *not* a quadratic cost function (i.e., it is not linear in σ). Thus, albeit dealing with linear systems with Gaussian noise, we cannot resort to optimization methods borrowed from classical LQG control problems [9].

General results.—the main analytical result of this Letter is presented here. Its proof may be found in the appendix.

Proposition 1 (maximal entanglement).—Let ϱ be a steady state achievable by continuous Gaussian measurements and linear driving for a system of any number of bosonic modes subject to losses and to a Hamiltonian matrix H. The logarithmic negativity $E_{\mathcal{N}}(\varrho)$ of any 1 versus $(N-1)$ modes or bisymmetric bipartition of ϱ is bounded by

$$E_{\mathcal{N}}(\varrho) \leq \max\left[0, -\tfrac{1}{2}\log_2(\alpha_1^{\uparrow}\alpha_2^{\uparrow})\right], \tag{5}$$

where $\{\alpha_j^{\uparrow}\}$ are the (strictly positive) eigenvalues of $(-A - A^{\mathsf{T}})$ in increasing order, and $A = \frac{1}{2}(\Omega H - \mathbb{1})$.

Inequality (5) corresponds to

$$\tilde{\nu}_-^2 \geq \alpha_1^{\uparrow}\alpha_2^{\uparrow}, \tag{6}$$

in terms of the smallest partially transposed symplectic eigenvalue of the Gaussian state ϱ.

The bound above applies to both conditional and unconditional states. In practice, only unconditional states are of interest since, although the first moments of the conditional states are in principle known, they fluctuate so fast (on the time scale of the environment's dynamics) that the actual experimental state is the unconditional, average one. This is where the linear driving plays its crucial role in preserving the entanglement. Since the entanglement (for us, the logarithmic negativity) only depends on the second moments and decreases under LOCC, and since the second moments of the conditional states do not depend on the linear drive, the optimal choice for the linear driving is the one, always existing, that keeps the first moments fixed (say, at zero). In this way, the linear drive's action guarantees that the unconditional state is at all times a conditional state—satisfying Ineq. (4)—with vanishing first moments. Hence, the optimal entangling strategy only depends on the optimal unravelling matrix U.

Applications.—Our theoretical result applies in general to all bosonic systems subject to losses and quadratic Hamiltonians. Here, we focus on optical modes oscillating in a damped cavity and interacting through a parametric $\chi^{(2)}$ crystal or more general nonlinear media (a "nondegenerate, multifrequency optical parametric oscillator" [15]). Parametric interactions are the state of the art technology to generate continuous variable entanglement. Also, optical bosonic systems can be interfaced with atomic systems [16], so that the feedback scheme could be used to control atomic entanglement as well.

The parametric interaction between modes j and k is described by the Hamiltonian $\chi(\hat{q}_j\hat{p}_k + \hat{p}_j\hat{q}_k)$ [17]. We will assume equal interaction strengths $\chi \geq 0$ between each pair of modes, consider a $(n+n)$-mode bipartition, and describe analytically the scaling of the control of the entanglement with the number of modes n (we also define $N = 2n$). Our bound in this case is tight, and yields the actual optimal entanglement achievable by continuous filtering. Because of the symmetry of the system under the exchange of any two modes, the entanglement between the n-modes subsystems can be reduced to two-mode entanglement [18]: a local symplectic transformation exists that turns the matrix A into an equivalent two-mode drift matrix $\bar{A}$, plus a direct sum of irrelevant decoupled single-mode matrices. The matrix $\bar{A}$ reads

$$\bar{A} = \begin{pmatrix} (n-1)\chi & 0 & n\chi & 0 \\ 0 & -(n-1)\chi & 0 & -n\chi \\ n\chi & 0 & (n-1)\chi & 0 \\ 0 & -n\chi & 0 & -(n-1)\chi \end{pmatrix} - \frac{\mathbb{1}}{2}. \tag{7}$$

For the system to be stable one must require: $\chi < \frac{1}{2(N-1)}$ (unstable systems, although in principle capable of generating substantial entanglement, are in practice not controllable and certainly undesirable). As $\bar{A}$ is symmetric and invertible, the "free" steady-state CM σ_f can be promptly determined from Eq. (2): $\sigma_f = -\bar{A}^{-1}/2$. Its logarithmic negativity is given by $\frac{1}{2}\log_2[(1+2\chi)(1+2(N-1)\chi)]$. Instead, the bound of Inequality (5) for any steady-state CM σ with continuous feedback control reads

$$E_{\mathcal{N}} \leq -\tfrac{1}{2}\{\log_2(1-2\chi) + \log_2[1-2(N-1)\chi]\}. \tag{8}$$

This upper bound is attained by the CM $\sigma_{\text{opt}} = R^{\mathsf{T}}\text{diag}(\alpha_2, 1/\alpha_2, 1/\alpha_1, \alpha_1)R$, where R is the orthogonal transformation that diagonalizes $\bar{A}$ and $\{\alpha_j\}$ are the eigenvalues of $-2\bar{A}$ in increasing order. This solution also saturates the Ineqs. (4) and (1). Both the free asymptotic entanglement and the optimal one under continuous filtering have thus been obtained analytically. Once the optimal achievable state is known as is the case here, the "optimal unravelling" U_{opt}, and hence the optimal feedback scheme, can be straightforwardly derived since $U_{\text{opt}} = E(\bar{A}\sigma_{\text{opt}} + \sigma_{\text{opt}}\bar{A}^{\mathsf{T}} + \mathbb{1})E^{\mathsf{T}}$, where $E = (2\bar{C}\sigma_{\text{opt}} - \bar{C})$ [9]. For two modes, this rigorously proves that the schemes considered in Ref. [4] are indeed optimal.

Local control.—Such an optimal entanglement is in general achieved by filtering the system through *global* measurements on the environment, as no restrictions were assumed for the unravelling matrix U. This applies to situations where the output channels of the two local subsystems can be combined before being measured (like,

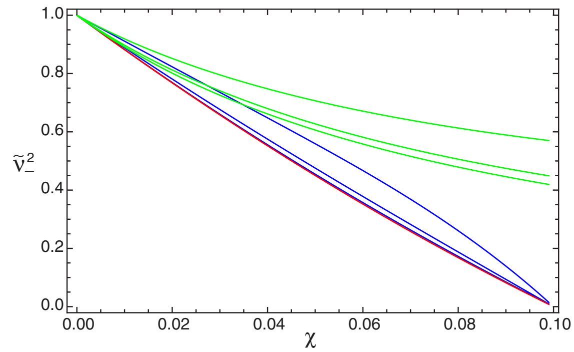

FIG. 1 (color online). Squared symplectic eigenvalue $\tilde{\nu}_-^2$ at steady state for a system of 6 modes ($\tilde{\nu}_- \to 0$ implies infinite entanglement). Green (lighter) curves depict $\tilde{nu}_-^2$ in the absence of control (from top to bottom: 1:5, 2:4, and 3:3 modes bipartition); blue (darker) curves refer to numerically optimized local feedback (from top to bottom: 1:5, 2:4, and 3:3 modes bipartition); the red curve is the analytical lower bound (6).

e.g., for a parametric crystal in a cavity). We intend now to provide a lower bound on the entanglement achievable under *local* control, where the environmental degrees of freedom pertaining to the separate subsystems cannot be combined, and compare it to the upper bound we obtained above. To this end, we will adopt direct (Markovian) feedback [7] and set $\mathbf{u}(t) = F\mathbf{y}(t)$. The unconditional evolution of the system is then described by

$$d\sigma/dt = A'\sigma + \sigma A'^{\mathsf{T}} + D', \tag{9}$$

with drift and diffusion matrices modified as $A' = \bar{A} + BFC$ and $D' = \mathbb{1} - C^{\mathsf{T}}F^{\mathsf{T}}B^{\mathsf{T}} - BFC + 2BFF^{\mathsf{T}}B^{\mathsf{T}}$. We also choose a specific form of U and BF. Since in the free dynamics, governed by the drift matrix of Eq. (7), the quadratures $\hat{p}_1$ and $\hat{p}_2$ are less noisy than $\hat{q}_1$ and $\hat{q}_2$, it is advantageous to monitor locally $\hat{p}_1$ and $\hat{p}_2$ and drive with the respective currents the quadratures $\hat{q}_2$ and $\hat{q}_1$. However, due to the possible asymmetry of the two subsystems for $m \neq n$, we have to consider different driving amplitudes μ_1 and μ_2 for their quadratures. All this corresponds to setting $U_{33} = U_{44} = 1$, $\sqrt{2}(BF)_{24} = \mu_2$, $\sqrt{2}(BF)_{43} = \mu_1$, and all other entries of U and BF vanishing. We can then find the steady-state solution of Eq. (9) as a function of the two feedback amplitudes μ_1 and μ_2, and evaluate its logarithmic negativity. It turns out that the maximum logarithmic negativity at steady state is attained for $\mu_2 = \mu_1 n/m$. Hence, we are left with the entanglement depending on one parameter, over which we minimize numerically in the stable region, determined by $(A' + A'^{\mathsf{T}}) < 0$. As a case of study, we have considered a system of 6 modes and summarized the results in Fig. 1. Because of the symmetry of the Hamiltonian, local control is very close to optimal global control in the case of a balanced bipartition. However, the more unbalanced the bipartition, the more degraded the control, although numerics indicate that arbitrarily large entanglement can always be retrieved approaching the instability.

Before concluding, let us further emphasize the usefulness of feedback control by describing the practical case of two modes with interaction strength to loss factor ratio $\chi = 0.45$. Without control, this system would generate 0.93 ebits of logarithmic negativity at steady state. The optimal feedback control would rise this value to 3.32 ebits. The Markovian local control discussed here, instead, allows one to reach 2.12 ebits showing that, in this instance, about half of the entanglement retrievable by measuring can be recovered from the environment by local measurements.

Conclusion.—We derived a bound on the entanglement achievable, at steady state and for various bipartitions, in multimode linear bosonic systems under continuous feedback control. When applied to the practical case of symmetric parametric interactions, our bound also allows one to determine the measurement strategy maximizing the steady-state entanglement, which is relevant to optimize the experimental generation of continuous variable entanglement, and hence useful for countless quantum information protocols [8]. More generally, our investigation yields a technique for the optimization of nonlinear figure of merits in bosonic quantum systems, with a broad range of applications in quantum information processing and state engineering.

Appendix—proof of proposition 1.—Henceforth, $|v\rangle$ will stand for a unit vector in the phase space Γ and $\{\lambda_j^\uparrow\}$ ($\{\lambda_j^\downarrow\}$) will be the $2N$ increasingly ordered (decreasingly ordered) eigenvalues of an N-mode CM σ. For each $|v\rangle$, one can define the unit vector $|w\rangle = \tilde{\Omega}\sigma^{1/2}|v\rangle/\sqrt{\langle v|\sigma|v\rangle}$, such that $\langle v|\sigma^{1/2}|w\rangle = 0$ (since $\tilde{\Omega} = -\tilde{\Omega}^\top$) and

$$\tilde{\nu}_-^2 \geq \min\langle v|\sigma|v\rangle\langle w|\sigma|w\rangle = \lambda_1^\uparrow\lambda_2^\uparrow. \qquad (10)$$

with the min taken over $|v\rangle$, $|w\rangle$ satisfying $\langle v|\sigma^{1/2}|w\rangle = 0$.

We will further denote by $|v_j\rangle$ the eigenvectors corresponding to the increasingly ordered eigenvalues of σ: $\sigma|v_j\rangle = \lambda_j^\uparrow|v_j\rangle$. Then, by using the Robertson-Schrödinger inequality and the Poincaré inequality [19], one can show that a vector $|w\rangle$ must exist in $\Omega\Gamma_k$ (defined as the subspace spanned by the k orthogonal vectors $\Omega|v_k\rangle$) for which $\langle w|\sigma|w\rangle \leq \lambda_k^\downarrow$, and such that

$$\lambda_k^\uparrow\lambda_k^\downarrow \geq 1. \qquad (11)$$

Now, let σ_∞ be a conditional CM at steady state obtained under continuous measurements, pure losses and a Hamiltonian matrix H. Applying Ineq. (4) to the eigenvectors corresponding to $\lambda_1^\downarrow$ and $\lambda_2^\downarrow$, one has for the two largest eigenvalues $\lambda_1^\downarrow$ and $\lambda_2^\downarrow$ of σ:

$$\lambda_1^\downarrow\lambda_2^\downarrow \leq \frac{1}{\alpha_1^\uparrow\alpha_2^\uparrow}, \qquad (12)$$

where $\{\alpha_j^\uparrow\}$ are the (strictly positive) eigenvalues of $(-A - A^\top)$ in increasing order. The chain of Ineqs. (10)–(12) leads to (6) for the partially transposed symplectic eigenvalue of the conditional state.

Finally, as we have seen previously, $\varrho = L(\varrho_0)$, where L is a LOCC operation and ϱ_0 a Gaussian state with a CM which is a stabilizing solution of (2). Hence $E_{\mathcal{N}}(\varrho) = E_{\mathcal{N}}(L(\varrho_0)) \leq E_{\mathcal{N}}(\varrho_0) \leq \max[0, -\log_2(\alpha_1^\uparrow\alpha_2^\uparrow)/2]$, where (6), the formula $E_{\mathcal{N}} = -\log_2(\tilde{\nu}_-)$, and the monotonicity of $E_{\mathcal{N}}$ under LOCC [20] have been invoked. ■

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[1] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, New York, 2010).
[2] W. P. Smith *et al.*, Phys. Rev. Lett. **89**, 133601 (2002); M. A. Armen *et al.*, *ibid.* **89**, 133602 (2002); S. Chaudhury *et al.*, *ibid.* **99**, 163002 (2007); R. L. Cook, P. J. Martin, and J. M. Geremia, Nature (London) **446**, 774 (2007); B. L. Higgins *et al.*, *ibid.* **450**, 393 (2007); B. L. Higgins *et al.*, Phys. Rev. Lett. **103**, 220503 (2009).
[3] S. Mancini and J. Wang, Eur. Phys. J. D **32**, 257 (2005); S. Mancini, Phys. Rev. A **73**, 010304(R) (2006).
[4] S. Mancini and H. M. Wiseman, Phys. Rev. A **75**, 012330 (2007).
[5] M. M. Wolf, J. Eisert, and M. B. Plenio, Phys. Rev. Lett. **90**, 047904 (2003); N. Schuch, M. M. Wolf, and J. I. Cirac, *ibid.* **96**, 023004 (2006).
[6] Ch. Hill and J. Ralph, Phys. Rev. A **77**, 014305 (2008).
[7] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. **70**, 548 (1993); Phys. Rev. A **49**, 1350 (1994).
[8] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[9] H. M. Wiseman and A. C. Doherty, Phys. Rev. Lett. **94**, 070405 (2005).
[10] A. S. Holevo, IEEE Trans. Inf. Theory **21**, 533 (1975); R. Simon, N. Mukunda, and B. Dutta, Phys. Rev. A **49**, 1567 (1994).
[11] H. M. Wiseman and L. Diósi, Chem. Phys. **268**, 91 (2001).
[12] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control* (Prentice-Hall, New Jersey, 1996).
[13] Partial transposition on a $(m+n)$-mode bipartition is described in phase space by $T = \mathbb{1} \oplus \sigma_z^{\oplus n}$, where the identity act on the first m modes while the direct sum of reflections σ_z acts on the other n modes. One then has $\tilde{\Omega} = T\Omega T$.
[14] For such bipartitions, only one partially transposed symplectic eigenvalue contributes to the logarithmic negativity. See A. Serafini, Phys. Rev. Lett. **96**, 110402 (2006).
[15] S. A. Diddams *et al.*, Opt. Lett. **24**, 1747 (1999); A. B. Matsko *et al.*, Phys. Rev. A **66**, 043814 (2002).
[16] See, e.g., B. Julsgaard *et al.*, Nature (London) **432**, 482 (2004).
[17] M. D. Reid, Phys. Rev. A **40**, 913 (1989).
[18] A. Serafini, G. Adesso, and F. Illuminati, Phys. Rev. A **71**, 032349 (2005).
[19] $\forall$ k-dimensional subspace Σ_k and hermitian M, $\exists$ $|v\rangle \in \Sigma_k$ such that $\langle v|v\rangle = 1$ and $\langle v|M|v\rangle \leq \lambda_k^\downarrow(M)$. See, e.g., R. Bhatia, *Matrix Analysis* (Springer, New York, 1996).
[20] M. B. Plenio, Phys. Rev. Lett. **95**, 090503 (2005).