NLEVP: A Collection of Nonlinear Eigenvalue Problems

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Abstract

We present a collection of 46 nonlinear eigenvalue problems in the form of a MATLAB toolbox. The collection contains problems from models of real-life applications as well as ones constructed specifically to have particular properties. A classification is given of polynomial eigenvalue problems according to their structural properties. Identifiers based on these and other properties can be used to extract particular types of problems from the collection. A brief description of each problem is given. NLEVP serves both to illustrate the tremendous variety of applications of nonlinear eigenvalue problems and to provide representative problems for testing, tuning, and benchmarking of algorithms and codes.

Categories and Subject Descriptors: G.4 [Mathematical Software]; G.1.3 [Numerical Linear Algebra]: Eigenvalues and eigenvectors (direct and iterative methods)

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1 Introduction

In many areas of scientific computing collections of problems are available that play an important role in developing algorithms and in testing and benchmarking software. Among the uses of such collections are

- tuning an algorithm to optimize its performance across a wide and representative range of problems;
- testing the correctness of a code against some measure of success, where the latter is typically an error or residual whose nature is suggested by the underlying problem;
- measuring the performance of a code—for example, speed, execution rate, or again an error or residual;
- measuring the robustness of a code, that is, the behaviour in extreme situations, such as for very badly scaled and/or ill conditioned data;
- comparing two or more different codes with respect to the factors above.

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A collection ideally combines problems artificially constructed to reflect a wide range of possible properties with problems representative of real applications. Problems for which something is known about the solution are always particularly attractive.

The practice of reproducible research, whereby research is published in such a way that the underlying numerical (and other) experiments can be repeated by others, has a growing number of adherents [25], [61]. Reproducible research is aided by the availability of well documented and maintained benchmark collections.

Two areas that have historically been well endowed with collections of problems implemented in software are linear algebra and optimization. In linear algebra an early collection is ACM Algorithm 694 [40], which contains parametrized, mainly dense, test matrices, most of which were later incorporated into the MATLAB `gallery` function. The University of Florida Sparse Matrix Collection is a regularly updated collection of sparse matrices [21], [22], with over 2200 matrices from practical applications. Matrix Market [66] also provides access to several collections of matrices, though at the time of writing it has not been updated for several years. Both the latter collections include the Harwell–Boeing collection [26] of sparse matrices and the NEP collection [3] of standard and generalized eigenvalue problems. The CONTEST toolbox [75] produces adjacency matrices describing random networks. In optimization we mention just the collections in the widely used Cute and Cuter testing environments [9], [33], though various other, sometimes more specialized, collections are available.

The growing interest in nonlinear eigenvalue problems has created the need for a collection of problems in this area. The standard form of a nonlinear eigenvalue problem is $F(\lambda)x = 0$, where $F : \mathbb{C} \to \mathbb{C}^{m\times n}$ is a given matrix-valued function and $\lambda \in \mathbb{C}$ and the nonzero vector $x \in \mathbb{C}^n$ are the sought eigenvalue and eigenvector, respectively. Rational and polynomial functions are of particular interest, the most practically important case being the quadratic $Q(\lambda) = \lambda^2 A + \lambda B + C$, which corresponds to the quadratic eigenvalue problem. For recent surveys on nonlinear eigenproblems see [67] and [80]. Associated with an $n \times n$ matrix quadratic $Q(\lambda)$ are the matrix equations $X^2A + XB + C = 0$ and $AX^2 + BX + C = 0$, where the unknown $X \in \mathbb{C}^{n\times n}$ is called a solvent [24], [31] [42]. Thus a matrix polynomial $P(\lambda)$ defines both an eigenvalue problem and two matrix equations.

We have built a collection of nonlinear eigenvalue problems from a variety of sources. Some are from models of real-life applications, while others have been constructed specifically to have particular properties. Many of the matrices have been used in previous papers to test numerical algorithms. In order to provide focus and keep the collection to a manageable size we have chosen to exclude linear problems from the collection. The problems range from the old, such as the `wing` problem from the classic 1938 book of Frazer, Duncan, and Collar [30], to the very recent, notably several problems from research in 3D vision that are not yet well known in the numerical analysis community.

Nonlinear eigenvalue problems are often highly structured and it is important to take account of the structure both in developing the theory and in designing numerical methods. We therefore provide a thorough classification of our problems that records the most relevant structural properties.

We have chosen to implement the collection in MATLAB, as a toolbox, recognizing that it is straightforward to convert the matrices into a format that can be read by other languages by using either the built-in MATLAB I/O functions or those provided in Matrix Market. A criterion for inclusion of problems is that the underlying MATLAB code and data files are not too large, since we want to provide the toolbox as a single file that can be downloaded in a reasonable time.

The NLEVP toolbox is available, as both a zip file and a tar file, from

`http://www.mims.manchester.ac.uk/research/numerical-analysis/nlevp.html`

For details of how to install and use the toolbox see [7].

In Section 2 we explain how we classify the problems through identifers that can be used to extract specific types of problem from the collection. The main features of the problems are described in Section 3, while Section 4 describes the design of the toolbox. Conclusions are given in Section 5.

2 Identifiers

We give in Table 1 a list of identifiers for the types of problems available in the collection and in Table 2 a list of identifiers that specify the properties of problems in the collection. These properties can be used to extract specialized subsets of the collection for use in numerical experiments. All the identifiers are case insensitive. In the next two subsections we briefly recall some relevant definitions and properties of nonlinear eigenproblems.

2.1 Nonlinear Eigenproblems

The **polynomial eigenvalue problem** (PEP) is to find scalars λ and nonzero vectors x and y satisfying $P(\lambda)x = 0$ and $y^*P(\lambda) = 0$, where

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \qquad A_i \in \mathbb{C}^{m\times n}, \quad A_k \neq 0 \tag{1}$$

is an $m \times n$ matrix polynomial of degree k. Here, x and y are right and left eigenvectors corresponding to the eigenvalue λ. The **reversal** of the matrix polynomial (1) is defined by

$$\mathrm{rev}\big(P(\lambda)\big) = \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i.$$

A PEP is said to have an eigenvalue ∞ if zero is an eigenvalue of $\mathrm{rev}(P(\lambda))$.

A **quadratic eigenvalue problem** (QEP) is a PEP of degree $k = 2$. For a survey of QEPs see [80]. Polynomial and quadratic eigenproblems are identified by `pep` and `qep`, respectively, in the collection (see Table 1), and any problem of type `qep` is automatically also of type `pep`.

The matrix function $R(\lambda) \in \mathbb{C}^{m\times n}$ whose elements are rational functions

$$r_{ij}(\lambda) = \frac{p_{ij}(\lambda)}{q_{ij}(\lambda)}, \qquad 1 \le i \le m, \quad 1 \le j \le n,$$

where $p_{ij}(\lambda)$ and $q_{ij}(\lambda)$ are scalar polynomials of the same variable and $q_{ij}(\lambda) \not\equiv 0$, defines a **rational eigenvalue problem** (REP) $R(\lambda)x = 0$ [56]. Unlike for PEPs there is no standard format for specifying REPs. For the collection we use the form

$$R(\lambda) = P(\lambda)Q(\lambda)^{-1},$$

where $P(\lambda)$ and $Q(\lambda)$ are matrix polynomials, or the less general form (often encountered in practice)

$$R(\lambda) = A + \lambda B + \sum_{i=1}^{k-1} \frac{\lambda}{\sigma_i - \lambda} C_i, \tag{2}$$

where A, B, and the C_i are $m \times n$ matrices, and the σ_i are the poles. Which form is used is specified in the help for the M-file defining the problem. Rational eigenproblems are identified by `rep` in the collection.

As mentioned in the introduction, PEPs and REPs are special cases of **nonlinear eigenvalue problems** (NEPs) $F(\lambda)x = 0$, where $F : \mathbb{C} \to \mathbb{C}^{m\times n}$. A convenient general form for expressing an NEP is

$$F(\lambda) = \sum_{i=0}^{k} f_i(\lambda) A_i, \tag{3}$$

where the $f_i : \mathbb{C} \to \mathbb{C}$ are nonlinear functions and $A_i \in \mathbb{C}^{m\times n}$. Any problem that is not polynomial, quadratic, or rational is identified by `nep` in the collection (see Table 1).

Table 1: Problems available in the collection and their identifiers.

`qep`	quadratic eigenvalue problem
`pep`	polynomial eigenvalue problem
`rep`	rational eigenvalue problem
`nep`	other nonlinear eigenvalue problem

Table 2: List of identifiers for the problem properties.

`nonregular`	`symmetric`	`hyperbolic`
`real`	`hermitian`	`elliptic`
`nonsquare`	`T-even`	`overdamped`
`sparse`	`*-even`	`proportionally-damped`
`scalable`	`T-odd`	`gyroscopic`
`parameter-dependent`	`*-odd`	
`solution`	`T-palindromic`	
	`*-palindromic`	
	`T-anti-palindromic`	
	`*-anti-palindromic`	

2.2 Some Definitions and Properties

Nonlinear eigenproblems are said to be **regular** if $m = n$ and $\det(F(\lambda)) \not\equiv 0$, and **nonregular** otherwise. Recall that a regular PEP possesses nk (not necessarily distinct) eigenvalues [31], including infinite eigenvalues. As the majority of problems in the collection are regular we identify only nonregular problems, for which the identifier is `nonregular`.

The identifiers `real`, `hermitian`, and `symmetric` are defined in Table 3. For PEPs, the `real` identifier corresponds to P having real coefficient matrices, while `hermitian` corresponds to Hermitian (but not all real) coefficient matrices. Similarly, `symmetric` indicates (complex) symmetric coefficient matrices, and the `real` identifier is added if the coefficient matrices are real symmetric. For problems that are parameter-dependent the identifiers `real` and `hermitian` are used if the problem is real or Hermitian for real values of the parameter.

Definitions of identifiers for odd-even and palindromic-like square matrix polynomials, together with the special symmetry properties of their spectra (see [63]) are given in Table 4.

Gyroscopic systems of the form $Q(\lambda) = \lambda^2 M + \lambda G + K$ with M, K Hermitian, $M > 0$, and $G = -G^*$ skew-Hermitian are a subset of $*$-even (T-even when the coefficient matrices are real) QEPs and are identified with `gyroscopic`. Here, for a Hermitian matrix A, we write $A > 0$ to denote that A is positive definite and $A \geq 0$ to denote that A is positive semidefinite. When $K > 0$

Table 3: Some identifiers and the corresponding spectral properties. For parameter-dependent problems, the problem is classified as `real` or `hermitian` if it is so for real values of the parameter.

Identifier	Property of $F(\lambda) \in \mathbb{C}^{m\times n}$	Spectral properties
`real`	$\overline{F(\lambda)} = F(\bar{\lambda})$	eigenvalues real or come in pairs $(\lambda, \bar{\lambda})$
`symmetric`	$m = n$, $\big(F(\lambda)\big)^T = F(\lambda)$	none unless F is real
`hermitian`	$m = n$, $(F(\lambda))^* = F(\bar{\lambda})$	eigenvalues real or come in pairs $(\lambda, \bar{\lambda})$

Table 4: Some identifiers and the corresponding spectral symmetry properties.

Identifier	Property of $P(\lambda)$	Eigenvalue pairing
`T-even`	$P^T(-\lambda) = P(\lambda)$	$(\lambda, -\lambda)$
`*-even`	$P^*(-\lambda) = P(\lambda)$	$(\lambda, -\bar{\lambda})$
`T-odd`	$P^T(-\lambda) = -P(\lambda)$	$(\lambda, -\lambda)$
`*-odd`	$P^*(-\lambda) = -P(\lambda)$	$(\lambda, -\bar{\lambda})$
`T-palindromic`	$\mathrm{rev}P^T(\lambda) = P(\lambda)$	$(\lambda, 1/\lambda)$
`*-palindromic`	$\mathrm{rev}P^*(\lambda) = P(\lambda)$	$(\lambda, 1/\bar{\lambda})$
`T-anti-palindromic`	$\mathrm{rev}P^T(\lambda) = -P(\lambda)$	$(\lambda, 1/\lambda)$
`*-anti-palindromic`	$\mathrm{rev}P^*(\lambda) = -P(\lambda)$	$(\lambda, 1/\bar{\lambda})$

the eigenvalues of Q are purely imaginary and semisimple [27], [58] and the quadratic $Q(i\lambda)$ is hyperbolic.

A Hermitian matrix polynomial $P(\lambda)$ is **hyperbolic** if there exists $\mu \in \mathbb{R} \cup \{\infty\}$ such that $P(\mu)$ is positive definite and for every nonzero $x \in \mathbb{C}^n$ the scalar equation $x^*P(\lambda)x = 0$ has k distinct zeros in $\mathbb{R} \cup \{\infty\}$. All the eigenvalues of such a P are real, semisimple, and grouped in k intervals, each of them containing n eigenvalues [1], [43], [65]. These polynomials are identified in the collection by `hyperbolic`. **Overdamped** systems $Q(\lambda) = \lambda^2 M + \lambda C + K$ are particular hyperbolic QEPs for which $M > 0$, $C > 0$, and $K \geq 0$; they have the identifier `overdamped`. Finally, a QEP is said to be **proportionally damped** when M, C, and K are simultaneously diagonalizable by congruence or strict equivalence [60] (a sufficient condition for which is that $C = \alpha M + \beta K$ with M and K simultaneously diagonalizable, hence the name), and such a QEP is identified by `proportionally-damped`.

Hermitian matrix polynomials $P(\lambda)$ with even degree k that are **elliptic**, i.e., $P(\lambda) > 0$ for all $\lambda \in \mathbb{R}$ [65, §34], are identified by `elliptic`. Elliptic matrix polynomials have nonreal eigenvalues.

The identifier `sparse` is used if the defining matrices are stored in MATLAB's sparse format. Problems that depend on one or more parameters are identified with `parameter-dependent`. A separate identifier, `scalable`, is used to denote that the problem dimension (or a function of it) is a parameter; for such problems a default value of the parameter is provided, typically being a value used in previously published experiments.

For some problems a *supposed* solution (eigenvalues and/or eigenvectors) is returned via the last output parameter, being either an exactly known solution or an approximate or computed solution. These problems are identified with `solution`. The documentation for the matrix provides information on the nature of the supposed solution.

Tables 5–10 identify the QEPs, the PEPs that are of degree at least 3, the nonsquare PEPs, the REPs, and the nonlinear but non-polynomial and non-rational problems in the collection.

3 Collection of Problems

This section contains a brief description of all the problems in the collection. The identifiers for the problem properties are listed inside curly brackets after the name of each problem. The problems are summarized in Table 11.

We use the following notation. $A \otimes B$ denotes the Kronecker product of A and B, namely the block matrix $(a_{ij}B)$ [41, Sec. B.13]. The ith unit vector (that is, the ith column of the identity matrix) is denoted by e_i.

Acoustic wave 1D {`pep,qep,symmetric,*-even,parameter-dependent,scalable`}. This quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$ arises from the finite element discretization of the time-harmonic wave equation $-\Delta p - (2\pi f/c)^2 p = 0$ for the acoustic pressure p in a bounded domain, where the boundary conditions are partly Dirichlet ($p = 0$) and partly impedance ($\frac{\partial p}{\partial n} + \frac{2\pi i f}{\zeta} p = 0$)

Table 5: Quadratic eigenvalue problems.

acoustic_wave_1d	acoustic_wave_2d	bicycle	bilby
cd_player	closed_loop	concrete	damped_beam
dirac	foundation	gen_hyper2	intersection
hospital	metal_strip	mobile_manipulator	omnicam1
omnicam2	pdde_stability	power_plant	qep1
qep2	qep3	qep4	railtrack
railtrack2	relative_pose_6pt	schrodinger	shaft
sign1	sign2	sleeper	speaker_box
spring	spring_dashpot	surveillance	wing
wiresaw1	wiresaw2		

Table 6: Polynomial eigenvalue problems of degree 3 and higher.

butterfly	orr_sommerfeld	plasma_drift	relative_pose_5pt

Table 7: Nonsquare polynomial eigenvalue problems.

qep4	surveillance

Table 8: Nonregular polynomial eigenvalue problems.

qep4	surveillance

Table 9: Rational eigenvalue problems.

loaded_string

Table 10: Nonlinear (but not rational or polynomial) eigenvalue problems.

fiber	gun	hadeler

Table 11: Problems in NLEVP.

`acoustic_wave_1d`	Acoustic wave problem in 1 dimension.
`acoustic_wave_2d`	Acoustic wave problem in 2 dimensions.
`bicycle`	2-by-2 QEP from the Whipple bicycle model.
`bilby`	5-by-5 QEP from bilby population model.
`butterfly`	Quartic matrix polynomial with T-even structure.
`cd_player`	QEP from model of CD player.
`closed_loop`	2-by-2 QEP associated with closed-loop control system.
`concrete`	Sparse QEP from model of a concrete structure.
`damped_beam`	QEP from simply supported beam damped in the middle.
`dirac`	QEP from Dirac operator.
`fiber`	NEP from fiber optic design.
`foundation`	Sparse QEP from model of machine foundations.
`gen_hyper2`	Hyperbolic QEP constructed from prescribed eigenpairs.
`gun`	NEP from model of a radio-frequency gun cavity.
`hadeler`	NEP due to Hadeler.
`intersection`	10-by-10 QEP from intersection of three surfaces.
`hospital`	QEP from model of Los Angeles Hospital building.
`loaded_string`	REP from finite element model of a loaded vibrating string.
`metal_strip`	QEP related to stability of electronic model of metal strip.
`mobile_manipulator`	QEP from model of 2-dimensional 3-link mobile manipulator.
`omnicam1`	9-by-9 QEP from model of omnidirectional camera.
`omnicam2`	15-by-15 QEP from model of omnidirectional camera.
`orr_sommerfeld`	Quartic PEP arising from Orr-Sommerfeld equation.
`pdde_stability`	QEP from stability analysis of discretized PDDE.
`plasma_drift`	Cubic PEP arising in Tokamak reactor design.
`power_plant`	8-by-8 QEP from simplified nuclear power plant problem.
`qep1`	3-by-3 QEP with known eigensystem.
`qep2`	3-by-3 QEP with known, nontrivial Jordan structure.
`qep3`	3-by-3 parametrized QEP with known eigensystem.
`qep4`	3-by-4 QEP with known, nontrivial Jordan structure.
`railtrack`	QEP from study of vibration of rail tracks.
`railtrack2`	Palindromic QEP from model of rail tracks.
`relative_pose_5pt`	Cubic PEP from relative pose problem in computer vision.
`relative_pose_6pt`	QEP from relative pose problem in computer vision.
`schrodinger`	QEP from Schrodinger operator.
`shaft`	QEP from model of a shaft on bearing supports with a damper.
`sign1`	QEP from rank-1 perturbation of sign operator.
`sign2`	QEP from rank-1 perturbation of 2*sin(x) + sign(x) operator.
`sleeper`	QEP modelling a railtrack resting on sleepers.
`speaker_box`	QEP from model of a speaker box.
`spring`	QEP from finite element model of damped mass-spring system.
`spring_dashpot`	QEP from model of spring/dashpot configuration.
`surveillance`	21-by-16 QEP from surveillance camera callibration.
`wing`	3-by-3 QEP from analysis of oscillations of a wing in an airstream.
`wiresaw1`	Gyroscopic QEP from vibration analysis of a wiresaw.
`wiresaw2`	QEP from vibration analysis of wiresaw with viscous damping effect.

[19]. Here, f is the frequency, c is the speed of sound in the medium, and ζ is the (possibly complex) impedance. We take $c = 1$ as in [19]. The eigenvalues of Q are the resonant frequencies of the system, and for the given problem formulation they lie in the upper half of the complex plane.

On the 1D domain $[0, 1]$ the $n \times n$ matrices are defined by

$$M = -4\pi^2 \frac{1}{n}(I_n - \frac{1}{2} e_n e_n^T), \quad C = 2\pi i \frac{1}{\zeta}\, e_n e_n^T, \quad K = n \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}.$$

Acoustic wave 2D {`pep,qep,symmetric,*-even,parameter-dependent,scalable`}. A 2D version of Acoustic wave 1D. On the unit square $[0, 1] \times [0, 1]$ with mesh size h the $n \times n$ coefficient matrices of $Q(\lambda)$ with $n = \frac{1}{h}(\frac{1}{h} - 1)$ are given by

$$M = -4\pi^2 h^2 I_{m-1} \otimes (I_m - \tfrac{1}{2} e_m e_m^T), \quad D = 2\pi i \tfrac{h}{\zeta} I_{m-1} \otimes (e_m e_m^T),$$
$$K = I_{m-1} \otimes D_m + T_{m-1} \otimes (-I_m + \tfrac{1}{2} e_m e_m^T),$$

where $\otimes$ denotes the Kronecker product, $m = 1/h$, ζ is the (possibly complex) impedance, and

$$D_m = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 4 & -1 \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{m \times m}, \qquad T_{m-1} = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(m-1)\times(m-1)}.$$

The eigenvalues of Q are the resonant frequencies of the system, and for the given problem formulation they lie in the upper half of the complex plane.

Bicycle {`pep,qep,real,parameter-dependent`}. This is a 2×2 quadratic polynomial arising in the study of bicycle self-stability [69]. The linearized equations of motion for the Whipple bicycle model can be written as

$$M\ddot{q} + C\dot{q} + Kq = f,$$

where M is a symmetric mass matrix, the nonsymmetric damping matrix $C = vC_1$ is linear in the forward speed v, and the stiffness matrix $K = gK_0 + v^2 K_2$ is the sum of two parts: a velocity independent symmetric part gK_0 proportional to the gravitational acceleration g and a nonsymmetric part $v^2 K_2$ quadratic in the forward speed.

Bilby {`pep,qep,real,parameter-dependent`}. This 5×5 quadratic matrix polynomial arises in a model from [4] for the population of the greater bilby (*Macrotis lagotis*), an endangered Australian marsupial. Define the 5×5 matrix

$$M(g, x) = \begin{bmatrix} gx_1 & (1-g)x_1 & 0 & 0 & 0 & 0 \\ gx_2 & 0 & 0 & (1-g)x_2 & 0 & 0 \\ gx_3 & 0 & 0 & 0 & (1-g)x_3 & 0 \\ gx_4 & 0 & 0 & 0 & 0 & (1-g)x_4 \\ gx_5 & 0 & 0 & 0 & 0 & (1-g)x_5 \end{bmatrix}.$$

The model is a quasi-birth-death process some of whose key properties are captured by the elementwise minimal solution of the quadratic matrix equation

$$R = \beta(A_0 + RA_1 + R^2 A_2), \qquad A_0 = M(g, b), \quad A_1 = M(g, e - b - d), \quad A_2 = M(g, d),$$

where b and d are vectors of probabilities and e is the vector of ones. The corresponding quadratic matrix polynomial is $Q(\lambda) = \lambda^2 A + \lambda B + C$, where

$$A = \beta A_2^T, \quad B = \beta A_1^T - I, \quad C = \beta A_0^T.$$

We take $g = 0.2$, $b = [1,\ 0.4,\ 0.25,\ 0.1,\ 0]^T$, and $d = [0,\ 0.5,\ 0.55,\ 0.8,\ 1]^T$, as in [4].

Butterfly {`pep,real,parameter-dependent,T-even,scalable`}. This is a quartic matrix polynomial $P(\lambda) = \lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$ of dimension m^2 with T-even structure, depending on a 10×1 parameter vector c [68]. Its spectrum has a butterfly shape. The coefficient matrices are Kronecker products, with A_4 and A_2 real and symmetric and A_3 and A_1 real and skew-symmetric, assuming c is real. The default is $m = 8$.

CD player {`pep,qep,real`}. This is a 60×60 quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M = I_{60}$ arising in the study of a CD player control task [17], [18]. The mechanism that is modeled consists of a swing arm on which a lens is mounted by means of two horizontal leaf springs. This is a small representation of a larger original rigid body model (which is also quadratic).

Closed-loop {`pep,qep,real,parameter-dependent`}. This is a quadratic polynomial

$$Q(\lambda) = \lambda^2 I + \lambda \begin{bmatrix} 0 & 1+\alpha \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

associated with a closed-loop control system with feedback gains 1 and $1 + \alpha$, $\alpha \geq 0$. The eigenvalues of $Q(\lambda)$ lie inside the unit disc if and only if $0 \leq \alpha < 0.875$ [79].

Concrete {`pep,qep,symmetric,parameter-dependent,sparse`}. This is a quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + (1 + i\mu)K$ arising in a model of a concrete structure supporting a machine assembly [29]. The matrices have dimension 2472. M is real diagonal and low rank. C, the viscous damping matrix, is pure imaginary and diagonal. K is complex symmetric, and the factor $1 + i\mu$ adds uniform hysteretic damping. The default is $\mu = 0.04$.

Damped beam {`pep,qep,real,symmetric,scalable`}. This QEP arises in the vibration analysis of a beam simply supported at both ends and damped in the middle [44]. The quadratic $Q(\lambda) = \lambda^2 M + \lambda C + K$ has real symmetric coefficient matrices with $M > 0$, $K > 0$, and $C = c e_n e_n^T \geq 0$, where c is a damping parameter. Half of the eigenvalues of the problem are purely imaginary and are eigenvalues of the undamped problem ($C = 0$).

Dirac {`pep,qep,real,symmetric,parameter-dependent,scalable`}. The spectrum of this matrix polynomial is the second order spectrum of the radial Dirac operator with an electric Coulombic potential of strength α,

$$D = \begin{bmatrix} 1 + \dfrac{\alpha}{r} & -\dfrac{d}{dr} + \dfrac{\kappa}{r} \\ \dfrac{d}{dr} + \dfrac{\kappa}{r} & -1 + \dfrac{\alpha}{r} \end{bmatrix}.$$

For $-\sqrt{3}/2 < \alpha < 0$ and $\kappa \in \mathbb{Z}$, D acts on $L^2((0,\infty), \mathbb{C}^2)$ and it corresponds to a spherically symmetric decomposition of the space into partial wave subspaces [76]. The problem discretization is relative to subspaces generated by the Hermite functions of odd order. The size of the matrix coefficients of the QEP is $n + m$, corresponding to n Hermite functions in the first component of the L^2 space and m in the second component [11].

For $\kappa = -1$, $\alpha = -1/2$ and n large enough, there is a conjugate pair of isolated points of the second order spectrum near the ground eigenvalue $E_0 \approx 0.866025$. The essential spectrum, $(-\infty, -1] \cup [1, \infty)$, as well as other eigenvalues, also seem to be captured for large n.

Fiber {`nep,sparse,solution`}. This nonlinear eigenvalue problem arises from a model in fiber optic design based on the Maxwell equations [49], [54]. The problem is of the form

$$F(\lambda)x = (A + s(\lambda)B - \lambda I)x = 0,$$

where $A \in \mathbb{R}^{2400 \times 2400}$ is tridiagonal and $B = e_{2400} e_{2400}^T$. The scalar function $s(\lambda)$ is defined in terms of Bessel functions. The real, positive eigenvalues are the ones of interest.

Foundation {`pep,qep,symmetric,sparse`}. This is a quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$ arising in a model of reinforced concrete machine foundations resting on the ground [29]. The matrices have dimension 3627; M is real and diagonal, C is complex and diagonal, and K is complex symmetric.

Gen hyper2 {`pep,qep,real,symmetric,hyperbolic,parameter-dependent,scalable,solution`}. This example generates a hyperbolic quadratic matrix polynomial from a given set of eigenvalues and eigenvectors (λ_k, v_k), $k = 1\colon 2n$, such that with

$$\begin{aligned} \Lambda &= \mathrm{diag}(\lambda_1, \dots, \lambda_{2n}) =: \mathrm{diag}(\Lambda_1, \Lambda_2), \qquad \Lambda_1, \Lambda_2 \in \mathbb{R}^{n\times n}, \\ V &:= [\, v_1, \dots, v_{2n} \,] =: [\, V_1 \quad V_2 \,], \qquad\qquad V_1, V_2 \in \mathbb{R}^{n\times n}, \end{aligned}$$

$\lambda_{\min}(\Lambda_1) > \lambda_{\max}(\Lambda_2)$, V_1 is nonsingular, and $V_2 = V_1 U$ for some orthogonal matrix U. Then the $n \times n$ symmetric quadratic $Q(\lambda) = \lambda^2 A + \lambda B + C$ with

$$\begin{aligned} A &= \Gamma^{-1}, \qquad \Gamma = V_1 \Lambda_1 V_1^T - V_2 \Lambda_2 V_2^T, \\ B &= -A(V_1 \Lambda_1^2 V_1^T - V_2 \Lambda_2^2 V_2^T)A, \\ C &= -A(V_1 \Lambda_1^3 V_1^T - V_2 \Lambda_2^3 V_2^T)A + B\Gamma B, \end{aligned}$$

is hyperbolic and has eigenpairs (λ_k, v_k), $k = 1\colon 2n$ [1], [35]. The quadratic $Q(\lambda)$ has the property that A is positive definite and $-Q(\mu)$ is positive definite for all $\mu \in (\lambda_{\max}(\Lambda_2), \lambda_{\min}(\Lambda_1))$. If $\lambda_{\max}(\Lambda) < 0$ then B and C are positive definite and $Q(\lambda)$ is overdamped.

Gun {`nep,sparse`}. This nonlinear eigenvalue problem models a radio-frequency gun cavity. The eigenvalue problem is of the form

$$F(\lambda)x = \left[K - \lambda M + i(\lambda - \sigma_1^2)^{\frac{1}{2}} W_1 + i(\lambda - \sigma_2^2)^{\frac{1}{2}} W_2\right] x = 0,$$

where M, K, W_1, W_2 are real symmetric matrices of size 9956×9956. K is positive semidefinite and M is positive definite. In this example $\sigma_1 = 0$ and $\sigma_2 = 108.8774$. The eigenvalues of interest are the λ for which $\lambda^{1/2}$ is close to 146.71 [62, p. 59].

Hadeler {`nep,real,symmetric,scalable`}. This nonlinear eigenvalue problem, from Hadeler [38], has the form

$$F(\lambda)x = \left[(e^{\lambda} - 1)A_2 + \lambda^2 A_1 - \alpha A_0\right] x = 0,$$

where $A_2, A_1, A_0 \in \mathbb{R}^{n\times n}$ are symmetric and α is a scalar parameter. This problem satisfies a generalized form of overdamping condition that ensures the existence of a complete set of eigenvectors [73].

Hospital {`pep,qep,real`}. This is a 24×24 quadratic polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M = I_{24}$, arising in the study of the Los Angeles University Hospital building [17], [18]. There are 8 floors, each with 3 degrees of freedom.

Intersection {`pep,qep,real`}. This 10×10 quadratic polynomial arises in the problem of finding the intersection between a cylinder, a sphere, and a plane described by the equations

$$\begin{aligned} f_1(x,y,z) &= 1.6\text{e-}3\, x^2 + 1.6\text{e-}3\, y^2 - 1 = 0, \\ f_2(x,y,z) &= 5.3\text{e-}4\, x^2 + 5.3\text{e-}4\, y^2 + 5.3\text{e-}4\, z^2 + 2.7\text{e-}2\, x - 1 = 0, \\ f_3(x,y,z) &= -1.4\text{e-}4\, x + 1.0\text{e-}4\, y + z - 3.4\text{e-}3 = 0. \end{aligned} \tag{4}$$

Use of the Macaulay resultant leads to the QEP $Q(x)v = 0$, where

$$Q(x)v = [\, yf_1 \quad zf_1 \quad f_1 \quad yf_2 \quad zf_2 \quad f_2 \quad yzf_3 \quad yf_3 \quad zf_3 \quad f_3 \,]^T = (x^2 A_2 + xA_1 + A_0)v, \tag{5}$$

$$v = [\, y^3 \quad y^2 z \quad y^2 \quad yz^2 \quad z^3 \quad z^2 \quad yz \quad y \quad z \quad 1 \,]^T. \tag{6}$$

The matrix A_2 is singular and the QEP has only four finite eigenvalues: two real and two complex. Let (λ_i, v_i), $i = 1, 2$ be the two real eigenpairs. With the normalization $v_i(10) = 1$, $i = 1, 2$, $(x_i, y_i, z_i) = (\lambda_i, v_i(8), v_i(9))$ are solutions of (4) [64].

Loaded string {`rep,real,symmetric,parameter-dependent,scalable`}. This rational eigenvalue problem arises in the finite element discretization of a boundary problem describing the eigenvibration of a string with a load of mass m attached by an elastic spring of stiffness k. It has the form

$$R(\lambda)x = \left(A - \lambda B + \frac{\lambda}{\lambda - \sigma} C\right) x = 0,$$

where the pole $\sigma = k/m$, and $A > 0$ and $B > 0$ are $n \times n$ tridiagonal matrices defined by

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad B = \frac{h}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 4 & 1 \\ & & 1 & 2 \end{bmatrix},$$

and $C = k e_n e_n^T$ with $h = 1/n$ [74].

Metal strip {`pep,qep,real`}. Modelling the electronic behaviour of a metal strip using partial element equivalent circuits (PEEC's) [5] results in the delay differential equation

$$\begin{cases} D_1\dot{x}(t-h) + D_0\dot{x}(t) = A_0 x(t) + A_1 x(t-h)\,, & t \geq 0, \\ x(t) = \varphi(t)\,, & t \in [-h, 0), \end{cases}$$

where

$$A_0 = 100\begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \quad A_1 = 100\begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix},$$

$$D_1 = -\frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}, \quad D_0 = I, \quad \varphi(t) = \begin{bmatrix}\sin(t), & \sin(2t), & \sin(3t)\end{bmatrix}^T.$$

Assessing the stability of this delay differential equation by the method in [28], [52] leads to the quadratic eigenproblem $(\lambda^2 E + \lambda F + G)u = 0$ with

$$\begin{aligned} E &= (D_0 \otimes A_1) + (A_0 \otimes D_1), \quad G = (D_1 \otimes A_0) + (A_1 \otimes D_0), \\ F &= (D_0 \otimes A_0) + (A_0 \otimes D_0) + (D_1 \otimes A_1) + (A_1 \otimes D_1). \end{aligned}$$

This problem is PCP-palindromic [28], i.e., there is an involutory matrix P such that $E = P\overline{G}P$ and $F = P\overline{F}P$.

Mobile manipulator {`pep,qep,real`}. This is a 5×5 quadratic matrix polynomial arising from the modelling as a time-invariant descriptor control system of a two-dimensional three-link mobile manipulator [15, Ex. 14], [14]. The system in its second-order form is

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

where the coefficient matrices are 5×5 and of the form

$$M = \begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_0 & -F_0^T \\ F_0 & 0 \end{bmatrix},$$

with

$$M_0 = \begin{bmatrix} 18.7532 & -7.94493 & 7.94494 \\ -7.94493 & 31.8182 & -26.8182 \\ 7.94494 & -26.8182 & 26.8182 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1.52143 & -1.55168 & 1.55168 \\ 3.22064 & 3.28467 & -3.28467 \\ -3.22064 & -3.28467 & 3.28467 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 67.4894 & 69.2393 & -69.2393 \\ 69.8124 & 1.68624 & -1.68617 \\ -69.8123 & -1.68617 & -68.2707 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The quadratic $Q(\lambda) = \lambda^2 M + \lambda D + K$ is close to being nonregular [15], [45].

Omnicam1 {`pep,qep,real`}. This is a 9×9 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ arising from a model of an omnidirectional camera (one with angle of view greater than 180 degrees) [70]. The matrix A_0 has one nonzero column, A_1 has 5 nonzero columns and rank 5, while A_2 has full rank. The eigenvalues of interest are the real eigenvalues of order 1.

Omnicam2 {`pep,qep,real`}. This is a 15×15 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ arising from a model of an omnidirectional camera (one with angle of view greater than 180 degrees) [70]. The matrix A_0 has one nonzero column, A_1 has 5 nonzero columns and rank 5, while A_2 has full rank. The eigenvalues of interest are the real eigenvalues of order 1.

Orr-Sommerfeld {`pep,parameter-dependent,scalable`}. This example is a quartic polynomial eigenvalue problem arising in the spatial stability analysis of the Orr–Sommerfeld equation [79]. The Orr–Sommerfeld equation is a linearization of the incompressible Navier–Stokes equations in which the perturbations in velocity and pressure are assumed to take the form $\Phi(x, y, t) = \phi(y)e^{i(\lambda x - \omega t)}$, where λ is a wavenumber and ω is a radian frequency. For a given Reynolds number R, the Orr–Sommerfeld equation may be written

$$\left[\left(\frac{d^2}{dy^2} - \lambda^2\right)^2 - iR\left\{(\lambda U - \omega)\left(\frac{d^2}{dy^2} - \lambda^2\right) - \lambda U''\right\}\right]\phi = 0. \tag{7}$$

In spatial stability analysis the parameter is λ, which appears to the fourth power in (7), so we obtain a quartic polynomial eigenvalue problem. The quartic is constructed using a Chebyshev spectral discretization. The eigenvalues λ of interest are those closest to the real axis and $\mathrm{Im}(\lambda) > 0$ is needed for stability. The default values $R = 5772$ and $\omega = 0.26943$ correspond to the critical neutral point corresponding to λ and ω both real for minimum R [13], [72].

PDDE stability {`qep,pep,scalable,parameter-dependent,sparse,symmetric`}. This problem arises from the stability analysis of a partial delay-differential equation (PDDE) [28], [52, Ex. 3.22]. Discretization gives rise to a time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + A_2 x(t - h_2),$$

where $A_0 \in \mathbb{R}^{n \times n}$ is tridiagonal and $A_1, A_2 \in \mathbb{R}^{n \times n}$ are diagonal with

$$\begin{aligned}
(A_0)_{kj} &= \begin{cases} -2(n+1)^2/\pi^2 + a_0 + b_0 \sin\bigl(j\pi/(n+1)\bigr) & \text{if} \quad k = j, \\ \;(n+1)^2/\pi^2 & \text{if} \quad |k - j| = 1\,, \end{cases} \\
(A_1)_{jj} &= a_1 + b_1 \frac{j\pi}{n+1}\left(1 - e^{-\pi(1 - j/(n+1))}\right), \\
(A_2)_{jj} &= a_2 + b_2 \frac{j\pi^2}{n+1}\bigl(1 - j/(n+1)\bigr).
\end{aligned}$$

Here, the a_k and b_k are real scalar parameters and $n \in \mathbb{N}$ is the number of uniformly spaced interior grid points in the discretization of the PDDE. Asking for the delays h_1, h_2 such that the delay system is stable leads to the quadratic eigenvalue problem $(\lambda^2 E + \lambda F + G)\,v = 0$ of dimension $n^2 \times n^2$ with

$$E = I \otimes A_2\,, \quad F = \bigl(I \otimes (A_0 + e^{-i\varphi_1} A_1)\bigr) + \bigl((A_0 + e^{i\varphi_1} A_1) \otimes I\bigr), \quad G = A_2 \otimes I,$$

where i is the imaginary unit and $\varphi_1 \in [-\pi, \pi]$ is a parameter. (To answer the stability question, the QEP has to be solved for many values of φ_1.)

Following [52], [28] the default values are

$$n = 20,\ a_0 = 2,\ b_0 = 0.3,\ a_1 = -2,\ b_1 = 0.2,\ a_2 = -2,\ b_2 = -0.3,\ \varphi_1 = -\pi/2.$$

This problem has the following properties: it is PCP-palindromic [28], i.e., there is an involutory matrix P such that $E = P\overline{G}P$ and $F = P\overline{F}P$. Moreover, only the four eigenvalues on the unit circle are of interest. The exact corresponding eigenvectors can be written as $x_j = u_j \otimes v_j$ for $j = 1\colon 4$.

Plasma drift {`pep`}. This cubic matrix polynomial of dimension 128 or 512 results from the modeling of drift instabilities in the plasma edge inside a Tokamak reactor [81]. It is of the form $P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$, where A_0 and A_1 are complex, A_2 is complex symmetric, and A_3 is real symmetric. The desired eigenpair is the one whose eigenvalue has the largest imaginary part.

Power plant {`pep,qep,symmetric,parameter-dependent`}. This is a QEP $Q(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0$ describing the dynamic behaviour of a nuclear power plant simplified into an eight-degrees-of-freedom system [51], [80]. The mass matrix M and damping matrix D are real symmetric and the stiffness matrix has the form $K = (1 + i\mu)K_0$, where K_0 is real symmetric (hence $K = K^T$ is complex symmetric). The parameter μ describes the hysteretic damping of the problem. The matrices are badly scaled.

QEP1 {`pep,qep,real,solution`}. This is a 3×3 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ from [80, p. 250] with

$$A_2 = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_0 = I.$$

The six eigenpairs (λ_k, x_k), $k = 1\colon 6$, are given by

k	1	2	3	4	5	6
λ_k	$1/3$	$1/2$	1	i	$-i$	∞
x_k	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$

Note that x_1 is an eigenvector for both of the distinct eigenvalues λ_1 and λ_2.

QEP2 {`pep,qep,real,solution`}. This is the 3×3 quadratic matrix polynomial [80, p. 256]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$, and $\lambda_5 = \lambda_6 = \infty$. The Jordan structure is given by

$$X_F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J_F = \operatorname{diag}\left(-1, 1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

for the finite eigenvalues and and

$$X_\infty = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for the infinite eigenvalues (see [31] or [80, Sec. 3.6] for definitions of Jordan structure).

QEP3 {`pep,qep,real,parameter-dependent,solution`}. This is a 3×3 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ from [23, p. 89] with

$$A_2 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -1-\epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 2 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

The eigenpairs (λ_k, x_k), $k = 1\colon 6$, are given by

$$\begin{array}{c|cccccc} k & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \lambda_k & 0 & 1 & 1+\epsilon & 2 & 3 & \infty \\ x_k & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}1\\ \frac{\epsilon-1}{\epsilon+1}\\0\end{bmatrix} & \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix}. \end{array}$$

For the default value of the parameter, $\epsilon = -1 + 2^{-53/2}$, the first and third eigenvalues are ill conditioned.

QEP4 {`pep,qep,nonregular,nonsquare,real,solution`}. This is the 3×4 quadratic matrix polynomial [16, Ex.2.5]

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The eigensystem includes an eigenvalue $\lambda_1 = 0$ with right eigenvectors $[2\ 1\ 0\ -1]^T$ and e_1 and an eigenvalue $\lambda = \infty$ with right eigenvector $[0\ 0\ 1\ 0]^T$. The Jordan and Kronecker structure is fully described in [16, Ex. 2.5].

Railtrack {`pep,qep,t-palindromic,sparse`}. This is a T-palindromic quadratic matrix polynomial of size 1005: $Q(\lambda) = \lambda^2 A^T + \lambda B + A$ with $B = B^T$. It stems from a model of the vibration of rail tracks under the excitation of high speed trains, discretized by classical mechanical finite elements [46], [47], [50], [63]. This problem has the property that the matrix A is of the form

$$A = \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix} \in \mathbb{C}^{1005\times1005},$$

where $A_{21} \in \mathbb{C}^{201\times67}$, that is, A has low rank ($\mathrm{rank}(A) = 67$). Hence this eigenvalue problem has many eigenvalues at zero and infinity.

Railtrack2 {`pep,qep,t-palindromic,sparse,scalable, parameter-dependent`}. This is a T-palindromic quadratic matrix polynomial of size $705m \times 705m$: $Q(\lambda) = \lambda^2 A^T + \lambda B + A$ with

$$A = \begin{bmatrix} 0 & \cdots & 0 & H_1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} H_0 & H_1^T & & 0 \\ H_1 & H_0 & \ddots & \\ & \ddots & \ddots & H_1^T \\ 0 & & H_1 & H_0 \end{bmatrix} = B^T,$$

where $H_0, H_1 \in \mathbb{C}^{705\times705}$ depend quadratically on a parameter ω, whose default value is $\omega = 1000$. The default for the number of block rows and columns of A and B is $m = 51$. The structure of A implies that there are many eigenvalues at zero and infinity.

Like the problem Railtrack this problem is from a model of the vibration of rail tracks, but here triangular finite elements are used for the discretization [20], [37], [48]. The parameter ω denotes the frequency of the external excitation force.

Relative pose 5pt {`pep,real`}. The cubic matrix polynomial $P(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0$ $A_i \in \mathbb{R}^{10\times10}$ comes from the five point relative pose problem in computer vision [57]. In this problem the images of five unknown scene points taken with a camera with a known focal length from two distinct unknown viewpoints are given and it is required to determine the possible solutions for the relative configuration of the points and cameras. The matrix A_3 has one nonzero column, A_2 has 3 nonzero columns and rank 3, A_1 has 6 nonzero columns and rank 6, while A_0 is of full rank. The solutions to the problem are obtained from the last three components of the finite eigenvectors of P.

Relative pose 6pt {`pep,qep,real`}. The quadratic matrix polynomial $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$, where $A_i \in \mathbb{R}^{10\times 10}$, comes from the six point relative pose problem in computer vision [57]. In this problem the images of six unknown scene points taken with a camera of unknown focal length from two distinct unknown camera viewpoints are given and it is required to determine the possible solutions for the relative configuration of the points and cameras. The solutions to the problem are obtained from the last three components of the finite eigenvectors of P.

Schrodinger {`pep,qep,real,symmetric,sparse`}. The spectrum of this matrix polynomial is the second order spectrum, relative to a subspace $\mathcal{L} \subset H^2(\mathbb{R})$, of the Schrödinger operator $Hf(x) = f''(x) + (\cos(x) - e^{-x^2})f(x)$ acting on $L^2(\mathbb{R})$ [12]. The subspace $\mathcal{L}$ has been generated using fourth order Hermite elements on a uniform mesh on the interval $[-49, 49]$, subject to clamped boundary conditions. The corresponding quadratic matrix polynomial is given by $K - 2\lambda C + \lambda^2 B$ where

$$K_{jk} = \langle Hb_j, Hb_k\rangle, \qquad C_{jk} = \langle Hb_j, b_k\rangle \qquad \text{and} \qquad B_{jk} = \langle b_j, b_k\rangle.$$

Here $\{b_k\}$ is a basis of $\mathcal{L}$. The matrices are of size 1998.

The essential spectrum of H consists of a set of bands separated by gaps. The end points of these bands are the Mathieu characteristic values. The presence of the short-range potential gives rise to isolated eigenvalues of finite multiplicity. The portion of the second order spectrum that lies in the box $[-1/2, 2] \times [-10^{-1}, 10^{-1}]$ is very close to the spectrum of H.

Shaft {`pep,qep,real,symmetric,sparse`}. The quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M, C, K \in \mathbb{R}^{400\times 400}$, comes from a finite element model of a shaft on bearing supports with a damper [55, Ex. 5.6]. The matrix M has rank 199 and so contributes a large number of infinite eigenvalues. C has a single nonzero element, in the $(20, 20)$ position. The coefficients M, C and K are very sparse.

Sign1 {`pep,qep,hermitian,parameter-dependent,scalable`}. The spectrum of this quadratic matrix polynomial is the second order spectrum of the linear operator $Mf(x) = \operatorname{sign}(x)f(x) + a\widehat{f}(0)$ acting on $L^2(-\pi, \pi)$ with respect to the Fourier basis $\mathcal{B}_n = \{e^{-inx}, \ldots, 1, \ldots, e^{inx}\}$, where $\widehat{f}(0) = (1/2\pi)\int_{-\pi}^{\pi} f(x)\,dx$ [10]. The corresponding QEP is given by $K_n - 2\lambda C_n + \lambda^2 I_n$ where

$$K_n = \Pi_n M^2 \Pi_n, \qquad C_n = \Pi_n M \Pi_n$$

and I_n is the identity matrix of size $2n + 1$. Here Π_n is the orthogonal projector onto $\operatorname{Span}(\mathcal{B}_n)$.

As n increases, the limit set of the second order spectrum is the unit circle, together with two real points: $\lambda_\pm$. The intersection of this limit set with the real line is the spectrum of M. The points $\lambda_\pm$ comprise the discrete spectrum of M.

Sign2 {`pep,qep,hermitian,parameter-dependent,scalable`}. This problem is analogous to problem Sign1, the only difference being that the operator is $Mf(x) = (2\sin(x) + \operatorname{sign}(x))f(x) + a\widehat{f}(0)$.

Near the real line, the second order spectrum accumulates at $[-3, -1] \cup [1, 3] \cup \{\lambda_\pm\}$ as n increases. The two accumulation points $\lambda_\pm \approx \{-0.7674, 3.5796\}$ are the discrete spectrum of M.

Sleeper {`pep,qep,real,symmetric,scalable,proportionally-damped,solution`}. This QEP describes the oscillations of a rail track resting on sleepers [59]. The QEP has the form

$$Q(\lambda) = \lambda^2 I + \lambda(I + A^2) + A^2 + A + I,$$

where A is the circulant matrix with first row $[-2,\ 1,\ 0,\ \ldots,\ 0,\ 1]$. The eigenvalues of A and corresponding eigenvectors are explicitly given as

$$\mu_k = -4\sin^2\left(\frac{(k-1)\pi}{n}\right), \quad x_k(j) = \frac{1}{\sqrt{n}}\exp\left(\frac{-2i\pi(j-1)(k-1)}{n}\right), \qquad k = 1\colon n.$$

The eigenvalues of Q can be determined from the scalar equations

$$\lambda^2 + \lambda(1 + \mu_k^2) + (1 + \mu_k + \mu_k^2) = 0.$$

Due to the symmetry, manifested in $\sin(\pi - \theta) = \sin(\theta)$, there are several multiple eigenvalues.

Speaker box {`pep,qep,real,symmetric`}. The quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$, with $M, C, K \in \mathbb{R}^{107\times 107}$, is from a finite element model of a speaker box that includes both structural finite elements, representing the box, and fluid elements, representing the air contained in the box [55, Ex. 5.5]. The matrix coefficients are highly structured and sparse. There is a large variation in the norms: $\|M\|_2 = 1$, $\|C\|_2 = 5.7\times 10^{-2}$, $\|K\|_2 = 1.0\times 10^7$.

Spring {`pep,qep,real,symmetric,proportionally-damped,parameter-dependent,scalable`}. This is a QEP $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$ arising from a linearly damped mass-spring system [77]. The damping constants for the dampers and springs connecting the masses to the ground, and those for the dampers and springs connecting adjacent masses, are parameters. For the default choice of the parameters, the $n\times n$ matrices K, C, and M are

$$M = I, \quad C = 10T, \quad K = 5T, \quad T = \begin{bmatrix} 3 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 3 \end{bmatrix}.$$

Spring dashpot {`pep,qep,real,parameter-dependent,scalable`}. Gotts [32] describes a QEP arising from a finite element model of a linear spring in parallel with Maxwell elements (a Maxwell element is a spring in series with a dashpot). The quadratic matrix polynomial is $Q(\lambda) = \lambda^2 M + \lambda D + K$, where the mass matrix M is rank deficient and symmetric, the damping matrix D is rank deficient and block diagonal, and the stiffness matrix K is symmetric and has arrowhead structure. This example reflects the structure only, since the matrices themselves are not from a finite element model but randomly generated to have the desired properties of symmetry etc. The matrices have the form

$$M = \mathrm{diag}(\rho\widetilde{M}_{11}, 0), \qquad D = \mathrm{diag}(0, \eta_1\widetilde{K}_{11}, \ldots, \eta_m\widetilde{K}_{m+1,m+1}),$$

$$K = \begin{bmatrix} \alpha_\rho\widetilde{K}_{11} & -\xi_1\widetilde{K}_{12}, & \ldots & -\xi_m\widetilde{K}_{1,m+1} \\ -\xi_1\widetilde{K}_{12} & e_1\widetilde{K}_{22} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\xi_m\widetilde{K}_{1,m+1} & 0 & 0 & e_m\widetilde{K}_{m+1,m+1} \end{bmatrix},$$

where $\widetilde{M}_{ij}$ and $\widetilde{K}_{ij}$ are element mass and stiffness matrices, ξ_i and e_i measure the spring stiffnesses, and ρ is the material density.

Surveillance {`pep,qep,real,nonsquare,nonregular`}. This is a 21×16 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ arising from calibration of a surveillance camera using a human body as a calibration target [71]. The eigenvalue represents the focal length of the camera. This particular data set is synthetic and corresponds to a 600×400 pixel camera.

Wing {`pep,qep,real`}. This example is a 3×3 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ from [30, Sec. 10.11], with numerical values modified as in [58, Sec. 5.3]. The eigenproblem for $Q(\lambda)$ arose from the analysis of the oscillations of a wing in an airstream. The matrices are

$$A_2 = \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}.$$

Wiresaw1 {`pep,qep,real,t-even,gyroscopic,parameter-dependent,scalable`}. This gyroscopic QEP arises in the vibration analysis of a wiresaw [82]. It takes the form $Q(\lambda)x = (\lambda^2 M + \lambda C + K)x = 0$, where the $n \times n$ coefficient matrices are defined by

$$M = I_n/2, \quad K = \operatorname*{diag}_{1\le j\le n}\big(j^2\pi^2(1-v^2)/2\big),$$

and

$$C = -C^T = (c_{jk}), \quad \text{with} \quad c_{jk} = \begin{cases} \dfrac{4jk}{j^2-k^2}v, & \text{if } j+k \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Here, v is a real nonnegative parameter corresponding to the speed of the wire. Note that for $0 < v < 1$, K is positive definite and the quadratic

$$G(\lambda) := -Q(-\imath\lambda) = \lambda^2 M + \lambda(\imath C) - K$$

is hyperbolic (but not overdamped).

Wiresaw2 {`pep,qep,real,parameter-dependent,scalable`}. When the effect of viscous damping is added to the problem in **Wiresaw1**, the corresponding quadratic has the form [82]

$$\widetilde{Q}(\lambda) = \lambda^2 M + \lambda(C + \eta I) + K + \eta C,$$

where M, C, and K are the same as in **Wiresaw1** and the damping parameter η is real and nonnegative.

4 Design of the Toolbox

The problems in the NLEVP collection are accessed via a single MATLAB function `nlevp`, which is modelled on the MATLAB `gallery` function. This function calls those that actually generate the problems, which reside in a `private` directory located within the `nlevp` directory. This approach avoids the problem of name clashes with existing MATLAB functions and also provides an elegant interface to the collection.

All problems are invoked with same syntax, which returns the coefficient matrices defining the problem (as specified in Section 2.1) in a cell array. To illustrate, the following example sets up the Omnicam2 problem, finds its eigenvalues and eigenvectors with `polyeig`, and then prints the largest modulus of the eigenvalues:

```
>> coeffs = nlevp('omnicam2')
coeffs =
    [15x15 double]    [15x15 double]    [15x15 double]
>> [X,e] = polyeig(coeffs{:}); max(abs(e))
ans =
  3.6351e-001
```

The nonlinear function $F(\lambda)$ in (3) can be evaluated by calling `nlevp` with `eval` as its first argument. This is useful for evaluating the residual of an approximate eigenpair, for example:

```
>> lam = e(end); x = X(:,end); Fx = nlevp('eval','omnicam2',lam)*x; norm(Fx)
ans =
  5.8137e-032
```

The second output argument from `nlevp` is a function handle that enables the nonlinear scalar functions $f_i(\lambda)$ in (3) and their derivatives to be evaluated. This facilitates the use of numerical methods that require derivatives, especially for the non-polynomial problems, for which obtaining the derivatives can be nontrivial. For example, the following code evaluates $f_i(0.5)$, $i = 1\colon 3$, and the first two derivatives (denoted `fp`, `fpp`), for the Fiber problem:

```
>> [coeffs,fun] = nlevp('fiber');
>> [f,fp,fpp] = fun(0.5)
f =
  1.0000e+000 -5.0000e-001 -7.0746e-001
fp =
            0 -1.0000e+000 -7.0725e-001
fpp =
            0            0  7.0696e-001
```

Problems and their properties are stored in a simple database made from cell arrays. The database is accessed with the `query` function in the `private` directory, which is invoked using the `query` argument to `nlevp`. For example, the properties for the Butterfly problem are returned in a cell array by the following call (whose syntax illustrates the command/function duality of MATLAB [39, Sec. 7.5]):

```
>> nlevp query butterfly
ans =
    'pep'
    'real'
    'parameter-dependent'
    'T-even'
    'scalable'
```

A more sophisticated example finds the names of all PEPs of degree 3 or higher:

```
>> pep = nlevp('query','pep'); qep = nlevp('query','qep');
>> pep_cubic_plus = setdiff(pep,qep)
pep_cubic_plus =
    'butterfly'
    'orr_sommerfeld'
    'plasma_drift'
    'relative_pose_5pt'
```

The cell array `pep_cubic_plus` can then easily be used to extract these problems. For example, the first problem in `pep_cubic_plus` can be solved using

```
coeffs = nlevp(pep_cubic_plus{1}); [X,e] = polyeig(coeffs{:});
```

Table 5–10 were generated automatically in MATLAB using appropriate `nlevp('query',...)` calls.

The toolbox function `nlevp_example.m` provides a test that the toolbox is correctly installed. It solves all the PEPs in the collection of dimension less than 500 using MATLAB's `polyeig` and then plots the eigenvalues. It produces Figure 1 and output to the command window that begins as follows:

```
NLEVP contains 46 problems in total,
of which 42 are polynomial eigenvalue problems (PEPs).
Run POLYEIG on the PEP problems of dimension at most 500:

             Problem   Dim  Max and min magnitude of eigenvalues
             -------   ---  ------------------------------------
    acoustic_wave_1d    10  3.14e+000, 4.59e-001
    acoustic_wave_2d    30  2.61e+000, 6.83e-001
             bicycle     2  1.41e+001, 3.23e-001
               bilby     5        Inf, 3.92e-016
           butterfly    64  2.01e+000, 3.59e-001
           cd_player    60  1.87e+006, 2.23e-004
         closed_loop     2  1.07e+000, 3.31e-001
            concrete  2472  is a PEP but is too large for this test.
                 ...
```

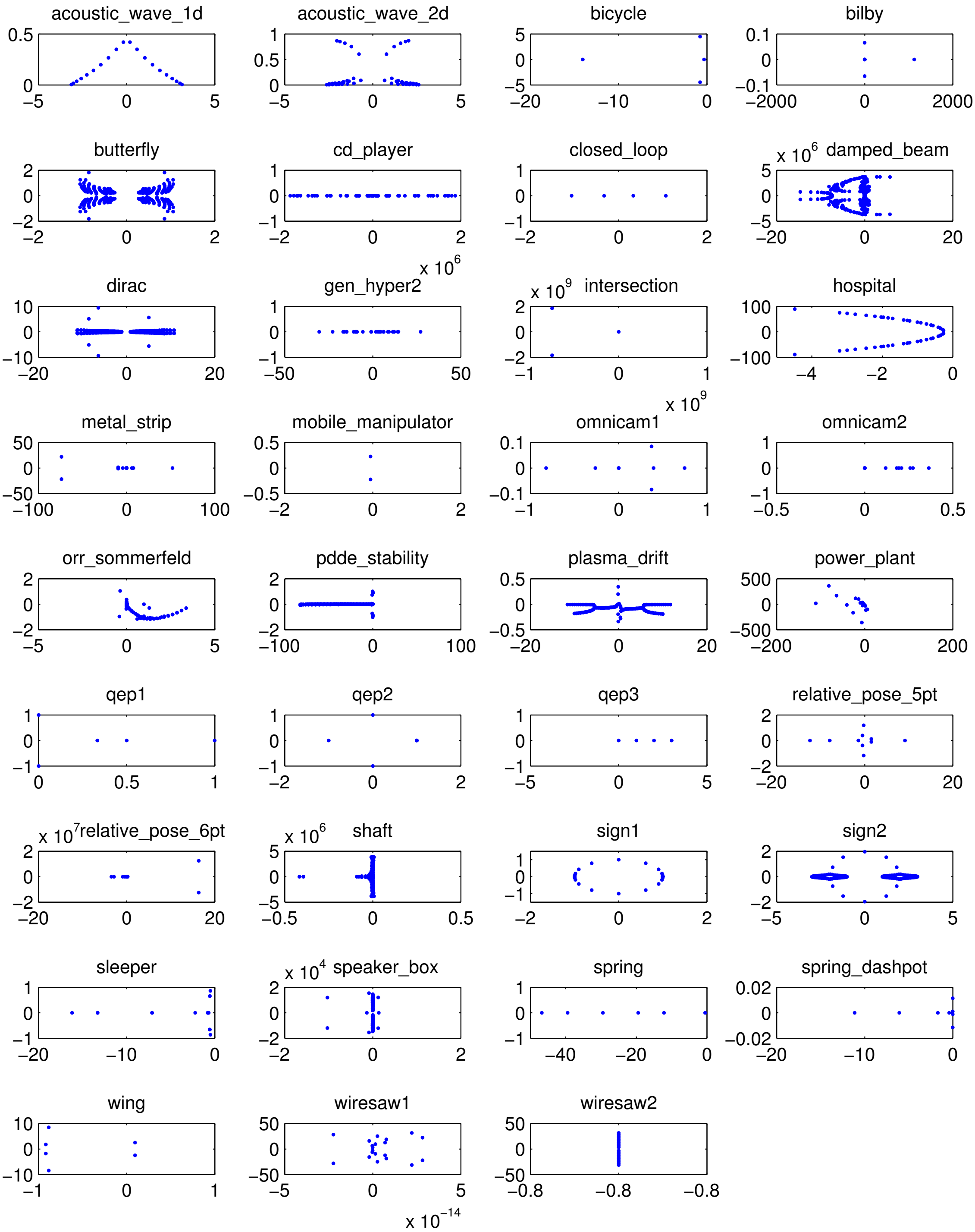

Figure 1: Eigenvalue plots for PEP problems produced by `nlevp_example.m`.

The `nlevp_example.m` function can be used as a template by the user wishing to test a given solver on subsets of the NLEVP problems.

The toolbox function `nlevp_test.m` automatically tests that the problems in the collection have the claimed properties. It is primarily intended for use by the developers as new problems are added, but it can also be used as a test for correctness of the installation. While many of the tests are straightforward, some are less so. For example, we test for hyperbolicity of a Hermitian matrix polynomial by computing the eigensystem and checking the types of the eigenvalues, using a characterization in [1, Thm. 3.4, P1]. To test for proportional damping we use necessary and sufficient conditions from [60, Thms. 2, 4]. We reproduce part of the output:

```
>> nlevp_test
Testing the NLEVP collection
Testing generation of all problems
Testing T-palindromicity
Testing *-palindromicity
...
Testing proportionally damping
Testing given solutions
NLEVP collection tests completed.
***  Errors: 0
```

5 Conclusions

The NLEVP collection demonstrates the tremendous variety of applications of nonlinear eigenvalue problems and provides representative problems for testing, provided in the form of a MATLAB toolbox. Version 1.0 of the toolbox was released in 2008 and the current version is 2.0. The toolbox has already proved useful in our own work and that of others [2], [6], [8], [34], [36], [53], [78] and we hope it will find broad use in developing, testing, and comparing new algorithms. By classifying important structural properties of nonlinear eigenvalue problems, and providing examples of these structures, this work should also be useful in guiding theoretical developments.

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