

Scaling and universality in the micro-structure of urban space

Rui Carvalho ¹

Alan Penn

*The Bartlett School of Graduate Studies
1-19 Torrington Place
University College London, Gower Street
London WC1E 6BT, United Kingdom*

Abstract

We present a broad, phenomenological picture of the distribution of the length of open space linear segments, l , derived from maps of 36 cities in 14 different countries. By scaling the Zipf plot of l , we obtain two master curves for a sample of cities, which are not a function of city size. We show that a third class of cities is not easily classifiable into these two universality classes. The cumulative distribution of l displays power-law tails with two distinct exponents, $\alpha_B = 2$ and $\alpha_R = 3$. We suggest a link between our data and the possibility of observing and modelling urban geometric structures using Lévy processes.

Key words: Fractals, Urban planning, Scaling laws, Universality.
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1 Introduction

The morphology of urban settlements and its dynamics has captured the interest of physicists [1,2,3,4,5,6,7] as it may shed light on Zipf's law for cities [8,9,10,11], challenge theoretical frameworks for cluster dynamics or improve predictions of urban growth [2,4,5].

The search for a unified theory of urban morphology has focused on the premise that cities can be conceptualized at several scales as fractals. At the

¹ Corresponding author. Email address: rui.carvalho@ucl.ac.uk

regional scale, rank-order plots of city size follow a fractal distribution [1] and population scales with city area as a power-law [12]. More recently, it has been observed that the area distribution of satellite cities, towns and villages around large urban centres also obeys a power-law with exponent ≈ 2 [2,4]. At the scale of transportation networks, railway networks appear to have a fractal structure [13]. At the scale of the neighbourhood, it has been suggested that urban space resembles a Sierpinsky gasket [1,12]. These scales are inter-related as summed up in [1, pp. 241] (author's translation from French): '*Polycentric growth, which is connected to the non-homogeneous distribution of pre-urban cores and the birth of a hierarchy of sub-centres, influences the morphology of the transport network, which plays in itself an important role in axial growth and therefore for the future spatial development of the urbanised area*'.

The fractal dimensions of US cities and international cities have values ranging from 1.2778 (Omaha, [14]) to 1.93 (Beijing, [1]), where the fractal dimension of large cities tends to cluster around the latter value [1,12,15,14]. Studies of urban growth of London between 1820 and 1962 show that fractal dimensions for this period vary from 1.322 to 1.791 [12]. The fractal dimensions for the growth of Berlin in 1875, 1920 and 1945 are 1.43, 1.54 and 1.69, respectively [1]. The fractal dimension of urban aggregates is a global measure of areal coverage, but detailed measures of spatial distribution are clearly needed to complement adequately the description of the morphology of an urban area [5]. Further, current approaches to data collection and modelling identify cities as fractal only on the urban periphery of the giant urban cluster that grows around the city core (or central business district), as clusters become compact at distances close to the centre of the city [12,4]. Although remote sensing techniques are promising in extracting urban morphology with greater detail, available studies have, to our knowledge, been limited to individually selected, medium scale cities (see e.g. [16]).

Hillier and Hanson [17] suggest an underlying structure to urban open space that is determined by the complexity of buildings which bound the space [18]. Urban space available for pedestrian movement, excluding by definition physical obstacles, is relatively linear. To extract this linearity, a street can be approximated by a set of straight lines. The global set of lines, the so-called *axial map*, is defined as the least number of longest straight lines. An axial map can be derived by drawing the longest possible straight line on a city map, then the next longest line, so-called axial line, until the open space is crossed and 'all axial lines that can be linked to other axial lines without repetition are linked' [17,19]. Figure 3 shows several axial maps.

Axial maps are relevant to urban theorists, as they encode the structure of open space in urban settlements and provide a simplified signature of the growth process. One could hope to extract the spatio-temporal dynamics of axial map growth by analysing a sequence of aerial photographs of the urban

periphery over a period of growth. Conversely, one could hope to model urban growth as the trajectory of N walkers on a plane, the step of the walk being axial line length.

Here we show that we can rescale axial line length and rank to obtain two distinct rank-order curves that provide a classification for several cities independently of city size. We also show that there is a class of cities that does not obey this classification. The collapse of curves suggests that spatial fluctuations in the length of urban linear structures, differing in size and location, are governed by similar statistical rules and supports the hypothesis that the linear dimension of large scale structures in cities reflects generic properties of city growth [20].

2 Structure of urban space

2.1 Intermittency in urban space

Let $\mathbf{l}_i = \{\mathbf{l}_{i,j}\}$, $j = 1, \dots, N_i$, be the N_i axial lines associated with city i . Each axial line, $\mathbf{l}_{i,j}$ ($j = 1, \dots, N_i$) is defined by the coordinates of its extremities

$$\mathbf{l}_{i,j} = \left\{ \left(x_{(i,j),1}, y_{(i,j),1} \right), \left(x_{(i,j),2}, y_{(i,j),2} \right) \right\}$$

The axial map of city i , \mathcal{C}_i , is thus a set of N_i points on a fourth dimensional space, $\mathcal{C}_i = \{(\rho_{i,j}, \theta_{i,j}, l_{i,j}, \varphi_{i,j})\}$, where (ρ_i, θ_i) are the polar coordinates of the axial line geometric centre, $\mathbf{s}_{i,j} = \left(\frac{x_{(i,j),1} + x_{(i,j),2}}{2}, \frac{y_{(i,j),1} + y_{(i,j),2}}{2} \right)$, and $\left(\pm \frac{l_{i,j}}{2}, \varphi_{i,j} \right)$ are the polar coordinates of the axial line's extremities on its geometric centre reference system, $\mathbf{d}_{i,j} = \pm \left(\frac{|x_{(i,j),1} - x_{(i,j),2}|}{2}, \frac{|y_{(i,j),1} - y_{(i,j),2}|}{2} \right)$.

Coordinates ρ and θ encode the geographic location of axial lines. The unconditional distribution of φ is multimodal for rather general families of urban settlements. This occurs, for example, when land is partitioned in clusters of randomly oriented orthogonal grids. Nevertheless, the unconditional distribution of l is unimodal and skewed to the right (see Figure 1), and, thus, the only coordinate which is a good candidate for inspection of intermittency in urban space. In what follows we will analyse the statistics of line length for a database of cities. We start by fitting the data for Tokyo to a stretched exponential distribution [21, pp. 153-154] in Figure 1 (a), but verify that the fit is unsuitable to describe the large events.

2.2 Inverse square and cubic laws for the distribution of line length

We analyse the unconditional probability distribution of (axial) line length of 36 cities in 14 different countries (see Table 1). In our analysis we use the rank-order technique [21]. To interpret the apparently unsystematic data in Figure 2(a) effectively, it is instructive to scale the data. Since the rank ranges between 1 and $\max(\text{rank}_i)$, we define a scaled relative rank for city i , $\widehat{r}_{i,j} \equiv \text{rank}_j / \max(\text{rank}_j)$. Similarly, for the ordinate, it is useful to define a scaled line length by $\widehat{l}_{i,j} = l_j / \langle l_j \rangle$ [22].

As shown in Figure 2(d), there is relatively good collapse of the data sets onto two master curves for 28 of the 36 cities under study (the cities plot in red and blue). The other 8 cities do not collapse clearly onto a single curve (see Figure 2(b)). Figure 2(c) is a plot of the exponents from a least-squares fit to the data of Figure 2(b) for $\log \widehat{l}_{i,j} > 0.2$, where the data are visually the most linear. The relatively small error bars (95% confidence bounds) in Figure 2(c) corroborate our choice of the cutoff. The fits on the rank order plot lead to straight lines with slope $-1/\alpha$, which suggest that the line length probability density may have a power-law tail, $P(l) \sim l^{-1-\alpha}$ with exponents close to $\alpha_B \simeq 2$ (cities in blue) or $\alpha_R \simeq 3$ (cities in red). The inverse square and cubic laws have diverging higher moments (larger than 2 and 3, respectively) and are not stable distributions.

Figure 3 is a plot of several axial maps, where we only plot lines with $\log \widehat{l}_{i,j} > 0.2$ (the range of data used in the least-squares fit of Figure 2). We suggest that urban growth can be regarded as a process where axial lines are added mainly to the urban periphery (this could, in principle, be monitored through remote sensing techniques for cities undergoing rapid urbanization) and modelled by N walkers, which jump along the corresponding N axial lines, extending the city. More needs to be known on the distribution of the walkers' waiting time to correctly model the dynamics of urban growth. Nevertheless, the walkers would generate a non-stable process, as the exponents α_B and α_R are, apparently, outside of the Lévy stable region.

3 Discussion

We have found that the length of urban open space structures displays universal features, largely independent of city size, and is self-similar across morphologically relevant ranges of scales (2 orders of magnitude) with exponents $\alpha_B \simeq 2$ (cities in blue) and $\alpha_R \simeq 3$ (cities in red). Our results are unexpected as two universality classes appear for a wide range of cities. The power-law tails of the pdfs support the hypothesis that urban space has a fractal struc-

ture [1,12], but the parallelism to a Sierpinski gasket [1,12] may be too simple for an accurate description.

Our findings show that it is important to model in detail the open space geometry of urban aggregates. They also support the hypothesis that it is more useful to model urban morphology as random rather than as the outcome of rational decisions, as previously suggested [12,2,4].

Cities with exponents $\alpha_B \simeq 2$ (cities in blue) display open space alignments which cross the whole structure, whilst cities with $\alpha_R \simeq 3$ (cities in red) tend not to. We propose that the large scale linear structures of the two classes of cities can be explained by two distinct non-stable Lévy processes, where the walker's jump has a tail that goes as $\alpha_B \simeq 2$ (cities in blue) or $\alpha_R \simeq 3$ (cities in red). The walker's jumps are much larger for a process with $\alpha_B \simeq 2$ (cities in blue) than for a process with $\alpha_R \simeq 3$ (cities in red), leading to the dominance of global geometric structures for the former as opposed to local geometric structures for the latter. The probability of a large axial line occurring for cities with $\alpha_G < 2$ (subset of cities in green) is larger than for the other cities (see, e.g., Las Vegas in Figure 2). We suggest that cities in green with $2 < \alpha_G < 3$ have mixed influences of global and local structures and that cities in green with $\alpha_G < 2$ have very strong patterns of global geometric structures. We note that the Lévy processes which, we propose, imprint the dominant global geometric structures of a city may appear at a late stage of urban growth, but change the city structure drastically (e.g. by generating long axial transportation routes).

Physicists have found exponents $\alpha \approx 3$ when studying the distribution of normalized returns in financial markets, both for individual companies [23,24] and for market indexes [25]. Our results for the class of cities plot in red on Figure 2 is reminiscent of these studies. We believe that parallels between urban growth and finance may not be too far fetched, as both processes seem to be largely dominated by geometric phenomena [6,26]. Indeed, there may be similarities between the dynamics of price fluctuations and urban growth, and we propose that axial lines may be seen as the urban equivalent of economic returns.

As more data becomes available through remote sensing, quantitative analyses should provide an improved view of the spatio-temporal dynamics of urban growth, particularly in squatter settlements, where time-scales for growth are much shorter than in conventional cities and one could hope to model growth against observed data. Further studies are still required, but it seems that the impact of local controls on growth (e.g. the green belt policy for London) is, at most, spatially localized. Indeed, at a 'macro' level, cities display a surprising degree of universality.

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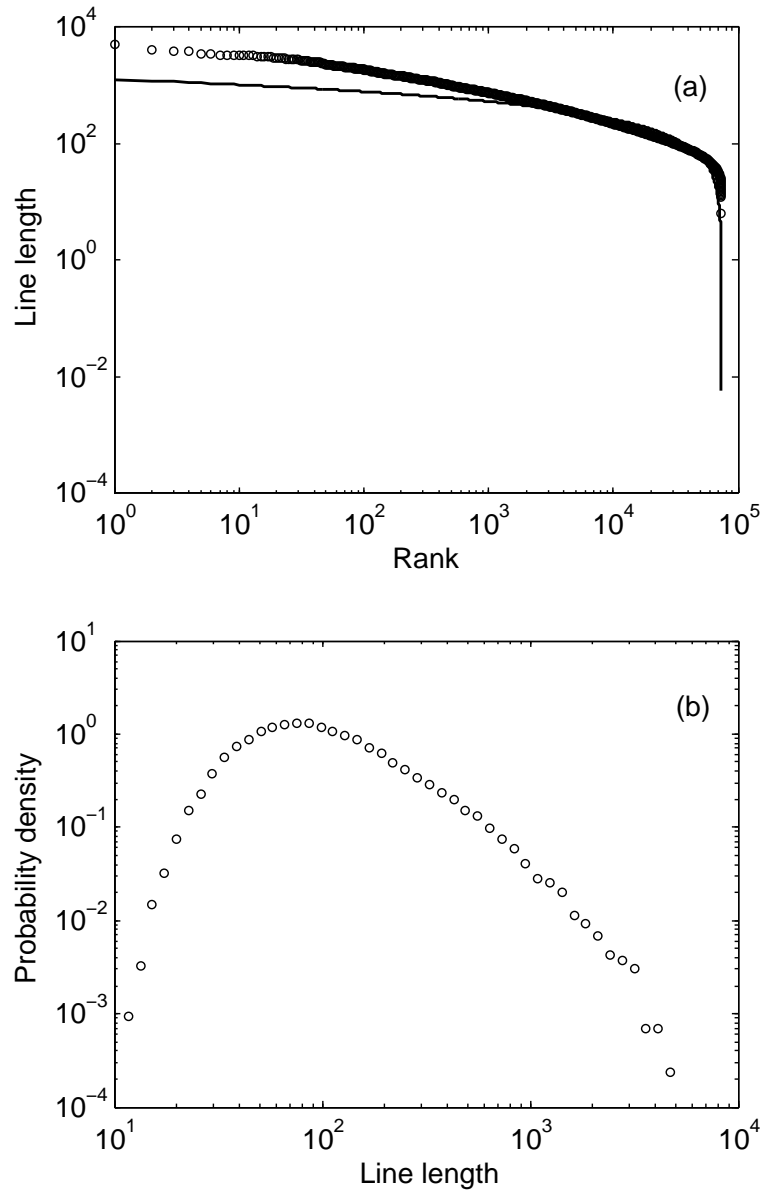


Fig. 1. Data are shown for the city of Tokyo. (a) Rank-order plot of line length (circle points) together with a fit of the data to a stretched exponential pdf (solid line). (b) Unconditional probability density of line length.

Table 1
Geographical location and number of lines of the cities analysed.

Country	City	Number of lines
Japan	Tokyo	73753
U.S.A.	Chicago	30571
Chile	Santiago	26821
Thailand	Bangkok	24223
Greece	Athens	23329
Turkey	Istanbul	21798
U.S.A.	Seattle	20213
U.K.	London	15969
U.S.A.	Baltimore	11636
Netherlands	Amsterdam	9619
U.K.	Bristol	7028
U.S.A.	Las Vegas	6909
Iran	Shiraz	6258
Cyprus	Nicosia	6023
Netherlands	Eindhoven	5782
U.K.	Milton Keynes	5581
Spain	Barcelona	5575
U.K.	Wolverhampton	5423
India	Ahmenabad	4876
U.S.A.	New Orleans	4846
Iran	Kerman	4372
U.K.	Nottingham	4365
U.K.	Manchester	4308
U.S.A.	Pensacola	4296
Iran	Hamadan	3855
Iran	Qazvin	3723
Netherlands	The Hague	3350
U.K.	Norwich	2119
U.S.A.	Denver	2092
Iran	Kermanshah	1870
U.K.	York	1773
Iran	Semnan	1770
Bangladesh	Dhaka	1566
Hong Kong	Hong Kong	916
U.K.	Hereford	854
U.K.	Winchester	616

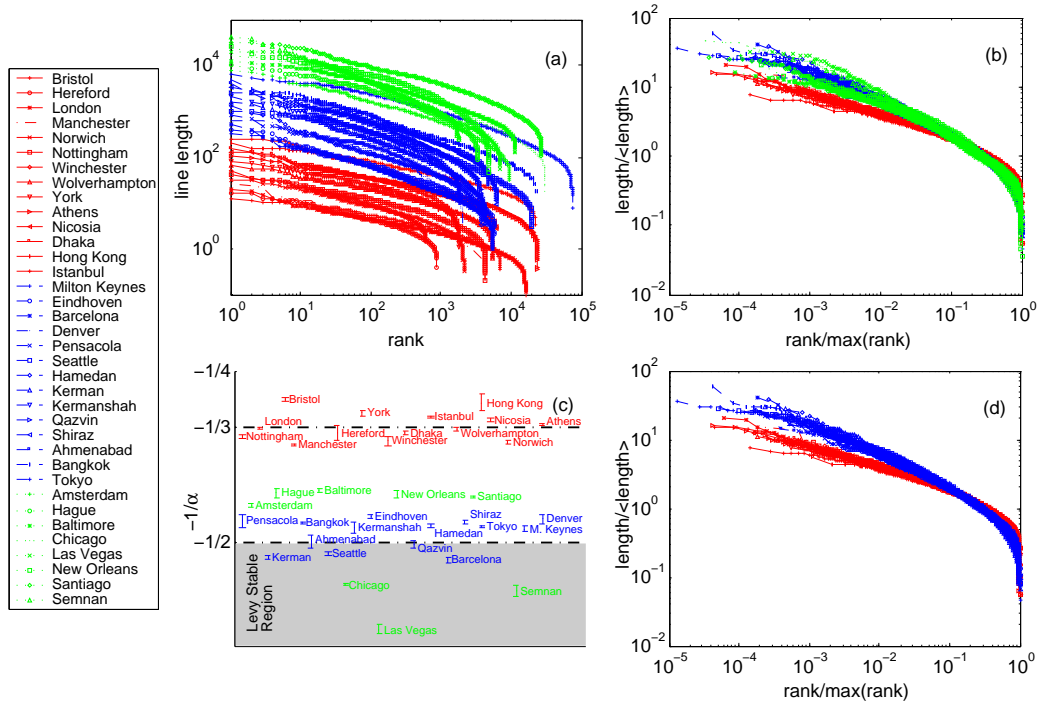


Fig. 2. (a) Rank-order plot of line length versus rank. Consecutive curves have been vertically shifted for clarity. (b) Data in (a) in scaled units. (c) Exponents determined from least squares fits to the log-log data in (b) for $10^{0.2} < y$. Error bars are 95% confidence bounds. (d) Data in (b) excluding cities in green. Cities are coloured according to their ordinate in (c), and we have coloured a group of cities in green as they deviate considerably from the two universality classes in (d).

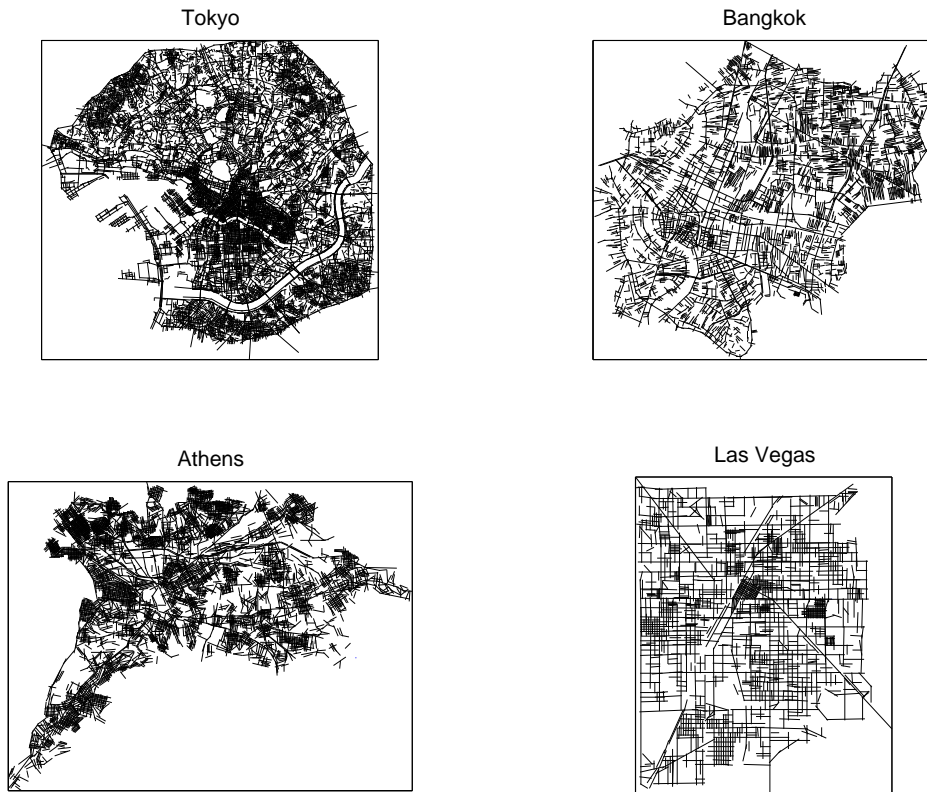


Fig. 3. Axial maps of a sample from Figure 2 of cities coloured in red (Athens), blue (Tokyo and Bangkok) and green (Las Vegas). Clockwise, the probability of occurrence of longer lines increases from bottom left (Athens, local geometric structures), to upper graphs (Tokyo and Bangkok, global geometric structures), to bottom right (Las Vegas, very strong global geometric structures).