

RESEARCH ARTICLE

Beyond the Hodge theorem: Curl and asymmetric pseudodifferential projections

Matteo Capoferri^{1,2} | Dmitri Vassiliev³

¹Maxwell Institute for Mathematical Sciences & Department of Mathematics, Heriot-Watt University, Edinburgh, UK

²Dipartimento di Matematica “Federigo Enriques”, Università degli Studi di Milano, Milano, Italy

³Department of Mathematics, University College London, London, UK

Correspondence

Matteo Capoferri, Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milano, Italy.

Email: matteo.capoferri@unimi.it

Funding information

Leverhulme Trust, Grant/Award Number: RPG-2019-240; London Mathematical Society; EPSRC, Grant/Award Number: EP/X01021X/1

Abstract

We develop a new approach to the study of spectral asymmetry. Working with the operator curl $:= *d$ on a connected oriented closed Riemannian 3-manifold, we construct, by means of microlocal analysis, the asymmetry operator — a scalar pseudodifferential operator of order -3 . The latter is completely determined by the Riemannian manifold and its orientation, and encodes information about spectral asymmetry. The asymmetry operator generalises and contains the classical eta invariant traditionally associated with the asymmetry of the spectrum, which can be recovered by computing its regularised operator trace. Remarkably, the whole construction is direct and explicit.

MSC 2020

58J50 (primary), 35P20, 35Q61, 47B93, 47F99, 58J28, 58J40 (secondary)

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1 | MAIN RESULTS

Let (M, g) be a connected closed Riemannian manifold of dimension $d \geq 2$ and let $\Omega^k(M)$ be the Hilbert space of real-valued k -forms, $1 \leq k \leq d - 1$. Hodge's theorem [42, Corollary 3.4.2] tells us that $\Omega^k(M)$ decomposes into a direct sum of three orthogonal closed subspaces

$$\Omega^k(M) = d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M) \oplus \mathcal{H}^k(M), \quad (1.1)$$

where $d\Omega^{k-1}(M)$, $\delta\Omega^{k+1}(M)$ and $\mathcal{H}^k(M)$ are the Hilbert subspaces of exact, coexact and harmonic k -forms, respectively. The overarching idea of our paper is that in the special case when $d = 3$, $k = 1$ and M is oriented the space $\delta\Omega^{k+1}(M)$ admits a further decomposition into a distinguished pair of orthogonal Hilbert subspaces $\delta\Omega_{\pm}^{k+1}(M)$. Moreover, this further decomposition is effectively described in terms of pseudodifferential projections for which the symbols can be written down explicitly in terms of curvature and its covariant derivatives. Remarkably, this leads to a new approach to the study of *spectral asymmetry*.

The study of spectral asymmetry, that is, the difference in the distribution of positive and negative eigenvalues of (pseudo)differential operators, has a long and noble history, initiated by the seminal series of papers by Atiyah, Patodi and Singer [2–5]. In a nutshell, the classical approach goes as follows. One considers the *eta function* of the operator at hand — say, curl or Dirac — defined as

$$\eta(s) := \sum_{\lambda_k \neq 0} \operatorname{sgn}(\lambda_k) |\lambda_k|^{-s}, \quad s \in \mathbb{C}, \quad (1.2)$$

where λ_k are the non-zero eigenvalues of the operator. The complex function $\eta(s)$ can be easily shown to be holomorphic for $\operatorname{Re} s > d$; one then tries to give a meaning to the quantity $\eta(0)$, called *eta invariant*, by examining the meromorphic extension of (1.2) for $\operatorname{Re} s < d$ and showing

that there is no pole at $s = 0$. The motivation for considering $\eta(0)$ as a measure of spectral asymmetry is that, when the operator in question is simply a Hermitian matrix, $\eta(0)$ is precisely the number of positive eigenvalues minus the number of negative eigenvalues. Classically, it has been shown that $\eta(0)$ is a geometric invariant which can be computed resorting to complex analysis and algebraic topology.

Perhaps surprisingly, there is almost no literature on the spectrum of curl on a 3-manifold with or without boundary. Indeed, if one looks carefully, the overwhelming majority of existing papers deal, effectively, with curl^2 as opposed to curl itself. Many authors [8, 9, 22, 32, 35, 56] have studied Maxwell's equations on domains in \mathbb{R}^3 subject to appropriate boundary conditions, which leads to the analysis of spectral problems of the form

$$\begin{pmatrix} 0 & i \text{curl} \\ -i \text{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} = \lambda \begin{pmatrix} E \\ B \end{pmatrix}. \quad (1.3)$$

One would think that the spectral problem (1.3) reduces to the spectral problem for curl by means of a unitary transformation

$$\begin{pmatrix} E \\ B \end{pmatrix} \mapsto \begin{pmatrix} v_+ \\ v_- \end{pmatrix} := \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}, \quad (1.4)$$

but this argument fails because physically meaningful boundary conditions do not agree with (1.4). The presence or absence of a boundary is a major issue in the subject.

To the best of our knowledge, one of the few papers dedicated to the study of the spectrum of curl on a closed oriented Riemannian manifold — at least, in the way it is understood in the current paper — is [7]. In [7], the author examined the qualitative properties of the spectrum of curl depending on the dimension d and established one-term asymptotic formulae (Weyl law) for the (global) eigenvalue counting functions, alongside a number of explicit examples. Although not entirely aligned in spirit with the aims of the current paper, we refer the reader to [21, 27–29, 33, 41, 43, 47, 48] and references therein, examining the spectrum of curl (or curl^2) from different perspectives.

Let us now formulate our problem and state our main results, postponing proofs and detailed arguments until later sections. In the rest of this paper, unless otherwise stated, the dimension d is 3 and the manifold is orientable and oriented. Let $*$ be the Hodge dual (see Appendix A for our sign convention), and define curl to be the differential expression

$$\text{curl} := *d \quad (1.5)$$

acting in $\Omega^1(M)$.

We will show in Section 2 — more precisely, in Theorem 2.1 — that the differential expression (1.5) gives rise to a self-adjoint operator (denoted by the same symbol) in the space of coexact 1-forms $\delta\Omega^2(M)$. The latter has discrete spectrum, accumulating to both $+\infty$ and $-\infty$, and not necessarily symmetric about zero. As a way of illustrating that spectral asymmetry actually occurs, in Appendices C and D, we write down explicitly the spectrum of curl on a Berger sphere and compute the corresponding (classical) eta invariant.

Note that the definition of curl does not involve the concept of connection. In this respect, curl is one of the most fundamental operators of mathematical physics, like the Laplace–Beltrami operator. Furthermore, curl lies at the heart of (homogeneous) Maxwell's equations. Namely, seeking

solutions harmonic in time reduces Maxwell’s equations on $M \times \mathbb{R}$ to the spectral problem for curl. And the solution of the Cauchy problem for Maxwell’s equations can be expressed in terms of eigenvalues and eigenforms of curl, see formula (E.8). Finally, it is worth emphasising that the sign of the eigenvalues of curl has a physical meaning: it coincides with the sign of electromagnetic chirality of time-harmonic polarised solutions of Maxwell’s equations generated by the corresponding curl eigenpair (see (E.6) and (E.7)). We refer the reader to Appendix E for further details and a self-contained exposition of this fact. These are but few of the reasons why a more detailed examination of the spectrum of curl — and in particular of its spectral asymmetry — is very timely, if not overdue.

As a first step, we introduce the following definitions.

Definition 1.1. The operator P_0 is defined as the extension to $\Omega^1(M)$ of the identity operator on $d\Omega^0(M)$. Here, the extension is specified by the requirement that the orthogonal complement of $d\Omega^0(M)$ in $\Omega^1(M)$ maps to zero.

Definition 1.2. Let $\theta : \mathbb{R} \rightarrow \mathbb{R}$,

$$\theta(z) := \begin{cases} 0 & \text{if } z \leq 0, \\ 1 & \text{if } z > 0 \end{cases} \tag{1.6}$$

be the Heaviside function. The operators P_{\pm} are defined as the extensions to $\Omega^1(M)$ of the operators $\theta(\pm \text{curl})$ on $\delta\Omega^2(M)$. Here, the extensions are specified by the requirement that the orthogonal complement of $\delta\Omega^2(M)$ in $\Omega^1(M)$ maps to zero.

The operator P_0 is the orthogonal projection onto the kernel of curl with harmonic 1-forms removed, whereas the operators P_{\pm} are the positive (+) and negative (-) spectral projections of curl.

Further on, we use the notation

$$\|\xi\| := \sqrt{g^{\mu\nu}(x) \xi_{\mu} \xi_{\nu}}, \tag{1.7}$$

and we denote by $|\cdot|$ the Euclidean norm of vectors. Throughout this paper, we use Greek letters for tensor indices.

Theorem 1.3.

- (a) *The operators P_0, P_+ and P_- are pseudodifferential of order 0.*
- (b) *Their principal symbols read*

$$[(P_0)_{\text{prin}}]_{\alpha}^{\beta}(x, \xi) = \|\xi\|^{-2} \xi_{\alpha} g^{\beta\gamma}(x) \xi_{\gamma}, \tag{1.8}$$

$$[(P_{\pm})_{\text{prin}}]_{\alpha}^{\beta}(x, \xi) = \frac{1}{2} \left[\delta_{\alpha}^{\beta} - [(P_0)_{\text{prin}}]_{\alpha}^{\beta}(x, \xi) \pm i \|\xi\|^{-1} E_{\alpha}^{\gamma\beta}(x) \xi_{\gamma} \right], \tag{1.9}$$

where

$$E_{\alpha\beta\gamma}(x) := \rho(x) \varepsilon_{\alpha\beta\gamma}, \tag{1.10}$$

ρ is the Riemannian density and ε is the totally antisymmetric symbol, $\varepsilon_{123} := +1$.

(c) *Their subprincipal symbols, defined in accordance with (3.6), are zero:*

$$(P_0)_{\text{sub}} = 0, \quad (P_{\pm})_{\text{sub}} = 0. \tag{1.11}$$

Euclidean versions of formulae (1.8) and (1.9) appeared in [46]. On the other hand, the notion of subprincipal symbol for operators acting on 1-forms was never previously defined in full generality. We address this matter in detail, including a discussion of known results, in Section 3.

Remarkably, the pseudodifferential projections P_{\pm} can be constructed explicitly. In Proposition 5.3, we will provide an algorithm for the calculation of their full symbols. The latter proposition and Theorem 1.3 give a detailed description of the structure of the projection operators P_0 and P_{\pm} , which is our first main result.

The projections P_+ and P_- are the key ingredients of our new approach to spectral asymmetry. One starts by observing that, at a formal level, we have

$$\#\{\text{positive eigenvalues}\} - \#\{\text{negative eigenvalues}\} = \text{Tr}(P_+ - P_-). \tag{1.12}$$

Here and further on, Tr stands for operator trace, whereas tr stands for matrix trace. Unfortunately, one is immediately presented with two major issues: (i) the LHS of (1.12) is a difference of two infinities and (ii) the (matrix) operator $P_+ - P_-$ in the RHS of (1.12) is not of trace class. Our strategy is to assign a meaning to the RHS of (1.12) by decomposing the procedure of taking operator trace into two steps:

- (1) take the pointwise matrix trace of the matrix pseudodifferential operator $P_+ - P_-$,
- (2) compute the operator trace of the resulting scalar pseudodifferential operator.

Step 1 above defines a scalar self-adjoint pseudodifferential operator which we call the *asymmetry operator* and denote by A — see Definition 6.1. This operator is defined uniquely, up to the addition of an integral operator with infinitely smooth kernel vanishing in a neighbourhood of the diagonal $M \times M$, and it is determined by the Riemannian 3-manifold (M, g) and its orientation.

The operator A is *prima facie* a pseudodifferential operator of order 0. However, taking the matrix trace brings about unexpected cancellations, so that the operator A turns out, in fact, to be of order -3 . This is our second main result.

Theorem 1.4. *The asymmetry operator A is a pseudodifferential operator of order -3 and its principal symbol reads*

$$A_{\text{prin}}(x, \xi) = -\frac{1}{2\|\xi\|^5} E^{\alpha\beta\gamma}(x) \nabla_{\alpha} \text{Ric}_{\beta}{}^{\rho}(x) \xi_{\gamma} \xi_{\rho}, \tag{1.13}$$

where Ric is the Ricci tensor.[†]

Remark 1.5. Note that formula (1.13) can be equivalently rewritten as

$$A_{\text{prin}}(x, \xi) = -\frac{1}{2\|\xi\|^5} E^{\alpha\beta\gamma}(x) \nabla_{\alpha} \left(\text{Ric}_{\beta}{}^{\rho}(x) - \frac{1}{3} \delta_{\beta}{}^{\rho} \text{Sc}(x) \right) \xi_{\gamma} \xi_{\rho},$$

[†] The Riemann curvature tensor Riem has components $\text{Riem}^{\kappa}{}_{\lambda\mu\nu}$ defined in accordance with

$$\text{Riem}^{\kappa}{}_{\lambda\mu\nu} := dx^{\kappa}(\text{Riem}(\partial_{\mu}, \partial_{\nu})\partial_{\lambda}) = \partial_{\mu}\Gamma^{\kappa}{}_{\nu\lambda} - \partial_{\nu}\Gamma^{\kappa}{}_{\mu\lambda} + \Gamma^{\kappa}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\lambda} - \Gamma^{\kappa}{}_{\nu\eta}\Gamma^{\eta}{}_{\mu\lambda},$$

the Γ 's being Christoffel symbols. The Ricci tensor is defined as $\text{Ric}_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}$ and $\text{Sc} := g^{\mu\nu} \text{Ric}_{\mu\nu}$ is scalar curvature.

that is, one can replace the Ricci tensor with its trace-free part. In other words, the principal symbol of the asymmetry operator does not feel scalar curvature.

A sufficient condition for a self-adjoint pseudodifferential operator to be of trace class and to have continuous integral kernel is for its order to be strictly less than $-d$, where d is the dimension of the manifold. See also [59, §12.1]. In the case at hand, the operator A has order -3 and the dimension of the manifold is 3, so we are looking at a borderline situation. One would hope that with a mild regularisation one could cross the finish line and be able to define a notion of regularised trace.

Indeed, we will establish in Theorem 7.1 that the integral kernel $\mathfrak{a}(x, y)$ of A decomposes as $\mathfrak{a}(x, y) = \mathfrak{a}_d(x, y) + \mathfrak{a}_c(x, y)$, where \mathfrak{a}_c is continuous, whereas \mathfrak{a}_d is possibly discontinuous but bounded, and integrates to zero over small spheres centred at either x or y .

This leads us to our third main result.

Theorem 1.6. *The integral kernel $\mathfrak{a}(x, y)$ of A is a bounded function, smooth outside the diagonal. Furthermore, for any $x \in M$, the limit*

$$\psi_{\text{curl}}^{\text{loc}}(x) := \lim_{r \rightarrow 0^+} \frac{1}{4\pi r^2} \int_{\mathbb{S}_r(x)} \mathfrak{a}(x, y) dS_y$$

exists and defines a continuous scalar function $\psi_{\text{curl}}^{\text{loc}} : M \rightarrow \mathbb{R}$. Here $\mathbb{S}_r(x) = \{y \in M \mid \text{dist}(x, y) = r\}$ is the sphere of radius r centred at x and dS_y is the surface area element on this sphere.

We call the function $\psi_{\text{curl}}^{\text{loc}}(x)$ and the number $\psi_{\text{curl}} := \int_M \psi_{\text{curl}}^{\text{loc}}(x) \rho(x) dx$ the local and global regularised trace of A , see Definitions 7.7 and 7.8. These two quantities are geometric invariants, in that they are determined by the Riemannian 3-manifold and its orientation. In particular, the global regularised trace is a real number measuring the asymmetry of the spectrum of curl.

The natural question is how are our geometric invariants $\psi_{\text{curl}}(x)$ and ψ_{curl} related to classical eta invariants. We comprehensively address this matter in our companion paper [20] where we prove that our $\psi_{\text{curl}}(x)$ and ψ_{curl} are precisely the classical local and global eta invariants for curl.

Remark 1.7. Pseudodifferential projections are very natural tools in the study of the spectral properties of systems. At first glance, one may be tempted to think that they are unnecessarily complicated, especially for operators acting on trivial bundles, in that one could try and diagonalise the operator along the lines of [14] instead, thus reducing the problem at hand to the examination a collection of scalar operators. However, after examining the matter more carefully, diagonalisation turns out to be topologically obstructed in numerous examples of physical relevance. In particular, it was shown in [16, Section 3.3] that for the case of the operator curl, one cannot even diagonalise the principal symbol $\text{curl}_{\text{prin}}$ (2.1) in the whole cotangent fibre at a given point of M , let alone the operator itself.

All in all, our results offer a new approach to the subject of spectral asymmetry, one that possesses the following elements of novelty.

1. We characterise asymmetry in terms of a pseudodifferential operator of negative order, as opposed to a single number. Here the fact that the order is negative is key as it opens the way to defining the regularised trace.

2. Our approach is not operator-specific. It is versatile and can be deployed in a variety of different scenarios. For example, our strategy can be applied to the Dirac operator,[†] as we will show in a separate paper.
3. Our technique is explicit and based on direct computations. Moreover, it rigorously implements the intuitive notion of spectral asymmetry as difference between numbers of positive and negative eigenvalues, without relying on ‘black boxes’ such as analytic continuation from the very beginning, or using heat-kernel-type arguments (see, e.g. [10, Theorem 2.6 and Remark 3]).

There are, of course, many different approaches to the subject of spectral asymmetry. A notable one, for instance, relies on a local invariant known as *Wodzicki residue* [64], see also [13, 51] for a broader discussion. Since the literature on spectral asymmetry is vast, we decided to refrain from including a detailed literature review here, as it would inevitably be incomplete and would deserve a separate paper (if not a monograph).

Remark 1.8. Throughout this paper, we focus our analysis on the space of real-valued 1-forms $\Omega^1(M)$. One could perform a completely analogous analysis working in the space of real-valued 2-forms $\Omega^2(M)$; indeed, on a 3-manifold, $\Omega^1(M)$ and $\Omega^2(M)$ are isomorphic by Hodge duality.

1.1 | Structure of the paper

Our paper is structured as follows.

In Section 2, we put curl on a rigorous operator-theoretic footing, and state its basic properties. Although most of the results from this section are known at least in some form, detailed proofs, often hard to find in the literature, are provided in Appendix B in full and in a self-contained fashion, for the convenience of the reader and future reference. In doing so, we shall rely on the auxiliary operator curl_E , called *extended curl*, which can be viewed as an elliptic ‘extension’ of curl.

In Section 3, we temporarily move away from dimension 3 and develop an invariant calculus for pseudodifferential operators acting on 1-forms over a Riemannian closed d -manifold. In particular, we define a notion of subprincipal symbol for these operators, and write down the composition formula.

In Section 4, we prepare the ground for the formulation of our main results and examine the issue of taking the trace of pseudodifferential operators on 1-forms. The highlight of this section is the notion of matrix trace tr , defined by (4.13) and (4.14), which addresses the issue of taking the pointwise matrix trace of integral operators whose integral kernel is a two-point tensor.

In Section 5, we study the pseudodifferential projections P_0 , P_+ and P_- , and provide an explicit algorithm for the construction of their full symbols. This section contains the proof of Theorem 1.3.

In Section 6, we define the asymmetry operator A (Subsection 6.1), prove that it is a pseudodifferential operator of order -3 (Subsection 6.2) and compute its principal symbol A_{prin} (Subsection 6.3). This section establishes Theorem 1.4.

Section 7 is the centrepiece of our paper. After examining the singular structure of the integral kernel $\alpha(x, y)$ of A near the diagonal (Subsection 7.1), we devise a regularisation procedure which allows one to define the local and global trace of the asymmetry operator, geometric invariants measuring spectral asymmetry (Subsection 7.2). This yields Theorem 1.6.

In Section 8, we briefly elaborate on the challenges involved in extending our results to dimensions higher than 3, and why doing so could be interesting.

[†] We also refer the reader to [11] for related results.

The paper is complemented by a list of notation given at the end of this section, and by seven appendices. In Appendix A, we specify our notation and conventions on exterior calculus. In Appendix B, we provide proofs for the basic properties of the operator curl, summarised in Section 2. Appendices C and D are concerned with spectral theory on Berger 3-spheres: in Appendix C, we recall the definition of a Berger sphere and list the eigenvalues of the Laplace–Beltrami operator, whereas in Appendix D, we list the eigenvalues of curl on a Berger sphere as an illustration of spectral asymmetry, and compute the eta invariant by explicitly examining the eta function for curl. In Appendix E, we examine the relation between the spectral problem for curl and solutions of Maxwell’s equations, and discuss the physical meaning of the sign of the eigenvalues of curl. In Appendix F, we derive series expansions for the parallel transport maps. Finally, in Appendix G, we independently verify formula (1.13).

Notation

Symbol	Description
\sim	Asymptotic expansion
$*$	Hodge dual (A.1)
$\ \cdot \ $	Riemannian norm (1.7)
$ \cdot $	Euclidean norm
$\langle \cdot \rangle$	Japanese bracket (7.5)
A	Asymmetry operator, Definition 6.1
$A_{\text{diag}}, A_{\text{pt}}$	Local decomposition of A as per (6.7) and (6.8)
$\mathfrak{a}(x, y)$	Integral kernel of the asymmetry operator A
curl	Curl, as a differential expression (1.5) and as an operator (2.4)
curl_E	Extended curl, Definition B.1
$\text{curl}_{E,d}, \text{curl}_{E,\delta}, \text{curl}_{E,\mathcal{H}}$	Orthogonal summands of curl_E , Proposition B.4
d	Dimension of the manifold M , $d \geq 2$
d	Exterior derivative, Appendix A
δ	Codifferential, Appendix A
$\Delta := -\delta d$	(Non-positive) Laplace–Beltrami operator
$\mathbf{\Delta} := -(d\delta + \delta d)$	(Non-positive) Hodge Laplacian
dist	Geodesic distance
$e_j^\alpha(x), e^k_\beta(x)$	Framing and dual framing (3.8)
$\tilde{e}_j^\alpha(x), \tilde{e}^k_\beta(x)$	Levi-Civita framing and dual Levi-Civita framing, Definition 3.5
$\varepsilon_{\alpha\beta\gamma}$	Totally antisymmetric symbol, $\varepsilon_{123} = +1$
$E_{\alpha\beta\gamma}$	Totally antisymmetric tensor (1.10)
f_{x^α}	Partial derivative of f with respect to x^α
g	Riemannian metric
G	Einstein tensor
$\gamma(x, y; \tau)$	Geodesic connecting x to y , with $\gamma(x, y; 0) = x$ and $\gamma(x, y; 1) = y$
$\Gamma^\alpha_{\beta\gamma}$	Christoffel symbols

(Continues)

Symbol	Description
$\eta_Q(s)$	Eta function of the operator Q
$H^s(M)$	Generalisation of the usual Sobolev spaces H^s to differential forms
$\mathcal{H}^k(M)$	Harmonic k -forms over M
$(\lambda_j, u_j), j = \pm 1, \pm 2, \dots$	Eigensystem for curl
θ	Heaviside theta function (1.6)
I	Identity matrix
Id	Identity operator
$(\mu_j, f_j), j = 0, 1, 2, \dots$	Eigensystem for $-\Delta$
M	Connected oriented closed manifold
$\text{mod } \Psi^{-\infty}$	Modulo an integral operator with infinitely smooth kernel
P_0, P_{\pm}	Definitions 1.1 and 1.2
$p_{\pm}(x, y)$	Full symbol of P_{\pm}
Q_{prin}	Principal symbol of the pseudodifferential operator Q
$Q_{\text{prin},s}$	Principal symbol of Q , a pseudodifferential operator of order $-s$
Q_{sub}	Subprincipal symbol of Q , for operators on 1-forms see Definition 3.2
Riem, Ric, Sc	Riemann curvature tensor, Ricci tensor and scalar curvature
$\rho(x)$	Riemannian density
$\mathbb{S}_r(x)$	Geodesic sphere of radius r centred at $x \in M$
tr	Matrix trace (4.4)
\mathbf{tr}	Matrix trace defined in accordance with Definition 4.3
Tr	Operator trace (4.3)
TM, T^*M	Tangent and cotangent bundle
$\Omega^k(M)$	Differential k -forms over M
$\psi_{\text{curl}}^{\text{loc}}(x)$	Regularised local trace of A , Definition 7.7
ψ_{curl}	Regularised global trace of A , Definition 7.8
Ψ^s	Classical pseudodifferential operators of order s
$\Psi^{-\infty}$	Infinitely smoothing operators (5.1)
$\zeta_Q(s)$	Zeta function of the operator Q
Z	Parallel transport map (4.6), see also Appendix F

2 | THE OPERATOR CURL

In this section, we put curl on a rigorous mathematical foundation, defining it within the framework of the theory of unbounded self-adjoint operators in Hilbert spaces. This would provide the reader with an account of the fundamental properties of curl, in a self-contained fashion, postponing proofs until Appendix B.

The main challenge that one faces when dealing with curl is that it is not elliptic.[†] Indeed, the formula for the principal symbol of curl (which happens to coincide with its full symbol) reads

[†] Recall that, by definition, a matrix (pseudo)differential operator is elliptic if the determinant of its principal symbol is nowhere vanishing on $T^*M \setminus \{0\}$.

$$[\text{curl}_{\text{prin}}]_{\alpha}^{\beta}(x, \xi) = -i E_{\alpha}^{\beta\gamma}(x) \xi_{\gamma}, \quad (2.1)$$

where the tensor E is defined in accordance with (1.10). Although ξ is not a 1-form in its own right, at a formal level the RHS of (2.1) can be rewritten, concisely, as $-i * \xi$. A straightforward calculation shows that

$$\det(\text{curl}_{\text{prin}}) = 0. \quad (2.2)$$

The eigenvalues of $\text{curl}_{\text{prin}}$ are simple and read

$$h^{(0)}(x, \xi) = 0, \quad h^{(\pm)}(x, \xi) = \pm \|\xi\|, \quad (2.3)$$

for all $(x, \xi) \in T^*M \setminus \{0\}$. Consequently, standard elliptic theory does not apply and care is required in defining the appropriate Hilbert space, domain of the operator and establishing the mapping properties of curl .

Let $H^s(M)$, $s > 0$, be the space of differential forms that are square integrable together with their partial derivatives up to order s . We do not carry in our notation for Sobolev spaces the degree of differential forms: this will be clear from the context. Henceforth, to further simplify notation, we drop the M and write Ω^k for $\Omega^k(M)$ and H^s for $H^s(M)$.

To begin with, let us observe that curl maps smooth coexact 1-forms to smooth coexact 1-forms. Therefore, it is natural to define curl as an operator

$$\text{curl} = *d : \delta\Omega^2 \cap H^1 \rightarrow \delta\Omega^2, \quad (2.4)$$

where $\delta\Omega^2$ is the Hilbert space of real-valued coexact 1-forms with inner product

$$\langle u, v \rangle := \int_M *u \wedge v = \int_M u \wedge *v. \quad (2.5)$$

The basic properties of the operator (2.4) are summarised by the following theorem.

Theorem 2.1. *Let (M, g) be a connected oriented closed Riemannian 3-manifold.*

- (a) *The operator curl (2.4) is self-adjoint.*
- (b) *The spectrum of curl is discrete and accumulates to $+\infty$ and $-\infty$.*
- (c) *Zero is not an eigenvalue of curl .*
- (d) *The operator curl^{-1} is a bounded operator from $\delta\Omega^2 \cap H^s$ to $\delta\Omega^2 \cap H^{s+1}$ for all $s \geq 0$.*

Remark 2.2. Let us point out, once again, that throughout this paper, curl comes in two guises: as a differential expression (1.5) and as a self-adjoint operator (2.4). Both will be denoted by ‘ curl ’; which is which will be clear from the context.

We would like to emphasise the significance of property (c) in the above theorem. If we start deforming the metric the eigenvalues of curl , understood as those of a self-adjoint operator (2.4), will never turn to zero or cross zero (change sign). This is in stark contrast with the Dirac operator where zero may be an eigenvalue, depending on the choice of metric [37]. In fact, the study of zero modes of the Dirac operator (harmonic spinors) is an active area of research.

3 | PSEUDODIFFERENTIAL OPERATORS ACTING ON 1-FORMS

In this section, we develop an invariant calculus for pseudodifferential operators acting on 1-forms over a closed Riemannian manifold (M, g) of arbitrary dimension d .

Let Q be a classical polyhomogeneous pseudodifferential operator of order s acting on 1-forms, and let

$$Q : u_\alpha(x) \mapsto v_\alpha(x) = (2\pi)^{-d} \int e^{i(x-y)^\gamma \xi_\gamma} q_\alpha^\beta(x, \xi) u_\beta(y) dy d\xi, \quad (3.1)$$

$$q_\alpha^\beta(x, \xi) \sim [q_s]_\alpha^\beta(x, \xi) + [q_{s-1}]_\alpha^\beta(x, \xi) + \dots \quad (3.2)$$

be its representation in local coordinates (the same for x and y). Here, \sim stands for asymptotic expansion [59, § 3.3], and ‘polyhomogeneous’ means that $[q_{s-k}]_\alpha^\beta(x, \lambda \xi) = \lambda^{s-k} [q_{s-k}]_\alpha^\beta(x, \xi)$ for any positive λ and $k = 0, 1, 2, \dots$

Throughout the paper, we denote pseudodifferential operators with upper case letters, and their symbols and homogeneous components of symbols — with lower case. For the principal and subprincipal symbol, we use upper case letters and denote them by Q_{prin} and Q_{sub} , respectively, with the subscript indicating that we are looking at invariant (or covariant) objects ‘living’ on the cotangent bundle as opposed to the operator Q itself which ‘lives’ (acts) on the base manifold.

Later on, we will sometimes write a pseudodifferential operator as an integral operator with distributional integral kernel (Schwartz kernel), see, for example, formula (4.5). We will use lower case Fraktur font for the Schwartz kernel.

Returning to the pseudodifferential operator (3.1), (3.2), it is easy to see that

$$[Q_{\text{prin}}]_\alpha^\beta(x, \xi) := [q_s]_\alpha^\beta(x, \xi) \quad (3.3)$$

provides an invariant definition for the principal symbol Q_{prin} of Q .

However, the subleading term, $[q_{s-1}]_\alpha^\beta$, is *not* invariant under change of coordinates. The task at hand is to define and provide a formula for the subprincipal symbol of Q , by determining appropriate correction terms to $[q_{s-1}]_\alpha^\beta$.

The original definition of subprincipal symbol goes back to J.J. Duistermaat and L. Hörmander [26, Eqn. (5.2.8)]. They defined the subprincipal symbol for operators acting on half-densities.

Duistermaat and Hörmander’s definition of subprincipal symbol extends to operators acting on scalar fields because scalars can be identified with half-densities: it is just a matter of multiplying or dividing by $\sqrt{\rho}$. This leads to the appearance of an additional correction term. Namely, given a pseudodifferential operator Q of order s

$$Q : f(x) \mapsto (2\pi)^{-d} \int e^{i(x-y)^\gamma \xi_\gamma} q(x, \xi) f(y) dy d\xi,$$

$$q(x, \xi) \sim q_s(x, \xi) + q_{s-1}(x, \xi) + \dots$$

acting on scalar fields, its subprincipal symbol reads

$$Q_{\text{sub}} := q_{s-1} + \frac{i}{2} \frac{\partial^2 q_s}{\partial x^\gamma \partial \xi_\gamma} + \frac{i}{2} \frac{\partial \ln \rho}{\partial x^\gamma} \frac{\partial q_s}{\partial \xi_\gamma}, \quad (3.4)$$

where ρ is the Riemannian density. In formula (3.4), one can, in principle, instead of the Riemannian density use any other prescribed positive reference density. However, in this paper, we

always stick with the Riemannian density. In particular, this implies the identity

$$\frac{\partial \ln \rho}{\partial x^\gamma} = \Gamma^\alpha_{\gamma\alpha} . \tag{3.5}$$

The definition of subprincipal symbol (3.4) extends further, without change, to operators acting on m -columns of scalar fields (sections of the trivial \mathbb{R}^m - or \mathbb{C}^m -bundle over M). By choosing a global framing (an orientable 3-manifold is automatically parallelisable), one can then produce a coordinate-invariant reformulation of the classical construction of subprincipal symbol for operators on 1-forms. However, we should like to emphasise that the approach presented in this manuscript avoids the reliance on a specific global framing. For this reason, we believe that it is still of interest and worth presenting.

Note that a notion of subprincipal symbol for operators acting on 1-forms is available in the literature in the special case when the principal symbol is of the form fI , where f is a scalar function on $T^*M \setminus \{0\}$ and I is the $d \times d$ identity matrix, see, for example, [36, 40].

Theorem 3.1. *Let Q be a pseudodifferential operator of order s acting on 1-forms with symbol (3.2). Then the quantity*

$$[Q_{\text{sub}}]_{\mu}{}^{\nu} := \left([q_{s-1}]_{\mu}{}^{\nu} + \frac{i}{2} \frac{\partial^2 [q_s]_{\mu}{}^{\nu}}{\partial x^\gamma \partial \xi_\gamma} \right) + \frac{i}{2} \left(\Gamma^\alpha_{\gamma\alpha} \frac{\partial [q_s]_{\mu}{}^{\nu}}{\partial \xi_\gamma} - \Gamma^\alpha_{\gamma\mu} \frac{\partial [q_s]_{\alpha}{}^{\nu}}{\partial \xi_\gamma} - \Gamma^\nu_{\gamma\alpha} \frac{\partial [q_s]_{\mu}{}^{\alpha}}{\partial \xi_\gamma} \right) \tag{3.6}$$

is covariant under change of local coordinates.

Definition 3.2. We call (3.6) the *subprincipal symbol* of Q .

Remark 3.3. ‘Covariance’ in Theorem 3.1 means that one does not get derivatives of Jacobians under change of local coordinates. More precisely, let x and \tilde{x} be two local coordinate systems, and let Q_{sub} and \tilde{Q}_{sub} be the quantity (3.6) evaluated in coordinates x and \tilde{x} , respectively. Put $J^\alpha_\beta := \partial \tilde{x}^\alpha / \partial x^\beta$, $[J^{-1}]^\mu_\nu := \partial x^\mu / \partial \tilde{x}^\nu$, $\tilde{\xi}_\alpha := [J^{-1}]^\beta_\alpha \xi_\beta$. Then the statement of the theorem is that

$$[\tilde{Q}_{\text{sub}}]_{\alpha}{}^{\beta}(\tilde{x}, \tilde{\xi}) = [J^{-1}]^\mu_\alpha(\tilde{x}) [Q_{\text{sub}}]_{\mu}{}^{\nu}(x(\tilde{x}), \xi(\tilde{x}, \tilde{\xi})) J^\beta_\nu(x(\tilde{x})) . \tag{3.7}$$

The above transformation law also applies to the principal symbol (3.3).

Remark 3.4. Note that the RHS of formula (3.6) does not reduce to

$$[q_{s-1}]_{\mu}{}^{\nu} + \frac{i}{2} \nabla_\gamma \frac{\partial [q_s]_{\mu}{}^{\nu}}{\partial \xi_\gamma}$$

by formally treating $\frac{\partial [q_s]_{\mu}{}^{\nu}}{\partial \xi_\gamma}$ as a (2,1)-tensor on M . The issue here is that $\frac{\partial [q_s]_{\mu}{}^{\nu}}{\partial \xi_\gamma}$ is not a true tensor, and even if it were, the sign in front of the last Christoffel symbol is not what one would expect.

We now proceed to the proof of Theorem 3.1. One way of proving the theorem is by a lengthy explicit calculation establishing (3.7). However, we will present an alternative proof which provides insight into the origins of formula (3.6). In order to do so, we need to introduce first some geometric notions that will be useful here and later on in the paper.

By a *frame* at a point $x \in M$, we mean a basis of orthonormal vectors in $T_x M$. By a *framing* in an open set $\mathcal{U} \subset M$, we mean the choice of a frame at each point x of \mathcal{U} , depending smoothly on the base point x .

Given a framing $\{e_j^\alpha\}_{j=1}^d$, the corresponding *dual framing* $\{e^k_\beta\}_{k=1}^d$ is defined as

$$e^k_\beta(x) := \delta^{jk} g_{\alpha\beta}(x) e_j^\alpha(x), \tag{3.8}$$

δ being the Kronecker symbol. Here and further on, e_j^α denotes the α th component of the j th vector field and e^k_β denotes the β th component of the k th covector field.

Definition 3.5. Let $z \in M$ be given and let $\{V_j\}_{j=1}^d$ be a frame at z . Let \mathcal{U}_z be a sufficiently small neighbourhood of z . We define the *Levi-Civita framing* $\{\tilde{e}_j\}_{j=1}^d$ to be the framing in \mathcal{U}_z obtained by parallel-transporting $\{V_j\}_{j=1}^d$ along geodesics emanating from z .

Let us emphasise that the Levi-Civita framing is uniquely determined in the neighbourhood of every point $z \in M$ up to a rigid (constant) orthogonal transformation, which reflects the freedom in choosing the frame $\{V_j\}_{j=1}^d$ at the point z . By definition, the Levi-Civita framing and its dual satisfy

$$[\partial_\beta \tilde{e}_j^\alpha + \Gamma^\alpha_{\beta\gamma} \tilde{e}_j^\gamma] \Big|_{x=z} = 0, \tag{3.9}$$

$$[\partial_\beta \tilde{e}^j_\alpha - \Gamma^\gamma_{\beta\alpha} \tilde{e}^j_\gamma] \Big|_{x=z} = 0, \tag{3.10}$$

see also [17, Lemma 7.2].

Proof of Theorem 3.1. Let us fix a point $z \in M$. Let \mathcal{U}_z be a small open neighbourhood of z and let $\{\tilde{e}_j\}_{j=1}^d$ be a Levi-Civita framing centred at z .

Consider the operator S mapping, locally, 1-forms on \mathcal{U}_z to sections of the trivial \mathbb{R}^d -bundle over \mathcal{U}_z , defined in accordance with

$$S : u_\alpha \mapsto \tilde{e}_j^\beta u_\beta. \tag{3.11}$$

We claim that (3.6) is the map

$$(z, \xi) \mapsto \left[S^{-1}(\rho^{1/2} S Q S^{-1} \rho^{-1/2})_{\text{sub}} S \right]_\alpha^\beta(z, \xi), \tag{3.12}$$

where

$$(\rho^{1/2} S Q S^{-1} \rho^{-1/2})_{\text{sub}}$$

denotes the usual subprincipal symbol [26, Eqn. (5.2.8)] of the operator $Q_{1/2} := \rho^{1/2} S Q S^{-1} \rho^{-1/2}$ acting on d -columns of half-densities.

The task at hand is to show, by means of an explicit computation, that (3.12) and (3.6) coincide. As a by-product of the upcoming calculation and (3.12), it will follow that (3.6) is covariant under changes of local coordinates and independent of the choice of the Levi-Civita framing $\{\tilde{e}_j\}_{j=1}^d$.

The operator $Q_{1/2}$ acts as

$$(Q_{1/2}f)_j(x) = \frac{1}{(2\pi)^d} \int e^{i(x-y)^\gamma \xi_\gamma} [\tilde{q}_{1/2}]_j^k(x, y, \xi) f_k(y) dy d\xi,$$

where

$$[\tilde{q}_{1/2}]_j^k(x, y, \xi) := \left[\frac{\rho(x)}{\rho(y)} \right]^{1/2} \tilde{e}_j^\alpha(x) q_\alpha^\beta(x, \xi) \tilde{e}^k_\beta(y).$$

The left (polyhomogeneous) symbol $q_{1/2} \sim \sum_{k=0}^{+\infty} [q_{1/2}]_{s-k}$ of $Q_{1/2}$ is obtained by excluding the y -dependence from $\tilde{q}_{1/2}$ via the amplitude-to-symbol operator

$$S_{\text{left}} \sim \sum_{k=0}^{+\infty} S_{-k}, \quad S_0 := (\cdot)|_{y=x}, \quad S_{-k} := \frac{1}{k!} \left[\left(-i \frac{\partial^2}{\partial y^\gamma \partial \xi_\gamma} \right)^k (\cdot) \right]_{y=x}. \tag{3.13}$$

The terms positively homogeneous in ξ of degree s and $s - 1$ in

$$[q_{1/2}]_j^k(x, \xi) := S_{\text{left}}([\tilde{q}_{1/2}]_j^k(x, y, \xi))$$

read

$$[(q_{1/2})_s]_j^k(x, \xi) = \tilde{e}_j^\alpha(x) [q_s]_\alpha^\beta(x, \xi) \tilde{e}^k_\beta(x) \tag{3.14}$$

and

$$[(q_{1/2})_{s-1}]_j^k(x, \xi) = \tilde{e}_j^\alpha [q_{s-1}]_\alpha^\beta \tilde{e}^k_\beta - i \tilde{e}_j^\alpha \frac{\partial [q_s]_\alpha^\beta}{\partial \xi_\gamma} \frac{\partial \tilde{e}^k_\beta}{\partial x^\gamma} + \frac{i}{2} (\ln \rho)_{x^\gamma} \tilde{e}_j^\alpha \frac{\partial [q_s]_\alpha^\beta}{\partial \xi_\gamma} \tilde{e}^k_\beta, \tag{3.15}$$

respectively. In the RHS of (3.15), framings are evaluated at x , and homogeneous components of q and their derivatives at (x, ξ) .

On account of (3.14) and (3.15), one obtains

$$\begin{aligned} [(Q_{1/2})_{\text{sub}}]_j^k(z, \xi) &= \left([(q_{1/2})_{s-1}]_j^k(x, \xi) + \frac{i}{2} \frac{\partial^2 [(q_{1/2})_s]_j^k}{\partial x^\gamma \partial \xi_\gamma}(x, \xi) \right) \Big|_{x=z} \\ &= \tilde{e}_j^\alpha \left([q_{s-1}]_\alpha^\beta + \frac{i}{2} \frac{\partial^2 [q_s]_\alpha^\beta}{\partial x^\gamma \partial \xi_\gamma} \right) \tilde{e}^k_\beta + \frac{i}{2} \frac{\partial \tilde{e}_j^\alpha}{\partial x^\gamma} \frac{\partial [q_s]_\alpha^\beta}{\partial \xi_\gamma} \tilde{e}^k_\beta \\ &\quad - \frac{i}{2} \tilde{e}_j^\alpha \frac{\partial [q_s]_\alpha^\beta}{\partial \xi_\gamma} \frac{\partial \tilde{e}^k_\beta}{\partial x^\gamma} + \frac{i}{2} (\ln \rho)_{x^\gamma} \tilde{e}_j^\alpha \frac{\partial [q_s]_\alpha^\beta}{\partial \xi_\gamma} \tilde{e}^k_\beta. \end{aligned} \tag{3.16}$$

In the RHS of (3.16), framings are evaluated at z , and homogeneous components of q and their derivatives at (z, ξ) .

In view of the identities

$$\tilde{e}^j_\mu \frac{\partial \tilde{e}_j^\alpha}{\partial x^\gamma} \Big|_{x=z} = -\Gamma^\alpha_{\gamma\mu}(z), \quad \frac{\partial \tilde{e}^k_\beta}{\partial x^\gamma} \tilde{e}^j_\mu \Big|_{x=z} = \Gamma^\nu_{\gamma\beta}(z), \tag{3.17}$$

which follow from (3.9), (3.10) and the identity (3.5), a straightforward calculation tells us that the quantity

$$\tilde{e}^j{}_\mu(z) [(Q_{1/2})_{\text{sub}}]_j^k(z, \xi) \tilde{e}_k{}^\nu(z) \tag{3.18}$$

coincides with (3.6).

The quantity (3.18) is invariant under rigid orthogonal transformations of the Levi-Civita framing; hence, this quantity is a covariant object. As (3.18) coincides with (3.6), this implies that (3.6) is covariant. \square

Let us consider the special case when M is an oriented 3-manifold. In this case, we have the following result.

Lemma 3.6. *The subprincipal symbol of curl is zero,*

$$(\text{curl})_{\text{sub}} = 0. \tag{3.19}$$

Proof. It follows from (2.1) that

$$\frac{\partial [\text{curl}_{\text{prin}}]_\alpha{}^\beta}{\partial \xi_\gamma} = -i E_\alpha{}^{\beta\gamma}(x) = \frac{i}{\rho(x)} \varepsilon^{\gamma\beta\mu} g_{\mu\alpha}(x), \tag{3.20}$$

where the second equality is a straightforward consequence of (1.10) and the properties of the totally antisymmetric symbol.

In light of (3.5), the substitution of (3.20) and (2.1) into (3.6) gives us

$$\begin{aligned} 2\rho [\text{curl}_{\text{sub}}]_\alpha{}^\beta &= -\rho \frac{\partial [\rho^{-1} \varepsilon^{\nu\beta\mu} g_{\mu\alpha}]}{\partial x^\nu} - \Gamma^\lambda{}_{\nu\lambda} [\varepsilon^{\nu\beta\mu} g_{\mu\alpha}] + \Gamma^\lambda{}_{\nu\alpha} [\varepsilon^{\nu\beta\mu} g_{\mu\lambda}] + \Gamma^\beta{}_{\nu\lambda} [\varepsilon^{\nu\lambda\mu} g_{\mu\alpha}] \\ &= -\varepsilon^{\nu\beta\mu} \frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \varepsilon^{\nu\beta\mu} \Gamma^\lambda{}_{\nu\alpha} g_{\mu\lambda} \\ &= -\varepsilon^{\nu\beta\mu} (\Gamma^\lambda{}_{\nu\alpha} g_{\mu\lambda} + \Gamma^\lambda{}_{\nu\mu} g_{\alpha\lambda}) + \varepsilon^{\nu\beta\mu} \Gamma^\lambda{}_{\nu\alpha} g_{\mu\lambda} \\ &= 0. \end{aligned}$$

Alternatively, one can prove (3.19) by fixing an arbitrary point on M and carrying out the above calculations in normal coordinates, using the fact that the subprincipal symbol is covariant by Theorem 3.1. \square

Returning to the analysis of the general case of a Riemannian manifold of arbitrary dimension, let us examine the second order differential operators δd and $d\delta$ acting on 1-forms. Similarly to (3.19), one can show that

$$(\delta d)_{\text{sub}} = 0, \tag{3.21}$$

$$(d\delta)_{\text{sub}} = 0. \tag{3.22}$$

Of course, formulae (3.21) and (3.22) imply that the Hodge Laplacian $\Delta := -(\delta d + d\delta)$ acting on 1-forms has zero subprincipal symbol.

The inner product

$$\langle u, v \rangle := \int_M g^{\alpha\beta}(x) \overline{u_\alpha(x)} v_\beta(x) \rho(x) dx$$

allows us to define the formal adjoint Q^* of a pseudodifferential operator Q acting on 1-forms.

Lemma 3.7. *We have*

$$[(Q^*)_{\text{prin}}]_\mu{}^\nu = g_{\mu\beta} \overline{[Q_{\text{prin}}]_\alpha{}^\beta} g^{\alpha\nu}, \quad (3.23)$$

$$[(Q^*)_{\text{sub}}]_\mu{}^\nu = g_{\mu\beta} \overline{[Q_{\text{sub}}]_\alpha{}^\beta} g^{\alpha\nu}. \quad (3.24)$$

Proof. Formula (3.23) is obvious. As to formula (3.24), it can be established by a lengthy explicit calculation, which shows that all additional terms involving derivatives of the metric and the Riemannian density cancel out. However, a shorter way of proving it is to argue as in the alternative version of the proof of Lemma 3.6: fix an arbitrary point on M and carry out calculations in normal coordinates, using the fact that the subprincipal symbol is covariant by Theorem 3.1. \square

Theorem 3.8. *Let Q and R pseudodifferential operators acting on 1-forms. Then*

$$(QR)_{\text{sub}} = Q_{\text{prin}}R_{\text{sub}} + Q_{\text{sub}}R_{\text{prin}} + \frac{i}{2}\{\{Q_{\text{prin}}, R_{\text{prin}}\}\}, \quad (3.25)$$

where

$$\begin{aligned} \{\{Q_{\text{prin}}, R_{\text{prin}}\}\}_\alpha{}^\beta := & \left(\frac{\partial [Q_{\text{prin}}]_\alpha{}^\kappa}{\partial x^\gamma} - \Gamma^{\alpha'}{}_{\gamma\alpha} [Q_{\text{prin}}]_{\alpha'}{}^\kappa + \Gamma^\kappa{}_{\gamma\alpha'} [Q_{\text{prin}}]_\alpha{}^{\kappa'} \right) \frac{\partial [R_{\text{prin}}]_\kappa{}^\beta}{\partial \xi_\gamma} \\ & - \frac{\partial [Q_{\text{prin}}]_\alpha{}^\kappa}{\partial \xi_\gamma} \left(\frac{\partial [R_{\text{prin}}]_\kappa{}^\beta}{\partial x^\gamma} - \Gamma^{\kappa'}{}_{\gamma\kappa} [R_{\text{prin}}]_{\kappa'}{}^\beta + \Gamma^\beta{}_{\gamma\beta'} [R_{\text{prin}}]_\kappa{}^{\beta'} \right) \end{aligned} \quad (3.26)$$

is the generalised Poisson bracket.

Remark 3.9. Note that if we *formally* treat principal symbols as (1,1)-tensors on M and put

$$\tilde{\nabla}_\alpha [Q_{\text{prin}}]_\beta{}^\gamma := \frac{\partial [Q_{\text{prin}}]_\beta{}^\gamma}{\partial x^\alpha} - \Gamma^{\beta'}{}_{\alpha\beta} [Q_{\text{prin}}]_{\beta'}{}^\gamma + \Gamma^\gamma{}_{\alpha\gamma'} [Q_{\text{prin}}]_\beta{}^{\gamma'}$$

then formula (3.26) simplifies and takes the form

$$\{\{Q_{\text{prin}}, R_{\text{prin}}\}\}_\alpha{}^\beta = \tilde{\nabla}_\gamma [Q_{\text{prin}}]_\alpha{}^\kappa \frac{\partial [R_{\text{prin}}]_\kappa{}^\beta}{\partial \xi_\gamma} - \frac{\partial [Q_{\text{prin}}]_\alpha{}^\kappa}{\partial \xi_\gamma} \tilde{\nabla}_\gamma [R_{\text{prin}}]_\kappa{}^\beta.$$

Compare with Remark 3.4.

It is possible to introduce genuine covariant derivatives on the cotangent bundle and reformulate formulae (3.6) and (3.26) in terms of the latter; see, for example, [17, Remark 4.5]. We refrain from doing this because it would take us away from the core subject of our paper.

Proof of Theorem 3.8. Let us fix a point $z \in M$. Let \mathcal{U}_z be a small open neighbourhood of z and let $\{\tilde{e}_j\}_{j=1}^3$ be a Levi-Civita framing centred at z . Arguing as in the proof of Theorem 3.1, we denote by $Q_{1/2}$ and $R_{1/2}$ the pseudodifferential operators acting on half-densities obtained from the operators Q and R (which act on 1-forms), see formula (3.11) and following text.

The task at hand is to show that, when $x = z$, the RHS of (3.25) coincides with the quantity

$$\left(\tilde{e}^j{}_\alpha [(Q_{1/2}R_{1/2})_{\text{sub}}]_j{}^k \tilde{e}_k{}^\beta \right) \Big|_{x=z},$$

where $(Q_{1/2}R_{1/2})_{\text{sub}}$ denotes the usual subprincipal symbol for operators acting on half-densities and for which the composition formula is known

$$(Q_{1/2}R_{1/2})_{\text{sub}} = (Q_{1/2})_{\text{prin}}(R_{1/2})_{\text{sub}} + (Q_{1/2})_{\text{sub}}(R_{1/2})_{\text{prin}} + \frac{i}{2}\{(Q_{1/2})_{\text{prin}}, (R_{1/2})_{\text{prin}}\}.$$

Here,

$$\{(Q_{1/2})_{\text{prin}}, (R_{1/2})_{\text{prin}}\} := \frac{\partial(Q_{1/2})_{\text{prin}}}{\partial x^\gamma} \frac{\partial(R_{1/2})_{\text{prin}}}{\partial \xi_\gamma} - \frac{\partial(Q_{1/2})_{\text{prin}}}{\partial \xi_\gamma} \frac{\partial(R_{1/2})_{\text{prin}}}{\partial x^\gamma}$$

is the Poisson bracket on matrix functions.

We have

$$\begin{aligned} \left(\tilde{e}^j{}_\alpha [(Q_{1/2})_{\text{prin}}(R_{1/2})_{\text{sub}}]_j{}^k \tilde{e}_k{}^\beta \right) \Big|_{x=z} &= \left(\tilde{e}^j{}_\alpha [(Q_{1/2})_{\text{prin}}]_j{}^l [(R_{1/2})_{\text{sub}}]_l{}^k \tilde{e}_k{}^\beta \right) \Big|_{x=z} \\ &= \left(\tilde{e}^j{}_\alpha [(Q_{1/2})_{\text{prin}}]_j{}^l \tilde{e}_l{}^\kappa \tilde{e}^m{}_\kappa [(R_{1/2})_{\text{sub}}]_m{}^k \tilde{e}_k{}^\beta \right) \Big|_{x=z} \\ &= \left(\tilde{e}^j{}_\alpha [(Q_{1/2})_{\text{prin}}]_j{}^l \tilde{e}_l{}^\kappa \right) \Big|_{x=z} \left(\tilde{e}^m{}_\kappa [(R_{1/2})_{\text{sub}}]_m{}^k \tilde{e}_k{}^\beta \right) \Big|_{x=z} \\ &= ([Q_{\text{prin}}]_\alpha{}^\kappa) \Big|_{x=z} ([R_{\text{sub}}]_\kappa{}^\beta) \Big|_{x=z} = ([Q_{\text{prin}} R_{\text{sub}}]_\alpha{}^\beta) \Big|_{x=z}. \end{aligned} \tag{3.27}$$

Here, we used the fact that, according to (3.12), the expression $(\tilde{e}^m{}_\kappa [(R_{1/2})_{\text{sub}}]_m{}^k \tilde{e}_k{}^\beta) \Big|_{x=z}$ is precisely $([R_{\text{sub}}]_\kappa{}^\beta) \Big|_{x=z}$.

Similarly, we have

$$\left(\tilde{e}^j{}_\alpha [(Q_{1/2})_{\text{sub}}(R_{1/2})_{\text{prin}}]_j{}^k \tilde{e}_k{}^\beta \right) \Big|_{x=z} = ([Q_{\text{sub}} R_{\text{prin}}]_\alpha{}^\beta) \Big|_{x=z}. \tag{3.28}$$

In view of (3.27) and (3.28), the proof of the theorem reduces to establishing the identity

$$\left(\{ \{ Q_{\text{prin}}, R_{\text{prin}} \} \}^\beta \right) \Big|_{x=z} = \left(\tilde{e}^j{}_\alpha \{ (Q_{1/2})_{\text{prin}}, (R_{1/2})_{\text{prin}} \}_j{}^k \tilde{e}_k{}^\beta \right) \Big|_{x=z}. \tag{3.29}$$

Recalling that

$$[(Q_{1/2})_{\text{prin}}]_j{}^l = \tilde{e}_j{}^\mu [Q_{\text{prin}}]_\mu{}^\nu \tilde{e}^l{}_\nu, \quad [(R_{1/2})_{\text{prin}}]_l{}^k = \tilde{e}_l{}^\rho [R_{\text{prin}}]_\rho{}^\sigma \tilde{e}^k{}_\sigma,$$

a straightforward calculation gives us

$$\begin{aligned} & \left. \{(Q_{1/2})_{\text{prin}}, (R_{1/2})_{\text{prin}}\}_j^k \right|_{x=z} \\ &= \left[\tilde{e}_j^\mu [Q_{\text{prin}}]_\mu^\nu \frac{\partial \tilde{e}_\nu^l}{\partial x^\gamma} \tilde{e}_l^\rho \frac{\partial [R_{\text{prin}}]_\rho^\sigma}{\partial \xi_\gamma} \tilde{e}^k_\sigma - \tilde{e}_j^\mu \frac{\partial [Q_{\text{prin}}]_\mu^\nu}{\partial \xi_\gamma} \tilde{e}_\nu^l \frac{\partial \tilde{e}_l^\rho}{\partial x^\gamma} [R_{\text{prin}}]_\rho^\sigma \tilde{e}^k_\sigma \right. \\ & \quad + \frac{\partial \tilde{e}_j^\mu}{\partial x^\gamma} [Q_{\text{prin}}]_\mu^\nu \frac{\partial [R_{\text{prin}}]_\nu^\sigma}{\partial \xi_\gamma} \tilde{e}^k_\sigma - \tilde{e}_j^\mu \frac{\partial [Q_{\text{prin}}]_\mu^\nu}{\partial \xi_\gamma} [R_{\text{prin}}]_\nu^\sigma \frac{\partial \tilde{e}^k_\sigma}{\partial x^\gamma} \\ & \quad \left. + \tilde{e}_j^\mu \{Q_{\text{prin}}, R_{\text{prin}}\}_\mu^\sigma \tilde{e}^k_\sigma \right] \Big|_{x=z}, \end{aligned}$$

so that

$$\begin{aligned} & \left(\tilde{e}^j_\alpha \{(Q_{1/2})_{\text{prin}}, (R_{1/2})_{\text{prin}}\}_j^k \tilde{e}_k^\beta \right) \Big|_{x=z} \\ &= \left[[Q_{\text{prin}}]_\alpha^\nu \frac{\partial \tilde{e}_\nu^l}{\partial x^\gamma} \tilde{e}_l^\rho \frac{\partial [R_{\text{prin}}]_\rho^\beta}{\partial \xi_\gamma} - \frac{\partial [Q_{\text{prin}}]_\alpha^\nu}{\partial \xi_\gamma} \tilde{e}_\nu^l \frac{\partial \tilde{e}_l^\rho}{\partial x^\gamma} [R_{\text{prin}}]_\rho^\beta + \tilde{e}^j_\alpha \frac{\partial \tilde{e}_j^\mu}{\partial x^\gamma} [Q_{\text{prin}}]_\mu^\nu \frac{\partial [R_{\text{prin}}]_\nu^\beta}{\partial \xi_\gamma} \right. \\ & \quad \left. - \frac{\partial [Q_{\text{prin}}]_\alpha^\nu}{\partial \xi_\gamma} [R_{\text{prin}}]_\nu^\sigma \frac{\partial \tilde{e}^k_\sigma}{\partial x^\gamma} \tilde{e}_k^\beta + \{Q_{\text{prin}}, R_{\text{prin}}\}_\alpha^\beta \right] \Big|_{x=z}. \end{aligned} \tag{3.30}$$

On account of (3.17) and (3.26), formula (3.30) implies (3.29). □

Example 3.10. Let M be an oriented 3-manifold. Then Theorem 3.8 and Lemma 3.6 imply

$$(\text{curl}^2)_{\text{sub}} = 0. \tag{3.31}$$

Alternatively, formula (3.31) follows from (3.21) and the fact that $\text{curl}^2 = \delta d$.

4 | TRACE OF PSEUDODIFFERENTIAL OPERATORS ACTING ON 1-FORMS

The aim of this section is to extend, in a natural way, the notion of trace of a pseudodifferential operator acting on 1-forms. Given a pseudodifferential operator Q acting on 1-forms, we will define a scalar operator $\text{tr } Q$ is such a way that, if Q is of trace class, then the operator-theoretic trace of Q coincides with the operator-theoretic trace of $\text{tr } Q$. Later, in the paper, we will use this definition without assuming that Q is of trace class: it may happen that Q itself is not of trace class, but $\text{tr } Q$ is — which is the case, up to an additional minor regularisation, for $Q = \theta(\text{curl}) - \theta(-\text{curl})$.

In this section, as in the previous one, we work on a closed Riemannian manifold (M, g) of arbitrary dimension d .

Let us start with the simpler case when Q is a pseudodifferential operator of order s acting on m -columns of scalar fields. Suppose that

$$s < -d. \tag{4.1}$$

Then [59, §12.1] Q is an integral operator with continuous integral kernel,

$$Q : u_j(x) \mapsto \int_M \mathbf{q}_j^k(x, y) u_k(y) \rho(y) dy. \tag{4.2}$$

We introduced the factor $\rho(y)$ in the above integral in order to make the integral kernel $\mathbf{q}_j^k(x, y)$ a (matrix-valued) scalar function on $M \times M$. Furthermore, if Q is self-adjoint, then it is of trace class and we have

$$\text{Tr } Q = \int_M (\text{tr } \mathbf{q})(x, x) \rho(x) dx, \tag{4.3}$$

where $(\text{tr } \mathbf{q})(x, y) := \mathbf{q}_j^j(x, y)$ is the trace of the matrix function $\mathbf{q}_j^k(x, y)$. We define the scalar operator

$$\text{tr } Q : f(x) \mapsto \int_M (\text{tr } \mathbf{q})(x, y) f(y) \rho(y) dy \tag{4.4}$$

and call it *the matrix trace of the operator* Q . It is easy to see that

$$\text{Tr } Q = \text{Tr}(\text{tr } Q).$$

Note that (4.4) is well defined for any $s \in \mathbb{R}$, not necessarily satisfying condition (4.1). More precisely, if Q is of order s , it is easy to see that

- (i) $\text{tr } Q$ is also an operator of order s ,
- (ii) $(\text{tr } Q)^* = \text{tr}(Q^*)$, where the star refers to formal adjoints with respect to the natural inner products,
- (iii) $(\text{tr } Q)_{\text{prin}} = \text{tr}(Q_{\text{prin}})$,
- (iv) $(\text{tr } Q)_{\text{sub}} = \text{tr}(Q_{\text{sub}})$.

Of course, when condition (4.1) is not satisfied the integral kernel $\mathbf{q}_j^k(x, y)$ appearing in formula (4.2) should be understood in the distributional sense (Schwartz kernel), but this does not prevent us from taking its matrix trace. The distribution $(\text{tr } \mathbf{q})(x, y)$ will be the Schwartz kernel of the scalar operator (4.4).

We will now adapt the above construction to the case of a pseudodifferential operator

$$Q : u_\alpha(x) \mapsto \int_M \mathbf{q}_\alpha^\beta(x, y) u_\beta(y) \rho(y) dy \tag{4.5}$$

acting on 1-forms. Here, we encounter a problem: the quantity $\mathbf{q}_\alpha^\beta(x, y)$ is not a scalar on $M \times M$ because the tensor indices α and β in $\mathbf{q}_\alpha^\beta(x, y)$ ‘live’ at different points, x and y , respectively. More precisely, the problem here is that the Schwartz kernel $\mathbf{q}_\alpha^\beta(x, y)$ of the operator (4.5) is a two-point tensor: it is a covector at x in the tensor index α and a vector at y in the tensor index β . We overcome this impediment as follows.

Let x and y be two points on the manifold M which are sufficiently close. Then, there is a unique shortest geodesic $\gamma(x, y)$ connecting x and y . Given a point $z \in \gamma(x, y)$, put

$$t = \frac{\text{dist}(z, x)}{\text{dist}(x, y)},$$

which is the variable arc length of the geodesic normalised by its total length. The variable t provides a convenient parameterisation of our geodesic $\gamma(x, y; \cdot) : [0, 1] \rightarrow M$, so that $\gamma(x, y; 0) = x$ and $\gamma(x, y; 1) = y$.

We denote by

$$Z : T_x M \ni u^\alpha \mapsto u^\alpha Z_\alpha^\beta(x, y) \in T_y M \tag{4.6}$$

the linear map realising the parallel transport of vectors from x to y along the unique shortest geodesic connecting x and y . Note that the result of parallel transport is independent of the choice of the particular parameterisation of the curve along which it takes place.

In what follows we will be raising and lowering indices in the two-point tensor $Z_\alpha^\beta(x, y)$ using the Riemannian metric $g(x)$ in the first index and $g(y)$ in the second. Of course,

$$Z_\alpha^\kappa(x, y) Z_\kappa^\beta(y, x) = \delta_\alpha^\beta, \quad Z_\alpha^\kappa(x, y) Z^\kappa_\beta(y, x) = \delta^\alpha_\beta, \tag{4.7}$$

$$Z_\alpha^\kappa(x, y) Z^\beta_\kappa(x, y) = \delta_\alpha^\beta, \quad Z_\kappa^\alpha(x, y) Z^\kappa_\beta(x, y) = \delta^\alpha_\beta. \tag{4.8}$$

Formula (4.7) expresses the fact that if we parallel transport a vector/covector from x to y along the shortest geodesic connecting these two points, and then parallel transport it back from y to x along the same geodesic, we get the original vector/covector. Formula (4.8) is a consequence of the fact that the Levi-Civita connection is metric compatible.

Remark 4.1. For later use, let us observe that formulae (4.7) and (4.8) imply

$$Z_\beta^\alpha(x, y) = Z^\alpha_\beta(y, x). \tag{4.9}$$

Indeed, take the first identity (4.7), multiply it by $Z^\alpha_\gamma(x, y)$ and then use the second identity (4.8). This gives us $\delta^\kappa_\gamma Z_\kappa^\beta(y, x) = \delta_\alpha^\beta Z^\alpha_\gamma(x, y)$, which is equivalent to (4.9).

Let us introduce a one-parameter family of scalar distributions defined by

$$Z^\alpha_\kappa(x, \gamma(x, y; \tau)) q_\alpha^\beta(x, y) Z_\beta^\kappa(y, \gamma(x, y; \tau)), \tag{4.10}$$

where $q_\alpha^\beta(x, y)$ is the Schwartz kernel from (4.5) and $\tau \in [0, 1]$. What happens here is that tensor indices are parallel transported to the common point $z = \gamma(x, y; \tau)$ on the geodesic $\gamma(x, y)$ connecting x and y . Thus, the quantity (4.10) is a genuine scalar.

As we are dealing with distributions, we need to clarify the precise meaning of formula (4.10).

Consider a map from smooth scalar functions of two variables to smooth (1,1) two-point tensors defined as

$$C^\infty(M \times M) \ni f(x, y) \mapsto Z^\alpha_\kappa(x, \gamma(x, y; \tau)) Z_\beta^\kappa(y, \gamma(x, y; \tau)) f(x, y). \tag{4.11}$$

The quantity $q_\alpha^\beta(x, y)$ in (4.10) is a distribution (continuous complex-valued linear functional) acting on smooth (1,1) two-point tensors. Formula (4.10) should be understood as a composition of q and (4.11).

Lemma 4.2. *The quantity (4.10) does not depend on τ .*

Proof. The statement of the lemma is an immediate consequence of the fact that the Levi-Civita connection is metric compatible. □

Since the definition (4.10) is independent of the choice of τ , henceforth we will set, for convenience, $\tau = 0$, in which case formula (4.10) simplifies and reads

$$q_\alpha^\beta(x, y) Z_\beta^\alpha(y, x). \tag{4.12}$$

There is still one problem with formula (4.12): the linear operator Z appearing in this formula is defined only for x and y sufficiently close. In order to view (4.12) as a well-defined distribution, we need a cut-off. Let $\chi : [0, +\infty) \rightarrow \mathbb{R}$ be a compactly supported infinitely smooth scalar function such that $\chi = 1$ in a neighbourhood of zero. We modify formula (4.10) to read

$$q_\alpha^\beta(x, y) Z_\beta^\alpha(y, x) \chi(\text{dist}(x, y)/\epsilon), \tag{4.13}$$

where $\epsilon > 0$ is a small parameter which ensures that the quantity (4.13) vanishes when x and y are not sufficiently close.

Definition 4.3. The scalar operator

$$\text{tr } Q : f(x) \mapsto \int_M q_\alpha^\beta(x, y) Z_\beta^\alpha(y, x) \chi(\text{dist}(x, y)/\epsilon) f(y) \rho(y) \, dy \tag{4.14}$$

is called *the matrix trace* of the operator (4.5).

Thus, given a pseudodifferential operator Q acting on 1-forms, we defined the scalar pseudodifferential operator $\text{tr } Q$. The latter depends on the small parameter $\epsilon > 0$ and the cut-off χ , but, as we will see, the choice of this parameter and the cut-off does not affect the main results of our paper. Let us emphasise that the matrix trace (4.14) is defined uniquely, modulo the addition of a scalar integral operator whose integral kernel is infinitely smooth and vanishes in a neighbourhood of the diagonal.

Proposition 4.4.

(a) For Q self-adjoint and under the condition (4.1), we have

$$\text{Tr}(\text{tr } Q) = \text{Tr } Q. \tag{4.15}$$

(b) We have

$$(\text{tr } Q)^* = \text{tr}(Q^*), \tag{4.16}$$

where the star refers to formal adjoints with respect to the natural inner products on scalar functions and 1-forms, respectively.

(c) We have

$$(\text{tr } Q)_{\text{prin}} = \text{tr } Q_{\text{prin}}, \tag{4.17}$$

$$(\text{tr } Q)_{\text{sub}} = \text{tr } Q_{\text{sub}}. \tag{4.18}$$

Remark 4.5. Let us emphasise that, crucially, the left-hand sides of (4.15), (4.17) and (4.18) do not depend on ϵ .

Proof of Proposition 4.4.

- (a) Formula (4.15) follows from the fact that $Z_\alpha^\beta(x, x) = \delta_\alpha^\beta$ and $\chi(0) = 1$.
- (b) We have

$$\begin{aligned}
 (\mathbf{tr} Q)^* : f(x) &\mapsto \int_M \overline{(\mathbf{tr} q)(y, x)} f(y) \rho(y) dy \\
 &= \int_M \overline{q_\alpha^\beta(y, x)} Z_\beta^\alpha(x, y) \chi(\text{dist}(x, y)/\epsilon) f(y) \rho(y) dy, \quad (4.19)
 \end{aligned}$$

$$Q^* : u_\alpha(x) \mapsto \int_M \overline{q_\alpha^\beta(y, x)} u_\beta(y) \rho(y) dy,$$

$$\begin{aligned}
 \mathbf{tr}(Q^*) : f(x) &\mapsto \int_M \overline{q_\alpha^\beta(y, x)} Z_\beta^\alpha(y, x) \chi(\text{dist}(x, y)/\epsilon) f(y) \rho(y) dy \\
 &= \int_M \overline{q_\alpha^\beta(y, x)} Z_\beta^\alpha(y, x) \chi(\text{dist}(x, y)/\epsilon) f(y) \rho(y) dy. \quad (4.20)
 \end{aligned}$$

Formulae (4.19), (4.20) and (4.9) imply (4.16).

- (c) Formula (4.17) follows from the fact that $Z_\alpha^\beta(x, x) = \delta_\alpha^\beta$ and $\chi(0) = 1$.

The proof of (4.18) is more subtle. Formula (3.1) tells us that, modulo an infinitely smooth contribution, the Schwartz kernel $q_\alpha^\beta(x, y)$ of the operator (4.5) reads

$$q_\alpha^\beta(x, y) = \frac{1}{(2\pi)^d \rho(y)} \int e^{i(x-y)^\gamma \xi_\gamma} q_\alpha^\beta(x, \xi) d\xi, \quad (4.21)$$

where the symbol $q_\alpha^\beta(x, \xi)$ admits the asymptotic expansion (3.2). We also have

$$Z_\alpha^\beta(x, y) = \delta_\alpha^\beta - \Gamma_{\lambda\alpha}^\beta(x) (y - x)^\lambda + O(|y - x|^2). \quad (4.22)$$

Substituting (4.21) and (4.22) into (4.13), we arrive at a representation of the scalar pseudodifferential operator (4.14) with amplitude depending on x, y and ξ . In order to write down the (left) symbol of the operator (4.14), we need to exclude the dependence on y , which is achieved in the standard manner — recall (3.13). This produces the following expressions for the leading and subleading terms of the symbol $(\mathbf{tr} q)(x, \xi)$ of the scalar pseudodifferential operator (4.14):

$$(\mathbf{tr} q)_s = [q_s]_\alpha^\alpha, \quad (4.23)$$

$$(\mathbf{tr} q)_{s-1} = [q_{s-1}]_\alpha^\alpha - i \Gamma_{\gamma\beta}^\alpha \frac{\partial [q_s]_\alpha^\beta}{\partial \xi_\gamma}. \quad (4.24)$$

Note that the right-hand sides of (4.23) and (4.24) do not depend on the parameter ϵ .

Substituting (4.23) and (4.24) into (3.4) and using (3.5), we get

$$(\text{tr } Q)_{\text{sub}} = [q_{s-1}]_{\alpha}^{\alpha} + \frac{i}{2} \frac{\partial^2 [q_s]_{\alpha}^{\alpha}}{\partial x^{\gamma} \partial \xi_{\gamma}} + \frac{i}{2} \Gamma^{\beta}_{\gamma\beta} \frac{\partial [q_s]_{\alpha}^{\alpha}}{\partial \xi_{\gamma}} - i \Gamma^{\alpha}_{\gamma\beta} \frac{\partial [q_s]_{\alpha}^{\beta}}{\partial \xi_{\gamma}}. \tag{4.25}$$

But taking the trace of (3.6) gives the same expression as in the right-hand side of (4.25). This completes the proof of formula (4.18).

Alternatively, one can prove formula (4.18) by fixing an arbitrary point on M and carrying out calculations in normal coordinates, using the covariance of subprincipal symbols defined in accordance with formulae (3.4) and (3.6). With this approach all the Christoffel symbols disappear. \square

Proposition 4.4 part (b) immediately implies

Corollary 4.6. *If Q is formally self-adjoint then so is $\text{tr } Q$.*

5 | THE PROJECTION OPERATORS P_0 AND P_{\pm}

This section is devoted to the study of the operators P_0 and P_{\pm} . We will show that they are pseudodifferential operators whose full symbols can be constructed via an explicit algorithm. Furthermore, we will compute their principal and subprincipal symbols, thus proving Theorem 1.3.

By Ψ^s , we denote the space of classical pseudodifferential operators of order s with polyhomogeneous symbols acting on 1-forms, recall (3.1), (3.2). We define

$$\Psi^{-\infty} := \bigcap_s \Psi^s, \tag{5.1}$$

and we write $Q = R \text{ mod } \Psi^{-\infty}$ if $Q - R$ is an integral operator with (infinitely) smooth integral kernel.

Let us begin by proving parts (a) and (b) of Theorem 1.3. The matter of subprincipal symbols (part (c) of Theorem 1.3) and the construction of our operators will be addressed afterwards.

Proof of Theorem 1.3, (a) and (b). Let f_j be the orthonormalised eigenfunctions corresponding to the eigenvalues (B.12) of $-\Delta$. Put

$$v_j := \mu_j^{-1/2} df_j, \quad j = 1, 2, \dots \tag{5.2}$$

The v_j form an orthonormal basis in the Hilbert space $d\Omega^0(M)$ and the operator P_0 can be written as

$$P_0 := \sum_{j=1}^{+\infty} v_j \langle v_j, \cdot \rangle. \tag{5.3}$$

Let

$$\Delta^{-1} := - \sum_{j=1}^{+\infty} \mu_j^{-1} f_j \langle f_j, \cdot \rangle \tag{5.4}$$

be the pseudo-inverse of the Laplace–Beltrami operator Δ .

Combining formulae (5.2)–(5.4), we get an explicit formula for the projection operator P_0

$$P_0 = -d \Delta^{-1} \delta. \tag{5.5}$$

In particular, formula (5.5) tells us that P_0 is a pseudodifferential operator of order zero and its principal symbol is given by formula (1.8).

Consider the operator curl in the Hilbert space $\delta\Omega^2(M)$. Let λ_j be its eigenvalues and u_j its orthonormalised eigenforms. Here, we enumerate using positive integers j for positive eigenvalues and negative integers j for negative eigenvalues, so that

$$\dots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \dots,$$

with account of multiplicities. The u_j form an orthonormal basis in the Hilbert space $\delta\Omega^2(M)$ and the operators P_{\pm} can be written as

$$P_+ = \sum_{j=1}^{+\infty} u_j \langle u_j, \cdot \rangle,$$

$$P_- = \sum_{j=-1}^{-\infty} u_j \langle u_j, \cdot \rangle.$$

We have

$$P_+ + P_- = \text{Id} - P_0 - P_{\mathcal{H}^1}, \tag{5.6}$$

where Id is the identity operator on 1-forms and $P_{\mathcal{H}^1}$ is the orthogonal projection onto the finite-dimensional space of harmonic 1-forms. Of course, $P_{\mathcal{H}^1}$ is an integral operator with infinitely smooth integral kernel,

$$P_{\mathcal{H}^1} \in \Psi^{-\infty}. \tag{5.7}$$

Consider the Hodge Laplacian

$$\Delta := -(d\delta + \delta d) = - \sum_{j=1}^{+\infty} \mu_j v_j \langle v_j, \cdot \rangle - \sum_{j \in \mathbb{Z} \setminus \{0\}} \lambda_j^2 u_j \langle u_j, \cdot \rangle \tag{5.8}$$

acting on 1-forms. This is an elliptic self-adjoint non-positive differential operator which commutes with the differential operator curl. Given an $s \in \mathbb{R}$, the operator

$$(-\Delta)^s := \sum_{j=1}^{+\infty} \mu_j^s v_j \langle v_j, \cdot \rangle + \sum_{j \in \mathbb{Z} \setminus \{0\}} |\lambda_j|^{2s} u_j \langle u_j, \cdot \rangle \tag{5.9}$$

is a pseudodifferential operator of order $2s$. It is easy to see that

$$P_+ - P_- = (-\Delta)^{-1/2} \text{curl} = (-\Delta)^{-1/4} \text{curl} (-\Delta)^{-1/4} = \text{curl} (-\Delta)^{-1/2}. \tag{5.10}$$

Equations (5.6) and (5.10) are a system of two linear algebraic equations for the two unknowns P_+ and P_- , whose unique solution is

$$P_{\pm} = \frac{1}{2} \left[\text{Id} - P_0 - P_{\mathcal{H}^1} \pm (-\Delta)^{-1/4} \text{curl} (-\Delta)^{-1/4} \right]. \tag{5.11}$$

We have

$$[\text{Id}_{\text{prin}}]_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}, \tag{5.12}$$

$$[-\Delta_{\text{prin}}]_{\alpha}^{\beta} = \|\xi\|^2 \delta_{\alpha}^{\beta}. \tag{5.13}$$

Formulae (5.11)–(5.13), (1.8), (5.7) and (2.1) imply (1.9). □

Recall that, as explained in Section 2, $\text{curl}_{\text{prin}}$ has three simple eigenvalues $h^{(\aleph)}$, $\aleph \in \{0, +, -\}$, given by (2.3). Let us denote by $P^{(\aleph)}$, $\aleph \in \{0, +, -\}$, the corresponding eigenprojections. It is easy to see that these are exactly the principal symbols (1.8), (1.9) of our pseudodifferential operators P_{\aleph} , $\aleph \in \{0, +, -\}$.

Observe that the pseudodifferential operators P_{\aleph} , $\aleph \in \{0, +, -\}$, satisfy

$$P_{\aleph}^* = P_{\aleph}, \quad P_{\aleph} P_{\beth} = \delta_{\aleph \beth} P_{\aleph}, \quad \sum_{\aleph} P_{\aleph} = \text{Id} \pmod{\Psi^{-\infty}}, \quad [\text{curl}, P_{\aleph}] = 0, \tag{5.14}$$

where δ is the Kronecker delta.

Next, let us examine what happens if we relax (5.14), asking that they are satisfied not exactly, but only modulo $\Psi^{-\infty}$, cf. (5.6) and (5.7). Namely, let us seek operators $\tilde{P}_{\aleph} \in \Psi^0$, $\aleph \in \{0, +, -\}$, satisfying

$$(\tilde{P}_{\aleph})^* = \tilde{P}_{\aleph} \pmod{\Psi^{-\infty}}, \tag{5.15a}$$

$$\tilde{P}_{\aleph} \tilde{P}_{\beth} = \delta_{\aleph \beth} \tilde{P}_{\aleph} \pmod{\Psi^{-\infty}}, \tag{5.15b}$$

$$\sum_{\aleph} \tilde{P}_{\aleph} = \text{Id} \pmod{\Psi^{-\infty}}, \tag{5.15c}$$

$$[\text{curl}, \tilde{P}_{\aleph}] = 0 \pmod{\Psi^{-\infty}}. \tag{5.15d}$$

Theorem 5.1. *Let \tilde{P}_0, \tilde{P}_+ and \tilde{P}_- be pseudodifferential operators of order zero, acting on 1-forms, satisfying*

$$(\tilde{P}_{\aleph})_{\text{prin}} = P^{(\aleph)}, \quad \aleph \in \{0, +, -\}, \tag{5.16}$$

and (5.15). Then

$$\tilde{P}_{\aleph} = P_{\aleph} \pmod{\Psi^{-\infty}}, \quad \aleph \in \{0, +, -\}. \tag{5.17}$$

Proof. The claim follows from a straightforward adaptation of [18, Theorem 2.2] to the setting of the current paper. The original result from [18] was formulated under the assumption of ellipticity but ellipticity was never used in establishing the existence of projections. What was used is simplicity of eigenvalues of the principal symbol, which holds for curl. Furthermore, the original

result from [18] was formulated under the assumption that the operator acts on columns of half-densities, but this assumption was never used in an essential way; what was used is the presence of an inner product. \square

In fact, we have a stronger version of Theorem 5.1.

Theorem 5.2. *Let \tilde{P}_0, \tilde{P}_+ and \tilde{P}_- be pseudodifferential operators of order zero, acting on 1-forms, satisfying conditions (5.15d), (5.16) and*

$$(\tilde{P}_{\aleph})^2 = \tilde{P}_{\aleph} \pmod{\Psi^{-\infty}}, \quad \aleph \in \{0, +, -\}. \tag{5.18}$$

Then we have (5.17).

What happens here is that in Theorem 5.1 assumptions (5.15a) and (5.15c) are redundant, and (5.15b) can be relaxed to (5.18).

Proof of Theorem 5.2. The proof is analogous to that of Theorem 5.1, only additionally taking into account [18, Theorem 4.1]. \square

The overall strategy of our paper hinges on the key observation (5.17). Indeed, this gives us an effective way of constructing the full symbols of our projections P_{\aleph} , $\aleph \in \{0, +, -\}$, without resorting explicitly to the representations (5.5) and (5.11).

Let introduce refined notation for the principal symbol. Namely, we denote by $(\cdot)_{\text{prin},s}$ the principal symbol of the expression within brackets, regarded as an operator in Ψ^{-s} . To appreciate the need for such notation, consider the following example. Let Q and R be pseudodifferential operators in Ψ^{-s} with the same principal symbol. Then, as an operator in Ψ^{-s} , $Q - R$ has vanishing principal symbol: $(Q - R)_{\text{prin},s} = 0$. But this tells us that $Q - R$ is, effectively, an operator in Ψ^{-s-1} and, as such, it may have non-vanishing principal symbol $(Q - R)_{\text{prin},s+1}$. This refined notation will be used whenever there is risk of confusion.

Proposition 5.3. *The operators P_0, P_{\pm} can be explicitly constructed, modulo $\Psi^{-\infty}$, as follows.*

- **Step 1.** Choose three arbitrary pseudodifferential operators $P_{\aleph,0} \in \Psi^0$, $\aleph \in \{0, +, -\}$, such that $(P_{\aleph,0})_{\text{prin}} = P^{(\aleph)}$.
- **Step 2.** For $k = 1, 2, \dots$, define

$$P_{\aleph,k} := P_{\aleph,0} + \sum_{n=1}^k X_{\aleph,n}, \quad X_{\aleph,n} \in \Psi^{-n}.$$

Assuming we have determined the pseudodifferential operator $P_{\aleph,k-1}$, compute, one after the other, the following quantities:

- (a) $R_{\aleph,k} := -((P_{\aleph,k-1})^2 - P_{\aleph,k-1})_{\text{prin},k}$,
 - (b) $S_{\aleph,k} := -R_{\aleph,k} + P^{(\aleph)}R_{\aleph,k} + R_{\aleph,k}P^{(\aleph)}$,
 - (c) $T_{\aleph,k} := [P_{\aleph,k-1}, \text{curl}]_{\text{prin},k-1} + [S_{\aleph,k}, \text{curl}]_{\text{prin}}$.
- **Step 3.** Choose a pseudodifferential operator $X_{\aleph,k} \in \Psi^{-k}$ satisfying

$$(X_{\aleph,k})_{\text{prin}} = S_{\aleph,k} + \sum_{\beth \neq \aleph} \frac{P^{(\beth)}T_{\aleph,k}P^{(\beth)} - P^{(\beth)}T_{\beth,k}P^{(\aleph)}}{h^{(\beth)} - h^{(\aleph)}}.$$

Recall that $h^{(\cdot)}$ is defined in accordance with (2.3) and (1.7).

• **Step 4.** Then

$$P_{\mathfrak{N}} \sim P_{\mathfrak{N},0} + \sum_{n=1}^{+\infty} X_{\mathfrak{N},n}. \tag{5.19}$$

Here \sim stands for an asymptotic expansion in ‘smoothness’: truncated sums give an approximation modulo pseudodifferential operators of lower order.

Proof. The algorithm presented here is a straightforward adaptation of that from [18, subsection 4.3], which can be applied to the operator curl in view of Theorem 5.1. \square

Remark 5.4. The paper [18] offers two different algorithms for the construction of an orthonormal basis of pseudodifferential projections. The first algorithm, using the full set of conditions from Theorem 5.1, is given in [18, subsection 3.4]. The second algorithm, using the minimal set of conditions from Theorem 5.2, is given in [18, subsection 4.3]. Of course, the end result is the same, modulo $\Psi^{-\infty}$, but the latter is much simpler.

Remark 5.5. Let us stress once again that the algorithm from Proposition 5.3 does not require ellipticity of the original operator, but it *does* require the eigenvalues of the principal symbol to be simple — see Step 3.

We are now in a position to prove part (c) of Theorem 1.3, which will be quite useful in the analysis of the asymmetry operator.

Proof of Theorem 1.3(c). For starters, let us observe that the claim (1.11) is *invariant* under changes of local coordinates, because the $(P_{\mathfrak{N}})_{\text{sub}}$ are covariant under such transformations. Therefore, it suffices to fix a point $z \in M$, choose normal coordinates at z , and check that (1.11) is satisfied at $(z, \zeta) \in T^*M \setminus \{0\}$ for all ζ , in the chosen coordinate system.

Indeed, let us fix a point z and work in normal coordinates centred at z . Let us choose arbitrary pseudodifferential operators $P_{\mathfrak{N},0} \in \Psi^0$, $\mathfrak{N} \in \{0, +, -\}$, satisfying $(P_{\mathfrak{N},0})_{\text{prin}} = (P_{\mathfrak{N}})_{\text{prin}}$ (recall that the latter are given by (1.8) and (1.9)) and

$$(P_{\mathfrak{N},0})_{\text{sub}} = 0. \tag{5.20}$$

Then it is easy to see from the algorithm in Proposition 5.3 — in particular, from formula (5.19) — that

$$(P_{\mathfrak{N}})_{\text{sub}} = (X_{\mathfrak{N},1})_{\text{prin}}. \tag{5.21}$$

So proving (1.11) reduces to proving that

$$[(X_{\mathfrak{N},1})_{\text{prin}}(z, \zeta)]_{\alpha}^{\beta} = 0 \tag{5.22}$$

in the chosen coordinate system. Let us compute, one after the other, the quantities from Step 2 in Proposition 5.3.

Using (3.25) and (5.20), we get

$$[R_{\mathbb{N},1}]_{\alpha}^{\beta}(z, \zeta) = -[(P_{\mathbb{N}}^2)_{\text{sub}}]_{\alpha}^{\beta}(z, \zeta) = -\frac{i}{2}\{(P_{\mathbb{N}})_{\text{prin}}, (P_{\mathbb{N}})_{\text{prin}}\}_{\alpha}^{\beta}(z, \zeta) = 0. \tag{5.23}$$

In the last equality, we used the fact that first spatial derivatives of $(P_{\mathbb{N}})_{\text{prin}}$ and Christoffel symbols vanish at (z, ζ) in normal coordinates.

Formula (5.23) immediately implies $[S_{\mathbb{N},k}]_{\alpha}^{\beta}(z, \zeta) = 0$. Hence, on account of Lemma 3.6, we have

$$[T_{\mathbb{N},1}]_{\alpha}^{\beta}(z, \zeta) = \frac{i}{2}\{(P_{\mathbb{N}})_{\text{prin}}, \text{curl}_{\text{prin}}\}(z, \zeta) - \frac{i}{2}\{\text{curl}_{\text{prin}}, (P_{\mathbb{N}})_{\text{prin}}\}(z, \zeta).$$

But the latter also vanishes in normal coordinates, so one arrives at (5.22). □

Remark 5.6. Observe that (1.8)–(1.11) and (3.2) give us explicit formulae for the homogeneous components of degree -1 of the symbols of the operators P_0 and P_{\pm} in arbitrary local coordinates.

6 | THE ASYMMETRY OPERATOR

6.1 | Definition of the asymmetry operator

The remainder of this paper is devoted to the study of the pseudodifferential operator $P_+ - P_-$ which encodes information about the asymmetry of the Riemannian manifold (M, g) under change of orientation. In particular, one expects this information to be carried by the trace of $P_+ - P_-$, either in its pointwise or global (operator-theoretic) versions.

Unfortunately, one encounters a problem: the pseudodifferential operator $P_+ - P_-$ is of order zero and its spectrum is the set of three points $\{-1, 0, +1\}$, all three being eigenvalues of infinite multiplicity. Hence, it is not trace class. In order to circumvent this problem, acting in the spirit of Section 4, we introduce the following definition.

Definition 6.1. We define the *asymmetry operator* to be the (scalar) pseudodifferential operator

$$A := \text{tr}(P_+ - P_-), \tag{6.1}$$

where tr is defined in accordance with Definition 4.3.

The above definition warrants a number of remarks.

Remark 6.2.

- (a) The asymmetry operator depends, *a priori*, on the choice of the parameter ϵ and the cut-off χ . However, we suppress the dependence on ϵ and χ in our notation for the asymmetry operator because, as we shall see later on, our main results do not depend on these choices. The dependence on ϵ and χ is, in a sense, trivial: two operators A corresponding to different choices of ϵ and χ differ by an integral operator with infinitely smooth integral kernel which vanishes in a neighbourhood of the diagonal in $M \times M$. The trace of the latter is clearly zero.
- (b) The Schwartz kernel of $P_+ - P_-$ is real; hence, the Schwartz kernel of A is also real.

(c) The operator $P_+ - P_-$ is self-adjoint; hence, by Corollary 4.6, the asymmetry operator A is also self-adjoint.

6.2 | The order of the asymmetry operator

Definition 6.1 implies that, *a priori*, our asymmetry operator A is a pseudodifferential operator of order 0. The goal of this subsection is to show that, as a result of cancellations in the symbol, the order of A is in fact -3 . This will be done in three steps.

1. Firstly, we will show that A is of order -2 . This will follow easily from the facts already established earlier in the paper.
2. Next, we will show that, in normal coordinates, parallel transport does not contribute to homogeneous components of the symbol of A of degree -2 and -3 .
3. Finally, we will show that A is of order -3 . This is the subject of Theorem 6.6 which will come in the end of the subsection and is one of the main results of this paper.

Lemma 6.3. *The asymmetry operator A is a pseudodifferential operator of order -2 .*

Proof. The claim of the lemma is equivalent to the following two statements:

$$A_{\text{prin}} = 0, \tag{6.2}$$

$$A_{\text{sub}} = 0. \tag{6.3}$$

Formula (6.2) follows from (6.1), (4.17) and (1.9), whereas formula (6.3) follows from (6.1), (4.18) and (1.11). □

Let us now fix an arbitrary point $z \in M$ and work in normal coordinates centred at z . In our chosen coordinate system, the operator A (modulo $\Psi^{-\infty}$) reads

$$A : f(x) \mapsto \frac{1}{(2\pi)^3} \int e^{i(x-y)^\mu \xi_\mu} [p_+ - p_-]_\alpha^\beta(x, \xi) Z_\beta^\alpha(y, x) f(y) dy d\xi, \tag{6.4}$$

where

$$[p_\pm]_\alpha^\beta(x, \xi) \sim \sum_{j=0}^{+\infty} [(p_\pm)_{-j}]_\alpha^\beta(x, \xi), \quad [(p_\pm)_{-k}]_\alpha^\beta(x, \lambda\xi) = \lambda^{-k} [(p_\pm)_{-k}]_\alpha^\beta(x, \xi) \quad \forall \lambda > 0, \tag{6.5}$$

is the full symbol of P_\pm in the chosen coordinate system.

Next, we observe that (6.4) can be recast as

$$A = A_{\text{diag}} + A_{\text{pt}}, \tag{6.6}$$

where

$$A_{\text{diag}} := \frac{1}{(2\pi)^3} \int e^{i(x-y)^\mu \xi_\mu} [p_+ - p_-]_\alpha^\alpha(x, \xi) f(y) dy d\xi \tag{6.7}$$

and

$$A_{\text{pt}} := A - A_{\text{diag}}. \tag{6.8}$$

Here, the subscript ‘pt’ stands for ‘parallel transport’. The operator A_{pt} describes the contribution to the asymmetry operator arising from the fact that our original operator curl acts on 1-forms and not on 3-columns of scalar fields,

The decomposition (6.6) is not invariant, it relies on our particular choice of local coordinates but it is convenient for our purposes. Namely, in what follows we will show that A_{pt} will not contribute to the principal symbol of order -3 of the asymmetry operator A at the point z .

Let us denote by $a_{\text{pt}}(x, \xi) \sim \sum_{j=0}^{+\infty} (a_{\text{pt}})_{-j}(x, \xi)$ the full symbol of A_{pt} . Our task is to show that

$$(a_{\text{pt}})_{-j}(z, \xi) = 0 \quad \text{for } j = 0, 1, 2, 3. \tag{6.9}$$

In what follows, we will be using the expansion (F.8) for Z_α^β .

The claim (6.9) for $j = 0$ and $j = 1$ follows immediately from (F.8).

Lemma 6.4. *We have*

$$(a_{\text{pt}})_{-2}(z, \xi) = 0.$$

Proof. Upon integration by parts, we get from (F.8)

$$(a_{\text{pt}})_{-2}(z, \xi) = \frac{1}{6} \text{Riem}^\alpha_{\mu\kappa\nu}(z) \frac{\partial^2 [(p_+ - p_-)_0]_\alpha{}^\kappa(z, \xi)}{\partial \xi_\mu \partial \xi_\nu} = \frac{1}{6} \text{Riem}_{\alpha\mu\kappa\nu}(z) \frac{\partial^2 [i |\xi|^{-1} \varepsilon^{\alpha\gamma\kappa} \xi_\gamma]}{\partial \xi_\mu \partial \xi_\nu}, \tag{6.10}$$

where $|\cdot|$ is the Euclidean norm. By the symmetries of the Riemann tensor, we have

$$\text{Riem}_{\alpha\mu\kappa\nu} = \frac{1}{2} (\text{Riem}_{\alpha\mu\kappa\nu} + \text{Riem}_{\kappa\nu\alpha\mu}). \tag{6.11}$$

Since

$$\frac{\partial^2 [i |\xi|^{-1} \varepsilon^{\alpha\gamma\kappa} \xi_\gamma]}{\partial \xi_\mu \partial \xi_\nu} \tag{6.12}$$

is symmetric in the pair of indices μ and ν , we can replace (6.11) in (6.10) with its symmetrised version

$$\text{Riem}_{\alpha\mu\kappa\nu} = \frac{1}{2} (\text{Riem}_{\alpha\mu\kappa\nu} + \text{Riem}_{\kappa\nu\alpha\mu} + \text{Riem}_{\alpha\nu\kappa\mu} + \text{Riem}_{\kappa\mu\alpha\nu}). \tag{6.13}$$

But the quantity (6.13) is symmetric in the pair of indices α and κ , whereas (6.12) is antisymmetric in the same pair of indices. Therefore, (6.10) vanishes. \square

Next, we observe that formulae (1.11), (3.6) and (1.9) imply

$$[(p_+ - p_-)_{-1}]_\alpha{}^\beta(z, \xi) = 0. \tag{6.14}$$

This fact is needed in establishing the following lemma.

Lemma 6.5. *We have*

$$(a_{\text{pt}})_{-3}(z, \xi) = 0.$$

Proof. The expansion (F.8) and formula (6.14) imply, upon integration by parts,

$$\begin{aligned} (a_{\text{pt}})_{-3}(z, \xi) &= -\frac{i}{6} \frac{\partial^2 \Gamma_{\sigma\kappa}^\alpha}{\partial x^\mu \partial x^\nu}(z) \frac{\partial^3 [(p_+ - p_-)_0]_{\alpha^\kappa}(z, \xi)}{\partial \xi_\sigma \partial \xi_\mu \partial \xi_\nu} \\ &= -\frac{i}{6} \delta_{\alpha\beta} \frac{\partial^2 \Gamma_{\sigma\kappa}^\beta}{\partial x^\mu \partial x^\nu}(z) \frac{\partial^3 [i |\xi|^{-1} \varepsilon^{\alpha\gamma\kappa} \xi_\gamma]}{\partial \xi_\sigma \partial \xi_\mu \partial \xi_\nu}. \end{aligned} \quad (6.15)$$

We observe that in formula (6.15), one can replace the quantity $\delta_{\alpha\beta} \frac{\partial^2 \Gamma_{\sigma\kappa}^\beta}{\partial x^\mu \partial x^\nu}(z)$ with its symmetrised version in the three indices σ, μ and ν . It is known [12, Eqn. (11.9)] that the latter is equal to

$$\frac{1}{2} \nabla_\nu \text{Riem}_{\alpha\sigma\mu\kappa}(z). \quad (6.16)$$

Using the symmetries of the Riemann tensor, (6.16) can be equivalently rewritten as

$$\frac{1}{2} \nabla_\nu \text{Riem}_{\alpha\sigma\mu\kappa} = \frac{1}{4} [\nabla_\nu \text{Riem}_{\alpha\sigma\mu\kappa} + \nabla_\nu \text{Riem}_{\mu\kappa\alpha\sigma}]. \quad (6.17)$$

Now, when contracted with

$$\frac{\partial^3 [i |\xi|^{-1} \varepsilon^{\alpha\gamma\kappa} \xi_\gamma]}{\partial \xi_\sigma \partial \xi_\mu \partial \xi_\nu}, \quad (6.18)$$

the quantity (6.17) can be replaced with its symmetrised version in the pair of indices μ and σ :

$$\frac{1}{8} [\nabla_\nu \text{Riem}_{\alpha\sigma\mu\kappa} + \nabla_\nu \text{Riem}_{\mu\kappa\alpha\sigma} + \nabla_\nu \text{Riem}_{\alpha\mu\sigma\kappa} + \nabla_\nu \text{Riem}_{\sigma\kappa\alpha\mu}]. \quad (6.19)$$

But the quantity (6.19) is now symmetric in the pair of indices α and κ , whereas (6.18) is antisymmetric in the same pair of indices. Therefore, (6.15) vanishes. \square

Lemmata 6.4 and 6.5 give us (6.9).

We are now in a position to prove the main result of this subsection.

Theorem 6.6. *The asymmetry operator A is a pseudodifferential operator of order -3 .*

Remark 6.7. Before rigorously proving Theorem 6.6, let us give an intuitive explanation of why the claim should be true.

The principal symbol $A_{\text{prin},2}$ (i.e. the principal symbol of the asymmetry operator A viewed as an operator of order -2) should be proportional to

- Ricci curvature — because a_{-2} involves two derivatives of the metric and because in dimension 3 the Riemann curvature tensor can be expressed in terms of the Ricci tensor;
- the totally antisymmetric tensor E — because the asymmetry operator A is expected to feel orientation.

If one attempts to construct a scalar quantity out of the above two objects, the metric and momentum ξ , one realises that the only possible outcome is the scalar zero.

Proof of Theorem 6.6. Let us denote by a_{diag} the full symbol of A_{diag} . Formulae (1.9) and (6.14) immediately imply

$$[(a_{\text{diag}})_0](z, \xi) = 0 \tag{6.20}$$

and

$$[(a_{\text{diag}})_{-1}](z, \xi) = 0, \tag{6.21}$$

respectively.

Hence, in view of (6.6) and (6.9), proving the claim reduces to showing that

$$[(a_{\text{diag}})_{-2}](z, \xi) = [(p_+ - p_-)_{-2}]_\alpha^\alpha(z, \xi) = 0. \tag{6.22}$$

Our strategy for proving (6.22) is to check it explicitly by implementing the algorithm from Proposition 5.3 in chosen normal coordinates. As the latter is computationally challenging, we describe below how to do so in a cunning way.

In what follows $z = 0$ (the centre of our normal coordinates).

Simplification 1. It is well known that

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{1}{3} \text{Riem}_{\alpha\mu\beta\nu}(0) x^\mu x^\nu + O(|x|^3). \tag{6.23}$$

The approximation (6.23) is sufficiently accurate for the computation of (6.22).

Simplification 2. Express the Riemann tensor Riem in terms of the Ricci tensor Ric via the identity

$$\begin{aligned} \text{Riem}_{\alpha\beta\gamma\delta}(x) &= \text{Ric}_{\alpha\gamma}(x) g_{\beta\delta}(x) - \text{Ric}_{\alpha\delta}(x) g_{\beta\gamma}(x) + \text{Ric}_{\beta\delta}(x) g_{\alpha\gamma}(x) - \text{Ric}_{\beta\gamma}(x) g_{\alpha\delta}(x) \\ &+ \frac{\text{Sc}(x)}{2} (g_{\alpha\delta}(x) g_{\beta\gamma}(x) - g_{\alpha\gamma}(x) g_{\beta\delta}(x)), \end{aligned} \tag{6.24}$$

which holds in dimension three, where $\text{Sc}(x) := g^{\alpha\beta}(x) \text{Ric}_{\alpha\beta}(x)$ is scalar curvature. Then substitute (6.24) into (6.23) and discard cubic and higher order terms.

Simplification 3. It is clear that Ric will appear in $[(a_{\text{diag}})_{-2}](z, \xi) = [(p_+ - p_-)_{-2}]_\alpha^\alpha(0, \xi)$ in a linear fashion. This means that we can set

$$\text{Ric}(0) = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_4 & c_5 \\ c_3 & c_5 & c_6 \end{pmatrix} \tag{6.25}$$

and do calculations assuming that only one of the constants $c_j, j = 1, \dots, 6$, is non-zero. In particular, one can choose the non-zero c_j to be equal to 1. To establish (6.22), one just needs to run the algorithm six times.

Simplification 4. Without loss of generality, evaluate the final quantities (as soon as there are no more differentiations with respect to momentum) at

$$\xi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.26}$$

This is sufficient because

- one can rotate the normal coordinate system so that ξ aligns with the third spatial coordinate and
- one can scale ξ so as to normalise it, using the homogeneity of $[(a_{\text{diag}})_{-2}](0, \xi)$.

Moreover, one can write $\xi = \xi_0 + \eta$ and replace $\|\xi\|$ with its Taylor expansion in η . The latter simplifies differentiations in ξ .

Simplification 5. In implementing the algorithm, expand all quantities at the k th step in powers of x and η retaining only terms up to order $2 - k$, jointly in x and η . Here, we are assuming that x and η are of the same order.

Simplifications 1–5 reduce our algorithm to the manipulation of polynomials. Note that the vector spaces of polynomials in the variables x^α and η_α , $\alpha = 1, 2, 3$, of degree 0, 1 and 2 have dimension 1, 7 and 28, respectively.[†]

We implement below the algorithm in the special case

$$c_j = \delta_{1j}. \tag{6.27}$$

The other cases are analogous.

Formulae (6.23)–(6.25) and (6.27) imply that in the chosen coordinate system the metric tensor reads

$$g_{\alpha\beta}(x) = \begin{pmatrix} 1 - \frac{1}{6}(x^2)^2 - \frac{1}{6}(x^3)^2 & \frac{1}{6}x^1x^2 & \frac{1}{6}x^1x^3 \\ \frac{1}{6}x^1x^2 & 1 - \frac{1}{6}(x^1)^2 + \frac{1}{6}(x^3)^2 & -\frac{1}{6}x^2x^3 \\ \frac{1}{6}x^1x^3 & -\frac{1}{6}x^2x^3 & 1 - \frac{1}{6}(x^1)^2 + \frac{1}{6}(x^2)^2 \end{pmatrix} + O(|x|^3)$$

and

$$\rho(x) = 1 - \frac{1}{6}(x^1)^2 + O(|x|^3).$$

Let us now retrace, step by step, the algorithm from Proposition 5.3. In view of Lemma 6.3, let us assume that we have carried out the first iteration of the algorithm and have determined pseudodifferential operators $P_{\pm,1} \in \Psi^0$ satisfying

$$P_{\pm,1}^2 = P_{\pm,1} \pmod{\Psi^{-2}}, \quad P_{+,1}P_{-,1} = 0 \pmod{\Psi^{-2}}, \quad [P_{\pm,1}, \text{curl}] = 0 \pmod{\Psi^{-1}}.$$

This gives us

$$(p_{\pm,1})_{-1}(x, \xi_0 + \eta) = -\frac{1}{12} \begin{pmatrix} ix^3 & \pm x^3 & 0 \\ \pm x^3 & -ix^3 & 0 \\ \pm x^2 - 3ix^1 & \pm x^1 + ix^2 & 0 \end{pmatrix} + O(|x|^2 + |\eta|^2).$$

[†] The dimension of the vector space of polynomials of degree k in n variables is $\binom{k+n}{k}$.

Without loss of generality, we can assume that

$$(p_{\pm,1})_{-2} = 0. \tag{6.28}$$

The latter can be achieved by choosing $X_{\pm,1}$ appropriately. Therefore, proving the theorem reduces to showing that

$$[\text{tr}(X_{+,2} - X_{-,2})_{\text{prin}}](0, \xi_0) = 0. \tag{6.29}$$

The first quantity we need to compute is $R_{\pm,2}$. Due to (6.28), we have

$$R_{\pm,2} = -(P_{\pm,1}^2)_{-2},$$

where $(P_{\pm,1}^2)_{-2}$ stands for the homogeneous component of degree -2 of the symbol of the operator $P_{\pm,1}^2$. To compute the latter, one needs to use the formula for the symbol of the composition of pseudodifferential operators: given two operators A and B with symbols a and b , the symbol σ_{BA} of their composition BA is given by the formula

$$\sigma_{BA} \sim \sum_{k=0}^{\infty} \frac{1}{i^k k!} \frac{\partial^k b}{\partial \xi_{\alpha_1}^{\xi} \dots \partial \xi_{\alpha_k}^{\xi}} \frac{\partial^k a}{\partial x^{\alpha_1} \dots \partial x^{\alpha_k}}, \tag{6.30}$$

see [59, Theorem 3.4].

By performing the relevant calculations with the simplifications described above, one obtains

$$\begin{aligned} R_{\pm,2}(0, \xi_0) &= -\frac{1}{12} \begin{pmatrix} 1 & \mp 2i & 0 \\ \pm 2i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ S_{\pm,2}(0, \xi_0) &= -\frac{1}{12} \begin{pmatrix} 2 & \mp i & 0 \\ \pm i & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ T_{\pm,2}(0, \xi_0) &= \pm \frac{1}{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and finally,

$$(X_{\pm,2})_{\text{prin}}(0, \xi_0) = -\frac{1}{24} \begin{pmatrix} 4 & \mp 3i & 0 \\ \pm i & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.31}$$

Formula (6.31) implies (6.29). □

6.3 | The principal symbol of the asymmetry operator

This subsection is concerned with the proof of Theorem 1.4. As in the previous subsection, in what follows we also assume to have chosen normal coordinates centred at z .

Remark 6.8. Before moving on to the proof, let us try to reproduce the intuitive argument from Remark 6.7 at the next order of homogeneity.

The principal symbol $A_{\text{prin},3}$ (i.e. the principal symbol of the asymmetry operator A viewed as an operator of order -3) should be proportional to

- the covariant derivative of Ricci curvature — because a_{-3} involves three derivatives of the metric and because in dimension 3, the Riemann curvature tensor can be expressed in terms of the Ricci tensor;
- the totally antisymmetric tensor E — because the operator A is expected to feel orientation.

If one attempts to construct a scalar quantity out of the above two objects, the metric and momentum ξ , one realises that the only possible non-zero outcome, up to a constant scaling, is

$$E^{\alpha\beta\gamma}(x) \nabla_{\alpha} \text{Ric}_{\beta}{}^{\rho}(x) \frac{\xi_{\gamma} \xi_{\rho}}{\|\xi\|^5}.$$

Proof of Theorem 1.4. In view of Lemma 6.5, proving (1.13) reduces to showing that, in chosen normal coordinates, we have

$$[(p_{+} - p_{-})_{-3}]_{\alpha}{}^{\alpha}(z, \xi) = -\frac{1}{2|\xi|^5} \varepsilon^{\alpha\beta\gamma} \nabla_{\alpha} \text{Ric}_{\beta}{}^{\rho}(z) \xi_{\gamma} \xi_{\rho}. \tag{6.32}$$

The strategy for proving (6.32) consists in implementing the algorithm from Proposition 5.3 in chosen normal coordinates up to order -3 . Even with the help of modern computer algebra, doing this to the required accuracy is a highly non-trivial task. Therefore, as in the proof of Theorem 6.6, we describe below some helpful simplifications, which once again reduce all calculations to the manipulation of polynomials.

In what follows $z = 0$ (the centre of our normal coordinates).

Simplification 1. It is known, see [58, p. 211 formula (3.4)] or [55, formula (6)], that

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{1}{3} \text{Riem}_{\alpha\mu\beta\nu}(0) x^{\mu} x^{\nu} - \frac{1}{6} (\nabla_{\sigma} \text{Riem}_{\alpha\mu\beta\nu})(0) x^{\sigma} x^{\mu} x^{\nu} + O(|x|^4). \tag{6.33}$$

The approximation (6.33) is sufficiently accurate for the computation of (6.32).

Simplification 2. Express the Riemann tensor Riem in terms of the Ricci tensor Ric via the identity (6.24). Then substitute (6.24) into (6.33) and discard quartic and higher order terms.

Simplification 3. It is clear that Ric and ∇Ric will appear in $(p_{\pm})_{-3}$ in a linear fashion. We can choose (6.25) and

$$\begin{aligned} \nabla_1 \text{Ric}_{\alpha\beta}(0) &= \begin{pmatrix} c_7 & c_8 & c_9 \\ c_8 & c_{10} & c_{11} \\ c_9 & c_{11} & c_{12} \end{pmatrix}, & \nabla_2 \text{Ric}_{\alpha\beta}(0) &= \begin{pmatrix} c_{13} & c_{14} & c_{15} \\ c_{14} & c_{16} & c_{17} \\ c_{15} & c_{17} & c_{18} \end{pmatrix}, \\ \nabla_3 \text{Ric}_{\alpha\beta}(0) &= \begin{pmatrix} c_{19} & c_{20} & c_{21} \\ c_{20} & c_{22} & c_{23} \\ c_{21} & c_{23} & c_{24} \end{pmatrix}, \end{aligned} \tag{6.34}$$

and do calculations assuming that only one of the constants c_j , $j = 1, \dots, 24$, is non-zero. In particular, one can choose the non-zero c_j to be equal to 1. To establish (6.32), one just needs to run the algorithm 24 times.

Simplification 4. Without loss of generality, evaluate the final quantities (as soon as there are no more differentiations with respect to momentum) at the particular ξ_0 given by (6.26). This is sufficient due to rotational symmetry and homogeneity. Moreover, one can write $\xi = \xi_0 + \eta$ and replace $\|\xi\|$ with its Taylor expansion in η . The latter simplifies differentiations in ξ .

Simplification 5. In implementing the algorithm, expand all quantities at the k th step in powers of x and η retaining only terms up order $3 - k$, jointly in x and η . Here, we are assuming that x and η are of the same order.

Simplifications 1–5 reduce our algorithm to the manipulation of polynomials.

We implement below the algorithm in the special case

$$c_{11} = 1, \quad c_j = 0, \text{ for } j \neq 11. \tag{6.35}$$

The other cases are analogous.

Formulae (6.25) and (6.33)–(6.35) imply that in the chosen coordinate system the metric tensor reads

$$g_{\alpha\beta}(x) = \begin{pmatrix} 1 - \frac{x^1 x^2 x^3}{3} & \frac{(x^1)^2 x^3}{6} & \frac{(x^1)^2 x^2}{6} \\ \frac{(x^1)^2 x^3}{6} & 1 & -\frac{(x^1)^3}{6} \\ \frac{(x^1)^2 x^2}{6} & -\frac{(x^1)^3}{6} & 1 \end{pmatrix} + O(|x|^4)$$

and

$$\rho(x) = 1 - \frac{x^1 x^2 x^3}{6} + O(|x|^4).$$

Let us now retrace, step by step, the algorithm from Proposition 5.3. In view of Theorem 6.6, let us assume that we have carried out the first two iterations of the algorithm and have determined pseudodifferential operators $P_{\pm,2} \in \Psi^0$ satisfying

$$P_{\pm,2}^2 = P_{\pm,2} \text{ mod } \Psi^{-3}, \quad P_{+,2} P_{-,2} = 0 \text{ mod } \Psi^{-3}, \quad [P_{\pm,2}, \text{curl}] = 0 \text{ mod } \Psi^{-2}.$$

This gives us

$$(p_{\pm,2})_{-1}(x, \xi_0 + \eta) = \frac{1}{24} \begin{pmatrix} \mp 5(x^1)^2 - 4ix^1 x^2 & 6i(x^1)^2 \mp 3x^1 x^2 & 0 \\ -2i(x^1)^2 \pm x^1 x^2 & \mp 3(x^1)^2 & 0 \\ 2ix^2 x^3 & -6ix^1 x^3 & 0 \end{pmatrix} + O(|x|^3 + |\eta|^3)$$

and

$$(p_{\pm,2})_{-2}(x, \xi_0 + \eta) = -\frac{1}{24} \begin{pmatrix} \pm ix^3 & 6x^3 & 0 \\ -2x^3 & \pm 3ix^3 & 0 \\ \pm 9ix^1 - 2x^2 & 6x^1 \pm 3ix^2 & 0 \end{pmatrix} + O(|x|^2 + |\eta|^2).$$

Without loss of generality, we can assume that

$$(p_{\pm,1})_{-3} = 0.$$

The latter can be achieved by choosing $X_{\pm,2}$ appropriately. Therefore, proving (6.32) reduces to showing that

$$[\text{tr}(X_{+,3} - X_{-,3})_{\text{prin}}](0, \xi_0) = -\frac{1}{2}. \tag{6.36}$$

By performing the relevant calculations with the simplifications described above and with account of (6.30), one obtains

$$\begin{aligned} R_{\pm,3}(0, \xi_0) &= \frac{i}{8} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ S_{\pm,3}(0, \xi_0) &= \mp \frac{1}{8} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ T_{\pm,3}(0, \xi_0) &= \mp \frac{i}{4} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and, finally,

$$(X_{\pm,3})_{\text{prin}}(0, \xi_0) = \mp \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.37}$$

Formula (6.37) implies (6.36). □

Remark 6.9. Observe that neither in this section nor elsewhere in the paper our arguments make use of the second (differential) Bianchi identity. Of course, the second Bianchi identity implies that the number of independent components in formula (6.34) is 15, not 18. Namely, c_{21} , c_{23} and c_{24} can be expressed in terms of the remaining components.

The second Bianchi identity may be needed for the calculation of the subprincipal symbol of the asymmetry operator. This matter is outside the scope of the current paper.

Since formula (1.13) is one of the key results of this paper, we provide in Appendix G an independent verification relying on formula (5.10).

7 | THE REGULARISED TRACE OF THE ASYMMETRY OPERATOR

7.1 | Structure of the integral kernel near the diagonal

The asymmetry operator A can be written as an integral operator

$$(Af)(x) = \int_M \mathfrak{a}(x, y) f(y) \rho(y) \, dy. \tag{7.1}$$

Now, the operator A is not guaranteed to be of trace class, so one cannot apply formula (4.15) to it. The issue here is that in dimension 3, a sufficient condition for a pseudodifferential operator

to be of trace class is that its order be strictly less than -3 , whereas according to Theorem 1.4 for a generic Riemannian 3-manifold the order of A is exactly -3 . This means that we need to perform some sort of very basic regularisation in order to get over the finish line and define, in an invariant and geometrically meaningful manner, the trace of A .

In order to define an appropriate regularisation, we need to understand the structure of the integral kernel $\mathfrak{a}(x, y)$. More precisely, we need to characterise explicitly the way in which the latter fails to be continuous.

Theorem 7.1. *In a neighbourhood of the diagonal $\{(x, x) \mid x \in M\} \subset M \times M$ the integral kernel of A can be written in local coordinates as*

$$\mathfrak{a}(x, y) = \mathfrak{a}_d(x, y) + \mathfrak{a}_c(x, y), \tag{7.2}$$

where

$$\mathfrak{a}_d(x, y) := \frac{1}{12\pi^2} E^{\alpha\beta}{}_\gamma(x) \nabla_\alpha \text{Ric}_{\beta\rho}(x) \frac{(x - y)^\gamma(x - y)^\rho}{\text{dist}^2(x, y)}, \tag{7.3}$$

and \mathfrak{a}_c is continuous as a function of two variables. Recall that the tensor E is defined in accordance with (1.10).

Remark 7.2. The tensor $E^{\alpha\beta}{}_\gamma \nabla_\alpha \text{Ric}_{\beta\rho}$ appearing in formula (7.3) is trace-free,

$$E^{\alpha\beta}{}_\gamma \nabla_\alpha \text{Ric}_{\beta\rho} g^{\gamma\rho} = 0. \tag{7.4}$$

This is an important property which will be used on two occasions in the subsequent arguments in this section. We will see that property (7.4) implies that the singularity of the integral kernel of our asymmetry operator A is weaker than one would expect of a generic pseudodifferential operator of order -3 . For a generic pseudodifferential operator of order -3 , one would have a logarithm in the leading term of the singularity of the integral kernel, but there is no logarithm in (7.3).

In order to prove Theorem 7.1, we need to establish first a number of preparatory lemmata. Until further notice, we will be working in Euclidean space \mathbb{R}^3 equipped with Cartesian coordinates. In what follows we use the *Japanese bracket* notation

$$\langle \xi \rangle := (1 + |\xi|^2)^{1/2}. \tag{7.5}$$

Lemma 7.3. *We have*

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\langle \xi \rangle^5} e^{-iy^\mu \xi_\mu} d\xi = \frac{1}{6\pi^2} |y| K_1(|y|), \tag{7.6}$$

where K_1 is the modified Bessel function of the second kind [1, § 9.6, p. 374].

Proof. Let us observe that the function

$$y \mapsto \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\langle \xi \rangle^5} e^{-iy^\mu \xi_\mu} d\xi$$

is real and spherically symmetric. Therefore, we have

$$\begin{aligned} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\langle \xi \rangle^5} e^{-iy^\mu \xi_\mu} d\xi &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\langle \xi \rangle^5} \cos(|y| \xi_1) d\xi \\ &= \frac{2\pi}{(2\pi)^3} \int_{-\infty}^{+\infty} \cos(|y| \xi_1) \left(\int_0^{+\infty} \frac{r}{(1 + (\xi_1)^2 + r^2)^{5/2}} dr \right) d\xi_1 \quad (7.7) \\ &= \frac{1}{12\pi^2} \int_{-\infty}^{+\infty} \frac{\cos(|y| \xi_1)}{(1 + (\xi_1)^2)^{3/2}} d\xi_1, \end{aligned}$$

where at the second step, we performed the change of variable $(\xi_2, \xi_3) = (r \cos \theta, r \sin \theta)$. Formula (7.6) now follows by explicitly evaluating the integral in the single variable ξ_1 appearing in the RHS of (7.7):

$$\int_{-\infty}^{+\infty} \frac{\cos(|y| \xi_1)}{(1 + (\xi_1)^2)^{3/2}} d\xi_1 = 2|y| K_1(|y|).$$

The latter is known as Basset’s integral [23, (10.32.11)]. □

Lemma 7.4. *We have*

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\langle \xi \rangle^5} e^{-iy^\mu \xi_\mu} d\xi = \frac{1}{12\pi^2} |y|^2 \ln |y| + h(y), \quad \text{where } h \in C^3(\mathbb{R}^3). \quad (7.8)$$

Proof. It is known that the modified Bessel function of the second kind K_1 admits the asymptotic expansion

$$K_1(t) = \frac{1}{t} + \frac{t}{4}(2 \ln t + 2\gamma - 1 - \ln 4) + O(t^3 \ln t) \quad \text{as } t \rightarrow 0^+, \quad (7.9)$$

where γ is the Euler–Mascheroni constant — see, for example, [23, (10.31.11) and (10.25.2)]. By combining (7.9) and (7.6), one obtains (7.8). □

Lemma 7.5. *We have*

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\xi_\gamma \xi_\rho}{\langle \xi \rangle^5} e^{-iy^\mu \xi_\mu} d\xi = -\frac{1}{12\pi^2} \left[2 \frac{y^\gamma y^\rho}{|y|^2} + (1 + 2 \ln |y|) \delta_{\gamma\rho} \right] + h_{\gamma\rho}(y), \quad \text{where } h_{\gamma\rho} \in C^1(\mathbb{R}^3). \quad (7.10)$$

Proof. Formula (7.10) is a straightforward consequence of (7.8), once one observes that

$$\frac{\xi_\gamma \xi_\rho}{\langle \xi \rangle^5} e^{-iy^\mu \xi_\mu} = -\frac{\partial^2}{\partial y^\gamma \partial y^\rho} \left(\frac{1}{\langle \xi \rangle^5} e^{-iy^\mu \xi_\mu} \right). \quad \square$$

Proof of Theorem 7.1. In the previous section, we have established that the asymmetry operator A is a pseudodifferential operator of order -3 with principal symbol (1.13). Locally, the symbol $a(x, \xi)$ of A and its integral kernel $\alpha(x, y)$ are related via the (distributional) identity

$$\alpha(x, y) = \frac{1}{(2\pi)^3 \rho(y)} \int_{\mathbb{R}^3} e^{i(x-y)^\mu \xi_\mu} a(x, \xi) d\xi. \quad (7.11)$$

Since a pseudodifferential operator of order -4 on a 3-manifold has a continuous integral kernel, formulae (1.13), (1.10) and (7.11) imply that problem at hand reduces, in normal coordinates centred at x , to examining the quantity

$$-\frac{1}{16\pi^3} \varepsilon^{\alpha\beta\gamma} \nabla_\alpha \text{Ric}_{\beta\rho}(0) \int_{\mathbb{R}^3} e^{-iy^\mu \xi_\mu} \frac{\xi_\gamma \xi_\rho}{|\xi|^5} (1 - \chi(|\xi|)) d\xi,$$

where χ is a cut-off as in (4.13). But

$$\frac{\xi_\gamma \xi_\rho}{|\xi|^5} = \frac{\xi_\gamma \xi_\rho}{\langle \xi \rangle^5} + O\left(\frac{1}{|\xi|^5}\right),$$

so the required result follows from Lemma 7.5. Note that the term with $\delta_{\gamma\rho}$ from (7.10) vanishes upon contraction with $\varepsilon^{\alpha\beta\gamma} \nabla_\alpha \text{Ric}_{\beta\rho}(0)$ due to formula (7.4). \square

Theorem 7.1 tells us that the singularity of the integral kernel $\mathbf{a}(x, y)$ of our asymmetry operator A is very weak. We are looking at a bounded function of two variables with a discontinuity on the diagonal. The discontinuity of $\mathbf{a}(x, y)$ on the diagonal manifests itself in the fact that limit of $\mathbf{a}(x, y)$ as y tends to x depends on the direction along which y tends to x .

The bottom line is that working with the asymmetry operator A does not require the use of the theory of distributions or microlocal analysis. It is an integral operator whose integral kernel has a mild discontinuity on the diagonal.

7.2 | Local and global trace

The decomposition (7.2), (7.3) depends on the choice of local coordinates. However, the quantity

$$\psi_{\text{curl}}^{\text{loc}}(x) := \mathbf{a}_c(x, x) \tag{7.12}$$

does not depend on the choice of local coordinates, that is, is a true scalar function $\psi_{\text{curl}} : M \rightarrow \mathbb{R}$. Let us elaborate. Denote

$$f(x, y) := 12\pi^2 \mathbf{a}_d(x, y) \text{dist}^2(x, y) = E^{\alpha\beta}_\gamma(x) \nabla_\alpha \text{Ric}_{\beta\rho}(x) (x - y)^\gamma (x - y)^\rho. \tag{7.13}$$

The issue at hand is that the expression (7.13) is not a scalar because the quantities $(x - y)^\gamma$ and $(x - y)^\rho$ are not vectors based at the point x . Let us switch from local coordinates x to local coordinates \tilde{x} and let $\tilde{f}(\tilde{x}, \tilde{y})$ be the representation of the quantity (7.13) in these new coordinates. It is easy to see, by writing down the appropriate Jacobians, that

$$\tilde{f}(\tilde{x}(x), \tilde{y}(y)) - f(x, y) = O(\text{dist}^3(x, y)). \tag{7.14}$$

Now, let $\tilde{\mathbf{a}}_c(\tilde{x}, \tilde{y})$ be the representation of the quantity $\mathbf{a}_c(x, y)$ in the new coordinates. Formulae (7.2), (7.3), (7.13), (7.14) and the fact that $\mathbf{a}(x, y)$ is a true scalar in two variables imply

$$\tilde{\mathbf{a}}_c(\tilde{x}(x), \tilde{y}(y)) - \mathbf{a}_c(x, y) = O(\text{dist}(x, y)),$$

which, in turn, implies $\tilde{\mathbf{a}}_c(\tilde{x}(x), \tilde{x}(x)) = \mathbf{a}_c(x, x)$.

Using formulae (7.2), (7.3) and (7.12) for the calculation of the function ψ_{curl} is impractical. The following lemma provides an alternative equivalent representation.

Lemma 7.6. *We have*

$$\psi_{\text{curl}}^{\text{loc}}(x) = \lim_{r \rightarrow 0^+} \frac{1}{4\pi r^2} \int_{\mathbb{S}_r(x)} \mathbf{a}(x, y) \, dS_y, \tag{7.15}$$

where $\mathbf{a}(x, y)$ is the integral kernel of the asymmetry operator (6.1) defined in accordance with (7.1), $\mathbb{S}_r(x) = \{y \in M \mid \text{dist}(x, y) = r\}$ is the sphere of radius r centred at x and dS_y is the surface area element on this sphere.

Proof. Let us fix an $x \in M$ and choose normal coordinates centred at x . Theorem 7.1 tells us that

$$\mathbf{a}(x, y) = \mathbf{a}_c(x, y) + c_{\gamma\rho}(x) \frac{(x - y)^\gamma (x - y)^\rho}{|x - y|^2},$$

where

$$c_{\gamma\rho}(x) = \frac{1}{12\pi^2} E^{\alpha\beta}{}_\gamma(x) \nabla_\alpha \text{Ric}_{\beta\rho}(x).$$

We have

$$\int_{\mathbb{S}_r(x)} (x - y)^\gamma (x - y)^\rho \, dS_y = \begin{cases} \frac{4\pi r^2}{3} & \text{if } \gamma = \rho, \\ 0 & \text{if } \gamma \neq \rho, \end{cases}$$

hence,

$$\frac{1}{4\pi r^2} \int_{\mathbb{S}_r(x)} \mathbf{a}(x, y) \, dS_y = \frac{1}{4\pi r^2} \int_{\mathbb{S}_r(x)} \mathbf{a}_c(x, y) \, dS_y + \frac{c_{11}(x) + c_{22}(x) + c_{33}(x)}{3}. \tag{7.16}$$

But formula (7.4) implies $c_{11}(x) + c_{22}(x) + c_{33}(x) = 0$, so the second term in the right-hand side of (7.16) vanishes. Formulae (7.16) and (7.12) now imply (7.15) by the continuity of $\mathbf{a}_c(x, y)$. \square

The advantage of formula (7.15) is that it does not involve the use of local coordinates. Hence, it can be viewed as a more convenient and natural definition of the scalar function $\psi_{\text{curl}}^{\text{loc}}$.

Definition 7.7. We call the scalar continuous function $\psi_{\text{curl}}^{\text{loc}}(x), \psi_{\text{curl}}^{\text{loc}} : M \rightarrow \mathbb{R}$, defined in accordance with formulae (7.15) and (7.1) *the regularised local trace of the asymmetry operator*.

Definition 7.8. We call the number

$$\psi_{\text{curl}} := \int_M \psi_{\text{curl}}^{\text{loc}}(x) \rho(x) \, dx$$

the regularised global trace of the asymmetry operator.

The function $\psi_{\text{curl}}^{\text{loc}}(x)$ and the number ψ_{curl} are determined by the Riemannian 3-manifold and its orientation. This means that we are looking at geometric invariants.

8 | CHALLENGES IN HIGHER DIMENSIONS

Let (M, g) be a connected closed Riemannian manifold of dimension

$$d = 4k + 3, \quad k = 1, 2, \dots \quad (8.1)$$

In this case, one can define curl in the usual way (1.5) as an operator acting in $\Omega^{2k+1}(M)$. It is easy to see that curl remains formally self-adjoint, and at a formal level, everything appears to be similar to what happens in dimension 3.

Remark 8.1. One could have also considered the case $d = 4k + 1$, $k = 1, 2, \dots$, with curl acting in $\Omega^{2k}(M)$ in accordance with (1.5). However, in this case, curl is formally anti-self-adjoint, the spectrum is purely imaginary and there is no spectral asymmetry [7, Theorem 3.2]. As our interest is in spectral asymmetry, we do not discuss the case $d = 4k + 1$. Note also that the spectral problem for curl in dimension $d = 4k + 1$ cannot be formulated in terms of the spectral theory of self-adjoint operators in *real* Hilbert spaces: one can introduce the factor i in the right-hand side of (1.5) to make the operator formally self-adjoint (as was done in [7, Definition 2.2]), but the resulting operator will no longer be an operator acting in the vector space of *real* differential forms. The bottom line is that the two cases, $d = 4k + 3$ and $d = 4k + 1$, are fundamentally different. We stick with (8.1) because this gives the only genuine generalisation of the operator curl to higher dimensions.

Analysis of higher dimensions (8.1) presents two major challenges.

First challenge. We expect that in higher dimensions, the asymmetry operator will still be a pseudodifferential operator of order -3 , as it is in the three-dimensional case. However, a sufficient condition for a pseudodifferential operator to be of trace class is that its order be less than $-d$. This means that defining the regularised trace of the asymmetry operator becomes challenging: one would need to analyse singularities of the integral kernel and perform regularisation reducing the order of the singularity by $4k + 1$. In other words, one would need to perform $4k + 1$ steps of regularisation. Doing this in a geometrically invariant fashion will not be an easy task.

Second challenge. In the current paper, we used the results of [18] which apply to operators whose principal symbols have simple eigenvalues (see also [19]). In dimension 3, the operator curl does possess this property — its principal symbol has three simple eigenvalues (2.3). In higher dimensions, the eigenvalues of the principal symbol of the operator curl are still given by formulae (2.3), only now they have multiplicity: zero is an eigenvalue of multiplicity $\binom{4k+2}{2k}$, whereas each of the two non-zero eigenvalues has multiplicity $\frac{1}{2} \binom{4k+2}{2k+1}$. This means that addressing the issue of higher dimensions would require extending the results of [18] to operators whose principal symbols have multiple eigenvalues. This will be a difficult task as the construction presented in [18] relied heavily on the assumption that the eigenvalues of the principal symbol of the operator are simple. Overall, partial differential operators whose principal symbols have multiple eigenvalues are known to present a major challenge in micro-local analysis.

Despite the challenges involved, examining the asymmetry operator in higher dimension could be an piece of research worth pursuing in the future. For example, since the seminal works by Milnor and Kervaire [44, 53], the study of exotic smooth structures on 7-spheres has been an active area of research in differential geometry. Our asymmetry operator, being very sensitive to the underlying geometry, might help to make some progress in this area.

APPENDIX A: EXTERIOR CALCULUS

In this appendix, we set out our conventions on exterior calculus. Henceforth, we identify differential forms with covariant antisymmetric tensors, and M is an oriented 3-manifold equipped with Riemannian metric g and Levi-Civita connection ∇ .

It is well known that the metric g induces a canonical isomorphism between the tangent bundle TM and the cotangent bundle T^*M , the so-called *musical isomorphism*. We denote it by $\flat : TM \rightarrow T^*M$ (lowering of indices) and its inverse by $\sharp : T^*M \rightarrow TM$ (raising of indices).

Given a scalar field $f \in C^\infty(M)$, its exterior derivative df is defined as the gradient. Given a 1-form $u \in \Omega^1(M)$, its exterior derivative $du \in \Omega^2(M)$ is defined, componentwise, as

$$(du)_{\alpha\beta} = \partial_{x^\alpha} u_\beta - \partial_{x^\beta} u_\alpha.$$

We define the action of the Hodge star $*$: $\Omega^k(M) \rightarrow \Omega^{3-k}(M)$ on a rank k antisymmetric tensor as

$$(*v)_{\mu_{k+1}\dots\mu_3} := \frac{1}{k!} \sqrt{\det g_{\alpha\beta}} v^{\mu_1\dots\mu_k} \varepsilon_{\mu_1\mu_2\mu_3}, \quad (\text{A.1})$$

where ε is the totally antisymmetric symbol, $\varepsilon_{123} := +1$.

With regards to the totally antisymmetric symbol, we adopt the convention of raising indices using the Kronecker symbol, so that, for example,

$$\varepsilon^{\alpha_1\alpha_2\alpha_3} := \varepsilon_{\alpha_1\alpha_2\alpha_3}.$$

Let \wedge denote the exterior product. Given a pair of real-valued rank k covariant antisymmetric tensors u and v , we define their L^2 inner product as

$$\begin{aligned} \langle u, v \rangle &:= \int_M \frac{1}{k!} u_{\alpha_1\dots\alpha_k} v_{\beta_1\dots\beta_k} g^{\alpha_1\beta_1} \dots g^{\alpha_k\beta_k} \sqrt{\det g_{\mu\nu}} dx \\ &= \int_M u \wedge *v = \int_M *u \wedge v. \end{aligned}$$

For the sake of clarity, let us mention that the exterior product of 1-forms reads

$$(u \wedge v)_{\alpha\beta} = u_\alpha v_\beta - u_\beta v_\alpha.$$

Given $w \in \Omega^k(M)$ and $u \in \Omega^{k-1}(M)$, we define the action of the codifferential $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ in accordance with

$$\langle w, du \rangle = \langle \delta w, u \rangle.$$

In particular, when $u \in \Omega^1(M)$ and $w \in \Omega^2(M)$, we get in local coordinates

$$\delta u = -\nabla^\alpha u_\alpha,$$

$$(\delta w)_\alpha = \nabla^\beta w_{\alpha\beta}.$$

APPENDIX B: BASIC PROPERTIES OF THE OPERATOR CURL

In this appendix, we provide, in a self-contained fashion, proofs of the fundamental properties of curl stated in Section 2.

In order to prove the Theorem 2.1, let us introduce an auxiliary elliptic operator: *extended curl*.

B.1 | Extended curl

It is known that one can extend curl to a 4×4 elliptic operator acting in $\Omega^1 \oplus \Omega^0$.

Definition B.1. We define *extended curl* to be the operator

$$\text{curl}_E := \begin{pmatrix} \text{curl} & d \\ \delta & 0 \end{pmatrix} : (\Omega^1 \cap H^1) \oplus (\Omega^0 \cap H^1) \rightarrow \Omega^1 \oplus \Omega^0. \quad (\text{B.1})$$

The operator (B.1) appeared in [52], see also [2, p. 229], [3, pp. 44–45 and p. 66], [4, p. 405], [60, pp. 190–191] and [25, §4].

Remark B.2. Note that curl_E is an operator of Dirac type, namely, its square is $\delta d + d\delta$ on $\Omega^1 \oplus \Omega^0$ (minus the non-positive Hodge Laplacian). Furthermore, (B.1) is the restriction of the *signature operator* $d + \delta$ to differential forms of even degree, in the sense that

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \text{curl}_E \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} = (d + \delta)* : \Omega^2 \oplus \Omega^0 \rightarrow \Omega^2 \oplus \Omega^0.$$

A straightforward calculation shows that the principal symbol of extended curl reads

$$(\text{curl}_E)_{\text{prin}}(x, \xi) = \begin{pmatrix} [(\text{curl})_{\text{prin}}]_{\alpha}^{\beta}(x, \xi) & i\xi_{\alpha} \\ -ig^{\beta\gamma}(x)\xi_{\gamma} & 0 \end{pmatrix}. \quad (\text{B.2})$$

The eigenvalues of (B.2) are

$$\pm \|\xi\|, \quad (\text{B.3})$$

each with multiplicity two, so that we have

$$\det [(\text{curl}_E)_{\text{prin}}] = \|\xi\|^4, \quad (\text{B.4})$$

Compare formulae (B.3) and (B.4) with (2.3) and (2.2), respectively.

Formulae (B.4) and (B.1) imply that curl_E is elliptic and self-adjoint.

Definition B.3 [57, Definition 1.7]. Let $L : D(L) \rightarrow H$ be a self-adjoint (possibly unbounded) linear operator in the Hilbert space H . We say that a closed vector subspace $V \subseteq H$ is an *invariant subspace* of the operator L if

$$L(D(L) \cap V) \subseteq V.$$

Proposition B.4. The Hilbert space $\Omega^1 \oplus \Omega^0$ decomposes into a direct sum of three orthogonal Hilbert subspaces

$$\Omega^1 \oplus \Omega^0 = [d\Omega^0 \oplus \delta\Omega^1] \oplus [\delta\Omega^2 \oplus 0] \oplus [H^1 \oplus H^0] \quad (\text{B.5})$$

which are invariant subspaces of the operator curl_E in the sense of Definition B.3. Accordingly, the operator curl_E decomposes into a direct sum of three operators

$$\text{curl}_E = \text{curl}_{E,d} \oplus \text{curl}_{E,\delta} \oplus \text{curl}_{E,H}, \quad (\text{B.6})$$

where

$$\text{curl}_{E,d} : (d\Omega^0 \cap H^1) \oplus (\delta\Omega^1 \cap H^1) \rightarrow d\Omega^0 \oplus \delta\Omega^1, \tag{B.7}$$

$$\text{curl}_{E,\delta} : (\delta\Omega^2 \cap H^1) \oplus 0 \rightarrow \delta\Omega^2 \oplus 0, \tag{B.8}$$

$$\text{curl}_{E,\mathcal{H}} : \mathcal{H}^1 \oplus \mathcal{H}^0 \rightarrow \mathcal{H}^1 \oplus \mathcal{H}^0. \tag{B.9}$$

Proof. That the three subspaces in the RHS of (B.5) are mutually orthogonal and their direct sum gives the whole space $\Omega^1 \oplus \Omega^0$ follows immediately from the Hodge decomposition theorem. Let us show that they are invariant subspaces of curl_E . We will do this in several steps.

Step 1. Observe that curl_E maps $\mathcal{H}^1 \oplus \mathcal{H}^0$ to zero. Hence, $\mathcal{H}^1 \oplus \mathcal{H}^0$ is an invariant subspace of curl_E .

Step 2. The facts that

- $\mathcal{H}^1 \oplus \mathcal{H}^0$ is an invariant subspace of curl_E and
- $[d\Omega^0 \oplus \delta\Omega^1] \oplus [\delta\Omega^2 \oplus 0]$ is the orthogonal complement of $\mathcal{H}^1 \oplus \mathcal{H}^0$

imply that $[d\Omega^0 \oplus \delta\Omega^1] \oplus [\delta\Omega^2 \oplus 0]$ is an invariant subspace of curl_E .

Step 3. Choose a λ in the resolvent set $\rho(\text{curl}_E)$ of the operator curl_E . It will be convenient for us to choose a real λ ,

$$\lambda \in \mathbb{R}. \tag{B.10}$$

This can always be achieved because the operator curl_E is elliptic and self-adjoint; hence, its spectrum is real and discrete. Note that zero is an eigenvalue of curl_E (see Step 1 above), so

$$\lambda \neq 0. \tag{B.11}$$

Let $\Delta := -\delta d$ be the (non-positive) Laplace–Beltrami operator. Let

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \tag{B.12}$$

be the eigenvalues of $-\Delta$ enumerated in increasing order with account of multiplicities. It will be convenient for us to further restrict the choice of λ by imposing the condition

$$\lambda \notin \{\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \dots\}. \tag{B.13}$$

Step 4. Observe that in order to prove that the two subspaces $d\Omega^0 \oplus \delta\Omega^1$ and $\delta\Omega^2 \oplus 0$ are invariant subspaces of curl_E , it is sufficient to prove that these are invariant subspaces of the resolvent $\mathcal{R}(\lambda) := (\text{curl}_E - \lambda \text{Id})^{-1}$.

Step 5. In view of (B.10), the operator $\mathcal{R}(\lambda)$ is self-adjoint. The facts that

- $[d\Omega^0 \oplus \delta\Omega^1] \oplus [\delta\Omega^2 \oplus 0]$ is an invariant subspace of $\mathcal{R}(\lambda)$ and
- the subspaces $d\Omega^0 \oplus \delta\Omega^1$ and $\delta\Omega^2 \oplus 0$ are mutually orthogonal

imply that it suffices to prove that only one of the subspaces $d\Omega^0 \oplus \delta\Omega^1$ and $\delta\Omega^2 \oplus 0$ is an invariant subspace of $\mathcal{R}(\lambda)$. In what follows we prove that $\delta\Omega^2 \oplus 0$ is an invariant subspace of $\mathcal{R}(\lambda)$.

Step 6. Let Ω_∞^k denote the vector space of infinitely smooth real-valued k -forms and let

$$d\Omega_\infty^{k-1} := \{\omega \in \Omega_\infty^{k-1}\}, \quad (\text{B.14})$$

$$\delta\Omega_\infty^{k+1} := \{\omega \in \Omega_\infty^{k+1}\}. \quad (\text{B.15})$$

The Hilbert spaces Ω^k , $d\Omega^{k-1}$ and $\delta\Omega^{k+1}$ are defined as L^2 closures of the vector spaces Ω_∞^k , $d\Omega_\infty^{k-1}$ and $\delta\Omega_\infty^{k+1}$, respectively, see [42, Corollary 3.4.2]. This implies that the vector space $\delta\Omega_\infty^2 \oplus 0$ is dense in the Hilbert spaces $\delta\Omega^2 \oplus 0$. As $\mathcal{R}(\lambda)$ is a bounded operator, in order to prove that $\delta\Omega^2 \oplus 0$ is an invariant subspace of $\mathcal{R}(\lambda)$, it is sufficient to prove that

$$[\mathcal{R}(\lambda)](\delta\Omega_\infty^2 \oplus 0) \subseteq \delta\Omega_\infty^2 \oplus 0. \quad (\text{B.16})$$

Step 7. Let $z = \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \in \delta\Omega_\infty^2 \oplus 0$. Put

$$v := \mathcal{R}(\lambda)z = \begin{pmatrix} v_1 \\ v_0 \end{pmatrix} \in [d\Omega_\infty^0 \oplus \delta\Omega_\infty^1] \oplus [\delta\Omega_\infty^2 \oplus 0]. \quad (\text{B.17})$$

Here z_1 , v_1 are 1-forms and v_0 is a 0-form (scalar). The v_1 and v_0 are infinitely smooth because the operator curl_E is elliptic. Formula (B.17) can be rewritten more explicitly as

$$\text{curl } v_1 + dv_0 - \lambda v_1 = z_1, \quad (\text{B.18})$$

$$\delta v_1 - \lambda v_0 = 0. \quad (\text{B.19})$$

We have [63, Section 6.8, p. 223]

$$\Omega_\infty^k = d\Omega_\infty^{k-1} \oplus \delta\Omega_\infty^{k+1} \oplus \mathcal{H}^k, \quad (\text{B.20})$$

compare with (1.1). We already know (see Step 2) that the 1-form v_1 does not contain a contribution from harmonic 1-forms, so formulae (B.20), (B.14) and (B.15) give us

$$v_1 = v_{1d} + v_{1\delta}, \quad (\text{B.21})$$

where

$$v_{1d} = dw_0, \quad (\text{B.22})$$

$$v_{1\delta} = \delta w_2 \quad (\text{B.23})$$

for some infinitely smooth 0-form (scalar) w_0 and some infinitely smooth 2-form w_2 .

Step 8. Applying the operator δ to (B.18) and using formulae (B.21)–(B.23) as well as the fact that $\delta z_1 = 0$, we get $\Delta(\lambda w_0 - v_0) = 0$, which implies

$$v_0 = \lambda w_0 + C, \quad (\text{B.24})$$

where $C \in \mathbb{R}$ is a constant. Substituting (B.21) and (B.24) into (B.19) and using formulae (B.22), (B.23), we get

$$-\Delta w_0 - \lambda^2 w_0 = \lambda C. \quad (\text{B.25})$$

In view of (B.11) and (B.13) equation, (B.25) has the unique solution $w_0 = -\lambda^{-1}C$, which immediately implies

$$v_0 = 0, \quad v_{1d} = 0. \tag{B.26}$$

It only remains to observe that formulae (B.17), (B.21), (B.23) and (B.26) imply $v \in \delta\Omega_\infty^2 \oplus 0$, thus proving the inclusion (B.16). □

B.2 | Decomposition of the spectrum of extended curl

Recall that extended curl (B.1) is a self-adjoint elliptic operator; hence, its spectrum is discrete and eigenfunctions infinitely smooth.

Proposition B.5. *The spectrum of curl_E decomposes into three parts as*

$$\sigma(\text{curl}_E) = \sigma(\text{curl}) \cup \sigma(\text{curl}_{E,d}) \cup \sigma(\text{curl}_{E,\mathcal{H}}), \tag{B.27}$$

where curl , $\text{curl}_{E,d}$ and $\text{curl}_{E,\mathcal{H}}$ are the operators (2.4), (B.7) and (B.9), respectively. Formula (B.27) is understood as the (disjoint) union of spectra with account of multiplicities.

Proof. Proposition B.5 is an immediate consequence of Proposition B.4 and the observation that the operator $\text{curl}_{E,\delta}$ defined by formulae (B.1) and (B.8) is precisely our original operator curl defined by formula (2.4). □

Proposition B.5 tells us that the spectrum of curl sits inside the spectrum of extended curl and the two spectra differ by $\sigma(\text{curl}_{E,d}) \cup \sigma(\text{curl}_{E,\mathcal{H}})$. So, the task at hand is working out the spectra of $\text{curl}_{E,d}$ and $\text{curl}_{E,\mathcal{H}}$. The following two lemmata address this issue.

Lemma B.6. *We have $\sigma(\text{curl}_{E,\mathcal{H}}) = \{0\}$, zero being an eigenvalue of multiplicity $\dim \mathcal{H} + 1$.*

Proof. Lemma B.6 is an immediate consequence of formulae (B.1) and (B.9). □

Lemma B.7. *We have*

$$\sigma(\text{curl}_{E,d}) = \{\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \dots\}, \tag{B.28}$$

where the μ_j are the eigenvalues (B.12) of $-\Delta$.

Proof. Let $v = \begin{pmatrix} dw_0 \\ v_0 \end{pmatrix}$ be an eigenfunction of $\text{curl}_{E,d}$ corresponding to an eigenvalue λ . Here, w_0 and v_0 are some infinitely smooth 0-forms (scalars). We have

$$dv_0 = \lambda dw_0, \tag{B.29}$$

$$-\Delta w_0 = \lambda v_0. \tag{B.30}$$

Elementary analysis of (B.29) and (B.30) yields (B.28). □

Proposition B.5 and Lemmata B.6 and B.7 tell us that the spectrum of curl can be recovered from the spectrum of curl_E , provided that one knows the spectrum of the Laplace–Beltrami operator

Δ. Moreover, the eta functions of curl and curl_E are the same, because contributions to $\sigma(\text{curl}_E)$ coming from the Laplace–Beltrami operator are symmetric about zero.

B.3 | Proof of Theorem 2.1

Proof of Theorem 2.1.

- (a) Standard elliptic theory tells us that the operator curl_E defined by formula (B.1) is self-adjoint in the operator-theoretic sense. Proposition B.4 and formulae (B.1) and (B.8) tell us that our original operator curl defined by formula (2.4) is part of the operator curl_E ; here the concept of ‘part of an operator’ is understood in accordance with [57, Definition 1.7]. Self-adjointness of curl now follows from [57, Proposition 3.11].
- (b) Discreteness of the spectrum of curl follows from Proposition B.5. As to the statement that the spectrum of curl is unbounded both from above and from below, we prove it by writing down spectral asymptotics.

Let $\lambda_k, k \in \mathbb{Z} \setminus \{0\}$, be the eigenvalues of curl . The choice of particular enumeration is irrelevant for our purposes, but what is important is that eigenvalues are enumerated with account of their multiplicity. We define the positive (+) and negative (−) counting functions for curl as

$$N_{\text{curl}}^{\pm}(\lambda) := \begin{cases} 0 & \text{for } \lambda \leq 0, \\ \sum_{0 < \pm \lambda_k < \lambda} 1 & \text{for } \lambda > 0. \end{cases} \tag{B.31}$$

We define, in a similar fashion, counting functions $N_{\text{curl}_E}^{\pm}(\lambda)$ and $N_{\text{curl}_{E,d}}^{\pm}(\lambda)$ for the operators curl_E and $\text{curl}_{E,d}$, respectively. Proposition B.5 and Lemma B.6 tell us that

$$N_{\text{curl}}^{\pm}(\lambda) = N_{\text{curl}_E}^{\pm}(\lambda) - N_{\text{curl}_{E,d}}^{\pm}(\lambda). \tag{B.32}$$

Lemma B.7 tells us that the evaluation of $N_{\text{curl}_{E,d}}^{\pm}(\lambda)$ reduces to the evaluation of eigenvalues of the Laplace–Beltrami operator. The latter is a subject which has been extensively studied over the last 100 years. Application of [38, Theorem 1.1] gives the following asymptotic formula for $N_{\text{curl}_{E,d}}^{\pm}(\lambda)$ with sharp remainder term estimate:

$$N_{\text{curl}_{E,d}}^{\pm}(\lambda) = \frac{\text{Vol}(M)}{6\pi^2} \lambda^3 + O(\lambda^2) \quad \text{as } \lambda \rightarrow +\infty, \tag{B.33}$$

where $\text{Vol}(M)$ is the Riemannian volume of the manifold.

Application of [39, Theorem 0.1] gives the following asymptotic formula for $N_{\text{curl}_E}^{\pm}(\lambda)$:

$$N_{\text{curl}_E}^{\pm}(\lambda) = \frac{\text{Vol}(M)}{3\pi^2} \lambda^3 + O(\lambda^2) = 2N_{\text{curl}_{E,d}}^{\pm}(\lambda) + O(\lambda^2) \quad \text{as } \lambda \rightarrow +\infty. \tag{B.34}$$

The factor 2 appears in the RHS of (B.34) because the eigenvalues of the principal symbol of the operator curl_E , given by formula (B.3), have multiplicity two.

Formulae (B.32)–(B.34) imply Bär’s [7, Theorem 3.6] asymptotic formula

$$N_{\text{curl}}^{\pm}(\lambda) = \frac{\text{Vol}(M)}{6\pi^2} \lambda^3 + O(\lambda^2) \quad \text{as } \lambda \rightarrow +\infty. \tag{B.35}$$

Formula (B.35) shows that the spectrum of curl accumulates to $+\infty$ and $-\infty$.

(c) Suppose that zero is an eigenvalue of curl. Let

$$u \in \delta\Omega_\infty^2 \tag{B.36}$$

be the corresponding eigenform (zero mode). Then $\text{curl } u = 0$, which implies

$$du = 0. \tag{B.37}$$

But (B.36) implies

$$\delta u = 0. \tag{B.38}$$

Formulae (B.37) and (B.38) tell us that the 1-form u is harmonic. As $\delta\Omega^2 \cap \mathcal{H}^1 = \{0\}$, we conclude that $u = 0$, which is a contradiction.

(d) Proposition B.5, Lemmata B.6 and B.7 and part (c) of Theorem 2.1 tell us that zero is an eigenvalue of curl_E of multiplicity $\dim \mathcal{H}^1 + 1$. Let $P_{\mathcal{H}^1}$ and $P_{\mathcal{H}^0}$ be orthogonal projections onto the spaces of harmonic 1-forms and 0-forms, respectively. Put

$$\widetilde{\text{curl}}_E := \text{curl}_E + \begin{pmatrix} P_{\mathcal{H}^1} & 0 \\ 0 & P_{\mathcal{H}^0} \end{pmatrix}.$$

The operator $\widetilde{\text{curl}}_E$ is invertible. By standard elliptic theory, $\widetilde{\text{curl}}_E^{-1}$ is a bounded operator acting from the Sobolev space H^s to the Sobolev space H^{s+1} . The operator curl^{-1} is part of the operator $\widetilde{\text{curl}}_E^{-1}$; hence, it possesses the same mapping properties. \square

APPENDIX C: THE SPECTRUM OF THE LAPLACIAN ON A BERGER SPHERE

In preparation for the calculations that will appear in Appendix D, we list here, in the form of a theorem, the eigenvalues of the Laplacian on a Berger sphere. These were first obtained in [62, Lemma 4.1] and reproduced in [50, Theorem 5.5] and [45, Proposition 3.9].

C.1 | Definitions and notation

In Euclidean space \mathbb{R}^4 equipped with Cartesian coordinates \mathbf{x}^α , $\alpha = 1, 2, 3, 4$, let

$$\mathbb{S}^3 := \{\mathbf{x} \in \mathbb{R}^4 \mid \|\mathbf{x}\| = 1\}$$

be the 3-sphere. We use bold script for four-dimensional objects and normal script for three-dimensional objects.

We prescribe orientation of \mathbb{S}^3 as follows. We define local coordinates $y = (y^1, y^2, y^3)$ on \mathbb{S}^3 to be positively oriented if

$$\det \begin{pmatrix} \frac{\partial \mathbf{x}^1}{\partial y^1}(y) & \frac{\partial \mathbf{x}^1}{\partial y^2}(y) & \frac{\partial \mathbf{x}^1}{\partial y^3}(y) & \mathbf{x}^1(y) \\ \frac{\partial \mathbf{x}^2}{\partial y^1}(y) & \frac{\partial \mathbf{x}^2}{\partial y^2}(y) & \frac{\partial \mathbf{x}^2}{\partial y^3}(y) & \mathbf{x}^2(y) \\ \frac{\partial \mathbf{x}^3}{\partial y^1}(y) & \frac{\partial \mathbf{x}^3}{\partial y^2}(y) & \frac{\partial \mathbf{x}^3}{\partial y^3}(y) & \mathbf{x}^3(y) \\ \frac{\partial \mathbf{x}^4}{\partial y^1}(y) & \frac{\partial \mathbf{x}^4}{\partial y^2}(y) & \frac{\partial \mathbf{x}^4}{\partial y^3}(y) & \mathbf{x}^4(y) \end{pmatrix} < 0.$$

The above definition of orientation agrees with that from [31, Appendix A]. Spherical coordinates

$$\begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \\ \mathbf{x}^4 \end{pmatrix} = \begin{pmatrix} \cos y^1 \\ \sin y^1 \cos y^2 \\ \sin y^1 \sin y^2 \cos y^3 \\ \sin y^1 \sin y^2 \sin y^3 \end{pmatrix}, \quad y^1, y^2 \in (0, \pi), \quad y^3 \in [0, 2\pi),$$

are an example of positively oriented local coordinates on \mathbb{S}^3 .

Of course, if one is only interested in the study of the Laplace–Beltrami operator, orientation is irrelevant. However, with curl or the Dirac operator [17, subsection 9.1], prescribing orientation of the manifold is essential.

Consider the vector fields

$$\begin{aligned} \mathbf{V}_1 &:= -\mathbf{x}^4 \frac{\partial}{\partial \mathbf{x}^1} - \mathbf{x}^3 \frac{\partial}{\partial \mathbf{x}^2} + \mathbf{x}^2 \frac{\partial}{\partial \mathbf{x}^3} + \mathbf{x}^1 \frac{\partial}{\partial \mathbf{x}^4}, \\ \mathbf{V}_2 &:= \mathbf{x}^3 \frac{\partial}{\partial \mathbf{x}^1} - \mathbf{x}^4 \frac{\partial}{\partial \mathbf{x}^2} - \mathbf{x}^1 \frac{\partial}{\partial \mathbf{x}^3} + \mathbf{x}^2 \frac{\partial}{\partial \mathbf{x}^4}, \\ \mathbf{V}_3 &:= -\mathbf{x}^2 \frac{\partial}{\partial \mathbf{x}^1} + \mathbf{x}^1 \frac{\partial}{\partial \mathbf{x}^2} - \mathbf{x}^4 \frac{\partial}{\partial \mathbf{x}^3} + \mathbf{x}^3 \frac{\partial}{\partial \mathbf{x}^4}. \end{aligned} \tag{C.1}$$

The vector fields (C.1) are tangent to \mathbb{S}^3 , that is, they are orthogonal to the gradient of $\|\mathbf{x}\|$, or, equivalently, $\mathbf{V}_j(\|\mathbf{x}\|^2) = 0$, $j = 1, 2, 3$. For $j = 1, 2, 3$, let V_j be the restriction of \mathbf{V}_j to \mathbb{S}^3 .

Let $a \in \mathbb{R}$ be a positive parameter. Consider the differential operator

$$B := V_1^2 + V_2^2 + \frac{1}{a^2} V_3^2. \tag{C.2}$$

The *Berger metric* g on \mathbb{S}^3 is defined via the identity

$$B_{\text{prin}}(y, \eta) = -g^{\alpha\beta}(y) \eta_\alpha \eta_\beta \tag{C.3}$$

or, equivalently,

$$g^{\alpha\beta}(y) := -\frac{\partial^2 B_{\text{prin}}}{\partial \eta_\alpha \partial \eta_\beta}.$$

We call (\mathbb{S}^3, g) the *Berger sphere*.

The notion of a Berger sphere is well established in differential geometry, nevertheless, for the benefit of a wider audience, we provide here an explicit description in the simplest possible coordinate system. Working in the southern hemisphere $\mathbf{x}^4 < 0$, consider the positively oriented local coordinates

$$y^\alpha = \mathbf{x}^\alpha, \quad \alpha = 1, 2, 3,$$

so that

$$\mathbf{x}^4 = -\sqrt{1 - (\mathbf{x}^1)^2 - (\mathbf{x}^2)^2 - (\mathbf{x}^3)^2} = -\sqrt{1 - |y|^2}.$$

In these coordinates, the vector fields V_j read

$$\begin{aligned} V_1 &= \sqrt{1 - |y|^2} \frac{\partial}{\partial y^1} - y^3 \frac{\partial}{\partial y^2} + y^2 \frac{\partial}{\partial y^3}, \\ V_2 &= y^3 \frac{\partial}{\partial y^1} + \sqrt{1 - |y|^2} \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial y^3}, \\ V_3 &= -y^2 \frac{\partial}{\partial y^1} + y^1 \frac{\partial}{\partial y^2} + \sqrt{1 - |y|^2} \frac{\partial}{\partial y^3}. \end{aligned} \tag{C.4}$$

Formulae (C.2), (C.3) and (C.4) imply

$$g^{\alpha\beta}(y) = \begin{pmatrix} 1 - (y^1)^2 - \frac{a^2-1}{a^2}(y^2)^2 & -\frac{y^1 y^2}{a^2} & -y^1 y^3 + \frac{a^2-1}{a^2} y^2 \sqrt{1 - |y|^2} \\ -\frac{y^1 y^2}{a^2} & 1 - (y^2)^2 - \frac{a^2-1}{a^2}(y^1)^2 & -y^2 y^3 - \frac{a^2-1}{a^2} y^1 \sqrt{1 - |y|^2} \\ -y^1 y^3 + \frac{a^2-1}{a^2} y^2 \sqrt{1 - |y|^2} & -y^2 y^3 - \frac{a^2-1}{a^2} y^1 \sqrt{1 - |y|^2} & \frac{a^2-1}{a^2}(y^2)^2 + \frac{a^2-1}{a^2}(y^1)^2 - \frac{1-(y^3)^2}{a^2} \end{pmatrix}. \tag{C.5}$$

We summarise the properties of the vector fields V_j , $j = 1, 2, 3$, and the Berger metric g in the following lemma. The proof amounts to straightforward calculations and is omitted.

Lemma C.1. *We have the following properties.*

- (a) $[V_j, V_k] = -2\varepsilon_{jkl} V_l$, $j, k = 1, 2, 3$.
- (b) The vector fields V_j are positively oriented, that is, $\det V_j^\alpha > 0$, $j, \alpha = 1, 2, 3$. Here, the V_j^α are the components of the vector fields V_j with respect to the basis $\frac{\partial}{\partial y^\alpha}$.
- (c) The vector fields

$$\tilde{V}_1 := V_1, \quad \tilde{V}_2 := V_2, \quad \tilde{V}_3 := \frac{1}{a} V_3 \tag{C.6}$$

are orthonormal with respect to the metric g defined by (C.5).

- (d) The Riemannian density $\rho(y)$ generated by the Berger metric g reads

$$\rho(y) = a \rho_0(y),$$

where $\rho_0(y)$ is the standard Riemannian density of the round \mathbb{S}^3 .

- (e) The vector fields V_j , $j = 1, 2, 3$, viewed as differential operators are skew-Hermitian.
- (f) The vector fields V_j , $j = 1, 2, 3$, are Killing vector fields for the standard round metric g_0 on \mathbb{S}^3 .
- (g) The vector field V_3 is a Killing vector field for the Berger metric g .
- (h) The operator B defined in accordance with (C.2) is the Laplace–Beltrami operator (henceforth denoted by Δ) on the Berger sphere.
- (i) The operator curl on the Berger sphere can be represented as

$$\text{curl}_s = \begin{pmatrix} \frac{2}{a} & -\frac{1}{a} V_3 & V_2 \\ \frac{1}{a} V_3 & \frac{2}{a} & -V_1 \\ -V_2 & V_1 & 2a \end{pmatrix}. \tag{C.7}$$

Here, the subscript s indicates that the operator appearing in (C.7) acts on 3-columns of scalar functions. The latter can be obtained by projecting 1-forms onto the orthonormal framing (C.6).

The operators curl and curl_s are related as

$$\text{curl}_\alpha^\beta = \tilde{V}^j_\alpha [\text{curl}_s]_j^k \tilde{V}_k^\beta, \quad \tilde{V}^j_\alpha := \delta^{jl} \tilde{V}_l^\gamma g_{\alpha\gamma}.$$

See also the projection operator \mathbf{S} in [16, Section 5.1].

C.2 | The spectrum

Theorem C.2. *The eigenvalues of the operator $-\Delta$ on the Berger 3-sphere are*

$$n(n + 2) + (a^{-2} - 1)(n - 2l)^2, \quad n = 0, 1, 2, 3, \dots, \quad 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor \tag{C.8}$$

with

- (i) multiplicity 1
if $n = 0$,
- (ii) multiplicity $2n + 2$
if n is odd,
- (iii) multiplicity $2n + 2$
if n is even and $0 \leq l < \frac{n}{2}$, and
- (iv) multiplicity $n + 1$
if n is even and $l = \frac{n}{2}$.

For a proof of the above theorem, we refer the reader to [50, Theorem 5.5]. Somewhat more involuted proofs may also be found in [62, Lemma 4.1] and [45, Proposition 3.9].

Remark C.3. When $a = 1$, Theorem C.2 reduces to the usual spectrum of $-\Delta$ on the round 3-sphere: eigenvalues $n(n + 2)$, $n = 0, 1, 2, \dots$, with multiplicity $(n + 1)^2$.

APPENDIX D: THE SPECTRUM OF CURL ON A BERGER SPHERE

From a perturbative perspective, and in analogy with what happens for the massless Dirac operator, one expects the spectrum of curl to be asymmetric about zero for an open dense subset of the space of metrics on a given Riemannian 3-manifold — put differently, spectral asymmetry of curl is a generic condition. However, we were unable to identify a precise reference for this fact in the literature.

In this appendix, we will write down explicitly the eigenvalues and the η -invariant for the operator curl on a Berger sphere, using classical (as opposed to pseudodifferential) techniques. On the

one hand, this will serve as an illustration of the spectral asymmetry of curl. On the other hand, it shall serve as a useful reference for further explicit calculations in future work.

Throughout this appendix, we use the notation from Appendix C.1.

D.1 | The spectrum

The spectrum of curl (C.7) was first computed by Gibbons [34, Section 5]. The positive spectrum of curl (C.7) also features in [50, Proposition 5.6]. We give below, for future reference and in the form of a theorem, explicit formulae for the *full* spectrum of curl, in a slightly reformulated form. The proof is omitted.

Theorem D.1. *The spectrum of the operator curl on the Berger 3-sphere is the (disjoint) union of the following four series of eigenvalues:*

I. Eigenvalues

$$\frac{n}{a}, \quad n = 2, 3, \dots, \quad (\text{D.1})$$

with multiplicity

$$2n - 2. \quad (\text{D.2})$$

II. Eigenvalues

$$\frac{n + 2(a^2 - 1)}{a}, \quad n = 2, 3, \dots \quad (\text{D.3})$$

Here, the multiplicity is as follows.

(a) If $n = 2$, the multiplicity is

$$1. \quad (\text{D.4})$$

(b) If $n = 3, 4, \dots$, the multiplicity is

$$2n - 2. \quad (\text{D.5})$$

III. Eigenvalues

$$a + \sqrt{a^2 + n(n+2) + (a^{-2} - 1)(n-2l)^2}, \quad n = 2, 3, \dots, \quad 1 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad (\text{D.6})$$

Here, the multiplicity is as follows.

(a) If n is odd, the multiplicity is

$$2n + 2. \quad (\text{D.7})$$

(b) If n is even and $l < \frac{n}{2}$, the multiplicity is

$$2n + 2. \quad (\text{D.8})$$

(c) If n is even and $l = \frac{n}{2}$, the multiplicity is

$$n + 1. \quad (\text{D.9})$$

IV. Eigenvalues

$$a - \sqrt{a^2 + n(n+2) + (a^{-2} - 1)(n-2l)^2}, \quad n = 2, 3, \dots, \quad 1 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor, \quad (\text{D.10})$$

with multiplicities as in (D.7)–(D.9).

Remark D.2.

- (i) Series I, II and III form the positive spectrum, whereas series IV is the negative spectrum.
- (ii) For $a = 1$, the Berger sphere reduces to the standard 3-sphere. It is not hard to check that for $a = 1$, Theorem D.1 gives us eigenvalues

$$\pm n, \quad n = 2, 3, \dots, \quad (\text{D.11})$$

with multiplicity

$$n^2 - 1. \quad (\text{D.12})$$

In particular, the spectrum is symmetric about 0.

- (iii) Formulae (D.11) and (D.12) agree with [7, Theorem 5.2].
- (iv) Note that in (D.6), the lowest value the integer l can take is 1. The case $l = 0$ is special: in this case, the square root becomes superfluous and the expression (D.6) turns into (D.3). We have highlighted the special status of the case $l = 0$ by separating out the series of eigenvalues (D.3).
- (v) Observe that the expression (C.8) appears under the square root in formulae (D.6) and (D.10).

D.2 | The eta invariant

In this subsection, we will compute the eta invariant for the operator curl on a Berger 3-sphere directly, using the explicit formulae for the eigenvalues established above in Theorem D.1. Recall that the eta-function is defined, in general, in accordance with (1.2).

Theorem D.3. *Consider the operator curl on a Berger 3-sphere, defined as in Appendix C.1. We have*

$$\eta_{\text{curl}}(0) = \frac{2}{3}(a^2 - 1)^2. \quad (\text{D.13})$$

Remark D.4. The above theorem warrants a number of remarks.

- (i) The eta invariant for the Dirac operator on a Berger 3-sphere was computed by Hitchin [37, p. 34] and reads

$$\eta_{\text{Dirac}}(0) = -\frac{1}{6}(a^2 - 1)^2. \quad (\text{D.14})$$

Note that the parameter λ in [37] corresponds to our parameter a , and that Hitchin adopts the same (or rather, equivalent) conventions as we do when defining the Dirac operator. The latter can be seen if one compares the spectra of the Dirac operator in [37] and [31]: Hitchin's

[37, Proposition 3.2] with $p = q = 1$ and minus sign in front of the square root agrees with [31, formulae (6.5) and (6.8)].

The result (D.14) was obtained under the assumption $a < 4$. We do not need to impose any restriction on a because, unlike for Dirac operator, where the dimension of the space of harmonic spinors is highly dependent on the Riemannian metric, zero is never an eigenvalue of curl (see Theorem 2.1).

- (ii) For a general connected oriented closed Riemannian 3-manifold, the invariants $\eta_{\text{curl}}(0)$ and $\eta_{\text{Dirac}}(0)$ are related as

$$\eta_{\text{curl}}(0) = -4\eta_{\text{Dirac}}(0) + k \tag{D.15}$$

for some $k \in \mathbb{Z}$. This follows from abstract algebraic topology arguments involving Hirzebruch polynomials, see [2, Section 1] and [3, Theorems 4.2 and 4.14]. Theorem D.3 establishes that, in the case of a Berger 3-sphere, we have (D.15) with $k = 0$ for $a < 4$. The significance of the value $a = 4$ is that for this particular value of the parameter a , the Dirac operator has a zero mode (harmonic spinor).

- (iii) The properties of the Dirac operator on a Berger 3-sphere, including an alternative (to Hitchin’s [37]) derivation of the spectrum, were further studied by Bär in [6]. Note that Bär adopts the opposite convention when defining the Dirac operator, so that the eigenvalues of the Dirac operator in [6] differ from those in [31, 37] by sign.
- (iv) Formula (D.13) appears in [24, Problem 7].

In preparation for the proof of Theorem D.3, let us make the following observation.

Lemma D.5. *The function η_{curl} admits the representation*

$$\eta_{\text{curl}}(s) = \theta(s) + (2a)^{-s} + 4a^s \zeta(s - 1), \tag{D.16}$$

where

$$\theta(s) := \sum_{j=1}^{\infty} \left[\left(\sqrt{a^2 + \mu_j} + a \right)^{-s} - \left(\sqrt{a^2 + \mu_j} - a \right)^{-s} \right], \tag{D.17}$$

the μ_j are the eigenvalues (B.12) of $-\Delta$ enumerated in increasing order with account of multiplicities, and $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

Proof. Formulae (D.16) and (D.17) are a straightforward consequence of Theorems C.2 and D.1. \square

Definition D.6. We define

$$\zeta_{-\Delta}(s) := \sum_{j=1}^{\infty} \mu_j^{-s}$$

to be the zeta function[†] associated with (minus) the Laplace–Beltrami operator $-\Delta$. Similarly, we define

$$\zeta_{\sqrt{-\Delta}}(s) := \sum_{j=1}^{\infty} \mu_j^{-s/2} = \zeta_{-\Delta}(s/2).$$

It is known [54, p. 243] that the function $\zeta_{\sqrt{-\Delta}}(s)$ is meromorphic in \mathbb{C} with simple poles at $s = 3, 1, -1, \dots$.

Lemma D.7. *For a closed Riemannian 3-manifold (M, g) , we have*

$$\zeta_{\sqrt{-\Delta}}(s) = \frac{1}{2\pi^2} \left[\frac{\text{Vol}(M)}{s-3} + \frac{\int_M \text{Sc}(x) \rho(x) dx}{12(s-1)} + \dots \right]. \tag{D.18}$$

Proof. Let $N_{\sqrt{-\Delta}}(\lambda)$ be the counting function for the positive eigenvalues of $\sqrt{-\Delta}$, see, for example, [15, Appendix B]. The functions $\zeta_{\sqrt{-\Delta}}$ and $N_{\sqrt{-\Delta}}$ are related as

$$\zeta_{\sqrt{-\Delta}}(s) = \int_0^{+\infty} \lambda^{-s} N'_{\sqrt{-\Delta}}(\lambda) d\lambda, \tag{D.19}$$

where the prime stands for differentiation with respect to λ . By combining [15, equation (B.1) and Theorem B.2] with formula (D.19), we obtain (D.18). \square

Since for the Berger 3-sphere $\text{Vol}(M) = 2\pi^2 a$ and $\text{Sc} = 8 - 2a^2$, we have the following.

Corollary D.8. *For the Berger 3-sphere*

$$\zeta_{\sqrt{-\Delta}}(s) = a \left[\frac{1}{s-3} + \frac{4-a^2}{6(s-1)} + \dots \right]. \tag{D.20}$$

We are now in a position to prove Theorem D.3.

Proof of Theorem D.3. Since $\zeta(-1) = -\frac{1}{12}$, formula (D.16) implies that the proof of (D.13) reduces to the proof of

$$\theta(0) = \frac{2}{3} a^2(a^2 - 2), \tag{D.21}$$

where θ is the function (D.17). Formulae (D.21) and (D.17) reduce the analysis of the operator curl to the analysis of the Laplace–Beltrami operator.

To establish (D.21), we use a trick due to Hitchin [37, p. 34]. We expand the terms in the RHS of (D.17) in inverse powers of μ_j as

$$\begin{aligned} \left(\sqrt{a^2 + \mu_j \pm a} \right)^{-s} &= \mu_j^{-s/2} \left(\sqrt{1 + a^2 \mu_j^{-1} \pm a \mu_j^{-1/2}} \right)^{-s} \\ &= \mu_j^{-s/2} \left(1 \pm a \mu_j^{-1/2} + \frac{1}{2} a^2 \mu_j^{-1} + O\left(\mu_j^{-2}\right) \right)^{-s} \end{aligned}$$

[†] The function $s \mapsto \zeta_{-\Delta}(s)$ is often called the *Minakshisundaram–Pleijel zeta function*, see [54, equation (6)].

$$\begin{aligned}
 &= \mu_j^{-s/2} \left(1 - s \left[\pm a \mu_j^{-1/2} + \frac{1}{2} a^2 \mu_j^{-1} \right] + \frac{s(s+1)}{2} \left[a^2 \mu_j^{-1} \pm a^3 \mu_j^{-3/2} \right] \right. \\
 &\quad \left. \mp \frac{s(s+1)(s+2)}{6} a^3 \mu_j^{-3/2} + O(\mu_j^{-2}) \right) \\
 &= \mu_j^{-s/2} \left(1 \mp s a \mu_j^{-1/2} + \frac{s^2}{2} a^2 \mu_j^{-1} \pm \frac{s(1-s^2)}{6} a^3 \mu_j^{-3/2} + O(\mu_j^{-2}) \right). \tag{D.22}
 \end{aligned}$$

Substituting (D.22) into (D.17), we get

$$\theta(s) = -2s a \zeta_{\sqrt{-\Delta}}(s+1) + \frac{s(1-s^2)}{3} a^3 \zeta_{\sqrt{-\Delta}}(s+3) + h(s), \tag{D.23}$$

where h is analytic at $s = 0$ and

$$h(0) = 0. \tag{D.24}$$

Formulae (D.23), (D.20) and (D.24) imply

$$\lim_{s \rightarrow 0} \theta(s) = -2a^2 \frac{4-a^2}{6} + \frac{1}{3} a^4 = \frac{2}{3} a^2 (a^2 - 2),$$

which proves (D.21). □

APPENDIX E: MAXWELL'S EQUATIONS: ELECTROMAGNETIC CHIRALITY AND POLARISATION

The focus of our paper is the study of asymmetry between positive and negative eigenvalues of the operator curl. It is natural to ask the question: does the sign of an eigenvalue of curl have a physical meaning? We show in this appendix that it does by means of a rather elementary argument. The key here are the notions of *electromagnetic chirality* and *polarisation*, see Definitions E.3 and E.5. Let us emphasise that the physical interpretation we provide in this appendix is not new: it has appeared in the literature in various related, if more abstract and ‘higher-tech’, formulations — see, for example, [30, Sec. 2], [33, Sec. 1.1.2] and references therein.

Let (M, g) be a connected oriented closed Riemannian 3-manifold. Consider homogeneous vacuum Maxwell equations on $M \times \mathbb{R}$

$$\begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} E \\ B \end{pmatrix}, \tag{E.1}$$

$$\delta E = \delta B = 0, \tag{E.2}$$

where E (electric field) and B (magnetic field) are the unknown quantities, time-dependent real-valued 1-forms.

We list below two basic symmetries of Maxwell’s equations (E.1) and (E.2). All the results in this appendix are given without proof because the corresponding proofs are elementary.

Definition E.1. Given a pair of real-valued 1-forms E and B , we define the *duality transform* as the linear map

$$\text{dual} : \begin{pmatrix} E \\ B \end{pmatrix} \mapsto \begin{pmatrix} -B \\ E \end{pmatrix}. \quad (\text{E.3})$$

The duality transform can be interpreted as the action of the four-dimensional Lorentzian Hodge star on the electromagnetic tensor

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

Observe that the duality transform is a skew-involution, that is, applying it twice gives minus identity.

Lemma E.2. *Maxwell's equations (E.1) and (E.2) are invariant under the action of the duality transform (E.3), that is, the duality transform maps solutions to solutions.*

Definition E.3 [49, 61]. Given a pair of real-valued 1-forms E and B , we define *electromagnetic chirality* as the quadratic functional

$$\text{chir}(E, B) := \int_M (E \wedge dE + B \wedge dB). \quad (\text{E.4})$$

Lemma E.4. *Chirality (E.4) is a conserved quantity for Maxwell's equations (E.1), (E.2), that is, if (E, B) is a solution then $\text{chir}(E, B)$ does not depend on time t .*

We will be seeking harmonic (in the time variable t) solutions of Maxwell's equations.

Definition E.5. A harmonic solution (E, B) of Maxwell's equations (E.1), (E.2) is said to be *polarised* if $(\dot{E}, \dot{B}) = \pm\lambda \text{dual}(E, B)$. Here, the dot stands for derivative in t and λ is the angular frequency.

Recall that (λ_j, u_j) , $j = \pm 1, \pm 2, \dots$, is the eigensystem for curl. Consider a harmonic solution

$$\begin{pmatrix} E \\ B \end{pmatrix} = \begin{pmatrix} u_j \cos(\lambda_j t + \varphi) \\ -u_j \sin(\lambda_j t + \varphi) \end{pmatrix} \quad (\text{E.5})$$

of Maxwell's equations (E.1) and (E.2), where $\varphi \in \mathbb{R}$ is an arbitrary phase. Of course, (E.5) is a linear combination of two basic harmonic solutions

$$\begin{pmatrix} E \\ B \end{pmatrix} = \begin{pmatrix} u_j \cos(\lambda_j t) \\ -u_j \sin(\lambda_j t) \end{pmatrix} \quad (\text{E.6})$$

and

$$\begin{pmatrix} E \\ B \end{pmatrix} = \begin{pmatrix} u_j \sin(\lambda_j t) \\ u_j \cos(\lambda_j t) \end{pmatrix}. \quad (\text{E.7})$$

Let E_0 and B_0 be a pair of infinitely smooth real-valued 1-forms which do not depend on t and satisfy the condition $\delta E_0 = \delta B_0 = 0$. Consider the Cauchy problem

$$E|_{t=0} = E_0, \quad B|_{t=0} = B_0$$

for Maxwell’s equations (E.1) and (E.2). Then, (E.6) and (E.7) can be used to write the solution as a series

$$\begin{pmatrix} E \\ B \end{pmatrix} = \sum_{j \in \mathbb{Z} \setminus \{0\}} \left[\begin{pmatrix} u_j \cos(\lambda_j t) \\ -u_j \sin(\lambda_j t) \end{pmatrix} \langle E_0, u_j \rangle + \begin{pmatrix} u_j \sin(\lambda_j t) \\ u_j \cos(\lambda_j t) \end{pmatrix} \langle B_0, u_j \rangle \right] + \begin{pmatrix} P_{\mathcal{H}^1} E_0 \\ P_{\mathcal{H}^1} B_0 \end{pmatrix}, \quad (E.8)$$

where $P_{\mathcal{H}^1}$ is the orthogonal projection onto the space of harmonic 1-forms and $\langle \cdot, \cdot \rangle$ is the inner product (2.5).

The following proposition is the main result of this appendix.

Proposition E.6. *For (E.5), we have*

$$\text{chir}(E, B) = \lambda_j, \quad (\dot{E}, \dot{B}) = -\lambda_j \text{dual}(E, B). \quad (E.9)$$

Formula (E.9) tells us that the sign of an eigenvalue of curl has a physical meaning and it can be interpreted in two different ways — in terms of chirality of the corresponding harmonic solution (E.5) of Maxwell’s equations or in terms of its polarisation. Note that the concepts of chirality and polarisation are different: polarisation involves time, whereas chirality does not.

APPENDIX F: TAYLOR EXPANSIONS FOR THE OPERATOR OF PARALLEL TRANSPORT

This appendix is concerned with local expansions of parallel transport maps in normal coordinates. These can, in principle, be found in many guises and with varied degrees of accuracy in the literature. Nevertheless, for the convenience of the reader and for future reference, we provide here a concise self-contained derivation, one which agrees with the definitions and sign conventions set out in this paper.

Let us fix a point $x \in M$ and let us choose normal coordinates centred at x . Let $y \in M$ be a point in a (small) neighbourhood of x .

Recall that $Z_\alpha^\beta(x, y)$ is the map realising the parallel transport of vectors along the unique geodesic connecting x to y , whereas $Z^\alpha_\beta(x, y)$ is the map realising the parallel transport of covectors along the same geodesic, see (4.6)–(4.8).

Proposition F.1. *We have*

$$Z_\alpha^\beta(0, y) = \delta_\alpha^\beta + \frac{1}{6} \text{Riem}^\beta_{\mu\alpha\nu}(0) y^\mu y^\nu - \frac{1}{6} \frac{\partial^2 \Gamma^\beta_{\mu\alpha}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho + O(|y|^4) \quad (F.1)$$

and

$$Z^\alpha_\beta(0, y) = \delta^\alpha_\beta - \frac{1}{6} \text{Riem}^\alpha_{\mu\beta\nu}(0) y^\mu y^\nu + \frac{1}{6} \frac{\partial^2 \Gamma^\alpha_{\mu\beta}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho + O(|y|^4). \quad (F.2)$$

Furthermore, for all $\tau \in [0, 1]$, we have

$$Z_\alpha^\beta(y, \tau y) = \delta_\alpha^\beta + \frac{\tau^2 - 1}{6} \text{Riem}^\beta_{\mu\alpha\nu}(0) y^\mu y^\nu - \frac{\tau^3 - 1}{6} \frac{\partial^2 \Gamma^\beta_{\mu\alpha}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho + O(|y|^4) \quad (\text{F.3})$$

and

$$Z^\alpha_\beta(y, \tau y) = \delta^\alpha_\beta - \frac{\tau^2 - 1}{6} \text{Riem}^\alpha_{\mu\beta\nu}(0) y^\mu y^\nu + \frac{\tau^3 - 1}{6} \frac{\partial^2 \Gamma^\alpha_{\mu\beta}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho + O(|y|^4). \quad (\text{F.4})$$

Note that formula (F.1) agrees with [17, formula (7.1)].

Proof of Proposition F.1. Let us start with a vector v^α at the origin and let us parallel transport it along a straight line to y — the unique shortest geodesic in the chosen coordinate system. This means imposing the condition

$$y^\mu \nabla_\mu v^\beta = y^\mu \left(\partial_\mu v^\beta + \Gamma^\beta_{\mu\alpha} v^\alpha \right) = 0 \quad (\text{F.5})$$

along the straight line. Let us now parameterise our straight line with parameter $t \in [0, 1]$ as $z(t) = ty$. Then

$$y^\mu \partial_\mu v^\beta = \dot{v}^\beta. \quad (\text{F.6})$$

Formulae (F.5) and (F.6) imply

$$\begin{aligned} v^\beta \Big|_{t=1} &= v^\beta \Big|_{t=0} - \int_0^1 y^\mu \Gamma^\beta_{\mu\gamma}(ty) v^\gamma(ty) dt \\ &= v^\beta \Big|_{t=0} - \int_0^1 y^\mu \left[\frac{\partial \Gamma^\beta_{\mu\alpha}}{\partial y^\nu}(0) y^\nu t + \frac{1}{2} \frac{\partial^2 \Gamma^\beta_{\mu\alpha}}{\partial y^\nu \partial y^\rho}(0) y^\nu y^\rho t^2 + O(|y|^3) \right] [v^\alpha \Big|_{t=0} + O(|y|^2)] dt. \end{aligned} \quad (\text{F.7})$$

In the above equation, we used the fact that

$$\dot{v}^\alpha \Big|_{t=0} = 0,$$

which follows from (F.5) and (F.6) and the fact that Christoffel symbols vanish at the origin.

Using the elementary differential-geometric identity

$$\frac{\partial \Gamma^\beta_{\mu\alpha}}{\partial y^\nu}(0) = -\frac{1}{3} \left(\text{Riem}^\beta_{\mu\alpha\nu}(0) + \text{Riem}^\beta_{\alpha\mu\nu}(0) \right)$$

and performing integration in (F.7), we obtain

$$v^\beta \Big|_{t=1} = v^\beta \Big|_{t=0} + \frac{1}{6} \text{Riem}^\beta_{\mu\alpha\nu}(0) y^\mu y^\nu v^\alpha \Big|_{t=0} - \frac{1}{6} \frac{\partial^2 \Gamma^\beta_{\mu\alpha}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho v^\alpha \Big|_{t=0} + O(|y|^4).$$

The latter implies (F.1).

Combining (F.1) with (4.8), one immediately obtains (F.2).

In order to obtain (F.3), let us parallel transport along our straight line from y to τy . Arguing as above, we get

$$\begin{aligned} v^\beta \Big|_{t=\tau} &= v^\beta \Big|_{t=1} - \int_1^\tau y^\mu \Gamma^\beta_{\mu\gamma}(ty) v^\gamma(ty) ds \\ &= v^\beta \Big|_{t=1} - \int_1^\tau y^\mu \left[-\frac{1}{3} \text{Riem}^\beta_{\mu\alpha\nu}(0) y^\nu t + \frac{1}{2} \frac{\partial^2 \Gamma^\beta_{\mu\alpha}}{\partial y^\nu \partial y^\rho}(0) y^\nu y^\rho t^2 + O(|y|^3) \right] \left[v^\alpha \Big|_{t=1} + O(|y|^2) \right] ds \\ &= v^\beta \Big|_{t=1} + \frac{\tau^2 - 1}{6} \text{Riem}^\beta_{\mu\alpha\nu}(0) y^\mu y^\nu v^\alpha \Big|_{t=1} - \frac{\tau^3 - 1}{6} \frac{\partial^2 \Gamma^\beta_{\mu\alpha}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho v^\alpha \Big|_{t=1} + O(|y|^4). \end{aligned}$$

The latter implies (F.3) which, in turn, combined with (4.8) immediately gives us (F.4). □

Remark F.2. For $\tau = 0$, expansions (F.3) and (F.4) simplify to read

$$Z_\alpha^\beta(y, 0) = \delta_\alpha^\beta - \frac{1}{6} \text{Riem}^\beta_{\mu\alpha\nu}(0) y^\mu y^\nu + \frac{1}{6} \frac{\partial^2 \Gamma^\beta_{\mu\alpha}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho + O(|y|^4) \tag{F.8}$$

and

$$Z^\alpha_\beta(y, 0) = \delta^\alpha_\beta + \frac{1}{6} \text{Riem}^\alpha_{\mu\beta\nu}(0) y^\mu y^\nu - \frac{1}{6} \frac{\partial^2 \Gamma^\alpha_{\mu\beta}}{\partial y^\nu \partial y^\rho}(0) y^\mu y^\nu y^\rho + O(|y|^4). \tag{F.9}$$

Formulae (F.1), (F.2), (F.8) and (F.9) agree with (4.7) and (4.8).

APPENDIX G: AN ALTERNATIVE DERIVATION OF FORMULA (1.13)

In this appendix, we verify, using an alternative method, formula (1.13), one of the main results of our paper.

In what follows we work in normal coordinates and we assume that curvature (but not its covariant derivative) vanishes at the origin, so that

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{1}{6} (\nabla_\sigma \text{Riem}_{\alpha\mu\beta\nu})(0) x^\sigma x^\mu x^\nu + O(|x|^4). \tag{G.1}$$

We continue using the notation $\Delta = -(d\delta + \delta d)$ for the Hodge Laplacian on 1-forms (5.8); $|\xi|$ stands for the Euclidean norm of ξ , whereas $\|\xi\|$ stands for the Riemannian norm thereof.

Proposition G.1. *Let*

$$[s]_\alpha^\beta(x, \xi) \sim \|\xi\|^{-1} \delta_\alpha^\beta + [s_{-2}]_\alpha^\beta(x, \xi) + [s_{-3}]_\alpha^\beta(x, \xi) + [s_{-4}]_\alpha^\beta(x, \xi) + \dots$$

be the (left) symbol of the operator $(-\Delta)^{-1/2}$ (recall (5.9)). Then

$$A_{\text{prin}}(x, \xi) = -\varepsilon_\beta^{\alpha\gamma} \left(([s_{-3}]_\alpha^\beta)_{x^\gamma} + i \xi_\gamma [s_{-4}]_\alpha^\beta \right) + O(|\xi|^{-3} |x|). \tag{G.2}$$

Proof. The claim follows from (5.10), (6.9), (6.30) and the fact that the full symbol of curl is (2.1). □

In the remainder of this appendix, we will derive explicit formulae for $[s_{-3}]_\alpha^\beta$ and $[s_{-3}]_\alpha^\beta$. The proofs are straightforward, hence omitted.

Lemma G.2. *Let*

$$[q]_\alpha^\beta(x, \xi) \sim \|\xi\|^2 \delta_\alpha^\beta + [q_1]_\alpha^\beta(x, \xi) + [q_0]_\alpha^\beta(x, \xi)$$

be the (left) symbol of the operator $-\Delta$. We have

$$[q_1]_\alpha^\beta(x, \xi) = i a_{\alpha}^{\beta\gamma}{}_{\mu\nu} \xi_\gamma x^\mu x^\nu + O(|\xi| |x|^3), \quad (\text{G.3})$$

$$[q_0]_\alpha^\beta(x) = b_{\alpha}^{\beta}{}_{\nu} x^\nu + O(|x|^2), \quad (\text{G.4})$$

where

$$a_{\alpha}^{\beta\gamma}{}_{\mu\nu} := \left[\frac{1}{2} \nabla_\mu \text{Ric}^\gamma{}_\nu - \frac{1}{12} \nabla^\gamma \text{Ric}_{\mu\nu} \right] \delta_\alpha^\beta - \frac{1}{6} \left[\nabla_\alpha \text{Riem}^\gamma{}_{\mu}{}^\beta{}_\nu - 3 \nabla_\mu \text{Riem}^\gamma{}_{\alpha}{}^\beta{}_\nu + 5 \nabla_\nu \text{Riem}^\gamma{}_{\mu}{}^\beta{}_\alpha \right], \quad (\text{G.5})$$

$$b_{\alpha}^{\beta}{}_{\nu} := -\frac{1}{6} \nabla^\beta \text{Ric}_{\alpha\nu} + \frac{1}{2} \nabla_\alpha \text{Ric}^\beta{}_\nu + \frac{1}{2} \nabla_\nu \text{Ric}_\alpha{}^\beta. \quad (\text{G.6})$$

Lemma G.3. *The symbols*

$$[r]_\alpha^\beta(x, \xi) \sim \|\xi\| \delta_\alpha^\beta + [r_0]_\alpha^\beta(x, \xi) + [r_{-1}]_\alpha^\beta(x, \xi) + [r_{-2}]_\alpha^\beta(x, \xi) + \dots$$

and s_α^β of $(-\Delta)^{1/2}$ and $(-\Delta)^{-1/2}$ are expressed in terms of (G.3) and (G.4) via the following hierarchy of identities:

$$\begin{aligned} [r_0]_\alpha^\beta &= \frac{1}{2|\xi|} [q_1]_\alpha^\beta - \frac{1}{2i|\xi|^2} \xi^\mu (\|\xi\|)_{x^\mu} \delta_\alpha^\beta + O(|x|^3), \\ [r_{-1}]_\alpha^\beta &= \frac{1}{2|\xi|} [q_0]_\alpha^\beta - \frac{1}{2i|\xi|^2} \xi^\mu ([r_0]_\alpha^\beta)_{x^\mu} + \frac{1}{4|\xi|} (|\xi|)_{\xi_\mu \xi_\nu} (\|\xi\|)_{x^\mu x^\nu} \delta_\alpha^\beta + O(|\xi|^{-1} |x|^2), \\ [r_{-2}]_\alpha^\beta &= -\frac{1}{2i|\xi|^2} \xi^\mu ([r_{-1}]_\alpha^\beta)_{x^\mu} \\ &\quad + \frac{1}{4|\xi|} (|\xi|)_{\xi_\mu \xi_\nu} ([r_0]_\alpha^\beta)_{x^\alpha x^\beta} + \frac{1}{12i|\xi|} (|\xi|)_{\xi_\mu \xi_\nu \xi_\rho} (\|\xi\|)_{x^\mu x^\nu x^\rho} \delta_\alpha^\beta + O(|\xi|^{-2} |x|), \\ [s_{-2}]_\alpha^\beta &= -|\xi|^{-2} [r_0]_\alpha^\beta - \frac{1}{i|\xi|^2} \xi^\mu (\|\xi\|^{-1})_{x^\mu} \delta_\alpha^\beta + O(|\xi|^{-2} |x|^3), \\ [s_{-3}]_\alpha^\beta &= -|\xi|^{-2} [r_{-1}]_\alpha^\beta - \frac{1}{i|\xi|^2} \xi^\mu ([s_{-2}]_\alpha^\beta)_{x^\mu} \\ &\quad + \frac{1}{2|\xi|} (|\xi|)_{\xi_\mu \xi_\nu} (\|\xi\|^{-1})_{x^\mu x^\nu} \delta_\alpha^\beta + O(|\xi|^{-3} |x|^2), \quad (\text{G.7}) \\ [s_{-4}]_\alpha^\beta &= -|\xi|^{-2} [r_{-2}]_\alpha^\beta - \frac{1}{i|\xi|^2} \xi^\mu ([s_{-3}]_\alpha^\beta)_{x^\mu} \\ &\quad + \frac{1}{2|\xi|} (|\xi|)_{\xi_\mu \xi_\nu} ([s_{-2}]_\alpha^\beta)_{x^\mu x^\nu} + \frac{1}{6i|\xi|} (|\xi|)_{\xi_\mu \xi_\nu \xi_\rho} (\|\xi\|^{-1})_{x^\mu x^\nu x^\rho} \delta_\alpha^\beta + O(|\xi|^{-4} |x|). \quad (\text{G.8}) \end{aligned}$$

Remark G.4. Lemmata G.2 and G.3 are true in any dimension d , not only in dimension $d = 3$.

In order to verify our formula (1.13) for A_{prin} , it only remains to compute (e.g. with the help of computer algebra) the right-hand sides of (G.7) and (G.8), and substitute the resulting expressions into (G.2).

For instance, for the particular choice of metric determined by

$$\text{Ric}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x^1 \\ 0 & x^1 & 0 \end{pmatrix} + O(|x|^2)$$

(recall (6.24) and (G.1)), we get

$$s_{-3}|_{(x, \xi_0)} = -\frac{1}{12} \begin{pmatrix} 0 & 3x^3 & 3x^2 \\ -x^3 & 0 & 3x^1 \\ -x^2 & 3x^1 & 0 \end{pmatrix} + O(|x|^2), \quad s_{-4}|_{(x, \xi_0)} = -\frac{i}{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(|x|), \quad (\text{G.9})$$

where ξ_0 is given by (6.26). Substituting (G.9) into (G.2), we obtain

$$A_{\text{prin}}(0, \xi_0) = -\frac{1}{2},$$

which agrees with (1.13) — see also (6.36).

ACKNOWLEDGEMENTS

M. C. was partially supported by a Leverhulme Trust Research Project Grant RPG-2019-240, by a Research Grant (Scheme 4) of the London Mathematical Society, by a grant of the Heilbronn Institute for Mathematical Research (HIMR) via the UKRI/EPSC Additional Funding Programme for Mathematical Sciences and by EPSC Fellowship EP/X01021X/1.

We are grateful to Christian Bär, Bruno Colbois, Stuart Dowker, Michal Kwasigroch, Jason Lotay and Nikolai Saveliev for useful comments and suggestions. We also thank Michael Levitin for kindly sharing his Mathematica scripts for the Berger sphere.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Matteo Capoferri  <https://orcid.org/0000-0001-6226-1407>

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