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Dalal Al Ghanim, Ronnie Loeffen, Alexander R. Watson

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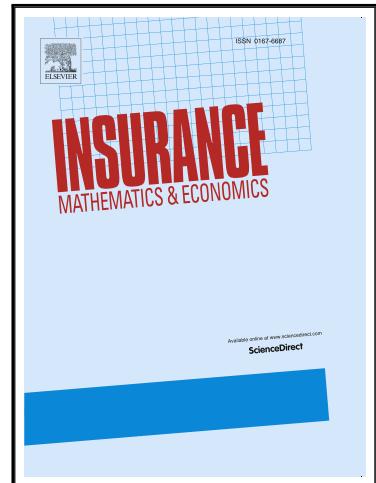
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# Stochastic optimal control of Lévy tax processes with bailouts

D. Al Ghanim\*      R. Loeffen†      A. R. Watson‡

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We consider controlling the paths of a spectrally negative Lévy process by two means: the subtraction of ‘taxes’ when the process is at an all-time maximum, and the addition of ‘bailouts’ which keep the value of the process above zero. We solve the corresponding stochastic optimal control problem of maximising the expected present value of the difference between taxes received and cost of bailouts given. Our class of taxation controls is larger than has been considered up till now in the literature and makes the problem truly two-dimensional rather than one-dimensional. Along the way, we define and characterise a large class of controlled Lévy processes to which the optimal solution belongs, which extends a known result for perturbed Brownian motions to the case of a general Lévy process with no positive jumps.

**Key words and phrases.** Risk process, tax process, spectrally negative Lévy process, capital injections, optimal control, perturbed Lévy process, Skorokhod reflection.

## 1 Introduction

Consider the capital of an insurance company as it evolves over time. A government considers two interventions: first, a loss carryforward taxation regime, in which some proportion, to be referred to as the tax rate, of the company’s increase in capital is taken whenever the level of capital reaches a new record; and second, a bailout system, in which the government injects money to the company in order to keep its capital level positive. We assume the government will bail the company out (instead of letting it go

\*Department of Mathematics, College of Science, King Saud University, P.O.Box 2455, Riyadh 11451, Saudi Arabia

†University of Liverpool, Institute for Financial and Actuarial Mathematics, Mathematical Sciences building, Liverpool L69 7ZL, ronnie.loeffen@liverpool.ac.uk

‡University College London, London, UK, alexander.watson@ucl.ac.uk

bankrupt); so the company is considered ‘too big to fail’. Though one can debate the use of such bailouts, they do occur, most notably during the 2008 financial crisis, and since they affect the tax strategy it is important to seriously consider models with this feature. Assume the government wants to choose the tax rate and the size and timing of the bailouts in order to maximise the expected discounted difference between tax revenue and cost of bailouts. In this work, we show that, when the company’s capital before government interventions is modelled by a spectrally negative Lévy process and the tax rate is at least  $\alpha \geq 0$  and at most  $\beta \in (0, 1)$ , this can be achieved using a threshold tax rate and minimal bailouts. Informally, a threshold tax rate means that tax is paid at rate  $\alpha$  when the capital level is below a threshold  $b \geq 0$ , and at rate  $\beta$  when it is above. By minimal bailouts, we mean that only the minimum amount of capital is injected in order to keep the company solvent. It makes sense that such a strategy might be optimal: bailouts are costly and so should be minimal, if the capital is large it is relatively safe to impose the highest tax rate, whereas if the capital level is low, then the risk of having to do costly bailouts is high which justifies reducing the tax burden of the company as much as possible. Note that the minimum and maximum tax rate  $\alpha$  and  $\beta$  are fixed and cannot be changed by the government.

There is some existing literature on optimal taxation of Lévy tax processes. In the setting where there are no government bailouts (so the company ceases to exist when its capital drops below 0), Albrecher et al. in [6, Section 4] start with a threshold tax rate (with the minimum tax rate being  $\alpha = 0$ ) and then look for the optimal threshold level  $b$  that maximises tax revenue (at any initial capital level); see also [3, Section 2.3] for the special case where the spectrally negative Lévy process is a compound Poisson risk process with exponentially distributed claims. Wang et al. [24] do the same but either include minimal bailouts (and their cost) or a terminal value at the time of bankruptcy. Wang and Hu [22] maximised tax revenue without bailouts over all  $[\alpha, \beta]$ -valued ‘latent’ tax rate functions (i.e., a tax rate which is a function of the running supremum of the uncontrolled capital level; see [2] for this terminology).

A closely related optimal control problem is the optimal dividend problem with mandatory capital injections (so bankruptcy of the company is not allowed) where money can be taken out, in an adapted way, to give as dividends to shareholders who in return cover the capital injections. This problem where one wants to maximise expected discounted value of the paid out dividends minus the cost of capital injections has been studied in [21], respectively [8], in the case where the uncontrolled risk process is a diffusion, respectively spectrally negative Lévy process. In the latter case it was shown in [8] (see also [21, Example 1] for the case of a Brownian motion with drift) that the optimal strategy is a dividend barrier strategy where dividends are paid out in a minimal way to keep the capital of the company below a certain level in combination with injecting capital in a minimal way to keep it positive. This mirrors our main result for the optimal taxation problem with mandatory bailouts.

We remark that in optimal dividend problems where no capital injections are allowed, or where capital injections are optional rather than mandatory, the optimal dividend strategy can be more exotic and one needs to assume a condition on the Lévy process in order for the dividend barrier strategy to be optimal, see [19] and [13]. This is consistent

with problems involving optimal taxation without bailout studied in the aforementioned papers [6, 22, 24] where results on optimality of the threshold tax rate strategy are provided under some conditions on the Lévy process. Indeed, such conditions are not required in [24] for the problem where minimal bailouts are present.

We now rigorously state our stochastic optimal control problem of interest. For any stochastic process  $Y = (Y_t)_{t \geq 0}$  whose sample paths has right and left limits, we write  $Y_{t-} = \lim_{s \uparrow t} Y_s$ ,  $Y_{t+} = \lim_{s \downarrow t} Y_s$  and  $\Delta Y_s := Y_{s+} - Y_{s-}$ . Further, for a measurable function  $f$  and an increasing and right-continuous function  $G$ , we define  $\int_{0+}^t f(s)dG(s)$  as the Stieltjes integral over  $(0, t]$ , and  $\int_0^t f(s)dG(s)$  as the Stieltjes integral over  $[0, t]$ ; that is,  $\int_0^t f(s)dG(s) = f(0)G(0) + \int_{0+}^t f(s)dG(s)$ . For reasons that will become clear, we need to, for a given path, define a running supremum with a prescribed starting value. To this end, for a path  $Y = (Y_t)_{t \geq 0}$  and  $\bar{x} \geq Y_0$  we define

$$\bar{Y}_t = \bar{x} \vee \sup_{0 \leq s \leq t} Y_s, \quad (1)$$

and call  $\bar{Y} = (\bar{Y}_t)_{t \geq 0}$  the *running supremum* of  $Y$  with *initial maximum level*  $\bar{x}$ . We let  $(\Omega, \mathcal{F})$  be a measurable space and on it we define a family of probability measures  $(\mathbb{P}_{x, \bar{x}})_{x \in \mathbb{R}, \bar{x} \geq x}$  and a stochastic process  $X = (X_t)_{t \geq 0}$  such that, under  $\mathbb{P}_{x, \bar{x}}$ ,  $X$  is a spectrally negative Lévy process starting at  $x$  and with initial maximum level  $\bar{x}$  (i.e.  $X_0 = x$  and  $\bar{X}_0 = \bar{x}$ ). We refer to Section 2.1 below for some background information on spectrally negative Lévy processes. Let  $(\mathcal{F}_t^\circ)_{t \geq 0}$  be the natural filtration of  $X$ , and  $\mathcal{F}_t := \bigcap_{s \geq t} \mathcal{F}_s^\circ$  its right-continuous enlargement. The random variable  $X_t$  represents the uncontrolled capital level at time  $t$  of an insurance company. We wish to understand the effect of controlling the capital level by introducing *taxation*, which yields revenue for the controller, and *bailouts* (or *capital injections*), which increase the process but have a cost for the controller. To this end, we fix a *lower tax rate bound*  $\alpha \geq 0$ , an *upper tax rate bound*  $\beta \in [\alpha, 1] \cap (0, 1)$ , and a *discount rate*  $q > 0$ . Next we define our class of controls.

**Definition 1.** A pair of stochastic processes  $(H, L) = ((H_t)_{t \geq 0}, (L_t)_{t \geq 0})$  is called an admissible control under  $\mathbb{P}_{x, \bar{x}}$  if the following holds: (i)  $H$  and  $L$  are left-continuous and adapted to  $(\mathcal{F}_t)_{t \geq 0}$ , (ii)  $H_t \in [\alpha, \beta]$  for all  $t \geq 0$ , (iii)  $L$  is increasing (in the weak sense) with  $L_0 = 0$ , (iv)  $\bar{X} + \bar{L}$  is continuous, (v)  $\mathbb{E}_{x, \bar{x}} \int_0^\infty e^{-qs} dL_s < \infty$ , (vi)  $\mathbb{P}_{x, \bar{x}}(U_{t+} \geq 0 \text{ for all } t \geq 0) = 1$  where  $U = (U_t)_{t \geq 0}$  is the controlled process defined by

$$U_t = X_t + L_t - \int_{0+}^t H_s d(\bar{X} + \bar{L})_s. \quad (2)$$

We write  $\Pi_{x, \bar{x}}$  for the set of all admissible controls under  $\mathbb{P}_{x, \bar{x}}$ .

For an admissible control  $(H, L)$ ,  $H_t$  represents the tax rate at time  $t$  and  $L_t$  represents the cumulative amount of bailout funds added to  $U$  up to time  $t$ . From (2) and Lemma 10 below, one can see that taxes are only paid when the controlled capital process  $U$  is at its maximum, i.e., at those times  $t$  such that  $\bar{U}_t = U_t$ . Regarding the conditions of an admissible control in Definition 1, (vi) ensures that the controlled capital level cannot be

strictly negative for any duration of time, which reflects that the controller/government is compelled to bail the company out. We highlight that, as, the bailout control  $L$  is assumed to be left-continuous (rather than right-continuous), the controlled process  $U$  can be strictly negative at a discrete set of time points. On the one hand, left-continuous controls are natural, as the controller can only react after an event, in particular a jump. On the other, this framework also extends easily to a version of the control problem in which bailouts are optional instead of mandatory; see, for example, [13, Definition 1]. Conditions (iv) and (v) in Definition 1 are present for technical reasons, though they also make sense from a practical point of view: (iv) avoids controls where tax has to be paid over a lump-sum bailout and (v) excludes controls where the expected discounted value of the total bailouts is infinity, which is a common assumption in these type of problems; we point once again to [13, Definition 1].

In order to state our optimality criterion, we fix a *bailout penalty factor*  $\eta \geq 0$ , an initial maximal capital  $\bar{x} \geq 0$  and an initial capital  $x \leq \bar{x}$  and we define, for  $(H, L) = \pi \in \Pi_{x, \bar{x}}$ , the *value function*

$$v^\pi(x, \bar{x}) = \mathbb{E}_{x, \bar{x}} \left[ \int_{0^+}^{\infty} e^{-qs} H_s d(\bar{X} + \bar{L})_s - \eta \int_0^{\infty} e^{-qs} dL_s^+ \right],$$

where  $L^+ = (L_t^+)_t \geq 0$  is the right-continuous version of  $L$ , i.e.  $L_t^+ = L_{t+}$ . In this definition, the first term represents the present value of taxation and the second term the present value of bailouts, with the factor  $\eta$  representing the a multiplicative additional cost of each bailout.

We wish to solve the *optimal control problem*

$$v^*(x, \bar{x}) = \sup_{\pi \in \Pi_{x, \bar{x}}} v^\pi(x, \bar{x}), \quad (3)$$

by finding  $v^*$  and for each pair  $(x, \bar{x})$  a choice of  $\pi$  which attains it.

Our contributions are the following. First, in Theorem 3, we solve (3) under a minor condition on the Lévy measure of  $X$  (equivalent to  $\Pi_{x, \bar{x}}$  being nonempty for some (or equivalently all)  $(x, \bar{x})$ ) in the case where  $\eta \geq 1$ , and show that an optimal control is given by a threshold tax rate, which corresponds to  $H_t = \alpha + (\beta - \alpha) \mathbf{1}_{\{\bar{U}_t > b\}}$  for some  $b \geq 0$ , in combination with minimal bailouts. A sample path of the controlled capital process associated with such threshold tax rate, denoted by  $V$ , is provided in Figure 1. The assumption  $\eta \geq 1$  makes sense because in practice there is a cost of capital. But in fact, Theorem 3 would no longer hold when we allow  $\eta < 1$ : an optimal control for (3) in that case might look very different, with minimal bailouts no longer being optimal.

Second, we observe that the optimally controlled capital process belongs to the class of natural tax processes with minimal bailouts with ‘natural’ meaning that the tax rate is a function of (the running supremum of) the controlled capital process (see also [2]). We give a rigorous definition of this class involving a construction and characterisation of its elements. If the initial maximum capital level  $\bar{x}$  is strictly positive, then this is not hard to do (see constructions in [4, 24] for specific natural tax rates), because the tax payments and bailouts occur at distinct times, and the existence and uniqueness of the

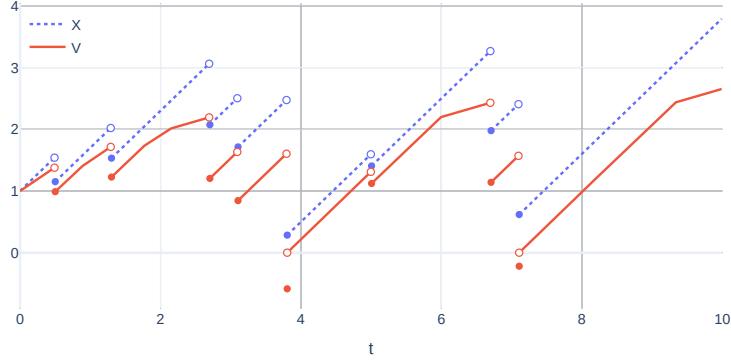


Figure 1: Illustration of the natural tax process  $V$  with threshold tax rate at  $b$  and minimal bailouts, where  $\alpha = 0.3$ ,  $\beta = 0.7$  and  $b = 2$ . The blue (dashed) line is the path of the background Lévy process  $X$ . The red line is the process  $V$ . Note that at times when  $V$  experiences a bailout, its instantaneous negative value is retained and the process is sent to the value zero immediately after.

process without bailouts has been shown in [2]. However, if  $\bar{x} = 0$  and the Lévy process is of unbounded variation, then tax payments and bailouts take place simultaneously within any time interval  $(0, \epsilon)$  with  $\epsilon > 0$ , and it is less clear how to construct the process. We do this by introducing a Skorokhod-type problem for càdlàg paths and use a contraction argument to prove the existence and uniqueness of its solution, which we call the tax-reflection transform. This extends known results for the special case where  $X$  is a Brownian motion; see [18, 10, 11, 12, 9] and Remark 7 below.

Third, whereas in [6, 24] (respectively [22] and [23]), the authors optimise over tax rate controls  $H$  of the form  $H_t = \beta \mathbf{1}_{\{\bar{U}_t > b\}}$  where  $b \geq 0$  (respectively  $H_t = f(\bar{X}_t)$  where  $f$  is an  $[\alpha, \beta]$ -valued measurable function), we allow  $H$  to be a more general adapted process. Given that a natural tax process with (or without) minimum bailouts is not Markov on its own but only when considered together with its running supremum (see Proposition 9 below) this requires one to consider the optimal value function  $v^*$  in (3) as a function of  $(x, \bar{x})$  and not just  $x$ . This makes it more challenging to establish a verification lemma for solving the optimal control problem in comparison to, e.g., optimal dividend problems (with or without capital injections) for which the optimally controlled process is on its own Markovian. Our third contribution is that we provide a rigorous framework for dealing with optimal taxation problems for this general class of tax rate controls, which, in principle, should be easily adaptable to handle other similar optimal taxation problems as well. Indeed, the first author has shown in [1] that the question of optimal taxation without bailouts can be handled using this approach.

The last contribution is that we provide an expression, in terms of the scale functions of the Lévy process, for the value function associated with the threshold tax rate and minimal bailouts using the novel approach developed in the first author's PhD thesis [1]. This involves a characterisation lemma which is similar in nature to the verification

lemma that we use for the optimal control problem. When  $x = \bar{x} > 0$  such an expression for the value function has been derived in [24, Equation (14)] in the case where  $\alpha = 0$  and can be derived from [5, Theorem 2] by taking  $\theta \downarrow 0$  there in the case where the tax rate is constant (i.e.  $\alpha = \beta$ ); both papers use different methods than ours.

The present article is an elaboration on the work done in the first author's PhD thesis [1], where a slightly weaker version of the main theorem was proved (in which taxation could not start from zero capital). During preparation of the present work, [23] was made available, which studies a version of the control problem where it is assumed that the tax rate controls are latent tax rates, bailouts are assumed to be minimal (so there is no optimisation with respect to the bailout strategy) and  $x = \bar{x} > 0$  in (3). The optimal control problem in [23] can be seen as a deterministic optimal control problem, because under these restrictions on the controls, one can obtain an analytic expression for the value function; this is done in [23] by making the connection with draw-down reflected Lévy processes, which is an alternative to our approach using a characterisation lemma.

The rest of the paper is structured as follows. In Section 2, after a few preliminaries on spectrally negative Lévy processes, we give an informal definition of the natural tax process with minimal bailouts in a general setting, and provide expressions for the net present value of tax and injections associated with a threshold tax rate. In Section 3.1, we state our main result about the solution of the control problem, whereas Section 3.2 covers an example which is computationally feasible and plots the optimal tax threshold and value function against various parameters. Most of the proofs are postponed to Section 4. Section 4.2 contains the rigorous definition and construction of the natural tax process with minimal bailouts. For this we need to introduce the tax-reflection transform of a càglàd path, which is done in Section 4.1. Section 4.3 consists of the verification lemma used for solving the control problem and the characterisation lemma used to prove, as well as the proof itself of, the expression for the value function associated with a natural tax process with a threshold tax rate and minimal bailouts. Finally, in Section 4.4 the main result is proved.

## 2 Natural tax processes with minimal bailouts

After some preliminary discussion on Lévy processes, we move on to describe the natural tax process with minimal bailouts, and compute a functional representing the net present value of taxation less bailout cost.

### 2.1 Preliminaries on spectrally negative Lévy processes

A stochastic process is called a *Lévy process* if it has stationary, independent increments and càdlàg paths. If additionally the paths have no positive jumps, and are not decreasing (in the weak sense), then  $X$  is called a *spectrally negative Lévy process*. Recall that  $X$  under  $\mathbb{P}_x$  is assumed to be such a process starting at  $x$ . The *Laplace exponent*  $\psi: [0, \infty) \rightarrow \mathbb{R}$  defined by  $\mathbb{E}_x[e^{sX_t}] = e^{sx+t\psi(s)}$ , has the following Lévy–Khintchine

representation:

$$\psi(s) = cs + \frac{1}{2}\sigma^2 s^2 + \int_0^\infty (e^{-s\theta} - 1 + s\theta \mathbb{1}_{\{\theta \leq 1\}}) \nu(d\theta), \quad s \geq 0.$$

Here,  $c \in \mathbb{R}$  incorporates any deterministic drift,  $\sigma \geq 0$  is the volatility of the Brownian component, and  $\nu$  is a measure on  $(0, \infty)$  satisfying  $\int_0^\infty \min\{1, \theta^2\} \nu(d\theta) < \infty$ , which represents the intensity and magnitude of the jumps. We call  $(c, \sigma, \nu)$  the *Lévy triplet* of  $X$ . If the Lévy triplet is such that  $\sigma = 0$  and  $\int_0^1 \theta \nu(d\theta) < \infty$  then  $X$  has paths of bounded variation; otherwise the paths of  $X$  have unbounded variation.

Along with the Laplace exponent, another quantity which appears occasionally is its right-inverse

$$\Phi(q) = \sup\{s \geq 0 : \psi(s) = q\}.$$

Associated with the spectrally negative Lévy process  $X$  are its so-called  $q$ -scale functions, given for any  $q \geq 0$ . We describe these in some detail here because they form part of the expression of the optimal value function. The first is a right-continuous function  $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$ , whose Laplace transform is given by

$$\int_0^\infty e^{-sx} W^{(q)}(x) dx = \frac{1}{\psi(s) - q}, \quad s > \Phi(q),$$

setting  $W^{(q)}(x) = 0$  for  $x < 0$ . With

$$\bar{W}^{(q)}(x) = \int_0^x W^{(q)}(y) dy, \quad x \geq 0,$$

the second is defined as

$$Z^{(q)}(x) = \begin{cases} 1 + q\bar{W}^{(q)}(x), & x \geq 0, \\ 1, & x < 0. \end{cases}$$

Finally, it will be notationally convenient to also define an antiderivative of  $Z$ ,

$$\bar{Z}^{(q)}(x) = \begin{cases} \int_0^x Z^{(q)}(y) dy, & x \geq 0, \\ x, & x < 0. \end{cases}$$

The scale function  $W^{(q)}$  is a strictly increasing on  $[0, \infty)$  and if  $X$  has paths of bounded variation, then  $W^{(q)}$  is continuous on  $\mathbb{R} \setminus \{0\}$ , whereas if  $X$  has paths of unbounded variation, then  $W^{(q)}$  is continuous on  $\mathbb{R}$  and continuously differentiable on  $\mathbb{R} \setminus \{0\}$ . Further, if  $q > 0$ , then

$$\lim_{x \rightarrow \infty} \frac{Z^{(q)}(x)}{W^{(q)}(x)} = \frac{q}{\Phi(q)}, \quad (4)$$

see [15, Lemma 3.3]. For more background information on spectrally negative Lévy processes and their scale functions, including their numerical computation, we refer to [16, Chapter 8] and [15].

## 2.2 Definition and functionals of natural tax processes with minimal bailouts

We now give a brief description of the process which will form the controlled process under the optimal strategy, postponing a full discussion with proofs until Sections 4.1 and 4.2. Suppose that  $\delta: [0, \infty) \rightarrow [0, 1)$  is a measurable function, such that the ordinary differential equation

$$y'(t) = 1 - \delta(y(t)), \quad t \geq 0, \quad y(0) = y_0, \quad (5)$$

has a solution for every  $y_0 \geq 0$ , in the sense that there is an absolutely continuous function  $y$  for which (5) holds for almost every  $t$ . The relevance of this differential equation arises from [2, Theorem 1]. Equation (5) has unique solutions when  $\delta$  is increasing [2, Example 2], and this will be the case for our candidate solutions. Let  $x \in \mathbb{R}$  and  $\bar{x} \geq x \vee 0$ . We work under the measure  $\mathbb{P}_{x, \bar{x}}$  given in the introduction, under which  $X$  is a spectrally negative Lévy process with  $(X_0, \bar{X}_0) = (x, \bar{x})$ . Recall also the notation (1) for running maxima, which depend on the variable  $\bar{x}$ . For such  $\delta$ , we will show in Section 4.2 that there exists a pair  $(V, K)$  of stochastic processes, with the property that, almost surely,  $V$  possesses left and right limits at every time;  $K$  is left-continuous;  $(V_0, \bar{V}_0, K_0) = (x, \bar{x}, 0)$ ;  $V_{t+} \geq 0$  for all  $t \geq 0$ ; and

$$\begin{aligned} V_t &= X_t + K_t - \int_{0+}^t \delta(V_s) d(\bar{X} + \bar{K})_s, \quad t \geq 0, \\ 0 &= \int_0^\infty \mathbb{1}_{\{V_t > 0\}} dK_{t+} \end{aligned} \quad (6)$$

Moreover,  $(V, K)$  is adapted to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ , and the bivariate process  $(V, \bar{V})$  is strong Markov in  $(\mathcal{F}_t)_{t \geq 0}$ . We call the process  $V$ , or the process  $(V, K)$ , the *natural tax process with minimal bailouts*, with the *(natural) tax rate*  $\delta$ . The interpretation of (6) is that when  $V$  is drifting at its running maximum at level  $v$ , a proportion  $\delta(v)$  is taken from the (uncontrolled) increase, whereas increases in  $K$  correspond to bailouts which cannot take place when  $V_t > 0$  and are there to ensure that the condition  $V_{t+} \geq 0$  is met.

As outlined in the introduction, the solution of the optimal control problem is such a tax process, with a specific threshold tax rate. The main result that we need on this type of process is a description of its value function. We will use this in the next section, varying the threshold, to obtain the solution of the control problem. For  $\gamma \in [0, 1)$  and  $x \geq 0$ , define

$$R_\gamma(x) = \begin{cases} \frac{\gamma}{1-\gamma} Z^{(q)}(x)^{\frac{1}{1-\gamma}} \int_x^\infty Z^{(q)}(y)^{-\frac{1}{1-\gamma}} (1 - \eta Z^{(q)}(y)) dy & \text{if } \gamma \in (0, 1), \\ 0 & \text{if } \gamma = 0, \end{cases} \quad (7)$$

noting that, when  $\gamma > 0$ , the integral above converges because, as  $q > 0$ , there exists  $s > 0$  such that  $\lim_{x \rightarrow \infty} e^{-sx} Z^{(q)}(x) = \infty$ , see (12) below.

The condition  $\int_1^\infty \theta \nu(d\theta) < \infty$  appearing in the following result is equivalent to the condition  $\mathbb{E}[X_1] = \psi'(0) > -\infty$ , and since the latter quantity appears in the value function, we need it to be finite. In the absence of this condition, the value of bailouts would be infinite.

**Proposition 2.** Assume that  $\int_1^\infty \theta \nu(d\theta) < \infty$ . Let  $(V, K)$  be the natural tax process with minimal bailouts with tax rate given by the threshold tax rate  $\delta_b$  with threshold level  $b \geq 0$ , which is defined by

$$\delta_b(z) = \alpha \mathbf{1}_{\{z \leq b\}} + \beta \mathbf{1}_{\{z > b\}}, \quad z \geq 0.$$

Then for  $(x, \bar{x}) \in \mathbb{R} \times [0, \infty)$  such that  $x \leq \bar{x}$  and for  $\eta \geq 0$ , the control  $\pi = (\delta_b(V), K)$  is admissible and

$$\begin{aligned} v^\pi(x, \bar{x}) &= \mathbb{E}_{x, \bar{x}} \left[ \int_0^\infty e^{-qs} \delta_b(V_s) d(\overline{X+K})_s - \eta \int_0^\infty e^{-qs} dK_s^+ \right] \\ &= \eta \left( \overline{Z}^{(q)}(x) + \frac{\psi'(0)}{q} \right) \\ &\quad + \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})} \left\{ R_\alpha(\bar{x}) + \left( \frac{Z^{(q)}(\bar{x})}{Z^{(q)}(\bar{x} \vee b)} \right)^{\frac{1}{1-\alpha}} (R_\beta(\bar{x} \vee b) - R_\alpha(\bar{x} \vee b)) \right\}. \end{aligned} \quad (8)$$

Note that (8) does not hold in the (excluded) case  $\alpha = \beta = 0$ , which corresponds to having no taxation at all. Proposition 2 will be proved in Section 4.3, after we have established Lemma 12, which gives a characterisation of the value function. We only mention for now that  $\delta_b$  satisfies the conditions in Theorem 6, see [2, Example 2].

### 3 Solution of the optimal control problem

#### 3.1 Main result

Before stating the main result we need one more piece of notation:

$$\begin{aligned} C(b) &= Z^{(q)}(b)^{-\frac{1}{1-\alpha}} (R_\beta(b) - R_\alpha(b)), \\ Q(b) &= Z^{(q)}(b)^{-\frac{1}{1-\alpha}} \left( \frac{Z^{(q)}(b)}{Z^{(q)'}(b)} (1 - \eta Z^{(q)}(b)) - R_\alpha(b) \right), \\ b^* &= \inf \left\{ b > 0 : R_\beta(b) < \frac{Z^{(q)}(b)}{Z^{(q)'}(b)} (1 - \eta Z^{(q)}(b)) \right\} \\ &= \inf \{ b > 0 : C(b) < Q(b) \}, \end{aligned} \quad (9)$$

where  $R_\gamma(\cdot)$  was defined in (7). Next we present our main result as well as a lemma and a follow-up remark containing some properties of  $b^*$ . The proof of the main result is postponed to Section 4.4.

**Theorem 3.** Assume that  $\int_1^\infty \theta \nu(d\theta) < \infty$  and that  $\eta \geq 1$ . Then,  $b^* < \infty$  and for each  $(x, \bar{x}) \in \mathbb{R} \times [0, \infty)$  with  $x \leq \bar{x}$  an optimal control for (3) is given by the pair  $(\delta_{b^*}(V), K)$  where  $(V, K)$  is the natural tax process with minimal bailouts and threshold tax rate  $\delta_{b^*}$ . The optimal value function  $v^*(x, \bar{x})$  is expressed by the right hand side of (8) with  $b = b^*$ .

Using the analogue of (25) for  $R_\gamma$ , we can derive the following ODE for  $C(\cdot)$ ,

$$C'(b) = \left( \frac{1}{1-\beta} - \frac{1}{1-\alpha} \right) \frac{Z^{(q)'}(b)}{Z^{(q)}(b)} (C(b) - Q(b)), \quad b > 0. \quad (10)$$

Equation (8) suggests that the threshold level that yields the highest value function amongst the threshold tax rate with minimal bailouts strategies is the one that maximises the function  $C(\cdot)$ . The next lemma shows that  $b^*$  is the unique maximiser of this function.

**Lemma 4.** *We have (i)  $b^* < \infty$ , (ii)  $C'(b) > 0$  and  $C(b) > Q(b)$  for  $b \in (0, b^*)$  and (iii)  $C'(b) < 0$  and  $C(b) < Q(b)$  for  $b \in (b^*, \infty)$ .*

*Proof.* We will first prove a monotonicity property for the function  $Q(\cdot)$  using arguments inspired by pp. 15–16 in [8]. We will need the following identity which follows from Theorem 1 in [7]:

$$\mathbb{E}_{0,0} \left[ e^{-q\hat{\tau}_a} \right] = Z^{(q)}(a) - W^{(q)}(a) \frac{qW^{(q)}(a)}{W^{(q)'}(a)} = Z^{(q)}(a) - \frac{Z^{(q)'}(a)^2}{Z^{(q)''}(a)}, \quad (11)$$

where  $\hat{\tau}_a := \inf\{t \geq 0 : \bar{X}_t - X_t > a\}$ ,  $Z^{(q)''}(y) = qW^{(q)'}(y)$  and  $W^{(q)''}(y)$  denotes the right-derivative of  $W^{(q)}(y)$ . Assume initially that  $\alpha > 0$ . We have by an integration by parts and (11),

$$\begin{aligned} Q(b) &= \int_b^\infty Z^{(q)}(y)^{-\frac{\alpha}{1-\alpha}} \left( \eta + \frac{Z^{(q)''}(y)}{Z^{(q)'}(y)^2} (1 - \eta Z^{(q)}(y)) \right) dy \\ &= -\eta \int_b^\infty Z^{(q)}(y)^{-\frac{\alpha}{1-\alpha}} \frac{Z^{(q)''}(y)}{Z^{(q)'}(y)^2} \left( \mathbb{E}_{0,0} \left[ e^{-q\hat{\tau}_y} \right] - \frac{1}{\eta} \right) dy, \end{aligned}$$

Since  $\mathbb{E}_{0,0} \left[ e^{-q\hat{\tau}_a} \right]$  is decreasing in  $a$ , it follows that there exist  $0 \leq a_1 \leq a_2 < \infty$  such that  $Q(\cdot)$  is strictly increasing on  $(0, a_1)$ , constant on  $(a_1, a_2)$  and strictly decreasing on  $(a_2, \infty)$ . If on the other hand  $\alpha = 0$ , then we have for a fixed  $a \in (0, \infty)$ ,

$$Q(b) = Q(a) - \eta \int_b^a \frac{Z^{(q)''}(y)}{Z^{(q)'}(y)^2} \left( \mathbb{E}_{0,0} \left[ e^{-q\hat{\tau}_y} \right] - \frac{1}{\eta} \right) dy$$

and so we can use the same argument as in the  $\alpha > 0$  case to deduce the same monotonicity property for  $Q(\cdot)$ .

Note that by (10),  $C'(b) > 0$  if and only if  $C(b) > Q(b)$  and  $C'(b) < 0$  if and only if  $C(b) < Q(b)$ . Using (4) together with l'Hôpital's rule in (9) yields

$$\lim_{b \rightarrow \infty} C(b) = \lim_{b \rightarrow \infty} Q(b) = \begin{cases} 0 & \text{if } \alpha > 0 \\ -\eta/\Phi(q) & \text{if } \alpha = 0. \end{cases}$$

Because  $Q(\cdot)$  is strictly decreasing on  $(a_2, \infty)$  it follows that  $C(b) < Q(b)$  for  $b > a_2$  as otherwise by (10),  $C(\cdot)$  will be from some point onwards increasing and thus larger than  $Q(\cdot)$  and then  $C(\cdot)$  and  $Q(\cdot)$  do not converge to the same limit at infinity. Consequently,

$b^* \leq a_2 < \infty$  and  $C(b) < Q(b)$  for all  $b > b^*$  since, assuming without loss of generality  $b^* < a_2$ ,  $Q(b) \geq Q(b^*)$  for  $b \in (b^*, a_2)$  and  $C'(b) < 0$  if  $C(b) < Q(b)$ . It remains to show that  $C(b) > Q(b)$  for  $b \in (0, b^*)$  which we prove by contradiction. Suppose this does not hold. Since by definition of  $b^*$ ,  $C(b) \geq Q(b)$  for  $b \in (0, b^*)$ , we must have then that there exists  $0 < b_1 < b_2 \leq b^*$  such that  $C(b) = Q(b)$  for all  $b \in [b_1, b_2]$ . Consequently, by (10),  $C'(b) = 0$  and thus  $Q'(b) = 0$  for all  $b \in (b_1, b_2)$ . Therefore,  $Q(\cdot)$  is decreasing on  $(b_1, \infty)$  by the monotonicity property of  $Q(\cdot)$ . So by (10) and recalling  $0 \leq \alpha < \beta < 1$ , we have for  $b > b_1$ ,

$$\begin{aligned} C'(b) &= C'(b) + \left( \frac{1}{1-\beta} - \frac{1}{1-\alpha} \right) \frac{Z^{(q)'}(b)}{Z^{(q)}(b)} (Q(b_1) - C(b_1)) \\ &= \left( \frac{1}{1-\beta} - \frac{1}{1-\alpha} \right) \frac{Z^{(q)'}(b)}{Z^{(q)}(b)} (C(b) - C(b_1) - (Q(b) - Q(b_1))) \\ &\geq \left( \frac{1}{1-\beta} - \frac{1}{1-\alpha} \right) \frac{Z^{(q)'}(b)}{Z^{(q)}(b)} (C(b) - C(b_1)). \end{aligned}$$

Hence by Grönwall's inequality,  $C(b) - C(b_1) \geq 0$  for all  $b > b_1$  and thus  $C(b) \geq C(b_1) = Q(b_1) \geq Q(b)$  for all  $b > b_1$  which forms a contradiction with  $b^* < \infty$ .  $\square$

*Remark 5.* From (9), Lemma 4 and its proof we can easily deduce the following properties of  $b^*$ .

- (i) If  $\eta = 1$ , then  $Q(b)$  is a strictly decreasing function on  $(0, \infty)$  and then so must  $C(b)$  be, which implies  $b^* = 0$ .
- (ii) If  $\eta > 1$  and  $X$  has paths of unbounded variation, then  $W^{(q)}(0) = 0$  and so  $\lim_{b \downarrow 0} \frac{Z^{(q)}(b)}{Z^{(q)'}(b)} (1 - \eta Z^{(q)}(b)) = -\infty$ , whereas  $R_\beta(0) \in \mathbb{R}$ , which implies  $b^* > 0$ .
- (iii) Further, if  $b^* > 0$  then it must be the unique point  $b \in (0, \infty)$  such that  $C(b) = Q(b)$  or equivalently  $R_\beta(b) = \frac{Z^{(q)}(b)}{Z^{(q)'}(b)} (1 - \eta Z^{(q)}(b))$ . We also remark that from the definition of  $b^*$  we see that it does not depend on the lower tax rate bound  $\alpha$ .

### 3.2 Example

In this section, we consider a specific example in which  $X$  is a Lévy process with Laplace exponent  $\psi$  given by

$$\psi(s) = (1 + \theta) \frac{2\lambda}{\mu} s + \frac{1}{2} \sigma^2 s^2 + \lambda \int_0^\infty (e^{-sx} - 1) \mu x e^{-\mu x} dx, \quad s \geq 0,$$

meaning that the process experiences negative jumps whose magnitude has a gamma (or Erlang) distribution with shape parameter 2 and rate  $\mu > 0$ . Jumps arrive at rate  $\lambda > 0$ , and the process also experiences Gaussian fluctuations with volatility  $\sigma > 0$ . The parameter  $\theta \in \mathbb{R}$  is the loading factor, and  $\mathbb{E}_0 X_1 = \psi'(0) = 2\theta\lambda/\mu$ . Unless otherwise mentioned, we will take  $\lambda = 1$ ,  $\mu = 2$ ,  $\theta = 0.1$  and  $\sigma = 1$  for the parameters associated with  $X$ , and  $\alpha = 0.3$ ,  $\beta = 0.6$  and  $\eta = 1.25$  for those associated with taxation.

Our goal is to investigate numerically the effect of changing the parameters  $\alpha$ ,  $\beta$ ,  $\eta$  and  $q$  on the optimal threshold  $b^*$  and the value function  $v$ . Computation of both of these requires evaluating not just the scale functions of the models in question, but also the function  $R_\gamma$  defined in (7). This function contains an integral over an unbounded domain, and established quadrature routines may struggle with it. Therefore, we first show that the integrand in question decays fast and hence show how to approximate it with a finite integral.

Taking [16, Corollary 8.9] and integrating yields the identity

$$\bar{W}^{(q)}(x) = \frac{\Phi'(q)}{\Phi(q)}(e^{\Phi(q)x} - 1) - \int_0^\infty e^{-qt} \mathbb{P}(X_t \in [-x, 0]) dt, \quad x \geq 0.$$

and bounding the probability by 0 or 1 gives a simple bound for  $\bar{W}^{(q)}(x)$  in terms of  $\Phi$  and its derivative. In particular, it implies that

$$Z^{(q)}(x) \geq q \frac{\Phi'(q)}{\Phi(q)}(e^{\Phi(q)x} - 1) \geq q \frac{\Phi'(q)}{\Phi(q)} e^{(\Phi(q)-\delta)x}, \quad x \geq \delta^{-1} \log 2, \quad (12)$$

for any  $0 < \delta < \Phi(q)$ . Applying this and similar arguments, we can obtain a bound for the integrand in the definition of  $R_\gamma$  for  $\gamma \in (0, 1)$ :

$$\begin{aligned} 0 \leq Z^{(q)}(x)^{-\frac{1}{1-\gamma}} (\eta Z^{(q)}(y) - 1) &\leq (\eta - 1) \left[ \frac{\Phi(q)}{q\Phi'(q)} e^{-(\Phi(q)-\delta)x} \right]^{\frac{1}{1-\gamma}} \\ &\quad + \eta \left[ \frac{\Phi(q)}{q\Phi'(q)} e^{-(\Phi(q)-\delta)x} \right]^{\frac{\gamma}{1-\gamma}}, \quad x \geq \delta^{-1} \log 2. \end{aligned} \quad (13)$$

Therefore, for example, if we set  $\epsilon > 0$  and select

$$M > \frac{1-\gamma}{\Phi(q)-\delta} \left[ \log(1-\gamma) - \log(\Phi(q)-\delta) + \frac{1}{1-\gamma} \log \left( \frac{\Phi(q)}{q\Phi'(q)} \right) + \log(\eta-1) - \log \epsilon \right],$$

then

$$\int_M^\infty (\eta - 1) \left[ \frac{\Phi(q)}{q\Phi'(q)} e^{-(\Phi(q)-\delta)x} \right]^{\frac{1}{1-\gamma}} dx < \epsilon,$$

provided that also  $M > \delta^{-1} \log 2$ . Similar considerations allow us to bound the integral of the second term in (13), as well as a similar integral appearing in the value function. In what follows, we have chosen  $\epsilon = 10^{-10}$  and used this approximation. On top of this, the quadrature routine that we used yielded an absolute error estimate of at most  $1.07 \times 10^{-6}$ .

Previous work on computing the value function in related models appears in [3, Section 3] and [5, p. 8], the risk model in both cases being a compound Poisson process with negative jumps having an exponential distribution. In the former case, this leads to an explicit expression for the value function in terms of hypergeometric and elementary functions; however, such a nice formula does not appear to be possible with Erlang jump distribution.

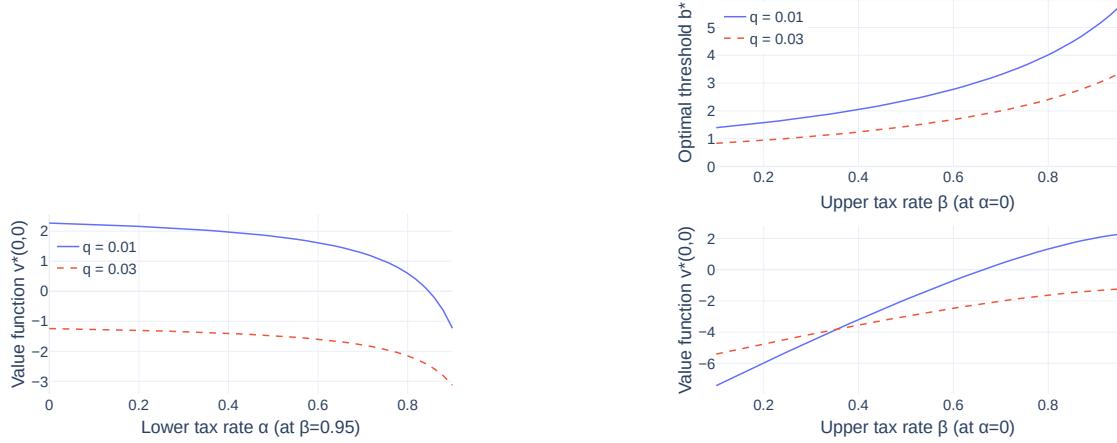


Figure 2: Effect of tax rate bounds on the optimal threshold  $b^*$  and the optimal value function at zero  $v^*(0,0)$ . The value of  $\alpha$  has no effect on  $b^*$ .

Returning to our specific example, it is readily seen that  $\psi$  is a rational function of  $s$ , which implies that the scale functions can be found by partial fraction decomposition and consist of mixtures of exponentials. However, the exact decomposition is quite complicated to write down, with both the rates and coefficients involving roots of polynomials, and for computation, we used Mathematica to compute the inverse Laplace transform required before importing the result into sage. We took a similar approach to the computation of  $\Phi(q)$ , required for our analysis above, which also involves roots of a polynomial depending on the characteristics of the process.

Figures 2 and 3 show the effect of varying the tax rates  $\alpha$  and  $\beta$ , and the effect of varying the bailout cost factor  $\eta$ , respectively. Where the optimal value function  $v^*(0,0)$  is shown, numerically  $10^{-8}$  is used for zero, since our implementations of  $v^*(x,x)$  are for the case  $x > 0$ . It is notable that both the threshold and the optimal value function at zero appear to be monotone in  $\eta$ , and that the optimal value function is not monotone in  $q$ , in that the graphs of  $v^*(0,0)$  for different values of  $q$  intersect. We also remark that the optimal value function  $v^*(x,x)$  appears to be concave, as may be expected from past work on the dividend model with capital injections [8, Lemma 3].

## 4 Proofs

### 4.1 The tax-reflection transform

In this section, we make no distributional assumptions about an underlying risk process, and instead explain rigorously how to introduce taxation and bailouts to any càdlàg path  $X = (X_t)_{t \geq 0}$ . This needs particular care when the path starts from level zero because then it is possible that taxation and bailouts are applied simultaneously. As we outlined

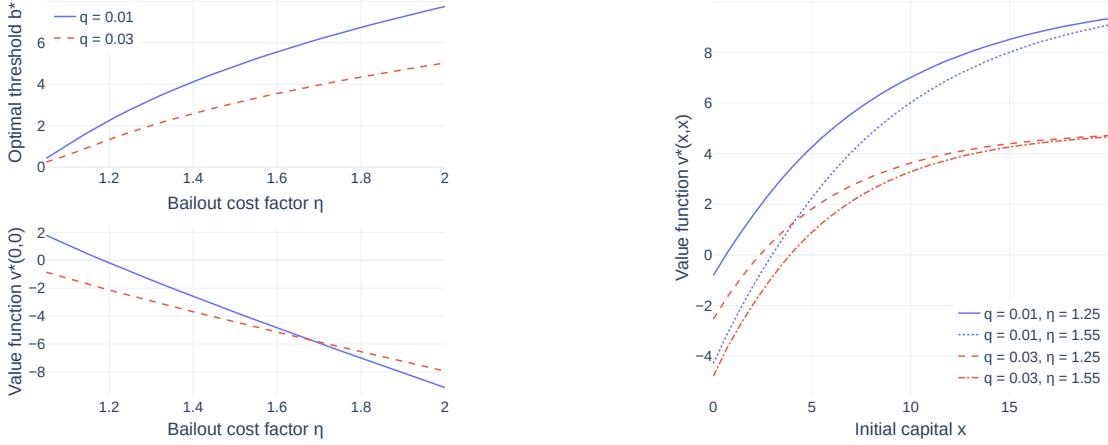


Figure 3: Effect of varying  $\eta$  on the optimal threshold  $b^*$  and the optimal value function at zero  $v^*(0,0)$ , and on the value function  $v^*(x,x)$ .

earlier, a key element here is the differential equation

$$y'(t) = 1 - \delta(y(t)), \quad t \geq 0, \quad y(0) = y_0, \quad (14)$$

where  $\delta: [0, \infty) \rightarrow [0, 1]$  is a measurable function. We call  $y$  a solution if it is absolutely continuous and satisfies (14) for almost every  $t$ . Recall once again the notation (1) in which the running maximum of paths depends on the variable  $\bar{x}$ .

**Theorem 6.** *Let  $\delta: [0, \infty) \rightarrow [0, 1]$  be a measurable function such that (14) has a unique solution for every  $y_0 \geq 0$ . Assume, in addition, that there exist some  $\epsilon > 0$  and  $\gamma \in [0, 1)$  such that  $\delta(x) = \gamma$  for  $x \in [0, \epsilon]$ . For every càdlàg path  $X$  with no positive jumps and for every  $\bar{x} \geq X_0 \vee 0$ , there exists a unique pair of càdlàg paths  $(V^+, K^+)$  with  $V^+$  positive and  $K^+$  increasing such that*

$$V_t^+ = X_t + K_t^+ - \int_{0^+}^t \delta(V_s^+) d(\overline{X + K^+})_s, \quad t \geq 0, \quad (15)$$

and the following complementarity condition is satisfied:

$$\int_0^\infty \mathbb{1}_{\{V_t^+ > 0\}} dK_t^+ = 0, \quad (16)$$

We call  $(V^+, K^+)$  the tax-reflection transform of the path  $X$  with tax rate  $\delta$  and initial maximum level  $\bar{x}$ .

*Proof.* Throughout the proof we use the notation  $\mathcal{S}(Y) = (\mathcal{S}_t(Y))_{t \geq 0}$  where  $\mathcal{S}_t(Y) = \sup_{0 \leq s \leq t} Y_s$  for a càdlàg path  $Y$  in order to distinguish it from the notation  $\overline{Y}_t$  in (1). We assume  $\bar{x} = X_0 \vee 0$  for now leaving the case  $\bar{x} > X_0 \vee 0$  until the end. We treat the cases  $X_0 > 0$  and  $X_0 \leq 0$  separately.

First assume  $\bar{x} = X_0 > 0$ . Then the unique pair of paths  $(V^+, K^+)$  satisfying the desired properties can be constructed recursively in a similar way as in the Appendix of [4]. To explain, given a càdlàg path  $Y$  with no positive jumps we define a number of transformations of  $Y$ . First, by Theorem 1(ii) and Lemma 4 in [2] and the first assumption in the theorem there exists a unique solution  $\Upsilon = \Upsilon(Y)$  to the equation

$$\Upsilon_t = Y_t - \int_{0^+}^t \delta(\Upsilon_s) d\mathcal{S}_s(Y), \quad t \geq 0.$$

Second,  $(\Psi, \Phi) = (\Psi, \Phi)(Y)$  is the reflection map defined by

$$\begin{aligned} \Psi_t &= \left( - \inf_{0 \leq s \leq t} Y_s \right) \vee 0, \\ \Phi_t &= Y_t + \Psi_t. \end{aligned}$$

Note that  $\Phi_t(Y)$  is the path  $Y$  reflected from below at 0. It is well-known (see, e.g., Section 3 of [25]), that, given  $Y$ ,  $(\Psi, \Phi)$  is the unique pair of paths with  $\Psi$  increasing and  $\Phi$  positive such that  $\Phi_t = Y_t + \Psi_t$  and the complementarity condition  $\int_0^\infty \mathbb{1}_{\{\Phi_t > 0\}} d\Psi_t = 0$  is satisfied. Using the transformations  $\Upsilon, \Psi, \Phi$  we construct a pair of paths  $(V^+, K^+)$  with  $V^+$  positive and  $K^+$  increasing and such that (15) and (16) are satisfied. It will be immediately clear from the construction that these properties are satisfied and that it is the only pair of paths satisfying these properties. With  $S_1 = \inf\{t \geq 0 : \Upsilon_t(X) < 0\}$ , we set  $(V_t^+, K_t^+) = (\Upsilon_t(X), 0)$ ,  $t \in [0, S_1]$ . If  $S_1 = \infty$  we are done. Otherwise, with  $Y_t = V_{S_1-}^+ + \Delta X_{S_1} + X_{S_1+t} - X_{S_1}$  and  $T_1 = S_1 + \inf\{t \geq 0 : \Phi_t(Y) = \bar{V}_{S_1-}^+\}$  we set  $(V_{S_1+t}^+, K_{S_1+t}^+ - K_{S_1-}^+) = (\Phi_t(Y), \Psi_t(Y))$ ,  $t \in [0, T_1 - S_1]$ . If  $T_1 = \infty$  we are done. Otherwise, with  $Y_t = V_{T_1-}^+ + X_{T_1+t} - X_{T_1}$  (note  $\Delta X_{T_1} = 0$  by lack of upward jumps) and  $S_2 = T_1 + \inf\{t \geq 0 : \Upsilon_t(Y) < 0\}$  we set  $(V_{T_1+t}^+, K_{T_1+t}^+ - K_{T_1-}^+) = (\Upsilon_t(Y), 0)$ ,  $t \in [0, S_2 - T_1]$ . In the obvious way we can continue this procedure to construct  $(V^+, K^+)$  on  $[S_2, T_2], [T_2, S_3], \dots$ , which yields the tax-reflection transform on the whole time horizon  $[0, \infty)$  provided the time points  $S_1, T_1, S_2, T_2, \dots$  do not accumulate. To see that this is the case, we argue by contradiction and assume  $T := \lim_{n \rightarrow \infty} T_n < \infty$ . Because we can identify, for  $n \geq 1$ ,  $T_n = \inf\{t > S_n : V_t^+ = \bar{V}_{S_n}^+\}$ , we have  $V_{T_n}^+ \geq V_0 = X_0 > 0$  while on the other hand  $V_{S_n}^+ = 0$  for any  $n \geq 1$ . Hence  $V^+$  does not have a right-limit at  $T$ . But (15) holds for all  $t \in [0, T)$  and so the limit  $\lim_{t \uparrow T} V_t^+$  must exist because all three terms on the right-hand side of (15) have a right-limit at  $T$  as  $X$  is càdlàg on  $[0, \infty)$  and  $K^+$  and the integral term are increasing on  $[0, T)$ . So we have reached a contradiction and we must have  $T = \infty$  and hence  $(V_t^+, K_t^+)$  is defined for all  $t \geq 0$ .

Now assume  $X_0 \leq 0$  and  $\bar{x} = 0$ . Then by (15)-(16) we must have  $K_0^+ = -X_0$  and so  $(\bar{X} + K^+)_t = \mathcal{S}_t(X + K^+)$  for all  $t \geq 0$ . Before tackling the general case we assume the special case where  $\delta(x) = \gamma$  for all  $x \geq 0$ . By properties of the reflection map,  $(V^+, K^+)$  is the tax-reflection transform of  $X$  with constant tax rate  $\gamma$  and initial maximum level 0 if and only if

$$(V^+, K^+) = \left( \Phi \left( X - \gamma \mathcal{S}(X + K^+) \right), \Psi \left( X - \gamma \mathcal{S}(X + K^+) \right) \right).$$

So if we can show there exists a unique càdlàg path  $K^+$  to the fixed point equation

$$K^+ = \Psi(X - \gamma \mathcal{S}(X + K^+)), \quad (17)$$

then existence and uniqueness of the tax-reflection transform of  $X$  with tax rate  $\gamma$  and initial maximum level 0 follows. To this end, let  $T > 0$ . We claim that for  $K_1, K_2$  two càdlàg paths,

$$\sup_{t \in [0, T]} |\mathcal{S}_t(X + K^1) - \mathcal{S}_t(X + K^2)| \leq \sup_{t \in [0, T]} |K_t^1 - K_t^2|. \quad (18)$$

Indeed, suppose  $t \in [0, T]$  is such that  $\mathcal{S}_t(X + K^1) \geq \mathcal{S}_t(X + K^2)$ . Then for any  $\epsilon > 0$  there exists  $t' \in [0, t]$  such that  $X_{t'} + K_{t'}^1 \geq \mathcal{S}_t(X + K^1) - \epsilon$  and so since  $\mathcal{S}_s(X + K^2)$  is increasing in  $s$ ,

$$\begin{aligned} |\mathcal{S}_t(X + K^1) - \mathcal{S}_t(X + K^2)| &\leq X_{t'} + K_{t'}^1 - \mathcal{S}_{t'}(X + K^2) + \epsilon \\ &\leq X_{t'} + K_{t'}^1 - (X_{t'} + K_{t'}^2) + \epsilon \\ &\leq \sup_{s \in [0, T]} |K_s^1 - K_s^2| + \epsilon. \end{aligned}$$

Since we can derive the same inequality for  $t \in [0, T]$  such that  $\mathcal{S}_t(X + K^1) \leq \mathcal{S}_t(X + K^2)$  for any given  $\epsilon > 0$ , (18) follows. Similarly we can show for two càdlàg paths  $Y^1$  and  $Y^2$

$$\sup_{t \in [0, T]} |\Psi_t(Y^1) - \Psi_t(Y^2)| \leq \sup_{t \in [0, T]} |Y_t^1 - Y_t^2|.$$

Hence for two càdlàg paths  $K^1$  and  $K^2$ ,

$$\begin{aligned} &\sup_{t \in [0, T]} |\Psi_t(X - \gamma \mathcal{S}(X + K^1)) - \Psi_t(X - \gamma \mathcal{S}(X + K^2))| \\ &\leq \sup_{t \in [0, T]} |X_t - \gamma \mathcal{S}_t(X + K^1) - (X_t - \gamma \mathcal{S}_t(X + K^2))| \leq \gamma \sup_{t \in [0, T]} |K_t^1 - K_t^2|. \end{aligned}$$

Since  $\gamma \in [0, 1)$ , we see that  $K \mapsto \Psi(X - \gamma \mathcal{S}(X + K))$  is a contraction mapping on the space of càdlàg functions on  $[0, T]$  under the uniform metric, which is a complete metric space. So by the Banach fixed point theorem there exists, for each  $T > 0$ , a unique càdlàg path  $(K_t^+)_t \in [0, T]$  such that  $K_t^+ = \Psi_t(X - \gamma \mathcal{S}(X + K^+))$  for  $t \in [0, T]$ . It follows that there exists a unique càdlàg solution to the fixed point equation (17).

When  $X_0 \leq 0$ ,  $\bar{x} = 0$  and the tax rate  $\delta$  is as in the statement of the proposition, then, with  $\epsilon$  as in the statement of the proposition,  $(V^+, K^+)$  on  $[0, T_0]$  where  $T_0 = \inf\{t \geq 0 : V_t^+ = \epsilon\}$ , is uniquely given by the tax-reflection transform of the path  $X$  with constant tax rate  $\gamma$  and initial maximum level 0 and, if  $T_0 < \infty$ , for  $t \geq 0$ ,  $(V_{T_0+t}^+, K_{T_0+t}^+ - K_{T_0-}^+)$  is uniquely given by the tax-reflection transform of the path  $s \mapsto \epsilon + X_{T_0+s} - X_{T_0}$  with tax rate  $\delta$  and initial maximum level  $\epsilon$ .

It remains to cover the case where  $\bar{x} > X_0 \vee 0$ . In this case one easily sees that  $(V^+, K^+)$  on  $[0, T_1]$  is uniquely given by the reflection map  $(\Phi(X), \Psi(X))$  with  $T_1 = \inf\{t \geq 0 : \Phi_t(X) = \bar{x}\}$  and, if  $T_1 < \infty$ , for  $t \geq 0$ ,  $(V_{T_1+t}^+, K_{T_1+t}^+ - K_{T_1-}^+)$  is uniquely given by the tax-reflection transform of the path  $s \mapsto \bar{x} + X_{T_1+s} - X_{T_1}$  with tax rate  $\delta$  and initial maximum level  $\bar{x}$ .  $\square$

*Remark 7.* The case where  $\delta(x) = \gamma$  for all  $x \geq 0$  with  $\gamma < 1$  and the path  $X$  is continuous with  $X_0 = 0$  in Theorem 6 has been well-studied in the literature. Namely, Le Gall and Yor [18]<sup>1</sup> remarked at the end of their paper that existence and uniqueness of the tax-reflection transform can be obtained via a fixed point argument (as we did above) in the case  $\gamma \in (-1, 1)$ . Further, Davis [10] (see also [11]) shows that existence and uniqueness also hold in the more difficult case where  $\gamma = -1$  and that existence holds but uniqueness fails when  $\gamma < -1$ . Despite that uniqueness does not hold in general when  $\gamma < -1$ , Davis [12] and Chaumont and Doney [9] were able to show that for  $\gamma < -1$  there still exists a unique solution (which is moreover adapted) to (15)-(16) if  $X$  is a Brownian motion. (Reflected) Brownian motions that are modified in this way when they reach a new maximum are known in the literature as perturbed Brownian motions.

## 4.2 Construction of the natural tax process with bailouts

In this section, we give a rigorous construction of the natural tax process with minimal bailouts  $(V, K)$ , which we described briefly in Section 2.2. This is based on the tax-reflection transform (Theorem 6) of a spectrally negative Lévy process  $X$ . Rather than construct it directly, we build a right-continuous version  $(V^+, K^+)$  and define  $(V, K)$  in terms of this.

**Definition 8.** Assume  $\delta : [0, \infty) \rightarrow [0, 1)$  satisfies the conditions of Theorem 6. Let  $(V^+, K^+)$  be the pair of stochastic processes given by the tax-reflection transform of the spectrally negative Lévy process  $X$  with  $X_0 = x$ , tax rate  $\delta$  and initial maximum level  $\bar{x} \geq x \vee 0$  as defined in Theorem 6. Then let  $K_0 = 0$  and  $K_t = K_{t-}^+$  for  $t > 0$ , to yield a left-continuous process  $K$ , and define  $V_0 = V_0^+ - K_0^+$  and  $V_t = V_t^+ - \Delta K_t^+$  for  $t > 0$ . We call both  $V$  and the pair  $(V, K)$  the *natural tax process with minimal bailouts and (natural) tax rate  $\delta$*  (under the measure  $\mathbb{P}_{x, \bar{x}}$ ).

The definition of  $(V, K)$  implies

$$V_t = X_t + K_t - \int_{0^+}^t \delta(V_s) d(\overline{X + K})_s, \quad t \geq 0. \quad (19)$$

Indeed, if  $t > 0$  is such that  $\Delta K_t^+ > 0$ , then  $-\Delta X_t \geq \Delta K_t^+$  by (15)-(16). Similarly, if  $K_0^+ > 0$ , then  $-X_0 \geq K_0^+$ . Therefore,  $(\overline{X + K})_s = (\overline{X + K^+})_s$  for all  $s \geq 0$  and the (countable) set  $\{s \geq 0 : V_s \neq V_s^+\}$  is a null set for the Lebesgue-Stieltjes measure  $d(\overline{X + K^+})_s$ . Hence (19) follows from (15).

**Proposition 9.** (i)  $(V, K)$  is adapted to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ .

(ii) The bivariate process  $(V, \overline{V})$ , with associated probability measures  $\mathbb{P}_{x, \bar{x}}$ , is strong Markov in  $(\mathcal{F}_t)_{t \geq 0}$ .

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<sup>1</sup>In [18]  $X$  is considered to be a Brownian motion, but their argument equally applies to a general continuous path  $X$ .

*Proof.* (i). We will begin by showing that  $(V^+, K^+)$ , as defined in Definition 8, is adapted to  $(\mathcal{F}_t)_{t \geq 0}$ . Considering the construction given in the proof of Theorem 6, if  $Y$  is an adapted stochastic process, then the processes  $\mathcal{S}(Y)$ ,  $\Upsilon(Y)$ ,  $\Phi(Y)$  and  $\Psi(Y)$  defined there are adapted; for the process  $\Upsilon(Y)$ , this can be shown using Theorem 1(ii) (in particular Equation (6)) and Lemma 4 in [2]. Furthermore, the random times  $S_1, T_1, S_2, T_2, \dots$  are stopping times with respect to  $(\mathcal{F}_t)_{t \geq 0}$ . This gives an immediate proof in the case that  $x = \bar{x} > 0$ . In the case where  $x \leq 0$ ,  $\bar{x} = 0$  and  $\delta(y) = \gamma$  for  $y \geq 0$  and  $\gamma \in [0, 1]$ , we analyse the proof of Theorem 6 in more detail. Let  $\Xi(L) = \Psi(X - \gamma\mathcal{S}(X + L))$ . We have already shown that this is a contraction mapping, and evidently it maps adapted paths to adapted paths. The fixed point of  $\Xi$  can be obtained by Picard iteration. Let  $L_t^{(0)} = 0$  for all  $t \geq 0$ , and define  $L^{(n)} = \Xi(L^{(n-1)})$  for  $n \geq 1$ . Then  $K^+$  is the limit of  $L^{(n)}$  as  $n \rightarrow \infty$ , and this limit holds uniformly on  $[0, T]$  for any  $T > 0$ . It follows that  $K^+$  is adapted, and that the same holds for  $V^+ = \Phi(X - \gamma\mathcal{S}(X + K^+))$ . Finally, the remaining cases ( $x \leq 0$ ,  $\bar{x} = 0$  and more general  $\delta$ ; and  $\bar{x} > x \vee 0$ ) can be resolved as in the proof of Theorem 6.

We have shown that  $(V^+, K^+)$  is adapted to  $(\mathcal{F}_t)_{t \geq 0}$ , and we turn now to  $(V, K)$ . Fix  $t > 0$ . Since  $K_t = \lim_{s \uparrow t} K_s^+$ , it is  $\mathcal{F}_t$ -measurable, and so  $K$  is adapted to  $(\mathcal{F}_t)_{t \geq 0}$ . Finally, the process  $V$  satisfies (19),  $X$  and  $K$  are adapted to  $(\mathcal{F}_t)_{t \geq 0}$ , and so is the integral  $t \mapsto \int_{0^+}^t \delta(V_s) d(\overline{X + K})_s$ , since it is continuous and adapted to  $(\mathcal{F}_t)_{t \geq 0}$ . It follows that  $V$  is adapted to  $(\mathcal{F}_t)_{t \geq 0}$ , and we are done.

(ii). Let  $T \geq 0$  be a finite stopping time with respect to  $(\mathcal{F}_t)_{t \geq 0}$ , and let  $(\tilde{V}^+, \tilde{K}^+)$  be the tax-reflection transform of

$$\tilde{X}_t := V_T + X_{T+t} - X_T, \quad t \geq 0,$$

with initial maximum level  $\overline{V}_T$ . Now define

$$\hat{V}_t^+ = \begin{cases} V_t^+, & 0 \leq t < T, \\ \tilde{V}_{t-T}^+, & t \geq T, \end{cases} \quad \hat{K}_t^+ = \begin{cases} K_t^+, & 0 \leq t < T, \\ K_{T-}^+ + \tilde{K}_{t-T}^+, & t \geq T, \end{cases}$$

where  $(V^+, K^+)$  is as in Definition 8. Since  $(\hat{V}^+, \hat{K}^+)$  is a tax-reflection transform of  $X$  with initial maximum level  $\bar{x}$ , and such a transform is unique, it follows that  $(V^+, K^+) = (\hat{V}^+, \hat{K}^+)$ . Consequently, with  $(\tilde{V}_0, \tilde{K}_0) := (\tilde{V}_0^+ - \tilde{K}_0^+, 0)$  and  $(\tilde{V}_t, \tilde{K}_t) := (\tilde{V}_t^+ - \Delta \tilde{K}_t^+, \tilde{K}_t^+)$  for  $t > 0$ , we have  $(V_{T+t})_{t \geq 0} = \tilde{V}$  and by Definition 8 and (19),

$$\tilde{V}_t = \tilde{X}_t + \tilde{K}_t - \int_{0^+}^t \delta(\tilde{V}_s) d(\overline{X + K})_s, \quad t \geq 0.$$

The homogeneous strong Markov property of Lévy processes [16, Theorem 3.1] means that, given  $V_T, \tilde{X}$  has the law of  $X$  under  $\mathbb{P}_{V_T, \overline{V}_T}$  and is conditionally independent of  $\mathcal{F}_T$ . It follows that, given  $(V_T, \overline{V}_T)$ , the pair  $(V_{T+t}, \overline{V}_{T+t})_{t \geq 0} = (\tilde{V}_t, \sup_{0 \leq s \leq t} \tilde{V}_s \vee \overline{V}_T)_{t \geq 0}$  is conditionally independent of  $\mathcal{F}_T$  and has the law of  $(V, \overline{V})$  under  $\mathbb{P}_{V_T, \overline{V}_T}$ . This establishes the strong Markov property of  $(V, \overline{V})$ .  $\square$

### 4.3 Verification lemma and proof of Proposition 2

We state the following result, a generalisation of Lemma 2.1 in [17] and Lemma 4 in [2], whose proof is however identical.

**Lemma 10.** *Fix  $\bar{x} \geq 0$  and let  $Y = (Y_t)_{t \geq 0}$  be a measurable real-valued path such that  $\bar{Y}$  is continuous. Let  $H = (H_t)_{t \geq 0}$  be a measurable path such that  $H_t \leq 1$  for all  $t \geq 0$ . Then the path  $U = (U_t)_{t \geq 0}$  defined by  $U_t = Y_t - \int_{0+}^t H_s d\bar{Y}_s$ , satisfies,  $\bar{U}_t = \bar{Y}_t - \int_{0+}^t H_s d\bar{Y}_s$ ,  $t \geq 0$ .*

Our approach is based on the following verification lemma for the control problem (3). In order to state the conditions for this, we define the operator

$$\mathcal{A}_q g(x) = \frac{1}{2} \sigma^2 g''(x) + c g'(x) + \int_0^\infty \left( g(x - \theta) - g(x) + g'(x) \theta \mathbb{1}_{\{\theta \leq 1\}} \right) \nu(d\theta) - q g(x),$$

with the understanding that the first term on the right hand side disappears when  $\sigma = 0$ , whenever  $g$  has sufficient regularity and integrability such that the right-hand side is well-defined and the integral against  $\nu$  converges absolutely.

The lemma involves a candidate optimal value function  $v(x, \bar{x})$ , where  $x$  and  $\bar{x}$  have the role of the initial capital and initial maximum level, respectively. We will write  $\partial_x v$  for the derivative of  $v$  in the first variable, and  $\partial_{\bar{x}} v$  for the derivative in the second variable.

**Lemma 11** (Verification lemma). *Assume that  $\int_1^\infty \theta \nu(d\theta) < \infty$  and  $\eta \geq 0$ . Let  $v: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  be a function of the form  $v(x, \bar{x}) = \sum_{i=1}^p g_i(x) h_i(\bar{x})$  for some  $p \geq 1$ , such that, for each  $i = 1, \dots, p$ :*

- (a)  *$g_i$  is continuous on  $\mathbb{R}$  and continuously differentiable on  $\mathbb{R} \setminus \{0\}$  with  $g'_i(0) := \lim_{x \downarrow 0} g'_i(x)$  well-defined and real. If  $X$  has paths of unbounded variation then, in addition,  $g_i$  is continuously differentiable on  $\mathbb{R}$  and twice continuously differentiable on  $(0, \infty)$ .*
- (b)  *$h_i$  is absolutely continuous on  $[0, \infty)$  with a locally bounded, left-continuous density denoted by  $h'_i$ .*

*If  $v$  satisfies in addition the conditions*

- (i)  $v(x, \bar{x}) = \eta x + v(0, \bar{x})$  for all  $x < 0$  and  $\bar{x} \geq 0$ ,
- (ii)  $\mathcal{A}_q v(x, \bar{x}) := \mathcal{A}_q[v(\cdot, \bar{x})](x) = 0$  for all  $x > 0$  and  $\bar{x} \geq 0$ ,
- (iii)  $\gamma \partial_x v(\bar{x}, \bar{x}) + (\gamma - 1) \partial_{\bar{x}} v(\bar{x}, \bar{x}) \geq \gamma$  for all  $\gamma \in [\alpha, \beta]$  and all  $\bar{x} \geq 0$ ,
- (iv)  $v(x, \bar{x})$  is bounded on the set  $\{x \geq 0, \bar{x} \geq 0 : x \leq \bar{x}\}$ ,
- (v)  $\partial_x v(x, \bar{x}) \leq \eta$  for all  $0 \leq x \leq \bar{x}$ ,

*then  $v(x, \bar{x}) \geq v^*(x, \bar{x})$  for all  $(x, \bar{x}) \in \mathbb{R} \times [0, \infty)$  such that  $x \leq \bar{x}$ .*

*Proof.* Before starting, we note that condition (ii) makes sense. This is true because, for a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the smoothness properties in part (a) above and such that, for some  $a, b \in \mathbb{R}$ ,  $g(y) = ay + b$  for  $y \leq 0$ , the function  $\mathcal{A}_q g : (0, \infty) \rightarrow \mathbb{R}$  is well-defined. This uses the assumption that  $\int_1^\infty \theta \nu(d\theta) < \infty$ .

Fix  $(x, \bar{x}) \in \mathbb{R} \times [0, \infty)$  such that  $x \leq \bar{x}$ . We work on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t^x)_{t \geq 0}, \mathbb{P}_{x, \bar{x}})$  where  $\mathcal{F}_t^x$  is the completion of  $\mathcal{F}_t$  under  $\mathbb{P}_x$  so that the filtration  $(\mathcal{F}_t^x)_{t \geq 0}$  satisfies the usual conditions. Fix  $\pi := (H, L)$  to be an admissible strategy, and write  $U$  for the controlled process. We let  $U^+$  and  $L^+$  to be the right-continuous modifications of  $U$  and  $L$ , respectively. Note that  $U_t^+ \geq 0$  for all  $t \geq 0$  by Definition 1(vi), and  $\bar{U}^+ := \overline{U^+}$  has continuous sample paths by Lemma 10 and Definition 1(iv). Fix  $\epsilon > 0$  and introduce the function  $v_\epsilon : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  defined by  $v_\epsilon(y, \bar{y}) = v(y + \epsilon, \bar{y} + \epsilon)$ . We further denote  $\partial_x v_\epsilon(y, \bar{y}) := \partial_x v(y + \epsilon, \bar{y} + \epsilon)$  and  $\partial_{\bar{x}} v_\epsilon(y, \bar{y}) := \partial_{\bar{x}} v(y + \epsilon, \bar{y} + \epsilon)$ . By the regularity assumptions, we can, for each  $i = 1, \dots, p$ , apply the Meyer-Itô formula (Theorem IV.70 and Corollary 1 in [20]) or the change of variables formula (see Theorem II.31 in [20])<sup>2</sup> for  $g_i(U^+ + \epsilon)$  depending on whether  $X$  has paths of unbounded or bounded variation and we can use the change of variables formula for  $h_i(\bar{U}^+ + \epsilon)$ . Applying this together with the integration by parts formula (see p.68, Theorems II.26 and II.28 in [20]) while noting that  $h_i(\bar{U}^+ + \epsilon)$  has continuous sample paths of bounded variation, we get, for  $t \geq 0$ ,

$$\begin{aligned}
& e^{-qt} v_\epsilon(U_t^+, \bar{U}_t^+) - v_\epsilon(U_0^+, \bar{U}_0^+) \\
&= - \int_{0^+}^t q e^{-qs} v_\epsilon(U_{s-}^+, \bar{U}_s^+) ds \\
&\quad + \int_{0^+}^t e^{-qs} \sum_{i=1}^p \left( g_i(U_{s-}^+ + \epsilon) dh_i(\bar{U}_s^+ + \epsilon) + h_i(\bar{U}_{s-}^+ + \epsilon) dg_i(U_s^+ + \epsilon) \right) \\
&= - \int_{0^+}^t q e^{-qs} v_\epsilon(U_{s-}^+, \bar{U}_s^+) ds + \int_{0^+}^t e^{-qs} \partial_x v_\epsilon(U_{s-}^+, \bar{U}_s^+) dU_s^+ \\
&\quad + \int_{0^+}^t e^{-qs} \partial_{\bar{x}} v_\epsilon(U_{s-}^+, \bar{U}_s^+) d\bar{U}_s^+ + \frac{1}{2} \int_{0^+}^t e^{-qs} \partial_{xx} v_\epsilon(U_{s-}^+, \bar{U}_s^+) d[U^+]_s^c \\
&\quad + \sum_{0 < s \leq t} e^{-qs} \left[ v_\epsilon(U_s^+, \bar{U}_s^+) - v_\epsilon(U_{s-}^+, \bar{U}_s^+) - \partial_x v_\epsilon(U_{s-}^+, \bar{U}_s^+) \Delta U_s^+ \right],
\end{aligned}$$

where  $\partial_{xx} v(y, \bar{y}) := \sum_{i=1}^p g_i''(y + \epsilon) h_i(\bar{y} + \epsilon)$  and, for any process  $Y$ ,  $[Y]^c$  stands for the continuous part of the quadratic variation process of  $Y$ . Since  $[U^+]_s^c = [X]_s^c = \sigma^2 s$  and

$$\begin{aligned}
v_\epsilon(U_s^+, \bar{U}_s^+) - v_\epsilon(U_{s-}^+, \bar{U}_s^+) &= v_\epsilon(U_{s-}^+ + \Delta X_s + \Delta L_s^+, \bar{U}_s^+) - v_\epsilon(U_{s-}^+, \bar{U}_s^+) \\
&= v_\epsilon(U_{s-}^+ + \Delta X_s, \bar{U}_s^+) - v_\epsilon(U_{s-}^+, \bar{U}_s^+) \\
&\quad + v_\epsilon(U_{s-}^+ + \Delta X_s + \Delta L_s^+, \bar{U}_s^+) - v_\epsilon(U_{s-}^+ + \Delta X_s, \bar{U}_s^+),
\end{aligned}$$

we get recalling the definition of the operator  $\mathcal{A}_q$ , (2) and using that  $U_0^+ = U_0 + L_0^+$ ,

<sup>2</sup>Note that the assumption of continuous differentiability in Theorem II.31 in [20] can be relaxed to absolute continuity with a locally bounded density.

$\bar{U}_0^+ = \bar{U}_0$  and  $\bar{U}_t^+ = (\bar{X} + \bar{L}^+)_0 + \int_{0^+}^t (1 - H_s) d(\bar{X} + \bar{L}^+)_s$  from Lemma 10,

$$\begin{aligned}
 & e^{-qt} v_\epsilon(U_t^+, \bar{U}_t^+) - v_\epsilon(U_0, \bar{U}_0) \\
 &= M_t + \int_{0^+}^t e^{-qs} \mathcal{A}_q v_\epsilon(U_{s-}^+, \bar{U}_s^+) ds + \int_{0^+}^t e^{-qs} \partial_x v_\epsilon(U_{s-}^+, \bar{U}_s^+) d(L_s^+)^c \\
 &+ \sum_{0 \leq s \leq t} e^{-qs} \left\{ v_\epsilon(U_{s-}^+ + \Delta X_s + \Delta L_s^+, \bar{U}_s^+) - v_\epsilon(U_{s-}^+ + \Delta X_s, \bar{U}_s^+) \right\} \\
 &- \int_{0^+}^t e^{-qs} \left[ \partial_x v_\epsilon(U_{s-}^+, \bar{U}_s^+) H_s - \partial_{\bar{x}} v_\epsilon(U_{s-}^+, \bar{U}_s^+) (1 - H_s) \right] d(\bar{X} + \bar{L}^+)_s, \quad (20)
 \end{aligned}$$

with the understanding that  $U_{0-}^+ := U_0$ ,  $\Delta X_0 := X_0$  and  $\Delta L_0^+ := L_0^+$  and where  $(L_t^+)^c = L_t^+ - \sum_{0 < s \leq t} \Delta L_s^+$  and  $M_t$  is the sum of  $M_t^{(1)}$  and  $M_t^{(2)}$  given by

$$\begin{aligned}
 M_t^{(1)} &= \int_{0^+}^t e^{-qs} \partial_x v_\epsilon(U_{s-}^+, \bar{U}_s^+) d \left[ X_s - cs - \sum_{0 < u \leq s} \Delta X_u \mathbf{1}_{\{|\Delta X_u| > 1\}} \right], \\
 M_t^{(2)} &= \sum_{0 < s \leq t} e^{-qs} \left\{ v_\epsilon(U_{s-}^+ + \Delta X_s, \bar{U}_s^+) - v_\epsilon(U_{s-}^+, \bar{U}_s^+) - \partial_x v_\epsilon(U_{s-}^+, \bar{U}_s^+) \Delta X_s \mathbf{1}_{\{|\Delta X_s| \leq 1\}} \right\} \\
 &- \int_{0^+}^t \int_{0^+}^\infty e^{-qs} \left\{ v_\epsilon(U_{s-}^+ - \theta, \bar{U}_s^+) - v_\epsilon(U_{s-}^+, \bar{U}_s^+) + \theta \partial_x v_\epsilon(U_{s-}^+, \bar{U}_s^+) \mathbf{1}_{\{\theta \leq 1\}} \right\} \nu(d\theta) ds.
 \end{aligned}$$

Note that  $M^{(1)}$  is a local martingale by the Lévy-Itô decomposition (see e.g. [16, Chapter 2]) and  $M^{(2)}$  is a local martingale as it is an integral of a predictable process against a compensated Poisson random measure, see Theorem II.1.8 and Proposition II.1.21 in [14]. By the regularity assumptions and conditions (i) and (v),  $y \mapsto v_\epsilon(y, \bar{y})$  is absolutely continuous on  $(-\infty, \bar{y})$  with density  $\partial_x v(y, \bar{y})$  bounded by  $\eta$ , which means in particular

$$v_\epsilon(U_{s-}^+ + \Delta X_s + \Delta L_s^+, \bar{U}_s^+) - v_\epsilon(U_{s-}^+ + \Delta X_s, \bar{U}_s^+) \leq \eta \Delta L_s^+,$$

where we note that  $U_{s-}^+ + \Delta X_s + \Delta L_s^+ \leq \bar{U}_s^+$  since otherwise  $\bar{U}_s^+$  is not continuous in  $s$ . Using this in combination with conditions (ii) (which implies  $\mathcal{A}_q v_\epsilon(y, \bar{y}) = 0$  for all  $0 \leq y \leq \bar{y}$  as  $\epsilon > 0$ ) and (iii) while noting that  $\{s > 0 : U_{s-}^+ < \bar{U}_s^+\}$  is a null set for the Lebesgue-Stieltjes measure  $d(\bar{X} + \bar{L}^+)_s$  by Lemma 10, we get from (20), for all  $t \geq 0$ ,

$$e^{-qt} v_\epsilon(U_t^+, \bar{U}_t^+) \leq v_\epsilon(U_0, \bar{U}_0) + M_t + \eta \int_0^t e^{-qs} dL_s^+ - \int_{0^+}^t e^{-qs} H_s d(\bar{X} + \bar{L}^+)_s. \quad (21)$$

Let  $(T_n)_{n \geq 1}$  be a localising sequence for the local martingale  $M$ . Then  $\mathbb{E}_{x, \bar{x}}[M_{t \wedge T_n}] = \mathbb{E}_{x, \bar{x}}[M_0] = 0$  and so,

$$\begin{aligned}
 v_\epsilon(x, \bar{x}) &\geq \mathbb{E}_{x, \bar{x}} \left[ \int_{0^+}^{t \wedge T_n} e^{-qs} H_s d(\bar{X} + \bar{L}^+)_s \right] - \mathbb{E}_{x, \bar{x}} \left[ \eta \int_0^{t \wedge T_n} e^{-qs} dL_s^+ \right] \\
 &+ \mathbb{E}_{x, \bar{x}} \left[ e^{-q(t \wedge T_n)} v_\epsilon(U_{t \wedge T_n}^+, \bar{U}_{t \wedge T_n}^+) \right].
 \end{aligned}$$

Letting  $t \rightarrow \infty$  and  $n \rightarrow \infty$  we have by the monotone convergence theorem and the dominated convergence theorem in combination with condition (iv),  $U_t^+ \geq 0$  and Definition 1(v),

$$v_\epsilon(x, \bar{x}) \geq \mathbb{E}_{x, \bar{x}} \left[ \int_{0^+}^{\infty} e^{-qs} H_s d(\overline{X + L^+})_s \right] - \mathbb{E}_{x, \bar{x}} \left[ \eta \int_0^{\infty} e^{-qs} dL_s^+ \right] = v^\pi(x, \bar{x}).$$

Taking  $\epsilon \downarrow 0$  and using the joint continuity of  $v(\cdot, \cdot)$  (which is implied by continuity of  $g_i$  and  $h_i$ ) yields  $v(x, \bar{x}) \geq v^\pi(x, \bar{x})$ . Since  $\pi$  and  $(x, \bar{x})$  were chosen arbitrarily the lemma follows.  $\square$

**Lemma 12** (Characterisation of the value function). *Assume that  $\int_1^\infty \theta \nu(d\theta) < \infty$  and  $b, \eta \geq 0$ . Let  $a > b$ . Let  $v_a: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  be a function satisfying the conditions of the function  $v(\cdot, \cdot)$  in Lemma 11 except that conditions (iii)-(v) are replaced by the following:*

(iii')

$$\begin{aligned} \alpha \partial_x v_a(\bar{x}, \bar{x}) + (\alpha - 1) \partial_{\bar{x}} v_a(\bar{x}, \bar{x}) &= \alpha, & \bar{x} \in [0, b], \\ \beta \partial_x v_a(\bar{x}, \bar{x}) + (\beta - 1) \partial_{\bar{x}} v_a(\bar{x}, \bar{x}) &= \beta, & \bar{x} \in (b, a], \end{aligned}$$

(iv')  $v_a(a, a) = 0$ .

Then for any  $\bar{x} \in [0, a]$  and  $x \leq \bar{x}$ ,

$$v_a(x, \bar{x}) = \mathbb{E}_{x, \bar{x}} \left[ \int_{0^+}^{\tau_a^+} e^{-qs} \delta_b(V_s) d(\overline{X + K})_s - \eta \int_0^{\tau_a^+} e^{-qs} dK_s^+ \right],$$

where  $(V, K)$  is the natural tax process with minimal bailouts and threshold tax rate  $\delta_b$ , and  $\tau_a^+ = \inf\{t \geq 0 : V_t \geq a\}$ .

*Proof of Lemma 12 (characterisation of the value function).* Fix  $a > b$ . We fix  $\epsilon > 0$  and let  $v_{a, \epsilon}: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  be the function defined by  $v_{a, \epsilon}(x, \bar{x}) = v_a(x + \epsilon, \bar{x})$ , though we purposely do not also shift the second argument by  $\epsilon$  as we did in the proof of Lemma 11. Because  $\pi = (\delta_b(V), K)$  satisfies the conditions in Definition 1 except (possibly) for item (v) (recall Definition 8 and adaptedness follows from Proposition 9(i)) and  $v_a$  satisfies the regularity conditions of  $v$  in Lemma 11, we deduce, by following the proof of Lemma 11, the identity (20) but with  $U$  replaced by  $V$ ,  $L$  replaced by  $K$ ,  $v_\epsilon$  by  $v_{a, \epsilon}$  and  $H$  by  $\delta_b(V)$ . Then replacing  $t$  by  $\tau_a^+ \wedge t \wedge T_n$  with  $(T_n)_{n \geq 1}$  a localising sequence for the local martingale  $(M_t)_{t \geq 0}$  and taking expectations as well as using condition (ii) in

Lemma 11, we get, for  $\bar{x} \geq 0$  and  $x \leq \bar{x}$ ,

$$\begin{aligned}
v_{a,\epsilon}(x, \bar{x}) &= \mathbb{E}_{x, \bar{x}} \left[ \int_{0^+}^{\tau_a^+ \wedge t \wedge T_n} e^{-qs} \right. \\
&\quad \times \left. \left[ \partial_x v_{a,\epsilon}(V_{s-}^+, \bar{V}_s^+) \delta_b(V_s) - \partial_{\bar{x}} v_{a,\epsilon}(V_{s-}^+, \bar{V}_s^+) (1 - \delta_b(V_s)) \right] d(\bar{X} + \bar{K}^+)_s \right. \\
&\quad - \sum_{0 \leq s \leq \tau_a^+ \wedge t \wedge T_n} e^{-qs} \left\{ v_{a,\epsilon}(V_{s-}^+ + \Delta X_s + \Delta K_s^+, \bar{V}_s^+) - v_{a,\epsilon}(V_{s-}^+ + \Delta X_s, \bar{V}_s^+) \right\} \\
&\quad \left. - \int_{0^+}^{\tau_a^+ \wedge t \wedge T_n} e^{-qs} \partial_x v_{a,\epsilon}(V_{s-}^+, \bar{V}_s^+) d(K_s^+)^c + e^{-q(\tau_a^+ \wedge t \wedge T_n)} v_{a,\epsilon}(V_{\tau_a^+ \wedge t \wedge T_n}^+, \bar{V}_{\tau_a^+ \wedge t \wedge T_n}^+) \right]. \tag{22}
\end{aligned}$$

By the smoothness assumptions,  $v_a(\cdot, \cdot)$ ,  $\partial_x v_a(\cdot, \cdot)$  and  $\partial_{\bar{x}} v_a(\cdot, \cdot)$  are all bounded on  $[0, a+1] \times [0, a]$  and right-continuous in the first argument. By the mean value theorem, for all  $\epsilon \in (0, 1)$ ,

$$\begin{aligned}
\sum_{0 \leq s \leq \tau_a^+ \wedge t \wedge T_n} e^{-qs} &\left| v_{a,\epsilon}(V_{s-}^+ + \Delta X_s + \Delta K_s^+, \bar{V}_s^+) - v_{a,\epsilon}(V_{s-}^+ + \Delta X_s, \bar{V}_s^+) \right| \\
&\leq \sup_{y \in [0, a+1], z \in [0, a]} \partial_x v_a(y, z) \sum_{0 \leq s \leq \tau_a^+ \wedge t \wedge T_n} e^{-qs} \Delta K_s^+.
\end{aligned}$$

Further,  $a \geq \bar{V}_{t \wedge \tau_a^+}^+ \geq (1 - \beta)(\bar{X} + \bar{K}^+)_t \geq (1 - \beta)K_{t \wedge \tau_a^+}^+$  where the second inequality is due to Lemma 10. From these observations it is justified to apply the dominated convergence and take  $\epsilon \downarrow 0$  in (22) to conclude that (22) holds with  $\epsilon = 0$ . By definition of  $(V^+, K^+)$ ,  $V_{s-}^+ + \Delta X_s + \Delta K_s^+ = 0$  if  $\Delta K_s^+ > 0$ . Hence  $v_a(V_{s-}^+ + \Delta X_s + \Delta K_s^+, \bar{V}_s^+) - v_a(V_{s-}^+ + \Delta X_s, \bar{V}_s^+) = \eta \Delta K_s^+$  by condition (i) of Lemma 11. Further,  $\{s > 0 : V_{s-}^+ > 0\}$  is a null set for the Lebesgue-Stieltjes measure  $d(K_s^+)^c$ . To see this, note that  $\{s > 0 : V_{s-}^+ > 0\} \subset \{s > 0 : V_s^+ > 0\} \cup \{s > 0 : \Delta V_s^+ > 0\}$  and  $\{s > 0 : V_s^+ > 0\}$  is a null set for  $dK_s^+$  by (16) and  $\{s > 0 : \Delta V_s^+ > 0\}$  is countable (because a càdlàg path has a countable number of jumps) and so is a null set for the continuous measure  $d(K_s^+)^c$ . We also note that  $\partial_x v_a(0, \bar{x}) = \eta$  if the spectrally negative Lévy process  $X$  has paths of unbounded variation by continuity of  $\partial_x v_a$  in that case, whereas  $(K_s^+)^c = 0$  for all  $s \geq 0$  if  $X$  has paths of bounded variation. Using these observations in combination with (22) for  $\epsilon = 0$  and (iii') (recall from the proof of Lemma 11 that  $\{s > 0 : V_{s-}^+ < \bar{V}_s^+\}$  is a null set for the measure  $d(\bar{X} + \bar{K}^+)_s$ ) yields

$$\begin{aligned}
v_a(x, \bar{x}) &= \mathbb{E}_{x, \bar{x}} \left[ \int_{0^+}^{\tau_a^+ \wedge t \wedge T_n} e^{-qs} \delta_b(V_s) d(\bar{X} + \bar{K}^+)_s - \eta \int_0^{\tau_a^+ \wedge t \wedge T_n} e^{-qs} dK_s^+ \right. \\
&\quad \left. + e^{-q(\tau_a^+ \wedge t \wedge T_n)} v_a(V_{\tau_a^+ \wedge t \wedge T_n}^+, \bar{V}_{\tau_a^+ \wedge t \wedge T_n}^+) \right].
\end{aligned}$$

With  $(\Psi, \Phi)$  the reflection map from the proof of Theorem 6,

$$\bar{V}_t^+ \geq (1 - \beta)(\bar{X} + \bar{K}^+)_t \geq (1 - \beta)(\bar{X} + \Psi_t(\bar{X})) = (1 - \beta)\bar{\Phi}_t(\bar{X}), \quad t \geq 0,$$

where the second inequality is due to  $K_t^+ \geq \Psi_t(\bar{X})$  since  $X_t \geq V_t^+$  and the equality is by definition of the reflection map. Because  $\bar{\Phi}_t(\bar{X}) \rightarrow \infty$  as  $t \rightarrow \infty$ , it follows that  $\tau_a^+ < \infty$ . Hence  $\tau_a^+ \wedge t \wedge T_n \rightarrow \tau_a^+$  as  $n, t \rightarrow \infty$ . So letting  $t \rightarrow \infty$  and  $n \rightarrow \infty$ , we get the desired identity by the dominated convergence theorem and using condition (iv') in combination with the process  $V^+$  not having upward jumps.  $\square$

*Proof of Proposition 2.* For  $\gamma \in [0, 1)$ ,  $a > 0$  and  $x \geq 0$ , define

$$R_{\gamma,a}(x) = \begin{cases} \frac{\gamma}{1-\gamma} Z^{(q)}(x)^{\frac{1}{1-\gamma}} \int_x^a Z^{(q)}(y)^{-\frac{1}{1-\gamma}} (1 - \eta Z^{(q)}(y)) dy & \text{if } \gamma \in (0, 1) \text{ and } x \leq a, \\ 0 & \text{if } \gamma = 0 \text{ or } x > a. \end{cases}$$

Fix  $a \in (b, \infty)$  and define  $v_a: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  by

$$\begin{aligned} v_a(x, \bar{x}) &= \eta \left( \bar{Z}^{(q)}(x) + \frac{\psi'(0)}{q} \right) \\ &+ \frac{Z^{(q)}(x)}{Z^{(q)}(\bar{x})} \left\{ R_{\alpha,a}(\bar{x}) + \left( \frac{Z^{(q)}(\bar{x})}{Z^{(q)}(\bar{x} \vee b)} \right)^{\frac{1}{1-\alpha}} (R_{\beta,a}(\bar{x} \vee b) - R_{\alpha,a}(\bar{x} \vee b)) \right\} \quad (23) \\ &- \eta \frac{Z^{(q)}(x)}{Z^{(q)}(a)} \left( \bar{Z}^{(q)}(a) + \frac{\psi'(0)}{q} \right) \left( \frac{Z^{(q)}(\bar{x})}{Z^{(q)}(\bar{x} \vee b)} \right)^{\frac{\alpha}{1-\alpha}} \left( \frac{Z^{(q)}(\bar{x} \vee b)}{Z^{(q)}(a)} \right)^{\frac{\beta}{1-\beta}}. \end{aligned}$$

We claim that for all  $(x, \bar{x}) \in \mathbb{R} \times [0, \infty)$  such that  $x \leq \bar{x} \leq a$ ,

$$v_a(x, \bar{x}) = \mathbb{E}_{x, \bar{x}} \left[ \int_0^{\tau_a^+} e^{-qs} \delta_b(V_s) d(\bar{X} + \bar{K})_s - \eta \int_0^{\tau_a^+} e^{-qs} dK_s^+ \right], \quad (24)$$

see also Remark 13 below. Once (24) is shown, Proposition 2 follows, first for the  $\eta = 0$  case and then for the general case, by taking  $a \rightarrow \infty$  and using the monotone convergence theorem and l'Hôpital's rule in combination with (4) (recall the assumption  $\beta > 0$ ). To prove (24) we show that  $v_a$  satisfies the conditions of Lemma 12. Regarding the smoothness conditions we have that  $v_a$  is of the form  $\sum_{i=1}^2 g_i(x)h_i(\bar{x})$  with  $g_1(x) = Z^{(q)}(x)$ ,  $g_2(x) = \eta \left( \bar{Z}^{(q)}(x) + \frac{\psi'(0)}{q} \right)$ ,  $h_2(\bar{x}) = 1$  and

$$\begin{aligned} h_1(\bar{x}) &= \frac{1}{Z^{(q)}(\bar{x})} \left\{ R_{\alpha,a}(\bar{x}) + \left( \frac{Z^{(q)}(\bar{x})}{Z^{(q)}(\bar{x} \vee b)} \right)^{\frac{1}{1-\alpha}} (R_{\beta,a}(\bar{x} \vee b) - R_{\alpha,a}(\bar{x} \vee b)) \right\} \\ &- \frac{\eta}{Z^{(q)}(a)} \left( \bar{Z}^{(q)}(a) + \frac{\psi'(0)}{q} \right) \left( \frac{Z^{(q)}(\bar{x})}{Z^{(q)}(\bar{x} \vee b)} \right)^{\frac{\alpha}{1-\alpha}} \left( \frac{Z^{(q)}(\bar{x} \vee b)}{Z^{(q)}(a)} \right)^{\frac{\beta}{1-\beta}}. \end{aligned}$$

Recall the regularity properties of  $W^{(q)}$  mentioned in Section 2.1. They imply that  $g_1$  satisfies the required smoothness conditions in Lemma 12 and then obviously so does  $g_2$ . Further, they imply that  $h_1$  is continuous on  $[0, \infty)$  and continuously differentiable on  $[0, \infty) \setminus \{b, a\}$  with the left-derivative serving as the left-continuous density  $h'_1$  on  $[0, \infty)$ . Therefore,  $h_1$  satisfies the required smoothness conditions in Lemma 12 as well. It is easy to see that  $v_a$  satisfies condition (i) in Lemma 11 by definition of  $Z^{(q)}$  and  $\bar{Z}^{(q)}$ . Next, it is well-known that  $\mathcal{A}_q Z^{(q)}(x) = 0$  and  $\mathcal{A}_q \bar{Z}^{(q)}(x) = \psi'(0)$  for all  $x > 0$ . This can be deduced by using martingale arguments and stochastic calculus (see, for instance, (3.7), Proposition 2 and Lemma 5 in [8]); or, alternatively, by taking Laplace transforms and using Fubini's theorem. So (ii) in Lemma 11 is satisfied as well. It is left to show that  $v_a$  satisfies conditions (iii') and (iv') in Lemma 12. One easily sees that (iv') holds. For (iii'), by observing that

$$R'_{\gamma,a}(x) = \frac{1}{1-\gamma} \frac{Z^{(q)'}(x)}{Z^{(q)}(x)} R_{\gamma,a}(x) - \frac{\gamma}{1-\gamma} (1 - \eta Z^{(q)}(x)), \quad x \in [0, a), \quad (25)$$

we get for  $\bar{x} \in (b, a]$ ,

$$\begin{aligned} & \gamma \partial_x v_a(\bar{x}, \bar{x}) + (\gamma - 1) \partial_{\bar{x}} v_a(\bar{x}, \bar{x}) \\ &= \gamma Z^{(q)'}(\bar{x}) h_1(\bar{x}) + \gamma \eta Z^{(q)}(\bar{x}) + (\gamma - 1) Z^{(q)}(\bar{x}) h'_1(\bar{x}) \\ &= R_{\beta,a}(\bar{x}) \frac{Z^{(q)'}(\bar{x})}{Z^{(q)}(\bar{x})} \left(1 - \frac{\gamma - 1}{\beta - 1}\right) + \beta \frac{\gamma - 1}{\beta - 1} + \eta Z^{(q)}(\bar{x}) \left(\gamma - \beta \frac{\gamma - 1}{\beta - 1}\right) \\ & \quad - \eta \frac{Z^{(q)'}(\bar{x})}{Z^{(q)}(a)} \left(\bar{Z}^{(q)}(a) + \frac{\psi'(0)}{q}\right) \left(\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(a)}\right)^{\frac{\beta}{1-\beta}} \left(\gamma + \frac{(\gamma - 1)\beta}{1 - \beta}\right). \end{aligned} \quad (26)$$

Similarly, for  $0 \leq \bar{x} \leq b$ ,

$$\begin{aligned} & \gamma \partial_x v_a(\bar{x}, \bar{x}) + (\gamma - 1) \partial_{\bar{x}} v_a(\bar{x}, \bar{x}) \\ &= \left( R_{\alpha,a}(\bar{x}) + \left(\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(b)}\right)^{\frac{1}{1-\alpha}} (R_{\beta,a}(b) - R_{\alpha,a}(b)) \right) \frac{Z^{(q)'}(\bar{x})}{Z^{(q)}(\bar{x})} \left(1 - \frac{\gamma - 1}{\alpha - 1}\right) + \alpha \frac{\gamma - 1}{\alpha - 1} \\ & \quad + \eta Z^{(q)}(\bar{x}) \left(\gamma - \alpha \frac{\gamma - 1}{\alpha - 1}\right) \\ & \quad - \eta \frac{Z^{(q)'}(\bar{x})}{Z^{(q)}(a)} \left(\bar{Z}^{(q)}(a) + \frac{\psi'(0)}{q}\right) \left(\frac{Z^{(q)}(b)}{Z^{(q)}(a)}\right)^{\frac{\beta}{1-\beta}} \left(\frac{Z^{(q)}(\bar{x})}{Z^{(q)}(b)}\right)^{\frac{\alpha}{1-\alpha}} \left(\gamma + \frac{(\gamma - 1)\alpha}{1 - \alpha}\right). \end{aligned} \quad (27)$$

From (26) and (27) we immediately see that condition (iii') in Lemma 12 is satisfied which completes the proof.  $\square$

*Remark 13.* We explain here briefly (details can be found in [1]) how we arrived at the guess (23) as the (correct) expression for the expectation in (24). To this end, denote, for  $\bar{x} \in [0, a]$  and  $x \leq 0$ , by  $\tilde{v}_a(x, \bar{x})$  the right hand side of (24). First, use known

fluctuation identities for spectrally negative Lévy processes reflected at their infimum to derive an expression for  $\tilde{v}_a(x, \bar{x})$  in terms of scale functions and  $\tilde{v}_a(\bar{x}, \bar{x})$ . Second, use this expression to compute  $\partial_x \tilde{v}_a(\bar{x}, \bar{x})$  and plug it into the equation (iii') in Lemma 12. This yields then an ordinary differential equation for  $\tilde{v}_a(\bar{x}, \bar{x})$  which, together with the boundary condition (iv') in Lemma 12, can be uniquely solved. This will then yield the guess (23) for  $\tilde{v}_a(x, \bar{x})$ . Note that this line of arguments does not constitute a proof of (24) because this would involve showing in addition that  $\tilde{v}_a(x, \bar{x})$  satisfies, a priori, (iii') in Lemma 12, which is non-trivial.

#### 4.4 Proof of Theorem 3

Let  $v : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  be such that  $v(x, \bar{x})$  is equal to the right hand side of (8) for all  $x \in \mathbb{R}$  and  $\bar{x} \geq 0$  and with  $b = b^*$ . Clearly  $v(x, \bar{x}) \leq v^*(x, \bar{x})$  for  $x \leq \bar{x}$  because, for  $x \leq \bar{x}$ ,  $v$  is the value function of the control  $(\delta_{b^*}(V), K)$  which is admissible (in particular Definition 1(v) is implied by Proposition 2). So Theorem 3 is proved once we show that  $v$  satisfies all conditions of the verification lemma (Lemma 11). Following the arguments in the proof of Proposition 2 we see that the smoothness conditions as well as conditions (i) and (ii) of Lemma 11 are satisfied. It remains to show that conditions (iii), (iv) and (v) of Lemma 11 are satisfied. We start with condition (iii). From (26) with  $a \rightarrow \infty$ , for  $\bar{x} > b^*$  and  $\gamma \in [\alpha, \beta]$ ,

$$\gamma \partial_x v(\bar{x}, \bar{x}) + (\gamma - 1) \partial_{\bar{x}} v(\bar{x}, \bar{x}) - \gamma = \left( \eta Z^{(q)}(\bar{x}) + R_\beta(\bar{x}) \frac{Z^{(q)'}(\bar{x})}{Z^{(q)}(\bar{x})} - 1 \right) \frac{\beta - \gamma}{\beta - 1} \geq 0,$$

where the inequality is due to  $C(\bar{x}) \leq Q(\bar{x})$  for  $\bar{x} > b^*$ , see Lemma 4, which implies  $R_\beta(\bar{x}) \leq \frac{Z^{(q)}(\bar{x})}{Z^{(q)'}(\bar{x})} (1 - \eta Z^{(q)}(\bar{x}))$  in combination with  $\frac{\beta - \gamma}{\beta - 1} \leq 0$  for all  $\gamma \in [\alpha, \beta]$ . On the other hand, by (27) (taking  $a \rightarrow \infty$ ), we have for  $\bar{x} \in [0, b^*]$  and  $\gamma \in [\alpha, \beta]$ ,

$$\begin{aligned} & \gamma \partial_x v(\bar{x}, \bar{x}) + (\gamma - 1) \partial_{\bar{x}} v(\bar{x}, \bar{x}) - \gamma \\ &= \left( \eta Z^{(q)}(\bar{x}) + \left\{ R_\alpha(\bar{x}) + \left( \frac{Z^{(q)}(\bar{x})}{Z^{(q)}(b^*)} \right)^{\frac{1}{1-\alpha}} (R_\beta(b^*) - R_\alpha(b^*)) \right\} \frac{Z^{(q)'}(\bar{x})}{Z^{(q)}(\bar{x})} - 1 \right) \frac{\alpha - \gamma}{\alpha - 1} \\ &= \left( \eta Z^{(q)}(\bar{x}) + \left\{ R_\alpha(\bar{x}) + Z^{(q)}(\bar{x})^{\frac{1}{1-\alpha}} C(b^*) \right\} \frac{Z^{(q)'}(\bar{x})}{Z^{(q)}(\bar{x})} - 1 \right) \frac{\alpha - \gamma}{\alpha - 1} \\ &\geq 0, \end{aligned}$$

where the inequality is due to  $C(b^*) \geq C(\bar{x}) \geq Q(\bar{x})$  for  $\bar{x} \in [0, b^*]$ , see Lemma 4, in combination with  $\frac{\alpha - \gamma}{\alpha - 1} \geq 0$  for all  $\gamma \in [\alpha, \beta]$ .

Next we look at condition (iv) of Lemma 11. Let us denote  $v(x, \bar{x}) = v^{(\eta)}(x, \bar{x})$  to indicate the dependence of  $v$  on  $\eta$ . Further let  $D = \{x \geq 0, \bar{x} \geq 0 : x \leq \bar{x}\}$ . We first deal with the case  $\eta = 0$ . For  $\bar{x} \geq b$  and  $x \leq \bar{x}$ ,  $0 \leq v^{(0)}(x, \bar{x}) \leq v^{(0)}(\bar{x}, \bar{x}) = R_\beta(\bar{x})$ . Since for  $\eta = 0$ ,

$$\lim_{x \rightarrow \infty} R_\beta(x) = \frac{\beta}{1 - \beta} \lim_{x \rightarrow \infty} \frac{\int_x^\infty Z^{(q)}(y)^{-\frac{1}{1-\beta}} dy}{Z^{(q)}(x)^{-\frac{1}{1-\beta}}} = \beta \lim_{x \rightarrow \infty} \frac{Z^{(q)}(x)}{q W^{(q)}(x)} = \frac{\beta}{\Phi(q)}$$

by l'Hôpital's rule in combination with (4) and since  $v^{(0)}(\cdot, \cdot)$  is continuous on  $D$ , it follows that  $v^{(0)}(\cdot, \cdot)$  is bounded on  $D$ . Let now  $\eta > 0$ . Then  $v^{(0)}(x, \bar{x}) - v^{(\eta)}(\bar{x}, \bar{x}) = \mathbb{E}_{x, \bar{x}}[\eta \int_0^\infty e^{-qs} dK_s]$  and it is easy to see that the right hand side is decreasing in  $x$  and decreasing in  $\bar{x}$  from the definition of  $(V, K)$  via the tax-reflection transform (recall Definition 8). So for all  $(x, \bar{x}) \in D$ ,  $|v^{(\eta)}(x, \bar{x})| \leq v^{(0)}(x, \bar{x}) + v^{(0)}(0, 0) - v^{(\eta)}(0, 0)$  and thus  $v^{(\eta)}(\cdot, \cdot)$  is bounded on  $D$ .

Finally, we check that condition (v) of Lemma 11 holds. Recall  $\bar{W}^{(q)}(x) = \int_0^x W^{(q)}(y) dy$  for  $x \geq 0$ . For  $\bar{x} \geq b^*$  and  $x \in [0, \bar{x}]$ ,

$$\begin{aligned} \partial_x v(x, \bar{x}) - \eta &= \eta(Z^{(q)}(x) - 1) + R_\beta(\bar{x}) \frac{Z^{(q)\prime}(x)}{Z^{(q)}(\bar{x})} \\ &\leq \eta q \bar{W}^{(q)}(x) + (1 - \eta Z^{(q)}(\bar{x})) \frac{Z^{(q)\prime}(x)}{Z^{(q)\prime}(\bar{x})} \\ &\leq \eta q \left( \bar{W}^{(q)}(x) + \bar{W}^{(q)}(\bar{x}) \frac{W^{(q)}(x)}{W^{(q)}(\bar{x})} \right) \\ &\leq 0, \end{aligned}$$

where the first inequality is since  $R_\beta(\bar{x}) \leq \frac{Z^{(q)}(\bar{x})}{Z^{(q)\prime}(\bar{x})}(1 - \eta Z^{(q)}(\bar{x}))$  by Lemma 4, the second inequality is due to  $\eta \geq 1$  and the last inequality is due to  $x \leq \bar{x}$  and the (strict) log-concavity of  $\bar{W}^{(q)}(\cdot)$  on  $(0, \infty)$ , where the latter follows from (11) as this identity implies

$$\bar{W}^{(q)}(a) \bar{W}^{(q)\prime\prime}(a) - \bar{W}^{(q)\prime}(a)^2 = \frac{1}{q} \left( \mathbb{E}_{0,0} \left[ e^{-q\hat{\tau}_a} \right] - 1 \right) \bar{W}^{(q)\prime\prime}(a) < 0, \quad a > 0.$$

On the other hand, for  $\bar{x} \in [0, b^*)$  and  $x \in [0, \bar{x}]$ ,

$$\begin{aligned} \partial_x v(x, \bar{x}) - \eta &= \eta(Z^{(q)}(x) - 1) + \left\{ R_\alpha(\bar{x}) + Z^{(q)}(\bar{x})^{\frac{1}{1-\alpha}} C(b^*) \right\} \frac{Z^{(q)\prime}(x)}{Z^{(q)}(\bar{x})} \\ &\leq \eta(Z^{(q)}(x) - 1) + \left\{ R_\alpha(b^*) + Z^{(q)}(b^*)^{\frac{1}{1-\alpha}} Q(b^*) \right\} \frac{Z^{(q)\prime}(x)}{Z^{(q)}(b^*)} \\ &= \eta q \bar{W}^{(q)}(x) + (1 - \eta Z^{(q)}(b^*)) \frac{Z^{(q)\prime}(x)}{Z^{(q)\prime}(b^*)} \\ &\leq \eta q \left( \bar{W}^{(q)}(x) + \bar{W}^{(q)}(b^*) \frac{W^{(q)}(x)}{W^{(q)}(b^*)} \right) \\ &\leq 0, \end{aligned}$$

where the last two inequalities follow by the same arguments as in the previous case and the first inequality follows because  $C(b^*) \leq Q(b^*)$  by Lemma 4 and because, via the

analogue of (25) for  $R_\gamma$  and the definition of  $Q(\cdot)$  in (9), for  $\bar{x} \in [0, b^*)$ ,

$$\begin{aligned} \frac{R_\alpha(\bar{x}) + Z^{(q)}(\bar{x})^{\frac{1}{1-\alpha}} C(b^*)}{Z^{(q)}(\bar{x})} &= \frac{R_\alpha(b^*) + Z^{(q)}(b^*)^{\frac{1}{1-\alpha}} C(b^*)}{Z^{(q)}(b^*)} \\ &\quad - \int_{\bar{x}}^{b^*} \frac{\alpha}{1-\alpha} \frac{Z^{(q)'}(y)}{Z^{(q)}(y)^2} Z^{(q)}(y)^{\frac{1}{1-\alpha}} (C(b^*) - Q(y)) dy \\ &\leq \frac{R_\alpha(b^*) + Z^{(q)}(b^*)^{\frac{1}{1-\alpha}} C(b^*)}{Z^{(q)}(b^*)}, \end{aligned}$$

where the inequality is due to  $C(b^*) \geq C(y) \geq Q(y)$  for  $y \in [0, b^*)$ , see Lemma 4.

## Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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