Infinite splitting in the syzygies of quaternionic groups

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Abstract

Let $\mathcal{F} = (\ldots \stackrel{\partial_{n+1}}{\to} \mathcal{F}_n \stackrel{\partial_n}{\to} \mathcal{F}_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\ldots \stackrel{\partial_1}{\to} \mathcal{F}_0 \to \mathfrak{R} \to 0)$ be a free resolution over the group ring $\mathfrak{R}[\Phi]$ where $\mathfrak{R}$ is commutative and Φ is finite. The n^{th} syzygy $\Omega_n^{\mathfrak{R}[\Phi]}$ is the stable class of $\mathrm{Im}(\partial_n)$ and has a tree structure with roots which do not extend infinitely downwards. We show that $\Omega_3^{\mathfrak{R}[Q_{8p}]}$ has infinitely many isomorphically distinct modules at the minimal level when $\mathfrak{R} = \mathbb{Z}[C_\infty]$ is the integral group ring of the infinite cyclic group and Q_{8p} is the quaternion group of order $8p$ where $p \geq 3$ is prime. This poses severe difficulties in attempting to solve the $D(2)$ problem of CTC Wall for the groups $C_\infty \times Q_{8p}$

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In [32] CTC Wall formulated the following fundamental problem in low dimensional topology:

D(2) : Let X be a finite connected complex of dimension three such that $H^3(X, \mathcal{B}) = H_3(\widetilde{X}; \mathbb{Z}) = 0$ for all local coefficient systems $\mathcal{B}$.

Is it true that X is homotopy equivalent to complex of dimension two ?

This is the $D(2)$ problem. It is parametrized by the fundamental group in the sense that every finitely presented group has its own quite distinct $D(2)$ problem. As will perhaps become clear, despite its topological origins it is essentially algebraic and combinatorial and its difficulty varies widely from case to case. In consequence there is no reasonable expectation that the problem can be solved simultaneously for all groups in uniform fashion.

In this paper we point out the apparent intractability of the $D(2)$ problem for groups of the form $C_\infty \times \mathcal{Q}$ where C_∞ is the infinite cyclic group and $\mathcal{Q}$ is finite of quaternionic type. There is a related question; thus suppose

(*) $$\mathcal{C} = (0 \to J \to \mathbb{Z}[G]^b \to \mathbb{Z}[G]^a \to \mathbb{Z} \to 0)$$

is an exact sequence of modules over the group ring $\mathbb{Z}[G]$. In [10], subject to a mild homological finiteness condition, later shown to be unnecessary by Mannan [22], the present author showed that, for a given finitely presented group G, the $D(2)$ problem is equivalent to the following realization problem:

R(2) : Is every such sequence $\mathcal{C}$ chain homotopy equivalent to the Cayley complex of a finite presentation of G ?

The nature of the module J is central; it plays the role of an 'algebraic π_2'. In cases where $\mathcal{C}$ is realised by a finite presentation $\mathcal{G}$ of G then $J = \pi_2(K_{\mathcal{G}})$, the second homotopy group of the geometric Cayley complex $K_{\mathcal{G}}$. Consequently to solve the $D(2)$ problem affirmatively for G one must first describe all $\mathbb{Z}[G]$-modules J which can possibly occur in the sequences **(*)**. In this paper we quantify the complexity of such descriptions for groups with a quaternionic factor.

We begin by considering groups $G = C_\infty \times \Phi$ where Φ is an arbitrary finite group; $\Lambda = \mathbb{Z}[G]$ will denote the integral group ring. We make the identification $\Lambda = \mathfrak{R}[\Phi]$ where $\mathfrak{R} = \mathbb{Z}[C_\infty]$ which we represent as the ring $\mathbb{Z}[t, t^{-1}]$ of integral Laurent polynomials in t. Observe that the trivial Λ-module $\mathfrak{R}$ has a free Λ-resolution of finite type

$$(\mathcal{F}) \;=\; (\ldots \stackrel{\partial_{n+1}}{\to} \mathcal{F}_n \stackrel{\partial_n}{\to} \mathcal{F}_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\ldots \stackrel{\partial_1}{\to} \mathcal{F}_0 \to \mathfrak{R} \to 0)$$

which decomposes canonically into short exact sequences:

$$\begin{array}{ccccccccccccccc}
 & \mathcal{J}_{n+1} & & & & \mathcal{J}_{n-1} & & & & \mathcal{J}_3 & & & \mathcal{J}_1 & & \\
 & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \quad \searrow & & \\
(\mathcal{F}) & \longrightarrow & \mathcal{F}_n & \xrightarrow{\partial_n} & \mathcal{F}_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots\cdots & \xrightarrow{\partial_3} & \mathcal{F}_2 & \xrightarrow{\partial_2} & \mathcal{F}_1 & \xrightarrow{\partial_1} & \mathcal{F}_0 & \longrightarrow \mathfrak{R} \longrightarrow & 0 \\
 & & & \searrow \quad \nearrow & & & & & & \searrow \quad \nearrow & & & & & \\
 & & & \mathcal{J}_n & & & & & & \mathcal{J}_2 & & & & &
\end{array}$$

where $\mathcal{J}_n = \mathrm{Ker}(\partial_{n-1}) = \mathrm{Im}(\partial_n)$. Given another such free Λ-resolution

$$(\mathcal{F}') \;=\; (\ldots \stackrel{\partial'_{n+1}}{\to} \mathcal{F}'_n \stackrel{\partial'_n}{\to} \mathcal{F}'_{n-1} \stackrel{\partial'_{n-1}}{\to} \ldots\ldots \stackrel{\partial'_1}{\to} \mathcal{F}'_0 \to \mathfrak{R} \to 0)$$

the module $\mathcal{J}'_n = \mathrm{Ker}(\partial'_{n-1})$ is *stably equivalent* to $\mathcal{J}_n$; that is

$$\mathcal{J}_n \oplus \Lambda^a \;\cong\; \mathcal{J}'_n \oplus \Lambda^b$$

for some positive integers a, b . The stability class $[\mathcal{J}_n]$ is the n^{th} *syzygy of* $\mathfrak{R}$, written $\Omega_n^{\mathfrak{R}[\Phi]} = [\mathcal{J}_n]$, and is independent of the particular choice of free resolution. Following Dyer and Sieradski [5] the stable class $[\mathcal{J}]$ of a finitely generated Λ-module has the structure of a directed tree obtained by writing $\mathcal{J}' \to \mathcal{J}''$ when $\mathcal{J}'' \cong \mathcal{J}' \oplus \Lambda$; then $[\mathcal{J}]$ has a minimal level corresponding to the roots of the tree. Taking Φ to be the quaternionic group $Q_{8p} = \langle x, y \mid x^{2p} = y^2;\ xyx = y\rangle$ of order $8p$ we shall prove :

Theorem I: For each prime $p \geq 3$ the minimal level of $\Omega_3^{\mathfrak{R}[Q_{8p}]}$ contains infinitely many isomorphically distinct modules.

As might be expected from analogous situations (cf [16]), Theorem I relies on an analysis of stably free modules over Λ. In [31] Swan considers the problem of classifying stably free modules over the integral group rings $\mathbb{Z}[Q_{4n}]$.

We adapt Swan's method to construct nontrivial stably free modules over group rings $\mathbb{Z}[C_\infty \times Q_{8p}]$ for primes $p \geq 3$. Theorem I then follows from:

Theorem II : For any odd prime p there is an infinite collection $\{\mathfrak{S}(\mu)\}_{\mu\geq 1}$ of pairwise non-isomorphic modules over $\Lambda = \mathbb{Z}[C_\infty \times Q_{8p}]$ which satisfy

$$\mathfrak{S}(\mu) \oplus \Lambda \cong \Lambda \oplus \Lambda.$$

We also consider free resolutions of $\mathbb{Z}$ over the group ring $\Lambda = \mathbb{Z}[C_\infty \times \Phi]$;

$$(\mathcal{E}) \ = \ (\ldots \stackrel{\partial_{n+1}}{\to} \mathcal{E}_n \stackrel{\partial_n}{\to} \mathcal{E}_{n-1} \ldots\ldots \stackrel{\partial_1}{\to} \mathcal{E}_0 \stackrel{\partial_0}{\to} \mathbb{Z} \to 0).$$

The stable class of $\mathrm{Im}(\partial_n)$ is the n^{th}-syzygy $\Omega_k^{\mathbb{Z}[G]}$ of $\mathbb{Z}$ over Λ. Choosing modules $\mathcal{J}_k$ in $\Omega_k^{\mathfrak{R}[\Phi]}$ we show:

Theorem III : $\quad \mathrm{Ext}^1_\Lambda(\mathcal{J}_{n-1}, \mathcal{J}_n) \ \cong \ \mathfrak{R}/|\Phi| \quad$ for $n \geq 1$.

We denote by $K_n(\mathcal{J}_{n-1}, \mathcal{J}_n, \alpha(t))$ the extension module

$$0 \to \mathcal{J}_n \to K_n(\mathcal{J}_{n-1}, \mathcal{J}_n, \alpha(t)) \to \mathcal{J}_{n-1} \to 0$$

classified by $\alpha(t) \in \mathfrak{R}/|\Phi| \ \cong \ \mathrm{Ext}^1_\Lambda(\mathcal{J}_{n-1}, \mathcal{J}_n)$. The syzygies $\Omega_n^{\mathbb{Z}[G]}$, $\Omega_{n-1}^{\mathfrak{R}[\Phi]}$ and $\Omega_n^{\mathfrak{R}[\Phi]}$ are then related by:

Theorem IV : $\quad K_n(\mathcal{J}_{n-1}, \mathcal{J}_n, t-1)$ is a representative of $\Omega_n^{\mathbb{Z}[G]}$.

The first part of the paper is taken up with those aspects of module extension theory required to prove Theorem III; the second part with the construction of stably free modules necessary for Theorem II. Theorem I is proved in section 12 and Theorem IV in section 13.

Finally, in §14 we give a brief survey of the current state of the $D(2)$ problem. Suffice to say here that, whilst there are no known examples of groups where the answer is negative, the class of groups for which the problem has been solved affirmatively is somewhat meagre and for the most part such groups are finite. We conclude by pointing out the significant difficulties that Theorems I and IV present in attempting to solve the $D(2)$ problem for the groups $C_\infty \times Q_{8p}$.

§1: Complete congruences :

Throughout Λ will denote an algebra, free of finite rank over a commutative Noetherian ring R; in particular, Λ is also Noetherian. $\mathcal{M}\mathrm{od}_\Lambda$ will denote the category of right Λ-modules. If $A, C \in \mathcal{M}\mathrm{od}_\Lambda$, $\mathcal{E}\mathrm{xt}^1_\Lambda(A, C)$ will denote the class of exact sequences of Λ-homomorphisms of the form

$$\mathcal{E} = \big(\ 0 \to \ \ C \to \ \ B \to \ \ A \to \ \ 0 \ \big)$$

If $\mathcal{E}, \mathcal{E}' \in \mathcal{E}\mathrm{xt}^1(A, C)$, then $\mathcal{E}$ and $\mathcal{E}'$ are *congruent*, written '$\mathcal{E} \equiv \mathcal{E}'$', when there is commutative diagram of Λ-homomorphisms of the form

$$\begin{array}{c} \mathcal{E} \\ \downarrow \varphi \\ \mathcal{E}' \end{array} = \left(\begin{array}{ccccccc} 0 \to & C & \stackrel{i}{\to} & B \stackrel{p}{\to} & A \to & 0 \\ & \mathrm{Id} \downarrow & & \downarrow \varphi_0 & \downarrow \mathrm{Id} & \\ 0 \to & C & \stackrel{i'}{\to} & B' \stackrel{p'}{\to} & A \to & 0 \end{array} \right).$$

We denote by $\mathcal{T}$ the *trivial extension* $\mathcal{T} = (0 \to C \stackrel{i_C}{\to} C \oplus A \stackrel{\pi_A}{\to} A \to 0)$ where $i_C(c) = \binom{c}{0}$ and $\pi_A \binom{c}{a} = a$. The extension $\mathcal{E}$ *splits* when it is congruent to the trivial extension ; $\mathcal{E}$ *splits on the right* when there exists a Λ-homomorphism $s : A \to B$ such that $p \circ s = \mathrm{Id}_A$ and *splits on the left* when there is a Λ-homomorphism $r : B \to C$ such that $r \circ j = \mathrm{Id}_B$; it is a standard exercise to show that:

(1.1) $\quad \mathcal{E}$ splits $\iff \mathcal{E}$ splits on the right $\iff \mathcal{E}$ splits on the left.

By the Five Lemma, congruence is an equivalence relation on $\mathcal{E}\mathrm{xt}^1(A, C)$. We denote by $\mathrm{Ext}^1_\Lambda(A, C)$ the set of equivalence classes under '$\equiv$'. Baer's Theorem ([1], [20], Chap III) is then:

(1.2) $\mathrm{Ext}^1_\Lambda(A, C)$ is an abelian group with respect to Baer sum.

In particular the zero element in $\mathrm{Ext}^1_\Lambda(A, C)$ is given by the trivial extension and the additive inverse of the extension $\mathcal{E} = (0 \to C \stackrel{i}{\to} X \stackrel{p}{\to} A \to 0)$ is $-\mathcal{E} = (0 \to C \stackrel{i}{\to} X \stackrel{-p}{\to} A \to 0)$. Eilenberg and Maclane reinterpreted Baer's Theorem thus ([20], p.89):

(1.3) $\mathrm{Ext}^1_\Lambda(A, C)$ is isomorphic to the module cohomology $H^1(A, C)$.

We denote by $\mathrm{Proj}^1(A, C)$ the subset of $\mathrm{Ext}^1_\Lambda(A, C)$, possibly empty, defined by extensions $\mathcal{P} = (0 \to C \to P \to A \to 0)$ where P is projective. In what follows $\mathcal{A}$, $\mathcal{C}$, $\mathcal{E}$ will denote short exact sequences of modules over Λ

$$\begin{aligned} \mathcal{A} &= (0 \to A' \xrightarrow{j_A} F \xrightarrow{\partial_A} A \to 0) \\ \mathcal{C} &= (0 \to C' \xrightarrow{j_C} E \xrightarrow{\partial_C} C \to 0) \\ \mathcal{E} &= (0 \to C \xrightarrow{i} B \xrightarrow{p} A \to 0) \end{aligned}$$

in which E, F are projective. We shall refer to $(\mathcal{A}, \mathcal{C}, \mathcal{E})$ as an *admissible triple*. By a *completion* $\mathfrak{C}$ of $(\mathcal{A}, \mathcal{C}, \mathcal{E})$ we shall mean a commutative diagram of Λ modules, as below, in which all rows and columns are exact:

$$
(\mathfrak{C}) \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C' & \xrightarrow{i'} & B' & \xrightarrow{p'} & A' & \to & 0 \\
 & & \downarrow{\scriptstyle j_C} & & \downarrow{\scriptstyle j_B} & & \downarrow{\scriptstyle j_A} & & \\
0 & \to & E & \xrightarrow{i_E} & G & \xrightarrow{p_F} & F & \to & 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \partial_B} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \to & C & \xrightarrow{i} & B & \xrightarrow{p} & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

If E and F are projective then the exactness of the middle row guarantees that G is projective. We regard $\mathcal{E}' = (0 \to C' \xrightarrow{i'} B' \xrightarrow{p'} A' \to 0)$ as a *first derivative* of $\mathcal{E}$, in furtherance of which viewpoint we note that $\mathcal{E}'$ is essentially unique. Thus suppose $\widetilde{\mathfrak{C}}$ below is also a completion of $(\mathcal{A}, \mathcal{C}, \mathcal{E})$:

$$
(\widetilde{\mathfrak{C}}) \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C' & \xrightarrow{\widetilde{i'}} & \widetilde{B'} & \xrightarrow{\widetilde{p'}} & A' & \to & 0 \\
 & & \downarrow{\scriptstyle j_C} & & \downarrow{\scriptstyle \widetilde{j_B}} & & \downarrow{\scriptstyle j_A} & & \\
0 & \to & E & \xrightarrow{\widetilde{i_E}} & \widetilde{G} & \xrightarrow{\widetilde{p_F}} & F & \to & 0 \\
 & & \downarrow{\scriptstyle \partial_C} & & \downarrow{\scriptstyle \widetilde{\partial_B}} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \to & C & \xrightarrow{i} & B & \xrightarrow{p} & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We say $\mathfrak{C}$ and $\widetilde{\mathfrak{C}}$ are *completely congruent*, written $\mathfrak{C} \equiv \widetilde{\mathfrak{C}}$, when there is an isomorphism of diagrams $h : \widetilde{\mathfrak{C}} \to \mathfrak{C}$ which restricts to the identity on the initial triple. It is straightforward to see that:

(1.4) Any admissible triple $(\mathcal{A}, \mathcal{C}, \mathcal{E})$ admits a completion; moreover, any two completions of $(\mathcal{A}, \mathcal{C}, \mathcal{E})$ are completely congruent.

A complete congruence $\mathfrak{C} \equiv \widetilde{\mathfrak{C}}$ induces a congruence $\mathcal{E}' \equiv \widetilde{\mathcal{E}}'$. We write $\tau_{\mathcal{A},\mathcal{C}}(\mathcal{E})$ for the congruence class $[\mathcal{E}']$ of the top row of any completion of $(\mathcal{A}, \mathcal{C}, \mathcal{E})$. It is now straightforward to see that:

(1.5) Let $(\mathcal{A}, \mathcal{C}, \mathcal{E})$, $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{E}})$ be admissible triples such that $\mathcal{A} \equiv \widetilde{\mathcal{A}}$, $\mathcal{C} \equiv \widetilde{\mathcal{C}}$ and $\mathcal{E} \equiv \widetilde{\mathcal{E}}$; then $\tau_{\mathcal{A},\mathcal{C}}(\mathcal{E}) = \tau_{\widetilde{\mathcal{A}},\widetilde{\mathcal{C}}}(\widetilde{\mathcal{E}})$.

It follows that the above mapping τ gives rise to a mapping of sets

$$\begin{array}{ccccccccc} \tau : \mathrm{Proj}^1(A,A') & \times & \mathrm{Proj}^1(C,C') & \times & \mathrm{Ext}^1_\Lambda(A,C) & \to & \mathrm{Ext}^1_\Lambda(A',C') \\ ([\mathcal{A}] & , & [\mathcal{C}] & , & [\mathcal{E}]) & \to & \tau_{\mathcal{A},\mathcal{C}}(\mathcal{E}). \end{array}$$

With fixed $\mathcal{A} \in \mathrm{Proj}^1(A,A')$ and $\mathcal{C} \in \mathrm{Proj}^1(C,C')$ then :

(1.6) $\tau_{\mathcal{A},\mathcal{C}} : \mathrm{Ext}^1_\Lambda(A,C) \to \mathrm{Ext}^1_\Lambda(A',C')$ is an additive homomorphism.

In what follows, we shall fix $\mathcal{A} \in \mathrm{Proj}^1(A,A')$ and $\mathcal{C} \in \mathrm{Proj}^1(C,C')$ to obtain a mapping, the '*derivative mapping for extensions*'

$$\begin{array}{rcl} \tau : \mathrm{Ext}^1_\Lambda(A,C) & \to & \mathrm{Ext}^1_\Lambda(A',C') \\ \tau([\mathcal{E}]) & = & \tau_{\mathcal{A},\mathcal{C}}(\mathcal{E}) \end{array}$$

§2 : Stable modules :

As Λ is Noetherian the following *surjective rank property* holds :

(2.1) If $\pi : \Lambda^n \to \Lambda^m$ is a surjective Λ homomorphism then $m \leq n$.

We denote by '$\approx$' the stability relation on Λ modules; that is

$$M_1 \approx M_2 \Longleftrightarrow M_1 \oplus \Lambda^{n_1} \cong M_2 \oplus \Lambda^{n_2}$$

for some integers $n_1, n_2 \geq 0$; then ' $\approx$ ' is an equivalence on isomorphism classes of Λ-modules. We denote by $[M]$ the *stable module* of M; that is, the set of isomorphism classes of modules N such that $N \approx M$; evidently M is finitely generated if and only if each $N \in [M]$ is finitely generated.

Henceforth M will denote a nonzero finitely generated Λ-module; then M is also finitely generated over $\mathfrak{R}$ from which it follows that:

(2.2) If $N \in [M]$ then for each integer $a > 0$, $N \oplus \Lambda^a \not\cong N$.

Following Dyer and Sieradski [5], when M is a finitely generated Λ-module we impose on $[M]$ the structure of a graph on in which the vertices are the isomorphism classes of modules $N \in [M]$ and where we draw an edge $N_1 \to N_2$ when $N_2 \cong N_1 \oplus \Lambda$. By (2.2) the graph of $[M]$ does not contain a nontrivial loop; consequently:

(2.3) If M is finitely generated over Λ then $[M]$ is an infinite directed tree.

If $N_1, N_2 \in [M]$ we write $g(N_1,N_2) = g \iff N_1 \oplus \Lambda^{a+g} \cong N_2 \oplus \Lambda^a$ where both a and $a+g$ are positive integers. It follows from (2.2) that there is a 'gap function' $g : [M] \times [M] \to \mathbb{Z}$ with the following properties: :

$$\begin{cases} g(N, N \oplus \Lambda^b) & = & b\ ; \\ g(N_2, N_1) & = & -g(N_1, N_2\ ); \\ g(N_1, N_3) & = & g(N_1,N_2) + g(N_2,N_3). \end{cases}$$

When M is nonzero we write $\mathrm{rk}_\Lambda(M) = a$ where a is the smallest positive integer for which there is a surjective homomorphism $\varphi : \Lambda^a \to M$; then

(2.4) If $K \in [M]$ is such that $0 \le g(K, M)$ then $g(K, M) \le \mathrm{rk}_\Lambda(M)$.

We say that $M_0 \in [M]$ is a *root module* for $[M]$ when $0 \le g(M_0, K)$ for all $K \in [M]$. Defining $\mu : [M] \to \mathbb{Z}$ by $\mu(K) = g(M, K)$ then as $g(M, K) = -g(K, M)$ it follows from (2.4) that μ is bounded below by $-\mathrm{rk}_\Lambda(M)$. If $\mu(M_0)$ is the minimum value of μ then M_0 is a root module; that is:

(2.5) If M is finitely generated then $[M]$ has a root module.

We define the height function $h : [M] \to \mathbb{N}$ by $h(L) = g(M_0, L)$ when M_0 is a root module. As $g(M_0, M_0 \oplus \Lambda^n) = n$ then h is surjective and so $[M]$ extends infinitely upwards. However, the existence of a root module shows that $[M]$ *does not extend infinitely downwards.*

§3: Syzygies :

Let M be finitely generated over Λ; by a *projective* 0-*complex* over M we mean an exact sequence of Λ-homomorphisms of the form

(3.1) $$0 \to J \to P \to M \to 0$$

where P is finitely generated and projective; we note the following ([19], p.145, [21], p. 97):

(3.2) (Schanuel's Lemma) If $(0 \to D_r \stackrel{i_r}{\to} P_r \stackrel{f_r}{\to} M \to 0)$ are projective 0-complexes for $r = 1, 2$ then $D_1 \oplus P_2 \cong D_2 \oplus P_1$.

If M is finitely generated then for some positive integer a there exists an exact sequence $0 \to J \to \Lambda^a \to M \to 0$; this special case is called a *free* 0-*complex*. Given another such free 0-complex $0 \to J' \to \Lambda^b \to M \to 0$ Schanuel's Lemma shows that $J \oplus \Lambda^b \cong J' \oplus \Lambda^a$; thus the stable class $[J]$ is independent of the free 0-complex chosen and depends only on M. The stable class $[J]$ is the *first syzygy* of M and is denoted by $\Omega_1^\Lambda(M)$ (abbreviated to $\Omega_1(M)$ when Λ is clear from context) . As Λ is Noetherian and M is finitely generated then J is also finitely generated. Consequently every module in $\Omega_1(M)$ is finitely generated. Suppose given a projective 0-complex $0 \to J \stackrel{i}{\to} S \stackrel{p}{\to} M \to 0$ in which S is stably free. If $S \oplus \Lambda^n \cong \Lambda^{m+n}$ where n is a positive integer we modify the exact sequence as follows:

$$0 \to J \oplus \Lambda^n \xrightarrow{\begin{pmatrix} i & 0 \\ 0 & \mathrm{Id} \end{pmatrix}} S \oplus \Lambda^n \xrightarrow{(p, 0)} M \to 0$$

As $S \oplus \Lambda^n \cong \Lambda^{m+n}$ then $J \in \Omega_1(M)$; hence:

(3.3) Let $0 \to J \to S \to M \to 0$ be an exact sequence of Λ-homomorphisms in which S is finitely generated and stably free; then $J \in \Omega_1(M)$.

The above argument generalises to give:

Proposition 3.4 : Let $0 \to J \xrightarrow{j} S \xrightarrow{\delta} F_{n-2} \xrightarrow{\partial_{n-2}} \dots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$ be an exact sequence of finitely generated Λ-modules where $n \geq 3$; if each F_r is free and S is stably free then $J \in \Omega_n(M)$.

Let $\mathcal{M}\mathrm{od}_\Lambda^{\mathrm{fin}}$ denote full subcategory of $\mathcal{M}\mathrm{od}_\Lambda$ consisting of finitely generated right Λ-modules. If $f, g : M \to N$ are morphisms in $\mathcal{M}\mathrm{od}_\Lambda^{\mathrm{fin}}$ we write '$f \sim g$' when $f - g = \xi \circ \eta$ for some Λ -homomorphisms $\eta : M \to P$ and $\xi : P \to N$ where P is projective. The *derived module category* $\mathcal{D}\mathrm{er}(\Lambda)$ is then the category whose objects are finitely generated right Λ-modules and where $\mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(M, N)$ is the quotient

$$\mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(M, N) = \mathrm{Hom}_\Lambda(M, N)/\sim$$

Then $\mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(M, N)$ has the natural structure of an abelian group. Moreover, if M_1, M_2 are finite generated Λ-modules then (cf [15], p.72)

(3.5) $$M_1 \cong_{\mathcal{D}\mathrm{er}(\Lambda)} M_2 \iff M_1 \oplus P_1 \cong_\Lambda M_2 \oplus P_2.$$

where P_1 , P_2 are finitely generated projectives. Evidently if $M_1 \approx M_2$ then $M_1 \cong_{\mathcal{D}\mathrm{er}(\Lambda)} M_2$ from which it follows that:

(3.6) The syzygy $\Omega_1(M)$ represents a single isomorphism class in $\mathcal{D}\mathrm{er}(\Lambda)$.

Let $(0 \to A' \xrightarrow{i} \Lambda^a \xrightarrow{p} A \to 0$ and $(0 \to B' \xrightarrow{j} \Lambda^b \xrightarrow{q} B \to 0)$ be exact sequences so that A' represents $\Omega_1(A)$ and B' represents $\Omega_1(B)$ and let $f : A \to B$ be a Λ-homomorphism. By the universal property of projective modules there exists a commutative diagram of the form

$$\begin{pmatrix} 0 \to & A' & \xrightarrow{i} & \Lambda^a & \xrightarrow{p} & A & \to 0 \\ & f' \downarrow & & \widehat{f} \downarrow & & f \downarrow & \\ 0 \to & B' & \xrightarrow{j} & \Lambda^b & \xrightarrow{q} & B & \to 0 \end{pmatrix}.$$

Proposition 3.7 : If $f \sim 0$ then $f' \sim 0$.

Proof : Let $f = \xi \circ \eta$ be a factorization of f through $\eta : A \to P$ and $\xi : P \to B$ where P is projective. As q is surjective then by the universal property of projective modules there exists a homomorphism $\widehat{\xi} : P \to \Lambda^b$ such that $q \circ \widehat{\xi} = \xi$. Define $F : \Lambda^a \to \Lambda^b$ by $F = \widehat{f} - \widehat{\xi} \circ \eta \circ p$. As $p \circ i = 0$ then :

(*) $$F \circ i = \widehat{f} \circ i.$$

Moreover, $q \circ F = q \circ \widehat{f} - q \circ \widehat{\xi} \circ \eta \circ p = q \circ \widehat{f} - \xi \circ \eta \circ p = q \circ \widehat{f} - f \circ p = 0$. Hence $\mathrm{Im}(F) \subset \mathrm{Ker}(q) = \mathrm{Im}(j)$. As j is injective we have a well defined homomorphism $j^{-1} \circ F : \Lambda^a \to B'$ and

(**) $\quad j \circ (f' - j^{-1} \circ F \circ i) \;=\; j \circ f' \;-\; F \circ i \;=\; j \circ f' \;-\; \widehat{f} \circ i \;=\; 0.$

As j is injective then $f' = j^{-1} \circ F \circ i$ and so $i : A' \to \Lambda^a$ and $j^{-1} \circ F : \Lambda^a \to B'$ is a factorization of f' through the free module Λ^a. □

In the above construction the class in $\mathcal{D}\mathrm{er}(\Lambda)$ of $f' : A' \to B'$ depends only on the class in $\mathcal{D}\mathrm{er}(\Lambda)$ of the homomorphism $f : A \to B$ and $[f']$ is a well defined morphism $[f'] : A' \to B'$ in $\mathcal{D}\mathrm{er}(\Lambda)$. Choosing for each Λ -module A a specific free 0-complex $(0 \to A' \xrightarrow{i} \Lambda^a \xrightarrow{p} A \to 0)$ and writing $\Omega_1(A) = A'$ then on defining $\Omega_1([f]) = [f']$ we obtain a functor

(***) $$\Omega_1 : \mathcal{D}\mathrm{er}(\Lambda) \to \mathcal{D}\mathrm{er}(\Lambda).$$

We iterate the construction by defining $\Omega_n = \Omega_1 \circ \Omega_{n-1}$.

§4 : Coprojective modules :

It follows from (1.3) and the additivity of $H^1(M, -)$ that:

(4.1) If M is finitely generated the following conditions are equivalent:

(i) $\mathrm{Ext}^1(M, \Lambda) = 0$;

(ii) $H^1(M, Q) = 0$ for any projective module Q ;

(iii) $H^1(M, \Lambda) = 0$.

Modules satisfying the equivalent conditions of (4.1) are said to be 1-*coprojective*. Let $\mathcal{F} = (0 \to J \xrightarrow{j} \Lambda^a \xrightarrow{p} M \to 0)$ be a free 0-complex. If $\alpha : J \to N$ is a Λ-homomorphism we denote by $\alpha_*(\mathcal{F})$ the pushout extension:

$$\begin{array}{c} \mathcal{F} \\ \\ \downarrow \\ \\ \alpha_*(\mathcal{F}) \end{array} \quad = \quad \left(\begin{array}{ccccccccc} 0 \longrightarrow & J & \xrightarrow{j} & \Lambda^a & \xrightarrow{p} & M & \longrightarrow & 0 \\ & \downarrow \alpha & & \downarrow \widehat{\alpha} & & \downarrow \mathrm{Id} & & \\ 0 \to & N & \xrightarrow{\widehat{j}} & \varinjlim(\alpha, j) & \xrightarrow{\widehat{p}} & M & \longrightarrow & 0 \end{array} \right).$$

where $\varinjlim(\alpha, j) = N \oplus \Lambda^a / \mathrm{Im}(\alpha \times -j)$ and $\widehat{j}$, $\widehat{\alpha}$ and $\widehat{p}$ are the canonical mappings so obtained. We showed in [13] (Proposition 5.28, p.104) that when M is 1-coprojective then for any Λ-module N there is an exact sequence

$$\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Lambda^a, N) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(J, N) \xrightarrow{\delta} \mathrm{Ext}^1(M, N) \xrightarrow{p_*} \mathrm{Ext}^1(\Lambda^a, N)$$

where $\delta(\alpha) = \alpha_*(\mathcal{F})$. As $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Lambda^a, N) \cong \mathrm{Ext}^1(\Lambda^a, N) = 0$ it follows that $\delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(J, N) \xrightarrow{\simeq} \mathrm{Ext}^1(M, N)$ is an isomorphism. As J represents $\Omega_1(A)$ we obtain the following *corepresentation theorem* for $\mathrm{Ext}^1(M, -)$:

(4.2) If M is 1-coprojective. then $\delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_1(M), -) \xrightarrow{\simeq} \mathrm{Ext}^1(M, -)$ is an isomorphism.

Theorem 4.3 : If A is 1-coprojective there is an isomorphism

$$\Omega_1 : \mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(A, B) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(\Omega_1(A), \Omega_1(B)).$$

Proof : To show Ω_1 is injective, suppose given a commutative diagram

$$\left(\begin{array}{ccccccc} 0 \to & A' & \stackrel{i}{\to} & \Lambda^a & \stackrel{p}{\longrightarrow} & A & \to 0 \\ & f' \downarrow & & \widehat{f} \downarrow & & f \downarrow & \\ 0 \to & B' & \stackrel{j}{\to} & \Lambda^b & \stackrel{q}{\longrightarrow} & B & \to 0 \end{array}\right).$$

where f' has a factorisation $f' = \xi \circ \eta$; $\xi : T \to B'$; $\eta : A' \to T$; with T projective. In the exact sequence $\mathrm{Hom}_\Lambda(\Lambda^a, T) \stackrel{i^*}{\to} \mathrm{Hom}_\Lambda(A', T) \stackrel{\delta}{\to} \mathrm{Ext}^1_\Lambda(A, T)$ as $\mathrm{Ext}^1_\Lambda(A, T) = 0$ then $i^* : \mathrm{Hom}_\Lambda(\Lambda^a, T) \stackrel{p^*}{\to} \mathrm{Hom}_\Lambda(A', T)$ is surjective. Choose $\widehat{\xi} : \Lambda^a \to T$ such that $i^*(\widehat{\xi}) = \xi$. Putting $r = \eta \circ \widehat{\xi} : \Lambda^a \to B'$ then the following diagram commutes

$$\begin{array}{ccc} A' & \xrightarrow{\quad i \quad} & \Lambda^a \\ \Big\downarrow f' & \swarrow r & \Big\downarrow \widehat{f} \\ B' & \xrightarrow[\quad j \quad]{} & \Lambda^b. \end{array}$$

It follows (cf [15], p.54) that the sequence $f'_*(\mathcal{A})$ splits on the left. Consequently $f'_*(\mathcal{A})$ splits on the right and so there exists a homomorphism $s : A \to \Lambda^b$ making the following diagram commute

$$\begin{array}{ccc} \Lambda^a & \xrightarrow{\quad p \quad} & A \\ \Big\downarrow \widehat{f} & \swarrow s & \Big\downarrow f \\ \Lambda^b & \xrightarrow[\quad q \quad]{} & B \end{array}$$

As f factors through Λ^b then $f \sim 0$ and $f \mapsto \Omega_1(f)$ is injective as claimed. To establish surjectivity, given a Λ-homomorphism $g : A' \to B'$ we must construct homomorphisms $\widehat{g} : \Lambda^a \to \Lambda^b$, $g_- : A \to B$ making the following diagram commute:

$$\left(\begin{array}{ccccccc} 0 \to & A' & \stackrel{i}{\to} & \Lambda^a & \stackrel{p}{\longrightarrow} & A & \to 0 \\ & g \downarrow & & \widehat{g} \downarrow & & g_- \downarrow & \\ 0 \to & B' & \stackrel{j}{\to} & \Lambda^b & \stackrel{q}{\longrightarrow} & B & \to 0 \end{array}\right).$$

In the exact sequence $\mathrm{Hom}_\Lambda(\Lambda^a, \Lambda^b) \stackrel{i^*}{\to} \mathrm{Hom}_\Lambda(A', \Lambda^b) \stackrel{\delta}{\to} \mathrm{Ext}^1_\Lambda(A, \Lambda^a)$, as $\mathrm{Ext}^1_\Lambda(A, \Lambda^a) = 0$ then $i^* : \mathrm{Hom}_\Lambda(\Lambda^a, \Lambda^b) \stackrel{p^*}{\to} \mathrm{Hom}_\Lambda(\Omega_1(A), \Lambda^b)$ is surjective. Choose $\widehat{g} : \Lambda^a \to \Lambda^b$ such that $i^*(\widehat{g}) = j \circ g$; then the following commutes:

$$\begin{array}{ccc} A' & \longrightarrow & \Lambda^a \\ \downarrow g & & \downarrow \widehat{g} \\ B' & \stackrel{j}{\longrightarrow} & \Lambda^b \end{array}$$

Hence $q \circ \widehat{g} \circ i = q \circ j \circ g = 0$ so that $\widehat{g}(\mathrm{Im}(i)) \subset \mathrm{Ker}(q)$. Consequently $\widehat{g}(\mathrm{Ker}(p)) \subset \mathrm{Ker}(q)$ thereby inducing $g_- : \Lambda^a/\mathrm{Ker}(p) \to \Lambda^b/\mathrm{Ker}(q)$ to make the following diagram commute:

$$\begin{array}{ccc} \Lambda^a & \stackrel{\natural}{\longrightarrow} & \Lambda^a/\mathrm{Ker}(p) \\ \downarrow \widehat{g} & & \downarrow g_- \\ \Lambda^b & \stackrel{\natural}{\longrightarrow} & \Lambda^b/\mathrm{Ker}(q) \end{array}$$

The result follows as $A \cong \Lambda^a/\mathrm{Ker}(p)$, $B \cong \Lambda^b/\mathrm{Ker}(q)$. □

More generally, we say that M is *n-coprojective* when $\mathrm{Ext}^k(M, Q) = 0$ for $1 \leq k \leq n$ and all projective modules Q. Iterating (4.3) we obtain:

(4.4) If A is n-coprojective then there is an isomorphism

$$\Omega_n : \mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(A, B) \stackrel{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(\Omega_n(A), \Omega_n(B)).$$

Taking $B = A$ we obtain the following as a special case of (4.4).

(4.5) If A is n-coprojective then Ω_n gives an isomorphism

$$\Omega_n : \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(A) \stackrel{\simeq}{\longrightarrow} \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\Omega_n(A)).$$

In particular, taking $(0 \to J \stackrel{j}{\to} \Lambda^a \stackrel{p}{\longrightarrow} M \to 0)$ to be a free 0-complex:

(4.6) If M is 1-coprojective then $\Omega_1 : \mathrm{End}_{\mathcal{D}\mathrm{er}}(M) \stackrel{\simeq}{\longrightarrow} \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ is an isomorphism.

Now suppose that A is 2-coprojective and consider the following diagram where $\tau = \tau_{\mathcal{A},\mathcal{B}}$ is the homomorphism of (1.6):

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(\Omega_1(A), B) & \xrightarrow{\Omega_1} & \mathrm{Hom}_{\mathcal{D}\mathrm{er}(\Lambda)}(\Omega_2(A), \Omega_1(B)) \\ \downarrow \delta & & \downarrow \delta \\ \mathrm{Ext}^1_\Lambda(A, B) & \xrightarrow{\tau} & \mathrm{Ext}^1_\Lambda(\Omega_1(A),\, \Omega_1(B)). \end{array} \tag{4.7}$$

As A is 2-coprojective then both A and $\Omega_1(A)$ are 1-coprojective so that both vertical arrows are isomorphisms. Furthermore, Ω_1 is also an isomorphism. We obtain the following *translation theorem for extension classes*:

(4.8) $\tau : \mathrm{Ext}^1_\Lambda(A, B) \xrightarrow{\simeq} \mathrm{Ext}^1_\Lambda(\Omega_1(A),\, \Omega_1(B))$ is an isomorphism if A is 2-coprojective.

Iteration of the above argument gives;

(4.9) $\tau^n : \mathrm{Ext}^1_\Lambda(A, B) \xrightarrow{\simeq} \mathrm{Ext}^1_\Lambda(\Omega_n(A),\, \Omega_n(B))$ is an isomorphism if A is $(n{+}1)$-coprojective.

Finally, taking $B = \Omega_1(A)$ we obtain the following:

(4.10) $\tau^{n-1} : \mathrm{Ext}^1_\Lambda(A, \Omega_1(A)) \xrightarrow{\simeq} \mathrm{Ext}^1_\Lambda(\Omega_{n-1}(A),\, \Omega_n(A))$ is an isomorphism if A is n-coprojective.

We note the following de-stabilization result (cf [13], p.97):

Proposition 4.11 : Let $0 \to J \oplus Q_0 \xrightarrow{j} Q_1 \to M \to 0$ be a projective 0-complex where Q_0 is also projective; if M is 1-coprojective then $Q_1/j(Q_0)$ is projective.

Proof : Let $i : J \to J \oplus Q_0$ be the inclusion, $i(x) = (x, 0)$, and let π be the projection $J \oplus Q_0 \to J$; $\pi(x, q) = x$. When $L = \varinjlim(i \circ \pi, j)$ we have a commutative diagram

$$\begin{array}{c} \mathcal{E} \\ \downarrow \nu(\alpha) \\ (i\circ\pi)_*(\mathcal{E}) \end{array} = \left(\begin{array}{ccccc} 0 \to & J \oplus Q_0 \xrightarrow{j} & Q_1 \to & M \to & 0 \\ & \downarrow i \circ \pi & \downarrow \nu & \downarrow \mathrm{Id} & \\ 0 \to & J \oplus Q_0 \to & L \to & M \to & 0 \end{array} \right)$$

where $\nu : Q_1 \to L = \varinjlim(i \circ \pi, j)$ is the natural map. As $H^1(M, Q_0) = 0$ it follows easily that $i_* \circ \pi_* = \mathrm{Id} : H^1(M, J \oplus Q_0) \to H^1(M, J \oplus Q_0)$. Let $c = c_{\mathcal{E}} \in H^1(M, J \oplus Q_0)$ be the element classifying $\mathcal{E}$. Then $(i \circ \pi)_*(\mathcal{E})$ is classified by $i_* \circ \pi_*(c) = c$. Thus $(i \circ \pi)_*(\mathcal{E})$ is congruent to $\mathcal{E}$, so that $L \cong Q_1$, and in particular, L is projective. Now put $S = \varinjlim(\pi, j)$. It is straightforward to check that $S = Q_1/j(Q_0)$, thus it suffices to show that S is projective. We have a commutative diagram

$$\begin{array}{ccc} \pi_*(\mathcal{E}) & & \\ \downarrow \nu(\alpha) & = & \\ (i \circ \pi)_*(\mathcal{E}) & & \end{array} \left(\begin{array}{ccccc} 0 \to & J \to & S \to & M \to & 0 \\ & \downarrow i & \downarrow \mu & \downarrow \mathrm{Id} & \\ 0 \to & J \oplus Q_0 \to & L \to & M \to & 0 \end{array} \right)$$

where $\mu : S \to L$ is the induced map on pushouts. We obtain a commutative diagram for any coefficient module B ;

$$\begin{array}{cccc} H^1(M, B) \to & H^1(L, B) \to & H^1(J \oplus Q_0, B) \to & H^2(M, B) \\ \downarrow \mathrm{Id} & \downarrow \mu^* & \downarrow i^* & \downarrow \mathrm{Id} \\ H^1(M, B) \to & H^1(S, B) \to & H^1(J, B) \to & H^2(M, B) \end{array} .$$

$\mathrm{Id} : H^k(M, B) \to H^k(M, B)$ is an isomorphism for $k = 1, 2$. As Q_0 is projective, i^* is an isomorphism. Thus $\mu^* : H^1(L, B) \to H^1(S, B)$ is surjective. As L is projective then $H^1(L, B) = 0$. Hence $H^1(S, B) = 0$ for all coefficient modules B and so $S = Q_1/j(Q_0)$ is projective. □

Now suppose given an exact sequence of Λ-modules

$$0 \to J \oplus \Lambda^d \xrightarrow{j} \Lambda^c \to \Lambda^b \to \Lambda^a \to M \to 0$$

in which M is 2-coprojective and split the sequence as follows :

$$0 \to J \oplus \Lambda^d \xrightarrow{j} \Lambda^c \to \Lambda^b \to M' \to 0 \quad ; \quad 0 \to M' \to \Lambda^a \to M \to 0.$$

As M is 2-coprojective then M' is 1-coprojective so that, by (4.11), $\Lambda^c/j(\Lambda^d)$ is projective. Consequently the sequence $0 \to \Lambda^d \to \Lambda^c \to \Lambda^c/j(\Lambda^d) \to 0$ splits to give an isomorphism $\Lambda^c/j(\Lambda^d) \oplus \Lambda^c \cong \Lambda^d$. Hence we have:

(4.12) Let $0 \to J \oplus \Lambda^d \xrightarrow{j} \Lambda^c \to \Lambda^b \to \Lambda^a \to M \to 0$ be exact; if M is 2-coprojective then $\Lambda^c/j(\Lambda^d)$ is stably free.

Suppose given a pair of projective 0-complexes:

$$\mathcal{E} = (0 \to K \xrightarrow{i} P \xrightarrow{p} M \to 0) \quad ; \quad \mathcal{F} = (0 \to K \xrightarrow{j} Q \xrightarrow{q} N \to 0)$$

and form the pushout square

$$\begin{array}{ccc} K & \xrightarrow{\;\;j\;\;} & Q \\ \downarrow i & & \downarrow \eta_Q \\ P & \xrightarrow{\;\;\eta_P\;\;} & \varinjlim(i, j) \end{array}$$

where $\varinjlim(i,j) = (P \oplus Q)/\mathrm{Im}(i \times -j)$. Taking the canonical projections

$$\pi_P : P \oplus Q \to P \quad ; \quad \pi_Q : P \oplus Q \to Q$$

then $p \circ \pi_P : P \oplus Q \to M$ vanishes on $\mathrm{Im}(i \times -j)$ giving an exact sequence

(4.13) $$0 \to Q \xrightarrow{\eta_Q} \varinjlim(i,j) \xrightarrow{\xi_P} M \to 0$$

where $\xi_P : \varinjlim(i,j) \xrightarrow{\xi_P} M$ is induced from $p \circ \pi_P$. Likewise $q \circ \pi_Q : P \oplus Q \to N$ vanishes on $\mathrm{Im}(i \times -j)$ and induces an exact sequence

(4.14) $$0 \to P \xrightarrow{\eta_P} \varinjlim(i,j) \xrightarrow{\xi_Q} N \to 0.$$

If $\mathrm{Ext}^1(M,Q) = 0$ then (4.13) splits and $\varinjlim(i,j) \cong M \oplus Q$. Similarly, if $\mathrm{Ext}^1(N,P) = 0$ then (4.14) splits so that $\varinjlim(i,j) \cong N \oplus P$. We obtain the following dual form of Schanuel's Lemma:

(4.15) Let $(0 \to K \xrightarrow{i} P \xrightarrow{p} M \to 0)$ and $(0 \to K \xrightarrow{j} Q \xrightarrow{q} N \to 0)$ be projective 0-complexes in $\mathcal{M}od_\Lambda$. If M and N are 1-coprojective then $M \oplus Q \cong N \oplus P$.

§5 : Lattices and duality:

Throughout G will denote the direct product $G = C_\infty \times \Phi$ where Φ is a finite group. We write $\Lambda = \mathbb{Z}[G]$ for the integral group ring of G and make the identification $\Lambda = \mathfrak{R}[\Phi]$ where $\mathfrak{R}$ is the integral group ring $\mathfrak{R} = \mathbb{Z}[C_\infty]$. We further identify $\mathfrak{R}$ with the ring $\mathbb{Z}[t, t^{-1}]$ of integral Laurent polynomials in t Then $\mathfrak{R}$ is a commutative Noetherian integral domain. Moreover, by the theorem of Sheshadri [19], [30]:

(**5.1**) Every finitely generated projective $\mathfrak{R}$-module is free.

If $i : \mathfrak{R} \to \Lambda$ denotes the canonical inclusion we have the 'extension of scalars' functor $i_* : \mathcal{M}od_\mathfrak{R} \to \mathcal{M}od_\Lambda$ and the 'restriction of scalars' functor $i^* : \mathcal{M}od_\Lambda \to \mathcal{M}od_\mathfrak{R}$. The following is clear:

(5.2) M is finitely generated over $\Lambda \Leftrightarrow i^*(M)$ is finitely generated over $\mathfrak{R}$.

A module $M \in \mathcal{M}od_\Lambda$ is said to be a Λ-*lattice* when $i^*(M)$ is a finitely generated free module over $\mathfrak{R}$. Such lattices are significant as they are precisely the Λ-modules defined by representations $\rho : \Phi \to GL_n(\mathfrak{R})$. The proofs of (5.3) to (5.6) below are straightforward

(5.3) Any finitely generated projective module over Λ is a Λ-lattice.

(5.4) Let M_1, M_2 be finitely generated Λ-modules which are stably equivalent; then M_1 is a Λ-lattice $\Longleftrightarrow$ M_2 is a Λ-lattice.

(5.5): Let $\mathcal{E} = (0 \to K \to L \to M \to 0)$ be an exact sequence of $\mathfrak{R}[\Phi]$-modules. If K and M are $\mathfrak{R}[\Phi]$-lattices then so also is L.

(5.6) Let $\mathcal{Q} = (0 \to \mathcal{J} \to Q_n \to \cdots \to Q_1 \to Q_0 \to M \to 0)$ be an exact sequence of $\mathfrak{R}[\Phi]$-modules in which M is an $\mathfrak{R}[\Phi]$-lattice and each Q_r is a finitely generated projective $\mathfrak{R}[\Phi]$-module. Then $\mathcal{J}$ is an $\mathfrak{R}[\Phi]$-lattice.

Taking $M = \mathfrak{R}$ then:

(5.7) The syzygy modules in a free $\mathfrak{R}[\Phi]$-resolution of $\mathfrak{R}$ over are lattices over $\mathfrak{R}[\Phi]$.

If M is a module over Λ we define the dual M^* by $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$. If M is a *right* Λ-module then M^* becomes a *left* Λ-module by writing

$$(\lambda \cdot \mathbf{a})(m) \quad = \quad \lambda \cdot \mathbf{a}(m).$$

A homomorphism $f : M \to N$ of right Λ-modules induces a homomorphism $f^* : N^* \to M^*$ of left Λ-modules by $f^*(\mathbf{a}) = \mathbf{a} \circ f$. To convert the left module M^* back to a right module we resort to a standard procedure from surgery theory (cf [33]) and assume that Λ is a ring with anti-involution θ by which we mean a self-inverse ring isomorphism $\theta : \Lambda \to \Lambda^{opp}$ to the opposite ring Λ^{opp} of Λ. When Λ has such an anti-involution θ we convert a left module $N = (N, \circ)$ to a right module $\overline{N} = (N, \diamond)$ as follows:

$$\begin{array}{rcl} \diamond : N \times \Lambda & \to & N \\ n \diamond \lambda & = & \theta(\lambda) \circ n \end{array}$$

$\overline{N}$ is the *conjugate* of N. We define the *conjugate dual* $M^\bullet$ of M by $M^\bullet = \overline{(M^*)}$. Repeating the duality functor has the same effect as repeating conjugate duality; that is:

(5.8) $\qquad M^{\bullet\bullet} \equiv M^{**} \qquad$ for any module $M \in \mathcal{M}\mathrm{od}_\Lambda$.

There is a homomorphism $\natural : M \to M^{\bullet\bullet}$ defined by $\natural(m)(\mathbf{a}) = \mathbf{a}(m)$ so that the correspondence $M \mapsto M^{\bullet\bullet}$ defines a natural transformation $\natural : \mathrm{Id} \to {}^{\bullet\bullet}$. We say that M is *reflexive* when $\natural : M \longrightarrow M^{\bullet\bullet}$ is an isomorphism.

The group algebra $\Lambda = \mathfrak{R}[\Phi]$ admits a canonical anti-involution '$\overline{}$ '

$$\overline{\sum_{g \in \Phi} a_g \cdot g} \quad = \quad \sum_{g \in \Phi} a_g \cdot g^{-1}$$

Now suppose that M is a Λ-lattice such that, with respect to an $\mathfrak{R}$ basis $\{e_i\}_{1 \le i \le n}$, M is described by the representation $\rho : G \to GL_n(\mathfrak{R})$. Then $M^\bullet$ has the dual basis $\{e_i^\bullet\}_{1 \le i \le n}$, defined by $e_i^\bullet(e_j) = \delta_{ij}$. with respect to which the representation $\rho^\bullet$ is given by conjugate transpose ; that is:

(5.9) $$\rho^\bullet(g) = \rho(g^{-1})^t.$$

In general modules are far from being reflexive; however $\rho^{\bullet\bullet}(g) = \rho(g)$ so that $\natural : M \longrightarrow M^{\bullet\bullet}$ is an isomorphism and hence:

(5.10) Any $\mathfrak{R}[\Phi]$-lattice is reflexive.

If M, N are $\mathfrak{R}[\Phi]$-modules then $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(M, N)$ is naturally an $\mathfrak{R}$-module. If, in addition, M, N are $\mathfrak{R}[\Phi]$-lattices we have canonical $\mathfrak{R}$-isomorphisms:

(5.11) $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(N^{\bullet}, M^{\bullet}) \cong \mathrm{Hom}_{\mathfrak{R}[\Phi]}(M, N)$.

The homomorphism $f : M \to N$ factors through the projective P if and only if $f^{\bullet} : N^{\bullet} \to M^{\bullet}$ factors through the projective $P^{\bullet}$; thus we have:

(5.12) $\mathrm{Hom}^{0}_{\mathfrak{R}[\Phi]}(N^{\bullet}, M^{\bullet}) \cong \mathrm{Hom}^{0}_{\mathfrak{R}[\Phi]}(M, N)$.

(5.13) $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(N^{\bullet}, M^{\bullet}) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, N)$.

In the regular representation ρ_{reg} of Φ, each $\rho_{\mathrm{reg}}(g)$ is a permutation matrix; hence $\rho_{\mathrm{reg}}(g) = \rho_{\mathrm{reg}}(g^{-1})^{t}$; thus the group ring $\mathfrak{R}[\Phi]$ is self-dual; that is;

(5.14) $$\mathfrak{R}[\Phi]^{\bullet} \cong \mathfrak{R}[\Phi].$$

If M is an $\mathfrak{R}[\Phi]$ lattice and N is free over $\mathfrak{R}$ then as Φ is finite we have the Eckmann-Shapiro relation $\mathrm{Ext}^{n}_{\mathfrak{R}[\Phi]}(M, i_{*}(N)) \cong \mathrm{Ext}^{n}_{\mathfrak{R}}(i^{*}(M), N)$. (cf [13] Appendix B). Taking $N = \mathfrak{R}$ then $i_{*}(N) = \mathfrak{R}[\Phi]$ and so

(5.15) $$\mathrm{Ext}^{n}_{\mathfrak{R}[\Phi]}(M, \mathfrak{R}[\Phi]) \cong \mathrm{Ext}^{n}_{\mathfrak{R}}(i^{*}(M), \mathfrak{R}).$$

If M is an $\mathfrak{R}[\Phi]$-lattice then $i^{*}(M)$ is free over $\mathfrak{R}$ and $\mathrm{Ext}^{n}_{\mathfrak{R}}(i^{*}(M), \mathfrak{R}) = 0$ for all $n \geq 1$. Thus $\mathrm{Ext}^{n}_{\mathfrak{R}[\Phi]}(M, \Lambda) = 0$ and we arrive at:

(5.16) If M is a lattice over $\mathfrak{R}[\Phi]$ then M is n-coprojective for all $n \geq 1$.

Evidently $\mathfrak{R}$ is an $\mathfrak{R}[\Phi]$-lattice. Thus we have:

(5.17) $\mathfrak{R}$ is n-coprojective for all $n \geq 1$.

We conclude this section by computing $\mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\mathfrak{R})$. As projective modules are direct summands of free modules it is enough to consider homomorphisms $f : \mathfrak{R} \to \mathfrak{R}$ which factor through $\Lambda^{(n)}$. Let $\epsilon : \Lambda \to \mathfrak{R}$ be the augmentation homomorphism, $\epsilon(x^{r}) = 1$. We note that $\mathrm{Hom}_{\Lambda}(\Lambda, \mathfrak{R}) \cong \mathfrak{R}$ generated by the augmentation homomorphism ϵ. If $\xi : \Lambda^{(n)} \to \mathfrak{R}$ is Λ linear then

(5.18) $$\xi = (\xi_{1}\epsilon, \cdots, \xi_{n}\epsilon) \text{ for some } (\xi_{1}, \cdots, \xi_{n}) \in \mathfrak{R}^{(n)}.$$

Let $\epsilon^{\bullet} : \mathfrak{R} \to \Lambda$ denote the Λ-dual of ϵ; then $\epsilon^{\bullet}(1) = \sum_{g \in \Phi} g$. Then $\mathrm{Hom}_{\Lambda}(\mathfrak{R}, \Lambda) \cong \mathfrak{R}$ generated by $\epsilon^{\bullet}$; if $\eta : \mathfrak{R} \to \Lambda^{(n)}$ is Λ linear then

(5.19) $$\eta = (\eta_{1}, \cdots, \eta_{n})^{t}\epsilon^{\bullet} \text{ for some } (\eta_{1}, \cdots, \eta_{n}) \in \mathfrak{R}^{(n)}.$$

If $f : \mathfrak{R} \to \mathfrak{R}$ admits a factorization $f = \xi \circ \eta$ through the free module $\Lambda^{(n)}$ then $f(1) = (\sum_{r=1}^{n} \xi_r \eta_r)\epsilon \circ \epsilon^{\bullet}(1)$. However $\epsilon \circ \epsilon^{\bullet}(1) = |\Phi|$ so that

$$\textbf{(5.20)} \qquad \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\mathfrak{R}) \cong \mathfrak{R}/|\Phi|.$$

For each $k \geq 1$ let $\mathcal{J}_k \in \Omega_k(\mathfrak{R})$. By (5.17), $\mathfrak{R}$ is n-coprojective for all $n \geq 1$. It follows from (4.10) that $\mathrm{Ext}^1_{\Lambda}(\mathcal{J}_{n-1}, \mathcal{J}_n) \cong \mathrm{Ext}^1_{\mathfrak{R}[\Phi]}(\mathfrak{R}, \mathcal{J}_1)$; by (4.2)

$$\mathrm{Ext}^1(\mathfrak{R}, \Omega_1(\mathfrak{R})) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\mathcal{J}_1, \mathcal{J}_1) = \mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathcal{J}_1)$$

whilst by (4.5) and (5.20), $\mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathcal{J}_1) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathfrak{R}) = \mathfrak{R}/|\Phi|$. We arrive at the following which is Theorem III of the Introduction.

(5.21) $\mathrm{Ext}^1_{\Lambda}(\mathcal{J}_{n-1}, \mathcal{J}_n) \cong \mathfrak{R}/|\Phi|$ for $n \geq 1$ where $\mathcal{J}_k \in \Omega_k(\mathfrak{R})$.

§6: An elementary matrix calculation:

Let R be an associative ring with unity. If $\alpha, \beta \in GL_2(R)$ we define

$$C(\alpha, \beta) = \alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \beta.$$

If $\alpha\beta + C(\alpha, \beta) = 0$ then $C(\alpha, \beta) = -\alpha\beta \in GL_2(R)$. This is a contradiction as $C(\alpha, \beta)$ is not invertible. Likewise if $C(\alpha, \beta) = 0$ then

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \alpha^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \beta^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is again a contradiction; thus it follows that:

(6.1) If $\alpha, \beta \in GL_2(R)$ then $\alpha\beta + C(\alpha, \beta) \neq 0$ and $C(\alpha, \beta) \neq 0$.

If $\Theta \in M_2(R[t, t^{-1}])$ we may write Θ as finite sum $\Theta = \sum \Theta_r t^r$ where each $\Theta_r \in M_2(R)$. We define $\chi(\Theta) \in \mathbb{N}$ by $\chi(\Theta) = |\{r \mid \Theta_r \neq 0\}|$; then:

$$\textbf{(6.2)} \qquad \chi(t^r \cdot \Theta \cdot t^s) = \chi(\Theta) \quad \text{for all } r, s \in \mathbb{Z}.$$

For each positive integer m define $P_m(t) \in M_2(R[t.t^{-1}])$ by

$$P_m(t) = \begin{pmatrix} 1 & 1 + t + \cdots + t^m \\ 0 & 1 \end{pmatrix}.$$

Writing $P_m(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \sum_{r=1}^{m} \begin{pmatrix} 0 & t^r \\ 0 & 0 \end{pmatrix}$ then $\chi(P_m(t)) = m + 1$ and $\alpha \cdot P_m(t) \cdot \beta = (\alpha\beta + C(\alpha, \beta)) + \sum_{r=1}^{m} C(\alpha, \beta) t^r$. It follows from (6.1) that:

$$\textbf{(6.3)} \qquad \text{If } \alpha, \beta \in GL_2(R), \ \chi(\alpha \cdot P_m(t) \cdot \beta) = m + 1$$

We define $T = \{\alpha \cdot t^r \mid \alpha \in GL_2(R) , r \in \mathbb{Z}\}$. As $\alpha \cdot t^t = t^r \cdot \alpha$ for $\alpha \in GL_2(R)$ and $r \in \mathbb{Z}$ then T is a subgroup of $GL_2(R[t,t^{-1}])$. There is an equivalence relation '$\approx$' on $GL_2(R[t,t^{-1}])$ given by

(6.4) $$X \approx Y \iff Y = \tau_1 \cdot X \cdot \tau_2 \quad \text{for some } \tau_1, \tau_2 \in T.$$

We denote by $\langle X\rangle$ the equivalence class of $X \in GL_2(R[t,t^{-1}])$ under '$\approx$':

$$\langle X\rangle \in T\backslash GL_2(R[t,t^{-1}]/T = GL_2(R[t,t^{-1}])/\approx .$$

Proposition 6.5: For any ring R the equivalence classes $\{\langle P_m(t)\rangle\}_{1\leq m}$ are pairwise distinct elements of $T\backslash GL_2(R[t,t^{-1}]/T$.

Proof : Suppose that $P_m(t) \approx P_n(t)$ so that $P_n(t) = \tau_1 P_m(t)\tau_2$ for $\tau_1, \tau_2 \in T$. Writing $\tau_1 = t^r \cdot \alpha$ and $\tau_2 = \beta \cdot t^s$ where $\alpha, \beta \in GL_2(R)$ and $r, s \in \mathbb{Z}$; then by (6.2) and (6.3) we see that:

$$\begin{aligned} \chi(P_n(t) &= \chi(\tau_1 \cdot P_m(t) \cdot \tau_2) \\ &= \chi(t^r \cdot \alpha \cdot P_m(t) \cdot \beta \cdot t^s) \\ &= \chi(\alpha \cdot P_m(t) \cdot \beta) \\ &= \chi(P_m(t)) \end{aligned}$$

Thus if $P_m(t) \approx P_n(t)$ then $\chi(P_m(t)) = \chi(P_n(t))$ and hence $m = n$. In the contrapositive, if $m \neq n$ then $P_m(t) \not\approx P_n(t)$. □

§7 : Fibre products and Milnor's patching construction :

A commutative diagram of ring homomorphisms

(7.1) $$\mathfrak{A} = \left\{ \begin{array}{ccc} A & \xrightarrow{\rho} & A_- \\ \downarrow \pi & & \downarrow \varphi_- \\ A_+ & \xrightarrow{\varphi_+} & A_0. \end{array} \right.$$

is said to be a *fibre square* when $\pi \times \rho$ maps A isomorphically onto the fibre product $A_+ \underset{\varphi_+ , \varphi_-}{\times} A_- = \{(\lambda_+, \lambda_-) \in A_+ \times A_- : \varphi_+(\lambda_+) = \varphi_-(\lambda_-)\}$.

Working in the category of 'modules over change of rings' we take a module P_0 over A_0 and, for $\sigma = +, -$, a module P_σ over A_σ and assume there are homomorphisms $\nu_\sigma : P_\sigma \to P_0$ over φ_σ inducing A_0-isomorphisms $\widehat{\nu_\sigma} : P_\sigma \otimes_{\varphi_\sigma} A_0 \to P_0$. Given a A_0-automorphism $\alpha : P_0 \to P_0$ we define the Milnor module $\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle$ by

$$\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle = P_+ \underset{\alpha\nu_+,\nu_-}{\times} P_-$$

and define an action of A on $\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle$ by

$$\begin{aligned} \bullet : \langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle \times A &\to \langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle \\ (x_+, x_-) \bullet a &= (x_+ \cdot \pi(a), x_- \cdot \rho(a)). \end{aligned}$$

In this notation the free module of rank n is described as

(7.2) $$A^n \;=\; \langle A_+^n,\, A_-^n,\, \varphi_+^n,\, \varphi_-^n,\, \mathrm{Id}\rangle.$$

Assuming that either φ_+ or φ_- is surjective and that $\alpha \in \mathrm{Aut}(P_0)$ then the following statements (7.3)-(7.7) below are true (c.f. [21],[25]) :

(7.3) $\pi_*(\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle) \cong P_+$;

(7.4) $\rho_*(\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle) \cong P_-$;

(7.5) If $\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle \cong \langle Q_+, Q_-, \mu_+, \mu_-, \beta\rangle$ then $P_+ \cong Q_+$ and $P_- \cong Q_-$;

(7.6) $\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle$ is projective $\iff$ P_+ , P_- are projective.

Denoting by $[\alpha]$ the class of α in $\nu_+(\mathrm{Aut}(P_+))\backslash\mathrm{Aut}(P_0)/\nu_-(\mathrm{Aut}(P_-))$, then if P_+, P_-, are projective :

(7.7) $$\langle P_+, P_-, \nu_+, \nu_-, \alpha\rangle \cong \langle P_+, P_-, \nu_+, \nu_-, \beta\rangle \iff [\alpha] = [\beta]$$

As a special case we may take $P_\sigma \;=\; A_\sigma^{(k)}$ $P_\sigma \otimes_{A_\sigma} A_0 \;=\; A_0^{(k)}$ so that $P_\sigma \otimes_{A_\sigma} A_0 \;=\; A_0^{(k)}$ and $\alpha \in GL_k(A_0)$. In this case we write

$$\mathcal{L}(\alpha) \;=\; (A_+^{(k)}, A_-^{(k)}, \varphi_+, \varphi_-, \alpha).$$

$\mathcal{L}(\alpha)$ is then said to be *locally free*[(1)] *of rank k with respect to $\mathfrak{A}$* or simply $\mathfrak{A}$-*locally free of rank* k.

We define $$\overline{GL_k(\mathfrak{A})} \;=\; \varphi_+(GL_k(A_+))\backslash GL_k(A_0)/\varphi_-(GL_k(A_-)).$$

When $\mathfrak{A}$ is a Milnor square, Milnor's classification theorem ([25], pp 20-24; see also Lemma A4 of [31], Appendix A) gives a bijection :

(7.8) $$\{\, \mathfrak{A}\text{-locally free modules of rank } k \,\} \quad \stackrel{\simeq}{\longleftrightarrow} \quad \overline{GL_k(\mathfrak{A})}.$$

The case $k = 1$ is of most interest to us. In this case $GL_1(A_\sigma) \;=\; U(A_\sigma)$, the unit group of A_σ for $\sigma \in \{+, 0, -\}$ and hence

(7.9) $$\{\mathfrak{A}\text{-locally free modules of rank } 1\,\} \stackrel{\simeq}{\longleftrightarrow} \varphi_+(U(A_+))\backslash U(A_0)/\varphi_-(U(A_-)).$$

---

(1) We stress that *local freeness* in the sense used here should not be confused with the notion of *local freeness at a prime p* which occurs frequently elsewhere in the literature; for example, in [31].

§8 : Constructing stably free modules :

We shall represent a typical fibre square $\mathfrak{A}$ of ring homomorphisms in the form of (7.1) above. In this section and the next we shall impose conditions upon $\mathfrak{A}$. If R is a ring we denote by $M_2(R)$ the ring of 2×2 matrices over R and by $i : R \to M_2(R)$ the injective ring homomorphism

$$i(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

If A is a commutative ring and $C_\infty^{(n)}$ is the free abelain group of rank n we represent the group ring $A[C_\infty^{(n)}]$ as $A[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, the ring of Laurent polynomials in $t_1, \dots, t_n$ with coefficients in A. We say that $\mathfrak{A}$ is *constrained* when conditions $\mathcal{C}(1)$, $\mathcal{C}(2)$, $\mathcal{C}(3)$ below are satisfied:

$\mathcal{C}(1)$: $\varphi_+ : A_+ \to A_0$ is surjective;

$\mathcal{C}(2)$: A_0 has a subring R such that there is an isomorphism of rings $\nu : M_2(R) \to A_0$ making the following diagram commute where i and j denote inclusion

$$\begin{array}{ccc} M_2(R) & \xrightarrow{\ \nu\ } & A_0 \\ {\scriptstyle i}\nwarrow & & \nearrow{\scriptstyle j} \\ & R & \end{array}$$

$\mathcal{C}(3)$: A_+, A_- are integral domains, possibly non-commutative, and every stably free module over $A_0[C_\infty^{(n)}]$ is free for all $n \geq 1$.

Until further mention we assume that $\mathfrak{A}$ is constrained. By Morita equivalence, if $A_0 \cong M_2(R)$ then every stably free $A_0[t, t^{-1}]$-module is free precisely when every stably free $R[t, t^{-1}]$-module is free. Given a matrix ring $M_2(R)$ for each $\lambda \in R$ we define

$$\overline{v_\lambda} = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \quad \text{and put} \quad \overline{\omega} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

We note the identities $\overline{v_\lambda} \cdot \overline{v_\lambda} = 0$; $\overline{v_\lambda} \cdot \overline{\omega} = 0$; $\overline{\omega} \cdot \overline{v_\lambda} = \overline{v_\lambda}$. We define $v_\lambda = \nu(\overline{v})$ and $\omega = \nu(\overline{\omega})$ and note that $v_\lambda = v_\mu \iff \lambda = \mu$. . It follows that:

(8.1) $\quad v_\lambda \cdot v_\lambda = 0; \quad v_\lambda \cdot \omega = 0; \quad \omega \cdot v_\lambda = v_\lambda.$

As $v_\lambda \cdot v_\lambda = 0$ then $(1 + v_\lambda)(1 - v_\lambda) = 1$; hence $1 + v_\lambda \in U(A_0)$ for each $\lambda \in R$. It follows easily from (8.1) that

$$\begin{pmatrix} 1+\upsilon_\lambda & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \upsilon_\lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\upsilon_\lambda & 1 \end{pmatrix}$$

$$= E(1,2;\omega)\, E(2,1;\upsilon_\lambda)\, E(1,2;-\omega)\, E(2,1;-\upsilon_\lambda).$$

As φ_+ is surjective then $E_2(A_0) \subset \mathrm{Im}(\varphi_+ : GL_2(A_+) \to GL_2(A_0))$ Consequently for each $\lambda \in R$ we have :

(8.2) $$\begin{pmatrix} 1+\upsilon_\lambda & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Im}(\varphi_+ : GL_2(A_+) \to GL_2(A_0)) \ .$$

It follows that in $GL_2(A_+)\backslash GL_2(A_0)/GL_2(A_-)$ we have equality

(8.3) $$\left[\begin{pmatrix} 1+\upsilon_\lambda & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

As $1+\upsilon_\lambda \in U(A_0)$ there is a projective A module $\mathfrak{S}(\lambda)$ defined by:

(8.4) $$\mathfrak{S}(\lambda) = \langle A_+, A_-, \varphi_+, \varphi_-, 1+\upsilon_\lambda\rangle.$$

It follows from (7.8) and (8.3) that $\mathfrak{S}(\lambda) \oplus A \cong A \oplus A$; to summarise:

Theorem 8.5 : If $\mathfrak{A}$ is constrained then for each $\lambda \in R$:

(i) $\mathfrak{S}(\lambda) \oplus A \cong A \oplus A$; in particular, $\mathfrak{S}(\lambda)$ is stably free of rank 1;

(ii) $\mathfrak{S}(\lambda) \cong \mathfrak{S}(\mu) \Longleftrightarrow 1+\upsilon_\lambda = \varphi_+(u_+)(1+\upsilon_\mu)\varphi_-(u_-)$ for some $u_\sigma \in U(A_\sigma)$;

(iii) $\mathfrak{S}(\lambda) \cong A \Longleftrightarrow 1+\upsilon_\lambda = \varphi_+(u_+)\varphi_-(u_-)$ for some $u_\sigma \in U(A_\sigma)$.

If G is a group and $\mathfrak{A}$ is a fibre square we denote by $\mathfrak{A}[G]$ the diagram

$$\mathfrak{A}[G] \qquad \begin{array}{ccc} A[G] & \xrightarrow{\rho} & A_-[G] \\ \downarrow \pi & & \downarrow \varphi_- \\ A_+[G] & \xrightarrow{\varphi_+} & A_0[G]. \end{array}$$

where the symbols ρ, π, φ_+ and φ_- are retained to represent the induced homomorphisms from the diagram $\mathfrak{A}$; then $\mathfrak{A}[G]$ is also a fibre square. Suppose that $\mathfrak{A}$ is constrained; as $\varphi_+ : A_+ \to A_0$ is surjective then so is $\varphi_+ : A_+[G] \to A_0[G]$. Also, as $M_2(R) \cong A_0$ then $M_2(R[G]) \cong A_0[G]$. Thus $\mathfrak{A}[G]$ satisfies conditions $\mathcal{C}(1)$ and $\mathcal{C}(2)$. In the special case where $G = C_\infty$ then $A_+[C_\infty]$ and $A_-[C_\infty]$ are integral domains. Moreover, as $(A_0[C_\infty])[C_\infty^{(n)}] \cong A_0[C_\infty^{(n+1)}]$ then $\mathfrak{A}[C_\infty]$ also satisfies $\mathcal{C}(3)$; that is:

(8.6) If $\mathfrak{A}$ is constrained then $\mathfrak{A}[C_\infty]$ is also constrained.

Now suppose that $\mathfrak{A}$ is a constrained fibre square; then $\mathfrak{A}[C_\infty]$ is also constrained. We define an equivalence relation ' $\sim$' on $GL_2(R[t,t^{-1}])$ by

$$X \sim Y \iff Y = \varphi_+(u_+)X\varphi_-(u_-) \text{ for some } u_\sigma \in U(A_\sigma[t,t^{-1}])$$

As A_+, A_- are integral domains then by Higman's Theorem ([9]) both $A_+[t,t^{-1}]$ and $A_-[t,t^{-1}]$ have only trivial units, thus:

(8.7) $$U(A_\sigma[t,t^{-1}]) = \{u_\sigma \cdot t^r \mid u_- \in U(A_\sigma)\}.$$

We define $T_\sigma = \varphi_-\sigma(U(A_\sigma[t,t^{-1}]) = \{\varphi_-\sigma(u_\sigma)\cdot t^r \mid u_\sigma \in U(A_\sigma),\ r \in \mathbb{Z}\}$. Hence T_σ is a subgroup of T; thus if $X,\ Y \in GL_2(R[t,t^{-1}])$ then:

$$X \sim Y \implies X \approx Y.$$

There is a surjective mapping $T_+\backslash GL_2(R[t,t^{-1}]/T_- \twoheadrightarrow T\backslash GL_2(R[t,t^{-1}]/T$; $[X] \mapsto \langle X\rangle$ where $[X]$ is the equivalence classes of X in $T_+\backslash GL_2(R[t,t^{-1}]/T_-$. It follows from (6.5) that:

Corollary 8.8: Given a constrained fibre square $\mathfrak{A}$ as above then the equivalence classes $\{[P_m(t)]\}_{1\le m}$ are pairwise distinct.

The induced isomorphism $A_0[t,t^{-1}] \cong M_2(R[t,t^{-1}])$ gives a bijection $1 + v_{p_m(t)} \longleftrightarrow P_m(t)$. In the notation of (8.4) put $\mathfrak{S}(m) = \mathfrak{S}(v_{p_m(t)})$. We arrive at the following:

Theorem 8.9

Let $$\mathfrak{A} = \left\{ \begin{array}{ccc} A & \xrightarrow{\rho} & A_- \\ \downarrow \pi & & \downarrow \varphi_- \\ A_+ & \xrightarrow{\varphi_+} & A_0 \end{array} \right.$$

be a constrained fibre square. Then there exists an infinite collection $\{\mathfrak{S}(m)\}_{m\ge 1}$ of pairwise isomorphically distinct stably free modules of rank 1 over $A[t,t^{-1}]$; furthermore for each m, $\mathfrak{S}(m) \oplus A[t,t^{-1}] \cong A[t,t^{-1}] \oplus A[t,t^{-1}]$.

§9 : Lifting stably free modules :

We continue to describe fibre squares in the form (7.1).

Theorem 9.1 : Let $\mathfrak{A}$ be a fibre square in which $\varphi_+ : A_+ \to A_0$ is surjective; if S is a stably free module over A_- then

(i) there exists a projective module $\widetilde{S}$ over A such that $\rho_*(\widetilde{S}) \cong S$; moreover

(ii) $\widetilde{S}$ may be chosen to be stably free if $S \otimes_{\varphi_-} A_0$ is free.

Proof : As S is stably free over A_- then, for some k, $S \oplus A_-^k \cong A_-^{n+k}$ and we may present S by means of a exact sequence

$$\mathcal{E} = (0 \to A_-^k \stackrel{j}{\to} A_-^{n+k} \stackrel{\pi}{\to} S \to 0)$$

which admits a splitting as S is projective. Let $\nu : S \to S \otimes_{\varphi_-} A_0$ be the mapping $\nu(x) = x \otimes 1$. Applying $\otimes_{\varphi_-} A_0$ we obtain a commutative diagram in which the downward arrows are homomorphisms over the ring homomorphism φ_-

(9.2)
$$\begin{array}{cccccccc} 0 \to & A_-^k & \stackrel{j}{\to} & A_-^{n+k} & \stackrel{\pi}{\to} & S & \to 0 \\ & \downarrow \varphi_-^k & & \downarrow \varphi_-^{n+k} & & \downarrow \nu & \\ 0 \to & A_0^k & \stackrel{j_*}{\to} & A_0^{n+k} & \stackrel{\pi_*}{\to} & S \otimes_{\varphi_-} A_0 & \to 0 \end{array}$$

As $\mathcal{E}$ admits a splitting, the bottom row is exact. By hypothesis $S \otimes_{\varphi_-} A_0$ is free. Let $h : S \otimes_{\varphi_-} A_0 \to A_0^n$ be an isomorphism and re-write (9.2) as

$$\begin{array}{cccccccc} 0 \to & A_-^k & \stackrel{j}{\to} & A_-^{n+k} & \stackrel{\pi}{\to} & S & \to 0 \\ & \downarrow \varphi_-^k & & \downarrow \varphi_-^{n+k} & & \downarrow h \circ \nu & \\ 0 \to & A_0^k & \stackrel{j_*}{\to} & A_0^{n+k} & \stackrel{h \circ \pi_*}{\to} & A_0^n & \to 0 \end{array}$$

where both rows remain exact and all downward arrows are isomorphisms. Let $\mathcal{A}_+$, $\mathcal{A}_-$ denote the standard exact sequences

$$\mathcal{A}_+ = (0 \to A_+^k \stackrel{i}{\to} A_+^{n+k} \stackrel{p}{\to} A_+^n \to 0) \ ; \ \mathcal{A}_0 = (0 \to A_0^k \stackrel{i}{\to} A_0^{n+k} \stackrel{p}{\to} A_0^n \to 0)$$

Then there is a morphism of exact sequences over φ_+

$$\begin{array}{cccccccc} 0 \to & A_0^k & \stackrel{i}{\to} & A_0^{n+k} & \stackrel{p}{\to} & A_0^n & \to 0 \\ & \uparrow {\varphi_+}^k & & \uparrow {\varphi_+}^{n+k} & & \uparrow {\varphi_+}^n & \\ 0 \to & A_+^k & \stackrel{i}{\to} & A_+^{n+k} & \stackrel{p}{\to} & A_+^n & \to 0. \end{array}$$

Moreover, there is a congruence

$$\begin{array}{cccccccc} 0 \to & A_0^k & \stackrel{j_*}{\to} & A_0^{n+k} & \stackrel{h \circ \pi_*}{\to} & A_0^n & \to 0 \\ & \uparrow \mathrm{Id} & & \uparrow \gamma & & \uparrow \mathrm{Id} & \\ 0 \to & A_0^k & \stackrel{i}{\to} & A_0^{n+k} & \stackrel{p}{\to} & A_0^n & \to 0. \end{array}$$

By composition, we obtain a commutative diagram

$$\begin{array}{cccccccc} 0 \to & A_-^k & \stackrel{j}{\to} & A_-^{n+k} & \stackrel{\pi}{\to} & S & \to 0 \\ & \downarrow \varphi_-^k & & \downarrow \varphi_-^{n+k} & & \downarrow h\nu & \\ 0 \to & A_0^k & \stackrel{i}{\to} & A_0^{n+k} & \stackrel{p}{\to} & A_0^n & \to 0 \\ & \uparrow {\varphi_+}^k & & \uparrow \gamma{\varphi_+}^{n+k} & & \uparrow {\varphi_+}^n & \\ 0 \to & A_+^k & \stackrel{i}{\to} & A_+^{n+k} & \stackrel{p}{\to} & A_+^n & \to 0. \end{array}$$

We thus obtain an exact sequence

$$0 \to \langle A_+^k, A_-^k, \varphi_+^k, \varphi_-^k, \mathrm{Id}\rangle \to \langle A_+^{n+k}, A_-^{n+k}, \varphi_+^{n+k}, {\varphi_-}^{n+k}, \gamma\rangle \to \langle A_+^n, S, \varphi_+^n, h\nu, \mathrm{Id}\rangle \to 0.$$

Put $\widetilde{S} = \langle A_+^n, S, \varphi_+^n, h\nu, \mathrm{Id}\rangle$; as in (7.2) $A^k = \langle A_+^k, A_-^k, \varphi_+^k, \varphi_-^k, \ \mathrm{Id}\rangle$ so the exact sequence becomes $0 \to A^k \to \langle A_+^{n+k}, A_-^{n+k}, \varphi_+^{n+k}, \varphi_-^{n+k}, \gamma\rangle \to \widetilde{S} \to 0$. Then $\widetilde{S}$ is projective by (7.6) and so $\widetilde{S} \oplus A^k \cong \langle A_+^{n+k}, A_-^{n+k}, \varphi_+^{n+k}, \varphi_-^{n+k}, \gamma\rangle$. However the congruence γ can be represented by an upper triangular matrix

$$\gamma = \begin{pmatrix} \mathrm{Id}_k & \alpha_0 \\ 0 & \mathrm{Id}_n \end{pmatrix}$$

where α_0 is a $k \times n$ matrix with values in A_0. Since $\varphi_+ : A_+ \to A_0$ is surjective we may choose a $k \times n$ matrix α_+ with values in A_+ such that $\varphi_+(\alpha_+) = \alpha_0$. Then putting

$$\widetilde{\gamma} = \begin{pmatrix} \mathrm{Id}_k & \alpha_+ \\ 0 & \mathrm{Id}_n \end{pmatrix}$$

we see that $\widetilde{\gamma} \in \mathrm{GL}_{n+k}(B)$ and that $\varphi_+(\widetilde{\gamma}) = \gamma$. Thus by (7.7)

$$\langle A_+^{n+k}, A_-^{n+k}, \varphi_+^{n+k}, \varphi_-^{n+k}, \widetilde{\gamma}\rangle \cong A^{n+k};$$

hence $\widetilde{S} \oplus A^k \cong A^{n+k}$ and $\widetilde{S}$ is stably free. Finally, $\rho_*(\widetilde{S}) = S$ by (7.3). □

For any ring Λ we denote by $\mathbf{SF}_n(\Lambda)$ the isomorphism classes of stably free Λ-modules of rank n; we say that the homomorphism $\psi : A \to B$ has the *lifting property for stably free modules* when $\psi_* : \mathbf{SF}_n(A) \to \mathbf{SF}_n(B)$ is surjective for all $n \geq 1$. It follows from (9.1) that:

Corollary 9.3 : Let $\mathfrak{A}$ be a fibre square as in (7.1); if $\mathfrak{A}$ is constrained then ρ has the lifting property for stably free modules.

§10: Stably free modules over quaternionic group rings:

The quaternionic group Q_{4n} of order $4n$ is given by the presentation

$$Q_{4n} = \langle x, y \mid x^n = y^2, \ yx = x^{-1}y\rangle$$

A subsidiary role is played by the cyclic and dihedral groups

$$C_n = \langle x, y \mid x^n = 1\rangle \quad ; \quad D_{2n} = \langle x, y \mid x^n = 1, y^2 = 1, \ yx = x^{-1}y\rangle.$$

When p is an odd prime we shall construct infinitely many stably free modules of rank 1 over the group algebras $\mathfrak{R}[Q_{8p}]$ where $\mathfrak{R} = \mathbb{Z}[t, t^{-1}]$. Essential to the construction are cyclic algebras which we now describe; a general reference for this construction is [29], Chapter 15.

By a *pointed involution* we mean a triple $(A, \theta, \mathbf{a})$ where A is a commutative ring, $\theta : A \to A$ is ring automorphism satisfying $\theta^2 = -\mathrm{Id}$ and $\mathbf{a} \in A$

satisfies $\theta(\mathbf{a}) = \mathbf{a}$. The cyclic algebra $\mathcal{C}(A, \theta, \mathbf{a})$ is then defined as the free A-module of rank 2 with basis $\{1, y\}$ and with multiplication

$$\begin{cases} y\lambda & = & \theta(\lambda)y \\ \\ y^2 & = & \mathbf{a} \end{cases} .$$

By a morphism of pointed involutions $f : (A_1, \theta_1, \mathbf{a}_1) \to (A_2, \theta_2, \mathbf{a}_2)$ we will mean a ring homomorphism $f : A_1 \to A_2$ such that $\theta_2 \circ f = f \circ \theta_1$ and $f(\mathbf{a}_1) = \mathbf{a}_2$. The cyclic algebra construction is functorial in the sense that a morphism of pointed involutions $f : (A_1, \theta_1, \mathbf{a}_1) \to (A_2, \theta_2, \mathbf{a}_2)$ gives rise to a ring homomorphism

$$f_* : \mathcal{C}(A_1, \theta_1, \mathbf{a}_1) \to \mathcal{C}(A_2, \theta_2, \mathbf{a}_2)$$

on taking $f(y_1^r \lambda) = y_2^r f(\lambda)$. On applying the cyclic algebra construction to a fibre square of pointed involutions

$$\begin{array}{ccc} (A, \theta, \mathbf{a}) & \stackrel{i_2}{\longrightarrow} & (A_2, \theta_2, \mathbf{a}_2) \\ \\ \downarrow i_1 & & \downarrow q_2 \\ \\ (A_1, \theta_1, \mathbf{a}_1) & \stackrel{q_1}{\longrightarrow} & (A_0, \theta_0, \mathbf{a}_0) \end{array}$$

we obtain a fibre square of ring homomorphisms:

$$\begin{array}{ccc} \mathcal{C}(A, \theta, \mathbf{a}) & \stackrel{(i_2)_*}{\longrightarrow} & \mathcal{C}(A_2, \theta_2, \mathbf{a}_2) \\ \\ \downarrow (i_1)_* & & \downarrow (q_2)_* \\ \\ \mathcal{C}(A_1, \theta_1, \mathbf{a}_1) & \stackrel{(q_1)_*}{\longrightarrow} & \mathcal{C}(A_0, \theta_0, \mathbf{a}_0). \end{array}$$

The group ring $\mathbb{Z}[\Gamma]$ of an abelian group Γ admits the canonical involution

$$\theta(\sum a_g g) = \sum a_g g^{-1}.$$

Evidently $(\mathbb{Z}[\Gamma], \theta, 1)$ is a pointed involution as $\theta(1) = 1$. The group ring $\mathbb{Z}[D_{2n}]$ is obtained by applying the cyclic algebra construction to $\mathbb{Z}[C_n]$ thus:

(10.1) $$\mathbb{Z}[D_{2n}] \cong \mathcal{C}(\mathbb{Z}[C_n], \theta, 1).$$

For clarity we write $C_{2n} = \langle x \mid x^{2n} = 1 \rangle$ and $C_n = \langle s \mid s^n = 1 \rangle$.

As $x^{2n} - 1 = (x^n - 1)(x^n + 1)$, the identifications $\mathbb{Z}[C_n] = \mathbb{Z}[s]/(s^n - 1)$ and $\mathbb{Z}[C_{2n}] = \mathbb{Z}[x]/(x^{2n} - 1)$ give the following fibre square of pointed involutions

$$
\begin{array}{ccc}
(\mathbb{Z}[C_{2n}], \theta, x^n) & \xrightarrow{\pi_-} & (\mathbb{Z}[x]/(x^n+1), \theta', -1) \\
\pi_+ \downarrow & & \downarrow \varphi_- \\
(\mathbb{Z}[C_n], \theta, 1) & \xrightarrow{\varphi_+} & (\mathbb{F}_2[C_n], \theta, 1).
\end{array} \tag{10.2}
$$

where $\theta_+(x) = s$ and $\varphi_+(\sum_{k=0}^{n-1} a_k s^k) = \sum_{k=0}^{n-1} [a_k] s^k = \varphi_-(\sum_{k=0}^{n-1} a_k x^k)$.

In each case θ denotes the canonical involution on the appropriate group ring and $\theta' : \mathbb{Z}[x]/(x^n+1) \to \mathbb{Z}[x]/(x^n+1)$ is the involution $\theta'(x^r) = x^{n-r}$.

Writing $\Sigma_x = \sum_{k=0}^{2n-1} x^k \quad ; \quad \Sigma_s = \sum_{k=0}^{n-1} s^k$

then the factorization $\sum_{j=0}^{2n-1} x^j = \sum_{k=0}^{n-1} x^k(1+x^n)$ shows that

(10.3) $\pi_+(\Sigma_x) = 2\Sigma_s \quad ; \quad \pi_-(\Sigma_x) = 0$

We define $\Theta(n) = \mathcal{C}(\mathbb{Z}[x]/(x^n+1), \theta', -1)$. Applying the cyclic algebra construction to (10.2) gives a fibre square where, by a slight abuse of notation, we use the same labels for the homomorphisms:

$$
\begin{array}{ccc}
\mathbb{Z}[Q_{4n}] & \xrightarrow{\pi_-} & \Theta(n) \\
\pi_+ \downarrow & & \downarrow \varphi_- \\
\mathbb{Z}[D_{2n}] & \xrightarrow{\varphi_+} & \mathbb{F}_2[D_{2n}].
\end{array} \tag{10.4}
$$

Writing $\Sigma_Q = \sum_{g \in Q_{4n}} g$ then $\Sigma_Q = \Sigma_x + \Sigma_x y$ where y denotes the variable from the cyclic algebra construction. Likewise writing $\Sigma_D = \sum_{\gamma \in D_{2n}} \gamma$ then $\Sigma_D = \Sigma_s + \Sigma_s y$. It now follows from (10.3) that

(10.5) $\pi_+(\Sigma_Q) = 2\Sigma_D \quad ; \quad \pi_-(\Sigma_Q) = 0$

However, $\mathbb{Z}[Q_{4n}]/(\Sigma_Q) \cong I^*(Q_{4n})$, the dual to the augmentation ideal. It follows that we have a fibre square

$$
\begin{array}{ccc}
I^*(Q_{4n}) & \xrightarrow{\pi_-} & \Theta(n) \\
\pi_+ \downarrow & & \downarrow \varphi_- \\
\mathbb{Z}[D_{2n}]/(2\Sigma_D) & \xrightarrow{\varphi_+} & \mathbb{F}_2[D_{2n}].
\end{array} \tag{10.6}
$$

We proceed to decompose $\Theta(2p) = \mathcal{C}(\mathbb{Z}[x]/(x^{2p}+1), \theta', -1)$ as a fibre product. For each integer $d \geq 1$ we denote by $c_d(x)$ the d^{th}-cyclotomic polynomial. If $d \geq 3$ each archimedean place of $\mathbb{Q}[x]/c_d(x)$ is complex. We denote by γ both the ring involution $\mathbb{Z}[x]/c_d(x) \to \mathbb{Z}[x]/c_d(x)$ induced by complex conjugation and also the involution $\mathbb{F}_p[x]/(x^2+1) \to \mathbb{F}_p[x]/(x^2+1)$, $a + bx \mapsto a - bx$ where $\mathbb{F}_p$ is the field with p elements. We note that $x^{2p} + 1 = (x^2 + 1)c_{4p}(x)$ where $c_{4p}(x) = \sum_{r=0}^{p-1}(-1)^r x^{2r}$. Let ζ be a primitive $4p$-th root of unity; then $c_{4p}(x)$ factorises as

$$c_{4p}(x) = \prod_{(r,\, 4p)=1} (x - \zeta^r).$$

As θ' is induced from the involution $x \mapsto x^{-1}$ then $\theta'(\zeta^r) = \zeta^{-r} = \overline{\zeta^r}$. In particular, under any imbedding $\mathbb{Q}[x]/c_{4p}(x) \to \mathbb{C}$, θ' corresponds to complex conjugation, so it will cause no confusion to replace θ' by γ. In addition

$$x^{2p} + 1 = (x^2 + 1)^p \pmod p$$

so that $c_{4p}(x) = (x^2+1)^{p-1} \pmod p$. This gives the following fibre square of pointed involutions:

$$\begin{array}{ccc}
(\mathbb{Z}[x]/(x^{2p}+1), \theta', -1) & \longrightarrow & (\mathbb{Z}[x]/c_{4p}(x), \gamma, -1) \\
\downarrow & & \downarrow \nu \\
(\mathbb{Z}[x]/(x^2+1), \gamma, -1) & \longrightarrow & (\mathbb{F}_p[x]/(x^2+1), \gamma, -1).
\end{array}$$

where $\nu : \mathbb{Z}[x]/c_{4p}(x) \to \mathbb{F}_p[x]/(x^2+1)^{p-1} \to \mathbb{F}_p[x]/(x^2+1)$ is the obvious composition. The cyclic algebra construction now gives the fibre square

(10.7)
$$\begin{array}{ccc}
\Theta(2p) & \longrightarrow & \mathcal{C}(\mathbb{Z}[x]/c_{4p}(x), \gamma, -1) \\
\downarrow & & \downarrow \nu \\
\mathcal{C}(\mathbb{Z}[x]/(x^2+1), \gamma, -1) & \longrightarrow & \mathcal{C}(\mathbb{F}_p[x]/(x^2+1), \gamma, -1).
\end{array}$$

It remains to describe the constituent rings in more familiar terms. Let A be a commutative ring ; if $a, b \in A$ we recall that the quaternion algebra

$$\left(\frac{a\ ,\ b}{A}\right)$$

is obtained by imposing on the free A-module of rank 4, with basis elements $\{1, i, j, k\}$ the (associative) multiplication determined by

$$i^2 = a \cdot 1; \ \ j^2 = b \cdot 1 \ ; \ k = ij = -ji.$$

There are excellent general references to quaternion algebras in, for example, [27] and [29]. However we are primarily interested in quaternion algebras of the form $\left(\frac{-1\ ,\ -1}{A}\right)$. Over the field $\mathbb{R}$ of real numbers $\left(\frac{-1\ ,\ -1}{\mathbb{R}}\right)$ is the original ring of Hamiltonian quaternions. If $\mathbb{F}$ is a subfield of $\mathbb{R}$ and $a, b \in \mathbb{F}$ satisfy $a < 0$ and $b < 0$ the quadratic form $Q(\mathbf{x}) \;=\; x_0^2 - ax_1^2 - bx_2^2 - abx_3^2$ is anisotropic over $\mathbb{F}$. A nonzero element

$$\mathbf{x} \;=\; x_0 \cdot 1 + x_1 \cdot i + x_2 \cdot j + x_3 \cdot k \in \left(\frac{a\ ,\ b}{\mathbb{F}}\right)$$

has multiplicative inverse $\mathbf{x}^{-1} \;=\; \dfrac{(x_0 \cdot 1 - x_1 \cdot i - x_2 \cdot j - x_3 \cdot k)}{Q(\mathbf{x})}$; hence:

(10.8) $\left(\frac{a\ ,\ b}{\mathbb{F}}\right)$ is a division ring if $\mathbb{F}$ is a subfield of $\mathbb{R}$ and $a < 0$, $b < 0$.

Over more general rings the conclusion of (10.8) fails; for example:

Proposition 10.9 : Let A be a commutative ring in which 2 is invertible; if there exist $\xi, \eta \in A$ such that $\xi^2 + \eta^2 = -1$ then

$$\left(\frac{-1, -1}{A}\right) \;\cong\; M_2(A).$$

Proof : The A-linear map $\theta : \left(\frac{-1, -1}{R}\right) \longrightarrow M_2(A)$ defined by

$$\theta(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;;\; \theta(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;;\; \theta(j) = \begin{pmatrix} \xi & \eta \\ \eta & -\xi \end{pmatrix} \;;\; \theta(k) = \begin{pmatrix} \eta & -\xi \\ -\xi & -\eta \end{pmatrix}$$

is a ring homomorphism and is bijective when 2 is invertible in A. □

Proposition 10.10 If p is an odd prime then $\left(\frac{-1, -1}{\mathbb{F}_p}\right) \;\cong\; M_2(\mathbb{F}_p)$.

Proof : By (10.9) it suffices to satisfy $\xi^2 + \eta^2 = -1$ for some $\xi, \eta \in \mathbb{F}_p$. There are two cases, according to whether or not -1 is a square in $\mathbb{F}_p$. If -1 is a square then put $\eta = 0$ to solve $\xi^2 + \eta^2 = -1$. In the case where -1 is not a square the mapping $\psi : \mathbb{F}_p \to \mathbb{F}_p$; $\psi(x) \;=\; x + 1$ has the property that $\mathbb{F}_p \;=\; \{\psi(1), \psi^2(1), \ldots, \psi^p(1)\}$. Now take the restriction of ψ to $(\mathbb{F}_p^*)^2$ and suppose that $\psi((\mathbb{F}_p^*)^2) \subset (\mathbb{F}_p^*)^2$; then for all $r \geq 1$, $\psi^r((\mathbb{F}_p^*)^2) \subset (\mathbb{F}_p^*)^2$. However $1 \in (\mathbb{F}_p^*)^2$ so that $\mathbb{F}_p \;=\; \{\psi(1), \psi^2(1), \ldots, \psi^p(1)\} \subset (\mathbb{F}_p^*)^2$. This is a contradiction as $0 \notin (\mathbb{F}_p^*)^2$. Thus there exists $\xi \in \mathbb{F}_p^*$ such that $\xi^2 + 1 \notin (\mathbb{F}_p^*)^2$. However, $-1 \notin (\mathbb{F}_p^*)^2$. As the set $(\mathbb{F}_p^*)^2$ of nonzero squares is a subgroup of index two in $\mathbb{F}_p^*$ then there exists $\eta \in \mathbb{F}_p$ such that $\eta^2 \;=\; \dfrac{(\xi^2 + 1)}{-1} \;=\; -\xi^2 - 1$. Either way, there exist $\xi, \eta \in \mathbb{F}_p$ such that $\xi^2 + \eta^2 = -1$ and the conclusion follows from (10.9). □

As $\mathcal{C}(\mathbb{F}_p[x]/(x^2+1), \gamma, -1) \;\cong\; \left(\dfrac{-1,-1}{\mathbb{F}_p}\right)$ it follows from (10.10) that:

(10.11) For any odd prime p, $\mathcal{C}(\mathbb{F}_p[x]/(x^2+1), \gamma, -1) \;\cong\; M_2(\mathbb{F}_p)$.

Similarly $\mathrm{Quat}(4) \;=\; \mathcal{C}(\mathbb{Z}[x]/(x^2+1), \gamma, -1)$ is more naturally described as the ring of integral quaternions; that is :

$$\mathrm{Quat}(4) \;\cong\; \left(\frac{-1,-1}{\mathbb{Z}}\right).$$

As it is a subring of the division ring of Hamiltonian quaternions then:

(10.12) Quat(4) is a noncommutative integral domain.

In general, Quat(4p) $=\; \mathcal{C}(\mathbb{Z}[x]/c_{4p}(x), \gamma, -1)$ is not a quaternion algebra. However, the corresponding rational algebra $\mathcal{C}(\mathbb{Q}[x]/c_{4p}(x), \gamma, -1)$ *is* a quaternion algebra. In fact, writing $\mu = 2\cos(\pi/2p)$ and $\sigma = 2\sin(\pi/2p)$ it is straightforward to see that :

(10.13) $\mathcal{C}(\mathbb{Q}[x]/c_{4p}(x), \gamma, -1) \cong \left(\dfrac{-\sigma^2,-1}{\mathbb{Q}(\mu)}\right).$

As $\mathbb{Q}(\mu)$ is a totally real field and $-\sigma^2 < 0$ then by (10.8) $\left(\dfrac{-\sigma^2,-1}{\mathbb{Q}(\mu)}\right)$ is a division algebra and contains Quat(4p) as a subring; thus:

(10.14) Quat$(4p)$ is a noncommutative integral domain.

We note that by Corollary 3.5 of [14] :

(10.15) If A is a finite ring every stably free module over $A[C_\infty^{(n)}]$ is free.

Thus it follows that:

(10.16) Every stably free module over $M_2(\mathbb{F}_p)[C_\infty^{(n)}]$ is free.

The fibre square (10.7) is now written as :

(10.17)
$$\begin{array}{ccc} \Theta(2p) & \longrightarrow & \mathrm{Quat}(4p) \\ \downarrow & & \downarrow \nu \\ \mathrm{Quat}(4) & \longrightarrow & M_2(\mathbb{F}_p). \end{array}$$

By (10.12), (10.14) and (10.16) the fibre square (10.17) is constrained. As before we put $\mathfrak{R} \;=\; \mathbb{Z}[t, t^{-1}]$ and write:

$$\textbf{(10.18)}\qquad \begin{cases} \Theta(2p)[t,t^{-1}] & \longrightarrow & \mathrm{Quat}(4p))[t,t^{-1}] \\ \downarrow & & \downarrow \nu \\ \mathrm{Quat}(4))[t,t^{-1}] & \longrightarrow & M_2(\mathbb{F}_p[t,t^{-1}]). \end{cases}$$

As (10.17) is constrained then (10.18) is also constrained. Hence by (8.9):

(10.19) There are infinitely many stably free modules of rank 1 over $\Theta(2p)[t,t^{-1}]$.

Applying $-\otimes_{\mathbb{Z}} \mathfrak{R}$ to (10.4) gives a fibre square labelled as follows:

$$\textbf{(10.20)}\qquad \begin{cases} \mathfrak{R}[Q_{8p}] & \xrightarrow{\pi_-} & \Theta(2p)[t,t^{-1}] \\ \pi_+ \downarrow & & \downarrow \varphi_- \\ \mathfrak{R}[D_{4p}] & \xrightarrow{\varphi_+} & (\mathfrak{R}/2)[D_{4p}] \end{cases}$$

in which $\varphi_+ : \mathfrak{R}[D_{4p}] \xrightarrow{\widetilde{\nu}} (\mathfrak{R}/2)[D_{4p}]$ is surjective. As $\mathbb{F}_2[D_{4p}]$ is finite then by(10.15), every stably free module over $(\mathfrak{R}/2)[D_{4p}] \cong \mathbb{F}_2[D_{4p}][t,t^{-1}]$ is free. Hence from (9.1) we see that :

(10.21) $\pi_- : \mathfrak{R}[Q_{8p}] \to \Theta(2p)[t,t^{-1}]$ has the lifting property for stably free modules.

Now by (10.19) let $\{\mathfrak{s}(m)\}_{m\geq 1}$ be an infinite collection of pairwise non-isomorphic modules over $\Theta(2p)[t,t^{-1}]$ which satisfy

$$\mathfrak{s}(m) \oplus \Theta(2p)[t,t^{-1}] \cong \Theta(2p)[t,t^{-1}] \oplus \Theta(2p)[t,t^{-1}].$$

As π_- has the lifting property for stably free modules then for each m we may choose a module $\mathfrak{S}(m)$ over $\mathfrak{R}[Q_{8p}]$ which satisfies $\rho_*(\mathfrak{S}(m)) \cong \mathfrak{s}(m)$ and $\mathfrak{S}(m) \oplus \mathfrak{R}[Q_{8p}] \cong \mathfrak{R}[Q_{8p}] \oplus \mathfrak{R}[Q_{8p}]$, We have proved the following which is Theorem II of the Introduction:

(10.22) For any odd prime p there is an infinite collection $\{\mathfrak{S}(m)\}_{m\geq 1}$ of pairwise non-isomorphic stably free modules of rank 1 over $\mathfrak{R}[Q_{8p}]$.

§11 : The stable class of $I^*[t,t^{-1}]$:

Let Φ be a finite group and let $\epsilon : \mathbb{Z}[\Phi] \to \mathbb{Z}$ denote the augmentation homomorphism. Recall there is a fibre square of ring homomorphisms

$$\textbf{(11.1)}\qquad \begin{matrix} \mathbb{Z}[\Phi] & \xrightarrow{\pi_-} & I^* \\ \downarrow \pi_+ & & \downarrow \nu \\ \mathbb{Z} & \xrightarrow{[\ ]} & \mathbb{Z}/|\Phi| \end{matrix}$$

where $I^* = \mathrm{Hom}_{\mathbb{Z}[\Phi]}(I, \mathbb{Z}[\Phi])$ is the $\mathbb{Z}[\Phi]$-dual of the augmentation ideal $I = \mathrm{Ker}(\epsilon)$. Tensoring (11.1) with $\mathfrak{R}$ gives the following fibre square

(11.2)
$$\begin{array}{ccc} \mathfrak{R}[\Phi] & \xrightarrow{\pi_-} & I^*[t, t^{-1}] \\ \downarrow \pi_+ & & \downarrow \nu \\ \mathfrak{R} & \xrightarrow{[\ ]} & (\mathbb{Z}/|\Phi|)[t, t^{-1}] \end{array}$$

As $\mathbb{Z}/|\Phi|$ is finite then by (10.15):

(11.3) Every stably free module over $(\mathbb{Z}/|\Phi|)[t, t^{-1}]$ is free.

With a mild confusion of notation we denote also by $\epsilon : \mathfrak{R}[\Phi] \to \mathfrak{R}$ the canonical augmentation homomorphism. It is straightforward to see that:

(11.4) $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}[\Phi], \mathfrak{R}) \cong \mathfrak{R}$, generated by ϵ.

More generally:

Proposition 11.5: If S is a stably free module of rank 1 over $\mathfrak{R}[\Phi]$ then

(i) there exists a surjective $\mathfrak{R}[\Phi]$-homomorphism $\eta : S \twoheadrightarrow \mathfrak{R}$;

(ii) $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R}) \cong \mathfrak{R}$ and ξ generates $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R})$ if and only if ξ is surjective;

(iii) if $\eta' : S \twoheadrightarrow \mathfrak{R}$ is also surjective then $\mathrm{Ker}(\eta') = \mathrm{Ker}(\eta)$.

Proof : Milnor's classification theorem [25] describes S as a fibre square

($\mathfrak{S}$)
$$\begin{array}{ccc} S & \xrightarrow{j_*} & S_- \\ \downarrow \eta & & \downarrow \nu_* \\ S_+ & \xrightarrow{[\ ]_*} & S_0 \end{array}$$

where S_+, S_- are stably free of rank 1 over $\mathfrak{R}$ and $I^*[t, t^{-1}]$ respectively and where $S_0 = S_- \otimes_\nu (\mathbb{Z}/|\Phi|)[t, t^{-1}]$. As $\nu : I^*[t, t^{-1}] \to (\mathbb{Z}/|\Phi|)[t, t^{-1}]$ is surjective and S_- is projective then $\nu_* : S_- \to S_0$ is surjective. As ($\mathfrak{S}$) is a fibre square it follows that $\eta : S \to S_+$ is also surjective. However by (5.1) $S_+ \cong \mathfrak{R}$ thereby proving i).

ii) As S is stably free of rank 1 then $S \oplus \mathfrak{R}[\Phi]^m \cong \mathfrak{R}[\Phi]^{m+1}$ for some $m \geq 1$. Hence $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R}) \oplus \mathrm{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}[\Phi], \mathfrak{R})^m \cong \mathrm{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}[\Phi], \mathfrak{R})^{m+1}$. As $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}[\Phi], \mathfrak{R}) \cong \mathfrak{R}$ then $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R}) \oplus \mathfrak{R}^m \cong \mathfrak{R}^{m+1}$; hence $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R})$ is a stably free $\mathfrak{R}$-module of rank 1. By Sheshadri's Theorem [30], $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R}) \cong \mathfrak{R}$ and so is generated over $\mathfrak{R}$ by a single element.

Let $\xi \in \operatorname{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R})$ be such a generator. If $\omega \in \operatorname{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R})$ there exists $c(\omega) \in \mathfrak{R}$ such that $\omega(x) = c(\omega)\xi(x)$ for all $x \in S$. If $\eta : S \twoheadrightarrow \mathfrak{R}$ is the surjective homomorphism constructed in i) then for all $x \in S$ we see that $\eta(x) = c(\eta)\xi(x)$. Choosing x so that $\eta(x) = 1$ it follows that $c(\eta)\xi(x) = 1$ and $c(\eta) \in \mathfrak{R}^*$. Thus $\xi(x) = c(\eta)^{-1}\eta(x)$ and ξ is surjective. Moreover η is then a generator. The same argument then shows that any surjective homomorphism $S \twoheadrightarrow \mathfrak{R}$ is a generator of $\operatorname{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R})$ thereby proving ii).

iii) If $\eta : S \to \mathfrak{R}$ and $\eta' : S \to \mathfrak{R}$ are both surjective then by ii) they both generate $\operatorname{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R})$ and there are units $c, c' \in \mathfrak{R}^*$ such that for all $x \in S$, $\eta'(x) = c\eta(x)$ and $\eta(x) = c'\eta'(x)$. Thus $\operatorname{Ker}(\eta') = \operatorname{Ker}(\eta)$. □

By iii) above a stably free $\mathfrak{R}[\Phi]$ -module S of rank 1 defines an $\mathfrak{R}[\Phi]$-module

$$J_S = \operatorname{Ker}(\eta)$$

where $\eta : S \to \mathfrak{R}$ is surjective. We have an exact sequence

(11.6) $$0 \to J_S \xrightarrow{j} S \xrightarrow{\eta} \mathfrak{R} \to 0.$$

For any $\mathfrak{R}[\Phi]$-module M we define $M^* = \operatorname{Hom}_{\mathfrak{R}[\Phi]}(M, \mathfrak{R})$; then:

Proposition 11.7 : When S is a stably free $\mathfrak{R}[\Phi]$-module S of rank 1 the sequence $0 \to \mathfrak{R} \xrightarrow{\eta^*} S^* \xrightarrow{j^*} J_S^* \to 0$ is exact and J_S^* is naturally a stably free module of rank 1 over $I^*[t, t^{-1}]$.

Proof : Applying $\operatorname{Hom}_{\mathfrak{R}[\Phi]}(-, \mathfrak{R})$ to (11.6) gives an exact sequence

$$0 \to \mathfrak{R} \xrightarrow{\eta^*} S^* \xrightarrow{j^*} J_S^* \to \operatorname{Ext}^1_{\mathfrak{R}[\Phi]}(\mathfrak{R}, \mathfrak{R}[\Phi]).$$

By (5.15) $\operatorname{Ext}^1_{\mathfrak{R}[\Phi]}(\mathfrak{R}, \mathfrak{R}[\Phi]) \cong \operatorname{Ext}^1_{\mathfrak{R}}(\mathfrak{R}, \mathfrak{R}) = 0$ and so

(*) $$0 \to \mathfrak{R} \xrightarrow{\eta^*} S^* \xrightarrow{j^*} J_S^* \to 0$$

is exact as claimed. As S is stably free of rank 1 then $S \oplus \mathfrak{R}[\Phi]^m \cong \mathfrak{R}[\Phi]^{m+1}$ for some $m \geq 1$. Hence

$$\operatorname{Hom}_{\mathfrak{R}[\Phi]}(S, \mathfrak{R}[\Phi]) \oplus \operatorname{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}[\Phi], \mathfrak{R}[\Phi])^m \cong \operatorname{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}[\Phi], \mathfrak{R}[\Phi])^{m+1}.$$

As $\operatorname{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}[\Phi], \mathfrak{R}[\Phi]) \cong \mathfrak{R}[\Phi]$ then $S^* \oplus \mathfrak{R}[\Phi]^m \cong \mathfrak{R}[\Phi]^{m+1}$ so that S^* is also stably free of rank 1. If $\mathbb{E}$ is the field of fractions of $\mathfrak{R}$ then

$$S^* \otimes_{\mathfrak{R}} \mathbb{E} \ \oplus \ \mathbb{E}[\Phi]^m \ \cong \ \mathbb{E}[\Phi]^{m+1}.$$

As Φ is finite then $\mathbb{E}[\Phi]$ is semisimple and so $S^* \otimes_{\mathfrak{R}} \mathbb{E} \cong \mathbb{E}[\Phi]$. Each of $\mathfrak{R}$, S^* and J_S^* is free of finite rank over $\mathfrak{R}$. Applying $- \otimes_{\mathfrak{R}} \mathbb{E}$ to the exact sequence (*) gives an exact sequence

(**) $$0 \to \mathbb{E} \xrightarrow{\eta^*} \mathbb{E}[\Phi] \xrightarrow{j^*} J_S^* \otimes_{\mathfrak{R}} \mathbb{E} \to 0.$$

Applying $-\otimes_{\mathfrak{R}} \mathbb{E}$ to the exact sequence $0 \to \mathfrak{R} \xrightarrow{\epsilon^*} \mathfrak{R}[\Phi] \xrightarrow{i^*} I^*[t, t^{-1}] \to 0$ defining $I^*[t, t^{-1}]$ gives an exact sequence

(***) $$0 \to \mathbb{E} \xrightarrow{\epsilon^*} \mathbb{E}[\Phi] \xrightarrow{j^*} I^*[t, t^{-1}] \otimes_{\mathfrak{R}} \mathbb{E} \to 0.$$

As $\mathbb{E}[\Phi]$ is semisimple then $J_S^* \otimes_{\mathfrak{R}} \mathbb{E} \cong I^*[t, t^{-1}] \otimes_{\mathfrak{R}} \mathbb{E}$. Put $\Sigma = \sum_{g \in \Phi} g$. Then Σ generates $\mathbb{E} \subset \mathbb{E}[\Phi]$. Hence Σ vanishes on $J_S^* \otimes_{\mathfrak{R}} \mathbb{E}$ and so Σ also vanishes on $J_S^* \subset J_S^* \otimes_{\mathfrak{R}} \mathbb{E}$. It follows that J_S^* is naturally a module over $I^*[t, t^{-1}] = \mathfrak{R}[\Phi]/(\Sigma)$. Let $\psi : S^* \oplus \mathfrak{R}[\Phi]^m \xrightarrow{\simeq} \mathfrak{R}[\Phi]^{m+1}$ be an isomorphism and consider the diagram

$$\begin{array}{ccccccc} 0 \to & \mathfrak{R} \oplus \mathfrak{R}^m & \xrightarrow{\eta^* \oplus \epsilon^*} & S^* \oplus \mathfrak{R}[\Phi]^m & \xrightarrow{j^* \oplus i^*} & J_S^* \oplus I^*[t, t^{-1}]^m & \to 0 \\ & & & \downarrow \psi & & & \\ 0 \to & \mathfrak{R}^{m+1} & \xrightarrow{\eta^* \oplus \epsilon^*} & \mathfrak{R}[\Phi]^{m+1} & \xrightarrow{i^*} & I^*[t, t^{-1}]^{m+1} & \to 0 \end{array}$$

As $\mathrm{Hom}_{\mathfrak{R}[\Phi]}(\mathfrak{R}, I^*[t, t^{-1}] = 0$ then $i^* \circ \psi \circ (\eta^* \oplus \epsilon^*) = 0$ and so ψ induces an $\mathfrak{R}[\Phi]$-homomorphism $\widehat{\psi} : J_S^* \oplus i^*[t, t^{-1}]^m \to I^*[t, t^{-1}]^{m+1}$ making the following commute:

$$\begin{array}{ccc} S^* \oplus \mathfrak{R}[\Phi]^m & \xrightarrow{j^* \oplus i^*} & J_S^* \oplus I^*[t, t^{-1}]^m \\ \downarrow \psi & & \downarrow \widehat{\psi} \\ \mathfrak{R}[\Phi]^{m+1} & \xrightarrow{i^*} & I^*[t, t^{-1}]^{m+1} \end{array}$$

As i^* and ψ are surjective so is $\widehat{\psi}$. However, both $J_S^* \oplus I^*[t, t^{-1}]^m$ and $I^*[t, t^{-1}]^{m+1}$ are free of rank $(m+1)(|\Phi| - 1)$ over $\mathfrak{R}$. As $\mathfrak{R}$ is Noetherian then $\widehat{\psi}$ is an isomorphism over $\mathfrak{R}$ and hence is bijective. Thus $\widehat{\psi}$ is an isomorphism over both $\mathfrak{R}[\Phi]$ and $I^*[t, t^{-1}]$ and so, as claimed, J_S^* is stably free of rank 1 over $I^*[t, t^{-1}]$. □

Theorem 11.8 : For any odd prime p there are infinitely many isomorphically distinct stably free modules $\{J(m)\}_{m \geq 1}$ of rank 1 over $I^*[t, t^{-1}]$.

Proof : Applying $-\otimes_{\mathbb{Z}} \mathfrak{R}$ to (10.6) we obtain a fibre square

(11.9) $$\begin{array}{ccc} I^*[t, t^{-1}] & \xrightarrow{\rho} & \Theta(2p)[t, t^{-1}] \\ \eta \downarrow & & \downarrow \nu \\ \mathfrak{R}[D_{4p}]/(2\Sigma) & \xrightarrow{\widetilde{\nu}} & (\mathfrak{R}/2)[D_{4p}] \end{array}$$

in which $\widetilde{\nu} : \mathfrak{R}[D_{4p}]/(2\Sigma) \longrightarrow (\mathfrak{R}/2)[D_{4p}]$ is surjective. As $\mathbb{F}_2[D_{4p}]$ is finite then by(10.15) every stably free module over $(\mathfrak{R}/2)[D_{4p}] \cong \mathbb{F}_2[D_{4p}][t, t^{-1}]$ is free. It follows from (9.3) that $\rho : I^*[t, t^{-1}] \to \Theta(2p)[t, t^{-1}]$ has the lifting property for stably free modules. By(10.19) there is an infinite collection $\{\boldsymbol{\mathfrak{S}}(m)\}_{m \geq 1}$ of pairwise non-isomorphic modules over $\Theta(2p)[t, t^{-1}]$ such that

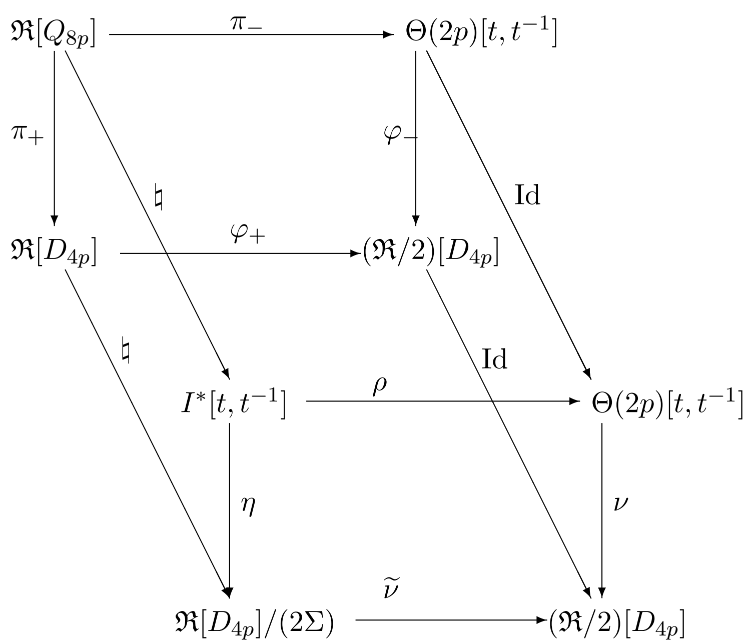
$$\boldsymbol{\mathfrak{S}}(m) \oplus \Theta(2p)[t, t^{-1}] \;\cong\; \Theta(2p)[t, t^{-1}] \oplus \Theta(2p)[t, t^{-1}].$$

As ρ has the lifting property for stably free modules then for each m we may choose a module $J(m)$ over $I^*[t, t^{-1}]$ such that

$$J(m) \oplus I^*[t, t^{-1}] \cong I^*[t, t^{-1}] \oplus I^*[t, t^{-1}]$$

and such that $\rho_*(J(m)) \cong \boldsymbol{\mathfrak{S}}(m)$. This completes the proof. □

Observe that the fibre squares (10.20) and (11.9) combine in the following commutative diagram where $\natural$ denotes the obvious surjection:



Re-tracing the steps in the proofs of (10.22) and (11.8) we see also that:

(11.10) The stably free modules $\mathfrak{S}(m)$ and $J(m)$ satisfy $\natural_*(\mathfrak{S}(m)) = J(m)$.

§12: Infinite splitting at the minimal level of $\Omega_3^{Q_{8p}}(\mathfrak{R})$: :

In this section Φ will denote a finite group and $\epsilon : \mathbb{Z}[\Phi] \to \mathbb{Z}$ will denote the canonical augmentation where Φ acts trivially on $\mathbb{Z}$; we denote by

I	:	the augmentation ideal $\mathrm{Ker}(\epsilon)$
I^*	:	the $\mathbb{Z}[\Phi]$ dual of I
$\Omega_k^{\Phi}(\mathbb{Z})$	:	the k^{th} syzygy of $\mathbb{Z}$ over $\mathbb{Z}[\Phi]$
$\Omega_k^{\Phi}(\mathfrak{R})$	:	the k^{th} syzygy of $\mathfrak{R}$ over $\mathfrak{R}[\Phi]$

The augmentation ϵ gives an exact sequence

(12.1) $$0 \to I \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0$$

The $\mathbb{Z}[\Phi]$ dual of the augmentation exact sequence has the form

(12.2) $$0 \to \mathbb{Z} \to \mathbb{Z}[\Phi] \to I^* \to 0$$

Let d be an integer $d \geq 2$; recall that Φ has *free period* $2d$ when there is an exact sequence of finitely generated $\mathbb{Z}[\Phi]$-modules

$$0 \to \mathbb{Z} \to F_{2d-1} \to F_{2d-2} \to .... \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

where each F_r is free.

Proposition 12.3: Let Φ be a finite group of free cohomological period four; then there exists an exact sequence of $\mathbb{Z}[\Phi]$-modules of the form

$$0 \to I^* \to \mathbb{Z}[\Phi]^{n_2} \to \mathbb{Z}[\Phi]^{n_1} \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0$$

for some positive integers n_1, n_2. In particular, $I^* \in \Omega_3^{\Phi}(\mathbb{Z})$.

Proof : By hypothesis there is an exact sequence of $\mathbb{Z}[\Phi]$ modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[\Phi]^c \to \mathbb{Z}[\Phi]^b \to \mathbb{Z}[\Phi]^a \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0.$$

We split this into two exact sequences

(I) $$0 \to \mathbb{Z} \to \mathbb{Z}[\Phi]^c \to L \to 0;$$

(II) $$0 \to L \to \mathbb{Z}[\Phi]^b \to \mathbb{Z}[\Phi]^a \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0.$$

We modify **(II)** to an exact sequence

(III) $$0 \to L \oplus \mathbb{Z}[\Phi] \to \mathbb{Z}[\Phi]^{b+1} \to \mathbb{Z}[\Phi]^a \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0.$$

Comparing **(I)** and (12.2) by (4.15) we see that $L \oplus \mathbb{Z}[\Phi] \cong I^* \oplus \mathbb{Z}[\Phi]^c$.

Substitution in **(III)** gives an exact sequence

$$0 \to I^* \oplus \mathbb{Z}[\Phi]^c \xrightarrow{j} \mathbb{Z}[\Phi]^{b+1} \to \mathbb{Z}[\Phi]^a \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0.$$

and hence an exact sequence

(IV) $$0 \to I^* \xrightarrow{j} S \to \mathbb{Z}[\Phi]^a \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0.$$

where $S = \mathbb{Z}[\Phi]^{b+1}/j(\mathbb{Z}[\Phi]^c)$. As $\mathbb{Z}$ is n-coprojective for all n it follows from (4.12) that S is stably free. Hence $S \oplus \mathbb{Z}[\Phi]^e \cong \mathbb{Z}[\Phi]^f$ for some positive integers e, f. Now modify **(IV)** to an exact sequence of the required form

$$0 \to I^* \to \mathbb{Z}[\Phi]^{n_2} \to \mathbb{Z}[\Phi]^{n_1} \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0.$$

where $n_1 = a + e$ and $n_2 = f$. □

Theorem 12.4: Let Φ be a finite group, let S be a stably free module of rank 1 over $\mathfrak{R}[\Phi]$ and let J_S^* be the corresponding stably free module of rank 1 over $I^*[t, t^{-1}]$ constructed in (11.8); if Φ has free period 4 then there exist positive integers m_1, m_2 and an exact sequence of $\mathfrak{R}[\Phi]$-modules of the form

$$0 \to J_S^* \to \mathfrak{R}[\Phi]^{m_2} \to \mathfrak{R}[\Phi]^{m_1} \to \mathfrak{R}[\Phi] \to \mathfrak{R} \to 0$$

In particular, J_S^* lies at the minimum level of $\Omega_3^{\Phi}(\mathfrak{R})$.

Proof : If S is a stably free module of rank 1 over $\mathfrak{R}[\Phi]$ then by (11.8) we have an exact sequence $0 \to \mathfrak{R} \xrightarrow{\eta^*} S^* \xrightarrow{j^*} J_S^* \to 0$. Applying $-\otimes_{\mathbb{Z}} \mathfrak{R}$ to (12.2) gives an exact sequence $0 \to \mathfrak{R} \xrightarrow{\epsilon^*} \mathfrak{R}[\Phi] \xrightarrow{i^*} I^*[t, t^{-1}] \to 0$ and comparison of the two via the dual Schanuel Lemma (4.15) gives an isomorphism

(*) $$I^*[t, t^{-1}] \oplus S^* \cong J_S^* \oplus \mathfrak{R}[\Phi].$$

As Φ has period 4 then by (12.3) there exists an exact sequence of $\mathbb{Z}[\Phi]$-modules $0 \to I^* \to \mathbb{Z}[\Phi]^{n_2} \to \mathbb{Z}[\Phi]^{n_1} \to \mathbb{Z}[\Phi] \to \mathbb{Z} \to 0$. Applying $- \otimes_{\mathbb{Z}} \mathfrak{R}$ gives an exact sequence

$$0 \to I^*[t, t^{-1}] \to \mathfrak{R}[\Phi]^{n_2} \to \mathfrak{R}[\Phi]^{n_1} \to \mathfrak{R}[\Phi] \to \mathfrak{R} \to 0.$$

which we modify to

$$0 \to I^*[t, t^{-1}] \oplus S^* \to \mathfrak{R}[\Phi]^{n_2} \oplus S^* \to \mathfrak{R}[\Phi]^{n_1} \to \mathfrak{R}[\Phi] \to \mathfrak{R} \to 0.$$

Substitution via (*) now gives an exact sequence

$$0 \to J_S^* \oplus \mathfrak{R}[\Phi] \xrightarrow{\iota} \mathfrak{R}[\Phi]^{n_2} \oplus S^* \to \mathfrak{R}[\Phi]^{n_1} \to \mathfrak{R}[\Phi] \to \mathfrak{R} \to 0$$

and hence an exact sequence

(**) $$0 \to J_S^* \xrightarrow{\iota} T \to \mathfrak{R}[\Phi]^{n_1} \to \mathfrak{R}[\Phi] \to \mathfrak{R} \to 0$$

where $T = (\mathfrak{R}[\Phi]^{n_2} \oplus S^*)/\iota(\mathfrak{R}[\Phi])$. It follows from (4.12) that T is stably free. Hence $T \oplus \mathfrak{R}[\Phi]^a \cong \mathfrak{R}[\Phi]^{a+b}$ for some positive integers a, b. As in (12.3) we may modify **(**)** to an exact sequence

(***) $$0 \to J_S^* \xrightarrow{\iota} \mathfrak{R}[\Phi]^{m_2} \to \mathfrak{R}[\Phi]^{m_1} \to \mathfrak{R}[\Phi] \to \mathfrak{R} \to 0$$

where $m_1 = n_1 + a$, $m_2 = a + b$. Evidently $J_S^* \in \Omega_3^\Phi(\mathfrak{R})$. As S^* is stably free of rank 1 it follows from (*) that J_S^* and $I^*[t, t^{-1}]$ lie at the same height within $\Omega_3^\Phi(\mathfrak{R})$. As $I^*[t, t^{-1}]$ lies at the minimal level, so also does J_S^*. □

We now specialise to the case $\Phi = Q_{8p}$. It is known (cf [3], Chap. 12, p.253) that Q_{8p} has free period 4 for any prime $p \geq 3$. From (11.8) and (12.4) we see the following which is Theorem I of the Introduction:

Theorem 12.5: For each prime $p \geq 3$ the minimal level of $\Omega_3^{\mathfrak{R}[Q_{8p}]}$ contains infinitely many isomorphically distinct modules.

§13 : $\Omega_n^{\mathbb{Z}[G]}$ as a class of extension modules:

We retain the previous notation namely that $\Lambda = \mathbb{Z}[G] = \mathfrak{R}[\Phi]$ where Φ is finite. In particular, it follows from (5.17) that :

(13.1) $\mathfrak{R}$ is k-coprojective for each $k \geq 1$.

As $\mathfrak{R}$ is 1-coprojective and $I_\mathfrak{R}(\Phi) \in \Omega_1(\mathfrak{R})$ then it follows from (4.2) that:

(13.2) $$\mathrm{Ext}^1_\Lambda(\mathfrak{R}, I_\mathfrak{R}(\Phi)) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(I_\mathfrak{R}(\Phi)).$$

However, by (4.5) and (5.20) we have:

(13.3) $$\mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(I_\mathfrak{R}(\Phi)) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\mathfrak{R}) \cong \mathfrak{R}/|\Phi|.$$

The nature of the isomorphism $\mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\mathfrak{R}) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(I_\mathfrak{R}(\Phi))$ is clear; given $\alpha \in \mathfrak{R}$, the diagram below commutes

$$\begin{array}{ccccccccc} 0 \to & I_\mathfrak{R}(\Phi) & \to & \Lambda & \stackrel{\eta}{\to} & \mathfrak{R} & \to 0 \\ & \downarrow \alpha_* & & \downarrow \alpha_* & & \downarrow \alpha_* & \\ 0 \to & I_\mathfrak{R}(\Phi) & \to & \Lambda & \stackrel{\eta}{\to} & \mathfrak{R} & \to 0 \end{array}$$

where in each case $\alpha_*(x) = \alpha \cdot x$. The correspondences $\alpha \mapsto \alpha_*$ thereby induce surjective ring homomorphisms

$$\mathfrak{R} \to \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\mathfrak{R}) \quad ; \quad \mathfrak{R} \to \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(I_\mathfrak{R}(\Phi))$$

which each have kernel equal to $|\Phi|\mathfrak{R}$ when Φ is finite. By (13.2) and (13.3) we may describe $\mathrm{Ext}^1_\Lambda(\mathfrak{R}, I_\mathfrak{R}(\Phi))$ either in terms of $\mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}\mathfrak{R})$ or $\mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(I_\mathfrak{R}(\Phi))$. In the present context it is more convenient to do the former. The isomorphism $\delta : \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\mathfrak{R})) \to \mathrm{Ext}^1_\Lambda(\mathfrak{R}, I_\mathfrak{R}(\Phi))$ takes the form $\delta(\alpha_*) = \alpha^\sharp(\mathcal{S})$ where $\alpha^\sharp(\mathcal{S})$ is the 'pullback' extension (cf [13], p.74)

$$\alpha^\sharp(\mathcal{S}) = (0 \to I_\mathfrak{R}(\Phi) \to \varprojlim(p, \alpha_*) \stackrel{\eta}{\to} \mathfrak{R} \to 0).$$

Thus given a module extension

$$(\mathcal{E}) \qquad 0 \to I_{\mathfrak{R}}(\Phi) \to X \xrightarrow{\eta} \mathfrak{R} \to 0.$$

the above considerations show that there is a congruence $\mathcal{E} \equiv \alpha^{\sharp}(\mathcal{S})$. The class $[\alpha]$ of α in $\mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\mathfrak{R})$ is the *k-invariant* of $\mathcal{E}$ (cf. [10], Chapter 6); when no confusion arises we shall simply refer to α as the *k-invariant* of $\mathcal{E}$.

Under the identification $\Lambda = \mathfrak{R}[\Phi]$ there are three augmentation homomorphisms to be considered. Firstly, we have the augmentation homomorphism $\epsilon : \Lambda \to \mathbb{Z}$ of the group ring of G over $\mathbb{Z}$; secondly, the augmentation homomorphism $\epsilon_{\mathfrak{R}} : \mathfrak{R}[\Phi] \to \mathfrak{R}$ of the group ring of Φ with coefficients in $\mathfrak{R}$; lastly, the augmentation homomorphism $\eta : \mathfrak{R} \to \mathbb{Z}$ of the group ring of C_∞ with coefficients in $\mathbb{Z}$. They are related by:

$$\textbf{(13.4)} \qquad \epsilon = \eta \circ \epsilon_{\mathfrak{R}}.$$

We obtain a commutative diagram with exact rows and columns as follows;

$$\textbf{(13.5)} \qquad \begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & I_{\mathfrak{R}}(\Phi) & \xrightarrow{\subset} & I_{\mathbb{Z}}(G) & \xrightarrow{\epsilon_{\mathfrak{R}}} & \mathfrak{R} & \to & 0 \\
 & & || & & \bigcap & & \downarrow{\scriptstyle t-1} & & \\
0 & \to & I_{\mathfrak{R}}(\Phi) & \xrightarrow{\subset} & \Lambda & \xrightarrow{\epsilon_{\mathfrak{R}}} & \mathfrak{R} & \to & 0 \\
 & & & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \eta} & & \\
 & & & & \mathbb{Z} & = & \mathbb{Z} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence :

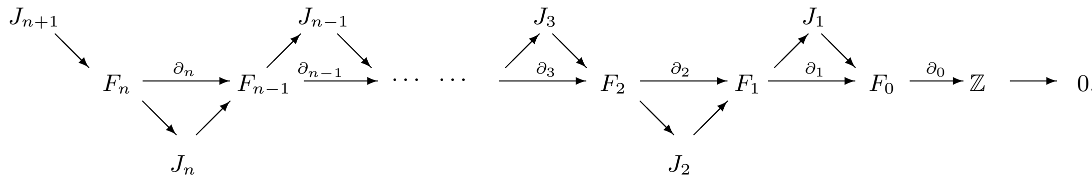
(13.6) The k-invariant of the extension $0 \to I_{\mathfrak{R}}(\Phi) \to I_{\mathbb{Z}}(G) \to \mathfrak{R} \to 0$ is the class of $t - 1 \in \mathfrak{R}/|\Phi|$.

In the above $I_{\mathfrak{R}}(\Phi)$ is a representative of $\Omega_1^{\mathbb{Z}[G]}$. In view of this, both the procedure and the convention are amenable to iteration. Thus suppose that $\mathfrak{R}$ is n-coprojective over $\Lambda = \mathfrak{R}[\Phi]$; then

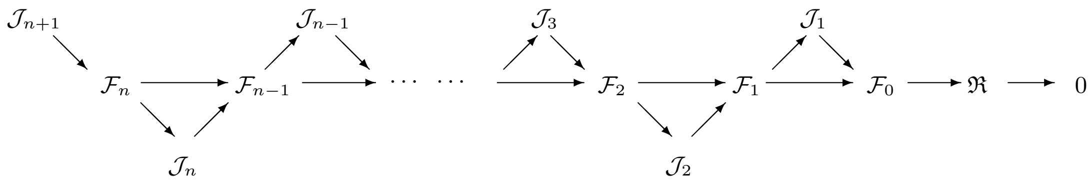
$$\textbf{(13.7)} \qquad \mathrm{Ext}^1_\Lambda(\Omega_{n-1}(\mathfrak{R}), \Omega_n(\mathfrak{R})) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\Omega_{n-1}(\mathfrak{R})) \cong \mathfrak{R}/|\Phi|.$$

The surjection $\mathfrak{R} \to \mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(\Omega_{n-1}(\mathfrak{R}))$ is induced by the correspondence $\alpha \mapsto \alpha_*$ where α_* denotes multiplication by α on any representing module; otherwise expressed, $\alpha \in \mathfrak{R}$ is mapped via the surjection $\mathfrak{R} \twoheadrightarrow \mathfrak{R}/|\Phi|$ to the k-invariant of the extension in $\mathrm{Ext}^1_\Lambda(\Omega_{n-1}(\mathfrak{R}), \Omega_n(\mathfrak{R})) \cong \mathfrak{R}/|\Phi|$ which it classifies.

Let Φ be a finite group; then taking $M = \mathbb{Z}$ we may construct a free $\mathbb{Z}[\Phi]$-resolution of $\mathbb{Z}$ thus:



As $\mathfrak{R}$ is free over $\mathbb{Z}$ then $- \otimes_{\mathbb{Z}} \mathfrak{R}$ is exact and gives a free Λ-resolution of $\mathfrak{R}$



where $\mathcal{F}_r = F_r \otimes_{\mathbb{Z}} \mathfrak{R}$ and $\mathcal{J}_r = J_r \otimes_{\mathbb{Z}} \mathfrak{R}$. From (13.3) and (13.5) we have

$$\mathrm{Ext}^1_{\mathbb{Z}[\Phi]}(J_{n-1}, J_n) \otimes_{\mathbb{Z}} \mathfrak{R} \;\cong_{\mathfrak{R}}\; \mathrm{Ext}^1_{\mathfrak{R}[\Phi]}(\mathcal{J}_{n-1}, \mathcal{J}_n) \;\cong\; \mathfrak{R}/|\Phi|.$$

By (5.5) it follows from the exact sequence $0 \to I_{\mathfrak{R}}(\Phi) \to I_{\mathbb{Z}}(G) \to \mathfrak{R} \to 0$ that $I_{\mathbb{Z}}(G)$ is a lattice over $\Lambda = \mathfrak{R}[\Phi]$. We now make specific choices $\mathcal{J}_{n-1} \in \Omega^{\mathfrak{R}[\Phi]}_{n-1}$, $\mathcal{J}_n \in \Omega^{\mathfrak{R}[\Phi]}_n$. It follows from (5.6) and (5.16) that $\mathcal{J}_{n-1}$ and $\mathcal{J}_n$ are k-coprojective for all $k \geq 1$. Now define $K_n(\mathcal{J}_{n-1}, \mathcal{J}_n, t-1)$ to be the extension module classified by $t-1$ in $\mathfrak{R}/|\Phi| \cong \mathrm{Ext}^1_{\mathfrak{R}[\Phi]}(\mathcal{J}_{n-1}, \mathcal{J}_n)$ thus:

$$0 \to \mathcal{J}_n \to K_n(\mathcal{J}_{n-1}, \mathcal{J}_n, t-1) \to \mathcal{J}_{n-1} \to 0.$$

$I_{\mathfrak{R}}(\Phi)$ is a representative of $\Omega^{\mathbb{Z}[G]}_1$; by (4.10) and the description of $I_{\mathbb{Z}}(G)$ in (13.5), we obtain the following which is Theorem IV of the Introduction:

(13.11) $\quad K_n(\mathcal{J}_{n-1}, \mathcal{J}_n, t-1)$ is a representative of $\Omega^{\mathbb{Z}[G]}_n$.

§14 : The intractability of the $D(2)$-problem:

As mentioned previously, in Appendix B of [10], subject to a mild homological finiteness condition the present author showed that for a given finitely presented group G, the $D(2)$ problem is equivalent to the realization problem $R(2)$. In fact for finite cyclic groups the R(2) problem was solved by Cockroft and Swan [4] some years before the publication of [32]. Subsequently the $R(2)$ problem was solved for finite abelian groups by Browning [2]. However, neither solution made any explicit connection with the $D(2)$ problem.

Since the publication of [10] and [11] there has been sporadic progress yielding a patchwork of results which the interested reader may find in [7], [8], [10], [12], [15], [23], [24],[26] and [28]. However, as yet there are no known examples of groups where the answer is negative and the rather meagre class of groups for which the problem has been solved affirmatively consists for the most part of finite groups.

For infinite groups very little is known. The affirmative solution for free groups is given in Appendix A of [10]. In his thesis, T.M. Edwards solved the problem affirmatively for the groups $C_\infty \times C_m$ ([6], [7]). The present paper will perhaps convince the reader of the difficulties involved in attempting to generalise Edwards' Theorem to the groups $C_\infty \times \Phi$ where Φ is finite quaternionic.

In fact, as is implicit in the calculations of Chapter 12 of [13], both Theorem I and Theorem II remain true if Q_{8p} is replaced by the general quaternionic group Q_{4n} where $n \geq 2$. An explicit account will appear in [17]. However the calculation for Q_{8p} employed here, which is adapted from the thesis of Kamali [18], is simpler than for the general case.

In conclusion we specify the difficulties posed by Theorems I and IV. Thus let $\{\mathcal{J}_3(\mu)\}_{\mu\in\mathbb{N}}$ be a faithful indexing of the minimal level of $\Omega_3^{\mathfrak{R}[Q_{8p}]}$, let $\mathcal{J}_2$ be a minimal element of $\Omega_2^{\mathfrak{R}[Q_{8p}]}$ and let $K_3(\mu)$ be the $\mathbb{Z}[G]$ module defined by the extension

$$\mathcal{K}_3(\mu) \quad = \quad (0 \to \mathcal{J}_3(\mu) \to K_3(\mu) \to \mathcal{J}_2 \to 0)$$

with extension class $(t-1) \in \mathfrak{R}/8p$. It follows from Theorem IV that $K_3(\mu) \in \Omega_3^{\mathbb{Z}[G]}$. The construction raises two questions:

Q1: Do the modules $\{K_3(\mu)\}_{\mu\in\mathbb{N}}$ lie at the minimal level of $\Omega_3^{\mathbb{Z}[G]}$?

As the collection $\{\mathcal{J}_3(\mu)\}_{\mu\in\mathbb{N}}$ is infinite then $\{\mathcal{K}_3(\mu)\}_{\mu\in\mathbb{N}}$ represents infinitely many *congruence classes* of extensions; what is less clear is the answer to :

Q2 : Do $\{K_3(\mu)\}_{m\in\mathbb{N}}$ represent infinitely many *isomorphism types* ?

Whilst an affirmative solution to **Q1** seems more than likely, **Q2** seems a long way from present technique. We note only that if both were to be answered affirmatively a positive solution to the $D(2)$ problem for $C_\infty \times Q_{8p}$ would require infinitely many minimal presentations of $C_\infty \times Q_{8p}$ to realise the modules $\{K_3(\mu)\}_{\mu\in\mathbb{N}}$.

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