Uniform spectral asymptotics for high-contrast periodic media

Ilia Kamotski, Valery Smyshlyaev and Shane Cooper

Abstract For an elliptic operator with high-contrast periodic matrix-inclusion coefficients in $\mathbb{R}^n$, $n \geq 2$, we obtain improved results on uniform approximations of typical Floquet-Bloch eigenvalues in terms of those of an explicit two-scale limit operator. As a result, we obtain not only improved rates for convergence of the spectra to the limit spectrum displaying band gaps, but also improved uniform estimates for an explicit asymptotics for the integrated density of states.

Key words: Floquet-Bloch spectrum; high-contrast ; two-scale operator approximations.
MSC2020: 35B40; 35B27.

1 Formulation and background

We consider a self-adjoint elliptic operator $\mathscr{A}_\tau = -\nabla \cdot A_\tau \nabla$ in $\mathbf{R}^n$, $n \geq 2$, depending on a positive parameter of contrast τ as follows. The coefficients $A_\tau(x)$ are assumed to be periodic with periodicity unit cell $\square = [-1/2, 1/2]^n$, i.e. $A_\tau(x+m) = A_\tau(x)$, $\forall x \in \mathbb{R}^n$ and $\forall m \in \mathbb{Z}^n$, and to adopt two different values in isolated periodic inclusions B and in their complements:

$$A_\tau(x) = \begin{cases} \tau, & x \in \square \backslash B, \\ 1, & x \in B. \end{cases} \tag{1}$$

Ilia Kamotski
Department of Mathematics, University College London, Gordon Street, London WC1E 6BT, UK
e-mail: i.kamotski@ucl.ac.uk

Valery Smyshlyaev
Department of Mathematics, University College London, Gordon Street, London WC1E 6BT, UK

Shane Cooper
Department of Mathematics, University College London, Gordon Street, London WC1E 6BT, UK

The boundary of inclusion B is assumed Lipschitz, $\overline{B} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ for simplicity, and the "matrix" (periodically extended $\square \backslash B$) is assumed connected. The interest is in the behaviour of operator $\mathscr{A}_\tau$ and its spectral characteristics for large contrasts τ.

According to Floquet-Bloch theory, for the spectrum of $\mathscr{A}_\tau$,

$$\mathrm{Sp}\,\mathscr{A}_\tau \;=\; \bigcup_{m=1}^{\infty} \; \bigcup_{\theta \in \square^*} \lambda_\tau^{(m)}(\theta),$$

where $\square^* = [-1/2, \pi/2]^n$ is the dual periodicity cell, and

$$0 \;\le\; \lambda_\tau^{(1)}(\theta) \le \lambda_\tau^{(2)}(\theta) \le \ldots \le \lambda_\tau^{(m)}(\theta) \le \ldots, \;\; \lambda_\tau^{(m)}(\theta) \to +\infty \;\text{ as } m \to \infty,$$

the "dispersion relations", are the eigenvalues of (non-negative self-adjoint) operators $\mathscr{A}_\tau(\theta) = -(\nabla + \mathrm{i}\theta) \cdot A_\tau(\nabla + \mathrm{i}\theta)$ in $L^2(\square)$ with the periodicity conditions.

It was shown in [1] that, as $\tau \to +\infty$, $\mathrm{Sp}\,\mathscr{A}_\tau$ converges, in the sense of Hausdorff (with unknown rate), to a limit spectrum displaying band gaps. (So the gaps do open for large enough τ in $\mathrm{Sp}\,\mathscr{A}_\tau$.) Zhikov [4], for a spectrally equivalent operator $\mathscr{B}_\varepsilon = -\varepsilon^2 \nabla \cdot A_{\varepsilon^{-2}}(\cdot/\varepsilon)\nabla$ with ε-periodic coefficients where $\varepsilon = \tau^{-1/2}$, has shown that the limit spectrum is associated with that of a non-negative self-adjoint two-scale limit operator $\mathscr{B}_0$ in a "bigger" (two-scale) Hilbert space $\mathbb{H}_0 \subset L^2(\mathbb{R}^n \times \square)$, with $\mathscr{B}_\varepsilon$ converging to $\mathscr{B}_0$ in the sense of (strong) "two-scale (pseudo-)resolvent convergence". In [2] and [5] certain estimates on asymptotics of integrated density of states

$$m_\tau(\lambda) \;:=\; (2\pi)^{-n} \sum_{m=1}^{\infty} \mathrm{meas}\left\{\theta \in \square^* : \lambda_\tau^{(m)}(\theta) \le \lambda\right\},$$

as $\tau \to \infty$ were obtained, demonstrating a concentration near the right ends of generic limit bands (which right ends are simple eigenvalues of the Dirichlet Laplacian in B, or "resonances", with a non-zero mean value of the corresponding eigenfunctions). Robust estimates on the rate of convergence of the spectra near the above typical resonances, explicit in terms of the inclusions' shapes and geometry, were obtained in [6] via decomposition of quasi-periodic and "electrostatic" solution operators with a subsequent analysis of related resonances by tools of layer potential theory.

In our recent work [7], we proposed a general approach for uniform two-scale type operator approximations for a broad class of spectral problems and applied it in particular to operators akin to $\mathscr{A}_\tau$, equivalently $\mathscr{B}_\varepsilon$. Resulting resolvent approximations and operator estimates in terms of the two-scale limit operator lead us, in particular, to the following approximations and estimates for $\lambda_{\varepsilon,\theta}^{(k)} := \lambda_{\varepsilon^{-2}}^{(k)}(\theta)$:

$$\left|\lambda_{\varepsilon,\theta}^{(k)} \,-\, \Lambda_{\theta/\varepsilon}^{(k)}\right| \;\le\; C\varepsilon, \quad \forall \theta \in \square^*, \quad \forall 0 < \varepsilon < 1, \tag{2}$$

with a constant C independent of ε and θ (although possibly depending on k).

In (2), the explicit approximations $\Lambda_\xi^{(k)}$, $\xi \in \mathbb{R}^n$, $k \in \mathbb{N}$ are as follow. They are eigenvalues of operator $\mathbb{L}_\xi$ entering a direct fiber-integral decomposition associated with the Zhikov's two-scale limit operator $\mathscr{B}_0$: $\mathscr{B}_0 = \mathscr{F}\mathbb{L}\mathscr{F}^{-1}$ where $\mathscr{F}$ is Fourier transform in the "macroscopic" variable, and

$$\mathbb{L} \,=\, \int_{\mathbb{R}^n}^{\oplus} \mathbb{L}_\xi \, \mathrm{d}\xi .$$

Here, for every $\xi \in \mathbb{R}^n$, $\mathbb{L}_\xi$ is a nonnegative self-adjoint operator in (complex) Hilbert space $\mathscr{H}_0 = \mathbb{C} \dotplus L^2(B)$ of functions from $L^2(\Box)$ which are constant outside B. Its spectrum $\mathrm{Sp}\,\mathbb{L}_\xi$ is discrete with eigenvalues $0 \le \Lambda_\xi^{(1)} \le \Lambda_\xi^{(2)} \le \ldots \Lambda_\xi^{(m)} \le \ldots$, $\Lambda_\xi^{(m)} \to +\infty$ as $m \to \infty$, with associated eigenfunctions $\psi_\xi^{(m)} = c_m + v_m \not\equiv 0$ such that $c_m \in \mathbb{C}$ and $v_m \in H^2(B) \cap H_0^1(B)$ solves

$$\begin{cases} \left(A^{\mathrm{h}}\,\xi \cdot \xi\right) c_m \;=\; \Lambda_\xi^{(m)} \left(c_m \,+\, \int_B v_m(x)\,\mathrm{d}x\right), \\ \quad -\Delta\, v_m(x) \;=\; \Lambda_\xi^{(m)} \left(c_m \,+\, v_m(x)\right), \qquad x \in B. \end{cases} \tag{3}$$

In (3), A^h is the "perforated domain" homogenized matrix, which is a positive definite symmetric $n \times n$ matrix with associated quadratic form

$$A^{\mathrm{h}}\,\xi \cdot \xi \;\;=\; \min_{u \in H^1_{\mathrm{per}}(\Box)} \int_{\Box \setminus B} |\xi \,+\, \nabla u(x)|^2 \mathrm{d}x,$$

where $H^1_{\mathrm{per}}(\Box)$ denotes $\Box$-periodic functions from $H^1_{\mathrm{loc}}(\mathbb{R}^n)$.

It follows form (3) that if $0 \neq \Lambda \in \mathrm{Sp}\,\mathbb{L}_\xi$, then either Λ is an eigenvalue of a Dirichlet Laplacian in B ($\Lambda \in \mathrm{Sp}\,(-\Delta_D)$) which has an associated eigenfunction ϕ with a zero mean over B ($\langle\phi\rangle := \int_B \phi(x)\mathrm{d}x = 0$), or $\Lambda \notin \mathrm{Sp}\,(-\Delta_D)$ and $\beta(\Lambda) = A^h\xi\cdot\xi$. Here $\beta(\Lambda)$ is the Zhikov's β-function [3, 4] well-defined on $\mathbb{R}\backslash\mathrm{Sp}\,(-\Delta_D)$:

$$\beta(\Lambda) \;:=\; \Lambda \,+\, \Lambda^2 \int_B b(x,\Lambda)\,\mathrm{d}x,$$

where $b(\cdot,\Lambda) = (-\Delta_D - \Lambda)^{-1}\,\mathbf{e}$ ($\mathbf{e}$ denotes the identical unity function) i.e. is the unique solution of

$$-\Delta b \,-\, \Lambda\, b \;=\; 1 \quad \text{in } B, \quad b = 0 \;\text{ on } \partial B.$$

Function $\beta(\Lambda)$ has spectral decomposition

$$\beta(\Lambda) \;=\; \Lambda \,+\, \Lambda^2 \sum_{m=1}^{\infty} \frac{|\langle\phi_m\rangle|^2}{\Lambda_m \,-\, \Lambda},$$

where ϕ_m are L^2-orthonormal eigenfunctions of $-\Delta_D$ corresponding to $\Lambda_m \in \mathrm{Sp}\,(-\Delta_D)$. Remark that, for $B = B_a$ a ball of radius $a < 1/2$, $\beta(\Lambda)$ is found explicitly in terms of trigonometric or Bessel functions. In particular, for $n = 3$,

$$\beta(\Lambda) \;=\; \Lambda\left(1-4\pi a^3/3\right) + 4\pi a\left(1 - a\Lambda^{1/2}\mathrm{cotan}\left(\Lambda^{1/2}a\right)\right),$$

see e.g. [8] p. 419.

The uniform spectral approximations and estimates (2) appear to also provide us, in particular, with some new results on the asymptotics of the integrated density of states (see Corollary 7.17 of [7]). These were obtained by us in [7] however as specialization of a general approach developed therein for the simple model with the high-contrast real scalar coefficients (1). An advantage of our general approach and on of the results based on it is that they apply with little or no change for more general model. In particular, the uniform estimate (2) and some of its corollaries apply for general measurable (i.e. with no regularity assumptions) complex Hermitian matrix-valued $A_\tau(x) = \tau A_1(x) + A_2(y)$ where A_1 and A_2 are supported in $\square\backslash B$ and $\overline{B}$ respectively and obey uniform boundedness and uniform ellipticity conditions. However, specifically for (1) (and in fact more generally for real-valued symmetric matrices A_1 and A_2) our results in [7] can be strengthened further as follows.

2 New results for model (1)

The following strengthening of estimate (2) holds.

Theorem 1 *Let $k=1$, or $k>1$ and the eigenvalue Λ_k of the Dirichlet Laplacian $-\Delta_D$ on B is simple with the associated eigenfunction ϕ_n having a non-zero mean over B: $\langle\phi_k\rangle := \int_B \phi_k(x)\mathrm{d}x \neq 0$. Let additionally, for $k>1$, the preceding (not necessarily simple) Dirichlet eigenvalue Λ_{k-1} has at least one eigenfunction ϕ_{k-1} such that $\langle\phi_{k-1}\rangle \neq 0$. Then*

$$\left|\lambda^{(k)}_{\varepsilon,\theta} \,-\, \Lambda^{(k)}_{\theta/\varepsilon}\right| \;\le\; C_k\,\varepsilon^2, \quad \forall\,\theta\in\square^*, \quad \forall\, 0<\varepsilon<1, \tag{4}$$

with a constant C_k independent of ε and θ (although possibly depending on k).

Notice that, compared to (2), the new estimate (4) displays improved rate ε^2 although under additional assumptions on the Dirichlet eigenvalue Λ_k. These assumptions however appear to hold generically in a certain sense, see e.g. a discussion in [2].

Theorem 1 leads in turn to the following strengthening of the estimates on the asymptotics of the integrated density of states $m_\tau(\lambda)$ (where $\tau=\varepsilon^{-2}$), compared to [7] Corollary 7.17:

Corollary 1 *Under the assumptions of Theorem 1, let C_k be the constant in the inequality* (4) *and let $mu_k := \Lambda_0^{(k)}$, i.e. $\Lambda_\xi^{(k)}$ for $\xi=0$. Then, for $\mu_k + 2C_k\tau^{-1} < \lambda < \Lambda_k - 2C_k\tau^{-1}$, $\beta(\lambda)>0$ and*

$$m_\tau(\lambda) \;=\; k-1+\,(2\pi)^{-n}\,\omega_n\tau^{-n/2}\,\frac{[\beta_B(\lambda)]^{n/2}}{\left(\mathrm{det}A^{\mathrm{hom}}_{\mathrm{pd}}\right)^{1/2}}\left\{1\,+\,R_\tau(\lambda)\right\}, \quad \textit{where} \tag{5}$$

ω_n is the volume of the unit ball in $\mathbb{R}^n$, and

$$|R_\tau(\lambda)| \;\leq\; D_k \, \frac{\tau^{-1}}{(\lambda-\mu_k)\,(\lambda_k-\lambda)} \;\; \textit{with a constant } D_k \textit{ independent of } \tau \geq 1 \textit{ and } \lambda. \tag{6}$$

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