Revisiting Distribution-Based Registration Methods

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***Abstract*— Normal Distribution Transformation (NDT) registration is a fast and learning-free pointcloud registration algorithm that works well in diverse environments. It uses a compact and discreet representation of pointclouds called NDT maps. However, because of discreetness in NDT maps, the global minima of the registration cost function does not always correlate to ground truth, particularly for rotational alignment. In this study, we examined the NDT registration cost function in depth and evaluated three modifications (Student-t likelihood function, heavy broaden tailed distribution, and overlapped NDT cells) that aim to reduce the impact of discreetness. The first two modifications make the NDT representation continuous by modifying the distribution to have broadened tails, while the last modification achieves continuity by creating overlap between NDT cells' distribution without increasing the number of NDT cells. We used the Pomerleau Dataset evaluation protocol for our experiments and found that using the heavily broadened tail NDT (HBT-NDT) (34.7% success rate) registration cost function and overlapped NDT cells (ONC-NDT) (33.5% success rate) resulted in significant improvements in registration results compared to the conventional NDT registration approach (27.7% success rate). However, no consistent improvement was observed by using Student-t likelihood-based registration cost function (22.2% success rate) over the NDT P2D registration cost function (23.7% success rate). Additionally, we also present the results of several other state-of-the-art registration algorithms for broader comparison.**

I. INTRODUCTION

Pointcloud registration is a method used in various computer vision tasks like point cloud matching, 3D reconstruction, localization and mapping, and odometry estimation [1][2]. In literature, several registration algorithms are available such as iterative closest point (ICP), which utilizes point [3] and point-normal [4] as features to find the correspondence between pointclouds. Normal distribution transform (NDT) registration is uses the distribution transform maps [5]. Furthermore, coherent point drift (CPD) solves pointcloud registration as a probability density estimation problem where pointcloud is assumed to be a Gaussian mixture model (GMM). Several other variants of these registration methods are available in the literature, and recently the focus is shifting to using deep learning for registration [2]. NDT registration is a fast and learning-free registration method that works well in diverse environments. which has been used in research and the industry for more than 15 years and is the main focus of this work.

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NDT registration uses a discreet and compact representation of pointcloud [6], a collection of normal distributions (μ_i, Σ_i) of the points in fixed-size grids called NDT maps. There are two types of NDT registration based on the use of NDT maps. NDT point-to-distribution (NDT-P2D) registration finds the pose variation between the NDT map of the previous scan and the current pointcloud, and NDT distribution-to-distribution (NDT-D2D) registration matches two NDT maps. Due to the discreetness of NDT maps, NDT registration has an inherent problem with the local minima of registration cost function not being at the ground truth posing variation between pointclouds. In this work, we have presented and evaluated three modifications in the NDT registration cost function to reduce the effect of discreetness in the NDT map which results in the following contributions.

- The effect of the normalization term of Gaussian distribution on the registration cost function which is considered a constant.
- Deriving and evaluating registration cost function based on Student-t likelihood function and NDT maps (Section III-A).
- Proposing and evaluating the modification in NDT registration based on cost function smoothing (Section III-B) and by creating a more continuous NDT map (Section III-C).
- Performance comparison of modified NDT registration with state-of-art registration methods using the Pomerleau dataset (Section IV-B).

II. RELATED WORK

A. NDT Registration

NDT Registration finds the transformation between two pointcloud using the NDT map representation of pointcloud. The NDT map is a collection of NDT cells created by subdividing the pointcloud in fixed-size non-overlapping grids. The NDT cells represent the normal distribution ($\mathcal{N}(\mu, \Sigma)$) of the points ($p_i = (x_i, y_i, z_i)^T, i = 1...n$) in the grids. In our experiments, we used NDT cells where $n_p >= 5$ but for cases where $n_p < 4$, probabilistic NDT method [12] could be used.

$$\mu = \frac{1}{n}\sum_{i}^{n} p_i \tag{1}$$

$$\Sigma = \frac{1}{n-1}\sum_{i}^{n}(p_i-\mu)(p_i-\mu)^T \quad (2)$$

Broadly, NDT registration cost functions are of two types, point-to-distribution(P2D) and distribution-to-distribution(D2D). In NDT registration, the cost function is minimized iteratively concerning the rigid transformation matrix Θ. The NDT P2D registration cost function is the approximation of the negative log-likelihood of the points in the current pointcloud ($\mathcal{X}$) belonging to the NDT cells in the NDT map ($\mathcal{M}$) of the previous pointcloud given by (3) where k_1 and k_2 are regularization parameters described in [13].

$$f_{p2d} = \sum_{x}^{\mathcal{X}}\sum_{\mu,\Sigma}^{\mathcal{M}} -k_1 e^{(\frac{-k_2}{2}(T(x,\Theta)-\mu)^T\Sigma^{-1}(T(x,\Theta)-\mu))} \quad (3)$$

The NDT D2D registration cost function represents the dissimilarity between the NDT representation of the current (M_{cur}) and previous (M_{prv}) pointcloud. The dissimilarity between NDT representation is represented as the summation of $L2$ distance between NDT cells of the current and previous NDT Map [14].

$$f_{d2d} = \sum_{i=1}^{N_{M_{cur}}}\sum_{j=1}^{N_{M_{prv}}} -k_1 e^{(\frac{-k_2}{2}d_{ij})} \quad (4)$$

where,

$$d_{ij} = \mu_{ij}^T\Sigma_{ij}^{-1}\mu_{ij},$$

$$\mu_{ij} = T(\mu_i,\Theta)-\mu_j, \quad \Sigma_{ij} = R^T\Sigma_i R + \Sigma_j$$

B. Background

The main reason for incorrectness in the NDT registration is the discreetness of NDT maps which is a well know issue and has been addressed in previous works. The two main approaches used in previous literature for tackling this issue are hierarchical registration and overlapped NDT cells. In [7][8], a hierarchical registration approach in which registration was done multiple times with NDT maps of different resolutions (coarse-to-fine grid size) was used. The result of coarse NDT registration becomes the initial guess for fine NDT registration for faster convergence. However, this approach takes longer as registration is done multiple times with NDT maps of different resolutions. The second approach to rectify the problem of discreetness used in [6] and [5] is to create overlapping NDT cells. This approach results in continuous NDT representation with an increased number of NDT cells which results in higher accuracy but also increases computation time significantly.

In [9], a new concept of dynamic scaling factors was introduced that expands the covariance of the NDT representation dynamically in each iteration to rectify the issue of NDT map discreetness and negative correlation of normal likelihood with rotation alignment. The claim of a negative correlation between normal likelihood and the rotation alignment of the covariance mentioned in the paper is incorrect and does not need rectification.

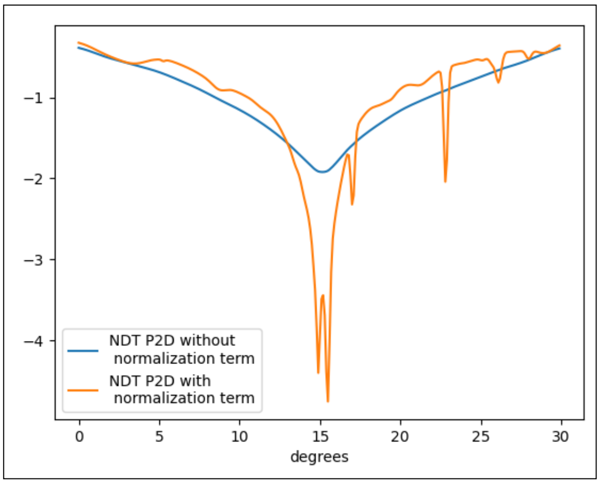

Fig. 1. Plot of NDT P2D cost function with (Orange) and without (Blue) the normalization term of the normal likelihood function.

Recently, several GMM-based registration approaches [10][11] have been proposed that used the Student-t distribution instead of the Normal distribution in the cost function. Thes GMM-based registration method with Student-t distribution methods results in better convergence due to the robustness against outliers and noise. However, no work has analyzed the use of Student-t likelihood as a registration cost function for NDT registration yet.

In this work, we empirically studied the NDT registration cost function in detail and introduced three different modifications to improve the pairwise registration results by reducing the effect of discreetness. The first modification uses Student-t likelihood as a registration cost function instead of Gaussian likelihood registration. The second modification, a heavily broadened-tailed NDT (HBD-NDT) registration cost function, was inspired by the broader-tailed Student-t likelihood function and has shown improvement in scan registration. In the third approach, an NDT map with overlapped NDT cells (ONC-NDT) is used without increasing the number of cells; hence, the computation time of registration does not increase while registration results improve.

III. Proposed Modifications

The likelihood of point (x) measured in multivariate normal distribution ($\mathcal{N}(\mu,\Sigma)$) with dimension d is calculated using (5).

$$\mathcal{L}(x) = \frac{1}{\sqrt{(2\pi)^d|\Sigma|}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right) \quad (5)$$

Both variants of NDT registration approximate the negative log-likelihood function, which can be summarized as the negative summation of the exponent of the square of Mahalanobis distance (d_{ij}) as given in (3) and (4) without the normalization term. For testing the effect of the normalization term on registration, we plotted the NDT P2D registration cost function with and without the normalization term as shown in Fig. 1. To plot the figure, we rotated a

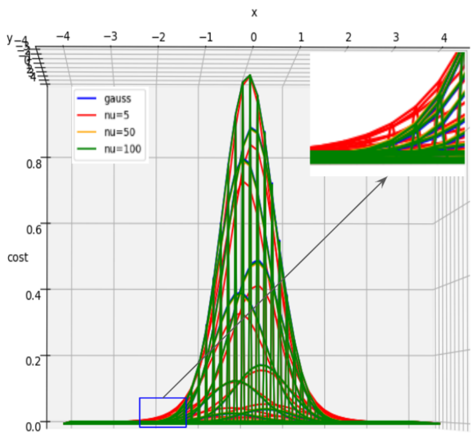

Fig. 2. Likelihood plots for 2D Gaussian, and Student-t distribution that shows broader tail for Student-t distribution.

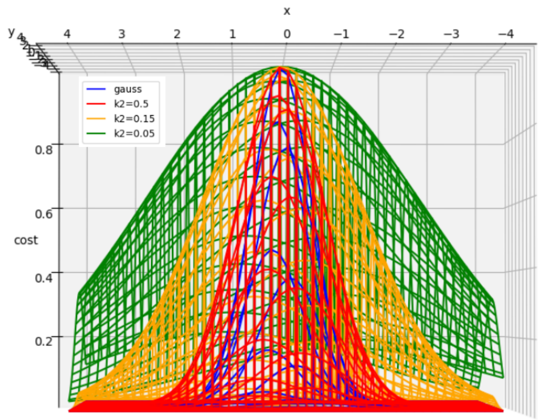

Fig. 3. Likelihood plot for 2D point distribution with heavily broadened tails. $k_1 = 1.0$.

pointcloud by $-15°$, and calculated registration cost by rotating the transformed pointcloud from $0°$ to $30°$ at an interval of $0.1°$. From Fig.1, we see that the normalization term of likelihood negatively impacts the registration cost function with more local minima in the cost function compared to the cost function without the regularization term. Also, the ground truth ($15°$) is not at the global minimum of the cost function plot.

A. Student-t (StDT) Registration Cost functions

The Student-t likelihood of point x being in the distribution ($\mathcal{N}(\mu, \Sigma)$) is given by (6). Then, the likelihood of point x being part of the NDT map $\mathcal{M}$ can be expressed as the summation of (6) and as shown in (7), and the likelihood of the pointcloud $\mathcal{X}$ at certain pose Θ being a part of an NDT map $\mathcal{M}$ can be given as the product of (7) and expressed as (8).

$$\mathcal{L}(x) = \frac{k}{|\Sigma|^{1/2}} \left[1 + \frac{1}{\nu}(x-\mu)^T \Sigma^{-1}(x-\mu)\right]^{-(\nu+p)/2} \tag{6}$$

$$\mathcal{L}(x|\mathcal{M}) = \sum_{i=1}^{n_M} \frac{k}{|\Sigma_i|^{1/2}} \left[1 + \frac{1}{\nu_i}(x-\mu_i)^T \Sigma_i^{-1}(x-\mu_i)\right]^{-(\nu_i+p)/2} \tag{7}$$

$$\mathcal{L}(X, \Theta|\mathcal{M}) = \prod_{j=1}^{n_X} \mathcal{L}(T(x_j, \Theta)|\mathcal{M}) \tag{8}$$

The best pose Θ that fits the pointcloud $\mathcal{X}$ to the NDT map $\mathcal{M}$ should maximize the likelihood function (8) or equivalently, minimize the negative log-likelihood (9).

$$-log(\mathcal{L}(\mathcal{X}, \Theta|\mathcal{M})) = -\sum_{j=1}^{n_X} log(\mathcal{L}(T(x_j, \Theta)|\mathcal{M})) \tag{9}$$

Similar to the NDT P2D registration cost function and the conclusions from the effect of normalization term on the registration cost function (Fig. 1), the StDT P2D registration algorithm will minimize the approximation of the negative log-likelihood as given in (10), over the space of transformation parameters Θ.

$$f_{StDT} = -\sum \left[1 + \tfrac{k_2}{\nu}(T(x,\Theta)-\mu)^T \Sigma^{-1}(T(x,\Theta)-\mu)\right]^{\frac{\nu+d}{2}} \tag{10}$$

where, $\nu = n_X - 1$ is the degree of freedom and d is dimension of x. By increasing the value of ν, the Student-t distribution approximates the Normal distribution. Hence, we used $\nu = 5$ for the registration and did not change based on population size. Fig. 2 shows the likelihood plot of Gaussian and Student-t distribution for 2d point distribution with $\nu = \{5, 50, 100\}$ for comparison and to justify the choice of $\nu = 5$. The number of points used to plot the distribution is 30.

B. Heavily Broaden Tailed NDT (HBT-NDT) Registration cost function

The aim of using the Student-t likelihood function was its robustness and capability of better distribution estimation for a small sample size because of broader tails compared to normal distribution. We have proposed a way to artificially broaden the tail for normal distribution to smoothen the cost function, which reduces the effect of discreetness. For a normal distribution, lowering the value of k_2 in Equation (3) and (4) makes the distribution's tail broad. Fig. 3 shows the effect of k_2 on tail broadness for normal distribution in the case of 2D point distribution. This can alternatively be interpreted as the scaling of the covariance matrix with a factor of $s = 1/k_2$.

C. Overlapping NDT cells

The NDT map's discreetness due to grids can be viewed as discreetness in the estimation of the surface geometry by cell covariance. Fig. 4 displays plots of covariance matrices as ellipsoids representing the approximate surface geometry for NDT map representations. In previous work, overlapping NDT cells were created by adding an NDT cell at the boundary of two regular NDT cells, which reduces the discreetness but increases the computation time dramatically due to an increase in the number of NDT cells. In our

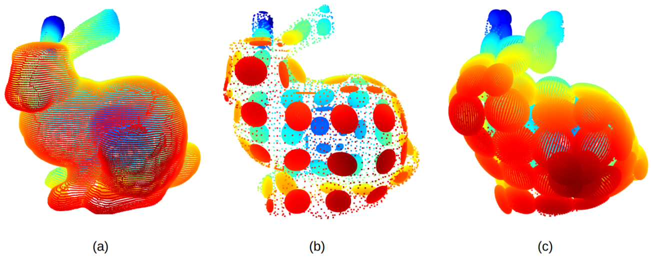

Fig. 4. NDT map representation for (b) non-overlapped NDT cells and (c) overlapped NDT cells for (a) Bunny pointcloud of Stanford Dataset.

approach, the computation of the mean and covariance for NDT cells do not change the number of cells; hence the computation time does not change much. Equations (11) and (12) show the calculation for distribution's mean μ' using the points inside the grid, and calculation of distribution's covariance Σ' using points in a box bigger than the grid size (gs) positioned at the center (c) of the cell. In Eq.12, a is a factor to increase the point search radius parameter in proportion to grid size for covariance calculation.

$$\mu' = \frac{1}{n}\sum_{i}^{n} p_i, \forall p_i : |p_i - c| < gs \tag{11}$$

$$\Sigma' = \frac{1}{n'-1}\sum_{i}^{n'}(p_i - \mu')(p_i - \mu')^T, \forall p_i : |p_i - c| < a \times gs \tag{12}$$

IV. Experiments and Results

A. Datasets and Evaluation

We evaluated the modified NDT registration methods using the Pomerleau dataset [16] following the protocol described in [17]. The Pomerleau dataset includes a protocol file and validation file for each scenario. The protocol file specifies the scan pair and initial pose to be used, while the validation file includes information on the initial pose difficulty, overlap ratio of the scan pair, and ground truth pose. We uniformly sub-sampled 10% of the scan pairs from the protocol file for each scenario. Additionally, we compared our modified NDT methods with several state-of-the-art registration algorithms including ICP, NDT, CPD, TEASER++, and FuzzyPSR registration to provide a comprehensive analysis of the registration methods. TEASER++ and FuzzyPSR registration methods are evaluated only for reference as these are feature-based registration methods.

The success rate of scan registration for rotation and translation is reported separately and compared for different registration algorithms. The registration was considered successful if the translation error (e_t) is less than 10cm and the rotation error (e_r) is less than $2.5°$. The translation and rotation error is calculated using (13) and (14) respectively, given the ground truth pose variation (ΔT) between two scans and estimated pose variation ($\Delta\hat{T}$) from registration.

$$e_t = \frac{||\delta_t||}{||\Delta t||} \times 100 \tag{13}$$

$$e_r = \cos^{-1}(\frac{1}{2}(trace(\delta_R) - 1)) \tag{14}$$

where,

$$\delta = \Delta\hat{T}^{-1}\Delta T = \begin{bmatrix} \delta_R & \delta_t \\ 0 & 1 \end{bmatrix}$$

For optimization of the registration cost function, we used Ceres Solver [18] which uses numerical differentiation, hence, the derivative of cost functions was not derived. In all cases, the initial guess of the pose (Θ) for optimization was the identity matrix. For all NDT and StDT registrations, all-to-all correspondence was used to get the best registration result.

The evaluation results for different registration algorithms for difficulty in pose and pointcloud overlap are shown in Fig. 5 and Fig. 6 respectively. The plots show the successful registration rate (%) vs different scenarios in the Pomerleau dataset for various registration algorithms.

B. Results

1) Comparison of NDT P2D and StDT P2D registration cost function: From Fig. 5 and Fig. 6, we observe that the overall successful registration rate for StDT P2D registration (13.17%) is close to the successful registration rate for NDT P2D registration (13.91%) for different pose difficulty or pointcloud overlap. The reason might be the similarity in the likelihood plots for both, Gaussian and Student-t distributions as shown in Fig. 2. The StDT likelihood has a broader distribution tail but the broadness is not enough to curb the discreetness in NDT maps thus no improvement in registration results can be seen for NDT-based registration.

2) Effect of Heavily Broaden Tailed Distribution: This modification involves reducing the value of k_2 in equations 3 and 4, resulting in a heavily broadened-tailed distribution, which in turn leads to a smoothened registration cost function. The smoothening effect due to the modification can also be viewed as inflating the covariance matrices resulting in increased continuity in the NDT maps. This continuity can result in improved registration results, as evidenced by the results in Fig. 5 and Fig. 6. The successful translation registration increased from 27.7% to 34.7% by this simple modification. It is worth noting that the improvement obtained with this modification was higher in translation alignment compared to rotation alignment for different pose difficulties and point cloud overlap. From the results, it can be concluded that the modification of reducing the value of k_2 in equations 3 and 4 can lead to improved registration results in the context of NDT pointcloud registration.

3) Effect of Overlapping NDT cells: In this modification, we create overlapped NDT cells using the criterion given in Eq.12 which results in a small increase in the number of NDT cells (∼1-2%) compared to the conventional NDT map. This small increase in the number of cells has only a minor effect on computation time while majorly improving the registration results compared to conventional NDT registration as evident from Fig. 5 and Fig. 6. Interestingly, the

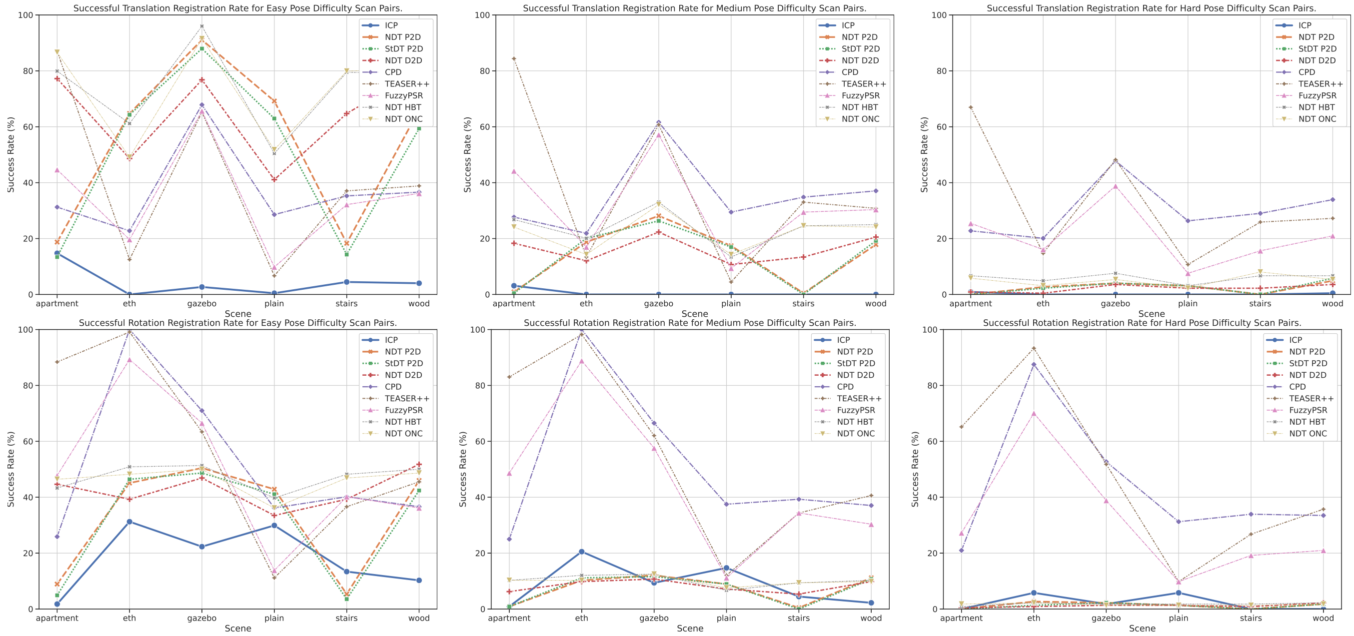

Fig. 5. Line graph showing the successful translation (Top Row) and rotation (Bottom Row) registration (%) vs. registration algorithms for easy (Column 1), medium (Column 2), and hard (Column 3) pose difficulty.

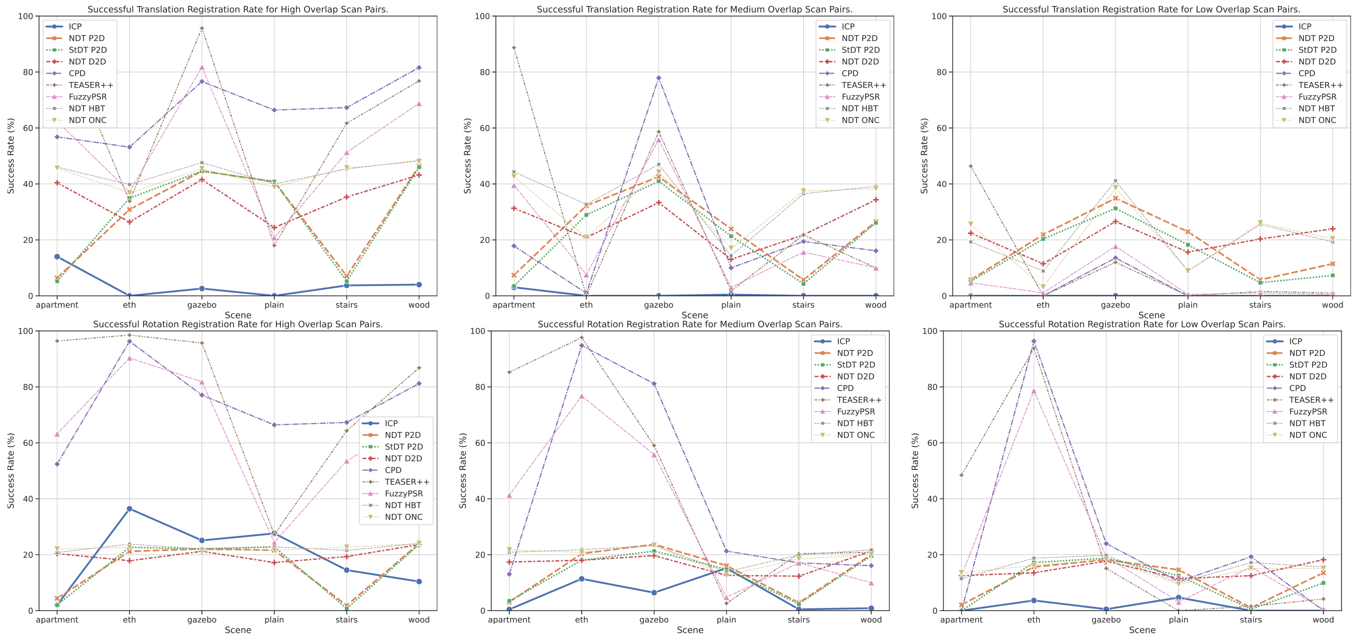

Fig. 6. Line graph showing the successful translation (Top) and rotation (Bottom) registration (%) vs. registration algorithms for high (a), medium (b), and low (c) pointcloud overlap.

TABLE I

OVERALL SUCCESS RATE FOR VARIOUS REGISTRATION METHODS ON POMERLEAU DATASET.

	apartment		eth		gazebo		plain		stairs		wood		Total	
	T	R	T	R	T	R	T	R	T	R	T	R	T	R
ICP	6.25	0.89	0.00	19.20	0.89	11.16	0.15	16.82	1.49	5.95	1.49	4.17	1.71	9.70
NDT P2D	6.55	3.27	28.72	19.35	41.07	21.58	29.91	17.71	6.25	1.93	29.61	19.64	23.68	13.91
StDT P2D	4.61	1.93	28.87	19.64	39.43	20.83	27.68	17.11	4.76	1.19	28.12	18.30	22.24	13.17
NDT D2D	32.14	17.11	20.39	16.67	34.23	19.64	18.01	13.99	26.79	15.18	34.67	21.28	27.70	17.31
NDT HBT	37.80	18.15	28.72	21.73	45.54	21.88	22.32	16.07	36.90	19.79	36.90	20.83	34.69	17.74
NDT ONC	38.84	19.49	22.17	20.24	43.01	21.28	22.77	15.03	37.50	19.20	36.76	20.09	33.50	19.22
CPD	27.23	23.96	21.58	95.83	59.08	63.39	28.12	34.97	33.04	37.80	35.86	35.71	34.15	48.61
TEASER++	79.46	78.87	13.54	96.88	58.04	59.08	7.29	11.01	31.99	32.59	32.29	40.62	37.10	53.17
FuzzyPSR	38.10	41.22	17.56	82.74	53.87	54.32	8.93	11.61	25.74	31.25	29.17	29.17	28.89	41.71

ONC-NDT registration method shows less performance improvement compared to the HBT-NDT registration method, ONC-NDT: 33.5% and HBT-NDT:34.7%. However, the successful rotation registration rate was slightly higher for ONC-NDT (19.2%) than HBT-NDT (17.7%). Overall, these results demonstrate the effectiveness of the proposed modification in improving the registration accuracy of NDT point clouds with only a small increase in NDT cells.

V. Discussion and Conclusion

In this study, we investigated the distribution-based registration approach to improve the registration results by modifying the current method. The first part of our investigation involved an analysis of the effect of the normalization term of the Gaussian distribution on the registration cost function. Adding this term increased the number of minimas in the cost function, making optimization more difficult. We then examined the use of the Student-t likelihood as an NDT-based registration cost function and found that the cost function smoothing due to the Student-t distribution's broader tail was not enough to improve the registration results.

Based on the broader tail concept, we introduced and evaluated the HBT-NDT cost function, which smoothened the cost function and resulted in better registration. We also evaluated the ONC-NDT, which reduced the discreteness in the NDT map, resulting in improved successful registration rates. However, none of the proposed modifications in NDT showed significant improvements in successful rotation alignment compared to successful translation alignment with cost function smoothening (HBT-NDT) having better results than NDT map continuity (ONC-NDT) overall. On the individual scan pair level, there were a few cases where one modification in NDT worked better while others did not however, it is hard to pinpoint the reason for this.

We also compared various registration algorithms and found that feature-based registration methods like TEASER++ and FuzzyPSR, and CPD, had the best results for successful rotation alignment. Overall, TEASER++ had the best performance regarding the percentage of successful registration, with consistent results for the initial pose difficulty and point cloud overlap. The best-performing registration algorithm for easy and hard pose difficulties was HBT-NDT and CPD, respectively. For high and low point cloud overlap, CPD and TEASER++ had the best performance, respectively. We did not compare the algorithms based on computation time, but the CPD algorithm was the most time-consuming as it computed all-to-all correspondence between points.

On the basis of scene complexity, TEASER++ has the best performance in the indoor environment (Apartment) by a huge margin with the worst performance in a repetitive environment (ETH-hall) or feature-less environments (ETH Plain). While the performance of NDT registration was consistent wrt to the scene complexity and heavily impacted by the initial pose difficulty.

In conclusion, our study provides insights into the distribution-based registration approach and proposes modifications to improve registration results. Our findings also provide a comparative analysis of various registration algorithms, which can guide researchers in selecting the best algorithm based on the specific requirements of their application.

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