Translations and Prawitz's Ecumenical system*

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Abstract. Since Prawitz proposal of his ecumenical system, where classical and intuitionistic logics co-exist in peace, there has been a discussion about the relation between *translations* and the ecumenical perspective. While it is undeniable that there exists a relationship, it is also undeniable that its very nature is controversial. The aim of this paper is to show that there are interesting relations between the Gödel-Gentzen translation and the ecumenical perspective. We show that the ecumenical perspective *cannot* be reduced to the Gödel-Gentzen translation, much less *be identified* with it.

1 Introduction

Ecumenical systems are formal codifications where two or more logics, even *rival* logics, can *co-exist in peace*, and this means that these logics accept and reject the same things, the same rules and the same basic principles. In [Pra15], Dag Prawitz proposed a natural deduction *ecumenical system*, here called Ec, where classical logic and intuitionistic logics are codified in the same system.[1] In this system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation and the constant for the absurd, but they would each have their own existential quantifier, disjunction and implication, with different meanings. Prawitz's main idea is that these different meanings are given by a semantical framework that can be accepted by both parties.

The language of Ec contains predicate variables $p, q, \ldots$ and the operators $\neg$, $\wedge$, $\bot, \forall, \rightarrow_i$, $\vee_i, \exists_i$, $\rightarrow_c, \vee_c, \exists_c$. The rules for the *intuitionistic* ($\rightarrow_i, \vee_i, \exists_i$) and

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[1] See also [PR17] and [PPdP21].

neutral operators $(\wedge, \neg, \bot, \forall)$ are the usual *Gentzen-Prawitz* natural deduction introduction and elimination rules. The rules for the *classical* operators are[2]:

$$\dfrac{\begin{array}{cc}[A]^n & [\neg B]^n\\ \multicolumn{2}{c}{\Pi}\\ \multicolumn{2}{c}{\bot}\end{array}}{A \to_c B}\ \to_c\text{-int},n,m \qquad \dfrac{A \to_c B \qquad A \qquad \neg B}{\bot}\ \to_c\text{-elim}$$

$$\dfrac{\begin{array}{cc}[\neg A]^n & [\neg B]^m\\ \multicolumn{2}{c}{\Pi}\\ \multicolumn{2}{c}{\bot}\end{array}}{A \vee_c B}\ \vee_c\text{-int},n,m \qquad \dfrac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot}\ \vee_c\text{-elim}$$

$$\dfrac{\begin{array}{c}[\forall x.\neg A(x)]^n\\ \Pi\\ \bot\end{array}}{\exists_c x.A(x)}\ \exists_c\text{-int},n \qquad \dfrac{\exists_c x.A(x) \qquad \forall x.\neg A(x)}{\bot}\ \exists_c\text{-elim}$$

It is undeniable that there is a relation between *translations* [FO10] and the ecumenical perspective. Prawitz himself observes in his paper that

> Highly relevant to these discussions are the well-known translations of classical predicate logic into intuitionistic predicate logic, first discovered by Gentzen and Gödel. Also of some relevance is the (less well-known) translation of intuitionistic predicate logic into quantified classical S4 established by Prawitz & Malmnäs (1968). These translations will not be dealt with here. The emphasis will instead be on meaning-theoretical considerations, but they can be seen to some extent as spelling out the philosophical significance of the fact that classical logic can be translated into intuitionistic logic.

But it is also undeniable that the very nature of this relationship is controversial. In the limit, we could even think that in fact there is nothing new in the ecumenical perspective: the classical operators could be eliminated by *definitions*, like $(A \to_c B) =_{\mathsf{def}} \neg(A \wedge \neg B)$. The aim of this paper is to show that there are interesting relations between the Gödel-Gentzen translation and the ecumenical perspective. We show that the ecumenical perspective *cannot* be reduced to the Gödel-Gentzen translation, much less *be identified* with it.

We shall use throughout the paper the following terminology:

- A formula is called *classical* (resp. *intuitionistic*) if and only if it has only neutral and classical (resp. neutral and intuitionistic) connectives.
- A set Γ of formulas is called *classical* (resp. *intuitionistic*) if and only if for every B in Γ, B is *classical* (resp. intuitionistic).

[2] We observe that in [Pra15], Prawitz presented also classical and intuitionistic constant predicates. We will present a more general formulation, with a single set of atomic propositions.

– IL/CL refers to first-order intuitionistic/classical logic.
– Cr is the system obtained from Ec by the addition of the *classical reductio* rule $\bot_c$:

$$\dfrac{\begin{array}{c}[\neg A]^n \\ \Pi \\ \bot\end{array}}{A}\ \bot_c, n$$

– Let Γ be a set of formulas. Then $\Gamma \vdash_{\mathsf{S}} A$ represents that A is provable in the system S with hypothesis in Γ.

2 On Krauss and *double-negation* translations

To our knowledge, the first ecumenical system, although without using the label *Ecumenical*, was proposed by Peter Krauss in 1992 in a *Technical Report* never published as a paper (see [Kra92]). Krauss defined a minimal system that has rules for $\vee_i, \to_i, \wedge_i, \forall_i, \exists_i, \vee_c, \to_c, \wedge_c, \forall_c$ and $\exists_c$.[3] For example, the rules for $\wedge_c$ and $\forall_c$ can be formulated in Prawitz's style as, respectively:

$$\dfrac{\begin{array}{c}[\neg A]^n \\ \Pi_1 \\ \bot\end{array} \qquad \begin{array}{c}[\neg B]^m \\ \Pi_2 \\ \bot\end{array}}{A \wedge_c B}\ \wedge_c\text{-int}, n, m$$

$$\dfrac{A \wedge_c B \qquad \neg A}{\bot}\ \wedge_c\text{-elim1} \qquad\qquad \dfrac{A \wedge_c B \qquad \neg B}{\bot}\ \wedge_c\text{-elim2}$$

and

$$\dfrac{\begin{array}{c}[\exists_i x.\neg A(x)]^n \\ \Pi \\ \bot\end{array}}{\forall_c x.A(x)}\ \forall_c\text{-int}, n \qquad\qquad \dfrac{\forall_c x.A(x) \qquad \neg A[y/x]}{\bot}\ \forall_c\text{-elim}$$

Krauss' aim was to provide a logical framework that help us identify where we need to use classical reasoning:

> We should rather like to persuade classical mathematicians to carry out their proofs *distinguishing* between intuitionistic and classical logic operators depending on what they actually prove. This way they may abstain from eliminating double negations without leaving familiar traditional tracks of reasoning. Moreover, this way their reasoning stays constructively valid and therefore preserves the possibility of a computational interpretation. For this more cautious form of classical reasoning we are proposing a formal framework presenting our system of first-order logic. ([Kra92], p.17)

[3] Krauss uses the symbols &, $\bigvee$, $\Longrightarrow, \forall, \exists$ for the intuitionistic operators and $\wedge, \vee, \to, \bigwedge, \bigvee$ for the classical ones. Negation is defined as $\neg A =_{\mathsf{def}} (A \Longrightarrow \bot)$.

Krauss correctly observes that we do not have a systematic way to identify in classical assertions the places we have to be classical and where we may stay intuitionistic, because these places depend on the proofs of the assertion:

> It should not come as a surprise that for this form of classical reasoning we cannot give a uniform method of describing the places in familiar classical assertions where intuitionistic logical operators are being used and where classical logical operators are being used, because this depends on how the assertion under consideration is actually being proved. There are several situations that may arise. Sometimes various options are available to be proved which all are logical equivalent, however the form of the resulting assertions may look quite different. We give some examples:
>
> – $\forall_i x(\varphi \to_i \exists_c y\psi)$
> – $\forall_i x(\varphi \to_c \exists_i y\psi)$
> – $\forall_c x(\varphi \to_i \exists_i y\psi)$
>
> ([Kra92], p.17)

In section 4 Krauss observes, as Prawitz did later in 2015, that

> Of course, our constructive refinement of classical logic is related to the Gödel-Gentzen Negative Translation (see, eg. Troelstra and van Dalen [TvD88], p.56). In this section we shall describe this relationship ([Kra92], p.19).

According to Krauss, the constructive interpretation of classical logic he proposes is not the Gödel-Gentzen negative translation[4] because

> It is not difficult to see that [this interpretation] interpolates the Gödel-Gentzen negative translation. ([Kra92], p.20)

But what does it mean to say that Krauss' constructive interpretation *interpolates* the Gödel-Gentzen negative translation? According to Krauss, we can define two functions $[\cdot]^\circ$ and $[\cdot]^-$ such that, if g corresponds to the Gödel-Gentzen translation[5], the following diagram commutes:

$$\begin{tikzcd}
& {[\Gamma]^\circ} \arrow[dr, "{[\cdot]^-}"] & \\
\Gamma \arrow[ur, "{[\cdot]^\circ}"] \arrow[rr, "g"] & & {[[\Gamma]^\circ]^-}
\end{tikzcd}$$

[4] See also [dPNdM01] pp.107-112.

[5] Actually, the translations proposed by Gödel and by Gentzen differ, in the propositional case, in the way they treat implication. The Go translation of Gödel is $Go[A \to B] = \neg(Go[A] \wedge \neg Go[B])$, while Gentzen's translation Ge is literal, *i.e.*, $Ge[A \to B] = (Ge[A] \to Ge[B])$. This difference has an important consequence: the elimination of $\to$ in the translation allows Gödel to prove that in the fragment $\{\neg, \wedge\}$ one cannot distinguish classical logic from intuitionstic logic with respect to theorems. We use Gödel's translation since it corresponds more closely to the form of the rules for classical implication $\to_c$.

The idea is that the function $[\cdot]^\circ$ eliminates occurrences of classical operators by their "intuitionistic interpretations"[6]:

- $[p]^\circ = p$, if p is atomic.
- $[\bot]^\circ = \bot$
- $[\neg A]^\circ = \neg[A]^\circ$
- $[A \wedge B]^\circ = [A]^\circ \wedge [B]^\circ$
- $[\forall x.A(x)]^\circ = \forall x.[A(x)]^\circ$
- $[A \vee_i B]^\circ = [A]^\circ \vee_i [B]^\circ$
- $[A \to_i B]^\circ = [A]^\circ \to_i [B]^\circ$
- $[\exists_i x.A(x)]^\circ = \exists_i x.[A(x)]^\circ$
- $[A \vee_c B]^\circ = \neg(\neg[A]^\circ \wedge \neg[B]^\circ)$
- $[A \to_c B]^\circ = \neg([A]^\circ \wedge \neg[B]^\circ)$
- $[\exists_c x.A(x)]^\circ = \neg\forall x.\neg[A(x)]^\circ$

On the other hand, the function $[\cdot]^-$ places double negations in front of atomic formulas variables:

- $[p]^- = \neg\neg p$, if p is atomic.
- $[\bot]^- = \bot$
- $[\neg A]^- = \neg[A]^-$
- $[\mathbf{Q}x.A(x)]^- = \mathbf{Q}x.[A(x)]^-$ for $\mathbf{Q} \in \{\forall, \exists_i, \exists_c\}$
- $[A \star B]^- = [A]^- \star [B]^-$ for $\star \in \{\wedge, \vee_i, \to_i, \vee_c, \to_c\}$

As usual, if Γ is a set of formulas, we denote by $[\Gamma]^\circ = \{[A]^\circ : A \in \Gamma\}$ (the same for $[\cdot]^-$ and $[\cdot]^g$, which will appear later on.).

The following result is straightforward.

Lemma 1. *Let A be a formula in Ec. Hence*

(a) $[A]^\circ$ *is an intuitionistic formula.*
(b) $\vdash_{\mathsf{Ec}} (A \leftrightarrow_i [A]^\circ)$

Proof. The proof of (a) is trivial, since the only operators that appear in A° are neutral or intuitionistic.
(b) is proved by structural induction over A.

- The base case where A is atomic or $\bot$ is trivial since $[\cdot]^\circ$ is the identity in these cases.
- If A has an intuitionistic or neutral main connective, then the result is immediate by the inductive hypothesis.
- If $A = B \to_c C$, then the result follows by the inductive hypothesis and the fact that $\vdash_{\mathsf{Ec}} (B \to_c C) \leftrightarrow_i \neg(B \wedge \neg C)$:

[6] The function $[\cdot]^\circ$ defined here is adapted to the meaning of the operators given by the rules of Prawitz's system, hence it is different from the one defined by Krauss.

$$\dfrac{\dfrac{\dfrac{[[B]^\circ \wedge \neg[C]^\circ]^2}{[B]^\circ}}{B}\,\text{Ind. Hyp.} \qquad \dfrac{\dfrac{\dfrac{[[B]^\circ \wedge \neg[C]^\circ]^2}{\neg[C]^\circ} \quad \dfrac{[C]^1}{[C]^\circ}\,\text{Ind. Hyp.}}{\bot}}{\neg C}\,1 \qquad B \to_c C}{\dfrac{\bot}{\neg([B]^\circ \wedge \neg[C]^\circ)}\,2}$$

$$\dfrac{\dfrac{\dfrac{[B]^2}{[B]^\circ}\,\text{Ind. Hyp.} \qquad \dfrac{\dfrac{\dfrac{[[C]^\circ]^1}{C}\,\text{Ind. Hyp.} \quad [\neg C]^3}{\bot}}{\neg[C]^\circ}\,1}{[B]^\circ \wedge \neg[C]^\circ} \qquad \neg([B]^\circ \wedge \neg[C]^\circ)}{\dfrac{\bot}{B \to_c C}\,2,3}$$

– The same reasoning holds for $A = B \vee_c C$, since $\vdash_{\mathsf{Ec}} (B \vee_c C) \leftrightarrow_i \neg(\neg B \wedge \neg C)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg[B]^\circ \wedge \neg[C]^\circ]^3}{\neg[B]^\circ} \quad \dfrac{[B]^1}{[B]^\circ}\,\text{Ind. Hyp.}}{\bot}}{\neg B}\,1 \qquad \dfrac{\dfrac{\dfrac{[\neg[B]^\circ \wedge \neg[C]^\circ]^3}{\neg[C]^\circ} \quad \dfrac{[C]^2}{[C]^\circ}\,\text{Ind. Hyp.}}{\bot}}{\neg C}\,2 \qquad B \vee_c C}{\dfrac{\bot}{\neg(\neg[B]^\circ \wedge \neg[C]^\circ)}\,3}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[[B]^\circ]^1}{B}\,\text{Ind. Hyp.} \quad [\neg B]^3}{\bot}}{\neg[B]^\circ}\,1 \qquad \dfrac{\dfrac{\dfrac{[[C]^\circ]^2}{C}\,\text{Ind. Hyp.} \quad [\neg C]^4}{\bot}}{\neg[C]^\circ}\,2}{\neg[B]^\circ \wedge \neg[C]^\circ} \qquad \neg(\neg[B]^\circ \wedge \neg[C]^\circ)}{\dfrac{\bot}{B \vee_c C}\,3,4}$$

– If $A = \exists_c x.B(x)$, then the result follows by the inductive hypothesis and the fact that $\vdash_{\mathsf{Ec}} \exists_c x.B(x) \leftrightarrow_i \neg(\forall x.\neg B(x))$:

$$\dfrac{\exists_c x.B(x) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[B[y/x]]^1}{[B[y/x]]^\circ}\,\text{Ind. Hyp.} \quad \dfrac{[\forall x.\neg[B(x)]^\circ]^2}{\neg[B[y/x]]^\circ}}{\bot}}{\neg B[y/x]}\,1}{\forall x.\neg B(x)}}{\dfrac{\bot}{\neg\forall x.\neg[B(x)]^\circ}\,2}$$

$$\dfrac{\dfrac{\neg\forall x.\neg[B(x)]^\circ \qquad \dfrac{\dfrac{\dfrac{\dfrac{[B[y/x]^\circ]^1}{B[y/x]}\,\text{Ind. Hyp.} \qquad \dfrac{[\forall x.\neg B(x)]^2}{\neg B[y/x]}}{\bot}}{\neg[B[y/x]]^\circ}\;2}{\forall x.\neg[B(x)]^\circ}}{\bot}}{\exists_c x.B(x)}\;2$$

Since the co-domain of $[\cdot]^\circ$ is the intuitionistic fragment of Ec, from now on, we will abuse the notation: whenever convenient we will identify $\to_i, \vee_i, \exists_i$ with IL's implication, disjunction and existential symbols $\to, \vee, \exists$.

Lemma 2. *$\Gamma \vdash_{\mathsf{Ec}} A$ if and only if $[\Gamma]^\circ \vdash_{\mathsf{IL}} [A]^\circ$.*

Proof. By Lemma 1 (b), $\Gamma \vdash_{\mathsf{Ec}} A$ implies that $[\Gamma]^\circ \vdash_{\mathsf{Ec}} [A]^\circ$. By the normalization theorem for Ec [PR17] and Lemma 1 (a), the only rules that are used in the derivation of $[A]^\circ$ from $[\Gamma]^\circ$ are rules for the intuitionistic operators. We can immediately conclude that this is a derivation in IL.

The other direction is direct, since Ec is a conservative extension of IL.

Lemma 3. *$\Gamma \vdash_{\mathsf{Cr}} A$ if and only if $[\Gamma]^\circ \vdash_{\mathsf{Cr}} [A]^\circ$.*

Proof. First of all, observe that $\vdash_{\mathsf{Cr}} A \leftrightarrow_i [A]^\circ$. In fact, if A is intuitionistic, then this holds trivially since $A = [A]^\circ$. If A's main connective is classical, we use the equivalences in Lemma 1, which are proved strictly in Ec, *i.e.* the rule $\bot_c$ is never applied. The main result then follows easily.

Observe that Cr collapses the ecumenical system into classical logic, since $\vdash_{\mathsf{Cr}} A \vee_i \neg A$. However, there is no proof of this formula if we restrict the application of $\bot_c$ to the atomic case. This can be achieved if we restrict the ecumenical formulas to the $\vee_i, \exists_i$-free ecumenical fragment.

Lemma 4. *Let Γ, A be $\vee_i, \exists_i$-free. If Π is a derivation of $[\Gamma]^\circ \vdash_{\mathsf{Cr}} [A]^\circ$, then every application of the classical reductio $\bot_c$ in Π can be restricted to the atomic case, i.e., with an atomic conclusion.*

Proof. The proof follows easily from the following reductions.[7]

1. $\wedge$-$\bot_c$-reduction: The derivation

$$\dfrac{\begin{matrix}[\neg(A \wedge B)]^1 \\ \Pi \\ \bot\end{matrix}}{A \wedge B}\;\bot_c,\,1$$

[7] This is an important part of the normalization strategy used by Prawitz in the monograph *Natural Deduction* [Pra65].

reduces to[8]

$$
\begin{prooftree}
\AxiomC{$[A \wedge B]^1$}
\UnaryInfC{A}
\AxiomC{$[\neg A]^2$}
\BinaryInfC{$\perp$}
\RightLabel{1}
\UnaryInfC{$[\neg(A \wedge B)]$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{$\perp$}
\RightLabel{$\perp_c$, 2}
\UnaryInfC{A}
\AxiomC{$[A \wedge B]^3$}
\UnaryInfC{B}
\AxiomC{$[\neg B]^4$}
\BinaryInfC{$\perp$}
\RightLabel{3}
\UnaryInfC{$[\neg(A \wedge B)]$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{$\perp$}
\RightLabel{$\perp_c$, 4}
\UnaryInfC{B}
\BinaryInfC{$A \wedge B$}
\end{prooftree}
$$

2. $\rightarrow_i$-$\perp_c$-reduction: The derivation

$$
\begin{prooftree}
\AxiomC{$[\neg(A \rightarrow_i B)]^1$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{$\perp$}
\RightLabel{$\perp_c$, 1}
\UnaryInfC{$A \rightarrow_i B$}
\end{prooftree}
$$

reduces to

$$
\begin{prooftree}
\AxiomC{$[A]^3$}
\AxiomC{$[A \rightarrow_i B]^1$}
\BinaryInfC{B}
\AxiomC{$[\neg B]^2$}
\BinaryInfC{$\perp$}
\RightLabel{1}
\UnaryInfC{$[\neg(A \rightarrow_i B)]$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{$\perp$}
\RightLabel{$\perp_c$, 2}
\UnaryInfC{B}
\RightLabel{3}
\UnaryInfC{$A \rightarrow_i B$}
\end{prooftree}
$$

3. $\neg$-$\perp_c$-reduction: The derivation

$$
\begin{prooftree}
\AxiomC{$[\neg\neg A]^1$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{$\perp$}
\RightLabel{$\perp_c$, 1}
\UnaryInfC{$\neg A$}
\end{prooftree}
$$

reduces to

$$
\begin{prooftree}
\AxiomC{$[A]^2$}
\AxiomC{$[\neg A]^1$}
\BinaryInfC{$\perp$}
\RightLabel{1}
\UnaryInfC{$[\neg\neg A]$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{$\perp$}
\RightLabel{2}
\UnaryInfC{$\neg A$}
\end{prooftree}
$$

[8] Let Σ be a set of derivations $\{\Sigma_1, \ldots, \Sigma_n\}$ such the end formula of Σ_i is A_i ($1 \leq i \leq n$), and let $\Gamma = \{A_1, \ldots, A_n\}$ be a set of undischarged assumptions in a derivation Π. We use the notation $\Sigma/[\Gamma]/\Pi$, to denote the result of replacing each assumption A_i in Π by the derivation Σ_i ($1 \leq i \leq n$). This is called the *concatenation operation* in [Pra71] pp. 251).

4. $\forall$-$\bot_c$-reduction: The derivation

$$\begin{array}{c} [\neg\forall x.A(x)]^1 \\ \Pi \\ \dfrac{\bot}{\forall x.A(x)}\ \bot_c,\, 1 \end{array}$$

reduces to

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{[\forall x.A(x)]^1}{A[y/x]} \qquad [\neg A[y/x]]^2}{\bot}}{[\neg\forall x.A(x)]}\ 1 \\ \Pi \\ \dfrac{\dfrac{\bot}{A[y/x]}\ 2}{\forall x.A(x)}\ y \text{ fresh} \end{array}$$

But the function $[\cdot]^\circ$ is more than simply a device to eliminate classical operators from the ecumenical language. In fact, according to Krauss, it is a *constructive* interpretation of classical reasoning in the *theory* of *stable atomic formulas* with IL.

Using the terminology introduced by Krauss, let STAT be defined as the set $\{(\neg\neg p \to_i p) : p \text{ is an atomic first-order formula}\}$. We can then prove that:

Theorem 1. *Let Γ and A be classical. Then $\Gamma \vdash_{\mathsf{Cr}} A$ if and only if* $\mathsf{STAT} + [\mathsf{\Gamma}]^\circ \vdash_{\mathsf{Ec}} [\mathsf{A}]^\circ$.[9]

Proof. By Lemma 3, $\Gamma \vdash_{\mathsf{Cr}} A$ if and only if $[\Gamma]^\circ \vdash_{\mathsf{Cr}} [A]^\circ$. By Lemma 4, if Π is a derivation of $[\Gamma]^\circ \vdash_{\mathsf{Cr}} [A]^\circ$, then every application of the classical reductio $\bot_c$ in Π is atomic, *i.e.*, with an atomic conclusion. We can then transform Π in Cr into a derivation Π' in $\mathsf{Ec} + \mathsf{STAT}$ by means of the following operation.

The derivation

$$\begin{array}{c} \Gamma \qquad [\neg p]^n \\ \Pi_1 \\ \dfrac{\bot}{p}\ \bot_c, n \\ \Pi_2 \\ C \end{array}$$

is transformed into

[9] This is an abuse of notation: while STAT may be an infinite set, only the finite subset of axioms involving the atomic subformulas of Γ, A is added to the context.

$$\dfrac{\dfrac{\begin{matrix}\Gamma & [\neg p]^n \\ \multicolumn{2}{c}{\Pi_1} \\ \multicolumn{2}{c}{\bot}\end{matrix}}{\neg\neg p}\, n \qquad \neg\neg p \to_i p}{\begin{matrix}[p] \\ \Pi_2 \\ C\end{matrix}}$$

And vice-versa.

Let us now consider the Gödel-Gentzen translation g adapted to Prawitz's system:

- $[p]^g = \neg\neg p$, if p is atomic.
- $[\bot]^g = \bot$
- $[\neg A]^g = \neg[A]^g$
- $[A \wedge B]^g = [A]^g \wedge [B]^g$
- $[\forall x.A(x)]^g = \forall x.[A(x)]^g$
- $[A \vee_i B]^g = [A]^g \vee_i [B]^g$
- $[A \to_i B]^g = [A]^g \to_i [B]^g$
- $[\exists_i x.A(x)]^g = \exists_i x.[A(x)]^g$
- $[A \vee_c B]^g = \neg(\neg[A]^g \wedge \neg[B]^g)$
- $[A \to_c B]^g = \neg([A]^g \wedge \neg[B]^g)$
- $[\exists_c x.A(x)]^g = \neg\forall x.\neg[A(x)]^g$

The next result highlights the fact that the provability of the translations collapses in Cr.

Lemma 5. *$\Gamma \vdash_{\mathsf{Cr}} A$ if and only if $[\Gamma]^g \vdash_{\mathsf{Cr}} [A]^g$. Hence, $[\Gamma]^\circ \vdash_{\mathsf{Cr}} [A]^\circ$ if and only if $[\Gamma]^g \vdash_{\mathsf{Cr}} [A]^g$.*

Proof. It is easy to show that $\vdash_{\mathsf{Cr}} (A \leftrightarrow_i [A]^g)$. In fact, for p atomic $[p]^g = \neg\neg p$ and $\vdash_{\mathsf{Cr}} (p \leftrightarrow_i \neg\neg p)$. The rest of the proof is similar to Lemma 3.

Obviously, the translation $[\cdot]^\circ$ *is not* the translation g, given that the former does not put double-negations in front of propositional variables. But if we have the translation $[\cdot]^-$ that places *double negations* in front of propositional variables, we can immediately see that:

Lemma 6. *For every ecumenical formula A, $[A]^g = [[A]^\circ]^-$.*

Theorem 2. *$[\Gamma]^g \vdash_{\mathsf{IL}} [A]^g$ if and only if $[[\Gamma]^\circ]^- \vdash_{\mathsf{IL}} [[A]^\circ]^-$*

Proof. Directly from Lemma 6.

It is in this sense that, according to Krauss, the constructive interpretation he proposes interpolates the Gödel-Gentzen translation.

3 New translations and the ecumenical perspective

As we saw, derivability is preserved if we replace the first order classical operators by their *constructive interpretation* given by the function $[\cdot]^\circ$.

But what can we say if we want to preserve *classical derivability* in the ecumenical system? Is it the case that if $\Gamma \vdash_{\mathsf{CL}} A$, then $\Gamma^* \vdash_{\mathsf{Ec}} A^*$, where A^* is the result of replacing every occurrence of $\vee, \to, \exists$ in A and in every formula B in Γ by their classical counterparts $\vee_c, \to_c, \exists_c$?

Clearly full preservation of derivability cannot be obtained, as the following simple example shows [PPdP21]:

$$\{p, (p \to q)\} \vdash_{\mathsf{CL}} q, \text{ but } \{p, (p \to_c q)\} \nvdash_{\mathsf{Ec}} q$$

where p, q are atomic.

In order to examine the relation between *classical derivability* and *ecumenical derivability* more closely, let us define the translation function T_c suggested above from the language of classical logic into the language of the ecumenical system.

1. $T_c[p] = p$, if p is atomic.
2. $T_c[\bot] = \bot$
3. $T_c[\neg A] = \neg T_c[A]$
4. $T_c[A \wedge B] = T_c[A] \wedge T_c[B]$
5. $T_c[A \vee B] = T_c[A] \vee_c T_c[B]$
6. $T_c[A \to B] = T_c[A] \to_c T_c[B]$
7. $T_c[\forall x.A(x)] = \forall x.T_c[A(x)]$
8. $T_c[\exists x.A(x)] = \exists_c x.T_c[A(x)]$

Although, as we saw above, we cannot get full preservation of derivability, we can get the following weaker result.

Theorem 3. *If $\Gamma \vdash_{\mathsf{CL}} A$, then $T_c[\Gamma] \vdash_{\mathsf{Ec}} \neg\neg T_c[A]$*

Proof. By induction on the length of the derivation Π of $\Gamma \vdash_{\mathsf{CL}} A$. The interesting cases are when Π ends with an application of $\to$-elimination, $\vee$-elimination or $\exists$-elimination.

1. Π ends with $\to$-elimination. Π is:

$$\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ A\end{matrix} \qquad \begin{matrix}\Gamma_2\\ \Pi_2\\ A \to B\end{matrix}}{B}$$

We can obtain directly from the induction hypothesis the following derivation Π':

$$\dfrac{\begin{matrix}T_c[\Gamma_1]\\ \Pi'_1\\ \neg\neg T_c[A]\end{matrix} \qquad \dfrac{\begin{matrix}T_c[\Gamma_2]\\ \Pi'_2\\ \neg\neg(T_c[A] \to_c T_c[B])\end{matrix}}{\neg\neg T_c[A] \to_c \neg\neg T_c[B]} \qquad \dfrac{\dfrac{[\neg T_c[B]]^2 \qquad [\neg\neg T_c[B]]^1}{\bot}}{\neg\neg\neg T_c[B]}\,1}{\dfrac{\bot}{\neg\neg T_c[B]}\,2}$$

Note that here we use the fact that $\neg\neg(A \rightarrow_c B) \vdash_{\mathsf{Ec}} \neg\neg A \rightarrow_c \neg\neg B$.

2. Π ends with $\vee$-elimination. Π is:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ A\vee B\end{array}\qquad \begin{array}{c}\Gamma_2\quad [A]^1\\ \Pi_2\\ C\end{array}\qquad \begin{array}{c}\Gamma_3\quad [B]^2\\ \Pi_3\\ C\end{array}}{C}\;1,2$$

We can obtain directly from the induction hypothesis the following derivation Π':

$$\dfrac{\dfrac{\begin{array}{c}T_c[\Gamma_1]\\ \Pi_1'\\ \neg\neg(T_c[A]\vee_c T_c[B])\end{array}}{T_c[A]\vee_c T_c[B]}\qquad \dfrac{\dfrac{\begin{array}{c}T_c[\Gamma_2]\quad [T_c[A]]^1\\ \Pi_2'\\ \neg\neg T_c[C]\end{array}\qquad [\neg T_c[C]]^3}{\bot}}{\neg T_c[A]}\;1\qquad \dfrac{\dfrac{\begin{array}{c}T_c[\Gamma_3]\quad [T_c[B]]^2\\ \Pi_3'\\ \neg\neg T_c[C]\end{array}\qquad [\neg T_c[C]]^4}{\bot}}{\neg T_c[B]}\;2}{\dfrac{\bot}{\neg\neg T_c[C]}\;3,4}$$

Here we use the fact that $\neg\neg(A \vee_c B) \vdash_{\mathsf{Ec}} A \vee_c B$.

3. Π ends with $\exists$-elimination. Π is:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \exists x.A(x)\end{array}\qquad \begin{array}{c}\Gamma_2\quad [A[y/x]]\\ \Pi_2\\ B\end{array}}{B}$$

We can obtain directly from the induction hypothesis the following derivation Π':

$$\dfrac{\dfrac{\begin{array}{c}T_c[\Gamma_1]\\ \Pi_1'\\ \neg\neg(\exists_c x.T_c[A(x)])\end{array}}{\exists_c x.T_c[A(x)]}\qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}T_c[\Gamma_2]\quad [T_c[A[y/x]]]^1\\ \Pi_2'\\ \neg\neg T_c[B]\end{array}\qquad [\neg T_c[B]]^2}{\bot}}{\neg T_c[A[y/x]]}\;1}{\forall x.\neg T_c[A(x)]}}{\dfrac{\bot}{\neg\neg T_c[B]}\;2}$$

Note that here we use the fact that $\neg\neg(\exists_c x.A) \vdash_{\mathsf{Ec}} \exists_c x.A(x)$.

As a direct corollary we obtain:

Corollary 1. *If* $\vdash_{\mathsf{CL}} A$, *then* $\vdash_{\mathsf{Ec}} \neg\neg T_c[A]$

But in Prawitz's ecumenical system we also have:

Lemma 7. *If* $\vdash_{\mathsf{Ec}} \neg\neg T_c[A]$ *then* $\vdash_{\mathsf{Ec}} T_c[A]$

Proof. Induction over the complexity of A.

1. Basis: A is atomic. Vacuously satisfied.

2. A is $B \wedge C$. The result follows directly from the induction hypothesis and the fact that $\vdash_{\mathsf{Ec}} \neg\neg T_c[B \wedge C]$ implies $\vdash_{\mathsf{Ec}} \neg\neg T_c[B] \wedge \neg\neg T_c[C]$.
3. A is $\forall x.B(x)$. The result follows directly from the induction hypothesis and the fact that $\vdash_{\mathsf{Ec}} \neg\neg\forall x.T_c[B(x)]$ implies $\vdash_{\mathsf{Ec}} \forall x.\neg\neg T_c[B(x)]$.
   In fact,
   $$\vdash_{\mathsf{Ec}} \neg\neg\forall x.T_c[B(x)] \text{ implies } \vdash_{\mathsf{Ec}} \forall x.\neg\neg T_c[B(x)] \text{ implies } \vdash_{\mathsf{Ec}} \neg\neg T_c[B[y/x]]$$
   By the induction hypothesis $\vdash_{\mathsf{Ec}} \neg\neg T_c[B[y/x]]$ implies $\vdash_{\mathsf{Ec}} T_c[B[y/x]]$, that finally implies $\vdash_{\mathsf{Ec}} \forall x.T_c[B(x)]$.
4. The main operator of A is classical. The result follows directly from the fact that classical operators satisfy the *classical reductio* $\perp_c$.

From Corollary 1 and Lemma 7 we obtain:

Theorem 4. *If* $\vdash_{\mathsf{CL}} A$*, then* $\vdash_{\mathsf{Ec}} T_c[A]$.

4 Conclusion

As we have seen, there are clear connections between the ecumenical perspective and translations, but these connections cannot be understood as a *reduction* of the former to the latter. In particular, Prawitz's ecumenical proposal *is not* a double-negation translation.

Translations clearly elucidate the behavior of the classical operators and how they interact with their intuitionistic counterparts. If one still wants to think of translations with respect to the ecumenical perspective, one should think of translations of derivations instead of translations between languages: the ecumenical perspective helps us to identify places where we have to be classical, and obviously we do not have to be classical at all times and everywhere.

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