

Nonlinearity Coefficients for Space-Division Multiplexing Fibers

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ABSTRACT

We review recent results on the nonlinearity coefficients of Manakov equations for multimode and multicore fibers in various regimes of linear coupling for space division multiplexing, and their implications on fiber design and data rates. We also propose novel semi-analytic expressions for the intermediate coupling regime.

Keywords: fiber optical communication, space-division multiplexing, nonlinear coupling.

1. INTRODUCTION

In the field of space-division multiplexing (SDM) the most common fiber structures are multicore fibers (MCFs) and multimode fibers (MMFs) [1]. It is still an open question which of the two technology, if any, will replace bundles of single mode fibers (SMFs) and for which applications. Significant efforts have been devoted to the study of linear and nonlinear effects, but the complexity of the fundamental modeling equations and of the system architectures hinders (to some extent) the understanding of the relation between the various fiber design parameters and the target performance metrics. Towards the goal of a clearer assessment of the potential of SDM, we review some of the most recent contributions in terms of averaged Kerr nonlinearity coefficients appearing in the Manakov equations in all regime of linear mode coupling: weak coupling regime (WCR), intermediate coupling regime (ICR), and strong coupling regime (SCR). For the ICR, we also present a novel formulation of the Manakov equation and of its nonlinearity coefficients.

2. CHANNEL MODELING

Propagation in an SDM fiber structure can be described by the following Manakov equations in all regimes of linear coupling [2]

$$\frac{\partial E_a}{\partial z} = L_a[\mathbf{E}] - j\gamma \sum_{b=1}^M \kappa_{ab} |E_b|^2 E_a \quad (1)$$

where $\mathbf{E} = [E_1, \dots, E_M]^T$ is the vector of modal envelopes, $L_a[\mathbf{E}]$ is the operator accounting for the linear effects (a stands for its a -th component) of attenuation, dispersion and linear coupling, M is the total number of modes (i.e., including polarizations), z is the coordinate along propagation, t is the temporal coordinate, $\beta_a^{(b)} := \left(\frac{d^b \beta_a(\omega)}{d\omega^b} \right)_{\omega_0}$ is the b -th order derivative with respect to the angular frequency ω of the propagation constant $\beta_a(\omega)$ of the a -th mode computed at the reference frequency ω_0 .

Note that Eq. (1) had already been proposed in the past for the SCR in [3], [4], [2] and for the WCR in [4], [2]. A related formulation for the ICR was proposed in [5], but with a different derivation and slightly different coefficients κ_{ab} . Eq. (1) is obtained by averaging the Kerr nonlinearity of the nonlinear Schrödinger equations over random orientation of $\mathbf{E}$ due to linear coupling of any intensity. Additionally, only nonlinear interactions between pair of modes for which the sum of the zeroth-order terms of the propagation constant $\beta(\omega)$ is zero have been kept. Indeed, it has emerged from experiments that $\beta_a^{(0)}$ fluctuate fast over distance, hence nonlinear terms for which the phase matching is achieved through a mismatch between the $\beta_a^{(0)}$ vanish [6], [7].

The second term on the right hand side of Eq. (1) accounts for the nonlinear effects and it depends on $\gamma = \frac{\omega_0 n_2}{c \mathcal{A}_{11}}$ – c is the speed-light in vacuum, n_2 is the nonlinear refractive index of the material, $\mathcal{A}_{11}$ is the fundamental mode effective area – and on a set of nonlinearity coefficients κ_{ab} , which in the SCR are [3], [2]

$$\kappa_{ab} = \kappa = \frac{4}{3} \frac{1}{M(M+1)} \sum_{\alpha=1}^M \sum_{\beta=1}^M \frac{\mathcal{A}_{11}}{\mathcal{A}_{\alpha\beta}} \quad (2)$$

in the ICR are

$$\kappa_{ab} = \sum_{c=1}^M \sum_{d=1}^M h_{cd} \frac{(E[|T_{db}|^2 |T_{ca}|^2] + E[T_{ca}^* T_{db}^* T_{da} T_{cb}])}{1 + \delta_{ba}} \quad (3)$$

and in the WCR are [2]

$$\kappa_{ab} = \begin{cases} \frac{4}{3} \frac{1}{M_A M_B} \sum_{\alpha \in I_A} \sum_{\beta \in I_B} \frac{\mathcal{A}_{11}}{\mathcal{A}_{\alpha\beta}}, & \text{if } B \neq A \\ \frac{4}{3} \frac{1}{(M_A+1)M_B} \sum_{\alpha \in I_A} \sum_{\beta \in I_A} \frac{\mathcal{A}_{11}}{\mathcal{A}_{\alpha\beta}}, & \text{if } B = A, \end{cases} \quad (4)$$

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where I_A is the set of mode indices of group A to which α belongs, M_A is the number of modes of group A , and [8]

$$h_{ab} = \begin{cases} \frac{A_{11}}{A_{aa}}, & \text{if } b = a \\ 2\frac{A_{11}}{A_{ab}}, & \text{if } b \neq a, \text{ but modes } a \text{ and } b \text{ have same polarization} \\ \frac{2A_{11}}{3A_{ab}}, & \text{if } b \neq a, \text{ and modes } a \text{ and } b \text{ have different polarization} \end{cases} \quad (5)$$

are nonlinearity coefficients appearing in the nonlinear Schrödinger equation and

$$A_{ab} = \frac{D_a D_b}{\int \int_{-\infty}^{\infty} |\mathbf{F}_a|^2 |\mathbf{F}_b|^2 dx dy} \quad (6)$$

is the intermodal effective area between modes $\mathbf{F}_a$ and $\mathbf{F}_b$, and $D_a = \int \int_{-\infty}^{\infty} \|\mathbf{F}_a(x, y, \omega_0)\|^2 dx dy$.

Note that the coefficients in the ICR given by Eq. (3) depend on the intensity of linear coupling through the moments of the elements of the matrix $\mathbf{T}$, which account for the frequency-independent part of linear mode coupling. The intensity of linear coupling between two mode groups can be quantified through the metric of crosstalk $XT_a = \sum_{b \neq a} P_b(z)/P_a(z=0)$ where $P_b(z)$ is the average power over the mode group b at position z when only the mode group a is excited. This means that Eq. (3) ideally is a universal expression for the nonlinearity coefficients, valid in any regime of linear coupling. This is depicted in Fig. 1a for a MMF with 2 mode groups of sizes 2 and 4, e.g., LP₀₁ and LP₁₁. If only intra-group coupling is present, Eq. (3) converges to Eq. (4). If both intra- and inter-group strong coupling exist, Eq. (3) converges to Eq. (2). If there is strong intra-group coupling and partial inter-group coupling, the coefficients assume values in between SCR and WCR. Observe from Fig. 1a that out of the 36 κ_{ab} elements, there are approximately only 3 different values for each XT level, except in the SCR for which they all converge to the same value. That is, each mode in a group has roughly the same behavior in any regime of coupling from the nonlinear perspective. We believe this to be the result of assuming an always-present intra-group coupling.

3. DEPENDENCE ON THE FIBER DESIGN PARAMETERS

The definitions of κ_{ab} provided in the previous section are intractable from an analytic perspective, and computationally expensive from a numerical perspective. Approximate closed-form expressions have been proposed in [9], [10], [11] for parabolic graded-index MMFs and MCFs, which depend only on fiber design parameters, in particular core radius R , core-cladding index contrast Δ (or, equivalently, numerical aperture NA), and, for MCFs, number of cores N_c . For MMFs in the SCR and in the WCR the approximate formulas are [9], [10]

$$\kappa_{ab} \approx \frac{4}{3} \frac{M_a}{M_a + \delta_{ab}} \frac{1}{\max\{a, b\}} \quad (7a)$$

$$\kappa \approx \frac{M}{M+1} \frac{7}{4\sqrt{M}} \quad (7b)$$

where M_a is the number of modes for group a , $a \in \{0, 1, \dots, G\}$, G is the total number of groups, and M is the total number of modes; $\delta_{ab} = 1$ if $a = b$ and 0 otherwise. In the WCR the number of groups and their sizes M_a are fully determined by M , as $M_a = 2a$ and $M = G(G+1)$ [12, Sec. 11.2.2]. Fig. 1b shows the agreement between the numerical and analytic results for the SCR case, Fig. 1c for the WCR one.

Approximate expressions for the nonlinearity coefficients of few-mode MCFs (FM-MCFs) in the WCR and in the SCR are [11]

$$\kappa_{ab} \approx \frac{4}{3} \frac{m_a}{m_a N_c + \delta_{ab}} \frac{1}{\max\{a, b\}} \quad (8a)$$

$$\kappa \approx \frac{7}{4} \frac{m}{(m N_c + 1)\sqrt{m}} \quad (8b)$$

where m_a is the number of modes per core for group a , m is the total number of modes per core, and N_c is the number of cores. For FM-MCFs, by WCR we mean strong coupling among correspondent mode groups of different cores and no coupling with other groups of the same or different core. By SCR we refer to strong coupling among all modes of all cores. Fig. 2a shows the agreement between numerical and analytic results for the SCR case, Fig. 2b for the WCR one.

As the figures emphasize, the nonlinearity coefficient in the SCR is a scalar, while in the WCR it becomes a square matrix whose number of rows corresponds to the number of mode groups, not the number of modes. The reason is that, loosely speaking, each mode group act as a single entity in the nonlinear regime, due to the effect of linear coupling. We also point out that κ_{ab} and κ depend only on M (and N_c in case of MCFs) and not on R or Δ separately. Hence, two fibers with different design, but same M (and N_c in case of MCFs) will have same nonlinearity coefficients.

Approximate closed-form expressions for γ are also provided in [9], [10], e.g.,:

$$\gamma \approx \frac{\omega n_2}{c} \frac{(NA k_0)^2}{4\pi\sqrt{m}} \quad (9a)$$

$$\gamma \approx \frac{\sqrt{m}}{\pi R^2} \quad (9b)$$

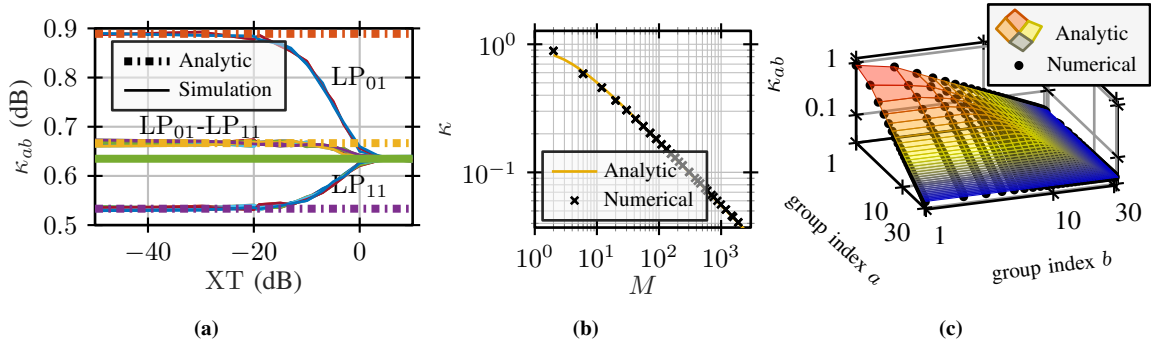


Figure 1: Fig. a: dependence of the nonlinearity coefficients κ_{ab} on the linear crosstalk for a MMF with two mode groups: LP_{01} and LP_{11} . The solid curves refer to (2) and (4), the dashed ones refer to the 36 values of κ_{ab} and are computed with (3) from the simulation of 500 realizations of a linear fiber channel. The label LP_{01} refers to the 4 values of κ_{ab} for the LP_{01} group, $LP_{01} - LP_{11}$ to the 16 values of κ_{ab} responsible for the nonlinear inter-group coupling, and LP_{11} to the 16 values of κ_{ab} for the LP_{11} group. Fig. b-c: comparison between numerical and theoretical results for κ and κ_{ab} for parabolic graded-index MMFs: in b) for the SCR, and in c) for the WCR. Fig. b: markers represent numerical results for different choices of R and NA, see [10]. The dashed line is the approximate closed-form expression Eq. (7b). Fig. c: the surface corresponds to the analytic κ_{ab} , Eq. (7a). The markers refer to the numerical values of κ_{ab} for a MMF with 31 mode groups.

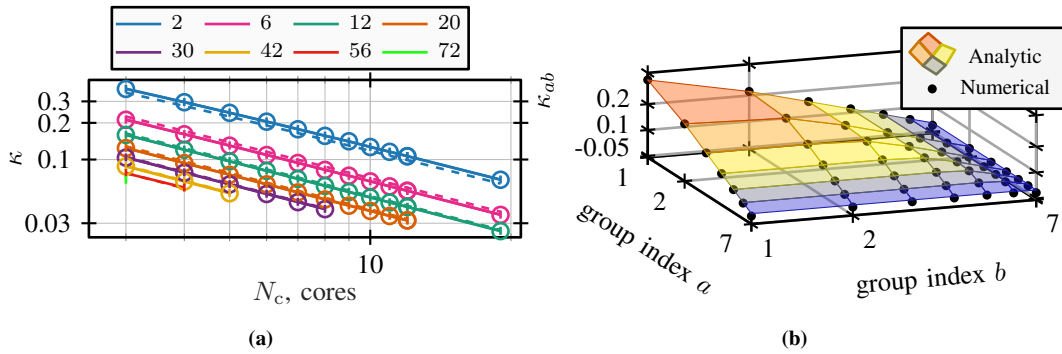


Figure 2: Comparison between numerical and theoretical results for κ and κ_{ab} for FM-MCFs: in a) for the SCR, and in b) for the WCR. Fig. a: each color refers to the value of m reported in the legend. Solid lines with circles refer to MCFs with fixed R and increasing Δ , solid lines with vertical bars refer to MCFs with fixed Δ and increasing R . The solid lines refer to the analytic formula Eq. (8b). Fig. c: the surface corresponds to the analytic κ_{ab} , Eq. (8a), while the markers to the numerical values of κ_{ab} for a MCF in the WCR with $N_c = 3$ cores and 7 mode groups ($\Delta = 0.37\%$, $R = 30.5 \mu\text{m}$) [11].

The difference between them lies in which design parameter, either R or NA, is made explicit.

Approximate expressions for the overall nonlinearity coefficients $\gamma\kappa_{ab}$ (and $\gamma\kappa$) can be obtained by multiplying one of the given expressions for γ with the relevant one for κ_{ab} (or κ).

4. IMPACT ON DATA RATES AND FIBER DESIGN

The nonlinearity coefficients $\gamma\kappa_{ab}$ are directly related to the variance of the nonlinear interference, assuming it to be an equivalent source of noise. In particular, focusing on the SCR, the nonlinear variance is σ^2 is proportional to $(\gamma\kappa)^2$, which boils down from the analytic expression for the generation of four-wave mixing components [13] [12, Eq.6.3.13]. In order to understand the potential of SDM structures to surpass bundles of SMFs, it is often of interest to assess the scaling of the nonlinearity and of the data rates with number of modes M . Assuming to fix the power per mode P , it has been shown with various perturbation models that $\sigma^2 = \eta(\gamma\kappa)^2 MP^3$ [2], [14], [15], [16]. Focusing on MMFs, given that the scaling of $(\gamma\kappa)^2$ with M is approximately in the range $[\frac{1}{M}, 1]$ [10], the scaling of the nonlinear variance is in the range $[\frac{1}{M}, M]$, assuming η to be constant with M , which is not true in general (see, e.g., [16], [17]), but accurate enough for our analysis. This result indicates that the scaling of $\gamma\kappa$ is critical in determining whether an SDM fiber brings about an increase or reduction in the data rate (per mode) expressed in bits/s/Hz/mode. The scaling of the data rate with M is displayed in Fig. 3 for two different scenarios of $\gamma\kappa$ – reducing as $1/M$, and as $1/\sqrt{M}$ – exploiting the perturbation model of [14]. As it appears from the previous discussion and from Fig. 3, the limiting condition for the rate of strongly-coupled MMFs not to reduce with respect to bundles of single-mode fibers is an approximate scaling of $\gamma\kappa$ as $1/\sqrt{M}$. The case of weakly-coupled MMFs it is quantitatively more involved to analyze due to the presence of a matrix of coefficients, but similar qualitative results and potentially higher rates can be obtained [17].

Ensuring a favorable scaling of $\gamma\kappa$ requires a careful design of MMFs and MCFs. In the former, it means increasing R rather than Δ , while in the latter it is also favorable to increase the number of cores N_c making sure that the regime of linear mixing among core modes is maintained [18].

Note that in the previous analysis we completely neglected the role of linear distortion effects and the constraint they impose on fiber design. Loosely speaking, a higher Δ tends to increase Rayleigh scattering and per-core differential mode delays, but it reduces the losses due to bends [19]. Also, increasing Δ , more modes can be guided. Thus, Δ

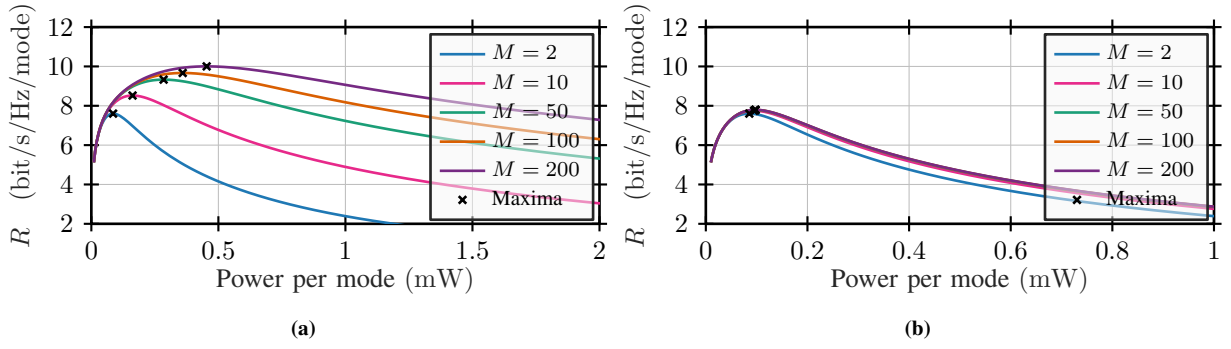


Figure 3: Comparison between the achievable data rates for a MMF in the SCR when **b)** $\gamma\kappa \propto 1/M$, and **c)** $\gamma\kappa \propto 1/\sqrt{M}$, within the perturbation model and the assumptions of [14]. See [10] for more details.

cannot be too small. A higher R allows to keep the per-core differential delays small, and to increase M . In case of MCFs, having multiple cores N_c with the right spacing allows to achieve the desired level of coupling [18]. In summary, designing a fiber requires a non-trivial tuning of the various parameters to trade-off linear and nonlinear effects and optimize data rates [17].

It is worth pointing out that even if the data rate per mode of an SDM fiber were not higher than bundles of SMFs, a benefit in terms of energy and cost per bit might still be achievable through component integration, in particular from amplifiers [1].

5. CONCLUSIONS

We reviewed relevant scaling trends and approximate closed-form expressions for the effective nonlinearity coefficients in multimode and few-mode multicore fibers. From the perspective of nonlinear effects, it has been observed that increasing the core-cladding index contrast has a negative impact on data rates, while it is helpful to increase core radius and number of cores. We provided a novel formulation of the Manakov equations and of the nonlinearity coefficients in the regime of intermediate linear coupling, which can be a starting point for further future analysis.

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