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COMPUTATIONAL UNIQUE CONTINUATION WITH FINITE DIMENSIONAL NEUMANN TRACE*

ERIK BURMAN†, LAURI OKSANEN‡, AND ZIYAO ZHAO‡

Abstract. We consider finite element approximations of unique continuation problems subject to elliptic equations in the case where the normal derivative of the exact solution is known to reside in some finite dimensional space. To give quantitative error estimates we prove Lipschitz stability of the unique continuation problem in the global H^1-norm. This stability is then leveraged to derive optimal a posteriori and a priori error estimates for a primal-dual stabilized finite element method.

Key words. unique continuation, conditional stability, finite dimension, Neumann boundary, finite element methods, stabilized methods, error estimates

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1. Introduction.

In this work we are interested in the approximation of a unique continuation problem subject to the Poisson equation. This means that the we look for the solution to the equation

$$-\Delta u = f \tag{1.1}$$

in $\Omega \subset \mathbb{R}^d$, and for some $f \in L^2(\Omega)$, when the boundary condition is unavailable on the boundary, or part of the boundary. In its stead some measured data is available. Typically, both Dirichlet and Neumann data are known on some part of the boundary (the elliptic Cauchy problem) or some measurement in the bulk. Both these situations can be handled using the arguments below, but for conciseness we will here concentrate on the second case. Therefore, we assume that for some $\omega \subset \Omega$ there is $q : \omega \to \mathbb{R}$ such that q is the restriction to ω of a solution to (1.1) and that we know q up to a quantifiable perturbation δq. The objective is then to reconstruct u using (1.1) and the a priori knowledge $u|_\omega = q$.

Unique continuation is an important model problem for many applications in control, data assimilation, or inverse problems. It is an ill-posed problem, so the assumption that the data q is associated to a solution is crucial for the solvability of the problem. It is, however, well known that if this is the case a unique solution exists

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†Department of Mathematics, University College London, 807b Gower Street, London, WC1E 6BT, UK (e.burman@ucl.ac.uk).
‡Department of Mathematics and Statistics, University of Helsinki, 00014 Helsinki, Finland (lauri.oksanen@helsinki.fi, ziyao.zhao@helsinki.fi).

and satisfies a conditional stability estimate [1]. If $B \subset\subset \Omega$, that is the set B does not intersect the boundary of Ω, then there holds for all $u \in H^1(\Omega)$,

$$\|u\|_{L^2(B)} \le C(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^{-1}(\Omega)})^{(1-\alpha)}(\|u\|_{L^2(\omega)} + \|\Delta u\|_{H^{-1}(\Omega)})^{\alpha},$$

where $\alpha \in (0,1)$. The coefficient α depends on the geometry of the domains Ω, ω, and B. As $\operatorname{dist}(\partial B, \partial\Omega) \to 0$, $\alpha \to 0$. In case B and Ω coincide the stability degenerates to logarithmic

$$\|u\|_{L^2(\Omega)} \le C\|u\|_{H^1(\Omega)} \log\left(\frac{\|u\|_{H^1(\Omega)}}{(\|u\|_{L^2(\omega)} + \|\Delta u\|_{H^{-1}(\Omega)})}\right)^{-\beta} \tag{1.2}$$

with $\beta \in (0,1)$. In a series of works [15, 17, 16] various finite element methods (FEM) have been designed and shown to satisfy bounds of the type

$$\|u - u_h\|_{L^2(B)} \le Ch^{\alpha k}(|u|_{H^{k+1}(\Omega)} + h^{-k}\|\delta q\|_{L^2(\omega)}), \tag{1.3}$$

or if $B = \Omega$ and $\delta q = 0$

$$\|u - u_h\|_{L^2(\Omega)} \le C|u|_{H^{k+1}(\Omega)} \log\left(Ch^{-k}\right)^{-\beta}.$$

In the recent contribution [21], the error bound on the form (1.3) was shown to be optimal. It can not be improved for general solutions and perturbations, regardless of the method. For moderately perturbed data and favorable subdomains ω and B this leads to sufficient accuracy, in some cases comparable to that of a well-posed problem. On the other hand, if the solution, or its normal derivative, is required on the boundary of the domain the above estimates are very poor. Indeed, there seems to be no results on how to approximate boundary traces accurately in unique continuation problems. In view of the result in [21], the only way to improve on the bounds is to have additional a priori knowledge. In the work [23] it was shown that if the Dirichlet boundary trace is close to some known finite dimensional space, then a FEM can be designed so that (1.3) holds with $\alpha = 1$, with an additional perturbation term measuring the distance of the true solution to the finite dimensional space. The assumption that the trace is close to a finite dimensional space holds in a variety of situations, for instance whenever it is a smooth perturbation of a constant, in optimal control with finite dimensional boundary control, or in engineering applications where strong modelling a priori knowledge is at hand, for example classes of admissible boundary profiles.

In the present work we consider the extension of these results to the case when the Neumann condition is in a finite dimensional space. The main result is Corollary 6.9, which gives optimal convergence rate for the finite element solution. Contrary to [23] we prove the stability underpinning the numerical analysis without resorting to the global stability (1.2) (see section 3). This makes the present analysis self contained. Although the proposed finite element method introduced in section 4 is similar to that of [23], the analysis differs in the Neumann case. Indeed the poorer regularity of the trace variable and the different functional analytical framework lead to some difficulties in the numerical analysis, that are handled in section 6, resulting in optimal a posterori and a priori error estimates.

1.1. Relation to previous work. Early work on computational unique continuation (UC) focused on rewriting the problem as a boundary integral [27, 33], while the earliest finite element reference appears to be [31]. The dominating regularization techniques are Tikhonov regularization [40] and quasi reversibility [37]. The literature on computational methods for the discretization of the regularized problem is very rich; see [34] and references therein. For references relevant for the present context we refer to [38, 4, 29, 8, 6, 7]. Iteration techniques using boundary integral formulations have been proposed in [35] and for methods using tools from optimal control, we refer to [36].

The first weakly consistent methods with regularization on the discrete level were introduced in [13], with the first analysis for ill-posed problems in [14] and then developed further in [15, 17, 22, 18, 16, 3]. This approach is related to previous work on finite element methods for indefinite problems based on least squares minimization in H^{-1} [9, 10]. More recent results using least squares minimization in dual norm for ill-posed problems can be found in [26, 25, 28]. Quantitative a priori error estimates have been derived in a number of situations with careful analysis of the effect of the physical parameters of the problem on the constants of the error estimates [18, 19, 20]. This has lead to a deeper understanding of the computational difficulty of recovering quantities via UC in different parameter regimes.

That Lipschitz stability can be recovered for finite dimensional target quantities has been known for some time in the inverse problem community; see, for example, [2, 5]. Nevertheless, it appears that the first time this property has been exploited in a computational method, leading to optimal error estimates, is [23].

2. Problem setting. Let $\mathcal{V}_N$ be a subspace of $L^2(\partial\Omega)$ that satisfies

1. For all $g \in \mathcal{V}_N$, there holds $\int_{\partial\Omega} g\ dx = 0$.
2. $\dim(\mathcal{V}_N) = N < \infty$.

We consider the following problems:

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u = q \text{ in } \omega, \\ \partial_\nu u|_{\partial\Omega} \in \mathcal{V}_N + \beta, \end{cases} \tag{2.1}$$

where $\Omega \in \mathbb{R}^d$ is an open, bounded polygonal domain, $\omega \subset \Omega$ is open, and nonempty. $f \in L^2(\Omega)$ and β is a constant satisfying $\beta\,|\partial\Omega| = \int_\Omega f\ dx$. We denote P to be a projection operator on $\mathcal{V}_N$, with $Q = 1 - P$.

3. Lipschitz stability. First, we define a continuous bilinear functional $l(\cdot,\cdot)$ on $H^1(\Omega) \times H^1(\Omega)$,

$$l(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)},$$

and for each $u \in H^1(\Omega)$, a continuous linear functional L_u on $H^1(\Omega)$,

$$L_u(v) := l(u,v).$$

Next, we introduce the space

$$H^1_\omega(\Omega) := \left\{ u \in H^1(\Omega) \,\middle|\, \int_\omega u\ dx = 0 \right\}.$$

Then, we have the following lemma.

Lemma 3.1. *For every $u \in H^1_\omega(\Omega)$, there holds*

$$\|u\|_{H^1(\Omega)} \lesssim \|L_u\|_{(H^1(\Omega))^*}.$$

Proof. We start from the inequality

$$\|u\|_{H^1(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)},$$

define a continuous linear operator

$$A\colon\ H^1(\Omega) \to \left[L^2(\Omega)\right]^n \times L^2(\omega),$$
$$Au = (\nabla u, u|_\omega),$$

and denote the natural imbedding from $H^1(\Omega)$ to $L^2(\Omega)$ by K. To simplify the notation, we denote

$$X := \left[L^2(\Omega)\right]^n \times L^2(\omega).$$

Then we have

$$\|u\|_{H^1(\Omega)} \lesssim \|Au\|_X + \|Ku\|_{L^2(\Omega)}.$$

Notice that K is compact and A is an injection. Indeed, suppose $Au_0 = 0$ for $u_0 \in H^1(\Omega)$, that is, $\nabla u_0 = 0$ and $u_0|_\omega = 0$. Since $\nabla u_0 = 0$ implies u_0 is a constant, $u_0 = 0$ in Ω follows immediately from $u_0|_\omega = 0$.

Therefore, by compactness-uniqueness [24, Lemma 9],

$$\|u\|_{H^1(\Omega)} \lesssim \|Au\|_X = \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\omega)}.$$

According to the Friedrich's inequality [11, Lemma 4.3.14], the condition $\int_\omega u \, d\mathbf{x} = 0$ implies that

$$\|u\|_{L^2(\omega)} \lesssim \|\nabla u\|_{L^2(\omega)} \le \|\nabla u\|_{L^2(\Omega)}.$$

Hence

$$\|u\|_{H^1(\Omega)} \lesssim \|\nabla u\|_{L^2(\Omega)}. \tag{3.1}$$

For arbitrary $\epsilon > 0$, there holds

$$\|\nabla u\|^2_{L^2(\Omega)} = L_u(u) \le \|L_u\|_{(H^1(\Omega))^*} \|u\|_{H^1(\Omega)} \le \epsilon^{-1} \|L_u\|^2_{(H^1(\Omega))^*} + \frac{\epsilon}{4} \|u\|^2_{H^1(\Omega)}. \tag{3.2}$$

Combining (3.1) and (3.2), and choosing ϵ small enough, we have

$$\|u\|_{H^1(\Omega)} \lesssim \|L_u\|_{(H^1(\Omega))^*}.$$ ∎

LEMMA 3.2. *Suppose $F \in (H^1(\Omega))^*$ satisfies*

$$F(\mathbf{c}) = 0$$

for all constant functions $\mathbf{c}$ on Ω. Then for the variational problem

$$l(u, v) = F(v) \ \ \forall v \in H^1(\Omega), \tag{3.3}$$

there exists a unique solution $u \in H^1_\omega(\Omega)$.

Proof. Since for any constant function $\mathbf{c}$ on Ω, $F(\mathbf{c})=0$ and $l(u,\mathbf{c})=0$, problem (3.3) amounts to finding $u\in H^1(\Omega)$ such that

$$l(u,v)=F(v)\ \ \forall v\in H^1_\omega(\Omega).$$

The coercivity of l on $H^1_\omega(\Omega)$ follows from (3.1). According to Lax–Milgram theorem [12, Corollary 5.8], there exists a unique $u\in H^1_\omega(\Omega)$ solves (3.3). ∎

For each $u\in H^1(\Omega)$ with $\partial_\nu u|_{\partial\Omega}\in L^2(\partial\Omega)$, we define a linear functional L^q_u on $H^1(\Omega)$

$$L^q_u(v):=L_u(v)-(P\partial_\nu u,v)_{L^2(\partial\Omega)},\ v\in H^1(\Omega).$$

By trace inequality, we can see that $L^q_u\in(H^1(\Omega))^*$.

THEOREM 3.3. *For $u\in H^1(\Omega)$ with $\partial_\nu u|_{\partial\Omega}\in L^2(\partial\Omega)$, there holds*

$$\|P\partial_\nu u\|_{L^2(\partial\Omega)}+\|u\|_{H^1(\Omega)}\lesssim\|u\|_{L^2(\omega)}+\|L^q_u\|_{(H^1(\Omega))^*}. \tag{3.4}$$

Proof. First, we assume that $u\in H^1_\omega(\Omega)$.

Write $u=v+w$, where $v,\ w\in H^1_\omega(\Omega)$ and satisfy

$$l(v,\varphi)=L^q_u(\varphi)\ \ \forall\varphi\in H^1(\Omega), \tag{3.5}$$

and

$$l(w,\varphi)=(P\partial_\nu u,\varphi)_{L^2(\partial\Omega)}\ \ \forall\varphi\in H^1(\Omega), \tag{3.6}$$

respectively. Since $L^q_u(\mathbf{c})=0$ and $(P\partial_\nu u,\mathbf{c})_{L^2(\partial\Omega)}=0$, (3.5) and (3.6) are both solvable by Lemma 3.2. We define an operator

$$\begin{gathered}A:\mathcal{V}_N\to L^2(\omega),\\ A(g):=w_g|_\omega,\ g\in\mathcal{V}_N,\end{gathered}$$

where $w_g\in H^1_\omega(\Omega)$ satisfies

$$l(w_g,\varphi)=(g,\varphi)_{L^2(\partial\Omega)}\ \ \forall\varphi\in H^1(\Omega).$$

Notice that $\|\Delta w_g\|_{H^{-1}(\Omega)}=0$. Then by (1.2), A is injective.

Since $A(\mathcal{V}_N)$ is a finite dimensional subspace of $L^2(\omega)$, there exist a norm on $A(\mathcal{V}_N)$ such that A is an isometry. As all norms are equivalent in the finite dimensional range of A, there holds

$$\|g\|_{L^2(\partial\Omega)}\lesssim\|Ag\|_{L^2(\omega)}.$$

Notice that $A(P\partial_\nu u)=w|_\omega$. We conclude that

$$\|P\partial_\nu u\|_{L^2(\partial\Omega)}\lesssim\|w\|_{L^2(\omega)}.$$

Applying Lemma 3.1 to v and w, there holds

$$\begin{aligned}\|u\|_{H^1(\Omega)}&\le\|v\|_{H^1(\Omega)}+\|w\|_{H^1(\Omega)}\lesssim\|v\|_{H^1(\Omega)}+\|P\partial_\nu u\|_{L^2(\partial\Omega)}\\ &\lesssim\|v\|_{H^1(\Omega)}+\|w\|_{L^2(\omega)}\lesssim\|v\|_{H^1(\Omega)}+\|u\|_{L^2(\omega)}\\ &=\|L^q_u\|_{(H^1(\Omega))^*}+\|u\|_{L^2(\omega)}.\end{aligned}$$

The above argument also gives

$$\|P\partial_\nu u\|_{L^2(\partial\Omega)} \lesssim \|L_u^q\|_{(H^1(\Omega))^*} + \|u\|_{L^2(\omega)}.$$

For $u \notin H^1_\omega(\Omega)$, there exist a $\tilde{u} \in H^1_\omega(\Omega)$ and a constant $C \in \mathbb{R}$ such that $u = C + \tilde{u}$. Then

$$\|u\|^2_{H^1(\Omega)} \leq 2\|\tilde{u}\|^2_{H^1(\Omega)} + 2C^2|\Omega|,$$
$$\|u\|^2_{L^2(\omega)} = \|\tilde{u}\|^2_{L^2(\omega)} + C^2|\omega|.$$

Therefore, we have

$$\begin{aligned}\|u\|^2_{H^1(\Omega)} &\leq 2\|\tilde{u}\|^2_{H^1(\Omega)} + 2C^2|\Omega| \lesssim \|L^q_{\tilde{u}}\|^2_{(H^1(\Omega))^*} + \|\tilde{u}\|^2_{L^2(\omega)} + C^2|\omega| \\ &= \|L^q_u\|^2_{(H^1(\Omega))^*} + \|u\|^2_{L^2(\omega)},\end{aligned}$$

and

$$\begin{aligned}\|P\partial_\nu u\|_{L^2(\partial\Omega)} = \|P\partial_\nu \tilde{u}\|_{L^2(\partial\Omega)} &\lesssim \|\tilde{u}\|_{L^2(\omega)} + \|L^q_{\tilde{u}}\|_{(H^1(\Omega))^*} \\ &\lesssim \|u\|_{L^2(\omega)} + \|L^q_u\|_{(H^1(\Omega))^*}.\end{aligned}$$

Combining the above two inequalities yields (3.4). ∎

4. Finite element method. Here we will introduce a finite element method for the approximation of (2.1). Some results detailing the continuity, stability and consistency properties of the method will then be proven, preparing the terrain for the error analysis in the next section.

Let $\mathcal{T}_h$ be a decomposition of Ω into shape regular simplices K that form a simplicial complex, and let $h = \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$ be the global mesh parameter. The trace inequality with scaling [11, eq. 10.3.8] reads

$$h^{1/2}\|u\|_{L^2(\partial K)} \lesssim \|u\|_{L^2(K)} + \|h\nabla u\|_{L^2(K)}, \quad u \in H^1(K). \tag{4.1}$$

On $\mathcal{T}_h$ we define the standard space of continuous finite element functions

$$V_h := \{v \in H^1(\Omega) \mid \ v|_K \in \mathbb{P}_k \text{ for } K \in \mathcal{T}_h\}.$$

Here $\mathbb{P}_k$ is the space of polynomial of degree at most $k \geq 1$ on K. For $m \geq 0$, we denote the broken semiclassical Sobolev seminorms and norms by

$$[u]^2_{H^m(\mathcal{T}_h)} = \sum_{K\in\mathcal{T}_h} \|(hD)^m u\|^2_{L^2(K)}, \quad \|u\|^2_{H^m(\mathcal{T}_h)} = \sum_{k=0}^m [u]^2_{H^k(\mathcal{T}_h)}.$$

The discrete inequality [30, Lemma 1.138] in $\mathbb{P}_k$ implies that for all integers $m \geq l \geq 0$

$$[u]_{H^m(\mathcal{T}_h)} \lesssim [u]_{H^l(\mathcal{T}_h)}, \quad u \in V_h. \tag{4.2}$$

Let F be a interior face between two simplices $K_1, K_2 \in \mathcal{T}_h$ such that $K_1 \cap K_2 = F$, then the jump over F is given by

$$[\![\nabla u]\!]_F = \nu_1 \cdot \nabla u|_{K_1} + \nu_2 \cdot \nabla u|_{K_2},$$

where ν_1 and ν_2 are the outward normal of K_1 and K_2, respectively.

Then we introduce the Lagrangian on $V_h \times V_h$,

$$\mathcal{L}(u,z) := \frac{h^2}{2}\|u-q\|^2_{L^2(\omega)} + a(u,z) - h^2(f,z)_{L^2(\Omega)} - h^2(\beta,z)_{L^2(\partial\Omega)} + \frac{1}{2}B(u) - h(hQ\partial_\nu u, h\beta)_{L^2(\partial\Omega)} + \frac{1}{2}[h^2\Delta u + h^2 f]^2_{H^0(\mathcal{T}_h)} + \frac{1}{2}J(u) - \frac{1}{2}S^*(z),$$

where

$$\begin{aligned} B(u) &= h\|hQ\partial_\nu u\|^2_{L^2(\partial\Omega)},\\ a(u,z) &= (h\nabla u, h\nabla z)_{L^2(\Omega)} - h\int_{\partial\Omega} h(P\partial_\nu u)z\, dx,\\ J(u) &= \sum_{K\in\mathcal{T}_h} h\|[\![h\nabla u]\!]\|^2_{L^2(\partial K\setminus\partial\Omega)},\\ S^*(z) &= h^2\|z\|^2_{H^1(\Omega)}. \end{aligned}$$

For

$$S(u) = J(u) + [h^2\Delta u]^2_{H^0(\mathcal{T}_h)},$$

we have the following lemma.

Lemma 4.1. *For $u \in H^1(\Omega)$ with $\Delta(u|_K) \in L^2(K)$ for all $K \in \mathcal{T}_h$, $\partial_\nu u|_{\partial\Omega} \in L^2(\partial\Omega)$, and $z \in H^1(\Omega)$, there holds*

$$a(u,z) \lesssim (S(u)^{1/2} + B(u)^{1/2})\|z\|_{H^1(\mathcal{T}_h)}.$$

Proof. An integration by parts reads

$$\begin{aligned} a(u,z) &= (h\nabla u, h\nabla z)_{L^2(\Omega)} - h\int_{\partial\Omega} h(P\partial_\nu u)z\, dx\\ &= -\sum_{K\in\mathcal{T}_h}\int_K h^2\Delta u z\, dx + \sum_{F\in\mathcal{F}_h} h\int_F [\![h\partial_\nu u]\!] z\, dx + h\int_{\partial\Omega}(Qh\partial_\nu u)z\, dx, \end{aligned}$$

where $\mathcal{F}_h$ is the set of elements faces in the interior of Ω.

Applying the trace inequality with scaling (4.1) on each face in $\mathcal{F}_h$ and $\partial\Omega$, we can obtain

$$\begin{aligned} &\sum_{F\in\mathcal{F}_h} h\int_F [\![h\partial_\nu u]\!] z\, dx\\ &\quad\le \sum_{K\in\mathcal{T}_h} h^{1/2}\|[\![h\partial_\nu u]\!]\|_{L^2(\partial K\setminus\partial\Omega)} h^{1/2}\|z\|_{L^2(\partial K\setminus\partial\Omega)}\\ &\quad\lesssim \sum_{K\in\mathcal{T}_h} h^{1/2}\|[\![h\partial_\nu u]\!]\|_{L^2(\partial K\setminus\partial\Omega)}\left(\|z\|_{L^2(K)} + \|h\nabla z\|_{L^2(K)}\right)\\ &\quad\le \left(\sum_{K\in\mathcal{T}_h} h\|[\![h\partial_\nu u]\!]\|^2_{L^2(\partial K\setminus\partial\Omega)}\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}\|z\|^2_{L^2(K)} + \|h\nabla z\|^2_{L^2(K)}\right)^{\frac{1}{2}}\\ &\quad\le J(u)^{1/2}\|z\|_{H^1(\mathcal{T}_h)}, \end{aligned}$$

and

$$h\int_{\partial\Omega}(Qh\partial_\nu u)z\, dx \le h^{1/2}\|Qh\partial_\nu u\|_{L^2(\partial\Omega)} h^{1/2}\|z\|_{L^2(\partial\Omega)} \lesssim B(u)^{1/2}\|z\|_{H^1(\mathcal{T}_h)}.$$

Finally, notice that

$$\sum_{K\in\mathcal{T}_h}\int_K h^2(\Delta u)z\,dx \le \sum_{K\in\mathcal{T}_h}[h^2\Delta u]_{L^2(K)}\|z\|_{L^2(K)}$$
$$\le [h^2\Delta u]_{H^0(\mathcal{T}_h)}[z]_{H^0(\mathcal{T}_h)} \le [h^2\Delta u]_{H^0(\mathcal{T}_h)}\|z\|_{H^1(\mathcal{T}_h)}.$$

Thus we conclude that

$$a(u,z) \lesssim \left([h^2\Delta u]_{H^0(\mathcal{T}_h)} + J(u)^{1/2} + B(u)^{1/2}\right)\|z\|_{H^1(\mathcal{T}_h)}$$
$$\lesssim (S(u)^{1/2} + B(u)^{1/2})\|z\|_{H^1(\mathcal{T}_h)}. \qquad \square$$

Finding saddle points of the Lagrangian $\mathcal{L}$ amounts to finding $(u,z)\in V_h\times V_h$ such that for all $(v,w)\in V_h\times V_h$, there holds

$$\begin{cases} a(u,w) - s^*(z,w) = h^2(f,w)_{L^2(\Omega)} + h^2(\beta,w)_{L^2(\partial\Omega)}, \\ h^2(u,v)_{L^2(\omega)} + b(u,v) + a(v,z) + s(u,v) = h^2(q,v)_{L^2(\omega)} + h^3(Q\partial_\nu v,\beta)_{L^2(\partial\Omega)} \\ +(h^2 f, h^2\Delta v)_{H^0(\mathcal{T}_h)}. \end{cases}$$

Here s^*, b, and s are the bilinear forms corresponding to S^*, B, and S, respectively. Solving the above system is equivalent to finding $(u,z)\in V_h\times V_h$ such that for all $(v,w)\in V_h\times V_h$, there holds

$$g(u,z,v,w) = h^2(f,w)_{L^2(\Omega)} + h^2(\beta,w)_{L^2(\partial\Omega)} + h^2(q,v)_{L^2(\omega)} + h^3(Q\partial_\nu v,\beta)_{L^2(\partial\Omega)} + (h^2 f, h^2\Delta v)_{H^0(\mathcal{T}_h)}, \tag{4.3}$$

where the bilinear form g is defined by

$$g(u,z,v,w) := h^2(u,v)_{L^2(\omega)} + b(u,v) + a(v,z) + s(u,v) + a(u,w) - s^*(z,w).$$

Remark 4.2. Recall that $f\in L^2(\Omega)$. Hence, if $u\in H^1(\Omega)$ solves (2.1), then $u\in H^2_{\mathrm{loc}}(\Omega)$. In particular, for all $F\in\mathcal{F}_h$ and all compact sets K in the interior of Ω there holds $[\![\nabla u]\!]=0$ on $F\cap K$. As K is arbitrary, the same holds on the whole set F, and $J(u)=0$. Therefore, $(u,0)$ solves (4.3) for all $(v,w)\in V_h\times V_h$ and the system (4.3) is consistent.

Preparing the terrain for the error analysis of the next section, we introduce the norm

$$|\!|\!|u,z|\!|\!|^2 = B(u) + h^2\|u\|^2_{L^2(\omega)} + S(u) + S^*(z).$$

According to Lemma 4.1, we have

$$\|L_u^q\|_{(H^1(\Omega))^*} = \sup_{v\in H^1(\Omega)}\frac{L_u^q(v)}{\|v\|_{H^1(\Omega)}} = \sup_{v\in H^1(\Omega)}\frac{a(u,v)}{h^2\|v\|_{H^1(\Omega)}}$$
$$\lesssim \sup_{v\in H^1(\Omega)}\frac{(S(u)^{1/2}+B(u)^{1/2})\|v\|_{H^1(\mathcal{T}_h)}}{h^2\|v\|_{H^1(\Omega)}} \lesssim \frac{1}{h^2}|\!|\!|u,0|\!|\!|.$$

By Theorem 3.3, we have

$$\|u\|_{H^1(\Omega)} \lesssim \frac{1}{h^2}|\!|\!|u,0|\!|\!|.$$

Notice that $|\!|\!|u,z|\!|\!|^2 = |\!|\!|u,0|\!|\!|^2 + |\!|\!|0,z|\!|\!|^2$, then $|\!|\!|u,z|\!|\!| = 0$ implies that $u=0$ and $z=0$. Hence $|\!|\!|\cdot,\cdot|\!|\!|$ is indeed a norm.

LEMMA 4.3.

$$\|\!|u,z|\!\| \lesssim \sup_{(v,w)\in V_h\times V_h} \frac{g(u,z,v,w)}{\|\!|v,w|\!\|}, \quad (u,z)\in V_h\times V_h.$$

Proof. The claimed inequality follows immediately from

$$\|\!|u,z|\!\|^2 = g(u,z,u,-z). \qquad \square$$

Due to Lemma 4.3, the linear system (4.3) admits a unique solution for any $f\in L^2(\Omega)$ and $q\in L^2(\omega)$. Denoting the unique solution by (u_h,z_h) and letting u solve (2.1), we have the Galerkin orthogonality

$$g(u_h-u,z_h,v,w)=0,\ (v,w)\in V_h\times V_h. \tag{4.4}$$

5. Necessity of regularization. One may wonder if the regularization introduced in section 4 is really needed. To illustrate necessity of a regularization of some form, let us consider a straightforward data fitting approach. To simplify the discussion, we suppose that $f=0$ and, therefore< $\beta=0$ in (2.1). Let $\{\psi_i\}_{i=1}^N$ be a basis of $\mathcal{V}_N$, and let $w_h^i\in V_h\cap H^1_\omega(\Omega)$ be the solution of

$$\int_\Omega \nabla w_h^i\cdot\nabla v\,dx = \int_{\partial\Omega}\psi_i v\,dx \ \ \forall v\in V_h.$$

Let W_h be the space spanned by $\{w_h^i\}_{i=1}^N\oplus\{1\}$ where 1 is the constant function that is identical to 1 in Ω. It feels natural to try to approximate the solution of (2.1) via

$$u_h = \underset{w_h\in W_h}{\arg\min}\|w_h-q\|_{L^2(\omega)}. \tag{5.1}$$

However, such u_h may not be unique. Indeed, if there exists $w_h\in W_h$ that satisfies $w_h|_\omega=0$, then (5.1) does not define u_h uniquely. In fact, this may happen even when $N=1$ if $h>0$ is not small enough, as shown by Proposition 5.2 below. In other words, the mesh size $h>0$ cannot be chosen independently of $\mathcal{V}$.

Presumably there is $h_0(\mathcal{V},\Omega,\omega)>0$ such that all $w_h\in W_h$ have the unique continuation property for

$$0\le h\le h_0(\mathcal{V},\Omega,\omega),$$

but we have opted not to try to find such a selection rule h_0. We expect that h_0 would depend on the implicit constant in (3.4). On the other hand, our method introduced in section 4 does not impose any constraints on the mesh size, as the system (4.3) has a unique solution for any $h>0$.

To simplify the discussion, we suppose that $\Omega:=[0,1]\times[0,1]$ and the mesh size $h=\frac{\sqrt{2}}{n-1}$ where $n>3$ is an integer. Let $\mathcal{T}_h$ be a uniform triangular decomposition of Ω as exemplified in Figure 1, and let the polynomial order of V_h be one. Then V_h is a n^2 dimensional space. The set of nodal basis functions $\{\phi_i\}_{i=1}^{n^2}$ forms a basis of V_h. We notice that $\{\phi_i\}_{i=1}^{n^2}$ consists of $4n-4$ boundary nodal basis functions and $(n-2)^2$ interior nodal basis functions.

Let $\{\phi_i\}_{i=1}^{4n-4}$ be boundary nodal functions. We denote the space of continuous functions with zero mean value on $\partial\Omega$ by Γ, that is,

$$\Gamma := \left\{ g\in C(\partial\Omega)\,\Big|\, \int_{\partial\Omega} g\,dx = 0\right\}.$$

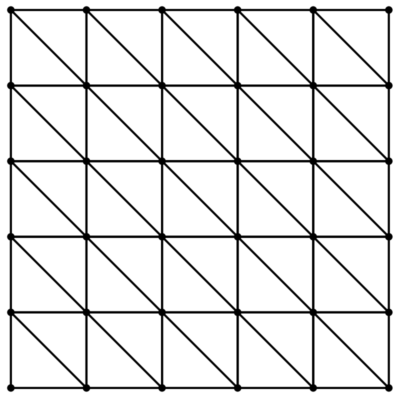


FIG. 1. *A uniform triangular decomposition on* $\Omega = [0,1] \times [0,1]$ *with* $n = 6$.

And we define the operator

$$\Lambda : \Gamma \to \mathbb{R}^{4n-4}$$

as

$$\Lambda(g) = \boldsymbol{b} = (b_1, b_2, \ldots, b_{4n-4}),\ g \in \Gamma,$$

where

$$b_i = \int_{\partial\Omega} g\phi_i \, dx, 1 \le i \le 4n-4.$$

Write the vector $\boldsymbol{e} = (\underbrace{1, 1, \cdots, 1}_{4n-4})^T$. Consider the functional it induced on $\mathbb{R}^{4n-4}$

$$T(\boldsymbol{a}) = \boldsymbol{e}^T \boldsymbol{a} = \sum_{i=1}^{4n-4} a_i,\ \boldsymbol{a} = (a_1, a_2, \ldots, a_{4n-4}) \in \mathbb{R}^{4n-4}.$$

We have the following lemma.

LEMMA 5.1. *Let* $R(\Lambda) := \{\Lambda g \mid g \in \Gamma\}$ *be the range of* Λ*, then we have*

$$T^{-1}(0) \subset R(\Lambda).$$

Proof. Rearrange the index such that ϕ_i and ϕ_{i+1} are two adjacent boundary nodal basis functions for $1 \le i \le 4n-5$. Without loss of generality, we assume that $\operatorname{supp}(\phi_i|_{\partial\Omega}) \cap \operatorname{supp}(\phi_{i+1}|_{\partial\Omega}) = [y_1, y_2]$ and

$$\phi_i|_{[y_1,y_2]} = 1 - \frac{y - y_1}{y_2 - y_1}, \quad \phi_{i+1}|_{[y_1,y_2]} = \frac{y - y_1}{y_2 - y_1}.$$

Notice that $[y_1, y_2] \cap \operatorname{supp}(\phi_j|_{\partial\Omega}) = \emptyset$ if $j \ne i, i+1$. Then for a nonzero smooth function g_i that is supported in (y_1, y_2) and that is odd corresponding to the midpoint of the interval $[y_1, y_2]$, we have

$$\int_{\partial\Omega} g_i \phi_i \, dx = -\int_{\partial\Omega} g_i \phi_{i+1} \, dx = c_i \ne 0,$$

and

$$\int_{\partial\Omega} g_i \phi_j \, dx = 0 \text{ if } j \neq i, \; i+1.$$

It follows that

$$\Lambda(g_i/c_i) = \boldsymbol{b}_i = (\underbrace{0, \ldots, 0}_{(i-1)\ zeros}, 1, -1, \underbrace{0, \ldots, 0}_{(4n-i-5)\ zeros}), \; 1 \leq i \leq 4n-5.$$

Observe that $\{\boldsymbol{b}_i\}_{i=1}^{4n-5}$ forms a basis of $T^{-1}(0)$, thus $T^{-1}(0) \subset R(\Lambda)$ follows immediately. ∎

PROPOSITION 5.2. *Suppose* $\Omega := [0,1] \times [0,1]$. *For any fixed integer* $n > 3$ *and* $h = \frac{\sqrt{2}}{n-1}$, *we set* $\omega := [\frac{1}{n-1}, 1 - \frac{1}{n-1}] \times [\frac{1}{n-1}, 1 - \frac{1}{n-1}]$. *Let* $\mathcal{T}_h$ *be the uniform triangular decomposition of* Ω, *and let the polynomial order of* V_h *be one. Then there exists a function* $g \in \Gamma$, *such that the solution* $u_h \in V_h \cap H^1_\omega(\Omega)$ *of*

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_{\partial\Omega} g v \, dx \;\; \forall v \in V_h \tag{5.2}$$

satisfies $u_h \neq 0$ *and* $u_h|_\omega = 0$.

Proof. Let $\{\phi_i\}_{i=1}^{n^2}$ be nodal basis functions of V_h and rearrange the index so that $\{\phi_i\}_{i=1}^{(n-2)^2}$ are the interior nodal basis functions while $\{\phi_i\}_{i=(n-2)^2+1}^{n^2}$ are the boundary nodal basis functions. Then we consider

$$\boldsymbol{A}\boldsymbol{\alpha} = \boldsymbol{b}, \tag{5.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n^2})^T$, the entries of $\boldsymbol{A} = (a_{ij})_{n^2 \times n^2}$ are $a_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx$, and the entries of $\boldsymbol{b} = (b_j)_{n^2}$ are $b_j = \int_{\partial\Omega} \psi\phi_j \, dx$ for some $\psi \in \Gamma$. Our first observation is that $b_j = 0$ for $j \leq (n-2)^2$ since $\{\phi_j\}_{j=1}^{(n-2)^2}$ are interior nodal basis functions. Then we decompose (5.3) into the following form:

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{b}_2 \end{pmatrix}.$$

Here $\boldsymbol{\alpha}_1 \in \mathbb{R}^{(n-2)^2}$, $\boldsymbol{\alpha}_2$, $\boldsymbol{b}_2 \in \mathbb{R}^{4n-4}$, $A \in \mathbb{R}^{(n-2)^2 \times (n-2)^2}$, $B \in \mathbb{R}^{(4n-4) \times (n-2)^2}$, and $D \in \mathbb{R}^{(4n-4) \times (4n-4)}$. For the interior nodal basis functions $\{\phi_j\}_{j=1}^{(n-2)^2}$,

$$\int_\Omega \nabla\phi_j \nabla\phi_i \, dx \neq 0 \text{ for some } (n-2)^2 + 1 \leq i \leq n^2$$

only if ϕ_j is an element adjacent to the boundary. For uniform triangular decomposition, there are only $4n-12$ interior elements adjacent to boundary elements. That is, at most $4n-12$ rows in B are nonzero.

Let $\boldsymbol{\alpha}_1 = 0$. Since the dimension of $\boldsymbol{\alpha}_2$ is $4n-4$ and the rank of B is $4n-12$, the linear system

$$\begin{cases} B\boldsymbol{\alpha}_2 = 0, \\ \boldsymbol{e}^T D\boldsymbol{\alpha}_2 = 0 \end{cases}$$

admits a 7 dimensional null space $X \subset \mathbb{R}^{4n-4}$. Select a nonzero $\boldsymbol{\alpha}_2 \in X$. According to Lemma 5.1, there exists a function $g \in \Gamma$ such that $\Lambda(g) = D\boldsymbol{\alpha}_2$. Recall that

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n^2})^T = \begin{pmatrix} 0 \\ \boldsymbol{\alpha}_2 \end{pmatrix}.$$

Set $u_h = \sum_{i=1}^{n^2} \alpha_i \phi_i$, then u_h solves (5.2). We notice that $u_h|_{[h,1-h]\times[h,1-h]} = 0$ since $\alpha_j = 0,\ 1 \le j \le (n-2)^2$, corresponding to the coefficients of interior nodal basis functions. ∎

6. Error analysis. The objective of this section is to leverage the stability of Theorem 3.3 to derive optimal error estimates. We start with a technical lemma.

LEMMA 6.1. *Suppose that* $u \in H^1(\Omega)$ *and* $(u_h, z_h) \in V_h \times V_h$ *solve* (2.1) *and* (4.3), *respectively. Then there holds*

$$h^{-1}\|h^2 L^q_{u-u_h}\|_{(H^1(\Omega))^*} \lesssim S(u-u_h)^{1/2} + B(u-u_h)^{1/2} + S^*(z_h)^{1/2}. \tag{6.1}$$

Proof. We have

$$\|h^2 L^q_{u-u_h}\|_{(H^1(\Omega))^*} = \sup_{w\in H^1(\Omega)} \frac{h^2 L^q_{u-u_h}(w)_{L^2(\Omega)}}{\|w\|_{H^1(\Omega)}} = \sup_{w\in H^1(\Omega)} \frac{a(u-u_h,w)}{\|w\|_{H^1(\Omega)}}.$$

Let $i_h : H^m(\Omega) \to V_h$ be an interpolator satisfying

$$\|u - i_h u\|_{H^m(\mathcal{T}_h)} \lesssim [u]_{H^m(\mathcal{T}_h)},\ u \in H^m(\Omega) \tag{6.2}$$

for all $m \ge 1$. The Scott–Zhang interpolator is a possible choice [39]. For any $w \in H^1(\Omega)$, (4.4) gives

$$g(u_h - u, z_h, 0, i_h w) = a(u_h - u, i_h w) - s^*(z_h, i_h w) = 0.$$

Applying Lemma 4.1, we have

$$\begin{aligned} a(u-u_h,w) &= a(u-u_h, w - i_h w) - s^*(z_h, i_h w) \\ &\lesssim (S(u-u_h)^{1/2} + B(u-u_h)^{1/2})\|w - i_h w\|_{H^1(\mathcal{T}_h)} + S^*(z_h)^{1/2} S^*(i_h w)^{1/2} \\ &\lesssim (S(u-u_h)^{1/2} + B(u-u_h)^{1/2})[w]_{H^1(\mathcal{T}_h)} \\ &\quad + S^*(z_h)^{1/2} h(\|i_h w - w\|_{H^1(\Omega)} + \|w\|_{H^1(\Omega)}) \\ &\lesssim h\left(S(u-u_h)^{1/2} + B(u-u_h)^{1/2} + S^*(z_h)^{1/2}\right)\|w\|_{H^1(\Omega)}, \end{aligned}$$

which implies (6.1). ∎

Then we can prove the a posteriori error estimate

THEOREM 6.2. *Suppose that* $u \in H^1(\Omega)$ *and* $(u_h, z_h) \in V_h$ *solves* (2.1) *and* (4.3), *respectively. Then there holds*

$$\begin{aligned} h\|u-u_h\|_{H^1(\Omega)} &\lesssim h\|u_h - q\|_{L^2(\omega)} + J(u_h)^{1/2} + h^{3/2}\|Q\partial_\nu u_h - \beta\|_{L^2(\partial\Omega)} \\ &\quad + [h^2\Delta u_h + h^2 f]_{H^0(\mathcal{T}_h)} + h\|z_h\|_{H^1(\Omega)} \lesssim \|\!|\!| u-u_h, z_h |\!|\!\|. \end{aligned} \tag{6.3}$$

Proof. Applying Theorem 3.3, we have

$$h\|u-u_h\|_{H^1(\Omega)} \lesssim h\|u_h - q\|_{L^2(\omega)} + h\|L^q_{u-u_h}\|_{(H^1(\Omega))^*}.$$

Here we recall $u|_\omega = q$. Using $-\Delta u = f,\ J(u) = 0,\ Q\partial_\nu u = \beta$, and Lemma 6.1, we have

$$\begin{aligned} h\|L^q_{u-u_h}\|_{(H^1(\Omega))^*} &\lesssim S(u-u_h)^{1/2} + B(u-u_h)^{1/2} + S^*(z_h)^{1/2} \\ &\lesssim J(u_h)^{1/2} + h^{3/2}\|Q\partial_\nu u_h - \beta\|_{L^2(\partial\Omega)} + [h^2\Delta u_h + h^2 f]_{H^0(\mathcal{T}_h)} \\ &\quad + h\|z_h\|_{H^1(\Omega)}. \end{aligned}$$

Furthermore, notice that the right-hand side of the first inequality in (6.3) can be written as

$$h\|u-u_h\|_{L^2(\omega)} + B(u-u_h)^{1/2} + J(u_h)^{1/2} + [h^2\Delta u_h + h^2 f]_{H^0(\mathcal{T}_h)} + h\|z_h\|_{H^1(\Omega)}$$

which is essentially identical to $\|\!|u-u_h, z_h\|\!|$. ◻

Next, we use the orthogonality and Lemma 4.3 to derive the best approximation in the norm $\|\!|\cdot\|\!|$. Before we present the lemma, we introduce the notation

$$\|u\|_{V_h^{-1/2}(\partial\Omega)} = \sup_{w\in V_h} \frac{(u,w)_{L^2(\partial\Omega)}}{\|w\|_{H^1(\Omega)}}.$$

LEMMA 6.3. *Suppose that $u\in H^1(\Omega)$ and $(u_h,z_h)\in V_h\times V_h$ solve* (2.1) *and* (4.3), *respectively. Then there holds*

(6.4)
$$\|\!|u-u_h, z_h\|\!| \lesssim \inf_{\tilde u\in V_h}\left(\|\!|u-\tilde u, 0\|\!| + \|h\nabla(u-\tilde u)\|_{L^2(\Omega)} + \|h\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)}\right).$$

Proof. For any $\tilde u\in V_h$, we have

$$\|\!|u_h-u, z_h\|\!| \le \|\!|u_h-\tilde u, z_h\|\!| + \|\!|\tilde u - u, 0\|\!|.$$

Applying Lemma 4.3 and Galerkin orthogonality (4.4), we get

$$\begin{aligned}\|\!|u_h-\tilde u, z_h\|\!| &\lesssim \sup_{(v,w)\in V_h\times V_h} \frac{g(u_h-\tilde u, z_h, v, w)}{\|\!|v,w\|\!|}\\ &= \sup_{(v,w)\in V_h\times V_h}\frac{g(u-\tilde u, 0, v, w)}{\|\!|v,w\|\!|}.\end{aligned}$$

Applying Cauchy–Schwarz inequality to each term in $g(u-\tilde u, 0, v, w)$, we have

$$\begin{aligned}g(u-\tilde u,0,v,w) &= h^2(u-\tilde u, v)_{L^2(w)} + b(u-\tilde u, v) + s(u-\tilde u, v) + a(u-\tilde u, w)\\ &\le \|\!|u-\tilde u, 0\|\!|\cdot\|\!|v,0\|\!| + a(u-\tilde u, w).\end{aligned}$$

Notice that for all $w\in V_h$, there holds

$$\begin{aligned}a(u-\tilde u, w) &= (h\nabla(u-\tilde u), h\nabla w)_{L^2(\Omega)} - h^2(P\partial_\nu(u-\tilde u), w)_{L^2(\partial\Omega)}\\ &\lesssim \|h\nabla(u-\tilde u)\|_{L^2(\Omega)}\cdot h\|w\|_{H^1(\Omega)} + h\|P\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)}\cdot h\|w\|_{H^1(\Omega)}\\ &\lesssim \left(\|h\nabla(u-\tilde u)\|_{L^2(\Omega)} + \|hP\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)}\right)\|\!|0,w\|\!|.\end{aligned}$$

Since P is a projection operator, we have

$$\|hP\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)} \lesssim \|h\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)}.$$

Thus

$$\begin{aligned}g(u-\tilde u,0,v,w) &\lesssim \|\!|u-\tilde u,0\|\!|\cdot\|\!|v,0\|\!|\\ &\quad + \left(\|h\nabla(u-\tilde u)\|_{L^2(\Omega)} + \|h\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)}\right)\|\!|0,w\|\!|\\ &\lesssim \left(\|\!|u-\tilde u,0\|\!| + \|h\nabla(u-\tilde u)\|_{L^2(\Omega)} + \|h\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)}\right)\|\!|v,w\|\!|.\end{aligned}$$

Hence for any $\tilde u\in V_h$,

$$\|\!|u_h-u, z_h\|\!| \lesssim \|\!|u-\tilde u, 0\|\!| + \|h\nabla(u-\tilde u)\|_{L^2(\Omega)} + \|h\partial_\nu(u-\tilde u)\|_{V_h^{-1/2}(\partial\Omega)},$$

which implies (6.4). ◻

Combining Theorem 6.2 and Lemma 6.3, we can obtain the a priori estimate.

THEOREM 6.4. *Suppose that $u \in H^1(\Omega)$ and $(u_h, z_h) \in V_h \times V_h$ solves* (2.1) *and* (4.3)*, respectively. Then there holds*

$$h\|u-u_h\|_{H^1(\Omega)} \lesssim \inf_{\tilde{u}\in V_h} \left(\|u-\tilde{u},0\| + \|h\nabla(u-\tilde{u})\|_{L^2(\Omega)} + \|h\partial_\nu(u-\tilde{u})\|_{V_h^{-1/2}(\partial\Omega)} \right).$$

Next, we will show that Theorem 6.4 gives the optimal convergence rate.

As in [17, Proposition 3.3], we decompose $\partial\Omega$ in disjoint and shape regular patches $\{F\}$ with diameter $\Theta(h)$ thus $|F| = \Theta(h^{d-1})$, and to each of them we associate a bulk patch T that extends $\Theta(h)$ into Ω such that $T\cap\partial\Omega = F$, and $T\cap T' = \emptyset$ if $T \neq T'$. On each patch T we can construct a function $\varphi_F \in H_0^1(T)$ with $\operatorname{supp}\varphi_F \subset T$ satisfying

$$\|\partial_\nu\varphi_F\|_{L^\infty(F)} \le C_\Omega,$$

where C_Ω is a constant that only depends on Ω, and

$$\int_F \partial_\nu\varphi_F \, dx = |F|.$$

When h is small enough, we can take $T\cap\omega = \emptyset$ for all T and $\varphi_F \in V_h$ such that $\varphi_F|_K \in \mathbb{P}_1$ for all $K \in \mathcal{T}_h$. A typical patch F and φ_F constructed on it is illustrated in Figure 2. We refer to [17, Appendix] for the construction in a more general case. Then we define an interpolation $\pi_h\colon\ H^m(\Omega) \to V_h$:

$$\pi_h u = i_h u + \sum_F \alpha_F(u)\varphi_F,$$

where $\alpha_F(u)$ is a functional defined as

$$\alpha_F(u) = \frac{1}{|F|}\int_F \partial_\nu(u - i_h u)\, dx.$$

Then we have the following lemma.

LEMMA 6.5. *Suppose $u \in H^m(\Omega)$, $m \ge 2$, and the polynomial order k of the finite element space V_h satisfies $k \ge m-1$, then there holds*

$$h^{1/2}\|h\partial_\nu(u-\pi_h u)\|_{L^2(\partial\Omega)} \lesssim [u]_{H^m(\Omega)}.$$

Proof. By the definition of π_h, there holds

$$h^{3/2}\|\partial_\nu(u-\pi_h u)\|_{L^2(\partial\Omega)} \le h^{1/2}\|h\partial_\nu(u-i_h u)\|_{L^2(\partial\Omega)} + h^{3/2}\left\|\sum_F \alpha_F(u)\partial_\nu\varphi_F\right\|_{L^2(\partial\Omega)}.$$

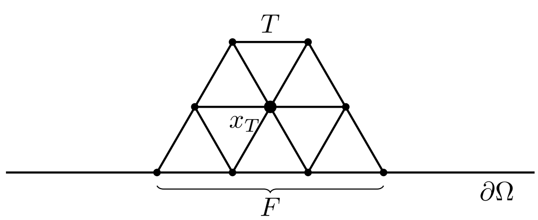


FIG. 2. *Patch F on $\partial\Omega$ together with associated bulk patch T. $\varphi_F|_K \in \mathbb{P}_1$ for all $K \in \mathcal{T}_h$, and $\varphi_F(x_T) = \Theta(h)$ and vanishes on other nodes.*

Notice that

$$\begin{aligned}
&\left\|\sum_F \alpha_F(u)\partial_\nu\varphi_F\right\|^2_{L^2(\partial\Omega)}\\
&= \left(\sum_F \alpha_F(u)\partial_\nu\varphi_F, \sum_{F'} \alpha_{F'}(u)\partial_\nu\varphi_{F'}\right)_{L^2(\partial\Omega)}\\
&= \sum_F\sum_{F'} \alpha_F(u)\alpha_{F'}(u)\left(\partial_\nu\varphi_F, \partial_\nu\varphi_{F'}\right)_{L^2(\partial\Omega)} = \sum_F \alpha_F^2(u)\|\partial_\nu\varphi_F\|^2_{L^2(F)}\\
&\le \sum_F \frac{C_\Omega}{|F|^2}\left(\int_F \partial_\nu(u-i_hu)\ dx\right)^2\cdot|F| \le \sum_F \frac{C_\Omega}{|F|}\int_F \mathbf{1}_F dx\int_F|\partial_\nu(u-i_hu)|^2dx\\
&\lesssim \|\partial_\nu(u-i_hu)\|^2_{L^2(\partial\Omega)}.
\end{aligned}$$

Thus

$$\begin{aligned}
h^{3/2}\|\partial_\nu(u-\pi_hu)\|_{L^2(\partial\Omega)} &\lesssim h^{1/2}\|h\partial_\nu(u-i_hu)\|_{L^2(\partial\Omega)}\\
&\lesssim [u-i_hu]_{H^1(\mathcal{T}_h)} + [u-i_hu]_{H^2(\mathcal{T}_h)}\\
&\lesssim \|u-i_hu\|_{H^m(\mathcal{T}_h)} \lesssim [u]_{H^m(\mathcal{T}_h)}.
\end{aligned}$$

Here the last inequality follows from the trace inequality and (6.2). ∎

PROPOSITION 6.6. *Suppose $u\in H^m(\Omega)$, $m\ge 2$, and the polynomial order k of the finite element space V_h satisfies $k\ge m-1$, then there holds*

$$\|h\partial_\nu(u-\pi_hu)\|_{V_h^{-1/2}(\partial\Omega)} \lesssim [u]_{H^m(\mathcal{T}_h)}. \tag{6.5}$$

Proof. By definition,

$$\|h\partial_\nu(u-\tilde{u})\|_{V_h^{-1/2}(\partial\Omega)} = \sup_{z\in V_h}\frac{(h\partial_\nu(u-u_h), z)_{L^2(\partial\Omega)}}{\|z\|_{H^1(\Omega)}}.$$

By trace inequality, for all $z\in V_h$, there holds $z|_{\partial\Omega}\in L^2(\partial\Omega)$. Then we define an operator $\pi_h^0: L^2(\partial\Omega)\to L^2(\partial\Omega)$:

$$\pi_h^0(z) = \sum_F\left(\frac{1}{|F|}\int_F z\ dx\right)\mathbf{1}_F.$$

Notice

$$\int_{\partial\Omega} z-\pi_h^0(z) = 0.$$

Applying the Poincaré's inequality on each F, we have

$$\begin{aligned}
\|z-\pi_h^0(z)\|^2_{L^2(\partial\Omega)} &\lesssim h^2\|\nabla_\partial z\|^2_{L^2(\partial\Omega)} \lesssim h^2\|\nabla z\|^2_{L^2(\partial\Omega)} \lesssim \sum_{K\in\mathcal{T}_h} h^2\|\nabla z\|^2_{L^2(\partial K)}\\
&\lesssim h\sum_{K\in\mathcal{T}_h}\left(\|\nabla z\|^2_{L^2(K)} + \|hD^2z\|^2_{L^2(K)}\right) \lesssim h\|\nabla z\|^2_{L^2(\Omega)}.
\end{aligned}$$

The last inequality follows from the discrete inverse inequality (4.2). Since

$$\int_F \partial_\nu(u-\pi_hu)dx = 0$$

for all F, there holds

$$
\begin{aligned}
(h\partial_\nu(u-\pi_h u), z)_{L^2(\partial\Omega)} &= (h\partial_\nu(u-\pi_h u), z-\pi_h^0(z))_{L^2(\partial\Omega)} \\
&\lesssim h\|\partial_\nu(u-\pi_h u)\|_{L^2(\partial\Omega)} \cdot \|z-\pi_h^0(z)\|_{L^2(\partial\Omega)} \\
&\lesssim h^{3/2}\|\partial_\nu(u-\pi_h u)\|_{L^2(\partial\Omega)} \cdot \|z\|_{H^1(\Omega)}.
\end{aligned}
$$

Then (6.5) follows immediately from Lemma 6.5. ∎

PROPOSITION 6.7. *Suppose $u \in H^m(\Omega)$, $m \geq 2$, and the polynomial order k of the finite element space V_h satisfies $k \geq m-1$, then there holds*

$$
\|h\nabla(u-\pi_h u)\|_{L^2(\Omega)} \lesssim [u]_{H^m(\mathcal{T}_h)}.
$$

Proof. By definition

$$
\|h\nabla(u-\pi_h u)\|_{L^2(\Omega)} \leq \|h\nabla(u-i_h u)\|_{L^2(\Omega)} + \left\| h\sum_F \alpha_F(u)\nabla\varphi_F \right\|_{L^2(\Omega)}.
$$

Notice

$$
\|h\nabla(u-i_h u)\|_{L^2(\Omega)} \leq \|u-i_h u\|_{H^2(\mathcal{T}_h)} \lesssim [u]_{H^2(\mathcal{T}_h)}.
$$

According to [17, eq. (3.2)], there holds

$$
\|\nabla\varphi_F\|_{L^2(\Omega)} \lesssim h^{\frac{d}{2}}.
$$

Thus

$$
\begin{aligned}
\left\|\sum_F \alpha_F(u)\nabla\varphi_F\right\|_{L^2(\Omega)}^2 &\lesssim \sum_F \alpha_F^2(u)\|\nabla\varphi_F\|_{L^2(\Omega)}^2 \lesssim \sum_F h^d\alpha_F^2(u) \\
&\lesssim h^d \sum_F \frac{1}{|F|^2}\left(\int_F \partial_\nu(u-i_h u)\, dx\right)^2 \\
&\lesssim h^d \sum_F \frac{1}{|F|}\int_F |\partial_\nu(u-i_h u)|^2\, dx \\
&\lesssim h^d\cdot h^{1-d}\sum_F \int_F |\partial_\nu(u-i_h u)|^2\, dx \lesssim h\|\partial_\nu(u-i_h u)\|_{L^2(\partial\Omega)}^2.
\end{aligned}
$$

Applying Lemma 6.5, we have

$$
h\left\|\sum_F \alpha_F(u)\nabla\varphi_F\right\|_{L^2(\Omega)} \lesssim h^{\frac{1}{2}}\|h\partial_\nu(u-i_h u)\|_{L^2(\partial\Omega)} \lesssim [u]_{H^m(\mathcal{T}_h)}.
$$

∎

PROPOSITION 6.8. *Suppose $u \in H^m(\Omega)$, $m \geq 2$, and the polynomial order k of the finite element space V_h satisfies $k \geq m-1$, then there holds*

$$
|\!|\!| u-\pi_h u, 0 |\!|\!| \lesssim [u]_{H^m(\mathcal{T}_h)}.
$$

Proof. By definition,

$$|||u-\pi_h u, 0||| \lesssim B(u-\pi_h u)^{1/2} + h\|u-\pi_h u\|_{L^2(\omega)} + S(u-\pi_h u)^{1/2}.$$

Since for the patch T introduced before Lemma 6.5 there holds $T \cap \omega = \emptyset$,

$$\|u-\pi_h u\|_{L^2(\omega)} = \|u - i_h u\|_{L^2(\omega)} \lesssim \|u - i_h u\|_{H^m(\mathcal{T}_h)} \lesssim [u]_{H^m(\mathcal{T}_h)}.$$

And there holds

$$\begin{aligned} B(u-\pi_h u)^{1/2} &= h^{\frac{1}{2}} \|hQ\partial_\nu(u-\pi_h u)\|_{L^2(\partial\Omega)} \lesssim h^{\frac{1}{2}} \|h\partial_\nu(u-\pi_h u)\|_{L^2(\partial\Omega)} \\ &\leq [u-\pi_h u]_{H^1(\mathcal{T}_h)} + [u-\pi_h u]_{H^2(\mathcal{T}_h)}. \end{aligned}$$

Notice $D^2\varphi_F = 0$, then there holds

$$[u-\pi_h u]_{H^2(\mathcal{T}_h)} = [u-i_h u]_{H^2(\mathcal{T}_h)}. \tag{6.6}$$

Applying Proposition 6.7, we have

$$B(u-\pi_h u)^{1/2} \lesssim [u]_{H^m(\mathcal{T}_h)} + \|u-i_h u\|_{H^m(\mathcal{T}_h)} \lesssim [u]_{H^m(\mathcal{T}_h)}.$$

Finally, according to (4.1), (6.6), and Proposition 6.7,

$$\begin{aligned} S(u-\pi_h u)^{1/2} &\lesssim [u-\pi_h u]_{H^1(\mathcal{T}_h)} + [u-\pi_h u]_{H^2(\mathcal{T}_h)} \\ &\lesssim \|u-i_h u\|_{H^m(\mathcal{T}_h)} \lesssim [u]_{H^m(\mathcal{T}_h)}. \end{aligned} \qquad \square$$

Combining Theorem 6.4 and the above three propositions, we can conclude the following corollary.

COROLLARY 6.9. *Suppose $u \in H^m(\Omega)$ and $(u_h, z_h) \in V_h \times V_h$ solves* (2.1) *and* (4.3)*, respectively. Here $m \geq 2$ and the polynomial order k of the finite element space V_h satisfies $k \geq m-1$. Then there holds*

$$\|u-u_h\|_{H^1(\Omega)} \lesssim h^{m-1}\|D^m u\|_{L^2(\Omega)}.$$

7. Perturbation analysis.

Consider the finite element method with perturbation $q_\delta \in L^2(\omega)$,

$$\begin{aligned} g(u,z,v,w) &= h^2(f,w)_{L^2(\Omega)} + h^2(\beta,w)_{L^2(\partial\Omega)} + h^2(q+q_\delta, v)_{L^2(\omega)} \\ &\quad + b(\beta, v) + (h^2 f, h^2\Delta v)_{H^0(\mathcal{T}_h)}, \end{aligned} \tag{7.1}$$

where $\|q_\delta\|_{L^2(\omega)} \leq \delta$. We have the following theorem.

THEOREM 7.1. *Suppose Ω is convex, $u \in H^2(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u = q \text{ in } \omega, \end{cases}$$

and there exists $p \in \mathcal{V}_N + \beta$ such that $\|\partial_\nu u - p\|_{H^{1/2}(\partial\Omega)} \leq \delta$. Let $(u_h, z_h) \in V_h \times V_h$ be the solution of (7.1) *with $k=1$ in V_h. Then there holds*

$$\|u-u_h\|_{H^1(\Omega)} \lesssim h\|D^2 u\|_{L^2(\Omega)} + \delta. \tag{7.2}$$

Proof. Let $u^* \in H^1(\Omega)$ solve

$$\begin{cases} -\Delta u^* = f \text{ in } \Omega, \\ \partial_\nu u^* = p \text{ on } \partial\Omega, \\ \int_\omega u^* \, dx = \int_\omega u \, dx, \end{cases}$$

and $u^\delta = u - u^*$. Notice that $u^\delta \in H^1_\omega(\partial\Omega)$ and satisfies

$$\begin{cases} -\Delta u^\delta = 0 \text{ in } \Omega, \\ \partial_\nu u^\delta = \partial_\nu u - p \text{ on } \partial\Omega. \end{cases}$$

Denote $\partial_\nu u - p$ by p_δ. According to the elliptic regularity [32, Corollary 2.2.2.6] and Lemma 3.1, there holds

$$\|u^\delta\|_{H^2(\Omega)} \lesssim \|u^\delta\|_{L^2(\Omega)} + \|p_\delta\|_{H^{1/2}(\partial\Omega)} \lesssim \|q_\delta\|_{L^2(\omega)} + \|p_\delta\|_{H^{1/2}(\partial\Omega)} \lesssim \delta.$$

Since $\partial_\nu u^* \in \mathcal{V}_N + \beta$, we can obtain the finite element approximation (u_h^*, z_h^*) of u^* by the method (4.3). Then Corollary 6.9 says that

$$\begin{aligned} \|u^* - u_h^*\|_{H^1(\Omega)} &\lesssim h\|D^2 u^*\|_{L^2(\Omega)} \lesssim h\|D^2 u\|_{L^2(\Omega)} + h\|D^2 u^\delta\|_{L^2(\Omega)} \\ &\lesssim h\|D^2 u\|_{L^2(\Omega)} + h\delta \end{aligned}$$

and

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} &\le \|u^\delta\|_{H^1(\Omega)} + \|u^* - u_h^*\|_{H^1(\Omega)} + \|u_h^* - u_h\|_{H^1(\Omega)} \\ &\lesssim \delta + h\|D^2 u\|_{L^2(\Omega)} + \|u_h^* - u_h\|_{H^1(\Omega)}. \end{aligned}$$

Notice that

$$g(u_h - u_h^*, z_h - z_h^*, v, w) = h^2(q_\delta + u^\delta, v)_{L^2(\omega)}.$$

By Lemma 4.3, we have

$$\begin{aligned} \|\!|u_h - u_h^*, z_h - z_h^*|\!\| &\lesssim \sup_{(v,w)\in V_h} \frac{g(u_h - u_h^*, z_h - z_h^*, v, w)}{\|\!|v, w|\!\|} \\ &\lesssim h \sup_{(v,w)\in V_h} \frac{(q_\delta + u^\delta, v)_{L^2(\omega)}}{\|v\|_{L^2(\omega)}} \lesssim h(\|q_\delta\|_{L^2(\omega)} + \|u^\delta\|_{L^2(\omega)}) \lesssim h\delta. \end{aligned}$$

Combining Theorem 3.3 and Lemma 4.1, there holds

$$\|u_h - u_h^*\|_{H^1(\Omega)} \lesssim h^{-1}\|\!|u_h - u_h^*, 0|\!\| \lesssim \delta,$$

which concludes (7.2) ∎

8. Numerical results. In order to simplify the implementation of the finite element method, in the numerical examples below, we take $\Omega = [0,1]^2$ and $\omega = [0.1, 0.9] \times [0.25, 0.75]$. We denote $\Gamma \in \partial\Omega$ to be the top edge of the unit square, that is

$$\Gamma = \{(x,y) \mid 0 < x < 1 \text{ and } y = 1\},$$

and let $\mathcal{V}_N$ be the subspace of $L^2(\partial\Omega)$ which is spanned by the basis $\{\phi_n\}$, $1 \le n \le N$, where ϕ_n is $\sqrt{2}\cos(n\pi x)$ on Γ and vanishes on $\partial\Omega \backslash \Gamma$. Furthermore, the polynomial order of the finite element space V_h is one.

Our first computational example studies the effect of the stabilizing term s in the computations. Without making any theoretical difference, we rescale s by a positive constant $\gamma > 0$. Choosing $N = 8$ for space $\mathcal{V}_N$, and the $H^1(\Omega)$ error for the artificial solution

$$u(x,y) = (e^y - y)\cos(\pi x) \tag{8.1}$$

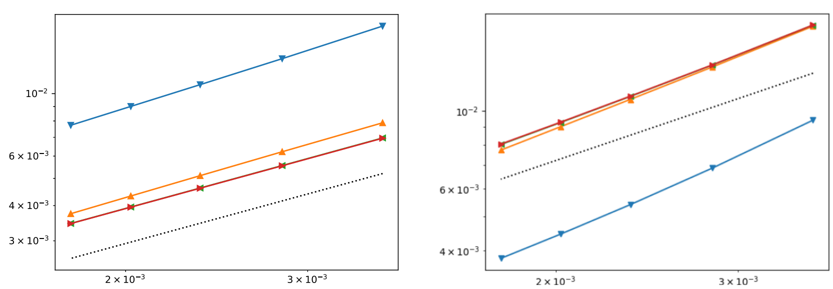
as a function of the mesh parameter h is illustrated for different values of γ in Figure 3a. It shows that the numerical results improve as $\gamma \to 0$, which implies that in this example the stabilizing term s can be omitted as the method still converges with the optimal rate without it.

Next, we study the effect of the dimension N of $\mathcal{V}_N$ in the computations, the $H^1(\Omega)$ for $u(x,y)$ in (8.1) as a function of the mesh size h is shown for different N in Figure 3b. It appears that when N is large enough, merely increasing N cannot affect the results.

Our second example illustrates the error estimate with perturbations. Taking

$$\tilde{u}(x,y) = (e^y - y)\cos(\pi x) + 0.025(e^y - y)\cos(2\pi x).$$

Figure 4 shows the stagnation of the convergence for $N = 1$ but not for $N = 2$ because $\partial_\nu \tilde{u}|_{\partial\Omega} \in \mathcal{V}_2$, which is compatible with our theory.



(a) (b)

Fig. 3. *The error $\|u - u_h\|_{H^1}$ as a function of mesh size h with the reference rate h presented by the dash line.* (a) *u_h is computed with parameter $\gamma = 1,\ 10^{-1},\ 10^{-2},\ 0$ with triangles pointing down, up, left, and right, respectively.* (b) *u_h is obtained with different $\mathcal{V}_N$ where $N = 1,\ 8,\ 16,\ 64$ with triangles pointing down, up, left, and square, respectively.*

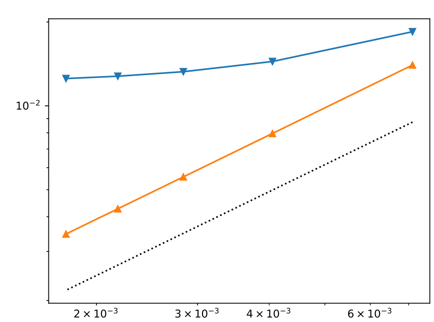


Fig. 4. *The error $\|u - u_h\|_{H^1}$ as a function of mesh size h. Here $N = 1, 2$ with triangles pointing down and up. Reference rate h is presented by the dash line.*

Then we study the effect of random perturbations of data. To this end, we denote the right-hand side of (4.3) by $F(v,w)$. Instead of solving (4.3), we numerically solve

$$g(u,z,v,w) = F(v,w) + \varepsilon\delta\frac{|F(v,w)|}{|\delta|},$$

where the random variables $\delta \sim \mathcal{N}(0,1)$ and $\mathcal{N}(0,1)$ represent the standard normal distribution. Figure 5 shows the stagnation of the convergence for $\varepsilon \neq 0$ compared with the consistent convergence without white noise.

To close this section, we define the ratio

$$C(u) = \frac{\|u-u_h\|_{H^1(\Omega)}}{h\|u\|_{H^2(\Omega)}},$$

and consider the functions

$$u_N(x,y) = (e^y - y)\cos(N\pi x), \quad N = 1,\ 2,\ 3,\ 4. \tag{8.2}$$

We use both the first-order and second-order finite element methods to compute u_{Nh}. The $H^1(\Omega)$ errors between u_N and u_{Nh} with first-order and second-order methods are plotted in Figures 6a and 6b, respectively.

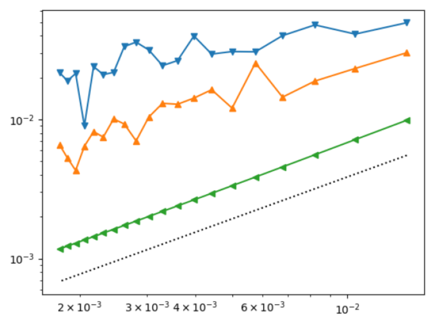


FIG. 5. *The error $\|u-u_h\|_{H^1}$ as a function of mesh size h. Here $\varepsilon = 0.12, 0.06, 0$ with triangles pointing down, up, and left. Reference rate h is presented by the dash line.*

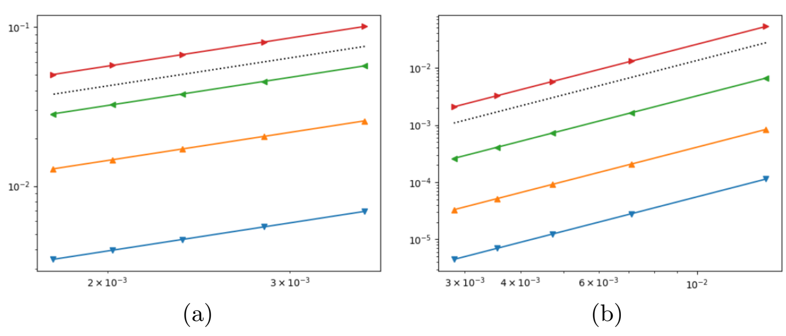


FIG. 6. (a) *The error $\|u_N - u_{Nh}\|_{H^1(\Omega)}$ as a function of mesh size h where u_{Nh} is computed by first-order method. Here $N = 1,\ 2,\ 3,\ 4$ with the triangles pointing down, up, left, and right, respectively. Reference rate h is dashed.* (b) *The error $\|u_N - u_{Nh}\|_{H^1(\Omega)}$ as a function of mesh size h^2 where u_{Nh} is computed by second-order method. Here $N = 1,\ 2,\ 3,\ 4$ with the triangles pointing down, up, left, and right, respectively. Reference rate h^2 is dashed.*

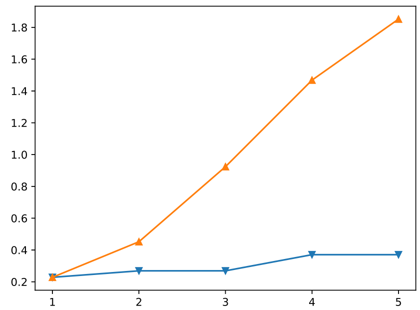


FIG. 7. *The ratio* $C(u)$ *as a function of* N*. Here* u *is as in* (8.1) *and* (8.2) *with triangles pointing down and up, respectively.*

Besides, the ratio $C(u)$ for u as in (8.1), and for u_N is shown in Figure 7 as a function of N. For any u, the ratio $C(u)$ gives a lower bound for the constant $C > 0$ in our a priori estimate. It appears that the constant grows as a function of N, as expected.

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