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Rethinking Affordances and Feedback in AR Environments to Foster Richer Mathematical Inquiry: Lessons from Touch the Derivative

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Abstract

Augmented reality (AR) technologies offer unique opportunities to explore fundamental ideas in calculus by blending digital and physical worlds, yet realising the full potential of this hybrid reality requires a degree of creativity as we grapple with existing theoretical constructs and seek new ones. This study centres on a single-participant case: Karim, a 15-year-old secondary-school student who used an AR prototype "Touch the Derivative" to investigate relationships between functions and their derivatives. In this paper, we examine how AR technologies can support rich mathematical inquiry by rethinking two interconnected elements: affordances and feedback. We analyse the crucial role of the physical world within AR environments through two intersecting perspectives: the spatial affordances enabled by six degrees of freedom (6DoF), and the physical, cognitive, and contextual dimensions of AR. We then examine how AR facilitates various forms of feedback—through Karim's interactions with both the researcher and the AR environment itself—highlighting the role of feedback to support understanding and engagement. We conclude by exploring how intentionally designed feedback mechanisms—enabled by analytics and automation—can amplify the affordances of AR and provide more impactful, inquiry-based learning experiences.

Keywords Augmented reality · Inquiry-based learning · Design affordances · Feedback · Calculus education

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Introduction

Augmented reality (AR) offers a new environment for the exploration of, and engagement with calculus, but designing AR tasks for students to explore its foundations is a highly complex and multifaceted endeavour (Akçayır & Akçayır, 2017). What insights can be gained from existing research on AR in calculus education, particularly when it comes to introducing students to differential calculus? The first thing to address is the technology itself; what are the affordances unique to AR, and in which contexts could AR be more effective than other digital technologies (Radu et al., 2023)? To help unpick this, it may help to understand the problem AR was trying to solve. AR was initially conceptualised as a "see-through" form of virtual reality (VR) (Caudell & Mizell, 1992), resulting in the creation of a unique environment which supplements the physical world, rather than completely replacing it (Azuma, 1997). This hybrid nature is precisely what sets AR apart from VR and other digital tools.

If AR environments are to be introduced into educational settings, do pedagogical approaches exist that are more conducive to AR, or is it time to develop new theories? Feedback is a vital component of any learning environment; physical, digital, or hybrid. Educators must design feedback that is timely, specific, and supportive, guiding students through their explorations without stifling their autonomy (Laurillard, 2016). This demands not only a deep understanding of direct feedback embedded in AR systems but also the way in which learners can receive feedback from other subjects within the shared AR learning environment (such as teachers and peers). How can we strike the perfect balance of knowing when to intervene to correct misconceptions and when to step back, allowing students to wrestle with the concepts and make their own "discoveries" (Gravemeijer & Doorman, 1999)? Indeed, the metaphorical point of "perfect balance" will differ for each student. By exploring these questions, we can better understand the future of calculus education and the role of AR in shaping it.

In this paper, we examine how AR technologies can support rich mathematical inquiry by rethinking two interconnected themes: affordances and feedback. While these can often be treated separately in educational design, we argue that their connection is vital—affordances shape what is possible within an AR environment, while feedback influences how learners interact with those possibilities. In this sense, inquiry acts as a natural bridge: for inquiry to flourish, learners must both perceive what actions are possible (affordances) and receive meaningful guidance on those actions (feedback). Guided by this view, we ask how feedback mechanisms can be designed to meaningfully exploit the unique affordances of AR and thus support deeper mathematical inquiry. We address the question through a case study of "Touch the Derivative", an AR prototype that enables a student, "Karim", to explore relationships between functions and their derivatives. Hence our reference to "the Karim case". After a brief literature review, we analyse the case from two complementary angles: (1) the spatial affordances of six degrees of freedom (6DoF) and the impact of situating artefacts in the "real" world and (2) the multi-level feedback that emerges from the AR system and interactions with the facilitator. We conclude by synthesising these analyses to offer design directions for future AR environments that seek to cultivate more impactful, inquiry-based learning experiences.

Literature Review

We begin by analysing how early calculus concepts are introduced in the English curriculum, then examine the use of digital tools (pre-AR) in teaching introductory calculus. The review transitions to recent AR studies, focusing on how this technology has been applied in calculus education, with particular attention to the spectrum of feedback mechanisms afforded by AR environments, such as physical artefacts, task-level, and process-level feedback.

Pedagogical Approaches Within the Teaching and Learning of Differential Calculus

The birth of calculus marked a pivotal moment in mathematical history. Since the early twentieth century, calculus has become a standard component of mathematics education for most 16-year-olds in European schools (Zuccheri & Zudini, 2014). The derivative, a central concept in calculus, is often defined as the instantaneous rate of change—the ratio of the instantaneous change in the dependent variable to that of the independent variable (Stewart, 2020). In mathematics classrooms, the independent and dependent variables are usually introduced in abstract form: x and y, respectively. Later, this concept can be extended to include velocities and the gradients of tangents (Strang & Herman, 2016)—typical contexts for applied mathematics problems, which often intersect with physics. As such, derivatives are critical in the interpretation of rates of change across natural sciences, engineering, and social sciences (Stewart, 2020; Strang & Herman, 2016).

In England, if teachers adhere strictly to the sequence outlined in the National Curriculum, students will start with a formal definition of the gradient, then move to the "First Principles Differentiation" (Department for Education, 2016). Although the new notation may be met with initial bewilderment, it seems procedural fluency is much easier to achieve than an understanding of the underlying concepts (Kaput, 1994). However, the mechanical "tricks" of calculus may be limited, and conceptual shortcomings can be exposed when students encounter unfamiliar contexts (Doorman & van Maanen, 2008). Furthermore, studies have shown that students often struggle to understand the relationship between a function and its derivative, particularly when interpreting their graphs or applying them to authentic real-life problems (Hilmi et al., 2021).

The "calculus reform movement" of the 1990s, which originated in the USA, initiated a significant shift in pedagogical approaches to calculus instruction. This movement advocated for a more conceptual approach, utilising multiple representations—graphical, numerical, and symbolic—to deepen student understanding (Doorman & van Maanen, 2008). This aligned with the arrival of graphic calculators as new mathematical tools, and the emphasis shifted towards fostering an intuitive grasp of fundamental concepts like derivatives, moving beyond traditional procedural instruction. The reform instigated the rollout of graphing calculators across leading curricula in the West: the Advanced Placement (AP), International Baccalaureate (IB), A-Levels (Hallett, 2006). In fact, some calculus questions across these curricula are so intricate that students would be at a disadvantage without the use of a calculator (Torres-Skoumal et al., 2014).

Pedagogically, the tension between direct instruction and inquiry-based learning has been a persistent theme in debates surrounding calculus education. Should educators lead students through a carefully structured progression of concepts, or should students be guided to "rediscover" the principles of calculus for themselves? Freudenthal (1991) advocated for guided reinvention, arguing that true understanding emerges not from passively receiving knowledge but from actively constructing it. In this case, Gravemeijer (2004) encourages instructional designers to shift their focus from decomposing expert knowledge into digestible pieces to imagining students elaborating, refining, and adjusting their current ways of knowing.

The debate around the use of context in calculus education is equally nuanced. Using everyday, real-world scenarios as contextual anchors may enable students to make sense of calculus, especially those who struggle to "see" mathematics through symbolic notation (Doorman, 2005). On the other hand, real-world examples may leave others overwhelmed by the extraneous details, leading to confusion rather than clarity (Akçayır & Akçayır, 2017). To understand what role context may play when first introducing students to calculus, Bisson et al. (2020) used comparative analysis to examine the learning outcomes of two teaching conditions: one group of students was introduced to differential calculus using decontextualised representations, whereas the other group engaged with contextualised problems such as projectile motion and optimisation puzzles. The researchers found that both groups achieved comparable learning outcomes, challenging the notion that one method holds a clear advantage over the other. Given the highly subjective nature of context, the interesting "contextual" problems designed by teachers may appear completely abstract to students (Gravemeijer & Doorman, 1999). If context-based tasks are intended to truly motivate students, then these contexts must serve as meaningful metaphors for each individual student (Papert, 1980), be it from real life or fantasy (Van den Heuvel-Panhuizen & Drijvers, 2014).

Exploring Differential Calculus with Digital Technologies

Historically, the teaching and learning of calculus has been bound to static methods—pen and paper, chalk and blackboard—leading many calculus classrooms to favour by-hand fluency over the use of digital technology (Clark-Wilson et al., 2020). As technologies are adopted by the wider education sector—from coloured ink to dynamic digital software—some mathematics educators have harnessed these tools to design tasks and activities to support teaching and learning. Design being the operative word: the efficacy of such technologies lies not in their intrinsic properties but in the manner in which they are deployed. In the case of differential calculus, coloured ink can help students distinguish a function (in one colour) from its derivative (in another colour), while dynamic tools such as GeoGebra allow for the design of tasks that enable students to trace a function's slope in real time (Hohenwarter et al., 2008).

Building on Bruner's representational system, Tall (2019) advocates for an embodied approach to calculus, using tools such as Mathematica, Maple, and GeoGebra to illustrate how functions look "locally straight" under magnification. These embodied approaches make advanced concepts more accessible and support the shift from physical to symbolic reasoning (Abrahamson et al., 2020; Verhoef et al., 2015). Using a computer-based activity to explore the unit circle and corresponding trigonometric graphs, Shvarts et al. (2021) argued that blending digital artefacts with bodily action can enhance students' conceptual understanding by fostering sensory-motor coordination and creating "body–artifact functional systems." Building on this, researchers are examining implicit feedback—such as gesture and gaze sensors—to better understand how embodied experiences may foster deeper conceptual understanding (Abrahamson et al., 2020; Shvarts et al., 2021). Wei et al. (2022) sought to categorise embodiment in technology-enhanced environments as follows: (1) embodiment not central (embodiment is implicit), (2) pseudo embodiment (designs which include "sort of" embodied elements such as dragging), and (3) embodiment (technology-enhanced tasks which are purposely designed for embodied experiences and provide feedback on such actions). It is worth noting that the "embodied approaches" in 2D screen-based environments differ significantly from the embodied experiences within 3D, AR environments. Finally, there is a concern that some digital tools will not only "take care" of the calculations but overshadow the conceptual underpinnings of calculus, particularly as technology becomes more sophisticated (Ferrara et al., 2006). Given most modern-day mathematics classrooms focus almost exclusively on historic symbolic manipulation, at best supported by static visuals, it seems we are some way off technological take-overs.

Exploring Differential Calculus with Augmented Reality Technologies

One of the most powerful features of AR is its ability to superimpose digital mathematical information (Tomaschko & Hohenwarter, 2019) onto real-world artefacts. Whether analysing the slope of a hill (Schacht & Swidan, 2019) or the curvature of a roller coaster (Bokhove et al., 2018), AR is able to make mathematics relevant and accessible, grounding abstract concepts in everyday experiences. By providing a more tangible and interactive learning experience, AR can help students grasp abstract concepts more effectively (Bos et al., 2022). Like Tall (2019) and Bos et al. (2022) created a learning environment for students to build their understanding of gradients through perception-based, action-based, and incorporation tasks; however, their environment was a 3D, AR environment—the AR sandbox—as opposed to a 2D computerised environment. In the AR sandbox, students can feel the sand (physical) and examine mathematical relationships (digital) unfold in real time. This sensorimotor engagement anchors abstract ideas in physical experience, creating a robust, intuitive understanding of concepts like the gradient. When exploring a classic calculus problem with 18- to 19-year-old Serbian students—maximising the volume of a box—Budinski and Lavicza (2019) designed a task which connected a hands-on approach (building a box using origami instructions) with a computer-based approach (constructing a box using GeoGebra 3D/AR).

They argued that adopting two approaches concurrently was better than separate, isolated approaches—as in a "dual approach" is greater than the sum of its parts. For example, the origami approach exploited the epistemic value of hands-on activities, whereas the AR task offered a "unique experience" (p. 389) for students since they were able to travel inside their virtual boxes and take screenshots from multiple points of views. If students can manipulate mathematical objects in an AR environment, moving around or even move inside the objects, the orientation perspectives are essentially limitless.

Not everyone can form mental images of functions from abstract algebraic notation, and AR can certainly "bridge" that process (Swidan et al., 2023). As such, many AR studies have been designed for students to visualise and interact with mathematical concepts, without the need for real-world counterparts. Construct 3D was an early AR system (headset and software) built for students to explore mathematical surfaces (Kaufmann, 2004). Virtual mathematical objects, planes, and points were constructed in "open space", not superimposed onto or alongside any concrete objects. Participants reported better understanding of 3D constructions when viewing these stereoscopically through the head-mounted device (as opposed to a 2D computer monitor), adding that many of the tasks would have been impossible to draw with pen and paper or existing CAD programs. Li et al. (2022) built an AR multi-representational learning environment to support students' understanding of linear functions. Their AR^+ Beijing travel plan consisted of a 3D animation of a vehicle travelling at constant speed (concrete), alongside the corresponding graph (semi-concrete) and algebraic expressions (abstract)—all realised in AR. Students with strong prior representational knowledge moved sequentially from concrete to abstract representations, while those with weaker backgrounds showed no clear learning path. Participants in Orozco et al.'s (2006) study described positive feelings when "holding" AR paraboloids in their hands. Similar "a-ha" moments were reported by Monteiro Paulo et al. (2021) as participants used GeoGebra AR to intersect planes and hyperbolic paraboloids. How might physical artefacts enhance these AR explorations, if at all? In the hyperbolic paraboloid exploration, could a physical artefact—say, a Pringles crisp (a physical artefact in the shape of a hyperbolic paraboloid)—enhance the AR task by merging digital and tactile representations of the surface? Of course, as with any educational innovation, integrating AR into the curriculum presents challenges. The necessity for adequate resources, teacher training, and student adaptation is a real concern, echoing the early days of computer-based learning. In line with Drijvers et al. (2014), Schutera et al. (2021) remind us that while AR holds tremendous potential, it requires careful planning and support to be truly effective.

The Role of Feedback Within AR Tasks

Finally, the role of instructional feedback is generally regarded as crucial for enhancing knowledge and skill acquisition and motivating learning (Shute, 2008). Instructional feedback is any information regarding aspects of one's performance or understanding communicated to the learner, intended to modify their thinking or behaviour to improve learning (Hattie & Timperley, 2007; Lipnevich & Smith, 2022; Narciss, 2008; Shute, 2008). Feedback might come from various sources, including teachers, peers, or the task itself (Lipnevich & Panadero, 2021). It may include information on the learner's current status, goals, and the steps and strategies required for improvement (Hattie & Timperley, 2007; Lipnevich & Panadero, 2021). AR technology can enhance the feedback process by overlaying simulated cues onto the physical world, creating immersive and interactive learning environments (Wu et al., 2013). The real-time interaction and 3D presentation features of AR provide visual and interactive feedback to foster students' sense of immediacy (Kotranza et al., 2009). This immediacy can help reinforce learning, as students can quickly see the results of their actions and make necessary adjustments (Bokhove et al., 2018). Visual feedback presents visual representations such as texts, images, models, and animations, making abstract concepts more concrete, highlighting errors or demonstrating correct procedures (Akçayir & Akçayir, (2017). Interactive feedback, delivered through simulations and hands-on activities, allows learners to receive feedback based on their actions and decisions within the AR environment (Yoon et al., 2018). This type of feedback helps learners understand the consequences of their actions in a simulated yet realistic context.

An influential typology categorises feedback into four levels—task, process, self-regulation, and self (Hattie & Timperley, 2007; Lipnevich & Panadero, 2021). Examples of embedding different types of feedback into AR systems can be found in previous research. Task-level feedback addresses how well tasks are understood or performed, including feedback on knowledge of results and correct responses, and typically takes the form of activating digital information, such as text, video, or models, based on the learner's actions to facilitate further exploration. For instance, in an AR application on probability, if a learner correctly identifies the coin pattern, a puppet appears on the coin, indicating the correct operation (Cai et al., 2020). In the research of Chen (2019) and Gecu-Parmaksiz and Delialioglu (2019), scanning a card with a specific pattern causes the corresponding geometric shape to appear, supporting further exploration. Additionally, virtual feedback can be based on student actions. Martin-Gonzalez et al. (2016) used the programme to generate vectors based on the positions of the learner's body and hands, allowing learners to explore vectors by altering their movements. Process-level feedback involves the key processes required to understand or complete tasks, guiding learners' problem-solving in a specific sequence of behaviours. Elaborated feedback, such as strategic hints, explanations, or worked examples, falls into this category. For example, AR feedback can include a video demonstrating the problem-solving steps for students (Kazanidis & Pellas, 2019). Self-regulation feedback encompasses higher-order comments related to self-monitoring and the regulation of actions and emotions. AR's visual and interactive feedback has been examined to be beneficial in helping students develop self-regulated learning skills (Cetintav & Yilmaz, 2023; Muali et al., 2020). Self-level feedback fosters positive attitudes by providing positive incentives or praise, such as rewarding users after they select the correct answer (Demitriadou et al., 2020). In this paper, we use these feedback levels to observe how they were manifested in the interaction of the student and researcher in the AR learning environment.

Two Complementary Lenses: Design Affordances and Feedback Analysis

In this section, we first introduce the basic details of Karim's AR-based activities on learning derivatives to provide the research context. We then analyse the case from the perspective of AR learning activity design and examine how these designs serve as feedback to support learning. Each analysis part begins with a brief overview of the theoretical frameworks, concepts, and constructs guiding our approach, followed by a detailed discussion. First, we examine the unique technical features of AR and assess how these features are emphasised or limited in the design of Karim's case, proposing alternative task designs where relevant. Finally, we evaluate the role of feedback within the AR environment, identifying potential benefits and challenges in practice, and offer recommendations for future designs.

Introducing the Karim Case

Touch the Derivative is an AR prototype designed for the Magic Leap device. Within the programme, there are four "families" of functions for users to explore: linear functions (A, B, and C), polynomials (parabolic (D) and cubic (E)), trigonometric (F), and exponential (G)—see Fig. 1. In the case study, a 15-year-old student, Karim, selected from seven function images—superimposed in a real room—to investigate the relationship between derivatives and their original functions. Karim could freely switch between these function images to test or verify his conjectures, making a total of 28 selections. In every session, Karim's hand movement controlled the position of an orange dot, which appeared through his headset to be in free space. When the orange dot was "placed" accurately on a blue function, the tangent line appeared and turned green, and a green point marked the corresponding value of the derivative as the dot was moved incrementally along the function (see Fig. 2). The derivative dots emerged in real-time and formed green derivative functions, despite being rather "dot-like" (instead of continuous). If the orange dot strayed from the blue function, the tangent line turned red. Karim was asked to discover the relationship between the blue function and the emerging green (derivative) dots, while a researcher guided his exploration. We adopt a micro-analytic lens on

Fig. 1 Seven 2D functions available in Touch the Derivative prototype

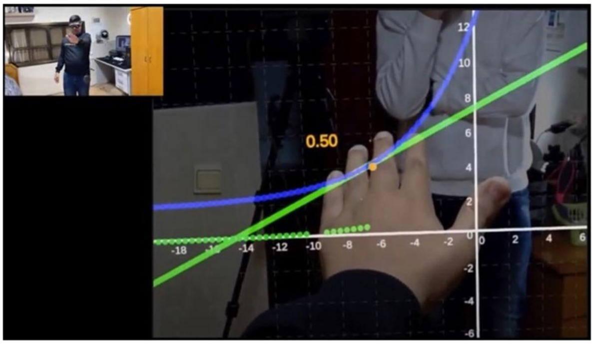


Fig. 2 Karim moves along the function, triggering the tangent line (green) and derivative points (green dots)

Karim's interaction within his AR environment—not to generalise about individual learners, but to trace how Touch the Derivative's design features and feedback mechanisms play out in real time. The Karim case thus serves as a vehicle for surfacing the affordances and constraints of AR in supporting mathematical inquiry.

Design Affordances

What Makes AR a Distinct Reality

Before delving into the framing ideas for this analysis, we first reflect on what makes AR a *distinct* reality. We start by articulating AR's 3D environment through the lens of 6DoF, a construct that describes spatial movement and orientation in physical space. As outlined in the introduction, AR originated from the need to address a real-world problem. The hybrid hallmark of AR—where the physical and digital elements *complement* one another (or at least that was the original design intention)—is precisely where AR offers great potential, straddling two worlds much like the dual origins of calculus.

In line with the early VR pioneers (Sutherland, 1965), we adopt 6DoF (or six degrees of movement) to describe motion in 3D space. 6DoF refer to three translational movements (up/down, right/left, back/forth) and three rotational movements (pitch, roll, and yaw)—essentially the rotations about the three axes (see Fig. 3a). Within 6DoF, there is no fixed x–y-z orientation; these axes are mathematical conventions rather than inherent properties of 3D movement.

The AR prototype, Touch the Derivative, allows for 6DoF; however, all the mathematical functions explored are two-dimensional (2D). In mathematics, it is conventional to represent a three-dimensional Cartesian coordinate system with the *y*-axis pointing towards the observer; however, this is not always the case in computer graphics (Foley & van Dam, 1982). To align with Touch the Derivative orientation, we constructed an example which is consistent with the negative parabola task (see Fig. 3b). This example demonstrates how a 2D parabola only requires two degrees of freedom—translation along the *y*-axis and translation along the *x*-axis. Translation along the *z*-axis and rotations about each axis are irrelevant, and perhaps even confusing.

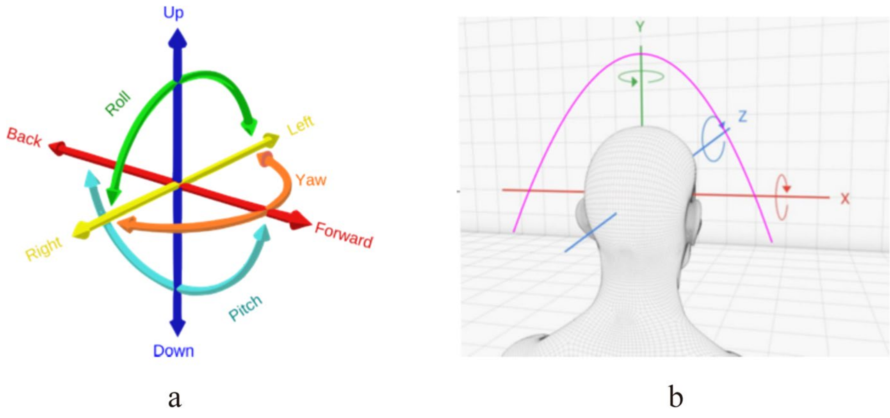


Fig. 3 **a** The six degrees of freedom (GregorDS, CC BY-SA 4.0). **b** 2D parabola in 3D space

AR Affordances in Terms of Physical, Cognitive, and Contextual Dimensions

To explore the affordances and limitations of AR within mathematics classrooms, Bujak et al. (2013) proposed a theoretical construct to guide their analysis: the "AR Manipulative". The researchers examine this hybrid tool through three dimensions: physical, cognitive, and contextual.

> Physical: The interaction with tangible objects supports learning both pragmatically and cognitively. Natural, tactile interactions reduce cognitive load by making abstract concepts easier to grasp (the pragmatic strand). The cognitive strand relates to embodied cognition; physical artefacts encourage epistemic actions and the formation of embodied representations, which can support learning.

> Cognitive: AR environments may help students bridge between symbolic abstraction and physical representation, by superimposing symbolic mathematical notation over concrete representations in a continuous, real-time manner. The tools may enable students to uncover salient mathematical structures.

> Contextual: AR not only situates mathematics in real-world contexts but also facilitates collaboration and personalisation. The digital part of AR is generic, but the physical part is highly personable. Given learners can see one another in a shared physical space, AR environments can afford face-to-face collaboration, whereas VR cannot.

Although this triple-lens framework is highly cited across the AR mathematics education literature (Gecu-Parmaksiz & Delialioglu, 2019; Swidan et al., 2023), its application in experimental studies remains limited. Thus, we mobilise the framework in two ways: first, to unpick the Karim case, tracing how its physical, cognitive, and contextual affordances unfold in situ; and second, to propose alternative AR tasks which consist of digital mathematical artefacts (e.g. a function) and purposeful corresponding physical artefacts, merged together in real time.

Examining Karim's Interactions Through Physical, Cognitive, and Contextual Dimensions

We now examine how Karim engages with these tasks through the physical, cognitive, and contextual strands outlined above.

Physical: Karim, a calculus novice, starts by examining the positive linear equation (task A). The task aims for Karim to "feel" the rate of change by moving his hand from left to right along the x-axis (1DoF) and experiencing corresponding changes in the y-axis (1DoF). Essentially, Karim is required to explore 2D functions within a 6DoF space, raising the question: is it beneficial to explore 2D functions (and their derivatives) in three-dimensional "free space"? In the first 10 min, the researcher repeatedly reminds Karim of the "rules" for navigating the functions:

> We should go along the functions one by one from the left to the right (0:01:44).
> You can only move from left to right, but not the other direction (0:03:34).
> It is forbidden to move from right to left, only from left to right (0:08:25).
> We must move on the graph in this way (0:08:38).

Karim's desire to "collect all the dots" (a pragmatic goal) may conflict with the epistemic goal of the task: to maintain the direction of hand movement as a proxy for experiencing the directional rate of change. Take $y = 2\times$ as an example. When Karim moves his hand from negative x to positive x, the function "feels" consistently uphill, matching the derivative of $+2$. Reversing the direction (from positive x to negative x) feels downhill, as if the slope were -2, even though the function's tangent never changes. This direction-dependent sensation reveals a subtle constraint of mathematical convention: a positive slope *looks* the same whether you trace it left-to-right or right-to-left, yet it only *feels* positive in one direction. Without explicit instructional guidance to only move from left to right, Karim—who has no prior knowledge of derivatives—might fail to distinguish between the invariant analytic slope and the variable embodied experience.

Cognitive: The tasks require Karim to connect the abstract notion of the derivative (rate of change) with the tangible experience of moving along functions from negative to positive x. While individuals may explore curves in both directions, in this learning task, movement from left to right aligns with a common pedagogical convention: presenting rate of change with respect to increasing x. However, in many real-world scenarios, the choice of reference axis is more adaptable. Physics offers a clear example: when exploring electric fields or gravitational fields, students learn how and why positive charges and masses "move downhill" to regions of lower potential. These directions of movement are not necessarily aligned with increasing x. Touch the Derivative may benefit from signposting to help unveil the implicit "move from left to right" rule underpinning derivatives

Fig. 4 **a** Feeling a linear function with finger. **b** Feeling a linear function with palm

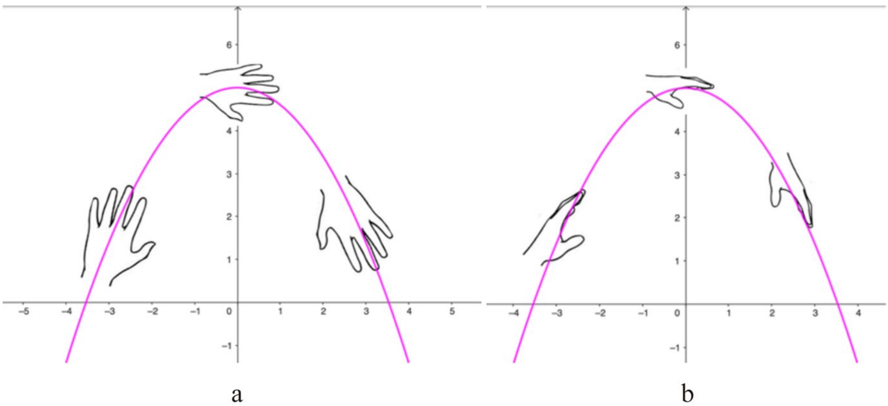


Fig. 5 **a** Feeling a parabola with finger. **b** Feeling a parabola with palm (in 3DoF)

and thus reduce students' cognitive load. When exploring the linear functions, Karim's hand remains in the x–y plane (see Fig. 4a). In contrast, when the derivative function is changing (as is the case for the parabolic, exponential and trigonometric functions), Karim adjusts the orientation of his hand. It seems he finds it easier to "feel" the function beneath the palm of his hand (see Fig. 4b).

As a thought experiment, consider the diagrams in Fig. 5a and b. Take your left hand and try to trace out the parabola as in Fig. 5a, where your hand is restricted to the x–y plane. Imagine your middle finger is the tangent as you move around the parabola. It may help to envisage doing this against a wall, to constrain your hand to 2DoF. Now change the orientation of your hand to mirror Fig. 5b, where you imagine you are "touching" the parabola with the palm of your hand. Does one feel more natural or instinctive?

To unpick this, we need to consider the wrist mobility and what parts of the hand we use to touch. Most wrists have a greater forward-back range (Fig. 5b) than left–right (Fig. 5a), which may make the motion depicted in Fig. 5a feel more awkward. Furthermore, Fig. 5a is using a finger to represent a tangent line, whereas Fig. 5b uses palm which is essentially mimicking a surface (adding an additional degree of freedom), rather than a line. A final point is that the

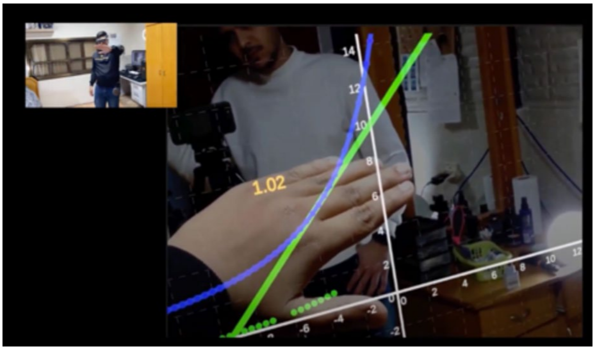


Fig. 6 Inclination of hand not aligned with the tangent

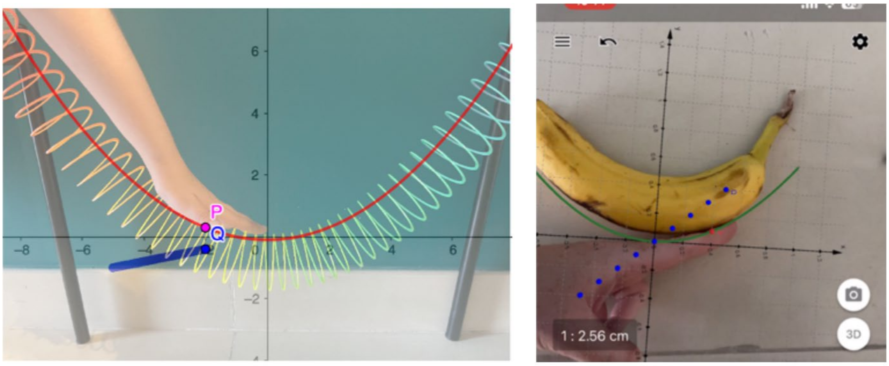
inclination of Karim's hand does not always reflect the green tangent (see Fig. 6). This mismatching of representations between the visual (green tangent) and concrete (inclination of Karim's hand) may lead to cognitive dissonance.

Contextual: The shared physical space in this AR environment allows for collaboration between Karim and the researcher. This shared reality is further supported by the presence of a screen that streams Karim's view enabling the researcher to observe and respond to Karim's interactions, in real-time. Exactly how the researcher responds will be elaborated on in the subsequent feedback analysis section. However, it is not clear as to what role (if at all) the rest of the physical environment plays. Karim can clearly see a wealth of objects in the background (wardrobe, desk, posters), but these could be classified as mathematically redundant information, or background visual "noise". The broader impact of this physical environment on learning remains unclear.

Constructing an AR Task for a Parabolic Function

Our epistemological and ontological approaches resonate with Touch the Derivative intervention: we are viewing mathematical learning as experiential, inquiry based. As such, we have taken the same digital information from task D (the parabola), but in our task, we have included purposeful physical artefacts. We propose that the physical artefacts may foster different cognitive and affective outcomes. Figure 7a and b presents two examples of a digital parabola being superimposed over two different artefacts: a helical spring toy (commonly known as a "slinky") and a banana. While these objects are not perfectly parabolic, they can serve as close approximations under certain conditions—for example, when a slinky exhibits shallow curvature (Cumber, 2024).

We have discussed the role of the physical environment—both implicit and explicit. As such, we start by examining the explicit impact physical artefacts may have, then move on to more implicit roles these physical artefacts may play. When a young child is learning to ride a bike, an adult or older child can reduce the degrees of freedom by adding stabilisers or starting with a balance bike (Shvarts et al., 2021). Physical objects could play a similar role when learning about 2D parabolas—reducing unnecessary degrees of freedom to scaffold understanding. Furthermore,



a b

Fig. 7 **a** AR slinky (AR helical spring toy). **b** AR banana

students can tangibly feel the changing slope of a slinky or banana. Supplementing the visual feedback with tactile reinforcement may encourage learners to form sensory-motor associations (Bos et al., 2022). We now turn to the implicit role tactile objects may play. If designed appropriately, so the digital and physical representations are correctly aligned, these "combined artefacts" could serve as fruitful mediators in the cognitive process. Suspended slinkies or bananas could potentially create points of engagement that ground abstract concepts in the physical world.

From a contextual standpoint, the physical artefacts may invite learners to form affective relationships with the material. We have chosen parabolic representations of natural phenomena: the curvature of the slinky and banana is wrapped up in respective gravitational and biological constraints (Powell, 2009), but man-made phenomena could also be used (Haas, 2021; Paul et al., 2024). Furthermore, our examples in Fig. 7a and b specifically opt for uncluttered physical backgrounds—a design decision that may reduce distraction and support focus. The hybrid nature of AR allows the abstract digital functions to be intertwined with artefacts which may hold personal meaning for each and every learner—perhaps the golden arches of the infamous McDonald's may resonate with some learners. This approach could see learners choosing to model *non-parabolic* objects, which some educators might find challenging. However, these instances can be harnessed as an opportunity to guide students to discover why their chosen object does not fit the parabolic model, thus deepening their understanding of mathematical concepts. The inquiry-based approach is most powerful when it straddles these dual aspects: providing both tactile anchors to simplify and scaffold cognitive load, while simultaneously allowing learners the freedom to imbue the learning context with personal meaning.

Critique Karim's Learning Through the Lens of Feedback

Focusing on Karim's learning process through the lens of feedback, we analyse the feedback he receives from the AR system and his conversations with the researcher separately and critically assess the role of this feedback to draw implications. As mentioned in Sect. 2.5 of the literature review, an influential typology categorises feedback into four levels—task, process, self-regulation, and self (Hattie & Timperley, 2007; Lipnevich & Panadero, 2021)—which will guide our analysis.

Feedback in the AR System

According to the design of the AR software, the programme consists of two scenes: graph selection and graph exploration. In each scene, visual interactive feedback is provided in response to the user's actions, functioning as task-level feedback when correct interactions occur. The feedback provided in each scene is analysed below.

In the graph selection scene, an orange dot is used to indicate the position of the user's hand. This dot accurately reflects which function image (A-G) corresponds to the user's hand position on the plane, serving as process-level feedback that guides the user in making their next step for the selection. As illustrated in Fig. 1, the orange dot appears on the A image, showing Karim's hand position in space. However, Karim could not enter the exploration by just putting the orange dot on the graph, as the dot did not reach the depth to trigger the change. As the dot did not indicate the distance between the user's hand and the trigger point for image selection, he continued to adjust his gestures, making small movements and reaching forward, until the scene finally changes.

After entering the graph exploration scene, the orange dot continues to appear when the user's hand is detected, adjusting its position in real time according to hand movements. However, it still does not indicate the distance between Karim's hand and the function graph. As shown in Fig. 8, Karim moved and adjusted his gestures but did not reach the blue line.

When Karim's hand touches the blue function graph, three task-level feedback elements are triggered: (a) a green line representing the tangent, (b) an orange number showing the derivative value, and (c) a green dot marking the derivative point. As Karim moves his hand along the function graph, the tangent line and derivative value continuously update, as new green derivative points are also generated. These green points do not disappear after triggering, allowing the user to observe the trend of the derivative graph, as depicted in Fig. 9. When the user's hand leaves the original function graph, the last green line will turn red, while the orange number and green points keep them the same.

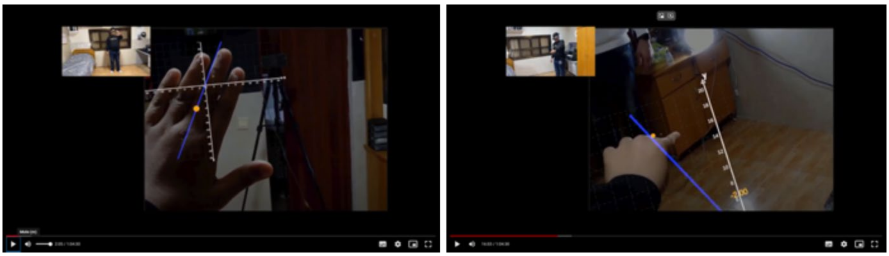


Fig. 8 Karim did not reach the blue line even though the orange dot is on the graph

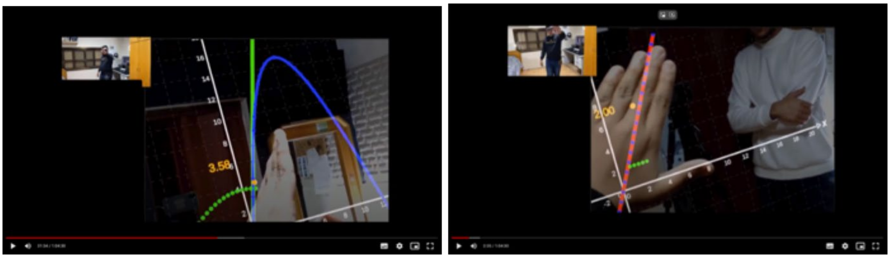


Fig. 9 The blue original function graph and three task-level feedback in the graph exploration scene

Despite the immediate responsiveness of these three feedback elements, there are some design limitations. First, none of the feedback elements is labelled, which could lead to user confusion about their meaning. For example, when Karim initially chose the image of a linear function, the tangent overlapped with the function itself, leading him to mistakenly believe that touching the graph would change the colour of the original function. He only realised that the green line represented the tangent after exploring other function graphs and then returned to re-examine the first one. Another issue affecting the exploration process was the limited field of view. At times, Karim was unable to see all the feedback elements as he moved his hand. As shown in Fig. 10, the derivative point and even the derivative value at the touch point sometimes fell outside his field of view. This made it difficult for Karim to verify the graph and value of the derivative simultaneously, limiting his ability to analyse the relationship between the derivative and the original function. He needed to step backward or turn his head to have an overview of the graphs.

Several additional interface details also deserve attention. First, the tangent is displayed as a finite-length line segment rather than an infinite line. In Fig. 11a, one endpoint of the segment is visible on the green tangent line, although mathematically, a tangent extends infinitely in both directions. Another issue concerns the incomplete display of numerical values. For example, in Fig. 11b, the derivative value is shown as ".00" instead of the full "0.00". Finally, Fig. 11c and d illustrates an unexpected disappearance of green derivative points: at 11:44 (Fig. 11c), multiple green points are visible, but at 11:45 (Fig. 11d), after Karim touches the function again and a new green tangent and point appear, the previously visible points

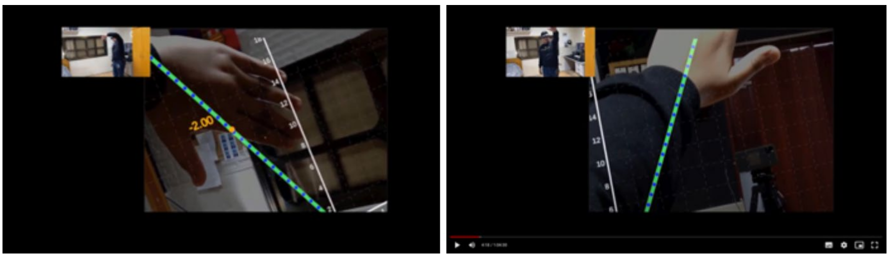


Fig. 10 Limited view of the graph exploration scene

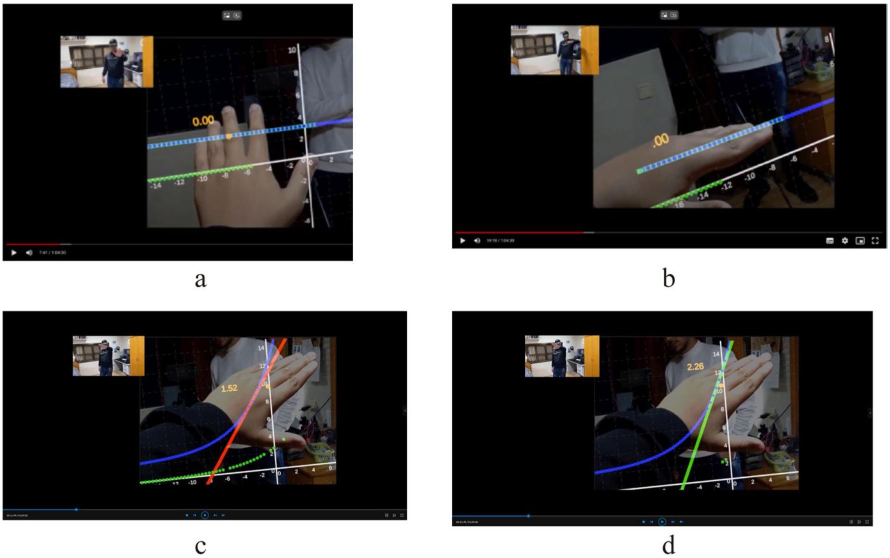


Fig. 11 **a** A finite-length tangent segment is displayed at the point of contact. **b** An example of an incomplete display of the derivative value. **c** At 11:44, multiple green derivative points are visible on the graph. **d** At 11:45, after a new touch, the previously visible green points disappeared

have disappeared. While these issues did not hinder Karim's exploration of the relationship between the derivative and the original function, they prompted questions during the process, which required clarification from the researcher.

Another instance where the researcher had to provide feedback as the supplement of the system occurred when entering the graph exploration scene. When Karim first selected a function graph to explore, no information was displayed, as shown in Fig. 12. Each time he entered the scene, only the function graph appeared, with no prompts guiding the user on possible actions. He stood still until the researcher explained how to position the function graph and interact with it by moving his hand from left to right. This absence of task-level and process-level feedback highlights the system's reliance on external feedback to support user interaction.

Fig. 12 The vacant scene after the first selection of the graph

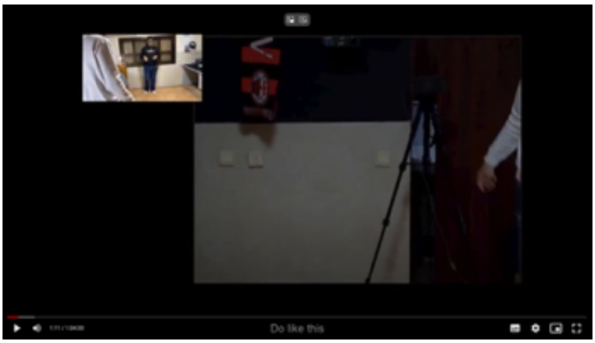


Table 1 Examples of the three types of feedback from the researcher

Self-level	"I saw you doing a nice movement." "Ah ha." "Okay."
Task-level	"What do you see?" "What did you explore?" "Did you find the relationship between the blue and green lines?"
Process-level	"Try to do, bring it in front of you." "With your finger, do like this." "You can examine another function if you want."

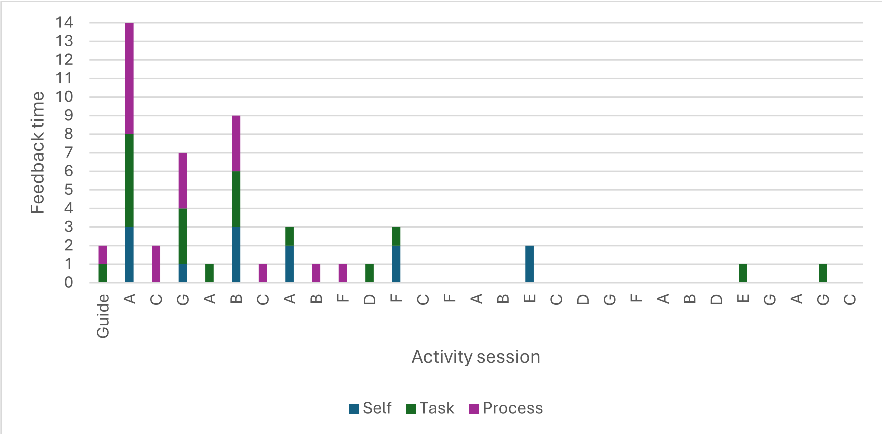


Fig. 13 The statistics on the feedback provided by the researcher

Feedback from the Researcher

According to the statistics on the feedback provided by the researcher, a total of 49 feedback events occurred throughout the process, including 13 instances of self-feedback, 18 of task-level feedback, and 18 of process-level feedback. Some examples of the feedback from the researcher are illustrated in Table 1. Figure 13 presents the distribution of the three types of feedback in the sessions. These feedback instances were not distributed evenly across the activities. The majority occurred during the guidance phase and in the initial sessions of the exploration. After the earlier phases, feedback was rarely provided. The figure also reveals that process-level feedback (how to use the AR system and what to do next) ceased after the tenth activity, indicating that Karim can operate the system and explore the graphs independently. In addition, some task-level feedback was given near the end of the exploration to support Karim's summary of what he learned in the activities.

Implication on Feedback

In Karim's case, the AR system primarily provides task-level feedback, designed to support his exploration of the relationship between the derivative and the original function. The feedback is triggered in response to Karim's correct actions, such as touching the original function, helping him understand the steps to take in the exploration. However, this type of feedback was not sufficient for him to ensure his performance, confirm his understanding of the knowledge content, or determine whether he was on the right path in his exploration. As a supplement, the researcher provided self-feedback, task-level feedback, and process-level feedback, helping Karim interact with the AR system, plan his exploration, and achieve the learning objectives. Notably, neither the AR system nor the researcher offered self-regulation feedback, aligning with findings by Lipnevich and Panadero (2021), who observed that most feedback in instructional settings tends to focus on task- and self-levels, despite the greater impact of process and self-regulation feedback. Thus, coordination of the AR system and the role of the researcher as a tutor are needed in the iteration to improve the feedback.

Prior work in exploratory environments highlights the importance of supporting students' exploratory processes in microworlds (Mavrikis et al., 2008)—insights that can inform AR designs aiming to foster rich mathematical inquiry. An AR system could shoulder some of the tutor's responsibilities by providing more context-aware feedback and support the tutor in achieving the role of the facilitator. Specifically, more feedback can be embedded in the system to direct students' attention explicitly on aspects of the environment, or even help them to set and work towards explicit goals. As the system supports the guide of the exploration, the tutor could turn out to be an "intelligent computer-based facilitator", prompting students to reflect on their actions, interpret visual feedback, and consider alternative solutions.

Back to Karim's case, integrating automated feedback combined with data-driven analysis into the AR system can be a potential solution to provide students with timely and personalised support, while strengthening the teacher's role as a facilitator (Mavrikis et al., 2010). For example, to create a more guided learning environment, the AR system can incorporate clear operational feedback, such as the step-by-step instructions outlined by de Ravé et al. (2016). These prompts can be used to help students place and interact with functions in an expected manner, ensuring they follow structured exploration pathways. At the task level, the system can introduce question scaffolding to remind students that the primary goal is to uncover the relationship between functions and their derivatives (Huang et al., 2015). This scaffolding can also record and update students' insights, providing personalised suggestions for the next function to explore through an automatic recommendation programme (Ley et al., 2010). Additionally, a point system can be implemented to automatically track students' correct actions and responses and offer self-feedback. By replying to correct responses, this feedback mechanism enhances students' self-efficacy and encourages sustained engagement in the learning process (Ortiz-Rojas et al., 2017).

Beyond supporting students directly, the system can capture and analyse data on students' interactions, including correct operations and conceptual responses. This data-driven approach allows teachers to gain deeper insights into students' progress and understanding (Mavrikis et al., 2019). Equipped with this information, teachers can offer self-regulation feedback, guiding students to reflect on their exploration process and fostering more meaningful engagement with the material. Compared to existing feedback in Touch the Derivative system, this automated, data-driven system could provide real-time, personalised guidance rather than simply confirming correct operations. It adapts dynamically to students' learning needs, helping them navigate learning challenges while allowing teachers to focus on high-level cognition rather than technology usage. By shifting from direct instruction to facilitation, teachers could better support students in developing self-regulation skills and deeper conceptual understanding.

Discussion

The visualisation power of AR is widely acknowledged in the literature (Orozco et al., 2006), particularly in relation to 3D functions, which are difficult to depict on paper and often more taxing to mentally visualise. Other studies highlight how AR enables learners not only to *see* mathematics but also to uncover its underlying structures (Kaufmann, 2004; Monteiro Paulo et al., 2021). However, Touch the Derivative presented 2D functions in an AR environment, which challenges the typical arguments used for AR in 3D spaces. Since 2D functions are easily represented on paper or computer screens, AR does not provide the same distinct advantage for such tasks. Nonetheless, under appropriate guidance, Karim successfully interpreted the green points as markers of the inclination of his hand, suggesting a meaningful interaction between human movement and the mathematical concept of gradient.

The Touch the Derivative prototype "floats" 2D graphs in free space, without anchoring to any physical counterparts in the "real world". In the "Design Affordances" section, we discussed how physical artefacts may support 2D mathematical explorations in 3D environments by reducing the degrees of freedom and providing tactile feedback. However, there are still issues around how to embody "local straightness" (Tall, 2019) when hands are representing tangents. Careful design considerations must be made; for example, strapping a pen-like object to a finger could reinforce the concept of a tangent, thereby aligning the sensory-motor action with the mathematical abstraction. Another potential solution could be to "pin" the function to a wall, providing a stable, clutter-free physical reference point. This example further demonstrates the cognitive scaffolding that physical objects may offer—again, reducing the degrees of freedom when necessary. Our analysis of physical artefacts within AR environments suggested how they may support the sense-making process by blending tangible objects with digital representations. This fusion creates opportunities for learners to both visualise and feel mathematical concepts. In this way, AR technology may provide environments for these approaches to co-exist, offering learners both concrete experiences and conceptual insights.

As this is not an experimental study, we cannot comment on the outcomes of affording learners' agency in choosing their own physical artefacts. Instead, we have presented how AR environments may be uniquely placed to blend affective (Papert, 1980), psychomotor (Bos et al., 2022; Shvarts et al., 2021), and cognitive learning experiences. Historically, affective and psychomotor dimensions of learning have been overshadowed by a focus on cognitive aspects (Helmer, 2023), but AR presents an opportunity to bring these dimensions into balance.

The feedback analysis underscores the critical importance of guided instruction, especially when engaging with new technologies. We explored various types of feedback, including task-level, process-level, and self-level feedback, and discussed also how to coordinate the system and the tutors on offering feedback to support student understanding and engagement based on previous work, as suggested by research in other exploratory learning contexts and adapted to the unique affordances of AR technology.

First, the results highlight the central role of task-level feedback in guiding learners through specific actions in the AR environment. As evidenced in Karim's case, the AR system was effective in providing immediate, task-oriented feedback related to his manipulation of the derivative and the original function. This real-time feedback allowed Karim to adjust his actions accordingly, reinforcing prior research emphasising the importance of responsive feedback in exploratory environments (Kotranza et al., 2009). However, while task-level feedback was well integrated into the AR system, the absence of process-level and self-regulation feedback limited deeper cognitive engagement. Literature suggests that while task feedback is crucial for guiding immediate actions, process-level feedback plays an important role in helping learners understand the strategies behind their actions, encouraging them to think more broadly about their problem-solving processes (Hattie & Timperley, 2007). This finding suggests that AR systems, like the one in Karim's case, would benefit from the incorporation of more process-level feedback to support strategic exploration and to scaffold learner reflection.

Moreover, the lack of self-regulation feedback in the AR system, which would have encouraged Karim to reflect on his learning process, is a notable gap. Self-regulation feedback is essential for fostering independence in learners, prompting them to monitor and assess their own progress and make adjustments where necessary (Lipnevich & Panadero, 2021). In this case, the researcher can play the role of a facilitator to help compensate for the system's shortcomings, inspiring Karim's reflections and aiding his adjustments (Mavrikis et al., 2008).

The findings also suggest that AR systems should be designed to support the coordination of the system and other elements (such as worksheets and tutors). As seen in Karim's exploration, the tutor had to step in frequently to guide his interactions with the system, particularly in the absence of clear instructions or prompts. The combination of the system feedback with the tutor's role has the potential to achieve the tutor's role in facilitating students' understanding and engagement (Mavrikis et al., 2008). In addition, offering some guidance through other materials, such as worksheets, might help bridge the interaction from AR to the curriculum (Geraniou & Mavrikis, 2015).

Conclusion

At its core, this paper asks: how can we rethink the affordances and feedback in AR environments to foster richer mathematical inquiry? Feedback, affordances, and the physical-cognitive-contextual framing are not separate strands but interwoven design considerations that determine how inquiry unfolds—and how much guidance learners receive as they confront new mathematical ideas. On first viewing of the Karim case, we were drawn to the central role the physical environment played—unsurprisingly, given AR is part physical, part digital. Equally important was the guiding presence of the researcher, who shaped the learning process by directing Karim's movements and attention. To deepen our understanding of Karim's interactions within the physical space and with the researcher, we adopted specific theoretical tools to unravel the intricate tapestry of threads that make up an AR learning experience—the digital interface, physical space, and human interaction.

In "Design Affordances" section, we used Bujak et al.'s (2013) framework to construct a comparable line of AR inquiry for the parabolic AR task. This allowed us to consider how physical objects might enhance the learning of 2D derivative functions by providing tactile feedback on the gradient—an aspect notably absent in Karim's AR tasks. By analysing the physical space using 6DoF, we identified challenges in exploring the gradients of 2D functions in 3D space—chiefly the moments of disconnect between Karim's hand movements and the digital tangent lines his hand was meant to trace out. This revealed a deeper issue in how AR environments articulate 2D mathematical concepts within 3D spaces. Physical artefacts may reduce the degrees of freedom in Karim's hand movements and provide a more grounded, intuitive exploration of derivatives. Moreover, these artefacts may play an affective role, fostering personal engagement and making the symbolic side of derivatives more tangible. While the theoretical tools illuminated key aspects of the learning process, they also underscored the need for more refined design strategies to better integrate physical space, cognitive processes, and emotional engagement.

Through the lens of feedback, it became clear that while the task-level feedback provided by the AR system was effective in facilitating Karim's understanding of specific actions, such as manipulating the derivative and the original function, the absence of process-level and self-regulation feedback limited deeper cognitive engagement. This finding highlights a key opportunity for AR system improvement. By incorporating more sophisticated feedback mechanisms, AR systems could prompt learners to reflect on their actions and think critically about the learning process, aligning with theories that emphasise the value of process and self-regulation feedback for fostering metacognitive skills and learner independence. Additionally, the coordinated role of the tutor emerged as essential, particularly when the system lacked clarity.

Taken together, the insights affirm AR's promise for enhancing calculus education, but also the challenges in designing environments that fully leverage the technology's unique affordances to support deeper learning. Yet important questions remain. What was Karim's experience like as he navigated the AR environment? Was he confused by non-verbal feedback? How intuitive or natural did he find the technology, and how did his hand feel as he navigated the unspoken "rules" underpinning the exploration? Future work could explore how feedback mechanisms might be enhanced or integrated to better support learners like Karim. For example, with the emergence of generative Artificial Intelligence, providing context-aware feedback and adapting to their interaction is increasingly feasible (Giannakos et al., 2024; Huang et al., 2021). Similarly, further research could explore how giving learners more agency in selecting their own physical artefacts influences both understanding and motivation. Moreover, AR presents an opportunity to balance affective, psychomotor, and cognitive dimensions of learning. Investigating how feedback in AR systems aligns with other instructional elements could reveal new strategies for AR environments like Touch the Derivative—ultimately helping to foster richer mathematical inquiry.

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Declarations

Ethics Approval The research has been approved by the Human Subjects Research Committee of Ben-Gurion University—Request num: 1659–1.

Conflict of Interests The authors declare no competing interests.

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