Conical singularities

in special holonomy and gauge theory

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Declaration

I, Enric Solé-Farré confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

In particular:

Chapter 2 of the thesis is based on [Sol24a]:

Stability of nearly Kähler and nearly parallel G_2 *manifolds*

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Chapter 3 of the thesis is based on [Sol24b]:

The Hitchin index in cohomogeneity one nearly Kähler structures

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Abstract

This thesis examines conical singularities in the context of special holonomy and gauge theory, with a focus on both analytical and variational aspects.

In the first chapter, we study instantons on metric cones and establish a new relation between the instanton deformation operator and the Bourguignon stability operator on the corresponding link. This framework is used to study instantons with isolated conical singularities, yielding an analytical construction of their moduli spaces. As a result, we give an explicit formula for their virtual dimension.

In the second chapter, we investigate generalisations of Hitchin's functionals, whose critical points correspond to nearly Kähler and nearly parallel G_2-structures. We study the gradient flow of these functionals and perform a spectral decomposition of their Hessians relative to natural indefinite inner products. This study leads to the definition of the *Hitchin index*, a Morse-like invariant that provides a lower bound for the Einstein co-index. We investigate the connection of this index with the deformation theory of G_2 and Spin(7)-conifolds.

In the third chapter, we investigate nearly Kähler manifolds under a cohomogeneity one symmetry assumption. This enables us to study and bound the cohomogeneity one contributions to the Hitchin index by reducing the PDE eigenvalue problem to an ODE eigenvalue problem. We focus our analysis on the inhomogeneous nearly Kähler structure on $S^3 \times S^3$ constructed by Foscolo and Haskins, and obtain non-trivial lower bounds for both the Hitchin and Einstein indices of the manifold, thereby addressing an open question posed by Karigiannis and Lotay.

Impact statement

This thesis contributes to the understanding of conical singularities in special holonomy and gauge theory. This work deepens our understanding of geometric analysis on singular spaces, particularly those exhibiting special holonomy, and provides concrete analytic tools for studying moduli spaces of instantons with isolated conical singularities, including a precise formula for their virtual dimension. These results address a gap in the literature relative to the relatively extensively studied cases of conical singularities in special holonomy metrics and calibrated submanifolds. These results create new pathways for exploring geometric structures with singularities, which play a central role in the mathematics underlying string theory and M-theory. In particular, the methods developed have applications to the analytic study of Calabi–Yau, G_2 and Spin(7) manifolds.

Another contribution is the introduction of the Hitchin index, a new invariant that captures the stability properties of critical points of Hitchin-type functionals. Its relationship to the Einstein co-index creates a framework for understanding rigidity and deformation in special holonomy geometries, which opens up new avenues in the Morse-theoretic approach to moduli spaces and their topology.

In a different direction, the use of cohomogeneity one symmetry to reduce partial differential equations to non-explicit ordinary differential equations, while still being able to prove interesting results, demonstrates that there remains considerable scope for investigating and establishing new results under a cohomogeneity one assumption.

Although the research is purely mathematical, its findings may influence future developments in other areas. The structures studied are relevant to theoretical physics, particularly in the pursuit of a unified framework for gravity and quantum field theory in the context of M-theory and string theories. The analytic tools developed could aid in modelling singularities that appear in those theories and in constructing compactification schemes for extra dimensions.

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La vida, perquè sigui vida, s'ha de viure a poc a poc...

Mercè Rodoreda (1908 – 1983)
from *'La Plaça del Diamant'*

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Contents

Introduction

The study of manifolds with special holonomy plays a central role in modern differential geometry and geometric analysis, with deep connections to topology, global analysis, and theoretical physics. These manifolds are distinguished by having a Riemannian holonomy group that is strictly smaller than the full orthogonal group. The resulting geometric structures often possess remarkable properties, such as Ricci-flatness or the existence of parallel differential forms, which lead to rich geometric structures.

The work by Berger in 1955 [Ber55] laid the foundation for the theory of special holonomy, identifying the possible holonomy groups that can arise in irreducible, simply connected, non-locally symmetric Riemannian manifolds:

Hol(g)	**dim(M)**	**Type of manifold**	**Curvature**	**Parallel 4-form**
SO(n)	n	Orientable	–	–
U(n)	$2n$	Kähler	–	$\frac{\omega^2}{2}$
SU(n)	$2n$	Calabi–Yau	Ricci-flat	$\frac{\omega^2}{2}$
Sp(n)	$4n$	Hyperkähler	Ricci-flat	$\frac{\omega_1^2}{2}, \frac{\omega_2^2}{2}, \frac{\omega_3^2}{2}$
Sp(n) $\cdot$ Sp(1)	$4n$	Quaternion-Kähler	Einstein $\lambda \neq 0$	Ω
G_2	7	Holonomy G_2	Ricci-flat	ψ
Spin(7)	8	Holonomy Spin(7)	Ricci-flat	Φ

Table 1: Berger's list of possible holonomy groups

The list, which has since been refined and confirmed through the construction of explicit examples, remains a guiding framework in the field. An additional candidate, Spin(9), was later shown by Alekseevski [Ale68] and Brown and Gray [BG72] to only occur as locally symmetric spaces. For the remaining groups, both compact and non-compact examples have been constructed, thanks notably to the contributions of Yau [Yau77] [Yau78], Bryant [Bry87], Bryant and Salamon [BS89] and Joyce [Joy96a] [Joy96b], amongst many others.

As noted in Table 1, all special holonomy manifolds admit a natural parallel 4-form Ω, due to their structure group being connected and contained in the normaliser of a semisimple Lie group (cf. Appendix A). The holonomy reduction ensures that the 4-form is parallel with respect to the Levi-Civita connection and therefore harmonic.

While the classical theory focuses mainly on smooth complete manifolds, modern developments —particularly those in geometric analysis, string theory, and gauge theory — require an extension of this framework to singular spaces, and in particular to manifolds with conical singularities. These spaces arise naturally as degenerations of smooth manifolds in moduli problems and therefore play a major role in compactifications of moduli spaces, glueing constructions, and as geometric flow singularities.

These considerations naturally lead to the central themes of this thesis: the analysis of conically singular spaces with special holonomy and the study of associated gauge-theoretic problems on such spaces. These problems not only retain many of the features of their smooth counterparts but also present new phenomena due to the singular geometry. In particular, the presence of a conical singularity affects the behaviour of differential operators, the moduli space structure of geometric objects like instantons, and the analytic techniques required to study them. An analogue study for calibrated submanifolds can be carried out and will be outlined in the introduction, but will not be examined in detail in the thesis.

An isolated conical singularity, in its simplest form, is a metric degeneration where a neighbourhood of a singular point is modelled on a metric cone

$$(C(\Sigma), g_C) = \left(\mathbb{R}_+ \times \Sigma^{n-1}, dr^2 + r^2 g_\Sigma\right) ,$$

where (Σ^{n-1}, g_Σ) is a Riemannian manifold called the link (cf. Definition 3.4) and r is the obvious coordinate in the $\mathbb{R}_+$ factor. In this thesis, we will only deal with the case where Σ is closed.

The geometry of the cone $(C(\Sigma), g_C)$ is intimately related to that of its link (Σ, g). If $C(\Sigma)$ admits a G-structure, then Σ naturally inherits an H-structure, with $H = \mathrm{Stab}^G(r\partial_r)$. When the cone metric is irreducible, its tangent space is an irreducible G-representation, and one has the identification

$$G/H \;\cong\; Gr^+(1,n) \;\cong\; S^{n-1}.$$

The metric cone carries a natural $\mathbb{R}_+$-action

$$\begin{aligned} \mathbf{t} : C(\Sigma) \times \mathbb{R}_+ &\longrightarrow C(\Sigma), \\ ((r,x),\lambda) &\longmapsto (\lambda r, x), \end{aligned}$$

which induces an action on smooth differential forms. A form is said to be homogeneous of rate $\lambda \in \mathbb{R}$ if it has weight λ under this action. Equivalently, a homogeneous k-form $\gamma \in \Omega^k(C(\Sigma))$ of weight λ can be expressed as

$$\gamma = r^\lambda \left(r^{k-1}\, dr \wedge \alpha \;+\; r^k\, \beta\right),$$

where $\alpha \in \Omega^{k-1}(\Sigma)$ and $\beta \in \Omega^k(\Sigma)$. Such homogeneous forms will play a central role in our analysis (see Section 3).

In the special holonomy setting, a cone $C(\Sigma)$ with special holonomy carries a canonical parallel 4-form Ω (see Table 1) that is invariant under dilations. The link Σ then inherits a pair of associated forms $\Xi \in \Omega^{n-4}(\Sigma)$ and $\Upsilon \in \Omega^{n-5}(\Sigma)$, defined by

$$*_C\Omega = r^{n-4}\left(\frac{dr}{r} \wedge \Upsilon + \Xi\right),$$

where $*_C$ denotes the Hodge star on the metric cone The fact that Ω is parallel for the Levi-Civita connection on the cone induces specific differential relations between Ξ and Υ.

$\mathbf{Hol}(g_C)$	$\boldsymbol{H}$	$\mathbf{dim}(\boldsymbol{\Sigma})$	**Type of manifold**	$\boldsymbol{\Upsilon}$	$\boldsymbol{\Xi}$
$\mathrm{SO}(n)$	$\mathrm{SO}(n-1)$	$n-1$	–	–	–
$\mathrm{U}(k+1)$	$\mathrm{U}(k)$	$2k+1$	Sasaki	$\eta \wedge \frac{\omega^{k-2}}{(k-2)!}$	$\frac{\omega^{k-1}}{(k-1)!}$
$\mathrm{SU}(k+1)$	$\mathrm{SU}(k)$	$2k+1$	Sasaki Einstein	$\eta \wedge \frac{\omega^{k-2}}{(k-2)!}$	$\frac{\omega^{k-1}}{(k-1)!}$
$\mathrm{Sp}(k+1)$	$\mathrm{Sp}(k)$	$4k+3$	3-Sasaki	$\eta_i \wedge \frac{\omega_i^{2k-1}}{(2k-1)!}$	$\frac{\omega_i^{2k-2}}{(2k)!}\left(\omega_i^2 - \sum_{(ijk)} \omega_i \wedge \eta_j \wedge \eta_k\right)$
G_2	$\mathrm{SU}(3)$	6	Nearly Kähler	ω	ρ
$\mathrm{Spin}(7)$	G_2	7	Nearly parallel G_2	φ	ψ

Table 2: Berger's list on metric cones

The notation for these induced forms on the link varies with the holonomy of the cone; for convenience, we summarise the most common conventions in Table 1.

The existence of the $(n-4)$-form Ξ induces a first order differential operator on Σ^{n-1}:

$$\begin{aligned} \mathrm{curl}_\Xi : \Omega^1 &\to \Omega^1 \\ X &\mapsto (-1)^n * (dX \wedge \Xi), \end{aligned} \tag{1}$$

generalising the usual 3-dimensional curl. We prove

Proposition (Prop. 2.10 & Prop. A.11)**.** *Let (Σ, g) be a closed Riemanninan manifold equipped with an H-structure admitting a compatible $(n-3)$-form Ξ. If Ξ is closed, then the curl operator is self-adjoint and fits in the complex*

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{\mathrm{curl}_\Xi} \Omega^1 \xrightarrow{d^*} \Omega^0 \to 0\ . \tag{2}$$

Let $\mathcal{E}_\lambda = \{X \in \Omega^1 |\ \mathrm{curl}_\Xi(X) = \lambda X\}$. If (Σ, g) is the link of a special holonomy Ricci-flat cone, then

(i) the Lie algebra of the automorphism group of the H-structure $\mathfrak{auto}(M, H)$ is given by

$$\mathfrak{auto}(M, H) \cong \mathcal{E}_{-2}\ ,$$

(ii) the Lie algebra of the isometry group of the metric $\mathfrak{isom}(M, g)$ satisfies

$$\mathfrak{isom}(M, g) \cong \mathcal{E}_{-2} \oplus \mathcal{E}_{n-2}\ ,$$

The current proof of the last claim requires a case-by-case discussion and forces one to exclude spaces of constant sectional curvature. However, we believe the result should extend to sphere quotients and admit a general proof that does not require case-by-case considerations.

Remark. *The complex* (2) *is, in general, not elliptic. The condition for the complex above to be elliptic is equivalent to the 3-form $*\Xi$ inducing a cross product, so it can only occur for $n-1=3$ and $n-1=7$. However, in the nearly Kähler and Sasaki cases, the complex still has finite cohomology $H^0 \cong H^3 \cong \mathbb{R}$ and $H^1 \cong H^2 \cong 0$ and all eigenvalues but one have finite multiplicity. In all cases, the curl operator can be viewed as part of the corresponding Dirac operator on Σ.*

Variational problems and special holonomy

Variational principles are foundational to modern differential geometry and mathematical physics. Many of the most geometrically significant structures can be realised as critical points of natural functionals. These functionals encode intrinsic or extrinsic geometric quantities, and their critical loci are characterised by a partial differential equation (PDE) known as the Euler—Lagrange (E–L) equation.

Let M^n denote a complete smooth manifold of dimension n. Among the most well-known examples of such variational problems are:

(i) Assume M is compact and $n \geq 3$. The Einstein-Hilbert functional

$$\begin{aligned}\mathcal{S} : \mathrm{Met}(M) &\to \mathbb{R} \\ g &\mapsto \frac{1}{n-1} \int_M s_g - \lambda(n-2)\, \mathrm{dvol}_g \ ,\end{aligned}$$

where $\mathrm{Met}(M)$ is the space of smooth metrics on M and s_g is the scalar curvature of the metric g (cf. Appendix C). The E–L equation of $\mathcal{S}$ is $\mathrm{Ric}_g = \lambda g$, and critical points are Einstein metrics of constant λ.

(ii) Let N^k be a closed manifold. The area functional

$$\begin{aligned}\mathcal{V} : \mathrm{Imm}(N, M) &\to \mathbb{R} \\ \iota &\mapsto \int_N \iota^*(\mathrm{dvol}_g) \ ,\end{aligned}$$

where $\mathrm{Imm}(N, M)$ is the space of immersions of N^k into M, equipped with a metric g. Critical points of $\mathcal{V}$ are called minimal submanifolds, and the E–L equation is $H^\Sigma = 0$, the vanishing of the mean curvature.

(iii) For a principal $\mathrm{U}(k)$-bundle P over (M, g), the Yang–Mills functional is

$$\begin{aligned}\mathcal{YM} : \mathcal{A} &\to \mathbb{R} \\ A &\mapsto \frac{1}{2} \int_M ||F_A||^2 \mathrm{dvol}_g \ ,\end{aligned}$$

where $\mathcal{A}(P)$ denotes the space of connections on P, and F_A is the curvature of the connection A. The critical points are called Yang–Mills connections and they satisfy the E–L equation $d_A^* F_A = 0$.

The Euler–Lagrange equations of these functionals are second-order elliptic PDEs (modulo gauge symmetries), and their solutions encode deep geometric and topological information of the underlying manifold.

In certain favourable situations, the geometric structure of M allows us to reduce these second-order PDEs to first-order PDEs. These reductions occur most notably when M admits a special holonomy structure. In this case, the existence of parallel differential forms (equivalently parallel spinors) allows one to rephrase the variational problem in terms of calibrated geometry or gauge-theoretic instanton equations.

	Euler–Lagrange Equation	Critical points	Symmetry group
Einstein–Hilbert	$\mathrm{Ric_g} = \lambda g$	λ - Einstein metrics	$\mathrm{Diff}(M)$
Area functional	$H^\Sigma = 0$	Minimal submanifolds	$\mathrm{Diff}(\Sigma)$
Yang–Mills	$d_A^* F_A = 0$	Yang–Mills connections	$\mathrm{Aut}(P)$

Table 3: Variational problems, their Euler–Lagrange equations and symmetries

For example, if M has holonomy contained in $G \subseteq \mathrm{SO}(n)$, where G is a semisimple group from Berger's list (i.e., G is either $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, G_2 or $\mathrm{Spin}(7)$), then the Ricci-flat condition is automatically satisfied. Furthermore, in the G_2 case, one may view the holonomy condition as the critical point condition of a Hitchin-type functional on 3-forms. A more thorough discussion of this viewpoint is presented in Section 6.

Let us now consider a special holonomy manifold (M, g) equipped with a parallel calibrating form $\theta \in \Omega^k$. That is, $\theta_\pi(p) \leq \mathrm{vol}_g(\pi(p))$ for any k-plane $\pi \subseteq T_pM$. The standard requirement that θ is closed follows from θ being parallel. Associated with θ, and for a fixed cohomology class $\alpha = [N] \in H_k(M, \mathbb{Z})$, we have the topological charge

$$c_\theta([\alpha]) = \langle \iota_* \alpha, [\theta] \rangle \ .$$

The calibrating condition implies that any immersion representing this homology class satisfies the inequality $c_\theta([\alpha]) \leq \mathcal{V}(\iota)$. An immersion ι is called a calibrated submanifold if it realises the equality above, and so calibrated submanifolds are absolute area minimisers within their homology class.

A similar situation arises in the context of gauge theory on special holonomy manifolds. Given a closed Riemannian manifold (M, g) with special holonomy and a principal bundle P over it, one defines the charge

$$c_\Omega(P) = -\frac{1}{2} \int_M \mathrm{Tr}(F_A \wedge F_A) \wedge *\Omega \ ,$$

where Ω is the associated parallel 4-form from Table 1. By Chern-Weil theory, it follows that c_Ω is a topological invariant of the principal bundle P. In all cases (cf. Section 1), one can show that $c_\Omega(P) \leq \mathcal{YM}$. Connections for which equality is satisfied are called Ω-instantons, and are absolute minimisers of the Yang–Mills functional.

Remark. *For minimal submanifolds, one can consider the more general setup of harmonic maps, where the area functional gets replaced by the energy functional*

$$\begin{aligned} \mathcal{E} : \mathcal{C}^\infty(N, M) &\to \mathbb{R} \\ u &\mapsto \int_M |du|^2 \mathrm{dvol}_g \ , \end{aligned}$$

for a pair of closed Riemannian manifolds (M, g) and (N, h). In the context of special holonomy, the notion of calibrated submanifolds is replaced by holomorphic maps in the Kähler case, and Smith maps (cf. [CKM23]) in general.

An alternative approach to this discussion would be from a spinorial perspective, but we will not explore it in this thesis.

Metric cones and Chern–Simons functionals

The previous discussion focused on variational problems in the compact setting, where the charges $c_\theta([\alpha])$ and $c_\Omega(P)$ are well-defined topological invariants. In the non-compact setting, the discussion becomes more complicated and requires the introduction of Chern–Simons type functionals that account for boundary contributions. We outline the calibrated submanifold case below. The instanton case is treated in detail in Section 2, and an analogue to this setup for G_2 and Spin(7) holonomy cones is the primary focus of Sections 7 and 8 respectively.

Let $\Theta \in \Omega^k(C(\Sigma))$ be a homogeneous parallel calibrating k-form on the cone, so it is given by

$$\Theta = r^k \left(\frac{dr}{r} \wedge \tau + \theta \right) \tag{3}$$

with $\tau \in \Omega^{k-1}(\Sigma)$ and $\theta \in \Omega^k(\Sigma)$. Fix a *suitable* reference immersion $\iota_1 \in \mathrm{Imm}(N^{k-1}, \Sigma^{n-1})$, and consider $\mathrm{Path}_\bullet(\mathrm{Imm}(N,\Sigma))$ the space of (smooth) paths in the space of immersions, based at ι_1. We define the functional:

$$\mathcal{C} : \mathrm{Path}_\bullet(\mathrm{Imm}(N,\Sigma)) \to \mathbb{R}$$
$$\iota_t \mapsto \int_{N\times[1,T]} \iota_t^*(\Theta)\ .$$

This corresponds to the calibration charge c_θ above evaluated over the compact manifold with boundary $N' = [1,T] \times N \subseteq \mathbb{R}_+ \times \Sigma$, but it is no longer a topological invariant. In particular, if we consider an infinitesimal variation of the boundary end-point, we obtain a Chern–Simons 1-form functional:

Lemma. *The Chern–Simons* 1*-form functional associated to* $\mathcal{C}$ *at an immersion* ι *is*

$$\widetilde{\mathcal{C}}_\iota : \Gamma(\mathrm{Nor}^{\iota(N)}) \to \mathbb{R}$$
$$X \mapsto \int_{\iota(N)} \mathcal{L}_X \tau = k \int_{\iota(N)} X \lrcorner \theta\ ,$$

where $\mathrm{Nor}^{\iota(N)}$ *denotes the normal bundle of* $\iota(N) \subseteq \Sigma$ *and* $\tau, \theta \in \Omega^\bullet(\Sigma)$ *are given by Equation* (3)*.*

Notice that this discussion is quite similar to the one outlined in Section 2 of Donaldson–Segal [DS11], where they consider cylinder metrics over special holonomy links (Σ, g_Σ), $\mathrm{Cyl}(\Sigma) = (\mathbb{R}\times\Sigma, dt^2 + g_\Sigma)$. In the Donaldson–Segal case, the Chern–Simons functional is the 1-form of a locally defined functional, whereas in our case, the functional $\mathcal{C}$ is globally well-defined modulo gauge.

We say that an immersion ι is a critical point of $\widetilde{\mathcal{C}}$ if $\widetilde{\mathcal{C}}_\iota$ vanishes for all vector fields. One might expect that critical points correspond to calibrated links and that gradient flows for $\mathcal{C}$ represent calibrated submanifolds on the metric cone.

While this holds in the nearly parallel G_2 case, it fails in the cases where (Σ, g) is a nearly Kähler or a Sasaki manifold, and is related to the lack of ellipticity of the curl complex (2). For instance, if N is a horizontal totally real submanifold of a Sasaki manifold, then $\widetilde{\mathcal{C}}$ will vanish, but it is not the link of a holomorphic cone inside $C(\Sigma)$. This mismatch suggests the need for additional geometric constraints.

To remedy this in the instanton case, we introduce a *cone constraint*. Importantly, we show that this constraint is preserved under gradient flow in all cases of interest, allowing us to recover the expected characterisation of instantons. While the geometric justification for the cone constraint remains case-dependent, it offers a coherent framework that is likely to extend to the calibrated submanifold case.

At a critical point ι of the Chern–Simons $\widetilde{\mathcal{C}}$ functional, its second variation characterises infinitesimal deformations of the calibrated cone condition. Along the linearised cone constraint and up to gauge fixing, the second variation can be identified with a Dirac-type operator $\widehat{D}_\iota$. Therefore, it has a discrete unbounded spectrum with finite multiplicities.

If ι_0 and ι_∞ are two distinct critical points of $\widetilde{\mathcal{C}}$ and ι_t is a gradient flow line connecting them, the associated spectral flow of the family $\widehat{D}_\iota$ provides a virtual count of the expected dimension of the moduli space of calibrated submanifolds asymptotic to the calibrated cones on each end. In analogy with finite-dimensional Morse theory, one would hope to define an index-like geometric quantity K_ι such that the spectral flow of the family of Dirac operators $\widehat{D}_{\iota_t}$ satisfies the relation

$$\mathrm{SpecFlow}(\widehat{D}_{\iota_t}) = K_{\iota_\infty} - K_{\iota_0} \ .$$

In the case of special Lagrangian and coassociative cones, we expect that this spectral index is related to the stability indices introduced by Joyce [Joy04] and Lotay [Lot07], respectively.

Summary of results and overview

This thesis investigates the analytic and variational aspects of instantons and special holonomy structures with isolated conical singularities (ICS). The overarching goal is to develop and understand a coherent deformation and moduli theory for such objects, extending the well-established compact theory to the singular setting. Ultimately, this work aims to provide tools that could contribute to the construction of new invariants and the development of enumerative theories in gauge theory and special holonomy metric, as well as laying the groundwork for understanding more geometric objects with higher codimension conical singularities.

Chapter I - Conically singular instantons

The first part of the thesis focuses on instantons with ICS, with particular emphasis on nearly parallel G_2 and nearly Kähler geometries. We begin with a review of instanton theory in general, highlighting new contributions and clarifications.

A first result concerns the ellipticity of the deformation complexes introduced by Reyes-Carrión [Rey98]. We provide a short and conceptual proof that these complexes are indeed elliptic under mild hypotheses (cf. Theorem 1.11). In addition, we present a new rigidity-type result as an application of the instanton charge:

Proposition (Prop. 1.9)**.** *Let (M^n, g, Ω) be a manifold carrying an admissible $N(H)$ structure with associated 4-form Ω closed, and let $E \to M$ a Hermitian vector bundle admitting an Ω-instanton. Let T^k be a flat torus of dimension k and denote by $\pi : M \times T^k \to M$ the trivial*

fibration. Assume the product metric on $M \times T$ admits an $N(\widehat{H})$-structure with characteristic 4-form $\widehat{\Omega}$, compatible with $N(H)$ and such that the difference $\widehat{\Omega} - \pi^*(\Omega)$ is exact.
Then the moduli spaces of irreducible instantons are related by

$$\mathcal{M}^{irred}_{\widehat{\Omega}}(\pi^*(E)) \cong \mathcal{M}^{irred}_{\Omega}(E) \times T^* \ ,$$

where T^* denotes the torus dual to T. In particular, to every $\widehat{\Omega}$-instanton on $\pi^*(E)$ we can associate an Ω-instanton on $E \to M$.

This result extends earlier work of Wang [Wan18b], who treated only the case $k = 1$. Notably, our proof relies on a direct argument using the topological charge, which is significantly shorter and more transparent than Wang's approach to the question.

Turning to metric cones, we introduce a Floer–type functional whose critical points coincide with instantons, thereby providing a variational framework naturally adapted to conical geometries. In this context, we define the notion of the cone bundle L, and establish a key link between the Hessians of the Yang–Mills functional S_A and the Floer–type functional D_A:

Proposition (Prop. 2.6)**.** *Let A be an Υ-instanton and L the associated cone bundle. There exists constants $C_i \in \mathbb{R}$ such that*

$$S_A(\alpha) = D_A^2(\alpha) - (n-4)D_A(\alpha) - d_A^* \sum_i C_i \left(\pi_{L_i^\perp}(d_A\alpha)\right) \ ,$$

where $L_i^\perp$ form the direct sum decomposition of irreducible $N(H)$-representations of $L^\perp$ and π_B is the bundle projection map to the corresponding bundle B.

This result generalises a formula of Waldron [Wal22] in the case of nearly parallel G_2-instantons. Building on this variational framework, we then move to the setting of connections with isolated conical singularities. We first establish that the instanton charge extends naturally to the ICS case. Using weighted analysis on conically singular manifolds, we generalise the Uhlenbeck gauge slice construction to an appropriate weighted gauge group. With this in hand, we can apply the implicit function theorem and obtain a natural description of the moduli spaces of instantons with isolated singularities.

We conclude the chapter with a virtual dimension formula for these moduli spaces, which decomposes into contributions from the geometry of the link and the global geometry of the manifold.

Chapter II - Hitchin Functionals and their index

The second part of the thesis focuses on Hitchin's functionals in dimensions six and seven. These functionals realise nearly Kähler and nearly parallel G_2 structures as their critical points, and they admit a natural interpretation as Chern–Simons-type functionals for the cone. Motivated by this perspective, we introduce two new Hitchin-type functionals, denoted $\mathcal{Q}$ and $\mathcal{T}$, which are defined on spaces of stable and exact forms carrying natural SU(3)– and G_2–structures, respectively. We summarise their main properties in the following two theorems:

Theorem (Prop. 7.12, Prop. 7.13, Prop. 7.16 & Prop. B.4)**.** *Let Σ^6 be a closed spinnable 6-manifold. Consider the space*

$$\mathcal{U} = \left\{\omega \in \Omega^2(\Sigma) \mid d\omega \text{ is stable, } \omega \text{ is stable and positive, } \omega^2 \text{ is exact}\right\} .$$

The new Hitchin functional $\mathcal{Q} : \mathcal{U} \to \mathbb{R}$ satisfies the following:

(i) Critical points are nearly Kähler structures [1]

(ii) The Einstein–Hilbert action is a lower bound for $\mathcal{Q}$. The two only coincide along rescalings of nearly Kähler structures.

(iii) The associated gradient flow is not parabolic, even after a DeTurck trick.

(iv) Critical points have a well-defined index with respect to a natural indefinite inner product, called the Hitchin index.

(v) The index provides a lower bound for the Einstein co-index.

(vi) The index corresponds to the count of solutions to the eigenvalue problem

$$\mathcal{E}_\lambda = \left\{(\beta, \gamma) \in \Omega^2_8 \times \Omega^3_{12} \;\middle|\; d\beta = \frac{\lambda}{4}\gamma \ \ , \ \ d^*\gamma = \frac{\lambda}{3}\beta\right\}$$

for $\lambda \in (0, 12)$.

(vii) There is an explicit connection between the spectrum of the second variation of $\mathcal{Q}$ and the spectrum of the second variation of Hitchin's original functional.

Theorem (Prop. 8.6, Prop. 8.7, Lemma 8.14 & Prop. 8.15)**.** *Let Σ^7 be a closed spinnable 7-manifold, with a given orientation. Consider $\mathcal{V} = \{\psi \in \Omega^4(\Sigma) \mid \psi \text{ is stable and exact}\}$. The new Hitchin functional $\mathcal{T} : \mathcal{V} \to \mathbb{R}$ satisfies the following:*

(i) Critical points are nearly parallel G_2 structures, up to orientation.

(ii) The Einstein–Hilbert action is a lower bound for $\mathcal{T}$. The two only coincide along rescalings of nearly parallel G_2 structures.

(iii) The associated gradient flow is third-order, in particular, not parabolic.

(iv) Critical points have a well-defined index with respect to a natural indefinite inner product, called the Hitchin index.

(v) The index provides a lower bound for the Einstein co-index.

(vi) The index corresponds to the count of solutions to the eigenvalue problem

$$\mathcal{E}_\lambda = \left\{\chi \in \Omega^4_{27} | \ d * \chi + \lambda\chi = 0\right\}$$

for $\lambda \in (0, 4)$.

[1] In general, a nearly Kähler structure is viewed as a pair of forms $(\omega, \rho) \in \Omega^2 \times \Omega^3$ satisfying a PDE. In our case, the 3-form ρ is determined by the 2-form ω.

(vii) *There is an explicit connection between the spectrum of $\delta^2\mathcal{T}$ and the spectrum of the second variation of Hitchin's original functional.*

The main motivation for introducing new Hitchin functionals and defining the Hitchin index was to provide a bridge between variational methods in G_2 and Spin(7) geometries and analytic contributions to moduli space formulas. More precisely, the idea was to relate the Hitchin index to the CS/AC analytic term appearing in the expected dimension formula for moduli spaces. From this perspective, the Hitchin index serves as a measure of instability. It plays a part in G_2 and Spin(7) geometries similar to Joyce's stability index for special Lagrangians and Lotay's index for coassociatives. In each of these settings, the index captures the instability of its singular model; model structures with a higher index should be regarded as less generic, appearing only in higher–codimension strata of the boundary of the moduli space. Thus, the Hitchin index, a variational invariant, becomes an object directly relevant to deformation problems and to the formulation of counting invariants.

Chapter III - A Cohomogeneity one computation

The final part of this thesis focuses on the study of the Hitchin index, introduced in Chapter II, in the context of the cohomogeneity one examples constructed by Foscolo and Haskins [FH17]. Building on their framework, we aim to construct cohomogeneity one solutions to the PDE associated with the Hitchin index. After recalling the general setup of Foscolo and Haskins, we derive the ODE system (110) obtained from our PDE under the cohomogeneity one ansatz. We then establish the following existence result:

Theorem (Thm. 11.7)**.** *Let $a, b > 0$, and consider the nearly Kähler halves Ψ_a and Ψ_b of Foscolo and Haskins [FH17], with singular orbits S^2 and S^3, respectively. Then, for every $\Lambda \in (0,\infty)$, there exists a unique (up to scale) solution to the ODE system* (110) *on the nearly Kähler half Ψ_a (resp. Ψ_b). Moreover, this solution depends continuously on the parameters a (resp. b) and Λ.*

By analysing the global behaviour of the ODE and applying an intermediate value argument, we prove:

Theorem (Thm. 12.6)**.** *The Hitchin index of the inhomogeneous nearly Kähler structure on $S^3 \times S^3$ is bounded below by 1. The Einstein co-index is bounded below by 4.*

The proof of this result is quite intricate, since the ODE, though linear, depends on the underlying nearly Kähler structure, which is itself determined by non-explicit functions.

The question of whether these bounds are sharp remains open and out of reach beyond the use of numerical methods. Similarly, there appears to be no clear path for treating the inhomogeneous S^6 case beyond numerical methods. This result stands in sharp contrast with the homogeneous nearly Kähler manifolds, all of which have vanishing Hitchin index. It remains an open question whether these bounds are sharp, and whether similar methods can be successfully applied to the inhomogeneous nearly Kähler structure on S^6, which appears to require a new analytic approach or numerical tools.

Outlook

The developments in this thesis, ranging from Floer-type functionals on cones and weighted analysis for conical instantons to the introduction of new Hitchin functionals and the Hitchin index, provide a comprehensive toolkit for making gauge theory and special holonomy metrics with singularities more analytically tractable. These methods are expected to play a key role in refined glueing constructions. We briefly list some open questions that naturally arise from this work.

On the gauge theory side, it is crucial to understand further the geometric and analytic constraints imposed by conical singularities, the cone condition, as well as to understand stability conditions for such singularities. Further understanding the virtual dimension of Spin(7)-instantons, in analogy with the algebraic side of Hermitian–Yang–Mills connections, remains a central problem. Constructing explicit examples of instantons with ICS via glueing constructions is a promising avenue, through glueing anti-self-dual connections along associative 3-folds or Cayley 4-folds, with singularities expected to appear at points where the Fueter section vanishes transversely, extending the early work of [Wal17].

In the direction of special holonomy and Hitchin-type functionals, one may ask whether a natural functional exists in the Calabi–Yau setting. More broadly, extending the work of Karigiannis–Lotay to the Spin(7) case and to manifolds having both CS and AC ends would advance understanding of the Hitchin functional and its spectral flow properties, while potentially offering new insights into the treatment of unstable singularities.

Finally, several natural directions emerge for the study of eigenvalue problems under a cohomogeneity one symmetry assumption. One may further investigate the stability of the Foscolo–Haskins nearly Kähler examples, potentially avoiding reliance on numerical methods, and more generally, extend the approach to other cohomogeneity one structures, such as Einstein manifolds and minimal submanifolds.

Gauge Theory

In 1997, Donaldson and Thomas [DT98] (cf. [DS11]) proposed a program to construct geometric invariants of exceptional holonomy manifolds in dimensions 7 and 8 using ideas from gauge theory in dimensions 3 and 4. The original proposal stems from Thomas' PhD thesis, where he introduced the Donaldson–Thomas (DT) invariants for compact Calabi–Yau threefolds as a holomorphic analogue of the three-dimensional Casson invariant.

The main obstacle to Donaldson and Thomas' proposal is that we are far from understanding how to compactify the moduli spaces of instantons in high dimensions. In particular, this is why DT invariants were defined using algebraic tools rather than analytic ones. In 2001, Tian [Tia00] published an influential paper outlining the main challenges one faces when constructing suitable compactifications for these moduli spaces: bubbling and singularity formation. Bubbling corresponds to an L^2-energy concentration. According to the work of Uhlenbeck (cf. [Weh04]), this process occurs in codimension four. In [Tia00], Tian showed that bubbling occurs along Ω-calibrated currents.

By singularity formation, we mean any other process for which the limit (up to a subsequence) of a sequence of connections $\{A_n\}_n$ whose curvature is bounded in L^2-norm might not exist. In [Tia00], Tian proved that this phenomenon must occur in codimension at least 5 for a sequence of Yang–Mills connections and conjectured that the codimension bound can be improved to 6 in the Ω-ASD case. We will concern ourselves with the case where singularities arise in codimension n, i.e. point singularities. In 2003, Baozhong Yang, a student of Tian, proved the first results on how these singularities behave.

Theorem 0.1 ([Yan03, Thm. 1 & 2])**.** *Let (M^n, g) be a complete Riemannian manifold with $n \geq 5$. Let A be a smooth, stationary Yang–Mills connection on the bundle E over $M \setminus x_0$ and let $E_\infty \to S^{n-1}$ be the induced bundle by the restriction of E in a neighbourhood of x_0. Assume that there exists a neighbourhood U of x_0 and a constant $C > 0$ such that*

$$|F_A|(x) \leq Cr^{-2} \ ,$$

where $r = \operatorname{dist}(x, x_0)$, the distance to the singularity. Then, the tangent cone of A exists and is unique and up to gauge. That this, there exists a smooth Yang–Mills connection A_∞ and a gauge transformation g_∞ on E_∞ around x_0 such that

$$||(g_\infty^*(A)(r) - A_\infty)\,||_{C^k(S^{n-1})} \leq C_k |log(r)|^\alpha$$

for some $C_k > 0$ and α depending on A. Furthermore, assume that A_0 is integrable, in the sense that every infinitesimal deformation of A_0 belongs to a one-parameter family of Yang-

Mills connections. Then one can drop the requirement of A being stationary, and the stronger estimate

$$|(g_\infty^*(A)(r) - A)|_{C^k(S^{n-1})} \leq C_k r^{\alpha-1-k}$$

holds, for some different $C_k > 0$ and α.

In contrast to the general setting, the behaviour of singularities of Hermitian Yang–Mills (HYM) is far better understood, due to the deep interaction between differential and algebraic geometry. The foundational result of Bando and Siu[BS94] shows that conically singular HYM connections correspond precisely to reflexive sheaves, a natural class of coherent sheaves that extend holomorphic bundles. Their result extends the well-known analytic–algebraic Donaldson–Uhlenbeck–Yau correspondence. Consequently, moduli spaces of smooth HYM connections can be compactified via reflexive sheaves.

The theory of Hermitian Yang–Mills (HYM) connections on reflexive sheaves has been extensively developed, particularly through the works of Chen and Sun [CS21] and Jacob, Sá Earp, and Walpuski [JSW18], among others. We refer the reader to their contributions for a comprehensive treatment of this subject. In light of these results, our focus shifts to the study of G_2 and Spin(7)-instantons, where the analytical framework is less well understood and there is no algebraic correspondence available.

It is worth noting that the natural inclusion SU(4) $\subseteq$ Spin(7) requires any theory of Spin(7)-instantons to be compatible with the case of reflexive sheaves on Calabi–Yau fourfolds, which leads to the consideration of $DT4$-invariants in the algebraic geometry literature. These sheaves provide a rich and well-understood class of examples against which we can test our constructions. However, we prove (cf. Proposition 1.7 and Corollary 4.2) that topological rigidity results imply we cannot get genuine Spin(7)-instantons on these singular holomorphic bundles associated with the reflexive sheaves.

1 Instantons

Let (M^n, g) be an n-dimensional Riemannian manifold without boundary, G a compact Lie group with Lie algebra $\mathfrak{g}$, and let $\pi : P \to M$ a principal G-bundle. We assume throughout that $G = \mathrm{U}(n)$ for simplicity. We denote by $\mathfrak{g}_P$ the vector bundle with fibre $\mathfrak{g}$, associated to P via the adjoint representation.

Recall that a connection H on P is the choice of a complement to the fibres of the tangent space of the G-orbits, i.e. a choice of splitting of the short exact sequence of vector bundles.

$$0 \to TG \to TP \to TM \to 0 \; .$$

Alternatively, a connection can be thought of as a 1-form $\theta \in \Omega^1(P, \mathfrak{g}_P)$ satisfying some G-equivariance properties, and such that $H = \ker\theta$.

One can ask whether a connection H is integrable, i.e. whether a submanifold in P whose tangent space is H locally exists. By Frobenius' theorem, the integrability failure is measured by the curvature 2-form $F = d_h\theta := \theta \circ h$, where $h : TP \to H$ is the projection map.

Theorem 1.1 (Structure equation and Bianchi identity). *Let θ be a connection in P and F its curvature. Then, it satisfies*

$$F = d\theta + \frac{1}{2}[\theta, \theta] \ , \tag{4}$$

where $[\theta, \theta](X, Y) = [\theta(X), \theta(Y)]$ and the bracket is induced by Lie bracket in $\mathfrak{g}$ under the pointwise identification $TG \cong \mathfrak{g}$. Moreover, F is covariantly closed, i.e. $d_h F = 0$.

Equation (4) is known as the Maurer–Cartan Equation or structure Equation. The condition of the curvature being closed is known as the (differential) Bianchi identity.

The G-equivariance of the connection and curvature forms allows us to (locally) identify them with objects in the base M. Thus, we have the following:

Proposition 1.2. *Let $U \subseteq M$ an open set over which the principal bundle P trivialises.*

(i) A connection $\theta \in \Omega^1(P)$ can be identified with a linear connection on the associated bundle $\mathfrak{g}_P$. Thus, it can locally be identified with a 1-form $A \in \Omega^1(U, \mathfrak{g}_P)$, where U is as above and the connection acts on $\Omega^k(U, \mathfrak{g}_P)$ as $d_A = d + A \wedge$.

(ii) The space of connections is naturally an affine space $\mathcal{A}$ modelled on $\Omega^1(M, \mathfrak{g}_P)$.

(iii) Under this identification, the curvature F is mapped to a (globally defined) 2-form $F_A \in \Omega^2(M, \mathfrak{g}_P)$.

The space of connections carries a natural action by principal bundle automorphisms. These form a group under composition, called the gauge group of P and denoted by $\mathcal{G}$. In the same spirit as above, its Lie algebra can be identified with $\Omega^0(M, \mathfrak{g}_P)$. For $g \in \mathcal{G}$, the action on a connection is given by $d_{A \cdot g}\alpha = g^{-1} d_A (g\alpha)$. In particular, the curvature 2-form F_A transforms as a tensor under the gauge group action, $F_{A \cdot g} = g^{-1} F_A g$. Finally, recall the Yang–Mills functional

$$\begin{aligned} \mathcal{YM} : \mathcal{A} &\to \mathbb{R} \\ A &\mapsto \frac{1}{2} \int_M \mathrm{Tr}(F_A \wedge *F_A) \end{aligned}$$

from the introduction. Its first variation is given by

$$\frac{\delta}{\delta\alpha} \mathcal{YM} = \int_\Sigma \mathrm{Tr}\left(d_A \alpha \wedge *F_A\right) \ , \tag{5}$$

By Stokes' theorem, the corresponding Euler-Lagrange equation is

$$d_A^* F_A = 0 \ , \tag{6}$$

called the Yang–Mills equation. The second variation is

$$\frac{\delta^2}{\delta\alpha\delta\beta} YM = \int_\Sigma \mathrm{Tr}\left(d_A\alpha \wedge *d_A\beta + \alpha \wedge \beta \wedge *F_A\right) = \langle \beta, (d_A^* d_A \alpha + \{F_A, \alpha\}) \rangle \ , \tag{7}$$

where $\{F_A, \alpha\} = *[*F_A, \alpha]$. The second variation operator $S_A(\alpha) = d_A^* d_A \alpha + \{F_A, \alpha\}$ is called the stability operator (cf. [BL81]).

The Yang–Mills functional is invariant under gauge transformations, which implies that both the Yang–Mills equation and its associated stability operator possess an infinite-dimensional kernel arising from the action of the gauge group. To obtain a well-posed variational problem, it is necessary to restrict our attention to a suitable complement of the gauge orbits. The standard approach is to impose a gauge-fixing condition, selecting a representative in each gauge orbit. The standard choice is the Coulomb gauge, where one restricts to the L^2-orthogonal complement of the infinitesimal gauge orbit $T_A\mathcal{G} \subseteq T_A\mathcal{A} \cong \Omega^1(M, \mathfrak{g}_P)$. Explicitly, this corresponds to imposing the condition $d_A^*\alpha = 0$ for $\alpha \in \Omega^1(M, \mathfrak{g}_P)$.

More conceptually, this procedure corresponds to working locally in the quotient space $\mathcal{A}/\mathcal{G}$. The slice theorem (cf. [FU84, Thm. 3.2.]) guarantees that near any irreducible connection A, the quotient $\mathcal{A}/\mathcal{G}$ is modelled locally as the product $\ker(d_A^*) \times \mathcal{G}$. We prove a slice theorem adapted to the conically singular case in Section 5.

Under the Coulomb gauge, the Yang–Mills equation and the stability operator become elliptic. In particular, the operator

$$\widehat{S}_A(\alpha) = S_A(\alpha) + d_A d_A^*\alpha = \Delta_A\alpha + \{F_A, \alpha\}$$

is strongly elliptic, so its eigenvalues are discrete, have finite multiplicity and are bounded below. In particular, the finite-dimensional notions of index and nullity generalise:

$$\mathrm{Ind}_A = \sum_{\lambda<0} \dim \left\{\alpha \in \Omega^1(\mathfrak{g}_P) |\ \widehat{S}_A(\alpha) = \lambda\alpha\right\} \tag{8a}$$

$$\mathrm{Nul}_A = \dim \left\{\alpha \in \Omega^1(\mathfrak{g}_P) |\ \widehat{S}_A(\alpha) = 0\right\} \tag{8b}$$

Instantons are an attempt to reduce the Yang–Mills equation above from a second-order PDE to a first-order PDE by exploiting some geometrical structure of the underlying base manifold (M, g). They were initially considered in the physics literature in the 4-dimensional case. In this case, the instanton equation reads $F_A = \pm * F_A$ and is known as the (anti)-self-dual equation. The study of solutions to the ASD equation led to numerous breakthroughs in low-dimensional topology in the 1980s through the works of Donaldson, Taubes, and Uhlenbeck, among others (cf. [DK90]).

We aim to explore higher-dimensional analogues of the ASD equation. The idea is to find such equations to exploit a reduction of the frame bundle of M. That is, we consider manifolds carrying a G-structure for some suitable group G (cf. Appendix A). This approach was initially systematised by Reyes-Carrión in [Rey98]. Harland and Nölle introduced a different approach using spin geometry in [HN12]. We follow the former approach throughout. First, we need the following technical result.

Proposition 1.3. *Let H be a semisimple subgroup of $\mathrm{SO}(n)$ with connected normaliser $N(H)$. Consider M^n a smooth manifold carrying an $N(H)$-structure. Then, there is a splitting*

$$\Lambda^2 T^*M = \mathfrak{h} \oplus \mathfrak{h}^\perp ,$$

where $\mathfrak{h}$ is the associated $N(H)$-vector bundle with fibres the Lie algebra of H.

Proof. The Lie algebra of the normaliser $N(\mathfrak{h})$ splits as the direct sum of $\mathfrak{h}$ and its centraliser $C(\mathfrak{h})$. Since $N(H)$ is connected, its action preserves this decomposition. □

The idea is to consider connections whose curvature F_A is a section of $\mathfrak{h} \otimes \mathfrak{g}_P$. In other words, we are interested in solving the equation $\pi^\perp(F_A) = 0$, where $\pi^\perp : \Lambda^2 \to \mathfrak{h}^\perp$. Thus, we need to characterise the subbundles $\mathfrak{h}$ and $\mathfrak{h}^\perp$. Since we assumed H is semisimple, its Killing form is an element of $S^2\mathfrak{h}$, invariant under $N(H)$, so it extends to a section $S^2(\mathfrak{h}) \subseteq S^2\Omega^2(M)$. Therefore, it naturally defines a 4-form Ω by composing with the alternating map $\mathrm{Alt} : S^2\Omega^2(M) \to \Omega^4(M)$.

We now assume that H is simple, rather than just semisimple. The discussion could be easily adapted to the semisimple case. However, we are not aware of any case of interest where the group H is semisimple but not simple.

Given the 4-form Ω there is an induced $N(H)$-equivariant map on 2-forms $\Lambda_\Omega(\beta) = *(\beta \wedge *\Omega)$. Since H is simple, by Schur's Lemma, the subbundle $\mathfrak{h}$ is an eigenbundle of $\Lambda_\Omega(\beta)$ of eigenvalue $\mu \in \mathbb{R}$. Assuming $\mu \neq 0$, we can rescale Ω to ensure $\mu = -1$ as needed. In all cases of interest, the following is true.

Lemma 1.4. *Let H be a group from Table 4. Then,*

(i) The bundle $\mathfrak{h}$ is the unique (-1)-eigenbundle of Λ_Ω.

(ii) All remaining eigenbundles of Λ_Ω have non-negative eigenvalues.

The proof is a case-by-case linear algebra exercise that we omit.

n	H	K	Ω	Geometry
4	SU(2)	SO(4))	vol	-
$2k$	SU(k)	U(k)	$\frac{\omega^2}{2}$	Almost hermitian
$4k$	Sp(k)	Sp(k)	$\frac{\omega_1^2+\omega_2^2+\omega_3^2}{6}$	Almost hyperhermitian
$4k$	Sp(k)	Sp(k)Sp(1)	Ω	Almost quaternionic hermitian
7	G_2	G_2	ψ	G_2
8	Spin(7)	Spin(7)	Φ	Spin(7)
$2k+1$	SU(k)	$\mathbf{1} \times \mathrm{U}(k)$	$\frac{\omega^2}{2}$	Transverse almost hermitian
$4k+3$	Sp(k)	$\mathbf{1}_3 \times \mathrm{Sp}(k)$	$\frac{\omega_1^2+\omega_2^2+\omega_3^2}{6}$	Transverse almost hyperhermitian

Table 4: Admissible geometries

More generally, we are unaware of any general criteria or argument that establishes which geometries the statement above holds for. Now that we have our candidate instanton equation, we can verify under which conditions it implies the Yang–Mills equation. Let A be a connection whose curvature satisfies the equation

$$F_A + *(F_A \wedge *\Omega) = 0 \; . \tag{9}$$

By the Bianchi identity, the Yang–Mills equation for a connection satisfying Equation (9) reduces to

$$d_A * (F_A) = -d_A(F_A \wedge *\Omega) = -F_A \wedge d * \Omega \; .$$

Therefore, we define

Definition 1.5. *Let* (M, g, Ω) *be a Riemannian manifold endowed with an* $N(H)$*-structure with associated* 4*-form* Ω*. Moreover, assume that*

$$\Gamma(\mathfrak{h}) \subseteq \left\{\beta \in \Omega^2 | \ \beta \wedge d * \Omega = 0\right\} \ . \tag{10}$$

Then Equation (9) *will be referred to as the* Ω*-instanton equation. A connection satisfying the* Ω*-instanton equation will be called an* Ω*-instanton.*

In particular, if the 4-form is coclosed, Lemma 1.4 implies

Proposition 1.6. *If* H *is one of the groups of Table 4 and* Ω *is coclosed,* Ω*-instantons are global minima of the Yang–Mills functional.*

To prove this, it is convenient to introduce the Ω-charge of the principal bundle P:

$$c_\Omega(P) = -\frac{1}{2}\int_M \mathrm{Tr}(F_A \wedge F_A) \wedge *\Omega \ .$$

Chern-Weil theory implies that $c_\Omega(P) = \frac{1}{16\pi^2}\left\langle (c_1^2(P) - 2c_2(P)) \cup [*\Omega], [M]\right\rangle$ is a topological quantity.

Proof. Let $-1 = \lambda_0 < \lambda_1 < ... < \lambda_k$ be the eigenvalues of $\Lambda_\Omega = *(\cdot \wedge *\Omega)$ by virtue of Lemma 1.4 and $\pi_i : \Lambda^2 T^*M \to \Lambda^2 T^*M$ the projection to the eigenbundle corresponding to λ_i. Then

$$c_\Omega(P) = \langle F_A, *(F_A \wedge *\Omega)\rangle_{L^2} = \sum_i \lambda_i ||\pi_i(F_A)||^2_{L^2} \geq -||F_A||^2_{L^2} \ .$$

Thus, $||F_A||^2_{L^2} \geq c_\Omega(P)$, with equality if and only if $\pi_i(F_A) = 0$ for $i \geq 1$, as needed. □

If the manifold carries multiple $N(H)$-reductions, we can define the charge difference, allowing us to obtain topological rigidity statements. The idea was first introduced by Lewis in his PhD thesis [Lew98]. Assume that (M, g) admits two compatible reductions, with groups $H_1 \subseteq H_2$ from Table 4 and such that $d(\Omega_2 - \Omega_1) = 0$. We define the relative charge of a bundle as

$$D(P) = \int_M \mathrm{Tr}(F_A \wedge F_A) \wedge (\Omega_2 - \Omega_1) \ . \tag{11}$$

Once more, Chern-Weil theory guarantees that $D(P)$ is a well-defined topological quantity. If both Ω_i are closed, then D is simply the difference of charges $c_{\Omega_2}(P) - c_{\Omega_1}(P)$. As a corollary, we have

Proposition 1.7. *Let* P *be a principal bundle such that* $D(P) = 0$*. Then,* Ω_2*-instantons and* Ω_1*-instantons coincide.*

Proof. Since we assumed the reductions were compatible, all H_2-representations carry an induced H_1-representation, and the symmetric operator $d(\beta) = *(\beta \wedge *(\Omega_2 - \Omega_1))$ can be decomposed into irreducible H_1-representations $d(\beta) = \sum_i \mu_i \pi_i(\beta)$.

If β is a (-1)-eigenform for Ω_2, then the μ_i that contribute to $d(\beta)$ are all strictly positive, since (-1) is the smallest eigenvalue for both Λ_{Ω_i}, by (ii) in Lemma 1.4.

Thus, if $D(P)=0$ and A is an Ω_2-instanton, we have

$$0 = D(P) = \sum_i \mu_i ||\pi_i(F_A)||^2 \geq 0 \ ,$$

so $||\pi_i(F_A)||^2 = 0$, and A must be an Ω_1-instanton. The converse is straightforward. $\square$

These topological rigidity statements are common in the literature, and they all follow a similar argument, for instance [DW19, Prop. 7.1] and [Joy00, Thm. 10.6.1] in the context of calibrated submanifolds and special holonomy, respectively. We give two applications of Proposition 1.7. The first one is straightforward:

Corollary 1.8 ([Lew98, Thm. 3.1])**.**

(i) Let (M^8, g, ω, ρ) be a Calabi-Yau fourfold, and $E \to M$ a vector bundle admitting a Hermite–Yang–Mills (HYM) connection. Then all Spin(7)*-instantons on E are HYM.*

(ii) Let (M^{4n}, g, ω_i) be a hyperkähler manifold and $E \to M$ a vector bundle admitting a hyper-Hermite–Yang–Mills (hHYM) connection. Then all HYM connections on E are hHYM.

Proposition 1.9. *Let (M^n, g, Ω) be a manifold carrying an admissible $N(H)$ structure with associated* 4*-form Ω closed, and let $E \to M$ a Hermitian vector bundle admitting an Ω-instanton. Let T^k be a flat torus of dimension k and denote by $\pi : M \times T^k \to M$ the trivial fibration. Assume the product metric on $M \times T$ admits an $N(\widehat{H})$-structure with characteristic* 4*-form $\widehat{\Omega}$, compatible with $N(H)$ and such that the difference $\widehat{\Omega} - \pi^*(\Omega)$ is exact.*
Then the moduli spaces of irreducible instantons are related by

$$\mathcal{M}^{irred}_{\widehat{\Omega}}(\pi^*(E)) \cong \mathcal{M}^{irred}_{\Omega}(E) \times T^* \ ,$$

where T^* denotes the torus dual to T. In particular, to every $\widehat{\Omega}$-instanton on $\pi^*(E)$ we can associate an Ω-instanton on $E \to M$.

A particular case of this Proposition is the main topic of Yuanqi Wang's paper [Wan18b], where he considers the cases $CY3 \times S^1$ and $G_2 \times S^1$. Other cases of interest, that have not appeared in the literature as far as we are aware, are $CYn \times T^2$ and $K3 \times T^3$.

Proof. Consider $\widehat{\Omega}$ and $\pi^*(\Omega)$-instantons on $M \times T^k$. Since $\widehat{\Omega} - \pi^*(\Omega)$ is assumed to be exact, $D(\pi^*(E))$ vanishes and Proposition 1.7 implies that the two instanton conditions are equivalent, so $\mathcal{M}_{\widehat{\Omega}} \cong \mathcal{M}_{\pi^*(\Omega)}$.
Let $d\theta_i \in \Omega^1(T)$ denote a basis of parallel 1-forms on the torus T^k. We have an injective map $\mathcal{M}_\Omega(E) \to \mathcal{M}_{\pi^*(\Omega)}(\pi^*(E))$ induced by pullback. More concretely, we have the map

$$\begin{aligned} \mathcal{M}^{irred}_{\Omega}(E) \times T^* &\to \mathcal{M}^{irred}_{\pi^*(\Omega)}(\pi^*(E)) \\ (A, \mu_i) &\mapsto \pi^*(A) + i \sum_{i=1}^k \mu_i d\theta_i \ . \end{aligned}$$

We construct an inverse to this map, assuming the connection A is irreducible. Let A be an irreducible $\pi^*(\Omega)$-instanton and ξ_i be the dual vector fields to $d\theta_i$, which are Killing since $d\theta_i$ are parallel and the torus is flat. We have that $\xi_i \lrcorner F_A$ from $\pi^*(\Omega)$-instanton equation.

Let $\widetilde{\xi_i}$ the horizontal A-lift of ξ_i, each of which generates an $\mathbb{R}$-action on P, so together they generate an abelian subgroup K inside the gauge group $\mathcal{G}$ since $[\widetilde{\xi_i}, \widetilde{\xi_j}] = F_A(\widetilde{\xi_i}, \widetilde{\xi_j}) = 0$. The group K preserves the connection A since

$$\mathcal{L}_{\widetilde{\xi_i}} A = \widetilde{\xi_i} \lrcorner F_A = 0 \; . \tag{12}$$

We need to understand the obstruction to the K-action inducing a T^k-action. For each of the generators $\widetilde{\xi_j}$, we have an associated gauge transformation[2] $g^j_A \in \mathcal{G}$ fixing A, which will act by multiplication $e^{i\mu_j}$ for $\mu \in \mathbb{R}^k$. The connection $\widetilde{A} = A - i\sum_{j=1}^k \mu_i d\theta_i$ also satisfies $\mathcal{L}_{\widetilde{\xi_i}} \widetilde{A} = 0$, and for each j we have $g_{\widetilde{A}} = \mathrm{id}_E$, so $\widetilde{A}$ descends to a connection on $E \to M$, as needed. □

1.1 Instanton deformation theory

We now turn to the deformation theory of Yang–Mills connections and instantons. Let A be a Yang–Mills connection (respectively an instanton). We want to understand under what conditions, for a perturbation α, the connection $A + \alpha$ is again a Yang-Mills connection (resp. an instanton)?

The general approach is to apply the Implicit Function Theorem (IFT) for Banach Manifolds (cf. Theorem 3.13) after choosing appropriate Banach completions of the domain and co-domain.

In the Yang–Mills case, we are interested in the map $f(A) = d_A^*(F_A)$. At a Yang–Mills connection, we can use the structure equation (4) to write f as

$$f(A+\alpha) = d_A^*(F_A + d_A\alpha + \frac{1}{2}[\alpha,\alpha]) = d_A^* d_A \alpha + \frac{1}{2} d_A^*([\alpha,\alpha]) \; ,$$

where the linear and quadratic terms are clear. To apply the IFT, we need the linear part to be Fredholm, which is false due to the gauge invariance of f. However, we have

Lemma 1.10. *Let A be a Yang–Mills connection. The map $d_A^* d_A \alpha$ fits in the elliptic complex*

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^* d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^*} \Omega^0(\mathfrak{g}_P) \to 0 \; .$$

The proof is standard and well-known. Notice that this complex is self-dual, so that any deformations will be a priori obstructed, and so one would expect generic moduli spaces to consist of points. However, if A happens to be an instanton, we can generally hope to have a moduli space of positive virtual dimension, as we will discuss now.

In the instanton case, we want to consider $f(A) = \pi^\perp(F_A)$. As before, the structure equation implies that, at an instanton A, we have

$$f(A+\alpha) = \pi^\perp(F_A + d_A\alpha + \frac{1}{2}[\alpha,\alpha]) = \pi^\perp(d_A\alpha) + \frac{1}{2}\pi^\perp([\alpha,\alpha]) \; .$$

Thus, we must prove that the map $\pi^\perp(d_A\alpha)$ is Fredholm. We have the following result, stemming from the work of Reyes-Carrión [Rey98]:

[2]This corresponds to the notion of broken gauge transformation in [Wan18b]

Theorem 1.11. *Let (M^n, g, Ω) be an n-dimensional Riemannian manifold with an $N(H)$-structure, where H satisfies Lemma 1.4, and corresponding associated 4-form Ω. Denote by $\widehat{\mathfrak{h}} \subseteq \Omega^*(M)$ the (graded) ideal generated by sections of $\mathfrak{h}$ and assume it is a differential ideal, i.e. $d\widehat{\mathfrak{h}} \subseteq \widehat{\mathfrak{h}}$. Then, for any Ω-instanton A:*

(i) The differential of $f(A) = \pi^\perp(F_A)$ is given by $df|_A(\alpha) = \pi^\perp(d_A\alpha)$. It fits into the complex

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{\pi^\perp \circ d_A} \Omega^2/\mathfrak{h}(\mathfrak{g}_P) \xrightarrow{\pi^\perp \circ d_A} \dots \tag{13}$$

(ii) The complex is elliptic.

The first claim is a main result of Reyes-Carrion [Rey98], whose proof we will sketch. The second claim is proved in many cases of interest in the literature (cf. [Rey98]) by a case-by-case computation. We present a short and simple general proof that, to the best of our knowledge, has not appeared in the literature before.

Proof. The first claim is precisely the discussion from Section 2 in [Rey98]. The idea is that the differential condition $d\widehat{\mathfrak{h}} \subseteq \widehat{\mathfrak{h}}$ is necessary and sufficient to have a short exact sequence of complexes

$$0 \to (\widehat{\mathfrak{h}}, d) \to (\Omega^*(M), d) \to (\Omega^*(M)/\widehat{\mathfrak{h}}, \pi^\perp \circ d) \to 0\ . \tag{14}$$

Explicitly, this corresponds to

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \Gamma(\mathfrak{h}) & \longrightarrow & \Gamma(\mathfrak{h}\wedge\Lambda^1) & \longrightarrow & \Gamma(\mathfrak{h}\wedge\Lambda^2) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & \Omega^4 & \longrightarrow & \dots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{\pi^\perp\circ d} & \Gamma(\mathfrak{h}^\perp) & \xrightarrow{\pi^\perp\circ d} & \Gamma((\mathfrak{h}\wedge\Lambda^1)^\perp) & \xrightarrow{\pi^\perp\circ d} & \Gamma((\mathfrak{h}\wedge\Lambda^2)^\perp) & \xrightarrow{\pi^\perp\circ d} & \dots
\end{array} \tag{15}$$

Suppose one twists the previous complexes by an Ω-instanton A. In that case, the bottom complex remains a complex, since $d_A^2 = F_A \in \Gamma(\mathfrak{h}) \subseteq \widehat{\mathfrak{h}}$ by assumption, and is precisely the complex we are interested in. We refer the reader to [Rey98] for further details.

To show ellipticity, we need to show that the complex of vector spaces $\big(\Lambda^*(T^*M)/\hat{\mathfrak{h}}, \pi^\perp(\xi\wedge\cdot)\big)$ is exact whenever $\xi \neq 0$, where $\hat{\mathfrak{h}}$ is the ideal generated by $\mathfrak{h}$. From the short exact sequence of complexes (14), we have

$$0 \to (\hat{\mathfrak{h}}, \xi\wedge\cdot) \to (\Lambda^*T^*M, \xi\wedge\cdot) \to \big(\Lambda^*T^*M/\hat{\mathfrak{h}}, \pi^\perp(\xi\wedge\cdot)\big) \to 0\ .$$

Using the 2-to-3 property, the claim is equivalent to proving that the leftmost term, $(\hat{\mathfrak{h}}, \xi\wedge\cdot)$, is exact. By induction, it suffices to check that the map $\mathfrak{h} \xrightarrow{\xi\wedge\cdot} \Lambda^3T^*M$ is injective for $\xi \neq 0$, i.e. $\mathfrak{h}$ has no decomposable elements.

Indeed, let $\beta = \xi\wedge\alpha \in \mathfrak{h}$ decomposable, so $\beta\wedge\beta = 0$. Since $\mathfrak{h}$ is the (-1)-eigenbundle of Λ_Ω, we have

$$0 = \beta\wedge\beta\wedge*\Omega = \langle\beta, \Lambda_\Omega(\beta)\rangle = -||\beta||^2\ . \qquad \square$$

The remaining question is under what conditions on the $N(H)$-structure do we have

(i) that the ideal $\widehat{\mathfrak{h}}$ is differential and,

(ii) that every $\beta \in \mathfrak{h}$ satisfies $\beta \wedge d * \Omega = 0$,

ensuring instantons are Yang–Mills connections. The following is an immediate first result:

Proposition 1.12. *If (M^n, g, Ω) has holonomy contained in $N(H)$, conditions (i) and (ii) above are satisfied.*

Proof. The holonomy condition implies that Ω is parallel, so $d * \Omega = \mathrm{Alt} \circ \nabla(*\Omega) = 0$, and condition (10) is trivially satisfied. Similarly, the bundle $\mathfrak{h}$ will be parallel, and differentiability of the ideal $\widehat{\mathfrak{h}}$ follows by the same argument. □

For a systematic approach to these questions, it is convenient to consider the intrinsic torsion associated with the $N(H)$-structure, discussed in the Appendix A. Using Lemma A.1, Reyes-Carrión proves

Proposition 1.13 ([Rey98, Prop. 8])**.** *The obstruction to the ideal $\widehat{\mathfrak{h}}$ being closed under the exterior derivative is given by the map*

$$\mathcal{O} : \Gamma(\mathfrak{h}) \xrightarrow{\nabla} \Omega^1 \otimes \Omega^2 \xrightarrow{\mathrm{Alt}} \Omega^3 \xrightarrow{\pi^\perp} \Gamma\left((\mathfrak{h} \wedge \Lambda^1)^\perp\right) ,$$

which depends only on the intrinsic torsion τ of the $N(H)$-structure.

We conclude the section by considering how the deformation complex looks in three special holonomy instances: when the holonomy is contained in Spin(7), G_2, and U(k), the latter being simply Kähler manifolds.

The case of Spin(7)-holonomy was first studied in detail in the PhD thesis of Lewis [Lew98]. Let Φ denote the associated parallel 4-form. The deformation complex is

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{\pi^\perp d_A} \Omega^2_7(\mathfrak{g}_P) \to 0 , \tag{16}$$

where the projection map can be written down explicitly as $\pi^\perp(\beta) = \frac{1}{4}[\beta + *(\Phi \wedge \beta)]$. As already remarked by Lewis, this can be identified with the twisted Dirac operator $\widehat{D}_A$. Since this is a 3-term complex, the virtual dimension of the moduli space of instantons coincides with minus the index of the complex (16).

For the G_2 case, let ψ denote the parallel 4-form. The deformation complex (13) is

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{\pi^\perp d_A} \Omega^2_7(\mathfrak{g}_P) \xrightarrow{\pi^\perp d_A} \Omega^3_1(\mathfrak{g}_P) \to 0 .$$

Using the isomorphisms $\Omega^2_7 \cong \Omega^1$ and $\Omega^3_1 \cong \Omega^0$ (cf. Lemma A.16), one shows it is isomorphic to

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{*(\psi \wedge d_A \cdot)} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^*} \Omega^0(\mathfrak{g}_P) \to 0 . \tag{17}$$

The complex above can be thought of as the linearisation of the G_2-monopole equation

$$*(F_A \wedge \psi) + d_A f = 0 \tag{18}$$

for a connection A and a Higgs field f. Akin to the case of 3-dimensional monopoles, we have

Lemma 1.14. *Let M^7 be a closed 7-manifold and consider $\varphi \in \Omega^3$ a coclosed G_2-structure ($d\psi = 0$) on M. Then the forgetful map $(f, A) \mapsto A$ mapping G_2-monopoles to G_2-instantons is surjective, and its fibre can be identified with infinitesimal automorphisms of A.*

Proof. Acting by d_A^* on the monopole equation, we get $\Delta_A f = 0$. Integrating by parts the condition $\langle \Delta_A f, f \rangle_{L^2} = 0$, we get $d_A f = 0$ as needed. □

Finally, for the Kähler setting, let ω be the Kähler form and $\Lambda : \Omega^k \to \Omega^{k-2}$ the adjoint to wedging with ω, $\Lambda(\beta) = *(\omega \wedge *\beta)$. The deformation complex can then be written as

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{\pi^{(2,0)}(d_A \cdot) + \Lambda d_A} \Omega^{(2,0)+(0,2)}(\mathfrak{g}_P) \oplus \Omega^0(\mathfrak{g}_P) \otimes \langle \omega \rangle \xrightarrow{\pi^{(3,0)}(d_A \cdot)} \dots$$

After complexification and using the Kähler identities, one finds that the complex above is isomorphic to the twisted Dolbeault complex:

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{\overline{\partial}_A} \Omega^1(0,1)(\mathfrak{g}_P) \xrightarrow{\overline{\partial}_A} \Omega^{(0,2)}(\mathfrak{g}_P) \xrightarrow{\overline{\partial}_A} \Omega^{(0,3)}(\mathfrak{g}_P) \to \dots \tag{19}$$

In the context of gauge theory, this first appeared in Kim's PhD thesis [Kim85]. In all the cases above, it will be useful to consider the "rolled-up" operator to the deformation complex, which coincides with the natural Dirac [3] operator twisted by A.

2 Instantons over cones

We now focus on studying the model version of our conically singular problem, instantons on a cone. For the remainder of the section, we take (Σ^{n-1}, g) a closed Riemannian manifold with $n > 4$, $(C(\Sigma), g_C)$ its associated metric cone and P a principal G-bundle over $C(\Sigma)$. We assume throughout that $G = \mathrm{U}(k)$ for simplicity, as before. Since $\mathbb{R}_+$ is contractible, we can identify P with a principal G-bundle over Σ, P_Σ (cf. Section 3) via a principal bundle isomorphism.

Given a connection $\boldsymbol{A}$ on P, we say that $\boldsymbol{A}$ is in temporal gauge if the local connection 1-form of $\boldsymbol{A}$ has no dr component, $\boldsymbol{A}_r = 0$, under the principal bundle isomorphism identification above. Given any connection on P, we can always find a family of bundle isomorphisms for which $\boldsymbol{A}$ is in a temporal gauge, by parallel transporting the gauge along the $\mathbb{R}_+$ direction. To do so, it suffices to find a gauge transformation g solving the ODE

$$g^{-1}\partial_r g + \boldsymbol{A}_r = 0\ .$$

For the remainder of this section, we will assume all connections are in temporal gauge. A connection $\boldsymbol{A}$ in temporal gauge should be interpreted as a family of connections $A_\Sigma(r)$ on P_Σ. The curvature of $\boldsymbol{A}$ in temporal gauge is

$$F_{\boldsymbol{A}} = dr \wedge \frac{\partial A}{\partial r} + F_A\ ,$$

where F_A is the curvature of A as a connection over P_Σ for each r. A connection A on P_Σ can be pulled back to a connection on P. In the temporal gauge, it should be viewed simply as the

[3]Recall that Spin(7) and G_2 manifolds carry a natural spin structure, and Kähler manifolds carry a natural spin^c-structure, so they all carry an associated Dirac operator.

constant path. This connection on the cone will be locally represented by a (-1)-homogeneous 1-form, and its curvature will be a (-2)-homogeneous 2-form, as defined in the introduction (cf. Definition 3.1).

Let us now study the instanton equation in the cone. We will assume that the cone $C(\Sigma)$ carries an admissible frame bundle reduction and that the associated 4-form Ω is co-closed. The reduction of the cone frame bundle induces an $N(H)$-reduction on the link Σ.

Recall that the $(n-4)$- form $*\Omega$ can be written as

$$*\Omega = r^{n-4}\left(\frac{dr}{r}\wedge\Upsilon+\Xi\right) ,$$

with $\Upsilon\in\Omega^{n-5}(\Sigma^{n-1})$ and $\Xi\in\Omega^{n-4}(\Sigma^{n-1})$. The condition $d*\Omega=0$ on the cone reduces to

$$d\Upsilon=(n-4)\Xi \qquad\qquad d\Xi=0 , \tag{20}$$

where the former condition implies the latter since $n>4$.

The Ω-instanton equation $*_C(F_{\boldsymbol{A}}\wedge *_C\Omega)=-F_{\boldsymbol{A}}$ in temporal gauge becomes

$$*(r\partial_r A)=\Xi\wedge F_A , \tag{21a}$$

$$-r\partial_r A\wedge\Xi=*F_A+F_A\wedge\Upsilon . \tag{21b}$$

In particular, we can combine the two to obtain the constraint:

$$F_A+*(F_A\wedge\Upsilon)=*\left(\Xi\wedge*(\Xi\wedge F_A)\right) . \tag{22}$$

This identity is purely linear, forcing a pointwise condition on the curvature. In other words, Equation (22) forces the curvature F_A to be a section of the bundle

$$L=\{\beta\in\Lambda^2(\Sigma)|\ \beta+*(\beta\wedge\Upsilon)=*\left(\Xi\wedge*(\Xi\wedge\beta)\right)\} . \tag{23}$$

We refer to L as the cone bundle and say that its sections satisfy the cone constraint.

We lack a priori geometric intuition for the role this subbundle plays in general and can only justify it through a case-by-case analysis. However, we have the following general observation, using the map:

$$\begin{aligned} L_\Xi : \Lambda^1 &\to \Lambda^2 \\ \alpha &\mapsto *(\Xi\wedge\alpha) \end{aligned}$$

Lemma 2.1. *Assume that Λ^1 does not contain any $N(H)$-representations isomorphic to $\mathfrak{h}$. Then the cone bundle L in (23) satisfies $\mathfrak{h}\subseteq L\subseteq\mathfrak{h}\oplus L_\Xi(\Lambda^1)$.*

Proof. The map $\Lambda_\Xi:\Lambda^2\to\Lambda^1$ given by $\beta\mapsto*(\Xi\wedge\beta)$ is the adjoint to L_Ξ, up to a constant. By the assumption, we have that $\mathfrak{h}$ must be contained in the kernel Λ_Ξ, so $\mathfrak{h}\subseteq L$.

Similarly, consider the decomposition $\Lambda^2=\mathfrak{h}\oplus\ L_\Xi(\Lambda^1)\oplus V$, with $V=[L_\Xi(\Lambda^1)\oplus\mathfrak{h}]^\perp$. Then, it follows that $\Lambda_\Xi(V)=0$. The claim follows from the $N(H)$-equivariance of the maps Λ_Ξ and Λ_Υ. □

In all three cases of interest, one has $L \cong \mathfrak{h} \oplus L_\Xi(\Lambda^1)$.

If we assume that $\boldsymbol{A}$ is a (-1)-homogeneous connection on P, the instanton equations (21) reduce to

$$*(F_A \wedge \Upsilon) = -F_A \ , \tag{24a}$$

$$F_A \wedge \Xi = 0 \ . \tag{24b}$$

Under Equation (24a) and the cone-closedness condition $d\Upsilon = (n-4)\Xi$, Equation (24b) is equivalent to the Yang–Mills equation on Σ:

$$d_A^* F_A = - * d_A * F_A = * d_A (F_A \wedge \Upsilon) = (n-4) * (F_A \wedge \Xi) = 0 \ .$$

In particular, we have that a (-1)-homogeneous solution to Equation (21) is a $*\Upsilon$-instanton.

Using the $(n-4)$-form Ξ on Σ, we can define a Chern-Simons–type functional. Fix a background connection A_0, and for a connection A, let $\alpha = A - A_0$. We define

$$CS_\Xi(A) = \int_\Sigma CS_0(A, A_0) \wedge \Xi = \frac{1}{n-4} \int_\Sigma \mathrm{Tr}\left(F_A^2 - F_{A_0}^2\right) \wedge \Upsilon \ , \tag{25}$$

with $CS_0(A, A_0) = \mathrm{Tr}\left(\alpha \wedge \left(F_A + \frac{1}{2} d_{A_0}\alpha + \frac{1}{3}\alpha \wedge \alpha \wedge \alpha\right)\right)$ the classical Chern-Simons 3-form. Notice that the functional is well-defined modulo gauge, unlike the 4-dimensional (cf. [Don02]) and cylinder case (cf. [DS11]).

The Chern-Simons functional had appeared in the literature for the nearly Kähler ([Xu09]) and nearly parallel G_2 ([Wal22]) cases, but their treatment is not as detailed as the one here. The first result that motivates the interest in this functional is the following:

Proposition 2.2.

(i) The gradient flow of CS_Ξ in (25) is given by $\partial_t A = (-1)^n * (F_A \wedge \Xi)$.

Assume further that the cone constraint (22) is preserved under the gradient flow.

(ii) A connection $\boldsymbol{A}$ on $C(\Sigma)$ is an Ω-instanton if and only if the induced family $A(r)$ on the link Σ evolves under the gradient flow of the Chern Simons, with the change of variable $t = log(r)$.

(iii) In particular, Υ-instantons are the critical points of CS_Ξ that satisfy the constraint (22).

Remark 2.3. *All relevant examples satisfy the condition that the cone constraint is preserved under the gradient flow. However, we have not succeeded in removing this assumption altogether or replacing it with a more geometrically natural condition.*

Proof.

(i) The first variation of CS_Ξ is given by

$$\frac{\delta}{\delta\alpha} CS_\Xi = \frac{1}{n-4} \int_\Sigma \mathrm{Tr}\left(d_A\alpha \wedge F_A \wedge \Upsilon\right) = \langle \alpha, (-1)^n * (F_A \wedge \Xi) \rangle \ .$$

(ii) Under the cone constraint (22), Equation (21a) implies (21b). The claim follows by considering the Chern-Simons flow under the change of variables $t = log(r)$. $\square$

At a critical point, one may study the second variation of the Chern-Simons and Yang–Mills functionals. First, recall

Lemma 2.4. *Let A be an Υ-instanton on (Σ, g). Then*

$$\frac{\delta^2}{\delta\alpha\delta\beta} CS_{\Xi} = \int_{\Sigma} \mathrm{Tr}\left(\beta \wedge d_A\alpha \wedge \Xi\right) = \langle \beta, (-1)^n * (d_A\alpha \wedge \Xi)\rangle \ , \tag{26}$$

Thus, we consider the associated endomorphisms

$$D_A(\alpha) = (-1)^n * (d_A\alpha \wedge \Xi) \tag{27a}$$

$$S_A(\alpha) = d_A^* d_A\alpha + \{F_A, \alpha\} \ . \tag{27b}$$

They satisfy the following relation:

Proposition 2.5. *Let A be an Υ-instanton, and consider the space*

$$\mathcal{C} = \left\{\alpha \in \Omega^1(\mathfrak{g}_P) | \ d_A\alpha \in \Gamma L\right\} \ ,$$

the space of infinitesimal deformations satisfying the cone condition (22)*. For $\alpha \in \mathcal{C}$, we have*

$$S_A(\alpha) = D_A^2(\alpha) - (n-4)D_A(\alpha) \ . \tag{28}$$

In particular, we have the bound $S_A|_{\mathcal{C}} \geq -\left(\frac{n-4}{2}\right)^2$, and the index and nullity of the gauged fixed operator $\widehat{S_A} = S_A + d_A d_A^*$ satisfies the lower bounds

$$\mathrm{Ind}_A \geq \sum_{\lambda \in (0, n-4)} d_\lambda \qquad\qquad \mathrm{Nul}_A \geq d_0 + d_{n-4} \ . \tag{29}$$

with $d_\lambda = \dim\left\{\alpha \in \mathcal{C} | \ D_A(\alpha) = \lambda\alpha \ , \ d_A^*\alpha = 0\right\}$.

Proof. The proof is a straightforward computation using the Leibniz rule, the cone condition $d\Upsilon = (n-4)\Xi$ and the cone constraint (22). Expanding D_A^2, we have

$$\begin{aligned} D_A^2(\alpha) &= * \left(\Xi \wedge d_A * (\Xi \wedge d_A\alpha)\right) = (-1)^n * d_A * *(\Xi \wedge *(\Xi \wedge d_A\alpha) \\ &= d_A^* \left[d_A\alpha + *(\Upsilon \wedge d_A\alpha)\right] = d_A^* d_A\alpha + (-1)^n * d_A(\Upsilon \wedge d_A\alpha) \\ &= d_A^* d_A\alpha + (-1)^n(n-4) * (\Xi \wedge d_A\alpha) - * \left(\Upsilon \wedge [F_A, \alpha]\right) \\ &= d_A^* d_A\alpha + (-1)^n(n-4) * (\Xi \wedge d_A\alpha) + * \left([*F_A, \alpha]\right) \\ &= S_A + (n-4)D_A \ , \end{aligned}$$

where we used the linearised cone constraint, i.e $\alpha \in \mathcal{C}$, from the first to the second line and the Υ-instanton condition in the second to last line. The lower bound for S_A follows from completing the square.

When restricted to $\mathcal{C}$, the operators D_A and S_A commute by the above computation, and so they admit a common basis of eigenvectors. The index and nullity estimates are simply the index and nullity of $\widehat{S}_A$ restricted to $\mathcal{C}$ via the eigenvalue count of D_A. $\square$

We have the following, more general computation:

Proposition 2.6. *Let A be an Υ-instanton. There exists constants $C_i \in \mathbb{R}$ such that*

$$S_A(\alpha) = D_A^2(\alpha) - (n-4)D_A(\alpha) - d_A^* \sum_i C_i \left(\pi_{L_i^\perp}(d_A\alpha)\right) \ , \tag{30}$$

where $L_i^\perp$ form the direct sum decomposition of irreducible $N(H)$-representations of $L^\perp$ and π_B is the bundle projection map to the corresponding bundle B.

Proof. The proof is again an application of Schur's Lemma. Since both maps $*(\Upsilon \wedge \cdot)$ and $*(\Xi \wedge \cdot)$ are $N(H)$-equivariant, there exist constants C_i such that for any $\beta \in \Lambda^2$

$$*\left(\Xi \wedge *(\Xi \wedge \beta)\right) = \beta + *(\Upsilon \wedge \beta) + \sum_i C_i \pi_{L_i^\perp}(\beta) \ .$$

Now, substituting in the previous proof, we have

$$D_A^2(\alpha) = d_A^* \left[d_A\alpha + *(\Upsilon \wedge d_A\alpha) + \sum_i C_i \pi_{L_i^\perp}(d_A\alpha) \right] \ .$$

The proof follows by reproducing the computations above. □

Remark 2.7. *The results above can be further generalised to arbitrary Yang–Mills connections by adding terms of the form $(1+\lambda_i)\{\pi_i F_A, \alpha\}$, where π_i are the projections to the irreducible representations orthogonal to $\mathfrak{h}$ and λ_i is the corresponding eigenvalue under the map $*(\Upsilon \wedge \cdot)$.*

Along $\mathcal{C}$, the above computations read like Weitzenböck formulae between Δ_A and $\left(D_A - \frac{n-4}{2}\right)^2$. This compares to the case when the manifold carries a special holonomy metric, and the Weitzenböck formulae relate Δ_A and D_A^2. For example, if we consider holomorphic deformations of a HYM connection, we have

$$\widehat{S}_A(\alpha) = 2\Delta_{\overline{\partial}_A}\alpha \ .$$

The challenge is therefore, going from $\mathcal{C}$ to $\Omega^1(\mathfrak{g}_P)$ and finding a way of working with the generalised Weitzenböck formula of Equation (30). We do not know any general criteria or strategy for doing so, and so our discussion comes down to a case-by-case study of Table 5 (cf. Table 2). Before proceeding, we present two additional applications of our Weitzenböck

Geometry	$\mathfrak{h}$	L
Sasakian	$\Lambda_0^{(1,1)}$	$\eta \wedge \Lambda^1 \oplus \Lambda^{(1,1)}$
Nearly Kähler	Λ_8^2	$\Lambda_6^2 \oplus \Lambda_8^2$
Nearly parallel G_2	Λ_{14}^2	Λ^2

Table 5: Characteristic bundles associated to special holonomy cones

formula. An interesting first corollary of the previous discussion arises by combining the results for the trivial connection with a Bochner–Weitzenböck identity. To this end, we first introduce the following operator.

Definition 2.8. *Let (Σ^{n-1}, g) be a closed Riemannian manifold. The divergence operator is*

$$\begin{aligned} \mathrm{div} : \mathrm{Sym}^2 &\to \Omega^1 \\ h &\mapsto c \circ \nabla h , \end{aligned}$$

where $c : \Omega^1 \otimes \mathrm{Sym}^2 \to \Omega^1$ is the usual contraction map.

The adjoint to the divergence is $\mathrm{div}^*(X) = -\frac{1}{2}\mathcal{L}_X g$. We have the following identities

Lemma 2.9 (Bochner formula)**.** *Let (Σ^{n-1}, g) be a closed Riemannian manifold. Then the Hodge Laplacian on 1-forms satisfies*

$$\Delta X = \nabla^*\nabla X + \mathrm{Ric}(X) = 2\,\mathrm{div}\,\mathrm{div}^*(X) + 2\,\mathrm{Ric}(X) - dd^*X \ . \tag{31}$$

Proposition 2.10. *Let $(\Sigma^{n-1}, g, \Upsilon, \Xi)$ be the closed link of a special holonomy Ricci-flat cone, so $\mathrm{Ric}(g) = (n-2)g$ and the condition $*(\Xi \wedge \beta) = 0$ for $\beta \in L^\perp$ holds. Let*

$$\mathcal{E}_\lambda = \{\alpha \in \mathcal{C} |\ \mathrm{curl}(\alpha) = \lambda\alpha\} \ .$$

Then,

(i) For $\lambda \neq 0$, $\mathcal{E}_\lambda \subseteq \Omega^1_{coclosed}$ and we have $\mathcal{E}_0 = d\Omega^0$.

(ii) For $\lambda \in (-2, n-2) \setminus \{0\}$, we have $\mathcal{E}_\lambda = 0$.

(iii) We have $\mathcal{E}_{-2} \oplus \mathcal{E}_{n-2} \cong \mathfrak{isom}(M, g) \cap \mathcal{C}$.

Proof. Recall that the curl operator respects the splitting $\mathcal{C} \oplus \mathcal{C}^\perp$ since $\mathrm{curl} : \mathcal{C} \to \mathcal{C}$ by definition of $\mathcal{C}$, and since it is self-adjoint, we must have $\mathrm{curl} : \mathcal{C}^\perp \to \mathcal{C}^\perp$. If $\lambda \neq 0$, the coclosed condition follows by direct differentiation, $\lambda d^*\alpha = *d(\Xi \wedge d\alpha) = 0$.

From Proposition 2.6, it follows that $\ker(\mathrm{curl}) \subseteq \mathcal{C}$. Now, Equation (28) implies that, for $\alpha \in \mathcal{E}_\lambda \subseteq \mathcal{C}$,

$$d^*d\alpha = D^2(\alpha) - (n-4)D(\alpha) + = \left[\lambda^2 - (n-4)\lambda\right] \alpha \ .$$

If $\lambda = 0$, we have $||d\alpha||^2 = 0$, and so $\alpha \in d\Omega^0$, since $H^1(\Sigma) = 0$ by Myers' theorem. If $\lambda \neq 0$, α is coclosed and taking norms on Equation (31), we get

$$\frac{1}{4}||\mathcal{L}_X g||^2 = ||\mathrm{div}^*(\alpha)||^2 = \left[\frac{\lambda^2 - (n-4)\lambda}{2} - (n-2)\right] ||\alpha||^2 \ .$$

Solving for λ, the second claim follows.

Finally, $\alpha \in \mathcal{C}$ is dual to a Killing field if and only if it solves the equation $d^*d\alpha = 2(n-2)\alpha$, and the claim $\mathcal{E}_{-2} \oplus \mathcal{E}_{n-2} \cong \mathfrak{isom}(M, g) \cap \mathcal{C}$ follows from the Weitzenböck formula (31) again. □

The fact that infinitesimal isometries are divided into two classes raises the question of the significance of each class. Indeed, we have the following:

Proposition 2.11 (Prop. A.11)**.** *Under the assumptions of Proposition 2.10, the space $\mathcal{E}_{-2}$ corresponds to $\mathfrak{aut}(\Sigma, g, \Upsilon, \Xi)$, the Lie algebra of infinitesimal automorphisms.*

A case-by-case discussion and proof are included in the Appendix for nearly Kähler, nearly parallel G_2 and Sasaki-Einstein structures in Lemma A.53, Lemma A.29 and Proposition A.68, respectively. Moreover, from that discussion, one can further conclude

Proposition 2.12. *Under the assumptions of Proposition 2.10, if* (Σ, g) *does not have constant sectional curvature,*

$$\mathfrak{isom}(M, g) \cong \mathcal{E}_{-2} \oplus \mathcal{E}_{n-2} \ .$$

The argument for this result is somewhat unsatisfactory, and we believe there should exist a general, direct proof that $\mathfrak{isom}(M, g) \subseteq \mathcal{C}$. However, we have not been able to find one.

Similarly, we can consider the following generalisation of Simons' result. For a connection A, consider the 1-forms given by $\alpha \lrcorner F_A \in \Omega^1(\Sigma, \mathfrak{g}_P)$ for $\alpha \in \Omega^1$.

Lemma 2.13. *Let A be a Yang–Mills connection. The 1-form $\alpha \lrcorner F_A$ satisfies the Coulomb gauge condition whenever α is closed. Similarly, if A is an instanton, it suffices that $d\alpha \in \Gamma(\mathfrak{h}^\perp)$.*

Proof. Using the condition $d_A^* F_A = 0$, we have

$$d_A^*(\alpha \lrcorner F_A) = (-1)^{n-2} * d_A\left(*F_A \wedge \alpha\right) = (-1)^{n-2} * (*F_A \wedge d\alpha) = (-1)^{n-2} \langle F_A, d\alpha \rangle \ .$$

If $d\alpha = 0$, the claim is clear. If A is an instanton for an $N(H)$-structure, $F_A \in \Gamma(\mathfrak{h} \otimes \mathfrak{g}_P)$. □

Let A be an Υ-instanton and $\alpha \in \Omega^1_{closed}$, and set $\hat{\alpha} = \alpha \lrcorner F_A$. By direct computation using the Leibniz rule, we have

$$\begin{aligned} D_A(\hat{\alpha}) &= (-1)^n * (\Xi \wedge d_A \hat{\alpha}) = *d_A[\Xi \wedge (\alpha \lrcorner F_A)] \\ &= (-1)^{n-1} * d_A[(\alpha \lrcorner \Xi) \wedge F_A] = (-1)^{n-1} * (\mathcal{L}_\alpha \Xi \wedge F_A) \\ &= (-1)^{n-1} * [(\nabla_\alpha \Xi + \mathcal{L}_{2\alpha} g)_* \, \Xi \wedge F_A] \\ &= (-1)^n * (\alpha \wedge \Upsilon \wedge F_A) + (-1)^{n-1} * [(\mathcal{L}_{2\alpha} g)_* \, \Xi \wedge F_A] \\ &= -\alpha \lrcorner * (\Upsilon \wedge F_A) + (-1)^{n-1} * [(\mathcal{L}_{2\alpha} g)_* \, \Xi \wedge F_A] \\ &= \hat{\alpha} + (-1)^{n-1} * [(\mathcal{L}_{2\alpha} g)_* \, \Xi \wedge F_A] \ , \end{aligned}$$

where we used the identity $\mathcal{L}_X \Xi - \nabla_X \Xi = (\nabla X)_*$ in the third line, and the fact that links of special holonomy cones satisfy $\nabla_X \Xi = -X \wedge \Upsilon$ in the fourth line.

Therefore, we are interested in the space

$$\mathcal{GCK} = \{\alpha \in \Omega^1_{closed}(\Sigma^{n-1}) |\ (\mathcal{L}_\alpha g)_* \Xi = f \Xi \ \text{ for } f \in \Omega^0\}$$

of generalised conformal Killing fields. We have proved that:

Proposition 2.14. *Let $\alpha \in \mathcal{GCK}$. Then $D_A(\hat{\alpha}) = -\hat{\alpha}$.*

In particular, if we further assume $\hat{\alpha} \in \mathcal{C}$, it follows that $S_A(\widehat{\alpha}) = (5 - n)\widehat{\alpha}$.

This recovers the well-known Yang-Mills stability result [BL81, Thm. 7.11] of Simons on the round sphere S^{n-1} for $n \geq 6$ by taking α a conformal Killing field.

Of course, one would like to have a direct proof that $\widehat{\alpha} \in \mathcal{C}$ for all $\alpha \in \mathcal{GCK}$. We could not find a general proof, but a proof for the nearly parallel G_2 and nearly Kähler cases follows from Corollaries 2.19 and 2.30 respectively.

One might wonder whether our discussion leads to new examples. Unfortunately, the following result due to Obata is rather discouraging:

Proposition 2.15 ([Oba71]). *Let (Σ^{n-1}, g) be either a nearly parallel G_2, nearly Kähler or Sasaki manifold with non-trivial conformal Killing fields. Then Σ is isometric to the round sphere S^{n-1}.*

In the nearly parallel G_2, we have $\mathcal{GCK} \cong \mathcal{CK}$, so there is nothing else to be done. In the nearly Kähler and Sasaki cases, the space $\mathrm{Sym}^2_0(TM)$ splits into multiple irreducible pieces, some of which act trivially on Ξ, so one might hope that $\mathcal{GCK}$ is in general non-trivial in these cases.

We conclude this section with one final application of Proposition 28. Assume the cone constraint is preserved under the gradient flow of the Chern–Simons functional. Consider $A(r)$ a gradient flow line with well defined Υ-instanton limits $A_0 = \lim_{r\to 0} A(r)$ and $A_\infty = \lim_{r\to\infty} A(r)$. We make the following definition:

Definition 2.16. *The spectral flow of the family $A(r)$ around k is the algebraic intersection number (with multiplicity) of the spectrum of the operator $D_{A(r)} - k$ with the zero axis. We denote it by* $\mathrm{SpecFlow}(A, k)$.

Then we have the following result:

Proposition 2.17 ([KM07, Prop. 14.2.1]). *The virtual dimension of the moduli space of Ω-instanton at $\boldsymbol{A} = A(r)$ with the same asymptotics is equal to* $\mathrm{SpecFlow}(A, 0)$.

By direct application of Proposition 28, we have

$$\mathrm{Ind}^{\mathcal{C}}_{A_\infty} - \mathrm{Ind}^{\mathcal{C}}_{A_0} = \mathrm{SpecFlow}(\boldsymbol{A}, 0) - \mathrm{SpecFlow}(\boldsymbol{A}, n-4) = \dim_{\mathrm{vir}}\left(\mathcal{M}_\Omega(\boldsymbol{A})\right) - \mathrm{SpecFlow}(\boldsymbol{A}, n-4)\ ,$$

where $\mathrm{Ind}^{\mathcal{C}}_A$ is the Yang–Mills index of A restricted to the subspace $\mathcal{C}$. Thus, we ask

Question 1. *Does the shifted spectral flow* $\mathrm{SpecFlow}(A, n-4)$ *carry any natural geometric significance?*

We proceed to specialise the preceding discussion to three cases of interest: nearly parallel G_2, nearly Kähler and Sasaki structures. We focus on the former two and include the latter as it presents some interesting differences.

2.1 Holonomy Spin(7) cones

We start with the case where the cone has holonomy contained in Spin(7), so (Σ^7, g) carries a nearly G_2-structure. Let A be a G_2-instanton on Σ. To be consistent with the existing G_2-geometry literature, the 3-form Υ will be denoted by φ and the 4-form Ξ as ψ for the remainder of this section (cf. Table 2). The Spin(7)-cone case is straightforward due to the following observation.

Proposition 2.18. *The cone constraint* (22) *is trivial. In particular, it is trivially preserved by the Chern-Simons gradient flow and Equations* (24a) *and* (24b) *are equivalent.*

Proof. In the terminology of Section A.2, we have $\Upsilon = \varphi \in \Omega^3(\Sigma^7)$ and $\Xi = \psi = *\varphi \in \Omega^4(\Sigma^7)$. Then, the cone constraint (22) become $\beta + *(\beta \wedge \varphi) = *(\psi \wedge *(\psi \wedge \beta))$, which is the identity in Lemma A.18. □

In particular, we have

Corollary 2.19. *The stability operator S_A is fully characterised by D_A via the relation*

$$S_A(\alpha) = D_A^2(\alpha) - 4D_A(\alpha) \; . \tag{32}$$

The index and nullity of $\widehat{S}_A = S_A + d_A d_A^*$ are given by the equality case of Equation (29).

Remark 2.20. *The relation* (32) *was first introduced in [Wal22] with a typo.*

To conclude this case, let us study the mapping properties of the deformation operator $\widehat{D}_A = (d_A^*, \pi_7 \circ d_A) : \Omega^1(\mathfrak{g}_P) \to \Omega^0(\mathfrak{g}_P) \oplus \Omega^2_7(\mathfrak{g}_P)$ on the cone. These computations were carried out in [Gho24] from a spinorial point of view. First, we need the following technical lemmas:

Lemma 2.21. *Let $\widehat{\beta} = rdr \wedge \alpha + r^2\beta$ be a homogeneous 2-form on a* Spin(7)*-cone. The map*

$$\begin{aligned} \widehat{\pi} : \Omega^2_{homo}(C) &\to \Omega^1(\Sigma) \\ \widehat{\beta} &\mapsto -4\alpha = -\frac{4}{r}\partial_r \lrcorner \widehat{\beta} \end{aligned}$$

is surjective, and 2-forms of type 21 *span its kernel. Moreover, it is a homothety on the image by a factor of* 1/4*, i.e. we have*

$$||\pi_7(\beta)||^2_C = \frac{1}{4}||\widehat{\pi}(\beta)||^2_\Sigma \; ,$$

where $\pi_7(\beta) = \frac{1}{4}[\beta + *(\Phi \wedge \beta)]$ is the projection map.

Proof. Clearly, the map is surjective. By dimensional count, its kernel has dimension 21. The map $\iota(\alpha) = rdr \wedge \alpha + r^2 * (\psi \wedge \alpha)$ is a right inverse for $\widehat{\pi}$ up to a constant; so it suffices to prove that its image is contained in $\Omega^2_7(C)$. From Lemma A.13, this is equivalent to proving that $*_C(\Phi \wedge \iota(\alpha)) = 3\iota(\alpha)$. Indeed

$$\begin{aligned} *_C(\Phi \wedge \iota(\alpha)) &= *_C[(r^3 dr \wedge \varphi + r^4\psi) \wedge (rdr \wedge \alpha + r^2 * (\psi \wedge \alpha)] \\ &= *_C\left[r^5 dr \wedge (\varphi \wedge *(\psi \wedge \alpha) + \psi \wedge \alpha) + r^6 \psi \wedge *(\psi \wedge \alpha))\right] \\ &= 3\left(rdr \wedge \alpha + r^2 * (\psi \wedge \alpha)\right) \; , \end{aligned}$$

where the last line follows from the G_2 identities (*iii*) and (*iv*) in Lemma A.17. Finally, by Lemma A.17 once more, we have

$$||\iota(\alpha)||^2_C = ||\alpha||^2_\Sigma + ||*(\psi \wedge \alpha)||^2_\Sigma = 4||\alpha||^2_\Sigma \; .$$

□

Remark 2.22. *Notice that the choice of constant -4 has no a priori geometric significance but is helpful to make $\widehat{D}_A$ manifestly self-adjoint.*

Using this identification, we can prove the following:

Proposition 2.23. *Let $\widehat{\alpha} = r^{\lambda}\left(f dr + r\alpha\right)$ be a λ-homogeneous section of $\Omega^1(C, \mathfrak{g}_P)$, with A a G_2 instanton on the link. Then, under the identification above, we have*

$$\begin{aligned} \widehat{D}_A : \Omega^0 \oplus \Omega^1(\Sigma, \mathfrak{g}_P) &\to \Omega^0 \oplus \Omega^1(\Sigma, \mathfrak{g}_P) \\ (f, \alpha) &\mapsto \begin{pmatrix} -(\lambda+7) & d_A^* \\ d_A & -*(\psi\wedge d_A) - (\lambda+1) \end{pmatrix} \begin{pmatrix} f \\ \alpha \end{pmatrix} \end{aligned} \tag{33}$$

Proof. From Lemma 3.2, we have

$$\begin{aligned} d_A\widehat{\alpha} &= r^{\lambda+1}\left[\frac{dr}{r}\wedge((\lambda+1)\alpha - d_A f) + d_A\alpha\right] \\ d_A^*\widehat{\alpha} &= r^{\lambda-1}\left(d_A^*\alpha - (\lambda+7)f\right) \ . \end{aligned}$$

Using the Cayley 4-form $\Phi = r^3 dr\wedge\varphi + r^4\psi$, we have

$$\begin{aligned} *_C\left(\Phi\wedge d_A\widehat{\alpha}\right) &= *_C\left[\left(r^3 dr\wedge\varphi + r^4\psi\right)\wedge\left(r^{\lambda-1}dr\wedge((\lambda+1)\alpha - d_A f) + r^{\lambda}d_A\alpha\right)\right] \\ &= r^{\lambda+5} *_C\left[\frac{dr}{r}\wedge(\varphi\wedge d_A\alpha + \psi\wedge[(\lambda+1)\alpha - d_A f]) + \psi\wedge d_A\alpha\right] \\ &= r^{\lambda+1}\left[\frac{dr}{r}\wedge *(\psi\wedge d_A\alpha) + *\left(\varphi\wedge d_A\alpha + \psi\wedge[(\lambda+1)\alpha - d_A f]\right)\right] \end{aligned}$$

Now, using the identification from Lemma 2.21, we have

$$\widehat{\pi}\circ\pi_7(d_A\alpha) = d_A f - (\lambda+1)\alpha - *(\psi\wedge d_A\alpha) \ .$$

□

The following gives a natural geometric interpretation of the kernel of $\widehat{D}_A$ in a certain range.

Lemma 2.24. *Let A be a G_2-instanton on Σ. For $\lambda \in (-7, -1)$, if $\widehat{\alpha} = r^{\lambda}(f dr + r\alpha)$ is a λ-homogeneous section in the kernel of $\widehat{D}_A$, then $f = 0$ and α is an eigenvector of $D_A = *(\psi\wedge d_A\cdot)$ of eigenvalue $-(\lambda+1)$.*

If A is the trivial connection, the same applies in the range $\lambda \in (-8, 0)$ with the difference that for $\lambda = 0$, f can be constant.

Proof. By Equation (33), we need to solve

$$d_A^*\alpha = (\lambda+7)f \ , \qquad d_A f = *(\psi\wedge d_A\alpha) + (\lambda+1)\alpha \ .$$

Acting by d_A^* on the equation on the left and substituting the right one, we have

$$\Delta_A f = -*(\psi\wedge[F_A, \alpha]) + (\lambda+1)d_A^*\alpha = (\lambda+1)(\lambda+7)f \ ,$$

where we used the instanton condition $F_A\wedge\psi = 0$, so the curvature term vanishes. The claim follows from the positivity of the Laplacian. If A is trivial, the improved bound is a corollary of Obata's theorem (cf. [Oba62]). □

Notice that the previous lemma is essentially a twisted version of Prop. A.32.

2.2 Holonomy G_2 cones

We now move on to the case where $C(\Sigma)$ is a metric cone with holonomy contained in G_2, so (Σ, g) carries a nearly Kähler structure, considered previously in the literature by [Xu09] and [CH16]. To be consistent with the existing literature on nearly Kähler manifolds, the 2-form Υ will be denoted by ω and the 3-form Ξ as ρ for the remainder of this section (cf. Table 2). We also set $\hat{\rho} = *\rho$, its Hodge dual, to maintain consistent notation with the rest of the thesis (cf. Section 7). From Lemma A.38, we have

Lemma 2.25. *The cone bundle is* $L = \left\{\beta \in \Omega^2(\mathfrak{g}_P) \,|\, \omega^2 \wedge \beta = 0\right\} \cong \Omega^2_8 \oplus \Omega^2_6 \cong \langle\omega\rangle^\perp$.

Proposition 2.26. *The Chern-Simons gradient flow preserves the cone constraint.*

Proof. The gradient flow is $\partial_t A = - * (F_A \wedge \rho)$. By the Maurer-Cartan formula, it follows that $\partial_t F_A = -d_A * (F_A \wedge \rho)$. Thus,

$$\frac{\partial}{\partial t}(F_A \wedge \omega^2) = -d_A * (F_A \wedge \rho) \wedge \omega^2 = -d_A\Big(* (F_A \wedge \rho) \wedge \omega^2\Big) = d_A(F_A \wedge \hat{\rho}) = -2F_A \wedge \omega^2 \ . \quad (34)$$

In particular, the cone constraint is preserved by the flow. ☐

We give a more geometrically satisfactory interpretation of the cone constraint.

The base manifold Σ is even-dimensional and carries a closed $(n-2)$-form $\omega^2/2$, so the space of connections is naturally equipped with a symplectic structure:

$$\mathcal{W}(\alpha, \beta) = \int_\Sigma \mathrm{Tr}(\alpha \wedge \beta) \wedge \frac{\omega^2}{2} \ . \quad (35)$$

This symplectic structure is preserved by the action of the gauge group $\mathcal{G}$ on the space of connections $\mathcal{A}$. In particular, it admits a well-known $\mathcal{G}$-equivariant moment map

$$\begin{aligned} \mu : \mathcal{A} \to &\Omega^6(M, \mathfrak{g}_P) \cong \Omega^0(M, \mathfrak{g}_P)^* \\ A \mapsto &F_A \wedge \frac{\omega^2}{2} \ . \end{aligned} \quad (36)$$

One can also consider a second Chern-Simons-type functional

$$CS_{\hat{\rho}}(A) = \int_\Sigma CS_0(A, A_0) \wedge \hat{\rho} \ .$$

Proposition 2.27. *The cone constraint* (22) *in the nearly Kähler case corresponds to the vanishing of the moment map* μ *in Equation* (36)*. Over* $\mu^{-1}(0)$*, the instanton equation on the cone reduces to the Hamiltonian flow of* $CS_{\hat{\rho}}$*.*

Proof. The first statement follows immediately from Lemma 2.25. The first variation of $CS_{\hat{\rho}}$ is

$$\frac{\delta}{\delta\alpha} CS_{\hat{\rho}} = \int_\Sigma Tr(\alpha \wedge F_A) \wedge \hat{\rho} \ .$$

Thus, the Hamiltonian flow with respect to $\mathcal{W}$ will be

$$\partial_t A = * \left(\frac{\omega^2}{2} \wedge *(F_A \wedge \hat{\rho})\right) = -J * (F_A \wedge \hat{\rho}) = - * (F_A \wedge \rho) \ ,$$

where we used Lemmas A.38 and A.39. ☐

As in the previous case, we have

Lemma 2.28. *The nearly Kähler instanton equations* (24a) *and* (24b) *are equivalent. In particular, critical points of the Chern-Simons functional CS_ρ are automatically nearly Kähler instantons.*

Proof. We need to show that the equations

$$*(F_A \wedge \omega) = -F_A \ , \qquad F_A \wedge \rho = 0$$

are equivalent. The first directly implies the second (cf. Lemma A.38). Now, the second is equivalent to $\pi_6(F_A) = 0$, and thus to $F_A \wedge \hat{\rho} = 0$. Differentiating and using the Bianchi identity, we have $F_A \wedge \omega^2 = 0$, and the claim follows. □

Finally, let us discuss what happens at the level of second variations. Since the cone bundle L corresponds to the kernel of the contraction operator $\Lambda : \Omega^2(\mathfrak{g}_P) \to \Omega^0(\mathfrak{g}_P)$, we have

Lemma 2.29. *Let A be a nearly Kähler instanton. The linearisation of the cone constraint is $d_A^*(J\alpha) = 0$ and $\mathcal{C}^\perp = \left\{ Jd_Af \middle| \, f \in \Omega^0(\mathfrak{g}_P) \right\}$. Moreover, the subspace $\mathcal{C}^\perp$ is an eigenspace of the operator D_A with eigenvalue* 4.

Proof. The first two claims are straightforward. Let us prove that $\mathcal{C}^\perp$ is an eigenspace of D_A. We have:

$$\begin{aligned} D_A(Jd_Af) &= - * (d_AJd_Af \wedge \rho) = *d_A(d_Af \wedge \hat{\rho}) \\ &= *([F_A, f] \wedge \hat{\rho}) + 4 * \left(d_Af \wedge \frac{\omega^2}{2} \right) \\ &= 4Jd_Af \ , \end{aligned}$$

where we used the relationship $JX \wedge \rho = -X \wedge \hat{\rho}$ in the first line and the instanton condition on the third. (cf. Proposition A.48). □

Thus, Proposition 2.6 takes the following form in the nearly Kähler case:

Corollary 2.30. *Let A be a nearly Kähler instanton. Let $\hat{\alpha} = \alpha + Jd_Af \in \Omega^1(\mathfrak{g}_P)$ with $\alpha \in \mathcal{C}$. Then*

$$\langle S_A(\hat{\alpha}), \hat{\alpha} \rangle = \langle D_A^2(\alpha) - 3D_A(\alpha), \alpha \rangle + 4||d_Af||^2 + ||\Lambda(d_AJd_Af)||^2 \ . \qquad (37)$$

The index and nullity of $\widehat{S}_A = S_A + d_Ad_A^*$ are given by the equality case of Equation (29).

Proof. Since $\mathcal{C}^\perp$ is an eigenspace of D_A, it follows easily that $\langle S_A(\alpha), Jd_Af \rangle = 0$. Thus, we only need to compute $\langle S_A(Jd_Af), Jd_Af \rangle$. From Proposition 2.6 and Lemma A.39, we have

$$\begin{aligned} \langle S_A(Jd_Af), Jd_Af \rangle &= \langle D_A^2(Jd_Af) - 3D_A(Jd_Af), Jd_Af \rangle + ||\Lambda d_AJd_Af||^2 \\ &= 4||d_Af||^2 + ||\Lambda(d_AJd_Af)||^2 \end{aligned}$$

where the second line simply follows from Jd_Af being an eigenform of D_A with eigenvalue 4.

Since $S_A(Jd_Af)$ is non-negative, and $S_A(Jd_Af) = 0$ implies $d_Af = 0$, the eigenspaces of non-positive eigenvalue are contained in $\mathcal{C}$, and the claim follows. □

Again, we finish with the study of the mapping properties of the deformation operator $\widehat{D}_A$: $\Omega^0(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P) \to \Omega^0(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P)$ on the cone. As in the Spin(7)-case, the computation was carried out in [Dri21] from a spinorial point of view. Using the identification

$$Omega^1(C) \cong \Omega^0(\Sigma) \oplus \Omega^1(\Sigma) \,, \tag{38}$$

we prove:

Proposition 2.31. *Let $\widehat{f} = r^\lambda f \in \Omega^0(\mathfrak{g}_P)$ and $\widehat{\alpha} = r^\lambda (gdr + r\alpha)$ be λ-homogeneous sections of $\Omega^0(C, \mathfrak{g}_P)$ and $\Omega^1(C, \mathfrak{g}_P)$ respectively. Then, $\widehat{D}_A(\widehat{f}, \widehat{\alpha})$ is given by*

$$\begin{aligned} \widehat{D}_A : \Omega^0 \oplus \Omega^0 \oplus \Omega^1(\Sigma, \mathfrak{g}_P) &\to \Omega^0 \oplus \Omega^0 \oplus \Omega^1(\Sigma, \mathfrak{g}_P) \\ (f, g, \alpha) &\mapsto \begin{pmatrix} 0 & -(\lambda+6) & d_A^* \\ \lambda & 0 & d_A^* J \\ d_A & -Jd_A & -*(\hat{\rho} \wedge d_A) + (\lambda+1)J \end{pmatrix} \begin{pmatrix} f \\ g \\ \alpha \end{pmatrix} \end{aligned} \tag{39}$$

Proof. As in the previous case, from Lemma 3.2, we know

$$\begin{aligned} d_A \widehat{\alpha} &= r^\lambda \left[\frac{dr}{r} \wedge ((\lambda+1)\alpha - d_A g) + d_A \alpha \right] \,, \\ d_A^* \widehat{\alpha} &= r^{\lambda-1} \left(d_A^* \alpha - (\lambda+6) g \right) \,, \\ d_A \widehat{f} &= r^\lambda \left[\lambda f \frac{dr}{r} + d_A f \right] \,. \end{aligned}$$

Plugging in $\psi = -r^3 dr \wedge \hat{\rho} + r^4 \frac{\omega^2}{2}$, we have

$$\begin{aligned} *_C (\psi \wedge d_A \widehat{\alpha}) &= *_C \left[\left(-r^3 dr \wedge \hat{\rho} + r^4 \frac{\omega^2}{2} \right) \wedge \left(r^{\lambda-1} dr \wedge ((\lambda+1)\alpha - d_A g) + r^\lambda d_A \alpha \right) \right] \\ &= r^{\lambda+4} *_C \left[\frac{dr}{r} \wedge \left(\frac{\omega^2}{2} \wedge [(\lambda+1)\alpha - d_A g] - \hat{\rho} \wedge d_A \alpha \right) + \frac{\omega^2}{2} \wedge d_A \alpha \right] \\ &= r^\lambda \left[\frac{dr}{r} \wedge * \left(\frac{\omega^2}{2} \wedge d_A \alpha \right) + * \left(\frac{\omega^2}{2} \wedge [(\lambda+1)\alpha - d_A g] - \hat{\rho} \wedge d_A \alpha \right) \right] \\ &= r^\lambda \left[\frac{dr}{r} \wedge d_A^*(J\alpha) + (\lambda+1) J\alpha - J d_A g - *(\hat{\rho} \wedge d_A \alpha) \right] \end{aligned}$$

where we used identities from Lemma A.38 and Proposition A.48 in the last line. □

As in the previous case, we have

Lemma 2.32. *Let A be a nearly Kähler instanton and $(\widehat{f}, \widehat{\alpha}) \in \Omega^0 \oplus \Omega^1(\mathfrak{g}_P)$ be λ-homogeneous solutions to $\widehat{D}_A(\widehat{f}, \widehat{\alpha})$. Under the identification $(\widehat{f}, \widehat{\alpha}) \cong (f, g, \alpha)$ from Equation (38), we have that $g = 0$ for $\lambda \in (-6, -1)$, and $f = 0$ for $\lambda \in (-5, 0)$. Moreover, if $\lambda \in (-5, -1)$, then α is an eigenvector of $D_A = -*(\rho \wedge d_A)$ of eigenvalue $-(\lambda+1)$.*

If A is the trivial connection, $g = 0$ for $\lambda \in (-7, 0) \setminus \{-6, -1\}$, $f = 0$ for $\lambda \in (-6, 1) \setminus \{-5, 0\}$ and α will be an eigenvector of D_A in the range $\lambda \in (-6, 0)$.

Proof. By Equation (39), we need to solve

$$d_A^* \alpha = (\lambda + 6) g \ , \tag{40a}$$

$$d_A^* J\alpha = -\lambda f \ , \tag{40b}$$

$$d_A f - J d_A g = *(\hat{\rho} \wedge d_A \alpha) - (\lambda + 1) J\alpha \ . \tag{40c}$$

Acting by d_A^* and by $d_A^* J$ on the last equation, we have

$$\begin{aligned} \Delta_A f &= *d_A \left(\hat{\rho} \wedge d_A \alpha\right) - (\lambda+1) d_A^*(J\alpha) \\ &= -4 * \left(\frac{\omega^2}{2} \wedge d_A \alpha\right) - * \left(\hat{\rho} \wedge [F_A, \alpha]\right) - (\lambda+1) d_A^*(J\alpha) \qquad (41a) \\ &= -(\lambda+5) d_A^*(J\alpha) \ , \\ \Delta_A g &= - * d_A(\rho \wedge d_A \alpha) + (\lambda+1) d_A^* \alpha = * \left(\rho \wedge [F_A, \alpha]\right) + (\lambda+1) d_A^*(\alpha) \\ &= (\lambda+1) d_A^* \alpha \ . \qquad (41b) \end{aligned}$$

where we use Lemma A.39, and the observation that, for any function h,

$$d_A^* J d_A h = (d_A^*)^2 (h\omega) = * \left([F_A, h] \wedge \frac{\omega^2}{2}\right) = 0 \ ,$$

since F_A is an instanton. Similarly, we have $\rho \wedge [F_A, \alpha] = 0 = \hat{\rho} \wedge [F_A, \alpha]$. Substituting Equations (40a) and (40b) above, we get

$$\Delta_A g = (\lambda+1)(\lambda+6) g \qquad \text{and} \qquad \Delta_A f = \lambda(\lambda+5) f \ .$$

The first claim follows from the positivity of the Laplacian. If both f and g vanish, Equation (40c) reduces to $*(\hat{\rho} \wedge d_A \alpha) = (\lambda+1) J\alpha$. Acting by J and using Lemma A.39, we see α satisfies $D_A(\alpha) = - * (\rho \wedge d_A \alpha) = -(\lambda+1)\alpha$, as needed.

If A is trivial, the improved bounds are a corollary of Obata's theorem (cf. [Oba62]). □

2.3 Kähler cones

Finally, let us discuss the case where the metric cone $C(\Sigma)$ carries a Kähler metric, so (Σ^{2k+1}, g) carries a Sasaki structure $(g, \eta, \omega, R, \Phi)$, as defined in Appendix A.4. In the notation of Sasakian geometry, we have $\Upsilon = \eta \wedge \frac{\omega^{k-2}}{(k-2)!}$ and $\Xi = \frac{\omega^{k-1}}{(k-1)!}$; and we verify

$$d\Upsilon = \frac{2}{(k-2)!} \omega^{k-1} = (2k-2) \frac{\omega^{k-1}}{(k-1)!} = (n-4)\Xi \ .$$

First, we have

Lemma 2.33. *The cone constraint* (22) *corresponds to the curvature being a section of the subbundle* $\left(\Lambda^{(2,0)+(0,2)}\right)^{\perp} \cong \eta \wedge \Lambda^1 H \oplus \Lambda^{1,1} \cong \eta \wedge \Lambda^1 H \oplus \langle \omega \rangle \oplus \Lambda_0^{1,1}$.

Proof. Let $\beta \in \Omega^2$. We can write it in terms of its irreducible representation components:

$$\beta = \eta \wedge \alpha + f\omega + \beta^{2,0} + \beta_0 \ ,$$

with $\beta^{2,0} \in \Omega^{(2,0)+(0,2)}$ and $\beta_0 \in \Omega_0^{1,1}$. Then, using the identities in a Sasaki manifold, we have

$$\begin{aligned}
*(\beta \wedge \Upsilon) &= *\left(\beta \wedge \eta \wedge \frac{\omega^{k-2}}{(k-2)!}\right) = (k-1)f\omega + \beta^{2,0} - \beta_0 \\
*\left(\Xi \wedge *(\Xi \wedge \beta)\right) &= *\left[\frac{\omega^{k-1}}{(k-1)!} \wedge *\left(\beta \wedge \frac{\omega^{k-1}}{(k-1)!}\right)\right] = *\left(\frac{\omega^{k-1}}{(k-1)!} \wedge (J\alpha + kf\eta)\right) \\
&= \eta \wedge \alpha + kf\omega \, .
\end{aligned}$$

Collecting the terms, it follows that the cone constraint is equivalent to $2\beta^{2,0} = 0$, as needed. □

As in the previous cases, we have

Proposition 2.34. *The Chern–Simons flow preserves the cone constraint.*

Proof. Denote by $\pi^{2,0} : \Lambda^2 \to \Lambda^{(2,0)+(0,2)}$ the projection to the $(2,0)+(0,2)$ component. We need to show that $\pi^{2,0}\left(\partial_t F_A\right) = 0$ assuming $\pi^{2,0}(F_A)$. Again, by the Maurer-Cartan Equation, we have $\partial_t F_A = d_A \partial_t A = d_A * (\Xi \wedge F_A)$.

Assume that at time t, F_A satisfies the cone constraint, so it admits the decomposition $F_A = \eta \wedge \alpha + f\omega + \beta_0$. Thus, we need to check that the following term vanishes:

$$\pi^{2,0}\left(\partial_t F_A\right) = \pi^{2,0}\left[d_A * (\Xi \wedge F_A)\right] = \pi^{2,0}\left[d_A\left(J\alpha + kf\eta\right)\right] = \pi^{2,0}\left(d_A J\alpha\right) \, .$$

Now, from the Bianchi identity, we have

$$d_A F_A = 2\omega \wedge \alpha - \eta \wedge d_A\alpha + d_A f \wedge \omega + d_A\beta_0 = 0 \, .$$

Now, looking at the irreducible (p,q)-parts, it follows that $\pi^{2,0}(d_A\alpha) = 0$. But this is equivalent to $\pi^{2,0}(d_A J\alpha) = 0$, as needed. □

Let us now focus on studying the critical points of the Chern-Simons flow. In contrast to the previous cases, the critical points of the Chern-Simons functional will not necessarily be Sasaki instantons if the cone constraint is dropped. This raises the question of whether general critical points of CS_Ξ have some geometric interpretation or significance. We have

Lemma 2.35. *Critical points of CS_Ξ are transverse connections with respect to the Reeb foliation.*

Proof. Let A be a critical point of CS_Ξ. We need to show that $R \lrcorner F_A = 0$. Now, since $\Xi = \frac{\omega^{k-1}}{(k-1)!}$ is horizontal (in fact, basic), we have that $0 = R \lrcorner (F_A \wedge \Xi) = (R \lrcorner F_A) \wedge \Xi$, and the claim follows since $L_\Xi : \Lambda^1 \to \Lambda^{2k-1}$ is an isomorphism. □

Thus, locally, we can think of A as a constant connection along the leaves. If the foliation were quasi-regular, with orbifold base $\mathcal{X}$, then $A = \pi^*(B) - \frac{k}{2}\eta$, where B is a connection on the base χ and k is the monodromy of the connection along the S^1-fibres. The following proposition is straightforward.

Proposition 2.36. *Let M be a quasi-regular Sasaki manifold and $A = \pi^*(B) - \frac{k}{2}\eta$ be a transverse connection. Then* *(i) The cone constraint is equivalent to B being a holomorphic connection.*

(ii) The connection A is a critical point of CS_Ξ if ΛF_B is constant. In this case k is the degree of B, $\deg(B) = \frac{i}{2\pi}\langle F_B, \omega\rangle$.

(iii) The connection A is a Sasaki instanton if and only if B is a Hermite–Yang–Mills connection.

Proof. The curvature of A will be $F_A = \pi^*(F_B) - k\omega - \frac{1}{2}dk \wedge \eta$. Thus,

(i) The cone condition becomes $0 = \pi^{(2,0)}(F_B)$, which is equivalent to B being a holomorphic connection.

(ii) Critical points of CS_Ξ correspond to $(\pi^*(F_B) - k\omega) \wedge \frac{\omega^{k-1}}{(k-1)!} = 0$. Contracting with ω and using the definition of the degree of E, the claim follows.

(iii) The two conditions above are precisely the Hermite–Yang–Mills equation. □

Even when the Sasaki structure is irregular, one can still make sense of the notions of A being transverse holomorphic. In all cases, one has a transverse analogue of the Hitchin–Kobayashi correspondence, as proved by [BH22].

3 Conical singularities and weighted Banach spaces

We describe the analytic framework necessary to study the deformation problems of interest. While standard analytical techniques on compact Riemannian manifolds break down in the non-compact setting, they can be adapted by prescribing appropriate behaviour near the non-compact ends. The key idea is to introduce weighted versions of the classical Hölder and Sobolev spaces. On those, we can define a suitable notion of (uniformly) elliptic operators. These will enjoy similar properties to their counterparts in the compact setup, including nice Fredholm and index theory, as well as good regularity properties. We begin by motivating this work by considering the study of harmonic forms on a Riemannian metric cone. We follow the approach outlined in [Che79] (cf. [FHN21, Appendix A]).

Recall the notion of homogeneous k-forms from the introduction:

Definition 3.1. *We will say a smooth k-form γ on the cone is homogeneous of rate λ if there exist $\alpha \in \Omega^{k-1}(\Sigma)$ and $\beta \in \Omega^k(\Sigma)$ such that*

$$\gamma = r^\lambda (r^{k-1} dr \wedge \alpha + r^k \beta) .$$

Equivalently, λ-homogeneous k-forms are elements of the representation of weight λ under the natural induced $\mathbb{R}_+$-action.

Denote by $*$ and $*_C$ the Hodge star operator on (Σ, g) and $(C(\Sigma), g_C)$ respectively. We are interested in understanding the mapping properties of the associated Laplace operator $\Delta_C = dd^{*_C} + d^{*_C} d$. The obvious approach to this problem is by separation of variables.

Let $\gamma = r^{\lambda+k}(\frac{dr}{r} \wedge \alpha + \beta)$ a λ-homogeneous k-form. We have the relation

$$*_C\gamma = r^{\lambda+n-k}\left(*\alpha + (-1)^k \frac{dr}{r} \wedge *\beta\right) , \tag{43}$$

where n is the dimension of the cone. Using this, we get

Lemma 3.2. *Let $\gamma = r^{\lambda+k}(\frac{dr}{r} \wedge \alpha + \beta)$ be a λ-homogeneous k-form. Then*

$$d\gamma = r^{\lambda+k}\left(\frac{dr}{r} \wedge ((\lambda+k)\beta - d\alpha) + d\beta\right) \tag{44a}$$

$$d^{*_C}\gamma = r^{\lambda+k-2}\left(-\frac{dr}{r} \wedge d^*\alpha + d^*\beta + (k-\lambda-n)\,\alpha\right) \tag{44b}$$

$$\begin{aligned}\Delta_C\gamma =& r^{\lambda+k-2}\frac{dr}{r} \wedge (\Delta\alpha - (\lambda+k-2)(\lambda+n-k)\alpha - 2d^*\beta) \\ &+ r^{\lambda+k-2}\left(\Delta\beta - (\lambda+k)(\lambda-k+n-2)\beta - 2d\alpha\right) .\end{aligned} \tag{44c}$$

Thus, one could try to solve the eigenvalue problem $\Delta_C\gamma = \nu^2\gamma$ by considering forms of the type $\gamma = \sum_i f_i(t)\gamma_i$, where $f_i(t)$ belong to a suitably chosen family of smooth functions and γ_i are homogeneous forms to solve. From the Lemma above, this leads to

$$\Delta_C(f_i\gamma_i) = f_i\Delta_C\gamma - f_i''\gamma_i - \frac{n-1}{r}f_i'\gamma_i = \nu^2 f_i\gamma_i . \tag{45}$$

Choosing γ_i to be eigenforms of the operator Δ_C leads to a Bessel-type equation for f. Thus, one should take $f_i(r)$ to be suitably rescaled multiples of the Bessel functions $J_\nu(r)$. In the case of harmonic form, $\nu^2 = 0$, it suffices to consider functions f of the form $f(r) = r^\lambda \log^k(r)$, rather than general Bessel functions. More concretely, we have the following result:

Theorem 3.3 ([FHN21, Thm. A.2, Prop. A.6])**.** *Let $\gamma = \sum_{j=0}^m \gamma_j\,(\log(r))^j$ with γ_j a λ-homogeneous k-form. If $\Delta_C\gamma = 0$ then $\Delta_C\gamma_j = 0$ for all j and either $m = 0$ or $m = 1$ and $\lambda = \frac{1-n}{2}$. Each $\gamma_j = r^{\lambda+k}\left(\frac{dr}{r} \wedge \alpha + \beta\right)$ solves the elliptic eigenvalue problem on the link:*

$$\begin{pmatrix}\Delta - (k-2)(n+1-k) & -2d^* \\ -2d & \Delta - k(n-k-1)\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \lambda(\lambda+n-1)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} . \tag{46}$$

Given the discussion above, we see that although it makes no sense to count solutions $\Delta_C\gamma = 0$ in general, one can do so if one prescribes the behaviour of the k-form at either infinity or the cone singularity. In that case, only finitely many λ will contribute, with the multiplicities given by the multiplicity of solutions in Equation (46). This behaviour of the Laplacian on the metric cone carries over to the case where we have a Riemannian manifold with singularities modelled on a metric cone.

The rigorous theory to treat problems of this nature was initially introduced in the works of Lockhart and McOwen [LO85], Bartnik [Bar86], and Lockhart [Loc87]. We will cover the main results of these papers, adapted to our context. Their study is carried out in the asymptotically cylindrical case, where the metric near the end converges (in a sense that will be made precise later) to $(\mathbb{R} \times \Sigma, dt^2 + g_\Sigma)$. We are interested in the case where the metric converges to that of a cone, $(\mathbb{R}_+ \times \Sigma, dr^2 + r^2 g_\Sigma)$. Since the two are conformally equivalent under the change $t \mapsto e^t$, up to a global factor e^{2t}, their results carry over to our case. A comprehensive discussion of their work and its adaptation to the conical case can be found in Marshall's PhD thesis [Mar02].

The section is organised as follows. We begin by defining the objects of interest, namely, manifolds and connections with conical singularities. We then discuss weighted Banach spaces and conclude with a review of the Fredholm properties of (uniform) elliptic operators between these Banach manifolds.

Manifold with conical singularities

Throughout this section, we assume M^n is a non-compact smooth n-manifold and Σ is a closed $(n-1)$-dimensional manifold. We say M is a manifold with end Σ if there exists a compact submanifold $\check{M} \subseteq M$ and a diffeomorphism $f : M \setminus \check{M} \to (a, b) \times \Sigma$.

The idea is to furnish the manifold with ends that have a Riemannian structure resembling a metric cone on each end. We focus on a particular family of examples.

Definition 3.4. *Consider $\overline{M}$ a compact connected Hausdorff topological space and $p_1, \dots, p_m \in \overline{M}$ such that $M = \overline{M} \setminus \{p_1, \dots, p_m\}$ carries a smooth n-dimensional Riemannian manifold structure, with metric g.*

We say (M, g) is a Riemannian manifold with isolated conically singularities (ICS), with singularities at $p_1, \dots, p_m$ with rates $\nu_1, \dots, \nu_m$, such that $\nu_i > 0$, if the following is satisfied for each $i \in \{1, \dots, m\}$: There exist $\epsilon > 0$, open disjoint neighbourhoods of p_i, U_i, Riemannian cones $C(\Sigma_i) = (\mathbb{R}_+ \times \Sigma_i, g_C)$ over closed Riemannian manifolds $(\Sigma_i^{n-1}, g_\Sigma)$ and diffeomorphisms $\Psi_i : (0, \epsilon) \times \Sigma_i \to U_i \setminus \{p_i\}$ such that

$$|\nabla_i^k \left(\Psi_i^*(g) - g_i\right)|_{g_{C_i}} = O(r_i^{\nu_i - k}) \qquad \text{as } r_i \to 0 \,, \tag{47}$$

for $k \geq 0$, where r_i is the radial coordinate of the cone and ∇_i is the induced Levi-Civita connection on the cone.

We say M is of rate ν if each p_i is a conical singularity of rate ν. The Riemannian manifold (Σ_i, g) is called the link of the singularity p_i.

Remark 3.5. *Given any closed Riemannian n-manifold M, one can produce an ICS manifold by removing any collection of points $\{p_1, \dots, p_m\}$. In this case, the link of each singular point is modelled on (S^{n-1}, g_{round}).*

Similarly, we can define the related notion of an asymptotically conical Riemannian manifold.

Definition 3.6. *Let (M^n, g) be a complete Riemannian n-manifold. We say (M, g) is asymptotically conical (AC) of rate $\nu < 0$ if there exists a compact set $K \subseteq M$, $R > 0$ and a Riemannian cone $C(\Sigma) = (\mathbb{R}_+ \times \Sigma, g_C)$ over a closed Riemannian manifold (Σ^{n-1}, g_Σ) such that there exists a diffeomorphism $\Psi : [R, \infty) \times \Sigma \to M \setminus K$ satisfying*

$$|\nabla_i^k (\Psi^*(g) - g)|_{g_C} = O(r^{\nu - k}) \qquad \text{as } r \to \infty \,, \tag{48}$$

for all $k \geq 0$, and terms as above.

Conically singular spaces admit a natural class of principal bundles. We state all definitions for the case of an ICS manifold. Analogue definitions follow for the AC case.

Definition 3.7. *Let $P \to M$ be a principal G-bundle over an ICS space M. We will say P is an admissible bundle if, for each p_i, there exist principal G-bundles Q_i over Σ_i and principal bundle isomorphisms*

$$F_i : \pi_i^* Q \to \Psi_i^* \left(P|_{U_i \setminus \{p_i\}} \right),$$

where $\pi_i : C(\Sigma_i) \to \Sigma_i$ are the natural projection map and Ψ_i are the diffeomorphisms of Definition 3.4. In this case we say P is framed by (Q_i, F_i).

This definition extends naturally to vector bundles. Given a framing, we can define the corresponding admissible objects, like bundle metrics and connections:

Definition 3.8. *Let (E, h, ∇) be a vector bundle E with a bundle metric h and a metric connection ∇ over an ICS manifold. We will say the triple (E, h, ∇) is admissible of rate $\nu > 0$ if E is framed by $E_i \to \Sigma_i$ and there exist bundle metrics h_i^∞ and metric connections ∇_i^∞ on each E_i such that*

$$|\nabla_i^k \left(F_i^*(h) - h_i^\infty \right)|_{g_C \otimes h_i^\infty} = O(r_i^{\nu_i - k}) \qquad |\nabla_i^k \left(F_i^*(\nabla) - \nabla_i^\infty \right)|_{g_C \otimes h_i^\infty} = O(r_i^{\nu_i - k}) \quad \text{as } r_i \to 0 \;,$$

where F_i are the induced bundle isomorphisms from Definition 3.7.

Remark 3.9. *One may define an admissible connection without choosing a metric; all that is needed is the choice of a compatible metric at the framing.*

As in the case of smooth Riemannian manifolds, smooth connections can be viewed as admissible connections, where the framing connection is the trivial one.

3.1 Weighted spaces

To study the Fredholm properties of various differential operators on M, we need to introduce suitable Banach spaces on which the operators act. For the remainder of this section, M will denote an n-dimensional ICS manifold and $E \to M$ an admissible vector bundle. We start by defining the usual and the weighted Banach spaces on sections of E, in the same spirit of [LO85]. First, we define a radius function to lighten the notation.

Definition 3.10. *Let M be an ICS manifold. We say $\rho : M \to (0, 1]$ is a radius function if we have constants $0 < c_1 < 1 < c_2$ such that for each i, we have*

$$c_1 r_i < \Psi_i^*(\rho) < c_2 r_i \tag{49}$$

on $(0, \epsilon) \times \Sigma_i$, where Ψ_i are the diffeomorphisms from Definition 3.4.

By using partitions of unity, it is clear that all ICS manifolds admit a radius function.

Definition 3.11. *Let (E, h, ∇) be an admissible triple over M and ρ a radius function. For all $p \geq 1$, $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\mu \in \mathbb{R}$, we define the weighted Sobolev space $W_\mu^{k,p}$ and the weighted*

Hölder space $C^{k,\alpha}_\mu$ of sections of E as the norm completion of $C^\infty_c(M,E)$ with respect to the norms

$$||\xi||_{W^{k,p}_\mu} = \left(\sum_{j=0}^{k} \int_M \left(\rho^{j-\mu}|\nabla^j \xi|\right)^p \rho^{-n}\, \mathrm{dvol} \right)^{1/p} \qquad ||\xi||_{C^{k,\alpha}_\mu} = \sum_{j=0}^{k} ||\rho^{j-\mu}\nabla^j\xi||_{C^0} + \left[\rho^{k-\mu}\nabla^k\xi\right]_\alpha \ ,$$

where the Hölder seminorm $[\cdot]_\alpha$ is defined using ∇-parallel transport to locally identify the fibres of E.

Some authors (cf. [Mar02]) prefer to consider the multi-index $\underline{\mu} \in \mathbb{R}^m$ to allow different decay rates at each singularity. We find no relevant advantage in working with these spaces, so we will always assume that the decay rate is the same around all singularities.

Notice that we have an isomorphism $W^{0,p}_{\frac{n}{p}} \cong L^p$. If we drop the Hölder seminorm, we get the usual spaces C^k_μ. Similarly, we set $C^\infty_\mu = \cap_{k\in\mathbb{N}} C^k_\mu$. By Definition 3.10, it is clear that different choices of radius function yield equivalent norms. As in the compact case, we get an analogue of the Sobolev embedding theorems, by adapting the results from Bartnik [Bar86, Thm. 1.2] and Lockhart and McOwen [LO85, Lemma 7.2].

Theorem 3.12 (Sobolev embeddings for weights spaces)**.** *Let M be an n-dimensional ICS manifold, equipped with an admissible triple (E,h,∇). Let $p,q \geq 1$, $\alpha,\beta \in (0,1)$, $k,l \in \mathbb{N}$ and consider the associated Sobolev and Hölder spaces of sections. Then:*

1. *If $k \geq l$, $k - \frac{n}{p} \geq l - \frac{n}{q}$, $p \leq q$ and $\mu \geq \mu'$, there is a continuous embedding $W^{k,p}_\mu \hookrightarrow W^{l,q}_{\mu'}$.*
2. *If $\mu > \mu'$ and $k - \frac{n}{p} \geq l + \alpha$, the embeddings $W^{k,p} \hookrightarrow C^{l,\alpha}_\mu \hookrightarrow W^{l,q}_{\mu'}$ are continuous.*
3. *If $\mu \geq \mu'$ and $k + \alpha \geq l + \beta$, the embeddings $C^{k+1}_\mu \hookrightarrow C^{k,\alpha}_\mu \hookrightarrow C^{l,\beta}_{\mu'} \hookrightarrow C^l_{\mu'}$ are continuous.*
4. *If $\mu_1 + \mu_2 \geq \mu$, the multiplication of smooth section extends to a continuous map*

$$C^{k,\alpha}_{\mu_1}(E) \times C^{k,\alpha}_{\mu_2}(E) \hookrightarrow C^{k,\alpha}_\mu(E \otimes E)\ .$$

The main purpose of introducing these weighted spaces is to make the Implicit Function Theorem for Banach spaces available to us, which we now recall:

Theorem 3.13 (Implicit Function Theorem (IFT), [Lan83, Thm. 2.1])**.** *Let X, Y be Banach spaces and let $f : X \to Y$ be a Fredholm map: its derivative $\mathcal{D}_x f$ is a Fredholm linear operator for all x, so the vector spaces $K = \ker(\mathcal{D}_x f)$ and $C = \mathrm{coker}(\mathcal{D}_x f)$ are finite-dimensional.*

Fix a point $x_0 \in X$, with $y_0 = f(x_0)$. Let $L = \mathcal{T}_{x_0} f$ be the derivative of f at x_0. There are charts (U,κ) for X, $(V,\tilde\kappa)$ for Y and a vector space B such that

$$\kappa : U \to B \oplus K \qquad \tilde\kappa : V \to B \oplus C\ ,$$

such that $\kappa(x_0) = 0$, $\tilde\kappa(x_0) = 0$ and the map $F = \tilde\kappa \circ f \circ \kappa^{-1} : B \oplus K \to B \oplus C$ is given by $F(b,n) = (L(b), \Phi(b,n))$ on an open neighbourhood $W \subseteq B \oplus K$, with $\Phi : W \to C$ a smooth map.

The map Φ is called the Kuranishi map or obstruction map. It essentially encodes the non-linear information of f. Since it holds that $f^{-1}(y_0) \cong F^{-1}(0,0) \cong \Phi^{-1}(0)$, if the obstruction map vanishes we get that $f^{-1}(y_0)$ is diffeomorphic to a neighbourhood of 0 in K.

3.2 Asymptotically conical operators

Let us now consider the relevant class of operators that possess good Fredholm and regularity properties, similar to those of their counterparts in compact manifolds.

Definition 3.14. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator of order k between sections of admissible vector bundles over an ICS manifold. Let $P_i^\infty : \Gamma(E_i^\infty) \to \Gamma(F_i^\infty)$ elliptic operators on the corresponding cone. We will say P is an admissible elliptic operator asymptotic to P_i^∞ if there exists $\mu > 0$ such that for each i and every $l \geq 0$*

$$|\nabla_\infty^l \left[F_i^*(Pu) - P_i^\infty(F_i^* u)\right]| = O(r^{\mu - l - k})$$

for every smooth section u of E on $U_i \setminus p_i$, where F_i are the maps from Definition 3.7.

A moment's thought suffices to realise that the set of admissible elliptic operators forms an algebra under composition. For instance, consider $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ an admissible connection (cf. Definition 3.8). Then any differential operator given by the composition of $\nabla^k : \Gamma(E) \to \Gamma(\otimes^k T^*M \otimes E)$ with a bundle map $\Gamma(\otimes^k T^*M \otimes E) \to \Gamma(F)$ with constant coefficients is an admissible elliptic operator. In particular, it follows that

Proposition 3.15. *Let A be an admissible connection. Then, the associated twisted Dirac operator D_A and twisted Laplacian Δ_A operators are admissible elliptic operators.*

Let us study some basic properties of admissible elliptic operators. First, integration by parts and Stokes' theorem yield

Lemma 3.16. *Let $P : \Gamma(E) \to \Gamma(F)$ be an admissible operator of order one and $P^* : \Gamma(F) \to \Gamma(E)$ its formal adjoint. Then for $u \in W_\mu^{1,2}$ and $v \in W_{\mu'}^{1,2}$ with $\mu + \mu' > 1 - n$, we have*

$$\langle Pu, v \rangle_{L^2} = \langle u, P^* v \rangle_{L^2} \quad .$$

Secondly, many of the local estimates for elliptic operators in $\mathbb{R}^n$ carry over to the ICS case, by adapting the scaling argument of Bartnik [Bar86, Thm. 1.2], which uses the uniform ellipticity property. The argument is the following.

Consider the compact set $K = \rho^{-1}([\epsilon, \infty))$. For $\epsilon > 0$ small enough, we can identify the complement of K with the disjoint collection $\{0 < r_i \leq \epsilon\}$ in $C(\Sigma_i)$ and P with its asymptotic model P_i^∞, up to a small error.

The region $\{0 < r_i \leq \epsilon\}$ can be decomposed into the annuli $\{2^{-k-1}\epsilon \leq r_i \leq 2^{-k}\epsilon\}$ for $k \in \mathbb{N}$. On each annulus, the weighted norms are all equivalent, up to a factor, to the norms on the standard annulus $\{1/2 \leq r_i \leq 1\}$ for each singularity. The desired inequalities follow by applying the standard inequalities in the rescaled annuli and rescaling back. For instance, this yields the weighted version of the standard elliptic regularity estimates.

Theorem 3.17 (Elliptic regularity)**.** *Let $P : \Gamma(E) \to \Gamma(F)$ be an admissible elliptic operator of order k. Suppose that $u, f \in L^2_{loc}$ such that u solves $Pu = f$ in the weak sense. If $u \in L^2_\mu(E)$ and $f \in C^{l,\alpha}_{\mu-k}(F)$, then $u \in C^{l+k,\alpha}_\mu(E)$ and u solves $Pu = f$ in the strong sense. Moreover, we have the estimate*

$$||u||_{C^{l+k,\alpha}_\mu} \leq C \left(||Pu||_{C^{l,\alpha}_{\mu-k}} + ||u||_{L^2_\mu} \right) \tag{50}$$

for $C > 0$ independent of u.

Now, let us deal with the mapping properties of admissible elliptic operators. The idea is that the operators will mimic the behaviour of the model operator on the cone, as in the case of the Laplacian outlined before. Thus, we make the following definitions:

Definition 3.18. *Let $C(\Sigma)$ be a metric cone and a hermitian vector bundle $(E^\infty, h^\infty) \to C(\Sigma)$.*

(i) We say a section $u \in \Gamma(E^\infty)$ is λ-homogeneous if $|u|_{h^\infty}$ is a homogeneous function of rate (or weight) λ.

Let $P_i^\infty : \Gamma(E_i^\infty) \to \Gamma(E_i^\infty)$ be a (formally) self-adjoint elliptic operator on the cone $C(\Sigma)$.

(ii) We say that $\lambda \in \mathbb{R}$ is an indicial root if there exists $u \in \Gamma(E_i^\infty)$ such that u is λ-homogeneous and satisfies $P_i^\infty(u) = 0$. We denote the indicial roots of P_i^∞ by $\mathcal{D}(P_i^\infty)$.

(iii) For $\lambda \in \mathcal{D}(P_i^\infty)$, consider the space $\mathcal{K}_\lambda(P_i^\infty)$ of sections $u \in \ker(P_i^\infty)$ of the form $u = \sum_{j=0}^m r^\lambda \log^j(r) u_j$, with each u_j a λ-homogeneous section.

(iv) For an admissible elliptic operator P on an ICS, a rate $\lambda \in \mathbb{R}$ is called a critical rate if λ lies in $\mathcal{D}(P_i^\infty)$ for some i. Set

$$\mathcal{D}(P) = \bigcup_{i=1}^m \mathcal{D}(P_i^\infty) \qquad \text{and} \qquad d(\lambda) = \sum_{i=1}^m \dim\, \mathcal{K}_\lambda(P_i^\infty) \ .$$

In general, one must allow for complex values of the critical rates. However, we will almost exclusively consider formally self-adjoint operators of order one, which guarantees that the critical rates are all real (cf. proof of Lemma 3.19).

The following lemma rules out the appearance of log terms in $\ker(P^\infty)$ and will be useful to us later on.

Lemma 3.19. *Let $P : \Gamma(E) \to \Gamma(E)$ be an admissible self-adjoint operator of order one, asymptotic to P_i^∞. Let $u = \sum_{j=0}^m r^\lambda \log^j(r) u_j \in \ker(P_i^\infty)$. Then $m = 0$.*
In particular, dim $\mathcal{K}_\lambda(P_i^\infty)$ *is equal to the multiplicity of λ as an indicial root.*

Proof. Let $u = \sum_{j=0}^m r^\lambda \log^j(r) u_j$ with u_j λ-homogeneous. Let P_{Σ_i} the operator induced on the link by P_i^∞, i.e $P_i^\infty = \partial_r + \frac{1}{r} P_{\Sigma_i}$. The condition that $P_i^\infty(u) = 0$ is equivalent to $P_{\Sigma_i}(u_j) = \lambda u_j$. Now, collecting the $\log^m(r)$ and $\log^{m-1}(r)$ terms of $P_i^\infty(u) = 0$, we have

$$P_{\Sigma_i}(u_m) = \lambda u_m \qquad P_{\Sigma_i}(u_{m-1}) + m u_m = \lambda u_{m-1} \ .$$

Projecting the latter equation to u_m and using the fact that P_{Σ_i} is self-adjoint, we have

$$m||u_m||^2 = \lambda\langle u_{m-1}, u_m\rangle - \langle P_{\Sigma_i}(u_{m-1}), u_m\rangle = \lambda\langle u_{m-1}, u_m\rangle - \langle u_{m-1}, P_{\Sigma_i}(u_m)\rangle = 0 \ ,$$

so $m = 0$, as needed. □

We can now extend the regularity result Theorem 3.17 to obtain an improved decay estimate for our solutions.

Proposition 3.20. *Let $P : \Gamma(E) \to \Gamma(F)$ be an admissible elliptic operator of order k and assume we have u and f as in Theorem 3.17. Moreover, let $\mu' > \mu$ such that $[\mu, \mu'] \cap \mathcal{D}(P) = \emptyset$, so there are no indicial roots of P contained in the interval. Then there exists $C > 0$ such that*

$$||u||_{C^{l+k,\alpha}_{\mu'}} \leq C \left(||Pu||_{C^{l,\alpha}_{\mu'-k}} + ||u||_{L^2_\mu} \right) \tag{51}$$

Proof. For a large enough compact set, one can study the boundary value problem $P(u-v) = 0$ on $M \setminus K$ with $u = v$ on ∂K with $v \in O(r^\mu)$ and $u \in O(r^{\tilde{\mu}})$. Since there are no indicial roots in $[\mu', \mu]$, one can prove (cf. [LO85]) the estimate

$$||u||_{C^{l+k,\alpha}_{\mu'}} \leq C \left(||Pu||_{C^{l,\alpha}_{\mu'-k}} + ||u||_{C^{k,\alpha}_{\mu}} \right) .$$

Combined with the elliptic regularity theorem above, the claim follows. □

As an immediate corollary, we have

Corollary 3.21. *The kernel of $P^{k+l,\alpha}_\mu : C^{k+l,\alpha}_\mu \to C^{l,\alpha}_{\mu-k}$ is independent of k (and thus α). Moreover, the kernel is also invariant from the decay rate μ, as long as we stay away from critical rates. That is $\ker(P)_\mu \cong \ker(P)_{\mu'}$ as long as $[\mu, \mu'] \cap \mathcal{D}(P) = \emptyset$.*

We can now state the main result that motivated this discussion

Theorem 3.22 ([LO85, Thm. 1.1])**.** *Let $P : \Gamma(E) \to \Gamma(F)$ be an admissible elliptic operator of order k. Then the set $\mathcal{D}(P)$ is discrete and the corresponding map $P^{k+l,\alpha}_\mu$ is Fredholm whenever $\mu \notin \mathcal{D}(P)$. If $\mu \in \mathcal{D}(P)$, the operator $P^{k+l,\alpha}_\mu$ fails to be Fredholm only because its image is not closed.*

A detailed proof can be found in [LO85] and [Mar02]. The theorem above implies, in particular, that we can make sense of the index of $P^{k+l,\alpha}_\mu$ for generic μ. As in the compact case, we have a version of the Fredholm Alternative:

Theorem 3.23. *Let $P : \Gamma(E) \to \Gamma(F)$ be an admissible elliptic operator of order k, and P^* its formal adjoint. Take $\lambda \notin \mathcal{D}(P)$, so $P^{k+l,\alpha}_\mu : C^{k+l,\alpha}_\mu \to C^{k,\alpha}_{\mu-k}$ is Fredholm. Then, there is an isomorphism $\operatorname{coker}(P)_\mu \cong \ker(P^*)_{k-n-\mu}$.*

Notice that $\ker(P^*)_{k-n-\mu}$ will be contained in the codomain of $P^{k+l,\alpha}_\mu$ whenever $\mu < k - \frac{n}{2}$. In this case, the isomorphism is an equality.

A natural question is how to compute the index of $P^{k,\alpha}_\mu$. We give two results in that direction. First, as in the cone case, the change of the index between two non-critical rates is accounted for by counting the solutions to the corresponding model problem on the cone:

Theorem 3.24 ([LO85])**.** *Let $P : \Gamma(E) \to \Gamma(E)$ be an admissible elliptic operator of order k and consider $\mu, \mu' \in \mathbb{R} \setminus \mathcal{D}(P)$, with $\mu' > \mu$. The index of the Fredholm operators $P^{l,\alpha}_\mu$ does not depend on l or α, and*

$$\operatorname{Ind}_{\mu'}(P) - \operatorname{Ind}_\mu(P) = - \sum_{\lambda \in \mathcal{D}(P)} d(\lambda) . \tag{52}$$

Therefore, we are interested in computing $\mathrm{Ind}_\mu(P)$ for some value μ and the set of indical roots $\mathcal{D}(P)$. If $P = \widehat{D}_A$ is a twisted Dirac operator, we get the following result by adapting the Atiyah–Patodi–Singer index theorem:

Theorem 3.25 (cf. [APS75, Thm. 4.2])**.** *Let $E \to M$ be an admissible hermitian operator over an ICS manifold with singularities $\{p_1, \dots, p_m\}$. Consider A an admissible unitary connection on E, framed by $\underline{A}^\infty = (A_1^\infty, \dots, A_m^\infty)$. Then the twisted Dirac operator $\widehat{D}_A$ is admissible, and satisfies*

$$\mathrm{Ind}_{\frac{1-n}{2}}(\widehat{D}_A) = \int_M \mathrm{ch}(E)\widehat{A}(M) + \frac{d(0) + \eta_{\underline{A}^\infty}(0)}{2} , \tag{53}$$

where $\mathrm{ch}(E)$ *and* $\widehat{A}(M)$ *denote the Chern character of E and the $\widehat{A}$-genus of M respectively; and $\eta_{\underline{A}^\infty}$ is the meromorphic continuation of the eta function*

$$\eta_{\underline{A}^\infty}(s) = \sum_{\lambda \in D(\widehat{D}_A)} \mathrm{sign}(\lambda)\lambda^{-s} . \tag{54}$$

Proof. (Idea). The standard L^2-index theorem of Atiyah, Patodi and Singer (APS) applies to manifolds with boundary, where we have a cylindrical collar $I \times Y$ on the boundary, and the operator P takes the shape $P = \partial_t + P_Y$ on the collar.

In our case, our conical ends are conformal to cylindrical ends. Since the index of the Dirac operator is a conformal invariant, the APS index theorem carries over to our case for the weight $\overline{\mu} = \frac{1-n}{2}$, which corresponds to L^2-sections under the conformal rescaling of the metric. A detailed discussion of how the result is adapted from the cylindrical to the conical setting is provided in [Moo17]. □

Remark 3.26. *If $\overline{\mu} = \frac{1-n}{2}$ is an indicial root, one needs to take closed images to compute the index, since the operator is not Fredholm.*

4 Instantons over spaces with conical singularities

We study instantons on spaces with isolated conical singularities (ICS). The asymptotically conical case has been previously treated for Spin(7)-instantons in [Pap22; Gho24], and for G_2-instantons in [Dri21]. Related results in the more general setting of conically singular spaces were also obtained by Yuanqi Wang [Wan18a; Wan19], using non-standard weighted function spaces. Our approach offers a more accessible framework that simplifies some aspects of his analysis.

For the remainder of this section, we consider a compact manifold (M^n, g) of dimension $n > 4$, with isolated conical singularities (ICS) at points $\{p_1, \dots, p_m\}$, and equipped with a closed $(n-4)$-form $*\Omega$. Let $P \to M \setminus \{p_1, \dots, p_m\}$ be an admissible principal G-bundle with fixed framing connections A_i^∞ near each singularity. We denote by ρ a radius function as in Definition 3.10.

We focus on connections on P that are asymptotic to the framing connections A_i^∞ at rate μ. Analogously to the compact setting, the space of such connections forms an affine space, denoted $\mathcal{A}_{\mu-1}$, modelled on the vector space $\Omega^1(\mathfrak{g}_P)_{\mu-1}$. Throughout this chapter, we fix $\mu > 0$

sufficiently small so that all admissible elliptic operators we encounter have no indicial roots in the interval $(0, \mu)$.

In the compact case, we proved the existence of a topological charge associated with a principal bundle P and a closed $(n-4)$-form $*\Omega$. This charge ensured that instantons are absolute minima of the Yang–Mills functional and provided specific topological rigidity results. This framework extends naturally to the conically singular setting. Specifically, we have the following:

Proposition 4.1. *Let $A \in \mathcal{A}_{\mu-1}$ be a connection on P asymptotic to the framing connections A_i^∞ at rate μ. Then:*

(i) There exists a well-defined topological charge $c_\Omega(P, A)$.

(ii) The charge is independent of the choice of connection within the class $\mathcal{A}_{\mu-1}$.

(iii) The charge is independent of the framing connections A_i^∞.

(iv) If (M^n, g) carries multiple suitable closed $n-4$ forms $*\Omega_k$, then there exists a well-defined charge difference $D(P, A)$, which only depends on the principal bundle P.

Proof. Let $A \in \mathcal{A}_{\mu-1}$. For $\epsilon > 0$, let $U_\epsilon = \{x \in M | \, \rho(x) > \epsilon\}$ and consider

$$c_\Omega(P) = \lim_{\epsilon \to 0} c_\Omega^\epsilon(P) = \lim_{\epsilon \to 0} \int_{U_\epsilon} F_A^2 \wedge *\Omega \, .$$

(i) Since A is in $\mathcal{A}_{\mu-1}$, we have that $|F_A| = O(\rho^{-2})$ and $\rho^{-2} \in L^2(M)$ since $n > 4$.

(ii) Now, let $A, B \in \mathcal{A}_{\mu-1}$. We know that $F_A^2 - F_B^2 = dCS(A, B)$, the Chern-Simons 3-form. As $*\Omega$ is closed by assumption, Stokes' theorem implies

$$c_\Omega^\epsilon(P)(A) - c_\Omega^\epsilon(P)(B) = \int_{U_\epsilon} \mathrm{Tr}\,(F_A^2 - F_B^2) \wedge *\Omega = \int_{\partial U_\epsilon} CS(A, B) \wedge *\Omega \, .$$

Since $A, B \in \mathcal{A}_{\mu-1}$, we have the estimate $|CS(A, B)|_{\{\rho=\epsilon\}} \leq C\epsilon^{-3+2\mu}$, and so

$$|c_\Omega^\epsilon(P)(A) - c_\Omega^\epsilon(P)(B)| \leq |CS(A, B)|_{\{\rho=\epsilon\}} \mathrm{vol}(\rho^{-1}(\epsilon)) \leq C\epsilon^{n-4+2\mu}$$

for $\epsilon > 0$ small enough.

(iii) Assume A and B are framed by A_i^∞ and B_i^∞ respectively. As above, we need to bound $|CS(A, B)|_{\{\rho=\epsilon\}}$. If $\epsilon > 0$ is small enough, we have

$$|CS(A, B)|_{\{\rho=\epsilon\}} \leq |CS(A, A_i^\infty)|_{\{\rho=\epsilon\}} + |CS(A_i^\infty, B_i^\infty)|_{\{\rho=\epsilon\}} + |CS(B_i^\infty, B)|_{\{\rho=\epsilon\}} \leq C\epsilon^{-3} \, .$$

(iv) The arguments used in items $(i) - (iii)$ carry verbatim for the charge difference. □

In particular, we have the following immediate corollary.

Corollary 4.2. *The topological rigidity statements in Proposition 1.7 hold in the case of a manifold with ICS.*

Remark 4.3. *In the case of AC manifolds, one cannot define analogue quantities $c_\Omega(P)$ and $D(P)$, but one can still make sense of the differences $c_\Omega(P,A)-c_\Omega(P,B)$ and $D(P,A)-D(P,B)$, provided $A, B \in \mathcal{A}_{\mu-1}$ for $\mu < \frac{4-n}{2}$. In particular, this is enough to obtain the rigidity analogues of Proposition 1.7(cf. [Pap22] and [MW24]).*

We have the following result to complete the parallelism with the compact case.

Proposition 4.4. *Assume M^n is a smooth manifold with $n > 5$ and P is an admissible ICS principal bundle, with structure group $G = \mathrm{SU}(2)$. Then $c_2(P)$ is well defined in $H^4(M,\mathbb{R})$, and the charge $c_\Omega(P)$ above coincides with the topological charge $\langle c_2(P) \cup [*\Omega], [M]\rangle$*

Proof. If P extends to a smooth principal bundle, there is nothing to prove. Otherwise, since $\pi_i(\mathrm{SU}(2))$ is a torsion group for all $i > 4$, there are only finitely many SU(2)-bundles over S^{n-1}. So, we can perform a surgery to change the admissible bundle P to a smooth bundle that will not change $c_2(P)$ in real cohomology. Alternatively, by Mayer-Vietoris, we have $H^4(M,\mathbb{R}) \cong H^4(M \setminus \{p_i\},\mathbb{R})$.

The standard Chern–Weil argument implies that the topological invariant $c_2(P)$ computed using a connection on the admissible bundle agrees with the one defined from a topological perspective; as both represent the same cohomology class in $H^4(M,\mathbb{R})$. □

5 A deformation problem

We now focus on the deformation theory of instantons modelled on the configuration space

$$\mathcal{A}^{k,\alpha}_{\mu-1} \cong \left\{ A + a \;\middle|\; a \in \Omega^1(\mathfrak{g}_P)^{k,\alpha}_{\mu-1} \right\} ,$$

where A is an admissible connection asymptotic to the framing connections A_i^∞ at each singular point.

To understand the local structure of the moduli space, we begin by studying the action of the gauge group and establishing a suitable slice theorem. Since the framing is fixed, we require the gauge transformations to act trivially at the singularities. Therefore, the space of infinitesimal gauge transformations is taken to be $\Omega^0(\mathfrak{g}_P)^{k+1,\alpha}_\mu$. As we shall see later, this is not the most suitable space to consider. For now, we have

Proposition 5.1. *The space $\Omega^0(\mathfrak{g}_P)^{k+1,\alpha}_\mu$ carries a natural Lie algebra structure under the pointwise bracket. Moreover, it is the Lie algebra of a Hilbert Lie group $\mathcal{G}^{k+1,\alpha}_\mu$ of gauge transformations acting smoothly on $\mathcal{A}^{k,\alpha}_{\mu-1}$.*

Proof. By the Sobolev embeddings in Theorem 3.12, we have a continuous map

$$\Omega^0(\mathfrak{g}_P)^{k+1,\alpha}_\mu \times \Omega^0(\mathfrak{g}_P)^{k+1,\alpha}_\mu \to \left(\Omega^0(\mathfrak{g}_P) \otimes \Omega^0(\mathfrak{g}_P)\right)^{k+1,\alpha}_\mu \xrightarrow{[\cdot,\cdot]} \Omega^0(\mathfrak{g}_P)^{k+1,\alpha}_\mu .$$

So the space $\Omega^0(\mathfrak{g}_P)^{k,\alpha}_\mu$ inherits the structure of a Lie algebra from $\mathfrak{g}$.

The remainder of the statement is to prove that the (usual) exponential map is well-defined and defines a smooth group action; and that $\mathcal{G}^{k+1,\alpha}_\mu$ acts smoothly on $\mathcal{A}^{k,\alpha}_{\mu-1}$. The remainder of the proof is verbatim to that of Freed and Uhlenbeck [FU84, Prop. A.2, A.3]. □

We now construct a slice for the action of the gauge group in Coulomb gauge, defined by the condition $d_A^* a = 0$. That is, given a perturbation $a \in \mathcal{A}_{\mu-1}^{k,\alpha}$, we want to find a gauge transformation $g \in \mathcal{G}_\mu^{k+1,\alpha}$ such that

$$S_A(g, a) := d_A^* \left(g^{-1} d_A g + g^{-1} a g \right) = 0.$$

The standard strategy is to apply the Implicit Function Theorem (Theorem 3.13) to solve this equation. The linearisation at $(\mathrm{id}, 0)$ is

$$\delta S_A = d_A^* d_A(\delta g) + \delta a.$$

To apply the IFT, we need the operator $d_A^* d_A$ to be invertible. We begin by showing it is injective.

Lemma 5.2. *The operator $d_A^* d_A : \Omega^0(\mathfrak{g}_P)_\mu^{k+1,\alpha} \to \Omega^0(\mathfrak{g}_P)_{\mu-2}^{k-1,\alpha}$ is injective.*

Proof. For $k \geq 2$, let $f \in \Omega^0(\mathfrak{g}_P)_\mu^{k,\alpha}$ lie in the kernel of $d_A^* d_A$. Then, by integration by parts (Lemma 3.16), we have that $||d_A f||_{L^2}^2 = \langle d_A^* d_A f, f \rangle_{L^2} = 0$, so f is constant. Since $\mu > 0$, we have $|f| \to 0$ near the singularities, and so $f = 0$. □

However, this operator is generally not surjective. We compute its cokernel

Proposition 5.3. *The cokernel of the operator $d_A^* d_A : \Omega^0(\mathfrak{g}_P)_\mu^{k+1,\alpha} \to \Omega^0(\mathfrak{g}_P)_{\mu-2}^{k-1,\alpha}$ is isomorphic to the direct sum of the Lie algebra stabilisers $\mathrm{Stab}(A_i^\infty)$ at each singular point p_i. In particular, the operator is invertible if and only if every framing connection A_i^∞ is irreducible.*

To prove this, we need to understand the indicial roots of the Laplacian:

Lemma 5.4. *Let $d_A^* d_A$ be the Laplace operator defined above. Then:*

(i) The set of indicial roots is given by

$$\mathcal{D}(d_A^* d_A) = \bigcup_{i=1}^m \left\{ \nu \in \mathbb{R} \,\middle|\, \nu(\nu + n - 2) \text{ is an eigenvalue of } d_{A_i^\infty}^* d_{A_i^\infty} \right\}.$$

In particular, $\mathcal{D}(d_A^* d_A) \cap (-n + 2, 0) = \emptyset$.

(ii) The value $\nu = 0$ is an indicial root if and only if A_i^∞ is reducible for some i.

(iii) Moreover, $d(0) = \sum_i \dim \mathrm{Stab}(A_i^\infty)$.

Proof. (i) Suppose $\nu \in \mathcal{D}(d_A^* d_A)$. Then, there exists a singular point p_i and $h \in \Omega^0(\mathfrak{g}_{P_i^\infty})$ such that $d_{A_i^\infty}^* d_{A_i^\infty}(r^\nu h) = 0$. Expanding this gives

$$d_{A_i^\infty}^* d_{A_i^\infty}(r^\nu h) = r^{\nu-2} \left(d_{A_i^\infty}^* d_{A_i^\infty} h - \nu(\nu + n - 2)h \right).$$

(ii) Setting $\nu = 0$, we want to count solutions $d_{A_i^\infty}^* d_{A_i^\infty} h = 0$, which corresponds to h generating an infinitesimal gauge symmetry of A_i^∞, i.e. $h \in \mathrm{Stab}(A_i^\infty)$.

(iii) This is equivalent to showing there are no $\log^j(r)$ contributions to the kernel of $d^*_{A_i^\infty} d_{A_i^\infty}$. From Lemma 5.2, we know that this kernel coincides with that of $d_{A_i^\infty}$, and the claim follows from Lemma 3.19. □

Proof (of Proposition 5.3). Since $P = d_A^* d_A$ is formally self-adjoint, we have $\operatorname{coker}(P)_\mu = \ker(P)_{2-n-\mu}$. In particular, $\operatorname{Ind}_{\frac{2-n}{2}}(P) = 0$. From part (i) of the lemma above, it follows that $\operatorname{Ind}_\delta(P) = 0$ for $\delta \in (-n+2, 0)$. Now, the index change formula (52) implies

$$-\operatorname{coker}(P)_\mu = \operatorname{Ind}_\mu(P) = -d(0) ,$$

and parts (ii) and (iii) of the lemma above finish the proof. □

Thus, for the gauge slice construction to work, we must enlarge our gauge group to account for the presence of reducible framing connections. For each singular point p_i, let χ_i be a smooth cut-off function satisfying

$$\chi_i(x) = \begin{cases} 1 & \text{if } \rho(x) < \frac{\varepsilon}{10}, \\ 0 & \text{if } \rho(x) > \frac{2\varepsilon}{10}, \end{cases}$$

for some small $\varepsilon > 0$. Define the finite-dimensional subspace

$$V_i = \{\chi_i g_0 \,|\, g_0 \in \mathfrak{stab}(A_i^\infty)\} ,$$

where $\mathfrak{stab}(A_i^\infty)$ is the Lie algebra of the stabiliser of the framing connection near the singularity p_i, and we identify $g_0 \in \Omega^0(\Sigma, \mathfrak{g}_P)$ with its pullback to the cone. We then define the extended Hölder Lie algebra

$$\widehat{\Omega^0}(\mathfrak{g}_P)^{k+1,\alpha}_\mu := \Omega^0(\mathfrak{g}_P)^{k+1,\alpha}_\mu \oplus \bigoplus_i V_i.$$

Remark 5.5. *Although the spaces V_i depend on the choice of cut-off function χ_i, the total space $\widehat{\Omega^0}(\mathfrak{g}_P)^{k+1,\alpha}_\mu$ is independent of these choices.*

The following proposition is now a direct consequence of our previous work.

Proposition 5.6. *We have:*

(i) The space $\widehat{\Omega^0(\mathfrak{g}_P)}\,^{k+1,\alpha}_\mu$ carries a Lie algebra structure, with $\Omega^0(\mathfrak{g}_P)^{k+1,\alpha}_\mu$ a Lie ideal.

(ii) There is an associated Hilbert Lie group $\widehat{\mathcal{G}}^{k+1,\alpha}_\mu$, which acts smoothly on the space of connections $\mathcal{A}^{k,\alpha}_{\mu-1}$.

(iii) The map $d_A^* d_A : \widehat{\Omega^0(\mathfrak{g}_P)}\,^{k+1,\alpha}_\mu \to \Omega^0(\mathfrak{g}_P)^{k-1,\alpha}_{\mu-2}$ is surjective. Its kernel is given by the Lie algebra of the stabiliser of the connection A.

Proof. Statement (i) follows from above, as does the associated Hilbert Lie group construction. In order to see that it acts smoothly on $\mathcal{A}^{k,\alpha}_{\mu-1}$, it suffices to notice that the map

$$d_A : V_i \to \Omega^1(\mathfrak{g}_P)^{k,\alpha}_{\mu-1}$$

is well-defined. Indeed,

$$d_A(\chi_i g_0) = d\chi_i g_0 + \chi_i d_A g_0 = d\chi_i g_0 + \chi_i(d_A - d_{A_i^\infty})g_0 \ , \tag{55}$$

where we used that $g_0 \in \mathfrak{stab}(A_i^\infty)$ and so $d_{A_i^\infty} g_0 = 0$, and we have omitted the principal bundle framing morphism to lighten notation. Now, the first term in (55) lies in $\Omega^1(\mathfrak{g}_P)^{k,\alpha}_{\mu-1}$ since we took $\chi \equiv 1$ in a neighbourhood of p_i. The condition that the second term lies in $\Omega^1(\mathfrak{g}_P)^{k,\alpha}_{\mu-1}$ is precisely asking that A converges to A_i^∞ at rate at least μ.

For (iii), surjectivity of $d_A^* d_A$ follows from the construction and Proposition 5.3. Indeed, for any $g_0 \in \mathfrak{stab}(A_i^\infty)$, we have

$$d_A^* d_A(\chi_i g_0) = \Delta\chi_i g_0 + 2\langle d\chi_i, d_A g_0\rangle + \chi_i d_A^* d_A g_0 \ ,$$

where the rightmost term is not in the image of the original gauge Lie algebra $\Omega^0(\mathfrak{g}_P)^{k,\alpha}_\mu$ under the map $d_A^* d_A$. This confirms that the extended terms account for the missing cokernel directions.

Finally, let $f \in \widehat{\Omega^0(\mathfrak{g}_P)}^{\,k+1,\alpha}_{\mu} \cap \ker(d_A^* d_A)$. As in the proof of Lemma 5.2, f is constant, and must take values in $\bigcap_i \mathfrak{stab}(A_i^\infty)$. So f is an infinitesimal gauge transformation preserving A, as needed. □

We have now arrived at

Theorem 5.7 (Slice theorem)**.** *Let $A \in \mathcal{A}^{k,\alpha}_{\mu-1}$ be an irreducible connection and consider the map $d_A^* : \Omega^1(\mathfrak{g}_P)^{k,\alpha}_{\mu-1} \to \Omega^0(\mathfrak{g}_P)^{k-1,\alpha}_{\mu-2}$. Then $\mathcal{A}^{k,\alpha}_{\mu-1}$ is locally diffeomorphic to $\ker(d_A^*) \times \widehat{\mathcal{G}}^{k+1,\alpha}_\mu$ in a neighbourhood of A.*

Now that we have identified a suitable gauge slice, we can proceed to study the deformation theory of instantons with isolated conical singularities. We focus on irreducible Spin(7) and G_2-instantons. The relevant moduli space is defined as

$$\mathcal{M}(\underline{A}^\infty)^{k,\alpha}_\mu := \frac{\{A \in \mathcal{A}^{k,\alpha}_{\mu-1} |\ A \text{ is an irreducible } *\Omega\text{−instanton}\}}{\widehat{\mathcal{G}}^{k+1,\alpha}_\mu} \ ,$$

where $\underline{A}^\infty = (A_1^\infty, \dots A_m^\infty)$ are the framing connections.

From Section 1, we know that the Coulomb gauge condition, together with the linearised instanton equation, fit into an elliptic complex. This complex provides a framework for analysing the deformation problem by applying the Inverse Function Theorem 3.13 in weighted Banach spaces. We obtain a Kuranishi model for the moduli space in a neighbourhood of a given instanton A. We have a pair of finite-dimensional spaces:

- The infinitesimal deformation space $I(A, \mu) := H^1_{A,\mu}$, representing solutions to the linearised problem modulo infinitesimal gauge transformations;
- The obstruction space $O(A, \mu) := H^2_{A,\mu}$, capturing the failure of surjectivity of the linearisation;

where $H^i_{A,\mu}$ are the cohomology groups associated to the corresponding deformation complex and implicitly depend on k and α. The Kuranishi map $F : H^1_{A,\mu} \to H^2_{A,\mu}$ determines the local structure of the moduli space as detailed in Theorem 3.13. More precisely, if the map F vanishes, a neighbourhood of A in $\mathcal{M}^{k,\alpha}_A$ will be diffeomorphic to a neighbourhood of $0 \in I(A,\mu)$. Thus, the virtual dimension of the moduli space is defined as $\dim_{\text{vir}}\mathcal{M} = \dim I(A,\mu) - \dim O(A,\mu)$. As in the compact case, we have the following regularity result:

Proposition 5.8. *For k large enough and μ generic, the inclusion map $\mathcal{M}(\underline{A}^\infty)^{k+1,\alpha}_\mu \hookrightarrow \mathcal{M}(\underline{A}^\infty)^{k,\alpha}_\mu$ is a homeomorphism.*

Proof. The proof in the compact case (cf. [DK90, Prop. 4.2.16]) carries over to this case, using the elliptic regularity estimates of Theorem 3.17. □

In particular, we may consider the limit of k large and drop the implicit dependency of k and α in the previous discussion, and consider the moduli $\mathcal{M}(\underline{A}^\infty)_\mu$.

In the Spin(7)-case, the elliptic complex we are considering is the 3-term complex

$$0 \to \widehat{\Omega^0(\mathfrak{g}_P)}^{k+1,\alpha}_\mu \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P)^{k,\alpha}_{\mu-1} \xrightarrow{\pi^\perp \circ d_A} \Omega^2_7(\mathfrak{g}_P)^{k-1,\alpha}_{\mu-2} \to 0 \ , \tag{56}$$

We can compute its virtual dimension using the Atiyah–Patodi–Singer index Theorem 3.25:

Theorem 5.9. *Let A be an irreducible admissible conically singular* Spin(7)*-instanton on a principal $U(k)$-bundle E, with model singularities $\underline{A}^\infty$. Then the instanton moduli space has virtual dimension*

$$\mathcal{M}^A_\mu = \int_M \mathrm{ch}(E)\widehat{A}(M) + \frac{\eta_{A^\infty}(0) - d(0)}{2} - \sum_{\lambda \in (0,5/2)} d(\lambda) \ ,$$

where, setting $D_A(\alpha) = *(\psi \wedge d_A\alpha)$, we have

$$d(\lambda) = \sum_i \dim \left\{ \alpha \in \Omega^1(\Sigma^7_i, \mathfrak{g}_P) \;\middle|\; D_{A^\infty_i}\alpha = \lambda\alpha \right\} \quad \text{for } \lambda > 0,$$

and

$$d(0) = \sum_i \dim \left\{ \alpha \in \Omega^1(\Sigma^7_i, \mathfrak{g}_P) \;\middle|\; D_{A^\infty_i}\alpha = 0 \right\} + \dim \mathfrak{stab}(A^\infty_i) \ .$$

Proof. We can identify the complex (56) with the twisted Dirac operator

$$\widehat{D_A} : \Omega^1(\mathfrak{g}_P)^{k,\alpha}_{\mu-1} \xrightarrow{(d^*_A \ , \ \pi^\perp \circ d_A)} \left(\Omega^0 \oplus \Omega^2_7\right)(\mathfrak{g}_P)^{k-1,\alpha}_{\mu-2} \ . \tag{57}$$

By Theorem 5.7 and the discussion above, we have that $\dim_{\text{vir}}\mathcal{M}_\mu = \mathrm{Ind}_\mu(D_A)$. From Theorem 3.25, we have

$$\mathrm{Ind}_{-7/2}(\widehat{D_A}) = \int_M \mathrm{ch}(E)\widehat{A}(M) + \frac{d(0) + \eta_{A^\infty}(0)}{2} \ .$$

Using the index change formula (52) from Theorem 3.24, we need a count of the indicial roots of this operator between $-7/2$ and $\mu - 1$. By Lemma 2.24, the claim follows. □

In the G_2 case, it is convenient first to consider the moduli space of G_2-monopoles. These are pairs $(f, A) \in (\Omega^0(\mathfrak{g}_P) \times \mathcal{A})^{k,\alpha}_{\mu-1}$ solving the monopole equation

$$d_A f + *(F_A \wedge \psi) = 0 \ . \tag{58}$$

We investigate the expected or virtual dimension of their moduli space. As in the previous case, the deformation problem can be identified with the twisted Dirac operator $\widehat{D_A}$ (cf. Equation (17)), acting on pairs of sections:

$$\begin{aligned} \widehat{D_A} : \left(\Omega^0 \oplus \Omega^1\right)(\mathfrak{g}_P)^{k,\alpha}_{\mu-1} &\to \left(\Omega^0 \oplus \Omega^1\right)(\mathfrak{g}_P)^{k-1,\alpha}_{\mu-2} \\ (f, \alpha) &\mapsto \left(d_A^* \alpha, d_A f + *(\psi \wedge d_A \alpha)\right) . \end{aligned} \tag{59}$$

Therefore, we have

Theorem 5.10. *Let A be an irreducible conically singular G_2-instanton with model singularities $\underline{A}^\infty$. The moduli space of irreducible G_2-monopoles has virtual dimension*

$$\mathcal{M}^A_\mu = -\sum_i \sum_{\lambda \in [0,2)} d^i_\lambda \ ,$$

where $d(\lambda)$ are defined as in Theorem 5.9.

Proof. Since D_A is formally self-adjoint, we have

$$\mathrm{Ind}_{-3}(\widehat{D_A}) = \ker(\widehat{D_A})_{-3} - \ker(\widehat{D_A}^*)_{-3} = 0 \ .$$

Using the index change formula (52) from Theorem 3.24, we need a count of the indicial roots of this operator between -3 and $\mu - 1$. By Lemma 2.32, the claim follows. ☐

Akin to the compact case, we have

Proposition 5.11. *Let (M^7, φ) be a holonomy G_2-manifold with isolated conical singularities. The forgetful map $(f, A) \mapsto A$ takes G_2-monopoles to G_2-instantons and is surjective. The fibres are the stabiliser of the connection A.*

The proof is the same as in the compact case, since we can integrate by parts by Lemma 3.16.

Remark 5.12. *In both the* Spin(7)*-instanton and G_2-monopole cases, the analytic correction term is bounded above, in absolute value, by the sum $\sum_i \mathrm{Ind}_{A_i^\infty} + \mathrm{Nul}_{A_i^\infty}$ by Corollaries 2.19 and 2.30 respectively; where $\mathrm{Ind}_{A_i^\infty}$ and $\mathrm{Nul}_{A_i^\infty}$ are the Yang–Mills index and nullity of A_i^∞, defined in Equation* (8)*.*

Hitchin functionals

We move to study the geometry of conifolds with special holonomy. We are motivated by the work on G_2-conifold moduli spaces, which have been studied extensively by Karigiannis and Lotay [KL20]. A similar discussion is expected in the case of Spin(7), although the details have not been fully worked out. Partial results corresponding to the asymptotically conical (AC) case can be found in [Leh21].

We have the following definition, in the spirit of Definition 3.4.

Definition 5.13. *Consider (M^n, g) a manifold with ICS; with singularities $p_1, \dots, p_m$, rates $\nu_1, \dots, \nu_m$ with $\nu_i > 0$ and links $\Sigma_1, \dots, \Sigma_m$. We say (M, g) is a holonomy G_2 manifold with ICS if M is equipped with torsion-free G_2-structure φ_M compatible with the metric g, and such that the cones $C(\Sigma_i)$ carry a compatible torsion-free G_2-structures φ_i satisfying*

$$|\nabla^k(\Psi_i^*(\varphi_M) - \varphi_i)| = O(r^{\nu_i - k})$$

on $(0, \varepsilon) \times \Sigma_i$ for each i, where Ψ_i are the diffeomorphisms of Definition 3.4.

In particular, as discussed in Section A.3, the links (Σ_i, g) carry a nearly Kähler structure.

The definitions for the holonomy G_2 asymptotically conical and holonomy Spin(7) cases follow the same logic. Using the techniques from Section 3, Karigiannis and Lotay [KL20] constructed the moduli spaces of ICS and AC holonomy G_2-manifolds and computed its virtual dimension:

Proposition 5.14 ([KL20] Cor. 5.35, Prop. 6.4 & Proposition 6.11)**.** *Let (M, φ) be an AC manifold with holonomy G_2, of generic rate $\nu \in (-3, 0)$ and link (Σ, ω, ρ). Then the moduli space of AC G_2-structures of rate ν has dimension*

$$\dim \mathcal{M}_\nu = b^3_{cs}(M) + \dim\,(\operatorname{im} \Upsilon^3) + \sum_{\lambda \in (-3, \nu)} \dim \mathcal{E}(\Sigma, \omega, \rho, \lambda)\ ,$$

where $b^3_{cs}(M)$ is the dimension of compactly supported harmonic 3-forms and $\Upsilon^3 : H^3(M, \mathbb{R}) \to H^3(\Sigma, \mathbb{R})$ is the map induced by the smooth embedding $\Sigma \hookrightarrow M$, and

$$\mathcal{E}(\Sigma, \omega, \rho, \lambda) = \left\{ \beta \in \Omega^2_{8,coclosed} | \ \Delta\beta = (\lambda + 3)(\lambda + 4)\beta \right\} .$$

Let (M, φ) be holonomy G_2 manifold with ICS; with singularities $p_1, \dots, p_n$, modelled on $\Sigma_1, \dots, \Sigma_n$. Fix $\nu > 0$ sufficiently close to 0. Then, there is a similar formula to the one above for the virtual dimension of the moduli space $\mathcal{M}_\nu$. Moreover, the dimension of the obstruction space $\mathcal{O}$ is bounded above by

$$\dim(\mathcal{O}) \leq n - 1 + \sum_{i=1}^{n} \sum_{\lambda \in (-3, 0)} \dim \mathcal{E}(\Sigma_i, \omega, \rho, \lambda)\ .$$

With the same approach, Lehmann constructs the moduli spaces of AC Spin(7) manifolds and computes their virtual dimension in [Leh21]. We are not aware of the corresponding computation for CS Spin(7) manifolds, but one expects it to be similar to Proposition 5.14.

Proposition 5.15 ([Leh21] Thm. 4.23)**.** *Let (M, Φ) be an AC* Spin(7) *manifold of generic rate $\nu \in (-4, 0)$ and link (Σ, φ). Then the moduli space of AC* Spin(7)*-structures of rate ν has dimension*

$$\dim \mathcal{M}_\nu = b^-_{cs} + \dim(\operatorname{im} \Upsilon^4) + \sum_{\lambda \in (-4,\nu)} \dim \mathcal{E}(\Sigma, \varphi, \lambda) ,$$

where $b^-_{cs}(M)$ is the dimension of compactly supported anti-self-dual harmonic 4-forms, $\Upsilon : H^4(M, \mathbb{R}) \to H^4(\Sigma, \mathbb{R})$ is the map induced by the smooth embedding $\Sigma \hookrightarrow M$ and

$$\mathcal{E}(\Sigma, \omega, \rho, \lambda) = \left\{ \chi \in \Omega^4_{27,exact} |\ d * \chi = -(\lambda + 4)\chi \right\} .$$

As expected, the virtual dimension formulae are comprised of two parts. On the one side, we have a topological term that depends exclusively on the cohomology of the conifold M. On the other side, we have an analytic term that records solutions to an elliptic PDE on the link of each cone singularity, and corresponds to the contributions of the indicial roots of the deformation operator that one considers when constructing the moduli space. In Section 6 of [KL20], Karigiannis and Lotay prove that

Proposition 5.16. *If there are no solutions to $\mathcal{E}(\lambda)$ for $\lambda \in (-3, 0]$, the moduli space of ICS G_2-structures is smooth.*

We provide a geometric interpretation of the spaces $\mathcal{E}$ in terms of the spectrum of the second variation of Chern-Simons type functionals and relate them to the Morse index of a related class of functionals.

6 Stable forms and Hitchin functionals

In the early 2000s, Nigel Hitchin [Hit00] [Hit01] showed how certain geometric structures can be realised as critical points of suitable functionals over a class of generic forms known as stable:

Definition 6.1. *Let V^n be an n-dimensional real vector space. A form $w \in \Lambda^p(V^*)$ is stable if the orbit of w under the induced* GL(V)*-action is open. The set of stable forms is denoted by $\Lambda^p_+(V^*)$.*

Hitchin classified all the possible cases in his original papers. Whenever the stabiliser of a stable form is a subgroup of SL(V), there is an invariant volume form associated with the stable form. Similarly, if the stabiliser is compact, there is an invariant inner product associated with it.
Let $w \in \Lambda^p_+(V^*)$ and assume that $\mathrm{Stab}(w) \subseteq \mathrm{SL}(V)$ for the remainder of the section. Assigning a stable form its invariant volume form defines a GL(V)-invariant map $\mathrm{vol} : \Lambda^p_+(V^*) \to \Lambda^n_+(V^*) \cong \mathbb{R}^*$. For any $\mu \in \mathbb{R}$, this map satisfies

$$\mathrm{vol}(\mu^p w) = \mu^n \, \mathrm{vol}(w) .$$

In other words, vol is homogeneous of degree n/p. Its derivative defines an invariant element $\widehat{w} \in (\Lambda^p V^*)^* \otimes \Lambda^n V^* \cong \Lambda^{n-p} V^*$,

$$\frac{\delta}{\delta\alpha}\operatorname{vol}(w) = \alpha \wedge \widehat{w}\ . \tag{60}$$

We call $\widehat{w}$ the Hitchin dual of w. Using Euler's formula, we obtain the relation

$$w \wedge \widehat{w} = \frac{n}{p}\operatorname{vol}(w)\ . \tag{61}$$

This discussion extends naturally to the setting of smooth manifolds. For convenience, we will reduce our discussion to the case where M is a closed, oriented manifold. We say a smooth p-form $\rho \in \Omega^p(M)$ is stable if it is pointwise stable. The existence of smooth stable forms is only obstructed by the reduction of the frame bundle to the corresponding stabiliser. The open space of stable forms will be denoted by $\Omega^p_+(M)$.

The induced volume map above extends to a smooth map $\operatorname{vol} : \Omega^p_+(M) \to \Omega^n(M)$, and we can define the corresponding volume functional $V : \Omega^p_+ \to \mathbb{R}$ by integrating against the fundamental class of the manifold. We refer to this functional as a Hitchin functional.

Since stable forms form an open set, one can study the variational properties of the functional V. The main result, due to Hitchin, is an application of Stokes' theorem.

Theorem 6.2 ([Hit00]). *A closed stable form $\rho \in \Omega^p_+(M)$ is a critical point of V within its cohomology class if and only if its Hitchin dual is closed; i.e. $d\widehat{\rho} = 0$.*

Let us look at two concrete instances of the Hitchin functional, described in detail in [Hit00].

Example 6.3 (Dimension 6, complex case, [Hit00]). *Let ρ be a stable 3-form on a 6-manifold with stabiliser $\mathrm{SL}(3,\mathbb{C})$. This form corresponds to the existence of a locally decomposable complex volume form $\rho + i\widehat{\rho}$. This complex 3-form defines an almost complex structure on the manifold. The critical point condition for the Hitchin functional implies that the complex volume form is closed, i.e., $d(\rho + i\widehat{\rho}) = 0$, which in turn implies the induced almost complex structure is integrable. Therefore, critical points correspond to complex manifolds equipped with a holomorphic volume form.*

Example 6.4 (Dimension 7 [Hit00]). *Let φ be a stable 3-form on a 7-manifold M with stabiliser G_2. The 3-form φ defines a Riemannian metric g_φ on M via the relation $g_\varphi(X,Y)\operatorname{vol}_\varphi = \frac{1}{6}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$, where $\operatorname{vol}_\varphi$ is the volume form determined by φ.*

The critical points of the corresponding Hitchin functional correspond to φ being both closed and co-closed, i.e. $d\varphi = 0 = d^*\varphi$. In this case, the metric g_φ has holonomy contained in G_2. Moreover, these critical points are local maxima of the Hitchin functional.

These examples illustrate the interest in studying these Hitchin functionals. As a further motivation, one can examine the gradient flow of the volume functional for G_2-structures along a fixed cohomology class. The corresponding flow is called the G_2-Laplacian flow. Bryant and Xu [BX11] proved the short-time existence and uniqueness of this flow. It remains a central object of study in special holonomy.

In the examples above, there is the critical assumption that the cohomology class over which we are trying to optimise is non-trivial. Otherwise, critical points cannot exist by standard Hodge theory.

If one wants to restrict to stable exact forms, one needs to impose a further non-degeneracy condition in the form of a Lagrange multiplier. In dimensions 6 and 7, Hitchin [Hit01] obtained two new functionals for the exact stable case and showed that the critical points of these functionals are nearly Kähler and nearly parallel G_2-structures, respectively. These notes further study these functionals and their variations, comparing them to two new examples of Hitchin-like functionals.

7 The nearly Kähler case

We adapt the previous discussion to realise and study nearly Kähler metrics as critical points of Hitchin-like functionals.

7.1 The nearly Kähler Hitchin functional

Let M^6 be a closed spinable manifold, so it admits an SU(3)-structure. In 6-dimensions, we have a non-degenerate pairing between Ω^3_{exact} and Ω^4_{exact}, defined as follows:

$$\begin{aligned} P : \Omega^3_{exact} \times \Omega^4_{exact} &\to \mathbb{R} \\ (\gamma, \chi) &\mapsto \int_M \beta \wedge \chi = -\int_M \gamma \wedge \xi \ , \end{aligned}$$

where $d\beta = \gamma$ and $d\xi = \chi$. This pairing follows from the Stokes' theorem and the identification $\Omega^3/\Omega^3_{closed} \cong \Omega^4_{exact}$. With it, one can construct an indefinite inner product on $\Omega^3_{exact} \times \Omega^4_{exact}$ given by $\{(\gamma_1, \chi_1), (\gamma_2, \chi_2)\} \coloneqq P(\gamma_1, \chi_2) + P(\gamma_2, \chi_1)$.

Let $\mathcal{R} \subseteq \Omega^3_{exact} \times \Omega^4_{exact}$ be the space of stable exact forms (ρ, σ), with $\omega = \widehat{\sigma}$ positive with respect to ρ, that is $\omega(\cdot, J_\rho\ \cdot) \geq 0$, where J_ρ is the almost complex structure induced by ρ.

In [Hit01], Hitchin introduced the functional that plays the analogue role for nearly Kähler structures as the examples in the previous section:

$$\begin{aligned} \mathcal{L} : \mathcal{R} &\to \mathbb{R} \\ (\rho, \sigma) &\mapsto 3\int_M \mathrm{vol}_\rho + 4\int_M \mathrm{vol}_\sigma - 12P(\rho, \sigma) \ . \end{aligned} \tag{62}$$

Hitchin showed that critical points of $\mathcal{L}$ are nearly Kähler structures. Let $\delta\rho = \gamma = d\eta \in \Omega^3_{exact}$ and $\delta\sigma = \chi = d\xi \in \Omega^4_{exact}$. Then $\delta\,\mathrm{vol}_\rho = \gamma \wedge \widehat{\rho}$ and $\delta\,\mathrm{vol}_\sigma = \chi \wedge \widehat{\sigma} = \chi \wedge \omega$, so

$$\delta\mathcal{L} = -3\int_M (\widehat{\rho} + 4\beta) \wedge \gamma + 4\int_M (\omega - 3\alpha) \wedge \chi = -3\int_M (d\widehat{\rho} + 4\sigma) \wedge \eta - 4\int_M (d\omega - 3\rho) \wedge \xi \ ,$$

where $d\alpha = \rho$ and $d\beta = \sigma$. Thus, the Euler–Lagrange equations are

$$d\widehat{\rho} = -4\sigma \qquad\qquad d\omega = 3\rho \ . \tag{63}$$

Proposition 7.1 (Thm. 6 [Hit01]). *The critical points of $\mathcal{L}$ are nearly Kähler structures.*

Proof. First, we need to check that Equations (63) imply that (ω, ρ) satisfy the SU(3)-conditions. Indeed, we have

$$\omega \wedge \rho = \frac{1}{3}\omega \wedge d\omega = \frac{1}{3}d\sigma = \frac{-1}{12}d^2\widehat{\rho} = 0 \ ,$$

so ω is of type $(1,1)$ with respect of the complex structure defined by ρ. Similarly, we have

$$\frac{\omega^3}{3!} = \frac{1}{3}\omega\wedge\sigma = \frac{-1}{12}\omega\wedge d\widehat{\rho} = \frac{-1}{12}\Big(d(\omega\wedge\widehat{\rho}) - d\omega\wedge\widehat{\rho}\Big) = \frac{1}{4}\rho\wedge\widehat{\rho}\,.$$

Thus, (ω,ρ) define an SU(3)-structure. The fact that the SU(3)-structure is a nearly Kähler structure follows from Proposition A.42. □

We find it convenient to work with the gradient flow of $\mathcal{L}$ with respect to the pairing $\{\cdot,\cdot\}$, rescaled by $(1/3, 1/4)$:

$$\frac{\partial\sigma}{\partial t} = -\left(d\widehat{\rho} + 4\sigma\right) \qquad\qquad \frac{\partial\rho}{\partial t} = \left(d\omega - 3\rho\right)\ . \tag{64}$$

We have no compelling argument for this rescaling beyond the fact that it possesses some desirable properties and enables us to motivate the study of this functional. Notice that a global rescaling can be obtained by suitably rescaling $\mathcal{L}$ (or the inner product $\{\cdot,\cdot\}$). However, the relevance of the rescaling lies in its distinction between 3-forms and 4-forms. We have

Proposition 7.2. *The rescaled gradient flow preserves the* SU(3)*-condition.*

Proof. Recall the decomposition of the intrinsic torsion τ into irreducible SU(3)-representations given in Proposition A.40.

Since $0 = d\omega^2/2 = \omega\wedge d\omega$, it follows that $\tau_1 = \pi_6(d\omega) = \pi_6(d\sigma) = 0$. Similarly, since ρ is exact, we have $\widehat{\tau_1} = \pi_6(d\widehat{\rho}) = J\pi_6(d\rho) = 0$. Thus

$$\frac{\partial}{\partial t}(\omega\wedge\rho) = \frac{\partial}{\partial t}\omega\wedge\rho + \omega\wedge\frac{\partial}{\partial t}\rho = -*\left(\pi_6(d\widehat{\rho})\right)\wedge\rho + \pi_6(d\omega)\wedge\omega = 0\ ,$$

proving the condition $\omega\wedge\rho = 0$ is preserved. Now, by Equation (61), we have $\mathrm{vol}_\rho = \frac{1}{2}\rho\wedge\widehat{\rho}$ and $\mathrm{vol}_\sigma = \frac{2}{3}\sigma\wedge\omega = \frac{1}{3}\omega^3$. Thus, it suffices to check that $\mathrm{vol}_\rho = \mathrm{vol}_\sigma$ is preserved under the flow. By Equation (60), we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{vol}_\rho &= \frac{\partial\rho}{\partial t}\wedge\widehat{\rho} = (d\omega - 3\rho)\wedge\widehat{\rho} = -d\widehat{\rho}\wedge\omega - 3\rho\wedge\widehat{\rho} \\ &= \frac{\partial\sigma}{\partial t}\wedge\omega + 4\sigma\wedge\omega - 3\rho\wedge\widehat{\rho} = \frac{\partial}{\partial t}\mathrm{vol}_\sigma + 6\left(\mathrm{vol}_\sigma - \mathrm{vol}_\rho\right)\ .\end{aligned}$$ □

The main result that motivates our study of the Hitchin functions is its relation to metric cones with special holonomy. Explicitly, we have

Proposition 7.3. *Let $(\rho(t),\sigma(t))$, $t\in(a,b)$, be a family of exact stable forms on M^6 defining a family of* SU(3)*-structures, with associated metric $g(t)$, and set $r = e^t$. The metric $\overline{g} = dr^2 + r^2 g(\log(r))$ in $(e^a, e^b)\times M$ has holonomy inside G_2 if and only if the* SU(3)*-structures satisfy the rescaled gradient evolution equations.*

Proof. Given an SU(3)-structure on Σ, we get a G_2-structure on the cone $C(\Sigma)$ by setting $\varphi = dr\wedge r^2\omega + r^3\rho$ and $\psi = *\varphi = -dr\wedge r^3\widehat{\rho} + r^4\sigma$. The condition $\mathrm{Hol}(g_\varphi)\subseteq G_2$ is equivalent to the 3-form φ being closed and coclosed. Thus, by differentiating, we get

$$0 = d\varphi = -dr\wedge r^2 d_\Sigma\omega + 3r^2 dr\wedge\rho + r^3 dr\wedge\frac{\partial\rho}{\partial r} \qquad\Longrightarrow\qquad r\tfrac{\partial\rho}{\partial r} = d_\Sigma\omega - 3\rho$$

$$0 = d\psi = dr\wedge r^3 d_\Sigma\widehat{\rho} + 4r^3 dr\wedge\sigma + r^4 dr\wedge\frac{\partial\sigma}{\partial r} \qquad\Longrightarrow\qquad r\tfrac{\partial\sigma}{\partial r} = -d_\Sigma\widehat{\rho} - 4\sigma\ .$$

where d_Σ is just the restriction of the exterior differential d along $\Lambda^* T^*\Sigma$ and we used that $d_\Sigma \rho = d_\Sigma \sigma = 0$. This condition corresponds precisely to the rescaled gradient flow equations under the change of variables $r = e^t$. The converse follows. □

Remark 7.4. *Theorem 8 in [Hit01] is very similar to the above propositions. The method is essentially the same, but Hitchin applies it to a different functional and considers an unweighted family of metrics, $\overline{g} = dt^2 + g_\Sigma(t)$. It is worth comparing the two. We can replace our Lagrange multiplier from 12 to 12λ and consider the metric cone with cone angle $2\pi\lambda$, with G_2-structure given by $\varphi = \frac{dr}{\lambda} r^2\omega + r^3\rho$ and metric $g_\lambda = \left(\frac{dr}{\lambda}\right)^2 + r^2 g_\Sigma$. The condition that the cone has holonomy in G_2 is then equivalent to the rescaled gradient flow that now depends on λ. In this case, the required relation between r and t becomes $r = e^{\lambda t}$. The resulting metric is conformal to the metric $dt^2 + g_\Sigma$ by a factor of $e^{2\lambda t}$. After suitable rescaling, the limiting metric $\lambda \to 0$ recovers Hitchin's result.*

Remark 7.5. *In his proof, Hitchin considers a Hamiltonian flow induced by the symplectic pairing induced by P rather than the gradient flow approach. With the Hamiltonian approach, one can see the vanishing condition $\omega \wedge \rho = 0$ as the vanishing of the moment map induced by the action of the diffeomorphism group. This approach would have worked equally well in our setup.*

We now focus on the second variation of $\mathcal{L}$:

Proposition 7.6. *Let $(\gamma_1, \chi_1), (\gamma_2, \chi_2) \in \Omega^3_{exact} \times \Omega^4_{exact}$, with $\gamma_i = d\eta_i$ and $\chi_i = d\xi_i$ for $i = 1, 2$. The second variation of $\mathcal{L}$ is given by*

$$\delta^2\mathcal{L} = \int_M -3(d\mathcal{I}\gamma_2 + 4\chi_2) \wedge \eta_1 - 4(d\mathcal{K}\chi_2 - 3\gamma_2) \wedge \xi_1 \ ,$$

where $\mathcal{I}$ and $\mathcal{K}$ are the linearisation of the Hitchin dual maps from Proposition A.46. In particular the Hessian of $\mathcal{L}$ at a critical point with respect to the pairing $\{\cdot, \cdot\}$ is

$$H^{\mathcal{L}}(\gamma, \chi) = \Big(4d\mathcal{K}\chi - 12\gamma, -3d\mathcal{I}\gamma - 12\chi\Big) \ .$$

Proof. By Proposition A.46, if $\delta\rho = \gamma$, then $\delta\widehat{\rho} = \mathcal{I}\gamma$ and if $\delta\sigma = \chi$, then $\delta\omega = \mathcal{K}\chi$. Combining this with our formula for the first variation yields the desired formula. The computation of the Hessian with respect to the pairing $\{\cdot, \cdot\}$ is now immediate. □

We want to study the spectral properties of $H^{\mathcal{L}}$. More concretely, the equations

$$-3d\mathcal{I}\gamma = (\mu + 12)\chi \quad (65a)$$

$$4d\mathcal{K}\chi = (\mu + 12)\gamma \ , \quad (65b)$$

for $\gamma \in \Omega^3_{exact}$ and $\sigma \in \Omega^4_{exact}$. Since the functional $\mathcal{L}$ is invariant under the action of the diffeomorphism group, it is convenient to work on a slice to the orbit of the diffeomorphism group. We use the same strategy as Foscolo in [Fos17].

Let (ω, ρ) be a nearly Kähler structure not isometric to the round S^6 and $\mathcal{O}$ be the orbit of $\mathrm{Diff}_0(M)$ in $\Omega^3_{exact} \times \Omega^4_{exact}$ going through (ω, ρ). The tangent space to this orbit is spanned by

$(\mathcal{L}_X\rho, \mathcal{L}_X\sigma)$, for $X \in \mathfrak{aut}^\perp \subseteq \Omega^1$, where aut is the space of vector fields preserving the nearly Kähler structure, and the complement is taken with respect to the L^2 metric. Using the Hodge decomposition of Theorem A.58, we can parametrise $(\gamma, \chi) \in \Omega^3_{exact} \times \Omega^4_{exact}$ explicitly by

$$\gamma = \mathcal{L}_X\rho + d(f\omega) + \gamma_0 \qquad \chi = \mathcal{L}_Y\sigma + d(g\widehat{\rho}) + \chi_0 \ ;$$

with $f, g \in \Omega^0$, $X, Y \in \mathfrak{aut}^\perp$, $\gamma_0 \in \Omega^3_{12,exact}$ and $\chi_0 \in \Omega^4_{8,exact}$. In particular, it follows that taking $X = 0$ or $Y = 0$ defines a complement to the tangent space of the diffeomorphism action. Let

$$\mathcal{W} = \{(d(f\omega) + \gamma_0, \mathcal{L}_Y\sigma + d(g\widehat{\rho}) + \chi_0)\} \subseteq \Omega^3_{exact} \times \Omega^4_{exact} \ ,$$

for $f, g \in \Omega^0$, $Y \in \mathfrak{aut}^\perp$, $\gamma_0 \in \Omega^3_{12,exact}$ and $\chi_0 \in \Omega^4_{8,exact}$. Taking the appropriate Hölder norm completions, we get

Proposition 7.7 ([Nor08] Theorem 3.1.4 & 3.1.7)**.** *There exists a slice to the diffeomorphism group action in $\Omega^3_{exact} \times \Omega^4_{exact}$, whose tangent space is given by $\mathcal{W}$.*

We can now study the spectral properties of the second variation of the functional $\mathcal{L}$. We have

Proposition 7.8. *Assume (M^6, ω, ρ) is not isometric to the round 6-sphere. Under the Hodge decomposition, Equations* (65) *are equivalent to*

$$-8g = (\mu + 12)f \ , \tag{66a}$$
$$-9f = (\mu + 12)g \ , \tag{66b}$$
$$Y + \frac{1}{3}dg = \frac{\mu + 12}{12}X \ , \tag{66c}$$
$$X - \frac{1}{4}df = \frac{\mu + 12}{12}Y \ , \tag{66d}$$
$$d * \gamma_0 = \frac{(\mu + 12)}{3}\chi_0 \ , \tag{66e}$$
$$d * \chi_0 = -\frac{(\mu + 12)}{4}\gamma_0 \ . \tag{66f}$$

Proof. Let

$$\gamma = \mathcal{L}_X\rho + d(f\omega) + \gamma_0 = \mathcal{L}_X\rho + df \wedge \omega + 3f\rho + \gamma_0 \ ,$$
$$\chi = \mathcal{L}_Y\sigma + d(g\widehat{\rho}) + \chi_0 = \mathcal{L}_Y\sigma + dg \wedge \widehat{\rho} - 4g\sigma + \chi_0 \ ,$$

with $f, g \in \Omega^0$, $X, Y \in \mathfrak{aut}^\perp$, $\gamma_0 \in \Omega^3_{12,exact}$ and $\chi_0 \in \Omega^4_{8,exact}$, by virtue of Theorem A.58. By the definition of $\mathcal{I}$ and $\mathcal{K}$, and Lemma A.47, we get

$$\mathcal{I}\gamma = \mathcal{L}_X\widehat{\rho} + Jdf \wedge \omega + 3f \wedge \widehat{\rho} - *\gamma_0 \qquad \mathcal{K}\chi = \mathcal{L}_Y\omega + dg\lrcorner\rho - 2g\omega - *\chi_0 \ .$$

Now, since (ω, ρ) define a nearly Kähler structure , we get

$$d\mathcal{I}\gamma = -4\mathcal{L}_X\sigma + d(Jdf \wedge \omega) + d(3f \wedge \widehat{\rho}) - d * \gamma_0 = -4\mathcal{L}_{X-\frac{1}{4}df}\sigma + d(3f \wedge \widehat{\rho}) - d * \gamma_0 \ ,$$
$$d\mathcal{K}\chi = 3\mathcal{L}_Y\rho + d(dg\lrcorner\rho) - d(2g\omega) - d(*\chi_0) = 3\mathcal{L}_{Y+\frac{1}{3}dg}\rho - d(2g\omega) - d(*\chi_0) \ .$$

Plugging these back in (65) and since the Hodge decomposition is orthogonal, the system (66a)-(66f) follows. $\square$

Proposition 7.9. *The eigenforms of $H^{\mathcal{L}}$ are multiples of $\left(\rho, \pm\sqrt{2}\,\sigma\right)$ and solutions to*

$$\Delta\gamma = \frac{(\mu+12)^2}{12}\gamma \tag{67}$$

for $\gamma \in \Omega^3_{12,exact}$. In particular, the spectrum of $\mathcal{H}^{\mathcal{L}}$ is discrete and has finite multiplicity for each μ.

Proof. First, equations (66a) and (66b) imply $72fg = (\mu+12)^2 fg$. The only solution to this equation with $fg \neq 0$ corresponds to $\mu = -12 \pm 6\sqrt{2}$. If we further impose the gauge fixing condition $X = 0$, equations (66c) and (66d) become

$$Y + \frac{1}{3}dg = 0 \qquad\qquad 3df \pm 6\sqrt{2}Y = \pm\frac{\sqrt{2}}{2}\left(Y - \frac{1}{3}dg\right) = 0\,,$$

since $f = \mp\frac{2\sqrt{2}}{3}g$ by equation (66b). Thus, $Y = df = dg = 0$, so f and $g = \pm\dfrac{3\sqrt{2}}{4}f$ must be constant, with associated eigenvalue $\mu = 12 \pm 6\sqrt{2}$. We have reduced our spectral problem to the PDE system (66e)-(66f)

$$d*\gamma_0 = \frac{(\mu+12)}{3}\chi_0 \qquad\qquad d*\chi_0 = -\frac{(\mu+12)}{4}\gamma_0\,, \tag{68}$$

with $(\gamma_0, \chi_0) \in \Omega^3_{12,exact} \times \Omega^4_{8,exact}$. If $\mu = -12$, (γ_0, χ_0) are harmonic exact forms and thus zero. Thus, we may assume $\mu \neq 12$. In this case, this PDE system is equivalent to (67). If γ_0 satisfies (68), then

$$\Delta\gamma_0 = dd^*\gamma_0 = -\frac{(\mu+12)}{3}d*\chi_0 = \frac{(\mu+12)^2}{12}\gamma_0\,.$$

Conversely, if γ_0 satisfies (67) and $\mu \neq -12$, the pair $(\gamma_0, \frac{3}{(\mu+12)}d*\gamma_0)$ satisfies (68):

$$d*\chi_0 = \frac{3}{(\mu+12)}d*d*\gamma_0 = \frac{-3}{(\mu+12)}\Delta\gamma_0 = -\frac{(\mu+12)}{4}\gamma_0\,. \qquad \square$$

Remark 7.10. *The case $\mu = 0$ corresponds to the nullity of $\mathcal{H}^{\mathcal{L}}$, i.e. infinitesimal deformations of the nearly Kähler structure. As expected, we recover the result of [MNS08] and [Fos17] on infinitesimal deformations of nearly Kähler structures.*

7.2 The closed Hitchin functional

The Euler–Lagrange equations associated with the functional $\mathcal{L}$ resemble the first variation of a Hamiltonian functional. This similarity suggests a natural approach: to seek out and analyse Lagrangians that correspond to the Hitchin functional $\mathcal{L}$ when interpreted in a Hamiltonian framework. In particular, we will treat the exact 3-form ρ as the moment variable within this setting. To formalise this approach, consider the map

$$\begin{aligned} Cl : \Omega^2 &\to \Omega^3 \times \Omega^4 \\ \omega &\mapsto \left(\frac{1}{3}d\omega, \frac{1}{2}\omega^2\right)\,, \end{aligned}$$

and let $\mathcal{U} := Cl^{-1}(\mathcal{R})$ be the preimage of exact stable forms (ρ, σ). The pullback of the Hitchin functional $\mathcal{L}$ under Cl will be the corresponding Lagrangian functional.

The space $\mathcal{U}$ is a priori quite mysterious. In particular, important questions to answer are under which conditions the space is non-empty, whether it is path-connected or simply connected. The following key result shows that $\mathcal{U}$ has a very natural geometric description:

Proposition 7.11. *There is a one-to-one map between $\mathcal{U}$ and the set of* SU(3)*-structures with torsion (cf. Proposition A.40) supported in the classes $\tau_0 = e^f$, $\widehat{\tau}_1$ and $\widehat{\tau}_2$, with $f \in \mathcal{C}^\infty(M)$. In particular, there is a well-defined map $F : \mathcal{U} \to \mathrm{Met}(M)$.*

This connection with SU(3)-structures justifies the choice of ρ as the moment variable, rather than σ, for which a result like Proposition 7.11 is not available.

Proof. Let $\omega \in \mathcal{U}$. Then the 3-form $\tilde{\rho} := \frac{1}{3} d\omega$ is stable and satisfies $\omega \wedge \tilde{\rho} = \frac{1}{6} d\omega^2 = 0$ since $\omega \in \mathcal{U}$. Thus, the pair $(\omega, \tilde{\rho})$ defines a U(3)-structure. Let $u = e^f \in \mathcal{C}^\infty(M)$ be the unique function such that

$$\frac{\omega^3}{3!} = \frac{1}{4u^2} \tilde{\rho} \wedge \widehat{\tilde{\rho}} = \frac{1}{4} \frac{\tilde{\rho}}{u} \wedge \widehat{\left(\frac{\tilde{\rho}}{u}\right)} .$$

Then the pair $(\omega, \rho) = (\omega, \frac{1}{u}\tilde{\rho}) = (\omega, \frac{1}{3u} d\omega)$ defines an SU(3)-structure. It follows easily that the torsion of this SU(3)-structure is given by $\tau_0 = u = e^f$, $\widehat{\tau}_1 = -df$ and $\widehat{\tau}_2$.

Conversely, given an SU(3)-structure (ω, ρ) with these torsion classes, it is clear that $d\omega = 3\tau_0\rho$ is stable, provided τ_0 is everywhere nonzero. $\square$

In particular, closed SU(3)-structures are a closed subset of $\mathcal{U}$, obtained by enforcing $f = 0$. Thus, one could think of $\mathcal{U}$ as some analogue of conformally closed SU(3)-structures.

Let us study the pullback of the Hitchin functional under the map Cl. We denote this pullback by $\mathcal{Q}$. We have

$$\mathcal{Q} = Cl^*\mathcal{L} = 3\int_M vol_{1/3d\omega} + 8\int_M vol_\omega - 12P\Big(\frac{1}{3}d\omega, \frac{\omega^2}{2}\Big) = \frac{1}{3}\int_M \mathrm{vol}_{d\omega} - 4\int_M \mathrm{vol}_\omega \ , \qquad (69)$$

where used the fact that $\mathrm{vol}_\sigma = 2\,\mathrm{vol}_\omega$ as a straightforward application of (61). Similarly, we can pull back the inner product. Let $[\cdot,\cdot] = \frac{1}{2} Cl^*\{\cdot,\cdot\}$. For $\alpha, \beta \in T\mathcal{U}$, we have

$$[\alpha, \beta] = \frac{1}{3}\int_M \alpha \wedge \beta \wedge \omega = \frac{1}{3}\int_M \alpha \wedge \mathcal{K}^{-1}(\beta) \ ,$$

where $\mathcal{K} : \Omega^4 \to \Omega^2$ is the linearisation of the Hitchin dual map from Proposition A.46 with respect to the SU(3)-structure from Proposition 7.11. This follows from noticing that, for any 4-form χ, the 2-form $\mathcal{K}(\chi)$ is the unique form that satisfies $\mathcal{K}(\chi) \wedge \omega = \chi$.

The following result further motivates the interest in the functional $\mathcal{Q}$.

Proposition 7.12. *Consider the map $\hat{\mathcal{S}} = F^*\mathcal{S}$ the pullback of the Einstein–Hilbert action* (145) *under the map $F : \mathcal{U} \to \mathrm{Met}(M)$ from Proposition 7.11.*

The Hitchin functional $\mathcal{Q}$ is bounded below by $\widehat{\mathcal{S}}$. Moreover, the two functionals coincide if and only if the SU(3)*-structure is a constant rescaling of a nearly Kähler structure.*

Proof. Using the formula for the scalar curvature in Lemma A.45, the pulled-back Einstein–Hilbert action (145) can be written as:

$$\widehat{\mathcal{S}}(\omega) = \frac{1}{5}\int_M s_g - 20\,\mathrm{vol}_g = \frac{1}{5}\int_M \Big(30\tau_0^2 - \frac{1}{2}|\widehat{\tau_2}|^2\Big) - 20\,\mathrm{vol}_g = \int_M 6\tau_0^2 - 4 - \frac{1}{10}|\widehat{\tau_2}|^2 \mathrm{vol}_g \ .$$

Similarly for $\mathcal{Q}$, we have $d\omega = 3\tau_0\rho$, and so, $\mathrm{vol}_{d\omega} = (3\tau_0)^2\,\mathrm{vol}_\rho$. Substituting in the definition of $\mathcal{Q}$:

$$\mathcal{Q} = \frac{1}{3}\int_M \mathrm{vol}_{d\omega} - 4\int_M \mathrm{vol}_\omega = \int_M 6\tau_0^2 - 4\,\mathrm{vol}_g \ ,$$

where we used $\mathrm{vol}_g = \frac{1}{2}\,\mathrm{vol}_\rho = \mathrm{vol}_\omega$. It follows that $\widehat{\mathcal{S}} \leq \mathcal{Q}$.

For the equality case, it is clear that one has $\widehat{\tau_2} = 0$. Thus, from Proposition A.40, we have

$$d\widehat{\rho} = -2e^f\omega^2 - df \wedge \hat{\rho} \ .$$

Differentiating one more, since $d\omega^2 = 0$, we have

$$0 = -2e^f df \wedge \omega^2 + df \wedge d\hat{\rho} = -4e^f df \wedge \omega^2$$

which implies $df = 0$, as needed. □

Let us study the variational properties of the Lagrangian functional $\mathcal{Q}$. The first variation of $\mathcal{Q}$ along β is

$$\delta\mathcal{Q} = \int_M \frac{1}{3} d\beta \wedge \widehat{d\omega} - 4\frac{\omega^2}{2} \wedge \beta = -\frac{1}{3}\int_M \Big(d(\widehat{d\omega}) + 12\frac{\omega^2}{2}\Big) \wedge \beta \ .$$

The gradient flow of $\mathcal{Q}$ with respect to the pairing $[\cdot,\cdot]$ is $\partial_t\omega = -\mathcal{K}(d(\widehat{d\omega}) - 6\omega$. This flow becomes slightly more enlightening if we consider the induced flow for $\sigma = \frac{\omega^2}{2}$:

$$\frac{\partial\sigma}{\partial t} = -d(\widehat{d * \sigma}) - 12\sigma = dd^*\sigma - 12\sigma = \Delta\sigma - 12\sigma \ , \tag{70}$$

since $\widehat{d\omega} = *d\omega$. We refer to this flow as the nearly Kähler Laplacian flow.

Proposition 7.13. *The critical points of $\mathcal{Q}$ are nearly Kähler structures.*

Proof. With respect to the induced SU(3)-structure, the fixed points of the gradient flow are

$$0 = \Delta\sigma - 12\sigma = -3d(\tau_0\widehat{\rho}) - 12\sigma = 12\tau_0^2\sigma - 3\tau_0\widehat{\tau_2} \wedge \omega - 12\sigma \ ,$$

which implies that the torsion of the underlying SU(3)-structure is $\tau_0 = 1$ and $\widehat{\tau_2} = \widehat{\tau_1} = 0$, as needed. □

The second variation of $\mathcal{Q}$ at a nearly Kähler structure is given by

$$\frac{\partial^2\mathcal{Q}}{\partial\alpha\partial\beta} = \frac{1}{3}\int_M d\alpha \wedge \mathcal{I}d\beta - 12\omega \wedge \alpha \wedge \beta = \frac{-1}{3}\int_M \alpha \wedge (d\mathcal{I}d\beta + 12\omega \wedge \beta) \ . \tag{71}$$

We can associate a symmetric endomorphism $\mathcal{H}^{\mathcal{Q}}$ to the second variation via the pairing $[\cdot,\cdot]$, which we refer to as the Hessian of $\mathcal{Q}$. Before studying the spectral properties of $\mathcal{H}^{\mathcal{Q}}$, it is convenient to get a more manageable description of $T_\omega\mathcal{U}$.

Proposition 7.14. *There is an isomorphism $T_\omega\mathcal{U} \cong \mathcal{K}(\Omega^4_{exact})$*

Proof. Recall that $\mathcal{U} = Cl^{-1}(\mathcal{R}) = \{\omega \in \Omega^2 | \ \omega \text{ is stable}, \ d\omega \text{ is stable}, \omega^2 \text{ is exact}\}$. The stability and positivity conditions are open, so we only need to study the constraint of ω^2 being exact. Its linearisation along $\delta\omega = \alpha$ is given by $\delta\omega^2 = 2\omega \wedge \alpha = 2\mathcal{K}^{-1}(\alpha)$. □

Since $\mathcal{K}$ is a pointwise linear isomorphism, we will instead study the spectral properties of $\overline{\mathcal{H}}^{\mathcal{Q}} := \mathcal{K}^{-1} \circ \mathcal{H}^{\mathcal{Q}} \circ \mathcal{K} : \Omega^4_{exact} \to \Omega^4_{exact}$. Explicitly, we want to solve the equation

$$d\mathcal{I}d\mathcal{K}\chi = -(\mu + 12)\chi \tag{72}$$

for $\mu \in \mathbb{R}$ and $\chi \in \Omega^4_{exact}$. Since the functional $\mathcal{Q}$ is invariant under the action of the diffeomorphism group, it is convenient to work on a slice to the orbit of the diffeomorphism group. Let $\omega \in \mathcal{U}$ be a nearly Kähler structure and $\mathcal{O}$ be the orbit of $\mathrm{Diff}_0(M)$ in $T\mathcal{U}$ going through ω. The tangent space to this orbit is spanned by $\mathcal{L}_X\omega$, for $X \in \mathfrak{aut}^\perp \subseteq \Omega^1$, where $\mathfrak{aut}$ is the set of vector fields preserving the nearly Kähler structure, and the complement is taken with respect to the L^2 metric. Under the isomorphism of Proposition 7.14, the image of the tangent space of the orbit is spanned by $\mathcal{K}^{-1}(\mathcal{L}_X\omega) = \mathcal{L}_X\sigma$, for $X \in \mathfrak{aut}^\perp \subseteq \Omega^1$, where we used Lemma A.47. By Theorem A.58, we can parametrise $\chi \in \Omega^4_{exact}$ by $\chi = \mathcal{L}_X\sigma + d(f\widehat{\rho}) + \chi_0$, where $f \in \Omega^0$, $X \in \mathfrak{aut}^\perp$ and $\chi_0 \in \Omega^4_{8,exact}$. In particular, taking $X = 0$, we get that $\mathcal{W} = \{d(f\widehat{\rho}) + \chi_0\} \subseteq \Omega^4_{exact}$ is a complement to the tangent space of the diffeomorphism orbit. Arguing as before and taking the appropriate Hölder norm completions, we can integrate $\mathcal{W}$ into a gauge slice, and we can prove

Theorem 7.15. *Assume (M^6, ω, ρ) is not isometric to the round 6-sphere. Under the Hodge decomposition and gauge fixing, Equation* (72) *is equivalent to*

$$(\mu + 6)f = 0 \ , \tag{73a}$$

$$df = 0 \ , \tag{73b}$$

$$\Delta\chi_0 = (\mu + 12)\chi_0 \ ; \tag{73c}$$

where $f \in \Omega^0$ and $\chi_0 \in \Omega^4_{8,exact}$. Solutions are $f = C$ with $C \in \mathbb{R}$ for $\mu = -6$ and the solutions to

$$\Delta\chi_0 = (\mu + 12)\chi_0 \qquad d\chi_0 = 0 \ .$$

In particular, the spectrum is discrete and has finite multiplicities.

Proof. Let

$$\chi = \mathcal{L}_X\sigma + d(f\widehat{\rho}) + \chi_0 \ ,$$

with $f \in \Omega^0$, $X \in \mathfrak{aut}^\perp$ and $\chi_0 \in \Omega^4_{8,closed}$. As in the proof of Proposition 7.8, we have

$$d\mathcal{K}\chi = 3\mathcal{L}_X\rho + d(df \lrcorner \rho) - d(2f\omega) - d(*\chi_0) = 3\mathcal{L}_{X + \frac{1}{3}df}\rho - 2df \wedge \omega - 6f \wedge \rho - d(*\chi_0) \ .$$

Since χ_0 is closed, $d(*\chi_0) \in \Omega^3_{12}$, by Corollary A.52. Thus, acting by $\mathcal{I}$, we get

$$\mathcal{I}d\mathcal{K}\chi = 3\mathcal{L}_{X + \frac{1}{3}df}\widehat{\rho} - 2Jdf \wedge \omega - 6f\widehat{\rho} + *d(*\chi_0) \ ,$$

and finally, acting by d once more, we get

$$d\mathcal{I}d\mathcal{K}\chi = -12\mathcal{L}_{X+\frac{1}{3}df}\sigma - d(2Jdf \wedge \omega) - 6d(f\widehat{\rho}) + d*d(*\chi_0) = -12\mathcal{L}_{X+\frac{1}{2}df}\sigma - 6d(f\widehat{\rho}) - \Delta\chi_0 \ .$$

Plugging this back in, and since the Hodge decomposition is unique, the system follows. □

As before, the case $\mu = 0$ recovers the infinitesimal deformations of the SU(3)-structure. Moreover, the following is a straightforward corollary of Proposition 7.9:

Proposition 7.16. *There is a correspondence between the eigenforms of $\mathcal{H}^{\mathcal{L}}$ and $\mathcal{H}^{\mathcal{Q}}$.*

This result motivates the following definition:

Definition 7.17. *Let (M^6, ω, ρ) be a nearly Kähler manifold. We define the Hitchin index of the nearly Kähler structure $\mathrm{Ind}_{(\omega,\rho)}$ as the number of negative eigenvalues of the Hessian endomorphism $\mathcal{H}^{\mathcal{Q}} : \mathcal{W} \to \mathcal{W}$ at (ω, ρ) minus one.*

Remark 7.18. *The Hitchin index does not account for constant rescalings on the structure, which always correspond to a negative eigenvalue of the Hessian. This justifies why we subtracted one from the count of the index of $\mathcal{H}^{\mathcal{Q}}$ in the definition above. In particular, notice that $\mathrm{Ind}_{(\omega,\rho)} \geq 0$.*

If we let $\mathcal{E}(\lambda) = \left\{\beta \in \Omega^2_{8,coclosed} \mid \Delta\beta = \lambda\beta\right\}$, the Hitchin index is

$$\mathrm{Ind}_{(\omega,\rho)} = \sum_{\lambda\in(0,12)} \dim \mathcal{E}(\lambda) \ . \tag{74}$$

Notice that a priori, this definition of the index is different from the usual Morse definition as the index of the quadratic form $\delta^2\mathcal{Q}$, since the pairing used to define the endomorphism $\mathcal{H}^{\mathcal{Q}}$ is indefinite. However, we will show that the two quantities are connected in this case. We prove

Proposition 7.19. *The Morse co-index of the second variation $\delta^2\mathcal{Q}$ restricted to $\Omega^4_{8,exact}$ is equal to the Hitchin index.*

Proof. Evaluate $\delta^2\mathcal{Q}(\beta, \beta)$ for $\beta \in T_\omega\mathcal{U}$. Equation (71) and Proposition 7.14 yield

$$\delta^2\mathcal{Q} = -\frac{1}{3}\int_M \mathcal{K}(\chi) \wedge (d\mathcal{I}d\mathcal{K}(\chi) + 12\chi) \ , \tag{75}$$

for $\chi \in \Omega^4_{exact}$. Fixing the diffeomorphism slice, we can take $\chi = d(f\widehat{\rho}) + \chi_0$ for $f \in \Omega^0$ and $\chi \in \Omega^4_{8,closed}$. By the computations of the proof of Proposition 7.8, we have

$$\begin{aligned}
\delta^2\mathcal{Q} &= -\frac{1}{3}\int_M \langle \Delta\chi_0 - 12\chi_0, \chi_0\rangle - 48\int_M f^2 \,\mathrm{vol}_g + 2\int_M (df \lrcorner \rho) \wedge [d(Jdf \wedge \omega) - df \wedge \widehat{\rho}] \\
&= -\frac{1}{3}\int_M \langle \Delta\chi_0 - 12\chi_0, \chi_0\rangle + 8\int_M \langle \Delta f - 6f, f\rangle \ .
\end{aligned}$$

Thus, the second variation has two distinct behaviours on the two subspaces of $\mathcal{W} = \{d(f\hat{\rho})\} \oplus \Omega^4_{8,exact}$, similar to the Einstein–Hilbert case (cf. Theorem C.1). Notice that the first subspace corresponds to conformal deformations of the metric, as expected. □

An interesting first result is the Hitchin stability of the homogeneous examples.

Theorem 7.20. *Let (M^6, ω, ρ) be one of the four homogeneous nearly Kähler manifolds. Then*

$$\mathrm{Ind}_{(\omega,\rho)} = 0 \,.$$

The main tools we need are a version of the Peter-Weyl theorem for naturally reductive homogeneous spaces and a comparison between the Hodge Laplacian and the canonical Laplacian. On a nearly Kähler structure, besides the Levi-Civita connection, there exists another metric connection, called the canonical connection, with the property that $\mathrm{Hol}\,(\nabla^{can}) \subseteq \mathrm{SU}(3)$. The relationship between these two connections is given explicitly by

$$\nabla^{can} = \nabla^{LC} - \frac{1}{2}\widehat{\rho}\,. \tag{76}$$

The canonical Laplacian is the connection Laplacian associated with this connection, $\Delta^{can} = (\nabla^{can})^* \nabla^{can}$. Both results above are due to Moroianu and Semmelmann. They are collected in [MS10; MS11] in their investigation of infinitesimal nearly Kähler and Einstein deformations of nearly Kähler manifolds.

Lemma 7.21. *[MS11, Prop. 4.5] Let (M^6, ω, ρ) be a nearly Kähler manifold, Δ^{can} the induced canonical Laplacian. For $\beta \in \Omega^2_8$, we have the Weitzenböck-type formula:*

$$(\Delta - \Delta^{can})\beta = (Jd^*\beta) \lrcorner \rho \,.$$

In particular, both Laplacians coincide on coclosed forms of type Ω^2_8.

Proposition 7.22. *[MS10, Lemmas 5.2 & 5.4] Let $(G/H, \omega, \rho)$ be a naturally reductive nearly Kähler manifold and consider $\rho : H \to \mathfrak{aut}(E)$ a representation of H, and $EM = G \times_\rho E$ the induced vector bundle. Then, the Peter-Weyl formalism and Frobenius reciprocity imply*

$$L^2(EM) = \overline{\bigoplus_{\gamma \in \mathrm{Irr}(G)} V_\gamma \times \mathrm{Hom}_H(V_\gamma, E)}\,,$$

where $\mathrm{Irr}(G)$ denotes the set of irreducible representations of G. Under this decomposition, the canonical Laplacian is given by $\Delta^{can} = -12\,\mathrm{Cas}^G_\gamma$, where Cas^G_γ is the Casimir of the representation V_γ, computed with respect to the Killing form.

Proof of Theorem 7.20. Using the computations of Moroianu and Semmelmann in [MS10], Karigiannis and Lotay [KL20, Prop. 6.3] showed that the homogeneous nearly Kähler structures on $\mathbb{C}P^3 \cong \mathrm{SO}(5)/\mathrm{U}(2)$, $S^3 \times S^3 \cong \mathrm{SU}(2)^3/\Delta\mathrm{SU}(2)$ and the flag manifold $F_{1,2} \cong \mathrm{SU}(3)/\mathrm{T}^2$ are stable. Thus, only the case of the round sphere $S^6 \cong G_2/\mathrm{SU}(3)$ remains. We start by computing the Casimir operator of G_2. Let ω_1 and ω_2 be the short and long fundamental weights respectively, so $V_{1,0}$ is the fundamental 7-dimensional G_2-representation and $V_{0,1}$ is its adjoint representation.

Since G is simple, the Freudenthal formula (cf. [MS10]) allows us to compute the value of the Casimir operator on a representation of highest weight γ. We have $\mathrm{Cas}^G_\gamma = -\langle \gamma, \gamma + 2\rho \rangle_B$, where ρ is the half-sum of positive roots and $\langle \cdot, \cdot \rangle_B$ is the Killing form. In the case of G_2-structure,

$\rho = \omega_1 + \omega_2$, and so, for an irreducible representation of highest weight (λ, μ), its Casimir operator is given by

$$\begin{aligned}\mathrm{Cas}_{G_2}(\lambda,\mu) &= -\lambda(\lambda+2)||\omega_1||_B^2 + \mu(\mu+2)||\omega_2||_B^2 + 2(\lambda\mu+\lambda+\mu)\langle\omega_1,\omega_2\rangle_B \\ &= -\frac{1}{12}\left(\lambda(\lambda+2)+3\mu(\mu+2)+3(\lambda\mu+\lambda+\mu)\right) .\end{aligned}$$

Therefore, by virtue of Lemma 7.21 and Proposition 7.22, the Hodge Laplacian on coclosed forms of type Ω^2_8 is given by

$$\Delta\beta = \sum_{(\lambda,\mu)\in\mathrm{Irr}(G)} \left(\lambda(\lambda+2)+3\mu(\mu+2)+3(\lambda\mu+\lambda+\mu)\right)\pi_\gamma(\beta) .$$

In particular, the only highest weight for which the eigenvalue of the canonical Laplacian is smaller than 12 is $(1,0)$, the fundamental 7-dimensional representation. The space of primitive $(1,1)$-forms can be identified with the adjoint representation of SU(3). By dimensional reasons it is clear that $\mathrm{Hom}_{\mathrm{SU}(3)}\left(V_{1,0}, \mathfrak{su}(3)\right) = 0$, and so (S^6, g_{round}) is stable. □

Remark 7.23. *Notice that a priori, this computation is only valid if one defines the Hitchin index using Equation* (74) *since the gauge slice is not valid in the round sphere case. However, one can retrace the proof of Theorem A.58 and show that the discussion can be adapted without significant changes. We omit the details.*

The next natural question is the study of the Hitchin functionals and the index problem for the remaining two known examples of nearly Kähler structures, due to Foscolo and Haskins [FH17]. We devote the final chapter of the thesis to this endeavour, where we prove

Theorem 7.24 (Thm. 12.6)**.** *Consider $(S^3\times S^3, g_{FH}, \omega_{FH}, \rho_{FH})$ the cohomogeneity one nearly Kähler structure on $S^3 \times S^3$ constructed by Foscolo and Haskins in [FH17]. Its Hitchin index is bounded below by 1, and the Einstein co-index is bounded below by 4.*

Finally, we outline the connection between the Hitchin functionals and the study of G_2-conifolds that we discussed at the start of the chapter, focusing on the Hitchin index. We conjecture there is an analogous discussion for Spin(7)-conifolds.

The expectation is that the Hitchin index acts as the stability index for conically singular G_2 manifolds. That is, the index measures the codimension of the singularity in the moduli space of conically singular G_2 manifolds. A first indication of this is the dimension bound of the obstruction space for G_2-conifold deformation that we saw in the introduction:

Proposition 7.25 ([KL20, Prop. 6.11])**.** *Let (M, φ) be a conically singular G_2-manifold with singularities $p_1, \ldots, p_n$, modelled on the $\Sigma_1, \ldots, \Sigma_n$. The dimension of the obstruction space to the deformation problem is bounded above by*

$$\dim(\mathcal{O}_{/\mathcal{W}}) \leq n - 1 + \sum_{i=1}^{n}\left(\mathrm{Ind}^{\Sigma_i}\right); .$$

Moreover, if $\mathrm{Ind}^{\Sigma_i} = 0$ for all i, the remaining obstruction space is ineffective.

To further pursue this discussion, it would be useful to study manifolds with both asymptotically conical (AC) and isolated conically singular (ICS) ends. Although Karigiannis and Lotay do not directly address this case, their methods should extend with minimal difficulty. In particular, for a manifold with only one conically singular point and an asymptotically conical end, we expect that the difference of the Hitchin indices will give the virtual dimension of the moduli space.

This expectation can be motivated by treating $\mathcal{L}$ as an analogue of the Chern–Simons functional in instanton Floer theory. Consider a family of SU(3)-structures $(\rho(t), \sigma(t))$ on Σ, evolving with the gradient flow of $\mathcal{L}$ and connecting two of its critical points. Proposition 7.3 implies there is an associated G_2 conifold with one ICS and an AC end. Following the Chern–Simons analogy, the virtual dimension of the moduli space of such conifold should be given by the spectral flow of the family of SU(3)-structures. In view of Proposition 7.16 (cf. Prop. 7.9), this corresponds to the index difference of the two nearly Kähler structures.

8 The nearly parallel G_2 case

We adapt the discussion from Section 6 to realise and study nearly parallel G_2-metrics as critical points of Hitchin-like functionals.

8.1 The nearly parallel G_2 Hitchin functional

Let M^7 be a closed spinable manifold, so it admits a G_2-structure. In dimension seven, we have a non-degenerate quadratic form on Ω^4_{exact}:

$$\begin{aligned} Q : \Omega^4_{exact} &\to \mathbb{R} \\ d\gamma &\mapsto \int_M d\gamma \wedge \gamma \ , \end{aligned} \tag{77}$$

induced by the isomorphism $(\Omega^4_{exact})^* \cong \Omega^4_{exact}$.

We consider the space $\mathcal{V} = \Omega^4_+ \cap \Omega^4_{exact}$ of stable exact 4-forms. Given a stable 4-form ψ and a fixed orientation on M, we consider the associated volume form $\mathrm{vol}_\psi = \frac{4}{7}\psi \wedge \widehat{\psi}$ and denote its Hitchin dual by $\varphi = \widehat{\psi}$. Comparing with the identity $\varphi \wedge \psi = 7\,\mathrm{vol}_g$, we get $\mathrm{vol}_g = \frac{1}{4}\,\mathrm{vol}_\psi$. In [Hit01], Hitchin introduced the functional

$$\begin{aligned} \mathcal{P} : \mathcal{V} &\to \mathbb{R} \\ \psi &\mapsto \int_M \mathrm{vol}_\psi - 2Q(\psi) \ , \end{aligned} \tag{78}$$

and showed that its critical points correspond to nearly parallel G_2-structures. Indeed, we have

Proposition 8.1. *The Euler–Lagrange equation of $\mathcal{P}$ is $d\varphi - 4\psi = 0$. In particular, critical points are nearly parallel G_2-structures. The gradient of $\mathcal{P}$ induced by Q is given by $\partial_t\psi = d\varphi - 4\psi$.*

Proof. Let $\delta\psi = \chi = d\eta \in \Omega^4_{exact}$. Then $\delta\,\mathrm{vol}_\psi = \chi \wedge \widehat{\psi} = \chi \wedge \varphi$ and so

$$\delta\mathcal{L} = \int_M \chi \wedge \varphi - 4\int_M \eta \wedge \psi = \int_M \eta \wedge (d\varphi - 4\psi) \ . \qquad \square$$

Again, we have a nice geometric interpretation of the gradient flow in terms of the induced metric.

Proposition 8.2. *Fix an orientation on M and let $\psi(t)$, $t \in (a,b)$, be a family of stable exact 4-forms and $g(t)$ the associated metric and set $r = e^t$. The induced metric $\overline{g} = dr^2 + r^2 g(\log(r))$ on $(e^a, e^b) \times M$ has holonomy contained in* Spin(7) *if and only if $\psi(t)$ satisfies the gradient flow equation of $\mathcal{P}$.*

Proof. From Proposition A.14, the condition $\mathrm{Hol}(g_\varphi) \subseteq \mathrm{Spin}(7)$ is equivalent to the 4-form $\Phi = dr \wedge r^3\varphi + r^4\psi$ being closed (and coclosed since it is self-dual). Thus, we get

$$0 = d\Phi = -dr \wedge r^3 d_M\varphi + 4r^3 dr \wedge \psi + r^4 dr \wedge \frac{\partial \psi}{\partial r} \qquad \Longrightarrow \qquad r\frac{\partial \psi}{\partial r} = d_M\varphi - 4\psi \ ,$$

where d_M is just the restriction of the exterior differential d along $\Lambda^* T^* M$ and we used that $d_M\psi = 0$, which is the gradient flow equations under the change of variables $r = e^t$. The converse follows. □

By replacing our Lagrange multiplier from 2 to 2λ and considering the limit as $\lambda \to 0$ of the induced conformal metric $e^{2\lambda t}(dt^2 + g_\Sigma(t))$, we recover the result of Hitchin for Spin(7)-metrics in [Hit01], as in the nearly Kähler case.

Similarly, we compute the second variation of $\mathcal{P}$.

Proposition 8.3. *Let $\chi_1, \chi_2 \in \Omega^4_{exact}$ and η_i such that $d\eta_i = \chi_i$. The second variation of $\mathcal{L}$ with respect to χ_1, χ_2 is*

$$\delta^2\mathcal{P} = \int_M (d\mathcal{J}\chi_2 - 4\chi_2) \wedge \eta_1 \ ,$$

where $\mathcal{J} : \Omega^4 \to \Omega^3$ is the linearisation of the Hitchin map from Lemma A.23. The Hessian of $\mathcal{P}$ with respect to the indefinite inner product induced by Q is given by

$$\mathcal{H}^{\mathcal{P}}(\chi) = d\mathcal{J}\chi - 4\chi \ .$$

Proof. By Proposition A.23, if $\delta\psi = \chi$, then $\delta\widehat{\psi} = \delta\varphi = \mathcal{J}\chi$. Combining this with our formula for the first variation and integrating it by parts, the expression follows. □

We study the spectral properties of $\mathcal{H}^{\mathcal{P}}$. Since the functional $\mathcal{L}$ is invariant under the action of the diffeomorphism group, it is convenient to work on a slice to the orbit of the diffeomorphism group.

Let (M^7, ψ) be a nearly parallel G_2-structure that is not isometric to the round S^7 and $\mathcal{O}$ be the orbit of $\mathrm{Diff}_0(M)$ in Ω^4_{exact} going through ψ. The tangent space to this orbit is spanned by $\mathcal{L}_X\psi$, for $X \in \Omega^1$.

Using the Hodge decomposition of Theorem A.34, we can parametrise $\chi \in \Omega^4_{exact}$ explicitly by

$$\chi = \mathcal{L}_X\psi + d(f\varphi) + \chi_0 \ ;$$

with $f \in \Omega^0$, $X \in \mathfrak{aut}^\perp$, and $\chi_0 \in \Omega^4_{8,exact}$. In particular, it follows that taking $X = 0$ defines a complement to the tangent space of the diffeomorphism action. As before, let

$$\mathcal{W} = \{d(f\varphi) + \chi_0\} \subseteq \Omega^4_{exact} \ ,$$

for $f \in \Omega^0$ and $\chi_0 \in \Omega^4_{27,exact}$. Taking the appropriate Hölder norm completions, we get

Proposition 8.4 ([Nor08] Theorem 3.1.4 & 3.1.7)**.** *There exists a slice to the diffeomorphism group action in $\Omega^3_{exact} \times \Omega^4_{exact}$, whose tangent space is given by $\mathcal{W}$.*

Going back to the study the spectral properties of $\mathcal{H}^{\mathcal{P}}$, we have

Proposition 8.5. *Assume (M^7, ψ) is not isometric to the round 7-sphere. Under the Hodge decomposition, the eigenvalue problem for the Hessian is equivalent to*

$$3f = (\mu + 4)f \ , \tag{79a}$$

$$\mu X - df = 0 \ , \tag{79b}$$

$$d * \chi_0 = -(\mu + 4)\chi_0 \ . \tag{79c}$$

for $f \in \Omega^0$, $X \in \mathfrak{aut}^\perp$ and $\chi_0 \in \Omega^4_{27,exact}$.

Proof. As above, let

$$\chi = \mathcal{L}_X\psi + d(f\varphi) + \chi_0$$

with $f \in \Omega^0$, $X \in \mathfrak{aut}^\perp$ and $\chi_0 \in \Omega^4_{27,exact}$, by virtue of Theorem A.34. By the definition of $\mathcal{J}$ and Lemma A.24, we get

$$\mathcal{J}\chi = \mathcal{L}_X\varphi + *(df \wedge \psi) + 3f\varphi - *\chi_0 \ .$$

Now, since the G_2-structure is nearly parallel, we get

$$d\mathcal{J}\chi = 4\mathcal{L}_X\psi - d(df \lrcorner \psi) + 3d(f\varphi) - d(*\chi_0) = 4\mathcal{L}_{X-\frac{1}{4}df}\psi + 3d(f\varphi) - d(*\chi_0) \ .$$

Now, substituting this in $\mathcal{H}^{\mathcal{P}}$, and since the Hodge decomposition is orthogonal, we get the required system of equations. □

The case $\mu = 0$ corresponds to the nullity of $\mathcal{H}^{\mathcal{P}}$, i.e. infinitesimal deformations of the nearly parallel G_2-structure. As expected, we recover the result of [AS12] on infinitesimal deformations of nearly parallel G_2-structures (cf. [NS21]). Notice that our functional approach does not detect the infinitesimal deformations arising from Killing fields that do not preserve the G_2-structure, that is, those arising from symmetries of the Sasaki-Einstein or 3-Sasaki structures (cf. Table 6).

8.2 The new G_2 Hitchin functional

We want to construct an analogue of the closed Hitchin functional. However, in this case, we cannot exploit any symplectic structure as in the case of the nearly Kähler Hitchin functional. Instead, we make a proposal imitating Proposition 7.12.

Recall that $\mathcal{V}$ is the space of stable exact 4-forms in M^7. Given a fixed orientation on M, the 4-form defines a G_2-structure on M, with torsion $d\varphi = \tau_0\psi + *\tau_3$. We define

$$\begin{aligned} \mathcal{T} : \mathcal{V} &\to \mathbb{R} \\ \psi &\mapsto \int_M \frac{7\tau_0^2 - 5}{4} \operatorname{vol}_\psi \ . \end{aligned} \tag{80}$$

Proposition 8.6. *Let $G : \mathcal{V} \to \mathrm{Met}(M)$ be the map taking a G_2-structure to its underlying metric. Consider $\widehat{\mathcal{S}} = G^*(\mathcal{S})$ the pullback of the Einstein–Hilbert action. The Hitchin functional $\mathcal{T}$ satisfies $\mathcal{T} \geq \widehat{\mathcal{S}}$, with equality if and only if the G_2-structure is a nearly parallel G_2-metric, up to rescaling and orientation.*

Proof. By Lemma A.22, we have

$$\widehat{\mathcal{S}} = \frac{1}{6}\int_M \left(42\tau_0^2 - \frac{1}{2}|\tau_3|^2\right) - 30\,\mathrm{vol}_g = \int_M 7\tau_0^2 - 5 - \frac{1}{12}|\tau_3|^2 \mathrm{vol}_g \ .$$

Using the relation $7\,\mathrm{vol}_g = \varphi \wedge \psi = \frac{7}{4}\,\mathrm{vol}_\psi$, the claim follows. □

Let us study the variations of $\mathcal{T}$. We have

Proposition 8.7. *The Euler–Lagrange equation of $\mathcal{T}$ is*

$$\tau_0 \mathcal{J} d\varphi + \frac{1}{2}\mathcal{J}(d\tau_0 \wedge \varphi) - \frac{7\tau_0^2 + 5}{4}\varphi = 0, \tag{81}$$

where $\mathcal{J} : \Omega^4 \to \Omega^3$ is the linearisation of the Hitchin dual map from Proposition A.23.
In particular, the critical points of $\mathcal{T}$ are nearly parallel G_2-structures, up to orientation. The gradient flow with respect to the quadratic form Q is

$$\partial_t \psi = d\left[\tau_0 \mathcal{J} d\varphi + \frac{1}{2}\mathcal{J}(d\tau_0 \wedge \varphi) - \frac{7\tau_0^2 + 5}{4}\varphi\right] \ . \tag{82}$$

First, we need the following technical result

Lemma 8.8. *The variation of τ_0 along $\delta\psi = \chi$ is*

$$\delta\tau_0\,\mathrm{vol}_\psi = \frac{1}{7}\left[d(\mathcal{J}\chi \wedge \varphi) + 2d\varphi \wedge \mathcal{J}\chi\right] - \tau_0 \varphi \wedge \chi \ . \tag{83}$$

Proof. Let $\delta\psi = \chi$. Then $\delta\varphi = \mathcal{J}\chi$ by Proposition A.23. Let us compute the variation of τ_0. By definition, we have $d\varphi \wedge \varphi = 4\tau_0 \psi \wedge \varphi = 7\tau_0\,\mathrm{vol}_\psi$. Taking the variation of this identity, we get

$$7\delta\tau_0\,\mathrm{vol}_\psi + 7\tau_0 \varphi \wedge \chi = d\mathcal{J}\chi \wedge \varphi + d\varphi \wedge \mathcal{J}\chi \ .$$

By the Leibniz rule, the claim follows. □

Proof of Proposition 8.7. Using the lemma above, we have

$$\begin{aligned}
\delta\mathcal{T} &= \frac{1}{4}\int_M 14\tau_0\delta\tau_0\,\mathrm{vol}_\psi + \left(7\tau_0^2 - 5\right)\varphi \wedge \chi \\
&= \frac{1}{4}\int_M 4\tau_0 d\varphi \wedge \mathcal{J}\chi - 2d\tau_0 \wedge \mathcal{J}\chi \wedge \varphi - \left(7\tau_0^2 + 5\right)\varphi \wedge \chi \\
&= \int_M \chi \wedge \left(\tau_0 \mathcal{J} d\varphi + \frac{1}{2}\mathcal{J}(d\tau_0 \wedge \varphi) - \frac{7\tau_0^2 + 5}{4}\varphi\right) \ ,
\end{aligned} \tag{84}$$

and the Euler–Lagrange equation follows. Let us study the critical points of $\mathcal{T}$. Using the torsion decomposition $d\varphi = 4\tau_0\psi + *\tau_3$, it is clear that $\tau_3 = 0$ and $\tau_0 = C \in \mathbb{R}$. We get an equation for τ_0:

$$12\tau_0^2 - \left(7\tau_0^2 + 5\right) = 0 \ ,$$

with solutions $\tau_0 = \pm 1$. The case $\tau_0 = 1$ is the nearly parallel G_2-structure condition. For $\tau_0 = -1$, we obtain a nearly parallel G_2-structure for the reversed orientation. The formula for the gradient flow follows from taking $\chi = d\eta$ and integrating by parts. □

Notice that, unlike the case of nearly Kähler structures, the flow depends explicitly on the torsion τ_0 and its derivatives. In particular, the flow is third order in ψ and thus non-parabolic.

Before studying the second variation, we have the following technical computation

Lemma 8.9. *Let (M, ψ) be a nearly parallel G_2-structure, and consider a variation $\delta\psi = \chi = f\psi + X \wedge \varphi + \chi_0$. We have*

$$\delta\tau_0 = \frac{1}{7}d^*X - \frac{1}{4}f\ .$$

Proof. From Equation (83), we get

$$\begin{aligned}\delta\tau_0 \operatorname{vol}_\psi &= \frac{1}{7}[d(\mathcal{J}\chi \wedge \varphi) + 2\mathcal{J}(4\psi) \wedge \chi] - \varphi \wedge \chi = \frac{1}{7}\left(d(\mathcal{J}\chi \wedge \varphi) + 6\varphi \wedge \chi\right) - \varphi \wedge \chi \\ &= \frac{1}{7}\left[d(*(X \wedge \varphi) \wedge \varphi) - f\psi \wedge \varphi\right] = -\frac{1}{7}\left(4d * X + f\psi \wedge \varphi\right) = \left(\frac{1}{7}d^*X - \frac{1}{4}f\right) \operatorname{vol}_\psi\ ,\end{aligned}$$

where we used the relation $7 \operatorname{vol}_g = \varphi \wedge \psi = \frac{7}{4} \operatorname{vol}_\psi$ once more. □

Proposition 8.10. *The second variation of $\mathcal{T}$ along $\delta\psi_i = \chi_i = f_i\psi + X_i \wedge \varphi + (\chi_0)_i$ is given by*

$$\delta^2\mathcal{T} = \int_M \chi_1 \wedge \left[\mathcal{J}d\mathcal{J}\chi_2 - 4\mathcal{J}\chi_2 + \frac{1}{14} * \left[d\left(d^*X_2 - \frac{7}{4}f_2\right) \wedge \varphi\right] - \frac{1}{14}\left(d^*X_2 - \frac{7}{4}f_2\right)\varphi\right]\ . \tag{85}$$

In particular, the Hessian with respect to the pairing Q defined in Equation (77) is given by

$$\mathcal{H}^{\mathcal{T}}(\chi) = d\left[\mathcal{J}d\mathcal{J}\chi - 4\mathcal{J}\chi + \frac{1}{14} * \left[d\left(d^*X - \frac{7}{4}f\right) \wedge \varphi\right] - \frac{1}{14}\left(d^*X - \frac{7}{4}f\right)\varphi\right]\ .$$

Proof. Notice that directly taking the variation of (84) would require us to understand the variation $\delta\mathcal{J}$. We avoid this by noticing that we can rewrite $\delta\mathcal{T}$ as

$$\begin{aligned}\delta\mathcal{T} &= \int_M \mathcal{J}\chi \wedge \left(\tau_0 d\varphi + \frac{1}{2}(d\tau_0 \wedge \varphi) - \frac{7\tau_0^2 + 5}{4}\mathcal{J}^{-1}\varphi\right) \\ &= \int_M \mathcal{J}\chi \wedge \left(\tau_0 d\varphi + \frac{1}{2}(d\tau_0 \wedge \varphi) - \frac{7\tau_0^2 + 5}{3}\psi\right)\ .\end{aligned}$$

So the right-hand side can be viewed as the variation of $\mathcal{T}$ for $\delta\varphi = \delta\widehat{\psi} = \mathcal{J}\chi \in \Omega^3$. Thus,

$$\begin{aligned}\delta^2\mathcal{T} &= \int_M \mathcal{J}\chi_1 \wedge \left[\tau_0 d\mathcal{J}\chi_2 + \delta\tau_0 d\varphi + \frac{1}{2}(d\delta_0\tau_0 \wedge \varphi) - \frac{14}{3}\tau_0\delta\tau_0\psi - \frac{7\tau_0^2 + 5}{3}\chi_2\right] \\ &= \int_M \mathcal{J}\chi_1 \wedge \left[d\mathcal{J}\chi_2 - \frac{2}{3}\delta\tau_0\psi + \frac{1}{2}(d\delta\tau_0 \wedge \varphi) - \frac{7\tau_0^2 + 5}{3}\chi_2\right]\ .\end{aligned}$$

Using the lemma, we can rewrite this as (85). From the definition of Q, the expression of the Hessian is straightforward. □

Let us study the spectrum of the Hessian. By Proposition 8.4, we can restrict ourselves to the tangent of a slice to the diffeomorphism orbit $\mathcal{W} = \{d(f\varphi) + \chi_0\} \subseteq \Omega^4_{exact}$ with $\chi_0 \in \Omega^4_{27,exact}$. We have

Proposition 8.11. *Let (M, ψ) be a nearly parallel G_2-manifold that is not isometric to the round S^7. The eigenvalue problem $(\mathcal{H}^{\mathcal{T}} - \mu) : \mathcal{W} \to \mathcal{W}$ is equivalent to the PDE*

$$\Delta\chi_0 + 4d * \chi_0 = \mu\chi_0 \tag{86}$$

for $\chi_0 \in \Omega^4_{27,exact}$ whenever $\mu \neq -5/2$. For $\mu = -5/2$, the eigenforms are additionally given by multiples of ψ.

Proof. As in the proof of Proposition 8.5, we can take $\chi \in \mathcal{W} \subseteq \Omega^4_{exact}$ as

$$\chi = d(f\varphi) + \chi_0 = 4f\psi + df \wedge \varphi + \chi_0$$

with $f \in \Omega^0$ and $\chi_0 \in \Omega^4_{27,exact}$, by virtue of Theorem A.34, and Proposition 8.4. We compute the four terms of the second variation separately. First,

$$d\mathcal{J}\chi = d\left(*(df \wedge \varphi) + 3f\varphi - *\chi_0\right) = 3d(f\varphi) - d(df \lrcorner \psi) - d(*\chi_0) = 3d(f\varphi) - \mathcal{L}_{df}\psi - d(*\chi_0)\ .$$

Thus, we have

$$\begin{aligned} d\mathcal{J}d\mathcal{J}\chi &= d\mathcal{J}\left[3d(f\varphi) - \mathcal{L}_{df}\psi - d(*\chi_0)\right] = d\left[3 * (df \wedge \varphi) + 9f\varphi - \mathcal{L}_{df}\varphi + *d * \chi_0\right] \\ &= 9d(f\varphi) - 7\mathcal{L}_{df}\varphi + \Delta\chi_0\ . \end{aligned}$$

Similarly, using the identity $d * (X \wedge \varphi) = -\mathcal{L}_X\psi$ once more, the third term in (85) becomes

$$\frac{1}{14}d * \left[d\left(\Delta f - 7f\right) \wedge \varphi\right] = -\frac{1}{14}\mathcal{L}_{d(\Delta f - 7f)}\psi\ .$$

The fourth term is simply $-\frac{1}{14}d\left[(\Delta f - 7f)\varphi\right]$. Putting all of these together and using the fact that the Hodge decomposition of A.34 is orthogonal, we have

$$\begin{aligned} -\frac{1}{14}\left(\Delta f - 7f\right) - 3f &= \mu f \\ -\frac{1}{14}d\left(\Delta f - 7f\right) - 3df &= 0 \\ \Delta\chi_0 + 4d * \chi_0 &= \mu\chi_0\ , \end{aligned}$$

Now, if $df \neq 0$, the first two equations combine to yield $\mu = 0$, which implies $f = 0$ since $(\Delta - 7)$ is strictly positive, by Obata's theorem [Oba62]. If $f = C \in \mathbb{R}$, it follows that $\mu = -5/2$. $\square$

Definition 8.12. *Let (M^7, φ) a nearly parallel G_2-manifold. We define the index of the nearly parallel G_2-structure Ind_φ as the number of negative eigenvalues of the Hessian endomorphism $\mathcal{H}^{\mathcal{T}} : \mathcal{W} \to \mathcal{W}$ at ψ minus one.*

Remark 8.13. *As in the nearly Kähler case, the Hitchin index does not account for constant rescalings on the structure, which always correspond to a negative eigenvalue of the Hessian. In particular, notice that $\mathrm{Ind}_\varphi \geq 0$.*

First, the following lemma shows that the index is well-defined.

Lemma 8.14. *The spectrum of $\mathcal{H}^{\mathcal{T}}$ is bounded below by -4. In particular, the index is well-defined.*

Proof. Let $\chi \in \Omega^4_{exact}$ an eigenvalue of $\mathcal{H}^{\mathcal{T}}$. If $\mu \neq -5/2$, we know that $\chi \in \Omega^4_{27}$, by Proposition 8.11. Taking the L^2-norm of $d * \chi + 2\chi$, we get

$$0 \leq \langle d * \chi + 2\chi, d * \chi + 2\chi \rangle = \langle d * d * \chi + 4d * \chi, \chi \rangle + 4||\chi||^2 = \langle \mathcal{H}^{\mathcal{T}}(\chi), \chi \rangle + 4||\chi||^2 . \qquad \square$$

Moreover, we have a relation between the spectrum of the Hessians $\mathcal{H}^{\mathcal{P}}$ and $\mathcal{H}^{\mathcal{T}}$:

Proposition 8.15. *Solutions to $(\mathcal{H}^{\mathcal{P}} - \lambda)\chi = 0$ with $\lambda \in \mathbb{R}$ for $\chi \in \Omega^4_{exact}$ are in correspondence with solutions to $\mathcal{H}^{\mathcal{T}}(\chi) = \mu\chi$ with $\mu \geq -4$. Moreover, the range $\lambda \in (-4, 0)$ is in two-to-one correspondence with the range $\mu \in (-4, 0)$, excluding multiples of the 4-form ψ.*

Proof. First, multiples of the canonical 4-form ψ are solutions to $(\mathcal{H}^{\mathcal{P}} - \lambda)\chi = 0$ for $\lambda = -1$ and to $(H^{\mathcal{T}} - \mu)\chi = 0$ for $\mu = -5/2$. Thus, we may assume that $\chi \in \Omega^4_{27,exact}$.
First, let χ be a solution to $d * \chi = -(\lambda + 4)\chi$. Then

$$\mathcal{H}^{\mathcal{T}}(\chi) = -(\lambda + 4)d * \chi - 4(\lambda + 4)\chi = [(\lambda + 4)^2 - 4(\lambda + 4)]\chi ,$$

which is negative in the interval $\lambda \in (-4, 0)$. Conversely, assume χ satisfies $\mathcal{H}^{\mathcal{T}}(\chi) = \mu\chi$ with $\mu \geq -4$. Let $\gamma_\pm = d * \chi - \lambda\chi$, for $\lambda_\pm = -2 \pm \sqrt{\mu + 4}$. Clearly, $\gamma \in \Omega^4_{27,exact}$. If $\gamma_\pm = 0$, we are done. Otherwise, we need to show that $\gamma_\pm$ is a non-trivial element of the kernel of $\mathcal{H}^{\mathcal{P}} - \lambda$. Substituting $\gamma_\pm$ in $\mathcal{H}^{\mathcal{T}}(\chi) - \mu\chi$, we have

$$0 = \Delta\chi + 4d * \chi - \mu\chi = d * (\gamma + \lambda\chi) + 4(\gamma + \lambda\chi) - \mu\chi = d * \gamma + (\lambda + 4)\gamma + (\lambda^2 + 4\lambda - \mu)\chi .$$

Our chosen values of λ are the roots of the rightmost term, so γ satisfies $\mathcal{H}^{\mathcal{P}}(\gamma) = \lambda\gamma$, as needed. $\square$

In particular, we have

Corollary 8.16. *Let (M, φ) be a nearly parallel G_2-manifold, and consider the spaces*

$$\mathcal{E}(\lambda) = \left\{\chi_0 \in \Omega^4_{27} | \; d * \chi_0 = \lambda\chi_0\right\} .$$

The Hitchin index of the nearly parallel G_2-structure is given by

$$\mathrm{Ind}_\varphi = \sum_{\lambda \in (-4,0)} \dim \, \mathcal{E}(\lambda) .$$

One could try to relate this to the Morse co-index of $\mathcal{T}$, as we did for the Hitchin index of nearly Kähler structures in Proposition 7.19. However, a moment of thought suffices to realise that both the index and the co-index of $\mathcal{T}$ are infinite, even when restricted to Ω^4_{27}. Indeed, taking $\mathcal{E}$ as above, we have

$$\delta^2\mathcal{T}\Big|_{\mathcal{E}(\lambda)} = \int_M \chi_1 \wedge [(\lambda + 4) * \chi_2] = (\lambda + 4)\langle \chi_1, \chi_2 \rangle .$$

As we did for the nearly Kähler structures, one could investigate the index of the known examples since they all possess some symmetry that would allow us to reduce the PDE to a simpler problem. We do not work out any examples explicitly, but provide an outline of how to compute, or rather bound, the Hitchin index.

(i) **Homogeneous examples:** The Peter-Weyl formalism for reductive spaces described above carries over verbatim. The case of nearly parallel G_2-structures is slightly more challenging since the differential operator is not simply a Laplacian. Thus, computations become more tedious. Some computations in this direction were carried out by Alexandrov and Semmelmann in [AS12] and Lehmann in [Leh21].

(ii) **Sasaki-Einstein examples:** Recall that the inclusion $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$ implies that every Sasaki-Einstein manifold carries a natural nearly parallel G_2-structure. Let us assume that the underlying Sasaki structure is quasi-regular, so the Reeb field integrates into an S^1 action. In this case, the PDE

$$\Delta\chi + 4d^*\chi = \mu\chi$$

is S^1-equivariant. Thus, one can try to use the Peter-Weyl (Fourier) formalism along the fibres to reduce this problem to a complex PDE on the leaf space and obtain a bound for the index in terms of Hodge numbers of the complex orbifold base, using the results of Nagy's PhD thesis [Nag01].

The added difficulty in this case is that, while the PDE above is S^1-invariant, the underlying G_2-structure is not (and thus neither is Ω^4_{27}), so one would need to check that the forms constructed above had the correct type.

When the Sasaki-Einstein is irregular, there is a higher-dimensional torus acting isometrically on the manifold. One might then try to generalise this approach, but the geometry of the orbit space becomes significantly more intricate and less tractable.

(iii) **Squashed examples:** The squashed nearly parallel G_2 metrics are constructed by rescaling the fibres of a 3-Sasaki manifold. In particular, the squashed metric has an isometric action by $\mathrm{SU}(2)$ with a 4-dimensional orbifold leaf space.

Thus, one can follow the same strategy of reducing the PDE to the 4-orbifold by using the Peter-Weyl formalism along the fibres. Similar ideas were presented in a recent preprint by Nagy and Semmelmann [NS23].

Cohomogeneity one nearly Kähler structures

We now turn to the study of the Hitchin and Einstein indices for the two known inhomogeneous nearly Kähler structures, constructed by Foscolo and Haskins [FH17].

Throughout this chapter, we consider 6-dimensional manifolds (M^6, ω, ρ) that admit a cohomogeneity one action by a compact Lie group $G \subseteq \mathrm{Aut}(M, \omega, \rho)$, such that the generic G-orbit has codimension one. These generic orbits, known as principal orbits, are diffeomorphic to the homogeneous space G/K, where K is the isotropy subgroup at a point on the orbit.

Since M will be a closed manifold with finite fundamental group, the general theory of cohomogeneity one manifolds (cf. [Bre72, Chapter IV, Theorem 8.2]) implies that the orbit space M/G is homeomorphic to a closed interval $[0, T]$. The preimage of the interior $(0, T)$ is an open dense subset $M^* \subset M$ that is G-equivariantly diffeomorphic to $(0, T) \times G/K$. At the endpoints of the interval $[0, T]$, the orbits degenerate to lower-dimensional submanifolds known as singular orbits, with isotropy subgroups $H_0, H_T \subset G$ respectively. These satisfy $K \subset H_i$, and the quotient spaces H_i/K are diffeomorphic to spheres.

The motivation for imposing a cohomogeneity one symmetry assumption is that it provides a dimensional reduction of the PDE into a system consisting of two parts: An algebraic problem on the space of invariant tensors, and an ODE on the orbit space $[0, T]$, that becomes singular at the endpoints.

We begin by reviewing the geometry of cohomogeneity one SU(3)-structures and the key elements of the Foscolo—Haskins construction. This includes their method of glueing nearly Kähler “halves” across a maximal volume orbit to obtain complete, inhomogeneous nearly Kähler manifolds. With this background, we then examine the eigenvalue problem associated with the Hitchin index under the assumption of a cohomogeneity one symmetry.

9 Cohomogeneity one SU(3)-structures

We study the structure induced on the principal orbits by the SU(3)-structure on M. By the work of Conti–Salamon [CS07], the frame bundle of any orientable hypersurface $\Sigma^5 \hookrightarrow M^6$ admits a reduction to a principal SU(2)-bundle. This is equivalent to the existence of a nowhere-vanishing 1-form η and a triple of 2-forms $(\omega_1, \omega_2, \omega_3)$ satisfying the conditions

(i) $\eta \wedge \omega_i \wedge \omega_j = 2\delta_{ij}\,\mathrm{vol}_\Sigma$,

(ii) $X \lrcorner \omega_1 = Y \lrcorner \omega_2 \implies \omega_3(X,Y) \geq 0$.

The forms ω_i pointwise span a subbundle of Λ^2, which we denote as Λ^2_+. Its orthogonal complement within $\ker(\eta)$ will be denoted by Λ^2_-. This notation is justified by observing that the Hodge star restricted to $\ker(\eta)$ acts as $\pm\,\mathrm{Id}$ on $\Lambda^2_\pm$. Let ν denote the positive unit normal vector field to Σ. The induced SU(2)-structure on Σ is given explicitly by

$$\eta = \nu \lrcorner \omega \qquad \omega_1 = \omega\,|_\Sigma \qquad \omega_2 + i\omega_3 = \nu \lrcorner (\hat{\rho} - i\rho)\ .$$

Conversely, given one parameter family of SU(2)-structures, we can define an SU(3)-structure on $\Sigma \times (a,b)$ by taking

$$\omega = \eta \wedge dt + \omega_1 \qquad \rho + i\hat{\rho} = (\omega_2 + i\omega_3) \wedge (\eta + idt)\ , \tag{87}$$

where t is the coordinate on the interval (a,b).

By the work of Podestà and Spiro [PS10], the only interesting cases of cohomogeneity one nearly Kähler manifolds occur when $G \cong \mathrm{SU}(2)^2$, and the principal orbit is always diffeomorphic to $S^2 \times S^3 \cong N_{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\triangle\mathrm{U}(1)$. Thus, we are interested in parametrising the set of invariant SU(2)-structures on it. On $N_{1,1}$, we have a distinguished invariant SU(2) structure: the Sasaki-Einstein structure coming from the Calabi-Yau conifold $V(z_1^2 + z_2^2 + z_3^2 + z_4^2) \subseteq \mathbb{C}^4$. We will denote the associated basis of invariant forms by $\eta^{se} \in \Omega^1$, $\omega_0^{se} \in \Omega^2_-$ and $\omega_1^{se}, \omega_2^{se}, \omega_3^{se} \in \Omega^2_+$, satisfying

$$d\eta^{se} = -2\omega_1^{se} \qquad d\omega_2^{se} = 3\eta^{se} \wedge \omega_3^{se} \qquad d\omega_3^{se} = -3\eta^{se} \wedge \omega_2^{se} \qquad d\omega_0^{se} = 0\ .$$

With respect to the Sasaki-Einstein structure, the space of invariant SU(2)-structures on $S^2 \times S^3$ is parametrised by $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{SO}_0(1,3)$. Given $(\lambda, \mu, A) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{SO}_0(1,3)$, the associated SU(2)-structure is given by

$$\eta = \lambda\eta^{se} \qquad \omega_i = \mu A \omega_i^{se}\ .$$

Remark 9.1. *The (left-invariant) Reeb field generates the subgroup of inner automorphisms of* SU(2) × SU(2) *that fixes* ΔU(1). *In terms of the invariant* SU(2)*-structure, the Reeb field induces a rotation in the* $(\omega_2^{se}, \omega_3^{se})$*-plane.*

We get the following formula from the structure equations for the Sasaki-Einstein metric. Let (λ, μ, A) denote an invariant SU(2)-structure. Then,

$$d\eta = -2\lambda\omega_1^{se} \qquad d\omega_i = \frac{\mu}{\lambda}\eta \wedge TA\omega_i^{se} \qquad d(\eta \wedge \omega_i) = d\eta \wedge \omega_i = -2\lambda\mu\langle A\omega_i, \omega_1^{se}\rangle\,\mathrm{vol}^{se}\ , \tag{88}$$

where $T \in \mathrm{End}(\mathbb{R}^{1,3})$ is given by $T(\omega_0^{se}) = T(\omega_1^{se}) = 0$, $T(\omega_2^{se}) = 3\omega_3^{se}$ and $T(\omega_3^{se}) = -3\omega_2^{se}$.

Remark 9.2. *Formula (2.17) in [FH17] contains two typos, which are corrected above.*

9.1 Local nearly Kähler conditions

We can ask under what conditions a family of SU(2)-structures on Σ gives rise to a nearly Kähler structure on $\Sigma \times (a,b)$. Using the definition of a nearly Kähler structure and (87), the

SU(3)-structure will be a nearly Kähler structure if and only if the SU(2)-structure satisfies the equations

$$d\omega_1 = 3\eta \wedge \omega_2 \qquad d(\eta \wedge \omega_3) = -2\omega_1^2 \,, \tag{89}$$

as well as the evolution equations

$$\partial_t \omega_1 = -3\omega_3 - d\eta \qquad \partial_t(\eta \wedge \omega_2) = -d\omega_3 \qquad \partial_t(\eta \wedge \omega_3) = d\omega_2 + 4\eta \wedge \omega_1 \,. \tag{90}$$

An SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ satisfying Equations (89) is called a nearly hypo SU(2)-structure. Equations (90) are called nearly hypo evolution equations. When restricting to the case of cohomogeneity one, Foscolo and Haskins introduce a change of variables to obtain an ODE system rather than a mixed differential and algebraic system. Their results are summarised in the following proposition:

Proposition 9.3 ([FH17, Prop. 3.9]). *Let $\Psi(t) = (\lambda, \underline{u}, \underline{v})$ be a solution of the ODE system*

$$\lambda \partial_t u_0 + 3v_0 = 0 \,, \tag{91a}$$
$$\lambda \partial_t u_1 + 3v_1 = 2\lambda^2 \,, \tag{91b}$$
$$\lambda \partial_t u_2 + 3v_2 = 0 \,, \tag{91c}$$
$$\partial_t v_0 - 4\lambda u_0 = 0 \,, \tag{91d}$$
$$\partial_t v_1 - 4\lambda u_1 = 0 \,, \tag{91e}$$
$$\partial_t v_2 - 4\lambda u_2 = -3\frac{u_2}{\lambda} \,, \tag{91f}$$
$$\lambda |u|^2 \partial_t \lambda^2 - \partial_t u_2^2 = -\lambda^4 u_1 \tag{91g}$$

defined on an interval $(a,b) \subseteq \mathbb{R}$ $u_2 < 0$, $\lambda, \mu^2 > 0$ and $u_1 v_2 - u_2 v_1 > 0$. Moreover, assume that there exists some $t_0 \in (a,b)$ for which the quantities

$$I_1(t) = \langle \underline{u}, \underline{v} \rangle \qquad I_2(t) = \lambda^2 |\underline{u}|^2 - u_2^2 \qquad I_3(t) = \lambda^2 |\underline{u}|^2 - |\underline{v}|^2 \qquad I_4(t) = v_1 - |\underline{u}|^2 \tag{92}$$

all vanish. Then $\psi_{\lambda,\mu,A}$ with $\mu = |\underline{u}|$ and

$$A = \frac{1}{\lambda \mu^2} \begin{pmatrix} u_1 v_2 - v_1 u_2 & \lambda \mu u_0 & 0 & \mu v_0 \\ u_0 v_2 - u_2 v_0 & \lambda \mu u_1 & 0 & \mu v_1 \\ u_1 v_0 - v_1 u_0 & \lambda \mu u_2 & 0 & \mu v_2 \\ 0 & 0 & -\lambda \mu^2 & 0 \end{pmatrix} = \begin{pmatrix} w_0 & x_0 & 0 & y_0 \\ w_1 & x_1 & 0 & \frac{\mu}{\lambda} \\ w_2 & -\lambda & 0 & y_2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{93}$$

defines an $\mathrm{SU}(2)^2$-invariant nearly Kähler structure on $(a,b) \times N_{1,1}$.

Remark 9.4. *The vanishing of I_1, I_2, I_3 and I_4 correspond to $\omega_1 \wedge \omega_3 = 0$, $\omega_1^2 = \omega_2^2$, and $\omega_1^2 = \omega_3^2$ and the second equation of* (89) *respectively. The ODE* (91g) *implies the vanishing of $I = (I_1, I_2, I_3, I_4)$ is a conserved quantity of the ODE.*

Corollary 9.5 ([FH17, Cor. 2.46]). *Let $\Psi_{\lambda,\mu,A}$ be an invariant nearly hypo structure on $N_{1,1}$ such that $w_1 = 0 = w_2$. Then, it is an invariant hypersurface of the sine-cone over the invariant Sasaki-Einstein.*

The ODE system (91) is invariant under various symmetries. Three of them will be key in the discussion ahead: time translation $t \mapsto t + t_0$, for $t_0 \in \mathbb{R}$ and the following two involutions:

$$\tau_1 : (\lambda, u_0, u_1, u_2, v_0, v_1, v_2, t) \mapsto (\lambda, -u_0, -u_1, u_2, v_0, v_1, -v_2, -t) \ , \tag{94a}$$

$$\tau_2 : (\lambda, u_0, u_1, u_2, v_0, v_1, v_2, t) \mapsto (\lambda, u_0, -u_1, u_2, -v_0, v_1, -v_2, -t) \ . \tag{94b}$$

A complete list of the symmetries of the ODE (91) and their geometric interpretation can be found in [FH17, Prop. 3.11].

There are four explicit examples of solutions to the ODE system (91): the three homogeneous examples S^6, $\mathbb{C}P^3$ and $S^3 \times S^3$; and the sine-cone over the homogeneous $N_{1,1}$ with its homogeneous Sasaki-Einstein structure.

Example 9.6 (The sine cone)**.**

$$\lambda = \sin(t) \qquad \mu = \sin^2(t) \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) \\ 0 & -\sin(t) & 0 & \cos(t) \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{95}$$

Example 9.7 (Homogeneous nearly Kähler on $S^3 \times S^3$)**.**

$$\lambda = 1 \qquad \mu = \frac{2\sqrt{3}}{3}\sin(\sqrt{3}t)$$

$$\mu A = \begin{pmatrix} \frac{2}{3}\left(\sin^2(\sqrt{3}t)+1\right) & \frac{\sqrt{3}}{3}\sin(2\sqrt{3}t) & 0 & \frac{2}{3}\left(2\sin^2(\sqrt{3}t)-1\right) \\ \frac{2}{3}\sin^2(\sqrt{3}t) & \frac{\sqrt{3}}{3}\sin(2\sqrt{3}t) & 0 & \frac{4}{3}\sin^2(\sqrt{3}t) \\ -\frac{2}{3}\cos(\sqrt{3}t) & -\frac{2\sqrt{3}}{3}\sin(\sqrt{3}t) & 0 & \frac{2}{3}\cos(\sqrt{3}t) \\ 0 & 0 & -\frac{2\sqrt{3}}{3}\sin(\sqrt{3}t) & 0 \end{pmatrix} \tag{96}$$

We collect some formulae and relations for general cohomogeneity one nearly Kähler manifolds:

Lemma 9.8. *Let (M^6, ω, ρ) be a cohomogeneity one nearly Kähler manifold, and let $(\eta, \omega_i) = \psi_{\lambda,\mu,A}(\eta^{se}, \omega_i^{se})$ the associated* SU(2)*-moving frame. Then, we have the following relations*

(i) $d\eta = 2\frac{\lambda w_1}{\mu}\omega_0 - 2\frac{\lambda x_1}{\mu}\omega_1 - 2\omega_3$,

(ii) $d\omega_0 = -3\frac{w_2}{\lambda}\eta \wedge \omega_2$,

(iii) $d(\eta \wedge \omega_0) = -2\frac{\lambda w_1}{\mu}\omega_1^2$,

(iv) $d\omega_2 = -\frac{3w_2}{\lambda}\eta \wedge \omega_0 - 3\eta \wedge \omega_1 + \frac{3y_2}{\lambda}\eta \wedge \omega_3$,

(v) $d\omega_3 = -3\frac{y_2}{\lambda}\eta \wedge \omega_2$,

(vi) $\partial_t \eta = \partial_t \log(\lambda)\eta$,

(vii) $\partial_t \omega_0 = \partial_t \log(\mu)\omega_0 - 2\frac{\lambda w_1}{\mu}\omega_1 - 3\frac{w_2}{\lambda}\omega_3$,

(viii) $\partial_t \lambda = 3y_2 - 2\frac{\lambda^2}{\mu}x_1$,

(ix) $\partial_t \mu = 2\lambda x_1$.

where the functions w_i, x_i and y_i are the components of the moving frame matrix A defined in Equation (93)*.*

Proof. All the identities follow from the above formulae and the fact that $A^{-1} = JA^tJ$ for $J = \mathrm{diag}(-1,1,1,1)$, since $A \in \mathrm{SO}_0(1,3)$. We give the full details of the derivation of (vii). We have

$$\partial_t\omega_0 = (\mu A)'\omega_0^{se} = \log(\mu)'\omega_0 + (A^tJA')\omega_0 = \log(\mu)'\omega_0 + \langle w', w\rangle\omega_0 + \langle w', x\rangle\omega_1 + \langle w', y\rangle\omega_3 \ ,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^{1,3}$. Using that $A \in \mathrm{SO}(1,3)$, we know that $\langle w, w\rangle = -1$, $\langle w, x\rangle = \langle w, y\rangle = 0$. First, $\langle w, w\rangle = -1$ implies $\langle w', w\rangle = 0$. For the ω_1 component, we have

$$\langle w', x\rangle = -\langle w, x'\rangle = -\frac{1}{\mu}\langle w, (\mu x)'\rangle = -\frac{1}{\mu}\langle w, u'\rangle = -\frac{1}{\mu}\left(-3\langle w, v\rangle + 2\lambda w_1\right) = -2\frac{\lambda}{\mu}w_1 \ ,$$

where we used the structure ODE (91a) - (91c) and the fact that $\langle w, v\rangle = 0 = \langle w, u\rangle$. Similarly,

$$\langle w', y\rangle = -\langle w, y'\rangle = -\frac{1}{\lambda\mu}\langle w, (\lambda\mu y)'\rangle = -\frac{1}{\lambda\mu}\langle w, v'\rangle = -\frac{1}{\lambda\mu}\left(4\lambda\langle w, v\rangle - 3\frac{u_2}{\lambda}w_2\right) = -3\frac{w_2}{\lambda} \ .$$

where we used (91d) - (91f), the fact that $\langle w, u\rangle = 0$ and $u_2 = -\mu\lambda$ on the last step. □

Similar relations could be obtained for the remaining forms, but are omitted since they are not needed in our discussion.

Lemma 9.9. *Let (M^6, ω, ρ) be a cohomogeneity one nearly Kähler structure, and let (η, ω_i) the associated* SU(2)*-moving frame. Then, the first column of A from Equation* (93) *satisfies the evolution equations*

$$\partial_t w_0 = -2\frac{\lambda x_0}{\mu}w_1 - 3\frac{y_0}{\lambda}w_2 \tag{97a}$$

$$\partial_t w_1 = -2\frac{\lambda x_1}{\mu}w_1 - 3\frac{\mu}{\lambda^2}w_2 \tag{97b}$$

$$\partial_t w_2 = 2\frac{\lambda^2}{\mu}w_1 - 3\frac{y_2}{\lambda}w_2 \tag{97c}$$

Proof. From the computations of Lemma 9.8, we know that

$$\langle w', w\rangle = 0 \qquad \langle w', x\rangle = -2\frac{\lambda}{\mu}w_1 \qquad \langle w', y\rangle = -3\frac{w_2}{\lambda} \ .$$

In other words, we have $A^{-1}w' = \begin{pmatrix} 0 \\ -2\frac{\lambda}{\mu}w_1 \\ -3\frac{w_2}{\lambda} \end{pmatrix}$, and the claim follows by matrix multiplication.

□

9.2 Smooth extensions over the singular orbit

To construct complete nearly Kähler manifolds, Foscolo and Haskins first construct two families of desingularisations of the cone singularity over $N_{1,1}$. We will refer to these as nearly Kähler halves and denote them by $\Psi_a(t)$ and $\Psi_b(t)$. In both cases, the parameter measures the size of the singular orbit.

Let us revise how the desingularising families are constructed. Due to the work of Eschenburg and Wang [EW00], we have a good understanding of the necessary and sufficient conditions for a cohomogeneity one tensor to extend smoothly over a singular orbit. In our case, this reduces to the following lemma:

Lemma 9.10 ([FH17, Lemma 4.1] and [PS10, Prop. 6.1]). *Let $\omega = F(t)\eta^{se} \wedge dt + G_0(t)\omega_0^{se} + G_1(t)\omega_1^{se} + G_2(t)\omega_2^{se} + G_3(t)\omega_3^{se}$ an $\mathrm{SU}(2)^2$-invariant 2-form on $(0,T) \times N_{1,1}$. Then*

(i) ω extends over a singular orbit $\mathrm{SU}(2)^2/\mathrm{SU}(2) \times \mathrm{U}(1) \cong S^2$ at $t = 0$ if and only if

- *(a) G_0, G_1, G_2, G_3 are even and F is odd;*
- *(b) $G_2(0) = G_3(0) = 0$ and $G_0(t) - G_1(t) = -\partial_t F(0)t^2 + O(t^4)$.*

(ii) ω extends over a singular orbit $\mathrm{SU}(2)^2/\triangle\mathrm{SU}(2) \cong S^3$ at $t = 0$ if and only if

- *(a) G_0, G_1, G_2 are odd and G_3, F are even;*
- *(b) $G_0(t) + G_2(t) = O(t^3)$, $G_3(t) = O(t^2)$ and $G_1(t) = 2F(0)t + O(t^3)$.*

Under the conditions of the lemma, the ODE system (91) gives rise to a singular ODE initial value problem. Foscolo and Haskins argue the existence and uniqueness of the solution to the ODE by formally solving it in terms of a power series and then applying a contraction mapping fixed point argument.

Theorem 9.11 ([FH17, Thm. 4.4 & 4.5]). *For each $a > 0$, there exists a unique solution to* (91) *that extends smoothly over the singular orbit $\mathrm{SU}(2)^2/\mathrm{SU}(2) \times \mathrm{U}(1)$, denoted by $\Psi_a(t)$. Similarly, for each $b > 0$, there exists a unique solution to* (91) *that extends smoothly over the singular orbit $\mathrm{SU}(2)^2/\triangle\mathrm{SU}(2)$, denoted by $\Psi_b(t)$.*

The first terms of each of the Taylor expansions were worked out by Foscolo and Haskins and are collected in Appendix D for convenience.

9.3 Complete nearly Kähler solutions

Once we have nearly Kähler halves, we need to match two such halves to construct a complete solution. In that direction, we have

Proposition 9.12 ([FH17, Prop. 5.15]). *Let $\Psi(t)$ be a solution to* (91) *which extends smoothly over the singular orbit. Then, $\Psi(t)$ has a unique maximal volume orbit: a unique T_* exists for which the nearly hypo structure on $N_{1,1}$, corresponding to $\Psi(T_*)$, has mean curvature zero.*

Thus, it is reasonable to match two nearly Kähler halves along their maximum volume orbits. Let Ψ^1 and $\Psi^2 = \widetilde{\Psi}$ be two solutions for the system (91) with maximum volume orbit at time T_*^1 and T_*^2, and assume that the two maximal volume orbits coincide, so $\left(\lambda(T_*^1), \mu(T_*^1)\right) = \left(\tilde{\lambda}(T_*^2), \tilde{\mu}(T_*^2)\right)$. In particular, the two solutions must coincide on the maximum volume orbit up to the action of the involutions (94). Acting by a time translation $\tau = T_1 + T_2 - t$ and τ_1 or τ_2, we consider

$$\Psi_2^{\pm}(t) = \left(\tilde{\lambda}(\tau), \mp\tilde{u}_0(\tau), -\tilde{u}_1(\tau), \tilde{u}_2(\tau), \pm\tilde{v}_0(\tau), \tilde{v}_1(\tau), -\tilde{v}_2(\tau)\right) .$$

We define the two solutions

$$\Psi(t) = \begin{cases} \Psi_1(t) & 0 \le t \le T_1 \\ \Psi_2^{+}(t) & T_1 \le t \le T_1 + T_2 \end{cases} , \tag{98a}$$

$$\Psi(t) = \begin{cases} \Psi_1(t) & 0 \le t \le T_1 \\ \Psi_2^{-}(t) & T_1 \le t \le T_1 + T_2 \end{cases} . \tag{98b}$$

If either solution is smooth, we will have a complete nearly Kähler manifold. The following lemmas outline the conditions necessary for this to occur.

Lemma 9.13 (Doubling lemma, [FH17, Lemmas 5.19 & 8.4])**.** *Let $a \in (0, \infty)$ and consider $\Psi_a(t)$ the corresponding nearly Kähler half with singular orbit S^2. Denote by T_a the time of maximum volume orbit.*

(i) If $w_1(T_a) = 0$, then (98b) *with $\Psi_1 = \Psi_2 = \Psi_a$ defines a smooth nearly Kähler structure on $\mathbb{C}P^3$.*

(ii) If $w_2(T_a) = 0$, then (98a) *with $\Psi_1 = \Psi_2 = \Psi_a$ defines a smooth nearly Kähler structure on $S^2 \times S^4$.*

Similarly, let $b \in (0, \infty)$ and consider $\Psi_b(t)$ the corresponding nearly Kähler half with singular orbit S^3. Denote by T_b the time of maximum volume orbit. If $w_1(T_b) = 0$ (resp. $w_2(T_b) = 0$), then (98b) *(resp.* (98a)*) with $\Psi_1 = \Psi_2 = \Psi_b$ defines a smooth cohomogeneity one nearly Kähler structure on $S^3 \times S^3$.*

Lemma 9.14 (Matching lemma, [FH17, Lemma 5.20, 8.4])**.**

(i) Suppose that there exist $a < a' \in (0, \infty)$ such that $(w_1(T_a), w_2(T_a)) = \pm\,(w_1(T_{a'}), -w_2(T_{a'}))$. Set $\Psi_1 = \Psi_a$ and $\Psi_2 = \Psi_{a'}$. Then either (98a) *defines a smooth nearly Kähler on $S^2 \times \mathcal{S}^4$ or* (98b) *defines one on $\mathbb{C}P^3$.*

(ii) Suppose that there exist $b < b' \in (0, \infty)$ such that $(w_1(T_b), w_2(T_b)) = \pm\,(w_1(T_{b'}), -w_2(T_{b'}))$. Then either (98a) *or* (98b) *defines a smooth nearly Kähler structure on $S^3 \times S^3$ for $\Psi_1 = \Psi_b$ and $\Psi_2 = \Psi_{b'}$.*

(iii) Suppose that there exist $a, b \in (0, \infty)$ such that $(w_1(T_a), w_2(T_a)) = \pm\,(w_1(T_b), -w_2(T_b))$. Then either (98a) *or* (98b) *defines a smooth nearly Kähler structure on S^6 for $\Psi_1 = \Psi_a$ and $\Psi_2 = \Psi_b$.*

We outline the proof of Foscolo and Haskins on the existence of an inhomogeneous nearly Kähler structure on $S^3 \times S^3$ using the result in Lemma 9.13. Consider the curve

$$\begin{aligned} \beta : (0, \infty) &\to \mathbb{R}^2 \\ b &\mapsto \left(w_1^b(T_b), w_2^b(T_b)\right) \ , \end{aligned}$$

For small b, the nearly Kähler half converges to the sine-cone, so $\lim_{b \to 0} \beta = (0,0)$. The homogeneous nearly Kähler structure on $S^3 \times S^3$ corresponds to $b = 1$ and $\beta(1) = \left(\frac{\sqrt{3}}{3}, 0\right)$. Foscolo and Haskins prove

Theorem 9.15 ([FH17, Thm. 7.12])**.** *There exists $b_* \in (0,1)$ such that $\beta(b_*) = (0, w_2(b_*))$. By Lemma 9.13, the nearly Kähler solution* (98b) *with $\Psi_1 = \Psi_2 = \Psi_{b_*}$ defines a smooth nearly Kähler structure on $S^3 \times S^3$.*

Their proof strategy first involves relating the zeros of w_1 and w_2 with those of v_0 and u_0, respectively. The functions u_0 and v_0 satisfy the system (91a)-(91d)

$$\lambda \partial_t u_0 = -3 v_0 \qquad\qquad \partial_t v_0 = 4 \lambda u_0 \ .$$

In particular, they are amenable to a Sturm comparison argument with the Legendre Sturm-Liouville problem

$$\sin(t) \partial_t \hat{u} = -3 \hat{v} \qquad\qquad \partial_t \hat{v} = 4 \sin(t) \hat{u} \ ,$$

which is the linearisation of the system (91) on the sine-cone. In their proof, Foscolo and Haskins can only prove that the curve β must cross the $w_1 = 0$ axis in the range $b \in (0,1)$ but cannot establish whether such crossing is unique, although they numerically conjecture this to be the case.

In any case, there exists $b_* \in (0,1)$ for which the curve β crosses the vertical axis for the last time before arriving at the homogeneous structure. For the remainder of the notes, we will refer to the corresponding Ψ_{b_*} (and its complete double) as **the** inhomogeneous nearly Kähler structure on $S^3 \times S^3$. We conclude this section by characterising this inhomogeneous nearly Kähler structure, which will be helpful when studying its index. Although we do not have an explicit expression for $w_1(t)$ and $w_2(t)$, we can characterise their qualitative behaviour.

Proposition 9.16. *Let $\Psi_{b_*}(t)$ be the nearly Kähler half corresponding to the inhomogeneous nearly Kähler structure on $S^3 \times S^3$ described in [FH17] with maximal volume orbit at time T_*. Then*

(i) $w_1(t) > 0$ for $t \in (0, T_*)$, and

(ii) $w_2(T_*) > 0$.

Proof. The solution Ψ_{b_*} corresponds to the last time the family $\beta(b)$ crosses the axis $w_1 = 0$ before the homogeneous solution (96). Since the homogeneous solution satisfies $w_1(t) > \delta > 0$ for $t \in (0, \frac{\pi\sqrt{3}}{6}]$ and $\delta > 0$; it follows that $w_1^{b_*}(t) > 0$ for $t \in (0, T_*)$, which implies that $w_1(T_*)$ is a zero with a non-positive slope. Thus, Equation (97b) reduces to

$$\partial_t w_1 \bigg|_{T_*} = -3 \frac{\mu}{\lambda^2} w_2 \leq 0 \ ,$$

which implies $w_2(T_*) \geq 0$. If it were zero, we would have $w_1(T_*) = w_2(T_*) = 0$, and we would be on the sine cone by Corollary 9.5, so $w_2(T_*) > 0$, as needed. $\square$

10 Hitchin functional in the cohomogeneity one setting

We consider the reduction of the closed nearly Kähler Hitchin functional introduced in Section 7.2 to the cohomogeneity one setting. Recall that this functional is defined as

$$\begin{aligned} \mathcal{Q} &: \mathcal{U} \to \mathbb{R} \\ \omega &\mapsto \frac{1}{3}\int_M \mathrm{vol}_{d\omega} - 4\int_M \mathrm{vol}_\omega \ , \end{aligned}$$

with $\mathcal{U} = \{\omega \in \Omega^2 | \ d\omega \text{ stable, } \omega \text{ stable and positive, } \omega^2 \text{ exact}\}$. It is instructive to investigate how the set $\mathcal{U}$ and the Hitchin functional $\mathcal{Q}$ restrict to the cohomogeneity one case. Consider $\omega = \lambda\eta^{se} \wedge dt + \underline{u}\omega^{se}$ a stable cohomogeneity one 2-form. The stability of ω corresponds to $\lambda|\underline{u}|^2 \neq 0$, and one obtains similar open conditions for the stability of $d\omega$ and the positivity of ω with respect to the induced almost complex structure. Finally, we find it convenient to weaken the condition of ω^2 being exact to $d\omega^2 = 0$. This corresponds to the evolution equation

$$\partial_t|\underline{u}|^2 = 4\lambda u_1 \ , \tag{99}$$

since $\omega^2 = 2|\underline{u}|^2\mathrm{vol}_h^{se} + 2\lambda\underline{u}\eta^{se} \wedge dt \wedge \omega^{se}$, and the claim follows by differentiation.

Proposition 10.1. *Let $\omega = \lambda(t)\eta^{se} \wedge dt + \underline{u}(t)\omega^{se}$ be a cohomogeneity one 2-form satisfying the evolution equation (99). The functional $\mathcal{Q}$ restricted to cohomogeneity one forms becomes*

$$\mathcal{Q}^{(1)}(\lambda, \underline{u}) = C\int_I 4\lambda^3 + \lambda|\partial_t\underline{u}|^2 - 4\lambda^2\partial_t u_1 + \frac{9}{\lambda}(u_2^2 + u_3^2) - 12\lambda|\underline{u}|^2 dt \ ,$$

for $C \in \mathbb{R}$ a constant and I the interval on which our tuple $(\lambda(t), \underline{u}(t))$ is defined.

Proof. As above, let $\omega = \lambda\eta^{se} \wedge dt + \underline{u}\omega^{se}$. Then

$$d\omega = -2\lambda dt \wedge \omega_1^{se} + (\partial_t\underline{u})\, dt \wedge \omega^{se} + 3u_2\eta^{se} \wedge \omega_3^{se} - 3u_3\eta^{se} \wedge \omega_2^{se} \ .$$

By Proposition 7.11, $\omega \in \mathcal{U}$ defines a natural associated SU(3)-structure, and $\widehat{d\omega} = *d\omega$ will be given by

$$\widehat{d\omega} = 2\lambda^2\eta^{se} \wedge \omega_1^{se} + (\partial_t u_0)\,\lambda\eta^{se} \wedge \omega_0^{se} - \sum_{i=1}^{3}(\partial_t u_i)\,\lambda\eta^{se} \wedge \omega_i^{se} + 3\frac{u_2}{\lambda}dt \wedge \omega_3^{se} - 3\frac{u_3}{\lambda}dt \wedge \omega_2^{se} \ .$$

Thus, we have

$$\mathrm{vol}_{d\omega} = 2\left(4\lambda^3 + \lambda|\partial_t\underline{u}|^2 - 4\lambda^2\partial u_1 + \frac{9}{\lambda}\left(u_2^2 + u_3^2\right)\right)\eta^{se} \wedge dt \wedge \mathrm{vol}_h^{se} \ .$$

Using that vol_ω is proportional to $\lambda|\underline{u}|^2$, the claim follows from the definition of $\mathcal{Q}$ and integration along the $N_{1,1}$ fibres. $\square$

It is convenient to introduce a change of basis. Recall that the Reeb field induces a rotation in the span $\langle \omega_2^{se}, \omega_3^{se} \rangle$ (cf. Remark 9.1). Thus, we find it suitable to introduce the new basis $(\underline{\hat{u}}, \theta) = (\hat{u}_0, \hat{u}_1, \hat{u}_2, \theta)$, related to $\underline{u}$ by

$$\hat{u}_0 = u_0 \qquad \hat{u}_1 = u_1 \qquad \hat{u}_2 = u_2 \cos(\theta) \qquad \hat{u}_3 = u_2 \sin(\theta) \ . \tag{100}$$

Since there is no risk of confusion, we abuse notation and set $\underline{u} = \underline{\hat{u}}$ from now on. Under this change of variables and rescaling, the functional $\mathcal{Q}^{(1)}$ becomes

$$\mathcal{Q}^{(1)}(\lambda, \underline{u}, \theta) = \int_I 4\lambda^3 + \lambda|\partial_t \underline{u}|^2 + \lambda(u_2 \partial_t \theta)^2 - 4\lambda^2 \partial_t u_1 + \frac{9}{\lambda} u_2^2 - 12\lambda |u|^2 dt \ , \tag{101}$$

with $\underline{u} = (u_0, u_1, u_2) \in \mathcal{C}^\infty(I, \mathbb{R}^{1,2})$.

Proposition 10.2. *The Euler-Lagrange equations for $\mathcal{Q}^{(1)}$ are*

$$\frac{\delta \mathcal{Q}^{(1)}}{\delta \lambda} \implies 12\lambda^2 + |\partial_t \underline{u}|^2 - 8\lambda \partial_t u_1 - \frac{9}{\lambda^2} u_2^2 - 12|\underline{u}|^2 = 0 \tag{102a}$$

$$\frac{\delta \mathcal{Q}^{(1)}}{\delta u_0} \implies \partial_t(\lambda \partial_t u_0) + 12\lambda u_0 = 0 \tag{102b}$$

$$\frac{\delta \mathcal{Q}^{(1)}}{\delta u_1} \implies \partial_t(\lambda \partial_t u_1) + 12\lambda u_1 - 4\lambda \partial_t \lambda = 0 \tag{102c}$$

$$\frac{\delta \mathcal{Q}^{(1)}}{\delta u_2} \implies \partial_t(\lambda \partial_t u_2) + 12\lambda u_2 - \frac{9}{\lambda} u_2 - \lambda(\partial_t \theta)^2 u_2 = 0 \tag{102d}$$

$$\frac{\delta \mathcal{Q}^{(1)}}{\delta \theta} \implies \partial_t \left(\lambda u_2^2 \partial_t \theta \right) = 0 \tag{102e}$$

By the Principle of Symmetric Criticality of Palais [Pal79], solutions to these Euler–Lagrange equations correspond to cohomogeneity one nearly Kähler solutions on $S^2 \times S^3 \times I$. In particular, together with Equation (99), they should be equivalent to the system of Foscolo and Haskins [FH17] above. We show this to be the case. First, we have

Lemma 10.3. *Equations* (102b)-(102e) *are equivalent to the system* (91a)-(91f) *of Foscolo–Haskins.*

Proof. First, Equation (102e) directly implies $\lambda u_2^2 \partial_t \theta = C$ for some $C \in \mathbb{R}$, but boundary conditions force $C = 0$, from which it follows that $\partial_t \theta = 0$. So, we can choose $\theta(t) = 0$ as in [FH17]. The converse is immediate. Differentiation by t of $\lambda \partial_t u_i$ implies that Equations (91a)-(91f) are equivalent to Equations (102b)-(102d). □

Thus, we are left with showing that Equation (102a) and the condition $\partial_t |\underline{u}|^2 = 4\lambda u_1$ are equivalent to (91g). Instead, we will show that the Euler-Lagrange equations are equivalent to the conservation of the quantities $I_i(t)$ from (92). We have the following:

Proposition 10.4. *Under Equations* (102b)- (102d) *and $\partial_t |\underline{u}|^2 = 4\lambda u_1$, the conditions $I_1(t) = 0$ and $I_4(t) = 0$ are automatically satisfied. Moreover, Equation* (102a) *is equivalent to the conditions $I_2(t) = 0$. The remaining condition $I_3(t) = 0$ follows from $I_2 = 0 = I_4$.*

Before we prove the proposition, we introduce the following auxiliary calculation.

Lemma 10.5. *Under Equations* (102b)- (102d) *and* $\partial_t|\underline{u}|^2 = 4\lambda u_1$*, we have*

$$|\partial_t\underline{u}|^2 = 2\lambda\partial_t u_1 + 12|\underline{u}|^2 - \frac{9}{\lambda^2}u_2^2 \ .$$

Proof. The proof is a combination of the Leibniz rule and the aforementioned equations.

$$\begin{aligned}|\partial_t\underline{u}|^2 &= \partial_t\langle\underline{u},\partial_t\underline{u}\rangle - \langle\underline{u},\partial_t\frac{1}{\lambda}(\lambda\partial_t\underline{u})\rangle = \partial_t(2\lambda u_1) + (2\lambda u_1)\frac{\partial_t\lambda}{\lambda} - \frac{1}{\lambda}\langle\underline{u},\partial_t(\lambda\partial_t\underline{u})\rangle \\ &= 2\lambda\partial_t u_1 + 4u_1\partial_t\lambda - \frac{1}{\lambda}\Big[-12\lambda|\underline{u}|^2 + 4\lambda u_1\partial_t\lambda + \frac{9}{\lambda}u_2^2\Big] = 2\lambda\partial_t u_1 + 12|\underline{u}|^2 - \frac{9}{\lambda^2}u_2^2 \ . \quad \square\end{aligned}$$

Proof of Proposition 10.4. We start by showing that the conditions $I_1(t) = 0 = I_4(t)$ are automatically satisfied. First, we have

$$-3I_1(t) = -3\langle\underline{u},\underline{v}\rangle = -\lambda u_0\partial_t u_0 + \lambda u_1\partial_t u_1 + 2\lambda^2 u_1 + \lambda u_2\partial_t u_2 = \frac{\lambda}{2}\partial_t|\underline{u}|^2 + 2\lambda^2 u_1 = 0 \ ,$$

by Equation (99). Similarly, for I_4, we have

$$3I_4(t) = 3(v_1 - |\underline{u}|^2) = 2\lambda^2 - \lambda\partial_t u_1 - 3|u|^2 \ .$$

Differentiating, and using Equations (102c) and (99), we have

$$3\partial_t I_4 = 4\lambda\partial_t\lambda - \partial_t(\lambda\partial_t u_1) - 3\partial_t|\underline{u}|^2 = 12\lambda f_1 - 3(4\lambda f_1) = 0 \ .$$

Thus, $I_4(t)$ is a constant, which must again be zero by the boundary conditions. Let us now study the relation between the Euler-Lagrange equation for λ and I_2. Using the Lemma 10.5, we have

$$\begin{aligned}\frac{\delta\mathcal{Q}^{(1)}}{\delta\lambda} &= 12\lambda^2 + |\partial_t\underline{u}|^2 - 8\lambda\partial_t u_1 - \frac{9}{\lambda^2}u_2^2 - 12|u|^2 = 12\lambda^2 - 6\lambda\partial_t u_1 - \frac{18}{\lambda^2}u_2^2 \\ &= 6\Big[2\lambda^2 - \lambda\partial_t u_1 - \frac{3}{\lambda^2}u_2^2\Big] = 6I_4(t) + 18\lambda^2 I_2(t) \ .\end{aligned}$$

Similarly, for I_3 we have

$$I_3(t) = \lambda^2|\underline{u}|^2 - |\underline{v}|^2 = \lambda^2|\underline{u}|^2 - \frac{\lambda^2}{9}\Big[|\partial_t\underline{u}|^2 - 4\lambda\partial_t u_1 + 4\lambda^2\Big] \ ,$$

and so, by a direct substitution in Equation (102a), we get

$$\begin{aligned}\frac{\delta\mathcal{Q}^{(1)}}{\delta\lambda} + \frac{9}{\lambda^2}I_3(t) &= \big(8\lambda^2 - 4\lambda\partial_t u_1 - 12|\underline{u}|^2\big) + \Big(9|\underline{u}|^2 - \frac{9}{\lambda^2}u_2^2\Big) \\ &= 8I_4(t) + \frac{9}{\lambda^2}I_2(t) \ .\end{aligned}$$

Therefore, $I_3(t)$ will vanish if and only if I_2 (and I_4) vanishes, as needed. $\square$

11 The Hitchin index in the cohomogeneity one setting

We now return to studying the eigenvalue problem associated with the Hitchin index, as introduced in Definition 7.17. Given a nearly Kähler manifold (M^6,ω,ρ), we are interested in 2-forms $\beta \in \Omega^2_8$ satisfying

$$\Delta\beta = \nu\beta \qquad\qquad d^*\beta = 0 \ , \tag{103}$$

for $0 < \nu < 12$, with the limiting case $\nu = 12$ corresponding to infinitesimal deformations of the nearly Kähler structure. There is a one-to-one correspondence with solutions to the 1st-order PDE system

$$d\beta = \frac{\Lambda}{4}\gamma \qquad\qquad d^*\gamma = \frac{\Lambda}{3}\beta\ , \tag{104}$$

with $\gamma \in \Omega^3_{12}$ and $\Lambda = \sqrt{12\nu}$. We restrict to the positive branch of the square root, so $\Lambda > 0$. If (β, γ) were a solution to (104) for Λ, then $(\beta, -\gamma)$ is a solution for $-\Lambda$ giving rise to the same solution of (103).

We will focus on the first-order PDE system (104) for $\Lambda \in (0, 12)$. While finding the complete set of solutions to this system seems currently out of reach, even in the cohomogeneity one case, we can restrict ourselves to finding solutions to the PDE system with the same cohomogeneity one symmetry as the underlying nearly Kähler structure. In other words, we are computing the Hitchin index of the functional $\mathcal{Q}^{(1)}$ introduced above.

For the remainder of the section, (M^6, ω, ρ) will denote a cohomogeneity one nearly Kähler manifold. Let us start by characterising cohomogeneity one forms of type 8 and 12.

Lemma 11.1. *Let (M, ω, ρ) be a cohomogeneity one nearly Kähler structure and let $\eta(t), \omega_i(t)$ be the associated moving frame for the underlying $SU(2)$ structure on $N_{1,1}$. Cohomogeneity one forms of type $\beta \in \Omega^2_8$ are parametrised by two functions, h_0 and h_1, so*

$$\beta = h_0\omega_0 + h_1(2\eta \wedge dt - \omega_1)\ .$$

Similarly, cohomogeneity one forms of type $\gamma \in \Omega^3_{12}$ are parametrised by four functions, f_0, f_2, f_3 and g_0, so

$$\gamma = f_0\eta \wedge \omega_0 + g_0 dt \wedge \omega_0 + f_2(\eta \wedge \omega_2 + dt \wedge \omega_3) + f_3(\eta \wedge \omega_3 - dt \wedge \omega_2)\ .$$

Proof. Let $\beta = h_i(t)\omega_i + V(t)\eta \wedge dt$ be an arbitrary cohomogeneity one 2-form. The condition that β is of type 8 is equivalent to $\omega \wedge \beta = - * \beta$. By direct computation, we have

$$*\beta = -h_0\eta \wedge dt \wedge \omega_0 + \frac{V}{2}\omega_1^2 + \sum_{i=1}^{3} h_i\eta \wedge dt \wedge \omega_i$$
$$\omega \wedge \beta = h_0\eta \wedge dt \wedge \omega_0 + (h_1 + V)\,\eta \wedge dt \wedge \omega_1 + h_2\eta \wedge dt \wedge \omega_2 + h_3\eta \wedge dt \wedge \omega_3 + h_1\omega_1^2\ ,$$

which implies $h_2 = h_3 = 0$ and $V = -2h_1$, as needed. Similarly, let $\gamma = f_i(t)\eta \wedge \omega_i + g_i dt \wedge \omega_i$ be an arbitrary cohomogeneity one 3-form. The condition that γ is of type 12 is equivalent to $\gamma \wedge \omega = 0$ and $\gamma \wedge \rho = 0 = \gamma \wedge \hat{\rho}$. Again, by direct computation, we have

$$\omega \wedge \gamma = f_1\eta \wedge \omega_1^2 + g_1 dt \wedge \omega_1^2 \qquad \rho \wedge \gamma = *(f_3 + g_2) \qquad \hat{\rho} \wedge \gamma = *(g_3 - f_2)\ ,$$

so, $f_1 = g_1 = 0$, $f_2 = g_3$ and $f_3 = -g_2$, as needed. □

We compute $d\beta$ and $d^*\gamma = - * d * \gamma$. Using Lemma 9.8, the defining equations (89) and the evolution equations (90), we can compute the exterior derivative of β:

$$
\begin{aligned}
d\beta &= \partial_t h_0 dt\wedge\omega_0 + h_0(d\omega_0 + dt\wedge\partial_t\omega_0) - \partial_t h_1 dt\wedge\omega_1 + h_1(2d\eta\wedge dt - d\omega_1 - dt\wedge\partial_t\omega_1)\\
&= \left(\frac{\partial_t(\mu h_0)}{\mu} + 6\frac{\lambda w_1}{\mu}h_1\right) dt\wedge\omega_0 - \left(\partial_t h_1 + 2h_0\frac{\lambda w_1}{\mu} + 6h_1\frac{\lambda x_1}{\mu}\right) dt\wedge\omega_1\\
&\quad - \left(\frac{3w_2}{\lambda}h_0 + 3h_1\right)(dt\wedge\omega_3 + \eta\wedge\omega_2)\ .
\end{aligned}
$$

Similarly, for $\gamma\in\Omega^3_{12}$ of cohomogeneity one, one computes

$$
*\gamma = -f_0 dt\wedge\omega_0 + g_0\eta\wedge\omega_0 + f_2(dt\wedge\omega_2 - \eta\wedge\omega_3) + f_3(dt\wedge\omega_3 + \eta\wedge\omega_2)\ .
$$

Again, using Lemma 9.8 and equations (89) and (90), we compute the exterior derivative

$$
\begin{aligned}
d*\gamma =& f_0 dt\wedge d\omega_0 + \partial_t g_0 dt\wedge\eta\wedge\omega_0 + g_0\left(dt\wedge\partial_t(\eta\wedge\omega_0) + d(\eta\wedge\omega_0)\right)\\
&+ \partial_t f_2\eta\wedge dt\wedge\omega_3 - f_2\left(dt\wedge d\omega_2 + d(\eta\wedge\omega_3) + dt\wedge\partial_t(\eta\wedge\omega_3)\right)\\
&- \partial_t f_3\eta\wedge dt\wedge\omega_2 - f_3\left(d(\eta\wedge\omega_2) + dt\wedge\partial_t(\eta\wedge\omega_2) - dt\wedge d\omega_3\right)\\
=& -\left(\partial_t g_0 + g_0\partial_t\log(\lambda\mu) + 6f_2\frac{w_2}{\lambda} + 3f_3\frac{w_2}{\lambda}\right)\eta\wedge dt\wedge\omega_0\\
&+ \left(2g_0\frac{\lambda w_1}{\mu} - 2f_2\right)\eta\wedge dt\wedge\omega_1 + \left(3f_0\frac{w_2}{\lambda} - \partial_t f_3 - 6f_3\frac{y_2}{\lambda}\right)\eta\wedge dt\wedge\omega_2\\
&+ \left(3g_0\frac{w_2}{\lambda} + \partial_t f_2 + 6f_2\frac{y_2}{\lambda} - 3f_3\frac{y_2}{\lambda}\right)\eta\wedge dt\wedge\omega_3 + \left(2f_2 - 2g_0\frac{\lambda w_1}{\mu}\right)\omega_1^2
\end{aligned}
$$

Since $\Lambda\neq 0$, we immediately get $f_0 = 0 = f_3$. Thus, the PDE system (104) reduces to

$$\partial_t(\mu h_0) + 6h_1\lambda w_1 = \frac{\Lambda}{4}\mu g_0\ , \tag{106a}$$

$$\partial_t h_1 + 2h_0\frac{\lambda w_1}{\mu} + 6\frac{\lambda x_1}{\mu}h_1 = 0\ , \tag{106b}$$

$$\partial_t(\lambda\mu g_0) + 6\mu w_2 f_2 = -\frac{\Lambda}{3}\lambda\mu h_0\ , \tag{106c}$$

$$3\frac{w_2}{\lambda}g_0 + \partial_t f_2 + 6\frac{y_2}{\lambda}f_2 = 0\ , \tag{106d}$$

$$3\frac{w_2}{\lambda}h_0 + 3h_1 + \frac{\Lambda}{4}f_2 = 0\ , \tag{106e}$$

$$2\frac{\lambda w_1}{\mu}g_0 - 2f_2 - \frac{\Lambda}{3}h_1 = 0\ . \tag{106f}$$

We distinguish two classes of equations. Equations (106a)-(106d) form a first order ODE system for h_0, h_1, g_0, f_2 whilst Equations (106e)-(106f) are of order zero and linear in h_1 and f_2. In particular, for $\Lambda\neq\sqrt{72}$, we can rewrite them as

$$h_1(t) = \frac{1}{\Lambda^2 - 72}\left(72\frac{w_2}{\lambda}h_0 + 6\Lambda\frac{\lambda w_1}{\mu}g_0\right)\ , \tag{107a}$$

$$f_2(t) = \frac{-1}{\Lambda^2 - 72}\left(12\Lambda\frac{w_2}{\lambda}h_0 + 72\frac{\lambda w_1}{\mu}g_0\right)\ . \tag{107b}$$

Since the last two equations are of order zero, it is reasonable to define the quantities

$$\Theta_1(t) = 3\frac{w_2}{\lambda}h_0 + 3h_1 + \frac{\Lambda}{4}f_2\ , \tag{108a}$$

$$\Theta_2(t) = 2\frac{\lambda w_1}{\mu}g_0 - 2f_2 - \frac{\Lambda}{3}h_1\ . \tag{108b}$$

As expected, these quantities Θ_i are conserved quantities of the system, so the system (106) is not overdetermined. More concretely, we have

Proposition 11.2. *Let* (h_0, h_1, g_0, f_2) *be a solution to* (106a)-(106d) *such that* $\Theta_1(t_0) = 0 = \Theta_2(t_0)$ *for some time* t_0*. Then* $\Theta_1(t) = 0 = \Theta_2(t)$ *for all time that* (h_0, h_1, g_0, f_2) *is defined.*

First, we state the following technical computation:

Lemma 11.3. *We have*

$$\partial_t\left(\frac{w_2}{\lambda}h_0\right) = \frac{\Lambda}{4}\frac{w_2}{\lambda}g_0 - 6\frac{w_1 w_2}{\mu}h_1 + 2\frac{\lambda w_1}{\mu}h_0 - 6\frac{y_2 w_2}{\lambda^2}h_0\,, \tag{109a}$$

$$\partial_t\left(\frac{\lambda w_1}{\mu}g_0\right) = -\frac{\Lambda}{3}\frac{\lambda w_1}{\mu}h_0 - 6\frac{w_1 w_2}{\mu}f_2 - 6\frac{\lambda^2 x_1 w_1}{\mu^2}g_0 - 3\frac{w_2}{\lambda}g_0\,. \tag{109b}$$

Proof. By direct computation,

$$\begin{aligned}
\partial_t\left(\frac{w_2}{\lambda}h_0\right) &= \frac{w_2}{\lambda\mu}\partial_t(\mu h_0) + h_0\left(\frac{1}{\lambda}\partial_t w_2 - \frac{w_2}{\lambda^2}\partial_t\lambda - \frac{w_2}{\lambda\mu}\partial_t\mu\right)\\
&= \frac{\Lambda}{4}\frac{w_2}{\lambda}g_0 - 6\frac{w_1 w_2}{\mu}h_1 + h_0\left[\left(2\frac{\lambda}{\mu}w_1 - 3\frac{y_2 w_2}{\lambda^2}\right) - \frac{w_2}{\lambda^2}\left(3y_2 - 2\frac{\lambda^2}{\mu}x_1\right) - 2\frac{w_2}{\mu}x_1\right]\\
&= \frac{\Lambda}{4}\frac{w_2}{\lambda}g_0 - 6\frac{w_1 w_2}{\mu}h_1 + 2\frac{\lambda w_1}{\mu}h_0 - 6\frac{y_2 w_2}{\lambda^2}h_0\,,
\end{aligned}$$

where we used Lemmas 9.8 and 9.9 in the second line. Similarly,

$$\begin{aligned}
\partial_t\left(\frac{\lambda w_1}{\mu}g_0\right) &= \frac{w_1}{\mu^2}\partial_t(\lambda\mu g_0) + g_0\left(\frac{\lambda}{\mu}\partial_t w_1 - 2\frac{w_1\lambda}{\mu^2}\partial_t\mu\right)\\
&= -\frac{\Lambda}{3}\frac{\lambda w_1}{\mu}h_0 - 6\frac{w_1 w_2}{\mu}f_2 + g_0\left[\left(-2\frac{\lambda^2 x_1}{\mu^2}w_1 - 3\frac{w_2}{\lambda}\right) - 4\frac{\lambda^2 x_1}{\mu^2}w_1\right]\\
&= -\frac{\Lambda}{3}\frac{\lambda w_1}{\mu}h_0 - 6\frac{w_1 w_2}{\mu}f_2 - 6\frac{\lambda^2 x_1 w_1}{\mu^2}g_0 - 3\frac{w_2}{\lambda}g_0\,. \qquad \square
\end{aligned}$$

Proof of Proposition 11.2. We start with Θ_1. Using the previous lemma and Equations (106b) and (106d), we have

$$\begin{aligned}
\frac{1}{3}\partial_t\Theta_1 &= \partial_t\left(\frac{w_2}{\lambda}h_0\right) + \partial_t h_1 + \frac{\Lambda}{12}\partial_t f_2 = -\frac{6}{\mu}(w_1 w_2 + \lambda x_1)h_1 - 6\frac{y_2 w_2}{\lambda^2}h_0 - \frac{\Lambda}{2}\frac{y_2}{\lambda}f_2\\
&= -\frac{6}{\mu}\left(w_1 w_2 + \lambda x_1 - \frac{\mu y_2}{\lambda}\right)h_1 - 2\frac{y_2}{\lambda}\Theta_1 = -2\frac{y_2}{\lambda}\Theta_1\,.
\end{aligned}$$

In the last line, we used that $w_1 w_2 + \lambda x_1 - \frac{\mu y_2}{\lambda}$ vanishes since it is the inner product of the second and third rows of the matrix $A \in \mathrm{SO}_0(1,3)$. Similarly, for Θ_2, we have

$$\begin{aligned}
\frac{1}{2}\partial_t\Theta_2 &= \partial_t\left(\frac{\lambda w_1}{\mu}g_0\right) - \partial_t f_2 - \frac{\Lambda}{6}\partial_t h_1 = \frac{6}{\mu}\left(\frac{\mu y_2}{\lambda} - w_1 w_2\right)f_2 - 6\frac{\lambda^2 x_1 w_1}{\lambda^2}h_0 + \Lambda\frac{\lambda x_1}{\mu}h_1\\
&= \frac{6}{\mu}\left(\frac{\mu y_2}{\lambda} - w_1 w_2 - \lambda x_1\right)f_2 - 3\frac{\lambda x_1}{\mu}\Theta_2 = -3\frac{\lambda x_1}{\mu}\Theta_2\,. \qquad \square
\end{aligned}$$

Thus, we can reduce ourselves to study the ODE system for $H = (\mu h_0, \lambda\mu g_0, h_1, f_2)$:

$$\partial_t H = \begin{pmatrix} 0 & \frac{\Lambda}{4\lambda} & -6\lambda w_1 & 0\\ -\frac{\Lambda\lambda}{3} & 0 & 0 & -6\mu w_2\\ -2\frac{\lambda w_1}{\mu^2} & 0 & -6\frac{\lambda x_1}{\mu} & 0\\ 0 & -3\frac{w_2}{\lambda^2\mu} & 0 & -6\frac{y_2}{\lambda}\end{pmatrix} H \tag{110}$$

with suitable initial conditions $H(t_0)$ satisfying (106e)-(106f). To lighten the notation and given the shape of the ODE system (110), we make the following change of variables for the remainder of the discussion:

$$\xi = \mu h_0 \qquad \chi = \lambda\mu g_0 \ .$$

Our ODE problem closely resembles the local nearly Kähler system (91), with the conserved quantities $I_i(t) = 0$ replaced by $\Theta_i(t) = 0$. This naturally raises the question of whether these Θ_i admit a geometric interpretation analogous to that of the I_i in the nearly Kähler case. Unfortunately, we do not currently have a satisfactory answer to this question.

To solve (110), we follow the same strategy for nearly Kähler structures: We solve (110) on a nearly Kähler half Ψ and then find suitable matching conditions along maximum volume orbits.

First, it is instructive to study the limiting case of the sine-cone, Example 9.6. In this case, the condition $w_1(t) = w_2(t) = 0$ and the conserved quantities yield the reduced system

$$\partial_t \begin{pmatrix} \xi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & \frac{\Lambda}{4\sin(t)} \\ \frac{\Lambda \ \sin(t)}{3} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \chi \end{pmatrix} \tag{111}$$

with $h_1 = f_2 = 0$ whenever $\Lambda \neq \sqrt{72}$. When $\Lambda = \sqrt{72}$, the system reduces to the ODE above but with $f_2 = -\sqrt{2}h_1(t) = C\sin^6(t)$.

The ODE (111) is the Legendre Sturm-Liouville problem under the change of variables $u = \sin(t)$. In particular, $\Lambda = 2$ and 6 are eigenvalue solutions to the Sturm-Lioville problem, each with a 2-dimensional eigenspace given by the corresponding Legendre polynomial of the first and second kind.

Notice that these solutions give rise to 2-forms β solving Equation (103), and decaying at rate -2, which is precisely the rate one would expect to see if we were trying to construct a solution close to the sine-cone, as it is the rate of harmonic forms on the Stenzel metric on T^*S^3 and the small resolution $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (cf. [FH17, Thm. 2.27]).

Remark 11.4. *The value $\Lambda = 12$ is also a solution to the Sturm-Liouville problem, corresponding to infinitesimal deformations of the nearly Kähler structure on the sine cone. Foscolo and Haskins used this for their Sturm comparison argument, discussed in Theorem 9.15.*

Existence of solutions over Nearly Kähler halves

We aim to solve the ODE system (110) on the nearly Kähler halves Ψ_a and Ψ_b discussed above. Explicitly, we want to find 2-forms

$$\beta = -h_0\omega_0 + h_1(\omega_1 - 2\eta \wedge dt) = -2\lambda h_1 \eta^{se} \wedge dt + \sum_{i=0}^{2} (-w_i\xi + u_i h_1)\,\omega_i^{se}$$

that extend smoothly over the singular orbits. The conserved quantities, Equations (106e) and (106f), guarantee the smoothness of the 3-form γ. By virtue of Lemma 9.10 and the Taylor expansions in Lemmas D.1 and D.2, we have the following:

Proposition 11.5.

(i) *The 2-form β extends over the singular orbit* $\mathrm{SU}(2)^2/U(1)\times\mathrm{SU}(2)\cong S^2$ *if and only if* ξ *is odd and* h_1 *is even. Equations* (106e) *and* (106f) *force* f_2 *to be even and* χ *to be odd. Their Taylor expansions are*

$$\xi = 6At + O(t^3) \qquad \chi = -A\Lambda t^3 + O(t^5) \qquad h_1 = -\frac{2\sqrt{3}}{3a}A + O(t^2) \qquad f_2 = Bt^2 + O(t^4)$$

for $A, B \in \mathbb{R}$.

(ii) *The 2-form β extends over the singular orbit* $\mathrm{SU}(2)^2/\triangle SU(2)$ *if and only if* ξ *and* h_1 *are even. Equations* (106e) *and* (106f) *imply* χ *is odd and* f_2 *is even. Their Taylor expansions are*

$$\xi = A\Lambda t^2 + O(t^4) \qquad \chi = 8bAt + O(t^3) \qquad h_1 = Bt^2 + O(t^4) \qquad f_2 = \frac{2A}{b} + O(t^2)$$

for $A, B \in \mathbb{R}$.

Proof.

(i) By Lemma 9.10, when the singular orbit is diffeomorphic to S^2, the coefficient functions G_i must be even, and F is odd. Since $w_i(t)$ are odd, ξ is odd too. Now, λ is odd, so h_1 is even, which is compatible with u_i being even. The conditions $G_2(t) = G_3(t) = 0$ are immediate from Lemma D.1. Finally, we have the condition $G_1(t) - G_0(t) = \partial_t F(0)t^2 + O(t^4)$. Let $\xi = At + O(t^3)$ and $h_1 = B + O(t^2)$. By the Taylor expansion in Lemma D.1, this last condition is equivalent to

$$\frac{3}{2}B + \frac{\sqrt{3}}{2a}A = -3B \qquad \Longrightarrow \qquad B = -\frac{\sqrt{3}}{9a}A\,,$$

as needed. Now, Equations (106e) and (106f) imply the parity of f_2 and χ. Let $\chi = Ct^3 + O(t^5)$ and $f_2 = D + O(t^2)$, then the first term of the Taylor expansion of (106e) and (106f) are, respectively,

$$-\frac{\Lambda}{4}D = 3B + \frac{\sqrt{3}}{3a}A = 0 \qquad\qquad C = \frac{\Lambda}{6}\frac{\sqrt{3}}{9a}B = -\frac{\Lambda}{6}A\,.$$

(ii) Similarly, in the case where the singular orbit is diffeomorphic to S^3, Lemma 9.10 implies that the coefficient functions G_i must be odd and F even. By the same argument as above, we conclude that ξ and h_1 must be even and have Taylor expansions of the form $\xi = At^2 + O(t^4)$ and $h_1 = Bt^2 + O(t^4)$ for $A, B \in \mathbb{R}$.

As before, Equations (106e) and (106f) imply the parity and decay of f_2 and χ. Let $\chi = Ct + O(t^3)$ and $f_2 = D + O(t^2)$. Then the first term of the Taylor expansion of (106e) and (106f) are, respectively,

$$-\frac{\Lambda}{4}D = 3\left(-\frac{1}{6b}\right)A \qquad \Longrightarrow \qquad D = \frac{2}{\Lambda b}A \qquad\qquad C = 4b^2 D = \frac{8b}{\Lambda}A\,. \qquad \square$$

We solve the ODE (110) such that these conditions are satisfied. To achieve this, we use the following technical result.

Proposition 11.6. *Consider the singular initial value problem*

$$\partial_t y(t) = \frac{1}{t} A_{-1} y(t) + A(t) y(t) \qquad y(0) = y_0 \ , \tag{112}$$

where y takes values in $\mathbb{R}^k$, A_{-1} is a $k \times k$ matrix, and $A(t)$ is a $k \times k$ matrix whose entries depend smoothly on t near 0. Then, the problem has a unique smooth solution whenever y_0 lies in the cone spanned by the eigenvectors of A_{-1} with non-negative eigenvalues. Furthermore, the solution $y(t)$ depends continuously on A_{-1}, $A(t)$ and y_0.

Proof. First, note that one could appeal to the general theory of first-order singular initial value problem (c.f. [FH17, Thm. 4.7]). However, we can solve the problem directly since our ODE is linear.
Let $\overline{A}(t) = \frac{1}{t} A_{-1} + A(t)$, so we can rewrite Equation(11.6) as $\partial_t y = \overline{A} y$. The solution to this ODE problem is simply $y(t) = \exp\left(\int_{t_0}^t \overline{A}(t)\right) y_0$. In the neighbourhood of 0, $B(t) = \displaystyle\int_{t_0}^t \overline{A}(s) ds$ can be put in Jordan canonical form $B = S^{-1} J S$, so $\exp \overline{A}(t) = S^{-1} (\exp J) S$. For simplicity, let us assume that $\overline{A}(t)$ is already diagonalised in a neighbourhood of 0; the general case follows. Then

$$y(t) = \exp\left(\int_{t_0}^t \overline{A}(t) dt\right) y_0 = \begin{pmatrix} t^{\lambda_1} & 0 & ... & 0 \\ 0 & t^{\lambda_2} & ... & 0 \\ & & \cdots & \\ 0 & 0 & 0 & t^{\lambda_n} \end{pmatrix} \exp\left(\int_{t_0}^t A(t) dt\right) y_0 \ ,$$

where λ_i are the eigenvalues of A_{-1}. Since $A(t)$ is smooth near 0, $y(t)$ will be smooth if and only if $Y(t) = \sum_i t^{\lambda_i} y_0^i$ is, where y_0^i are the coordinates of y_0 in the suitable eigenvector basis. The functions t^{λ_i} are smooth around 0 only if $\lambda_i \geq 0$. Therefore, picking the initial condition y_0 orthogonal to the negative eigenspace of A_{-1} is sufficient for $y(t)$ to be smooth. Continuous dependence on the parameters follows from standard ODE theory. □

We prove the main result of this section.

Theorem 11.7. *Let $a, b > 0$, and consider the nearly Kähler halves Ψ_a and Ψ_b of Foscolo and Haskins [FH17], with singular orbits S^2 and S^3, respectively. Then, for every $\Lambda \in (0, \infty)$, there exists a unique (up to scale) solution to the ODE system* (110) *on the nearly Kähler half Ψ_a (resp. Ψ_b). Moreover, the solution depends continuously on the parameters a (resp. b) and Λ.*

Proof. We consider the two cases separately.
Desingularisation over S^2: In view of Proposition 11.5, it is useful to consider $\overline{H} = \left(\overline{\xi}, \overline{\chi}, \overline{h_1}, \overline{f_2}\right) = \left(t^{-1}\xi, t^{-3}\chi, h_1, t^{-2} f_2\right)$. Under this reparameterisation, the ODE system (110) becomes

$$\partial_t \overline{H} = \begin{pmatrix} -\frac{1}{t} & \frac{\Lambda}{4\lambda} t^2 & -6\frac{\lambda w_1}{t} & 0 \\ -\frac{\Lambda\lambda}{3t^2} & -\frac{3}{t} & 0 & -6\frac{\mu w_2}{t} \\ -2\frac{\lambda w_1}{\mu^2} t & 0 & -6\frac{\lambda x_1}{\mu} & 0 \\ 0 & -3\frac{w_2}{\lambda^2\mu} t & 0 & -6\frac{y_2}{\lambda} - \frac{2}{t} \end{pmatrix} H \ .$$

Using the Taylor expansions in Lemma D.1, it is straightforward to check that we are under the hypotheses of Proposition 11.6, with singular term

$$A_{-1} = \begin{pmatrix} -1 & 0 & -3\sqrt{3}a & 0 \\ -\frac{\Lambda}{2} & -3 & 0 & 0 \\ -\frac{\sqrt{3}}{3a} & 0 & -3 & 0 \\ 0 & -\frac{2\sqrt{3}}{9a} & 0 & -6 \end{pmatrix} .$$

The matrix A_{-1} has three distinct negative eigenvalues: $-6, -4$ and -3, and a one-dimensional kernel, spanned by $\left(6, -\Lambda, -\frac{2\sqrt{3}}{3a}, \frac{\sqrt{3}\Lambda}{27a}\right)$.

Desingularisation over S^3: As before, we consider $\overline{H} = \left(\overline{\xi}, \overline{\chi}, \overline{h_1}, \overline{f_2}\right) = \left(t^{-2}\xi, t^{-1}\chi, t^{-2}h_1, f_2\right)$. Under this reparameterisation, the ODE system (110) becomes

$$\partial_t \overline{H} = \begin{pmatrix} -\frac{2}{t} & \frac{\Lambda}{4\lambda t} & -6\lambda w_1 & 0 \\ -\frac{\Lambda\lambda}{3}t & -\frac{1}{t} & 0 & -6\frac{\mu w_2}{t} \\ -2\frac{\lambda w_1}{\mu^2} & 0 & -6\frac{\lambda x_1}{\mu} - \frac{2}{t} & 0 \\ 0 & -3\frac{w_2}{\lambda^2\mu}t & 0 & -6\frac{y_2}{\lambda} \end{pmatrix} H .$$

Using the Taylor expansions in Lemma D.1, the singular term is

$$A_{-1} = \begin{pmatrix} -2 & \frac{\Lambda}{4b} & 0 & 0 \\ 0 & -1 & 0 & 4b^2 \\ -\frac{1}{2b} & 0 & -5 & 0 \\ 0 & \frac{1}{2b^2} & 0 & -2 \end{pmatrix} .$$

The matrix A_{-1} has three distinct negative eigenvalues: $-5, -3$ and -2, and a one dimensional kernel, spanned by $\left(\Lambda, 8b, -\frac{\Lambda}{10b}, \frac{2}{b}\right)$.

The statement follows from Proposition 11.6. □

Doubling and matching

We derive conditions under which our solutions over each nearly Kähler half can be matched along the maximum volume orbit to produce an element for the cohomogeneity one example. A pair $(\beta, \gamma) \in \Omega^2_8 \times \Omega^3_{12}$ solving the PDE (104) is given by

$$\beta = -2\lambda h_1 \eta^{se} \wedge dt + \sum_{i=0}^{2} \left(w_i \xi + u_i h_1\right) \omega_i^{se}$$

$$\gamma = \sum_{i=0}^{2} \left(w_i \chi + v_i f_2\right) \eta^{se} \wedge \omega_i^{se} - \mu f_2 dt \wedge \omega_3^{se} .$$

Recall from the discussion in Section 9.3, the cohomogeneity one complete nearly Kähler structure is constructed by matching a solution $\Psi(t)$ with another solution $\Psi^{\pm}(t)$, which is defined by a time translation and the appropriate action of the involutions (94). First, we have

Lemma 11.8. *Under the symmetries τ_1 and τ_2 we have*

$$\begin{pmatrix} w_0 & u_0 & v_0 \\ w_1 & u_1 & v_1 \\ w_2 & u_2 & v_2 \end{pmatrix} \xrightarrow{\tau_1} \begin{pmatrix} w_0 & -u_0 & v_0 \\ w_1 & -u_1 & v_1 \\ -w_2 & u_2 & -v_2 \end{pmatrix} \qquad \begin{pmatrix} w_0 & u_0 & v_0 \\ w_1 & u_1 & v_1 \\ w_2 & u_2 & v_2 \end{pmatrix} \xrightarrow{\tau_2} \begin{pmatrix} w_0 & u_0 & -v_0 \\ -w_1 & -u_1 & v_1 \\ w_2 & u_2 & -v_2 \end{pmatrix} .$$

We want to find conditions for which the pair (β,γ) can be matched along the maximum volume orbit in the doubling case. We need to understand how a solution $H(t)=(\xi,\chi,h_1,f_2)$ behaves under these symmetries.

Proposition 11.9. *Let $H(t)$ be a solution over a nearly Kähler half $\Psi(t)$ solving* (110) *for $\Lambda\in\mathbb{R}$. Then*

(i) The tuple $H^+(t)=(-\xi,\chi,h_1,f_2)$ is a solution to (110) *over the nearly Kähler half $\Psi^+(t)$.*

(ii) The tuple $H^-(t)=(\xi,-\chi,h_1,f_2)$ is a solution to (110) *over the nearly Kähler half $\Psi^-(t)$.*

Proof. Straightforward computation. □

We are now ready to match two solutions in the case of doubling a nearly Kähler half. Let $\beta^+(t)$ (resp. $\gamma^+(t)$) be the image of β (resp. γ) under the symmetry τ_1. Along the maximal volume orbit, i.e., at $t=T_*$, the functions w_2, u_0, u_1 and v_2 vanish, and so we have

$$\begin{aligned}
\beta(T_*) &= -2\lambda h_1\eta^{se}\wedge dt+(\xi w_0)\omega_0^{se}+(\xi w_1)\omega_1^{se}+(h_1u_2)\omega_2^{se}\ ,\\
\beta^+(T_*) &= -2\lambda h_1\eta^{se}\wedge dt-(\xi w_0)\omega_0^{se}-(\xi w_1)\omega_1^{se}+(h_1u_2)\omega_2^{se}\ ,\\
\gamma(T_*) &= (w_0\chi-v_0f_2)\eta^{se}\wedge\omega_0^{se}+(w_1\chi-v_1f_2)\eta^{se}\wedge\omega_1^{se}-\mu f_2dt\wedge\omega_3^{se}\ ,\\
\gamma^+(T_*) &= (w_0\chi-v_0f_2)\eta^{se}\wedge\omega_0^{se}+(w_1\chi-v_1f_2)\eta^{se}\wedge\omega_1^{se}-\mu f_2dt\wedge\omega_3^{se}\ .
\end{aligned}$$

It follows that the equation $(\beta,\gamma)=\alpha(\beta^+,\gamma^+)$ has two non-trivial solutions:

$$\alpha=1\implies\xi(T_*)=0\qquad\qquad\alpha=-1\implies\chi(T_*)=h_1(T_*)=f_2(T_*)=0\ .$$

Similarly, let $\beta^-(t)$ and $\gamma^-(t)$ be the images of β and γ under the involution τ_2. Along the maximal volume orbit, the functions w_1, u_1, v_0 and v_2 vanish, and so we have

$$\begin{aligned}
\beta(T_*) &= -2\lambda h_1\eta^{se}\wedge dt+(\xi w_0+u_0h_1)\omega_0^{se}+(\xi w_2+u_2h_1)\omega_1^{se}\ ,\\
\beta^-(T_*) &= -2\lambda h_1\eta^{se}\wedge dt+(\xi w_0+u_0h_1)\omega_0^{se}+(\xi w_2+u_2h_1)\omega_1^{se}\ ,\\
\gamma(T_*) &= (w_0\chi)\eta^{se}\wedge\omega_0^{se}+(v_1f_2)\eta^{se}\wedge\omega_1^{se}+(w_2\chi)\eta^{se}\wedge\omega_2^{se}-\mu f_2dt\wedge\omega_3^{se}\ ,\\
\gamma^-(T_*) &= -(w_0\chi)\eta^{se}\wedge\omega_0^{se}+(v_1f_2)\eta^{se}\wedge\omega_1^{se}+-(w_2\chi)\eta^{se}\wedge\omega_2^{se}-\mu f_2dt\wedge\omega_3^{se}\ .
\end{aligned}$$

As before, the equation $(\beta,\gamma)=\alpha(\beta^-,\gamma^-)$ has two non-trivial solutions:

$$\alpha=1\implies\chi(T_*)=0\qquad\qquad\alpha=-1\implies\xi(T_*)=h_1(T_*)=f_2(T_*)=0\ .$$

Thus, we have proved

Proposition 11.10. *Let Ψ_{b^*} be the nearly Kähler half corresponding to the inhomogeneous nearly Kähler structure in $S^3\times S^3$ of Foscolo and Haskins. A solution to* (110) *extends to the whole $S^3\times S^3$ if and only if $\chi(T_*)=0$ or $\xi(T_*)=h_1(T_*)=f_2(T_*)=0$.*

We can simplify the matching conditions by using the conserved quantities (106e) and (106f).

Lemma 11.11. *Assume that the nearly Kähler half doubles under τ_2, so $w_1(T_*)=0$. Then*

- *If $h_1(T_*) = 0$, we have $\xi(T_*) = f_2(T_*) = 0$.*
- *If $\Lambda \neq 0$ and $f_2(T_*) = 0$, we have $\xi(T_*) = h_1(T_*) = 0$.*
- *If $\Lambda \neq \sqrt{72}$ and $\xi(T_*) = 0$, we have $h_1(T_*) = f_2(T_*) = 0$.*

A similar statement holds for the involution τ_1.

Proof. We only show the details for the case $w_1(T_*) = 0$. The constraints (106e)-(106f) at $t = T_*$ reduce to

$$3\frac{w_2}{\lambda\mu}\xi + 3h_1 + \frac{\Lambda}{4}f_2 = 0\ , \tag{113a}$$

$$-2f_2 - \frac{\Lambda}{3}h_1 = 0\ . \tag{113b}$$

If $h_1(T_*) = 0$, the claim follows directly, since $w_2(T_*) \neq 0$, as otherwise we would be on the sine-cone by Corollary 9.5. If $\Lambda \neq 0$ and $f_2(T_*) = 0$, the second equation implies $h_1(T_*) = 0$, and so $\xi(T_*) = 0$. Finally, the determinant of the matrix

$$\begin{pmatrix} 3 & \frac{\Lambda}{4} \\ \frac{\Lambda}{3} & 2 \end{pmatrix}$$

is nonzero whenever $\Lambda \neq \sqrt{72}$, and the final claim follows. □

Similarly, one can investigate the matching conditions when the two halves are not isometric, as is the case for the inhomogeneous S^6 case. However, in this case, the matching conditions will depend on the values of $\Psi(T_*)$ on the maximum volume orbits, which are not explicit in the case of the inhomogeneous nearly Kähler structure on S^6. We do not investigate this case further here.

12 The index of the inhomogeneous nearly Kähler $S^3 \times S^3$

We study the Hitchin index of the inhomogeneous nearly Kähler structure on $S^3 \times S^3$ constructed in [FH17] and discussed in Section 9.3. We will prove that in the eigenvalue range $\Lambda \in (0, \sqrt{72})$, there exists a complete solution to Equation (110). In particular, the nearly Kähler Hitchin index is at least one.

We prove the claim by performing an analysis of the zeros of the functions $\xi(t, \Lambda), \chi(t, \Lambda), h_1(t, \Lambda)$ and $f_2(t, \Lambda)$, in the same spirit to the one used in Proposition 9.16 and using the intermediate value theorem. First, we exploit the dependence of the conserved quantities on Λ when restricted to the maximum volume orbit. We have

Lemma 12.1. *Let Ψ_{b_*} be the nearly Kähler half corresponding to the inhomogeneous nearly Kähler structure, with maximum volume orbit time T_*. Then,*

(i) the functions $h_1(T_*, \Lambda)$ and $f_2(T_*, \Lambda)$ always have opposite signs;

(ii) the functions $\xi(T_*, \Lambda)$ and $f_2(T_*, \Lambda)$ have the same sign if $\Lambda < \sqrt{72}$ and have opposite signs for $\Lambda > \sqrt{72}$.

Proof. Since $w_1(T_*) = 0$, the conserved quantities on the maximum volume orbit evaluate to

$$\frac{w_2}{\lambda\mu}\xi = -h_1 - \frac{\Lambda}{12} f_2 = \left(\frac{\Lambda^2}{72} - 1\right) h_1 \ , \tag{114a}$$

$$f_2 = -\frac{\Lambda}{6} h_1 \ . \tag{114b}$$

Thus, (i) follows. By Proposition 9.16 (ii), $w_2(T_*) > 0$, and the second claim follows. □

We prove a crucial result that will allow us to prove the main theorem of this section. The key idea is that $\Lambda = \sqrt{72}$ is a degenerate value for the conserved quantities that acts as a barrier.

Proposition 12.2. *Let $H(t, \Lambda) = (\xi, \chi, h_1, f_2)$ be a non-trivial solution to* (110) *over Ψ_{b_*}.*

(i) At $\Lambda = \sqrt{72}$, we have $\xi(T_*, \sqrt{72}) = 0$ and $f_2(T_*, \sqrt{72}) \neq 0$.

(ii) The zero of $\xi(T_*, \Lambda)$ at $\Lambda = \sqrt{72}$ is transverse. In particular $\chi(T_*, \sqrt{72}) \neq 0$.

(iii) The function $\xi(T_*, \Lambda)$ has no zeros for $\Lambda \in (0, \sqrt{72})$.

Proof. On the maximum volume orbit, the system (110) evaluates to

$$\partial_t \xi\Big|_{T_*} = \frac{\Lambda}{4\lambda}\chi \ , \tag{115a}$$

$$\partial_t \chi\Big|_{T_*} = -\frac{\Lambda\lambda}{3}\xi - 6\mu w_2 f_2 \ , \tag{115b}$$

$$\partial_t h_1\Big|_{T_*} = 0 \ , \tag{115c}$$

$$\partial_t f_2\Big|_{T_*} = -3\frac{w_2}{\lambda^2\mu}\chi \ , \tag{115d}$$

where all the functions on the right-hand side are evaluated at the maximum volume orbit time.

(i) By Equation (114a) we have $\xi(T_*, \sqrt{72}) = 0$. If $f_2(T_*, \sqrt{72}) = 0$, one of the two zeros would have to be degenerate by Lemma 12.1. If ξ (resp. f_2) had a non-transverse zero, Equation (115a) (resp. Equation (115d)) would imply $\chi(T_*, \sqrt{72}) = 0$, so $H(T_*, \sqrt{72}) = 0$. Since T_* is a smooth point of a linear first-order ODE for H, the uniqueness of solutions would force the solution to be trivial, leading to a contradiction.

(ii) By the previous item, $\xi(T_*, \Lambda)$ changes sign at $\sqrt{72}$. Thus, $\partial_t \xi\Big|_{T_*} = 0$ would force $\partial_t^2 \xi\Big|_{T_*} = 0$, By Equation (115a) and (115b), we have

$$\partial_t^2 \xi\Big|_{T_*} = \frac{\Lambda}{4\lambda}\partial_t \chi = -\frac{3\Lambda}{2\lambda}\mu w_2 f_2 \neq 0 \ .$$

(iii) Assume we had $\overline{\Lambda} \in (0, \sqrt{72})$ such that $\xi(T_*, \overline{\Lambda}) = 0$. Then, the conserved quantities would also force $f_2(T_*, \overline{\Lambda}) = h_1(T_*, \overline{\Lambda}) = 0$. By Lemma 12.1, $\xi(T_*, \Lambda)$ and $f_2(T_*, \Lambda)$ have the same sign for $\Lambda \in (0, \sqrt{72})$, the slopes at their zeros must have the same sign. Since $w_2(T_*) > 0$ by Proposition 9.16 (ii), Equations (115a) and (115d) force $\chi(T_*, \overline{\Lambda}) = 0$. Therefore $H(T_*, \sqrt{72}) = 0$, which is a contradiction as above. □

Let us study the behaviour of these functions for small Λ. First, we have

Lemma 12.3. *The function $\chi(t,\Lambda)$ is strictly positive for $t \in (0, T_*]$ and Λ small.*

Proof. For $\Lambda = 0$, corresponding to the harmonic case, the ODE system totally decouples. In particular, $\chi(t,0)$ is a solution to the singular ODE:

$$\partial_t \chi = -6\frac{w_1 w_2}{\mu}\chi \; . \tag{116}$$

The asymptotics in Proposition 11.5 imply that $\chi(t,0) > 0$ for any small time. Since Equation (116) is linear, $\chi(t,0) > 0$ for all time. In particular, since $\chi(t,\Lambda)$ is continuous on Λ, we have $\chi(t,\Lambda) > 0$ for Λ small. □

We can use this result to refine the last statement of Proposition 12.2:

Proposition 12.4. *The function $\xi(T_*,\Lambda)$ is strictly positive for $\Lambda \in (0,\sqrt{72})$.*

Proof. By virtue of Proposition 12.2 (*iii*), it suffices to prove this for Λ small enough. We argue by contradiction. Assume that $\xi(T_*,\Lambda) < 0$ for some small Λ. Lemma 12.1 implies $h_1(T_*,\Lambda) > 0$. By the smoothness conditions in Theorem 11.7, $h_1(t,\Lambda) < 0$ for t small enough. In particular, there exists $T < T_*$ for which $h_1(T,\Lambda) = 0$. Assume T is the smallest time for which this happens. Thus, h_1 has a non-negative slope at T. By Equation (106b), we have

$$\partial_t h_1\Big|_T = -2\frac{\lambda w_1}{\mu^2}\xi \geq 0 \; ,$$

By Proposition 9.16 (*i*), $w_1(t) > 0$ for $t \in (0,T_*)$, so $\xi(T) \leq 0$. Again, by the smoothness conditions, $\xi(t,\Lambda) > 0$ for t small and so, there exists $\overline{T} \leq T$ such that $\xi(\overline{T},\Lambda) = 0$ and $\partial_t \xi \mid_{\overline{T}} \leq 0$. But Equation (106a) would imply

$$\partial_t \xi\Big|_{\overline{T}} = \frac{\Lambda}{4\lambda}\chi - 6\lambda w_1 h_1 > 0 \; ,$$

since $\chi(t,\Lambda) > 0$ by the above lemma and $h_1 \leq 0$ since $\overline{T} \leq T$, which is a contradiction, and so we must have $\xi(T_*,\Lambda) > 0$ for Λ small. □

We have the tools to prove the existence of a complete solution for $\Lambda \in (0,\sqrt{72})$.

Proposition 12.5. *On the inhomogeneous nearly Kähler structure in $S^3 \times S^3$ of Foscolo and Haskins, there exists a cohomogeneity one solution to Equation* (104) *for $\Lambda \in (0,\sqrt{72})$.*

Proof. Since $\xi(T_*,\Lambda)$ is strictly positive for $\Lambda \in (0,\sqrt{72})$, the transverse zero at $\Lambda = \sqrt{72}$ from Proposition 12.2 (*i*) must have strictly negative slope, so $\chi(T_*,\sqrt{72}) < 0$ by Equation (115a). By Lemma 12.3, $\chi(T_*,0) > 0$, and by continuity on Λ, there exists $\Lambda_* \in (0,\sqrt{72})$ such that $\chi(T_*,\Lambda_*) = 0$.

The doubling conditions in Proposition 11.10 imply that $H(t,\Lambda_*)$ doubles to a solution of the system (106) on the whole manifold. □

Our main theorem is now straightforward.

Theorem 12.6. *The Hitchin index of the inhomogeneous nearly Kähler structure on $S^3 \times S^3$ is bounded below by 1. The Einstein co-index is bounded below by 4.*

Proof. Due to the relation between the PDEs (103) and (104), the proposition above implies that there exists a cohomogeneity one 2-form $\beta \in \Omega^2_{8,coclosed}$ solving $\Delta\beta = \nu\beta$ for $\nu \in (0, 6)$. The claim for the bound on the Hitchin index follows.

The Einstein co-index bound follows from the computation in Prop. C.3:

$$\mathrm{Ind}^{EH} = b^2(M) + b^3(M) + 3 \sum_{\nu \in (0,2)} \dim \mathcal{E}(\nu) + 2 \sum_{\nu \in (2,6)} \dim \mathcal{E}(\nu) + \sum_{\nu \in (6,12)} \dim \mathcal{E}(\nu) \ , \quad (117)$$

where $\mathcal{E}(\nu)$ are the corresponding eigenspaces; $\mathcal{E}(\nu) = \{\beta \in \Omega^2_8 \mid d^*\beta = 0 \ , \ \Delta\beta = \nu\beta\}$. □

Appendix

A G-structures

We collect some well-known results and identities for G-structures. On a smooth manifold M, a G-structure corresponds to a reduction of the frame bundle of M to a principal G-bundle. We restrict to the cases where $G \subseteq \mathrm{SO}(n)$, so a choice of G-structure always includes a choice of metric. With it, we canonically identify the space of 1-forms with smooth vector fields. Similarly, we consider the contraction on a k-form γ by an l-form α as

$$\alpha \lrcorner \gamma = *(*\gamma \wedge \alpha) \ , \tag{118}$$

which extends the usual vector field contraction.

We present two types of results. First, we have a collection of results in representation theory. The reduction of the frame bundle to a principal G-bundle means TM and all the associated vector bundles via natural constructions inherit an induced G-action so that they can be decomposed into irreducible G-representations. We will be interested in the bundles $\Lambda^* T^*M = \bigoplus_k \Lambda^k T^*M$ and $\mathrm{Sym}^2(T^*M)$.

Recall that we have an induced action of $\mathrm{End}(TM)$ on the space of forms given by

$$S_*(\Omega)(X_1, \dots, X_k) = -\sum_{i=1}^{k} \Omega(X_1, \dots, S(X_i), \dots X_k) \ . \tag{119}$$

for $\Omega \in \Lambda^k T^*M$. Under the metric, we identify $\mathrm{Sym}^2(T^*M) \subseteq T^*M \otimes T^*M \cong \mathrm{End}(TM)$, so we get an induced action of $\mathrm{Sym}^2(T^*M)$ on $\Lambda^* T^*M$. In particular, if Ω is an G-invariant k-form, it induces a map between G-representations of $\Lambda^k T^*M$ and $\mathrm{Sym}^2(T^*M)$.

The decompositions above will carry over to the spaces of smooth sections of each of the bundles. We denote an irreducible G-representation of dimension m in $\Lambda^k T^*M$ as $\Lambda^k_m T^*M$, and analogously $\Omega^k_m := \Gamma(\Lambda^k_m)$. In all cases of interest, any two representations of the same dimension are isomorphic as representations, so there are no ambiguities arising from our choice of notation.

The second class of results concerns differential identities on G-structures with reduced torsion. In particular, we will be interested in G-structures that correspond to (Ricci-flat) cones.

Recall that, on the frame bundle of M^n, $\pi : P \to M$, we have a canonical 1-form $\theta \in \Omega^1(P, TM)$, given by the differential of the projection map π. Thus, given h a connection on a reduced frame bundle P, with structure group $G \subseteq \mathrm{SO}(n)$, we have a natural 2-form $\widehat{\Theta} = d_h \theta = d\theta \circ h$.

Since π is G-invariant, we have θ is G-equivariant, and we can identify $\widehat{\Theta_h}$ with a 2-form on the base $\Theta_h \in \Omega^2(M, TM)$, the torsion of the connection h. The space of connections on P is an affine space modelled on $\Omega^1(M, \mathfrak{g}_P)$, so if we were to pick a different connection h', the difference $\Theta_{h'} - \Theta_h$ would be a section of $\Lambda^1(M) \otimes \mathfrak{g}$. Thus, the class

$$\tau = [\Theta] \in \frac{\Omega^1(M) \otimes \Omega^2(M)}{\Omega^1 \otimes \Gamma(\mathfrak{g})} \cong \Omega^1 \otimes \Gamma(\mathfrak{g}^\perp) \tag{120}$$

is independent of the choice of connection. The section τ is known as the intrinsic torsion of P, and it precisely captures the obstruction to the existence of a torsion-free connection on P, measuring the failure of the Levi-Civita connection to have holonomy contained in G (cf. [Bry87]).

Equivalently, the intrinsic torsion can be used to measure the failure of the bundles Λ^k_m of being parallel for the Levi-Civita connection. For example, on 2-forms, one has the map

$$\begin{aligned} \alpha : \Gamma(TM) \otimes \Gamma(\mathfrak{g}) &\to \Gamma(\mathfrak{g}^\perp) \\ (X, \beta) &\mapsto \mathrm{proj}_{\mathfrak{g}^\perp}(\nabla_X \beta) . \end{aligned} \tag{121}$$

An easy computation shows

Lemma A.1 ([Rey98, Lemma 6]). *For $X \in \Gamma(TM)$, we have*

$$\alpha_X(\beta) = \mathrm{proj}_{\mathfrak{g}^\perp}([\tau_X, \beta]) \ . \tag{122}$$

Finally, we collect some useful Riemannian geometry identities. Since the Levi-Civita connection is a metric connection, we have

$$2\nabla X = \mathcal{L}_X g + dX \ , \tag{123}$$

where we are interpreting X either as a 1-form or a vector field, as needed. Because the Levi-Civita connection is torsion-free, for any tensor S, we have

$$\mathcal{L}_X S - \nabla_X S = (\nabla X)_* S \tag{124}$$

If S is a G-invariant tensor, then, by the definition of the torsion, we have $\nabla_X S = (\tau_X)_* S$.

Lemma A.2. *Consider a G-structure characterised by a family of G-invariant tensors $S_1, \dots, S_n$. A vector field X is an infinitesimal automorphism of the G-structure if and only if X is Killing and*

$$(2\tau_X + dX)_* S_i = 0 \tag{125}$$

for every G-invariant tensor S_i.

Proof. Since the G-structure is characterised by the $S_1, \dots, S_n$, we must have $\mathcal{L}_X S_i = 0$ for all $i \in 1, \dots n$. The claim follows from combining Equations (123) and (124) and noticing that $2\tau_X + dX \in \Omega^2$ whilst $\mathcal{L}_X g \in \Gamma\left(\mathrm{Sym}^2\right)$. □

Special holonomy metrics

Given a G-structure on M with metric g, we say that g is special holonomy whenever $\tau = 0$. In this case, the subbundles Λ^*_p are parallel: for $\beta \in \Omega_p$, we have that $\nabla\beta \in \Gamma(T^*M \otimes \Lambda_p)$.

Composing with the alternating map, we get that d splits as a sum of first-order differential operators $d^p_q : \Omega_p \to \Omega_q$, that coincide with their symbol. These will satisfy some second-order relations induced by the condition $d^2 = 0$.

If g has non-vanishing torsion, the operators d^p_q are modified by zero-order terms that depend on the torsion. We will compute the exact form of d^p_q for nearly Kähler and nearly parallel G_2-structures in Sections A.2 and A.3.

On a manifold with special holonomy, the following result, originally due to Chern [Che57] (cf. [Sam73]), holds.

Theorem A.3. *Let (M^n, g) be a closed manifold equipped with a special holonomy metric. The space of harmonic forms is compatible with the induced G-representations described above. That is, every harmonic form splits as a sum of harmonic forms, each of which belongs to an irreducible G-representation Ω_p. Moreover, if $\Lambda^k_p \cong \Lambda^l_p$, then we have an induced isomorphism at the level of harmonic forms, $\mathcal{H}^k_p \cong \mathcal{H}^l_p$.*

Proof.(Sketch). The classic Weitzenböck formula for k-forms is

$$\Delta = \nabla^*\nabla + \widetilde{R}\,, \tag{126}$$

where $\widetilde{R}$ is an endomorphism associated to the Riemann curvature tensor. The proof follows by checking that the linear map $\widetilde{R}$ is a morphism of G-representations. □

We have the following result, which we attribute to Bonan [Bon66]:

Theorem A.4. *Let (M, g) be Riemannian manifold with holonomy contained in either* $\mathrm{SU}(n), G_2$ *or* $\mathrm{Spin}(7)$*. Then its Ricci curvature vanishes.*

Remark A.5. *Alternatively, one has that any manifold carrying a parallel spinor must be Ricci-flat. By the work of McKenzie Wang [Wan89], these are precisely the cases considered above.*

A useful extension of this is the following result:

Proposition A.6. *Let (M^n, g) be a manifold carrying a G-structure for $G =$* $\mathrm{SU}(n), G_2$ *or* $\mathrm{Spin}(7)$*. Then the Ricci curvature of g is fully determined by its intrinsic torsion.*

The cases of G_2 and SU(3)-structures were worked out explicitly by Bryant [Bry05] and Bedulli and Vezzoni [BV07], respectively.

One may study infinitesimal deformations of special holonomy metrics by using Lemma A.2, where we recover

Proposition A.7. *Let (M^n, g) be a closed Ricci-flat special holonomy manifold. Then*

$$\mathfrak{aut}(M, G) \cong \mathfrak{isom}(M, g) \cong \{X \in \Omega^1(M) |\ \nabla X = 0\} \cong \mathcal{H}^1\ ,$$

where $\mathfrak{aut}(M, G)$ *is the infinitesimal automorphsims of the special holonomy metrics.*

Proof. The isomorphisms $\mathfrak{isom}(M,g) \cong \{X \in \Omega^1(M) | \nabla X = 0\} \cong \mathcal{H}^1$ follows from the well-known Weitzenböck identity

$$\Delta X = \nabla^* \nabla X + \mathrm{Ric}(X) = 2 \operatorname{div} \operatorname{div}^*(X) + 2 \mathrm{Ric}(X) - dd^* X \ . \tag{127}$$

It is clear that $\mathfrak{aut}(M,G) \subseteq \mathfrak{isom}(M,g)$ since $G \subseteq \mathrm{SO}(n)$ by assumption, and equality follows from Lemma A.2 since $\tau = 0$. □

In particular, closed irreducible Ricci-flat manifolds must be infinitesimally rigid by applying the Cheeger-Gromoll splitting principle to the universal cover.

Links of special holonomy cones

Consider the case where we have a G-structure on (Σ^{n-1}, g), whose metric cone $(C(\Sigma), g_C) = (\mathbb{R}_{\geq 0} \times \Sigma, dr^2 + r^2 g_\Sigma)$ is a special holonomy manifold.
We discuss the analogous properties to the ones described above. Theorem A.3 in this case becomes:

Proposition A.8. *Let (Σ^{n-1}, g) be a closed manifold whose cone is a special holonomy manifold. The space of harmonic forms is compatible with the induced G-representations.*

The proof follows from a case-by-case analysis for nearly parallel G_2 (Prop.A.35) , nearly Kähler (Prop. A.59) and Sasaki ([BG08, Thm. 7.2.6 & Prop. 7.4.14] cases. Although we expect a general proof to exist, we have not yet been able to find one.

Remark A.9. *In this case, the bundles themselves are not parallel, so it is not true that, if $\Lambda^k_p \cong \Lambda^l_p$, we have an induced isomorphism $\mathcal{H}^k_p \cong \mathcal{H}^l_p$.*

Similarly, we can study the Lie algebra $\mathfrak{aut}(M,G)$ of infinitesimal deformations of the G-structure. We outline a general result for these infinitesimal deformations, which is proved in detail on a case-by-case basis in later sections. Recall that if (Σ^{n-1}, g) is the link of a special holonomy cone from Berger's list (cf. Table 2), the cone admits an associated invariant 4-form Ω, which induces an invariant 3-form $*\Xi$ on Σ. The existence of this invariant 3-form Ξ defines a map

$$\begin{aligned} L_\Xi : \Lambda^1 &\to \Lambda^2 \\ X &\mapsto X \lrcorner * \Xi = *(\Xi \wedge X). \end{aligned}$$

In all cases listed in Table 2, the torsion tensor can be identified with $*\Xi$; that is,

$$\tau_X = C\, X \lrcorner * \Xi = C * (\Xi \wedge X),$$

for some universal constant $C \in \mathbb{R}$. As before, we expect a general proof of this fact to exist, although we have not found one.
Let $X \in \Omega^1$, the dual of a Killing field. From Lemma A.2, we will have $X \in \mathfrak{aut}(M,G)$ if and only if $[2C * (\Xi \wedge X) + dX]_* S = 0$, for all invariant tensors S. Acting by the dual map to L_Ξ, $\beta \mapsto *(\beta \wedge \Xi)$, it follows that

Proposition A.10. *Let (Σ^{n-1}, g) be the link of a special holonomy cone. The 1-form X will correspond to an infinitesimal automorphism of the induced G-structure only if it is an eigenform of the curl operator* $\mathrm{curl}(X) = (-1)^n * (dX \wedge \Xi)$ *introduced in* (1).

In the case where the cone is Ricci-flat, the condition above becomes an if and only if condition; and the 1-form X must be of eigenvalue (-2), i.e.

Proposition A.11 (Prop. A.29, Lemma A.53 & Prop. A.68)**.** *Let (Σ^n, g) be the link of a special holonomy Ricci-flat cone. Then*

$$\mathfrak{aut}(\Sigma, G) \cong \{X \in \Omega^1 | \, \mathrm{curl}(X) = -2X\} \; . \tag{128}$$

We move on to analyse each particular case of interest in detail.

A.1 Spin(7)-structures

We recall some well-known results on Spin(7)-structures. All results are classic and have been collected for convenience. We refer the interested reader to the detailed notes of Salamon and Walpuski [SW17] for further details.

Definition A.12. *A* Spin(7)*-structure on a manifold M^8 is a reduction of its frame bundle to a* Spin(7)*-principal bundle. A manifold equipped with a choice of frame reduction is called a* Spin(7)*-manifold* [4].

Since $\mathrm{Spin}(7) \subseteq \mathrm{Spin}(8)$ is the stabiliser of any nonzero vector $v \in \mathbb{R}^8 \cong \mathbb{O}$, a Spin(7)-structure is equivalent to the choice of a spin structure together with a nowhere vanishing spinor.

We have the following decomposition into Spin(7)-representations:

Lemma A.13. *Let (M^8, g, Φ) be a* Spin(7)*-manifold. The spaces Λ^0 and Λ^1 are irreducible with respect to the induced* Spin(7)*-action. The spaces Λ^2, Λ^3 and Λ^4 decompose orthogonally as*

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21} \qquad \Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48} \; ,$$
$$\Lambda^4 = \Lambda^4_+ \oplus \Lambda^4_- \; , \qquad \Lambda^4_+ = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \; , \qquad \Lambda^4_- = \Lambda^4_{35} \; .$$

They are described by

$$\begin{aligned}
\Lambda^2_7 =& \{\beta \in \Lambda^2 | \, \star(\Phi \wedge \beta) = 3\beta\} \; , \\
\Lambda^2_{21} =& \{\beta \in \Lambda^2 | \, \star(\Phi \wedge \beta) = -\beta\} \cong \mathfrak{spin}(7) \; , \\
\Lambda^3_8 =& \{X \lrcorner \Phi | \, X \in \Gamma(TM)\} \; , \\
\Lambda^3_{48} =& \{\gamma \in \Lambda^3 | \, \gamma \wedge \Phi = 0\} \; , \\
\Lambda^4_1 =& \langle \Phi \rangle \; , \\
\Lambda^4_7 =& \{\xi_*(\Phi) | \, \xi \in \mathfrak{spin}(7)^\perp \subseteq \mathfrak{so}(8)\} \; , \\
\Lambda^4_{35} =& \{S_*(\Phi) | \, S \in \mathrm{Sym}^2_0(T^*M)\} \; ,
\end{aligned}$$

where $\mathrm{Sym}^2_0(T^*M)$ are the traceless symmetric endomorphisms. The remaining terms follow from the relation $\Lambda^k = *\Lambda^{8-k}$.

[4] The term Spin(7)-manifold is sometimes reserved in the literature to manifolds carrying a metric with holonomy contained in Spin(7).

Now, we have the following identification of the intrinsic torsion.

Proposition A.14 ([Fer86]). *Let (M^8, g, Φ) be a* Spin(7)*-structure. The intrinsic torsion τ is a section of $\Omega^2_7 \otimes \Omega^1 \cong \Omega^3 \cong \Omega^3_8 \oplus \Omega^3_{48}$. In particular, there exists forms $\tau_1 \in \Omega^1$ and $\tau_3 \in \Omega^3_{48}$ that fully characterize the intrinsic torsion τ of the* Spin(7)*-structure. They satisfy*

$$d\Phi = \tau_1 \wedge \Phi + *\tau_3 \,. \tag{129}$$

Proof. The isomorphism $\Omega^2_7 \otimes \Omega^1 \cong \Omega^3$ is a representation theory computation. Now, Equation (129) follows from the metric compatibility of the Levi-Civita, $d = \mathrm{Alt} \circ \nabla$. □

A.2 G_2-structures

We recall some well-known results on G_2-structures and nearly G_2-manifolds. Most of these results are classic and have been collected for convenience. The main new result is the discussion about the Dirac operator and Hodge decomposition at the end of the section (cf. [DS23]).

Definition A.15. *A G_2-structure on a manifold M^7 is a reduction of its frame bundle to a G_2-principal bundle. A manifold equipped with a choice of frame reduction is a G_2-manifold* [5].

Equivalently, M^7 is equipped with a smooth stable differential form $\varphi \in \Omega^3(M)$. Similarly, we could have chosen a stable 4-form ψ, where φ and ψ will be Hitchin duals to each other (cf. Equation (63)).

Moreover, since $G_2 \subset \mathrm{Spin}(7)$ is the stabiliser of any nonzero vector $v \in \mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$, a G_2-structure is equivalent to the choice of a spin structure together with a nowhere vanishing spinor.

Since $G_2 \subseteq \mathrm{SO}(8)$, the G_2-structure fully characterises the metric on M: for $X, Y \in \Gamma(M)$, we define the associated metric $g_\varphi : \mathrm{Sym}^2(TM) \to \mathbb{R}$ as

$$g_\varphi(X, Y)\,\mathrm{vol}_\varphi = \frac{1}{6}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi \,.$$

It is worth noting that, in some cases, the opposite orientation convention is chosen. We follow the same convention as Bryant [Bry05], Joyce [Joy00], Salamon and Walpuski [SW17] and Dwivedi and Singhal [DS23]. Bryant-Salamon [BS89], Harvey-Lawson [RL82] and Karigiannis and Lotay[KL20] follow the opposite convention.

Lemma A.16. *Let (M^7, φ) be a G_2 manifold. The spaces $\Lambda^0 T^*M \cong \mathbb{R}$ and $\Lambda^1 T^*M \cong \mathbb{R}^7$ are irreducible with respect to the induced G_2 action. The spaces Λ^2 and Λ^3 decompose orthogonally as*

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \qquad\qquad \Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} \,.$$

[5] The term G_2-manifold is sometimes reserved for manifolds carrying a metric with holonomy contained in G_2.

They are described by

$$\begin{aligned}
\Lambda^2_7 =& \{X \lrcorner \varphi |\ X \in \Gamma(TM)\} = \{\beta \in \Lambda^2 |\ \star(\varphi \wedge \beta) = 2\beta\} \ , \\
\Lambda^2_{14} =& \{\beta \in \Lambda^2 |\ \beta \wedge \psi = 0\} = \{\beta \in \Lambda^2 |\ \star(\varphi \wedge \beta) = -\beta\} \cong \mathfrak{g}_2 \ , \\
\Lambda^3_1 =& \Lambda^0 \cong \langle \varphi \rangle \ , \\
\Lambda^3_7 =& \{X \lrcorner \psi |\ X \in \Gamma(TM)\} \ , \\
\Lambda^3_{27} =& \{\gamma \in \Lambda^3 |\ \gamma \wedge \psi = 0, \gamma \wedge \varphi = 0\} \cong \{S_*(\varphi) |\ S \in \mathrm{Sym}^2_0(T^*M)\} \ .
\end{aligned}$$

The decomposition for Λ^k for $k > 3$ follows from $\Lambda^k = \star\Lambda^{7-k}$.

Using the metric we identify Λ^2_7 and Λ^3_7 with Λ^1 using the maps $X \mapsto X \lrcorner \varphi$ and $X \mapsto X \lrcorner \psi$. By Schur's Lemma, these maps are homotheties. The following lemma gives a precise characterisation:

Lemma A.17. *Let X, Y be 1-forms, then the following holds:*

(i) $*(\varphi \wedge *(\varphi \wedge X)) = -4X$,

(ii) $\psi \wedge *(\varphi \wedge X) = 0$,

(iii) $*(\psi \wedge *(\psi \wedge X)) = 3X$,

(iv) $\varphi \wedge *(\psi \wedge X) = 2\psi \wedge X$.

These statements are pointwise in nature, so it suffices to verify them in local coordinates. The corresponding computations are straightforward, involving only linear algebra, and are therefore omitted. Similarly, we have

Lemma A.18. *Let β be 2-form, then:*

(i) $*(\varphi \wedge *(\varphi \wedge \beta) = 2\beta + *(\varphi \wedge \beta)$;

(ii) $*(\psi \wedge *(\psi \wedge \beta)) = \beta + *(\varphi \wedge \beta)$.

In this case, the intrinsic torsion satisfies the following

Proposition A.19. *Let (M^7, φ) be a G_2-structure. The intrinsic torsion τ is a section of $\Omega^2_7 \otimes \Omega^1 \cong \Omega^2 \oplus \Omega^3$. In fact, the Ω^2_7 and Ω^3_7 terms coincide.*
Explicitly, there exists forms $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega^2_{14}$ and $\tau_3 \in \Omega^3_{27}$ that fully characterise the intrinsic torsion of the G_2-structure. They satisfy

$$\begin{aligned}
d\varphi &= \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3 \\
d\psi &= 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi
\end{aligned}$$

We characterise the torsion of a G_2-structure on the link of a Spin(7)- holonomy cone.

Proposition A.20. *Let $C(\Sigma)$ be a metric cone whose holonomy is contained in* Spin(7)*. Then the G_2-structure on the link Σ has vanishing torsion except for $\tau_0 = 4$.*

Proof. From Proposition A.14, the condition for $C(\Sigma)$ to have holonomy in Spin(7) can be rewritten as $d\Phi = 0$, where $\Phi = r^3 dr \wedge \varphi + r^4\psi$ is the characteristic 4-form of the Spin(7)-structure. Thus,

$$0 = d\Phi = r^3 dr \wedge (4\psi - d\varphi) + r^4 d\psi \, . \qquad \square$$

Definition A.21. *A G_2-structure with vanishing torsion except for $\tau_0 = 4$ is called a nearly parallel G_2-structure.*

From Proposition A.6, one can compute the following expression for the scalar curvature.

Lemma A.22 ([Bry05, Eq. (4.28)]). *Let (M, φ) be a G_2-structure. The scalar curvature of the associated metric is given by*

$$s_g = 42\tau_0^2 + 12d^*\tau_1 + 30|\tau_1| - \frac{1}{2}|\tau_2|^2 - \frac{1}{2}|\tau_3|^2 \, .$$

Finally, we have an explicit formula for the linearisation of Hitchin's duality map (cf. (60)) in terms of irreducible representations.

Proposition A.23 ([Hit00, Lemma 20], [Bry05, Sect. 6]). *Given $\psi \in \Omega^4(M)$ defining a G_2-structure, consider $\chi = \chi_1 + \chi_7 + \chi_{27} \in \Omega^4$ and consider $\psi_t = \psi + t\chi$. For t small, ψ_t is still a stable 4-form. Then the image $\widehat{\chi}$ of χ under the linearisation of Hitchin's duality map at ψ is*

$$\delta\varphi = \partial_t \widehat{\psi_t} \coloneqq \mathcal{J}(\chi) = \frac{3}{4} * \chi_1 + *\chi_7 - *\chi_{27} \, .$$

Similarly, the metric g_φ changes by

$$\delta g_\varphi = \frac{1}{2}\chi_1 + \frac{1}{2}\iota_\varphi(\chi_{27}) \, ,$$

where $\iota_\varphi : \Omega^3_{27} \to \Gamma(\mathrm{Sym}^2_0(TM))$ is the inverse of the map $S \mapsto S_*(\varphi)$.

The following lemma is a useful consequence of this result in combination with the Lie derivative.

Lemma A.24. *Let $\mathcal{J} : \Omega^4 \to \Omega^3$ be the linearisation of the Hitchin dual map defined above. For any $X \in \Omega^1$, we have*

$$\mathcal{L}_X\varphi = \mathcal{J}\mathcal{L}_X\psi \qquad \mathcal{L}_X g = \frac{1}{2}\pi_1(\mathcal{L}_X\varphi)g + \frac{1}{2}\iota_\varphi\left[\pi_{27}(\mathcal{L}_X\varphi)\right] \, . \tag{130}$$

Proof. Take $\delta\psi = \mathcal{L}_X\psi$. Then, by the Proposition A.23, we get $\delta\varphi = \mathcal{J}\mathcal{L}_X\psi$. However, by the definition of the Lie derivative, this must be equal to $\mathcal{L}_X\varphi$. Similarly, one computes the metric variation. $\square$

Nearly parallel G_2 identities

We now restrict ourselves to the case where the G_2-structure is a nearly parallel G_2-structure. We derive some useful identities for the exterior differential between the different irreducible representations, i.e. the operators d^p_q mentioned above. We provide a coordinate-free proof. A coordinate derivation of some of the same identities can be found in [DS23], following the approach of [KL20].

Since a nearly parallel G_2-manifold is the link of a special holonomy manifold, we have a curl-like operator (cf. Eq. (1)).

$$\begin{aligned} \mathrm{curl} : \Omega^1 &\to \Omega^1 \\ X &\mapsto *(dX \wedge \psi) \ . \end{aligned}$$

It was originally introduced in [Kar10], although the focus there is on metrics with holonomy G_2.

Proposition A.25. *Let $f \in C^\infty$ and $X \in \Omega^1$. We have*

(i) $dX = \frac{1}{3}\mathrm{curl}(X) \lrcorner \varphi + \pi_{14}(dX)$*;*

(ii) $d^*\mathrm{curl}(X) = 0$*;*

(iii) $\mathrm{curl}(df) = 0$*;*

(iv) $\mathrm{curl}(\mathrm{curl}(X)) = d^*dX + 4\mathrm{curl}(X)$*.*

Proof. Items (ii) and (iii) follow from the definition of the curl. To show (i), we have that $\pi_7(dX) = Y \lrcorner \varphi$. But, by virtue of Lemma A.17, we have

$$\mathrm{curl}(X) = *(dX \wedge \psi) = *(\pi_7(dX) \wedge \psi) = *(Y \lrcorner \varphi \wedge \psi) = 3Y \ .$$

Finally, for (iv), we have

$$\begin{aligned} \mathrm{curl}(\mathrm{curl}(X)) &= *\left(\psi \wedge d * (\psi \wedge dX)\right) = *d * *(\psi \wedge *(\psi \wedge dX)) = d^*\left[dX + *(dX \wedge \varphi)\right] \\ &= d^*dX + 4 * (dX \wedge \psi)) \ . \end{aligned}$$

□

Similarly, for 2-forms, we get

Proposition A.26. *Let $\beta = X \lrcorner \varphi + \beta_0$ for $X \in \Omega^1$ and $\beta_0 \in \Omega^2_{14}$. We get the following identities*

(i) $d(X \lrcorner \varphi) = -\frac{3}{7}(d^*X)\varphi + \left(\frac{1}{2}\mathrm{curl}(X) - 3X\right) \lrcorner \psi + 2i_\varphi(\mathcal{L}_X g)$*,*

(ii) $d^*(X \lrcorner \varphi) = \mathrm{curl}(X)$

(iii) $d\beta_0 = \frac{1}{4} d^*\beta_0 \lrcorner \psi + \gamma_0$ *for some* $\gamma_0 \in \Omega^3_{27}$*,*

where $\iota_\varphi : \Gamma(Sym^2_0(TM)) \to \Omega^3_{27}$ *from Lemma A.16.*

Proof. Wedging $d(X \lrcorner \varphi)$ with ψ and using Lemma A.17, we get

$$d(X \lrcorner \varphi) \wedge \psi = d(X \lrcorner \varphi \wedge \psi) = 3d(*X) = -3d^*X \ .$$

Using that $\varphi \wedge \psi = 7 \ vol$, we get the π_1-term. For the π_7, we can wedge with φ and use the Leibniz rule,

$$d(X \lrcorner \varphi) \wedge \varphi = d(X \lrcorner \varphi \wedge \varphi) - 4X \lrcorner \varphi \wedge \psi = 2d(X \wedge \psi) - 12 * X = 2 * \mathrm{curl}(X) - 12 * X \ .$$

Using the identity $\langle X \wedge \varphi, Y \wedge \varphi \rangle = 4\langle X, Y \rangle$, we get $\pi_7(d(X \lrcorner \varphi))$. Now, for the Ω^3_{27}-component, we use Proposition A.23. Let $\delta\varphi = \chi = \mathcal{L}_X\varphi$. Thus, $\delta g = 2\pi_1(\mathcal{L}_X\varphi) + 1/2\iota_\varphi\,[\pi_{27}(\mathcal{L}_X\varphi)]$. By the definition of the Lie derivative, we must also have $\delta g = \mathcal{L}_X g$. Thus,

$$\pi_{27}(d(X \lrcorner \varphi) = \pi_{27}(\mathcal{L}_X\varphi) = 2i_\varphi(\delta g) = 2\iota_\varphi(\mathcal{L}_X g)\ .$$

For (ii), we have

$$d^*(X \lrcorner \varphi) = *d*(X \lrcorner \varphi) = *(dX \wedge \psi) = \mathrm{curl}(X)\ .$$

Finally, for (iii), we have $d\beta_0 \wedge \psi = 0$ by the Leibniz rule again. Wedging with φ and using the Leibniz rule, we get

$$d\beta_0 \wedge \varphi = d(\beta_0 \wedge \varphi) = -d*\beta_0\ .$$

□

Finally, for 3-forms,

Proposition A.27. *For every $\gamma = f\varphi + *(X \wedge \varphi) + \gamma_0$ with $f \in C^\infty(M)$, $X \in \Omega^1$ and $\gamma_0 \in \Omega^3_{27}$, we get*

(i) $d(f\varphi) = df \wedge \varphi + 4f\psi$;

(ii) $d^*(f\varphi) = -*(df \wedge \psi) = -df \lrcorner \varphi$;

(iii) $d\pi_7(\gamma) = d*(X \wedge \varphi) = \frac{4}{7}(d^*X)\psi + (\frac{1}{2}\mathrm{curl}(X) + X) \wedge \varphi + 2*i_\varphi(\mathcal{L}_X g)$;

(iv) $d^*\pi_7(\gamma) = -*d(X \wedge \varphi) = *\left[(\frac{2}{3}\mathrm{curl}(X) + 4X) \wedge \psi\right] + \beta_0$ *for some* $\beta_0 \in \Omega^2_{14}$;

(v) $\pi_7(d^*\gamma_0) = \frac{4}{3}\pi_7(d\gamma_0)$.

Proof. Statements (i) and (ii) require no discussion. Let us prove (iii). Using Lemma A.24 and Cartan's magic formula, we get

$$d*(X \wedge \varphi) = -d(X \lrcorner \psi) = -\mathcal{L}_X\psi = -\mathcal{J}^{-1}\mathcal{L}_X\varphi = -\mathcal{J}^{-1}[d(X \lrcorner \varphi) + 4X \lrcorner \psi]\ .$$

Now, using Proposition A.26, we get

$$d*(X \wedge \varphi) = \frac{4}{7}(d^*X)\psi - \Big[-(\frac{1}{2}\mathrm{curl}(X) - 3X) - 4X\Big] \wedge \varphi + \pi_{27}d(X \lrcorner \varphi)\ .$$

For (iv), we use Proposition A.25. We have

$$d(X \wedge \varphi) = dX \wedge \varphi - 4X \wedge \psi = \left(\frac{1}{3}\mathrm{curl}(X) \lrcorner \varphi + \pi_{14}(dX)\right) \wedge \varphi - 4X \wedge \psi\ .$$

Multiplying by minus the Hodge star, we get

$$-*d(X \wedge \varphi) = *\Big((\frac{2}{3}\mathrm{curl}(X) + 4X) \wedge \psi\Big) + \beta_0\ .$$

where we used Lemma A.17 once again.

To prove (v), we combine Lemma A.24 with integration by parts:

$$4\int_M \langle X, \pi_7(d\gamma_0)\rangle = \int_M \langle X\wedge\varphi, d\gamma_0\rangle = \int_M d\gamma_0 \wedge *(X\wedge\varphi) = -\int_M \gamma_0\wedge \mathcal{L}_X\psi =$$
$$= \int_M \langle \gamma_0, dX\lrcorner\varphi\rangle = \int_M \langle d^*\gamma_0 \wedge *(X\wedge\psi)\rangle = 3\int_M \langle X, \pi_7(d^*\gamma_0)\rangle ,$$

where $\mathcal{J}^{-1}$ just acted as (-1) since $\gamma_0 \in \Omega^3_{27}$. Since X was arbitrary, the statement now follows. □

From the two lemmas above, we have the following corollary:

Corollary A.28. *For $\beta_0 \in \Omega^2_{14,coclosed}$ we have $d\beta_0 \in \Omega^3_{27}$. Similarly, for $\gamma_0 \in \Omega^3_{27,coclosed}$ we have $d\gamma_0 \in \Omega^4_{27}$.*

Finally, we have the following characterisation of Killing fields on a nearly parallel G_2-manifold.

Lemma A.29. *Let X be a Killing field for (M,φ). Then either $\mathrm{curl}(X) = -2X$ or $\mathrm{curl}(X) = 6X$. Moreover, $X \in \mathfrak{aut}(M, g, \varphi)$, it preserves the G_2-structure, if and only if $\mathrm{curl}(X) = -2X$.*

Proof. The Killing field statement is proved in Proposition 2.10. It follows from combining the Bochner characterisation of Killing 1-forms with Lemma A.25 (v).
Now, $\mathcal{L}_X\psi = d*(\varphi\wedge X)$ and the claim follows from Lemma A.27 (iii). □

Remark A.30. *The space $\{X \in \Omega^1 |\ \mathrm{curl}(X) = 6X\}$ appeared in [AS12] under the label D_1, corresponding to one of the pieces of infinitesimal deformation of the nearly parallel G_2 structure (cf. Table 6).*

Dirac operator and Hodge decomposition

The purpose of this section is to obtain a Hodge-type decomposition of forms on nearly G_2 manifolds. We obtain it by studying a Dirac-type operator and its mapping properties. The results of this section closely follow the ideas in [Fos17]. In [DS23], Dwivedi and Singhal also use twisted Dirac operators to obtain Hodge-like decompositions. The twisted Dirac here is different, and we obtain a different Hodge decomposition that is more suitable for our purposes.

The choice of a G_2-structure is equivalent to the choice of a spin structure, together with the choice of a unit spinor. By the work of Bär[Bär93], from the point of view of spin geometry, the nearly parallel condition can be rephrased as the unit spinor Φ satisfying the real Killing spinor condition:

$$\nabla_X\Phi = \frac{1}{2}X\cdot\Phi\ , \tag{131}$$

where $\cdot$ denotes Clifford multiplication and ∇ is the connection induced by the Levi-Civita connection on the spinor bundle.

In terms of G_2-representations, we can identify the real spinor bundle $\$$ with $\Lambda^0\oplus\Lambda^1$, where the isomorphism follows is given by

$$(f, X) \mapsto f\Phi + X\cdot\Phi\ .$$

Let us compute the Dirac operator $\not{D}$ under this isomorphism. Since Φ satisfies (131), $\not{D}\Phi = -\frac{7}{2}\Phi$. Thus,

$$\not{D}(f\Phi) = -\frac{7}{2}f\Phi + \nabla f \cdot \Phi \ ,$$

and

$$\not{D}(X \cdot \Phi) = \sum_{i=1}^{7} e_i \cdot \nabla_{e_i} X \cdot \Phi - X \cdot \Phi - X \cdot \not{D}\Phi = dX \cdot \Phi + (d^* X)\Phi + \frac{5}{2}X \cdot \Phi \ .$$

To complete this computation, we need to understand the Clifford action of 2-forms.

Lemma A.31. *For any 2-form $\beta = Y \lrcorner \varphi + \beta_0$, we have*

$$\beta \cdot \Phi = 3Y \cdot \Phi \ .$$

Proof. First, we have that $\beta_0 \cdot \Phi = 0$. Now, using Section 4.2 from [Kar10], we get

$$(Y \lrcorner \varphi) \cdot \Phi = -\frac{1}{2}(Y \cdot \varphi + \varphi \cdot Y) \cdot \Phi = 3Y \cdot \Phi \ . \qquad \square$$

Using that $dX = \frac{1}{3}\mathrm{curl}(X) \lrcorner \varphi + \pi_{14}(dX)$, and collecting the computations above, we get

$$\not{D}(f\Phi + X \cdot \Psi) = (d^* X - \frac{7}{2}f)\Phi + (\mathrm{curl}(X) + df + \frac{5}{2}X) \cdot \Phi \ . \tag{132}$$

Now, consider the operator

$$\check{D} : \Omega^3_1 \oplus \Omega^3_7 \to \Omega^4_1 \oplus \Omega^4_7$$
$$\gamma = (f\varphi, X \lrcorner \varphi) \mapsto (\pi_1(d\gamma), \pi_7(d\gamma)) \ .$$

Using the identities in Lemma A.27, we identify $\check{D}$ with the operator $D : \Omega^0 \oplus \Omega^1 \to \Omega^0 \oplus \Omega^1$

$$D(f, X) = \left(\frac{4}{7}d^* X + 4f, df + \frac{1}{2}\mathrm{curl}(X) + X\right) \ .$$

First, notice that D is an elliptic self-adjoint operator since D and $\not{D}$ coincide up to rescaling and a self-adjoint term of order zero. We compute its kernel.

Proposition A.32. *Let (M^7, φ) be a complete nearly parallel G_2–manifold that is not isometric to the round 7–sphere. Then* $\ker(D) = \mathfrak{aut}(M, g, \varphi) = \{X \in \Omega^1 | \ \mathrm{curl}(X) = -2X\}$.

Proof. Let $(f, X) \in \ker(D)$. Then

$$d^* X = -7f \qquad df = -\frac{1}{2}\mathrm{curl}(X) - X \ .$$

Acting by d^* on the right equation and combining with the left one, we arrive at

$$\Delta f = -d^* X = 7f \ .$$

By Obata's theorem, $f = 0$ under the assumption that (M^7, φ) is not isometric to (S^7, g_{round}). The remaining equation is $\mathrm{curl}(X) = -2X$, which we know corresponds to an infinitesimal automorphism of the G_2-structure by Lemma A.29. $\square$

Remark A.33. *For the round 7–sphere, the kernel of D consists of elements of the form $(f, X - \nabla f)$, where X satisfies $\mathcal{L}_X\varphi = 0$ and f satisfies $\Delta f = 7f$.*

Now, since D is a self-adjoint elliptic operator, we have the usual Hodge-type decomposition:

Theorem A.34. *Let (M^7, ψ) be a nearly parallel G_2-manifold that is not isometric to the round 7-sphere. The following holds.*

(i) $\Omega^4 = \{X \wedge \varphi |\ X \in \mathfrak{aut}(M, \varphi)\} \oplus d\Omega^3_{1\oplus 7} \oplus \Omega^4_{27}$. *More concretely, for every $\chi \in \Omega^4$, there exists unique $X \in \mathfrak{aut}(M, \varphi)$, $Y \in \mathfrak{aut}(M, \varphi)^{\perp}$, $f \in \Omega^0$ and $\chi_0 \in \Omega^4_{27}$ such that*

$$\chi = (X \wedge \varphi) + d(f\varphi + *(Y \wedge \varphi) + \chi_0 \ ,$$

where $\mathfrak{aut}(M, \varphi)^{\perp}$ is the L^2-complement to infinitesimal automorphism of the G_2-structure.

(ii) There is an L^2-orthogonal decomposition $\Omega^4_{exact} = d\Omega^3_{1\oplus 7} \oplus \Omega^4_{27,exact}$.

Proof. Statement (i) follows from the identification of $\check{D}$ with $\not{D}$ up to 0th order terms and Proposition A.32. Now, (ii) follows from (i). Notice that, for X an infinitesimal automorphism, we have $d^*(X \wedge \varphi) = - * \mathcal{L}_X\psi = 0$, so $\{X \wedge \varphi |\ X \in \mathfrak{aut}(M, \varphi)\}$ is L^2-orthogonal to exact forms and pointwise to Ω^4_{27}. Orthogonality follows from Corollary A.28, in that if χ_0 is closed, then $d^*\chi_0 \in \Omega^3_{27}$. □

We conclude by proving

Proposition A.35 ([DS23, Thm 3.8 & Thm 3.9])**.** *Harmonic* 2*-forms are of type* 14 *and harmonic* 3*-forms of type 27.*

Proof. Let $\beta = X \lrcorner \varphi + \beta_0$ a harmonic 2-form. Then Lemma A.26 implies

$$d^*X = 0 \qquad \mathrm{curl}(X) + d^*\beta_0 = 0 \qquad \frac{1}{2}\mathrm{curl}(X) - 3X + \frac{1}{4}d^*\beta_0 = 0 \ .$$

Together, they imply that X is harmonic by *(iv)* in Lemma A.25, which forces $X = 0$ by Myers' theorem.

Similarly, consider γ a harmonic 3-form. We can assume that (M^7, ψ) is not diffeomorphic to S^7, since there are no non-trivial harmonic 3-forms in that case. Let γ be a closed and coclosed 3-form. By Theorem A.34, we have

$$\gamma = *(X \wedge \varphi) + d^*\chi + \gamma_0 \ ,$$

for $X \in \mathfrak{aut}(M, \varphi)$, $\chi \in \Omega^4_{1\oplus 7}$ and $\gamma_0 \in \Omega^3_{27}$. The condition $d^*\gamma = 0$ implies that $d^*\gamma_0 = *d(X \wedge \varphi) = -4X \lrcorner \varphi - \pi_{14}(dX)$, and so we have

$$\langle d^*\gamma_0, X \lrcorner \varphi \rangle = \langle \gamma_0, d(X \lrcorner \varphi) \rangle = -4\langle \gamma_0, X \lrcorner \psi \rangle = 0 \ ,$$

So $X = 0 = d^*\gamma_0$ and so $d\gamma_0 \in \Omega^4_{27}$, by Corollary A.28. Now, the condition $d\gamma = 0$ implies

$$0 = \langle d\gamma, \chi \rangle = \langle \gamma, d^*\chi \rangle = ||d^*\chi||^2 + \langle d\gamma_0, \chi \rangle = ||d^*\chi||^2 \ ,$$

and so $d^*\chi = 0$ as needed. □

A.3 SU(3)-structures

We recall some well-known results on SU(3)-structures and nearly Kähler manifolds. These results are classic and have been collected for convenience.

Definition A.36. *An SU(3)-structure on a manifold M^6 is a reduction of its frame bundle to an SU(3) principal bundle. A manifold equipped with a choice of frame reduction is called an* SU(3)*-manifold.*

Equivalently, M^6 is equipped with a pair of stable differential forms $(\omega, \rho) \in \Omega^2(M) \times \Omega^3(M)$ satisfying the following algebraic constraints:

$$\omega \wedge \rho = 0 \qquad \frac{1}{3!}\omega^3 = \frac{1}{4}\rho \wedge \hat{\rho} \, . \tag{133}$$

Moreover, ω is positive with respect to the almost complex structure induced by ρ. Here we mean stability in the sense of Hitchin (cf. 6), so their orbit under the induced $\mathrm{GL}(6, \mathbb{R})$ is open. The 3-form $\hat{\rho} = *\rho$ is the Hitchin dual of ρ, as defined by Equation (63). The algebraic constraints and positivity of ω guarantee that the stabiliser of the pair is precisely $\mathrm{SU}(3) = \mathrm{Sp}(6, \mathbb{R}) \cap \mathrm{SL}(3, \mathbb{C})$. Similarly, one could have chosen a pair $(\rho, \sigma) \in \Omega^3 \times \Omega^4$ with Hitchin dual $\hat{\sigma} = \omega$ and satisfying the above conditions.

Moreover, since $\mathrm{SU}(3) \subset \mathrm{SU}(4) \cong \mathrm{Spin}(6)$ is the stabiliser of any $v \in \mathbb{C}^4 \backslash \{0\}$, an SU(3)-structure is equivalent to the choice of a spin structure on M, together with a nowhere vanishing spinor. As before, $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ and the metric on M can be reconstructed explicitly from the SU(3)-structure as follows. Let J be the almost complex structure induced by ρ. Then the condition $\omega \wedge \rho = 0$ is equivalent to ω is of type $(1,1)$ with respect to J. Then, since ω is positive, we have that $g := \omega(\cdot, J\cdot)$ defines our metric, and its induced volume form coincides with $\frac{1}{3!}\omega^3$.

The decomposition of $\Lambda^* T^* M$ is well-known and most commonly phrased in terms of the (p, q)-decomposition of the complexification $\Lambda^* T^* M \otimes \mathbb{C}$, induced by the almost complex structure J. However, we find it more convenient to work with the real irreducible representations.

Lemma A.37. *Let (M, J, ω, ρ) be an SU(3)-manifold. The spaces $\Lambda^2 T^* M$ and $\Lambda^3 T^* M$ decompose orthogonally as*

$$\Lambda^2 = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8 \qquad \Lambda^3 = \Lambda^3_{1+1} \oplus \Lambda^3_6 \oplus \Lambda^3_{12} \, .$$

They can be characterised by

$$\begin{aligned}
\Lambda^2_1 &= \langle \omega \rangle \, , \\
\Lambda^2_6 &= \{X \lrcorner \rho \mid X \in TM\} = \{\beta \in \Lambda^2 | \; *(\beta \wedge \omega) = \beta\} \, , \\
\Lambda^2_8 &= \{\beta \in \Lambda^2 | \; \beta \wedge \omega^2 = 0 = \beta \wedge \rho\} \cong \{\beta \in \Lambda^2 | \; *(\beta \wedge \omega) = -\beta\} \\
&\cong \{S_*(\omega) | \; S \in \mathrm{Sym}^2_+(T^*M)\} \, , \\
\Lambda^3_{1\oplus 1} &= \langle \rho, \hat{\rho} \rangle \, , \\
\Lambda^3_6 &= \{X \wedge \omega \mid X \in TM\} \, , \\
\Lambda^3_{12} &= \{\gamma \in \Lambda^3 | \; \gamma \wedge \omega = 0 = \gamma \wedge \rho\} = \{S_*(\rho) | \; S \in \mathrm{Sym}^2_-(T^*M)\} \, .
\end{aligned}$$

The symmetric endomorphisms splits as $\mathrm{Sym}^2 \cong \mathbb{R} \oplus \mathrm{Sym}^2_+ \oplus \mathrm{Sym}^2_-$*, with* $\mathrm{Sym}^2_+ = \{S \in \mathrm{Sym}^2_0(TM) \mid JS = SJ\}$ *and* $\mathrm{Sym}^2_- = \{S \in \mathrm{Sym}^2(TM) \mid JS = -SJ\}$*. As before, we get the decomposition for the remaining* Λ^k *using the identification* $\Lambda^k \cong *\Lambda^{6-k}$*.*

We identify Λ^2_6 and Λ^3_6 with Λ^1 via the maps $X \mapsto X \lrcorner \rho$ and $X \mapsto X \wedge \omega$ respectively. We have the following identities:

Lemma A.38. *In the decomposition of the previous lemma, the Hodge-$*$ operator is given by:*

(i) $*\omega = \frac{1}{2}\omega^2$;

(ii) $*(X \lrcorner \rho) = -JX \wedge \rho = X \wedge \hat{\rho}$;

(iii) $*X = JX \wedge \frac{\omega^2}{2}$;

(iv) $*(X \wedge \omega) = \frac{1}{2} X \lrcorner \omega^2 = JX \wedge \omega$;

(v) $*\rho = \hat{\rho}$ *and* $*\hat{\rho} = -\rho$;

(vi) $*\,(S_*\rho) = -S_*\hat{\rho} = (JS)_*\rho$;

From this lemma, one can characterise the different types of forms in terms of algebraic conditions and the Hodge star. The following two lemmas give the precise characterisation:

Lemma A.39. *Let* $\beta = \lambda\omega + X \lrcorner \rho + \beta_0 \in \Omega^2$ *with* $\beta_0 \in \Omega^2_8$*, then the following holds:*

(i) $*(\beta \wedge \omega) = -\beta_0 + 2\lambda\omega + X \lrcorner \rho$;

(ii) $*(\beta \wedge \beta \wedge \omega) = -\left|\beta_0\right|^2 + 6\lambda^2 + 2|X|^2$;

(iii) $*(\beta \wedge \rho) = 2JX$ *and* $*(\beta \wedge \hat{\rho}) = -2X$;

(iv) $*\left(\beta \wedge \omega^2\right) = 6\lambda$;

(v) $*(\hat{\rho} \wedge *(\hat{\rho} \wedge \beta)) = *(\rho \wedge *(\rho \wedge \beta)) = \beta + *(\beta \wedge \omega) - *(\beta \wedge \frac{\omega^2}{2})\omega$.

Similarly, let $\gamma = \lambda\rho + \mu\hat{\rho} + X \wedge \omega + \gamma_0 \in \Omega^3$ *with* $X \in \Omega^1$ *and* $\gamma_0 \in \Omega^3_{12}$*. The following holds:*

(i) $*(\gamma \wedge \omega) = 2JX$;

(ii) $*(\gamma \wedge \rho) = -4\mu$;

(iii) $*(\gamma \wedge \hat{\rho}) = 4\lambda$.

The proofs of both these lemmas are purely local and can be verified using local coordinates; we omit them for brevity.

For an SU(3)-structure, the characterisation of its intrinsic torsion was carried out by Gray and Hervella in [GH80] (cf. [CS02]). We have the following:

Proposition A.40. *Let (M^6, g, ω, ρ) be an SU(3)-structure. Then the intrinsic torsion is a section of*

$$\Omega^1 \otimes \Omega^2_{1+6} \cong \Omega^0 \oplus \Omega^0 \oplus \Omega^1 \oplus \Omega^1 \oplus \Omega^2_8 \oplus \Omega^2_8 \oplus \Omega^3_{12} \ .$$

That is, there exists forms $\tau_0, \widehat{\tau}_0 \in \Omega^0$, $\tau_1, \widehat{\tau}_1 \in \Omega^1$, $\tau_2, \widehat{\tau}_2 \in \Omega^2_8$ and $\tau_3 \in \Omega^3_{12}$ that fully characterise the torsion of the SU(3)*-structure. They satisfy*

$$\begin{aligned} d\omega &= 3\tau_0\rho + 3\widehat{\tau}_0\widehat{\rho} + \tau_1 \wedge \omega + \tau_3 \ , \\ d\rho &= 2\widehat{\tau}_0\omega^2 + \widehat{\tau}_1 \wedge \rho + \tau_2 \wedge \omega \ , \\ d\widehat{\rho} &= -2\tau_0\omega^2 + \widehat{\tau}_1 \wedge \widehat{\rho} + \widehat{\tau}_2 \wedge \omega \ . \end{aligned}$$

Given an SU(3)-structure, we are interested in the induced G_2-structures on its metric cone. In particular, we will be interested in the following three classes of SU(3)-structures.

Definition A.41. *Let $(\Sigma^6, g, \omega, \rho)$ be an* SU(3)*-manifold and consider $(C(\Sigma), \varphi) = (\Sigma \times \mathbb{R}_+, r^2 dr \wedge \omega + r^3\rho)$ the associated cone carrying a G_2-structure. Then*

(i) *Σ carries a closed* SU(3)*-structure if the G_2-structure on the cone is closed.*

(ii) *Σ carries a coclosed* SU(3)*-structure if the G_2-structure on the cone is coclosed.*

(iii) *Σ carries a nearly Kähler structure if the G_2-structure on the cone is parallel.*

Using the torsion decomposition of Gray and Hervella, we can phrase these conditions in terms of their intrinsic torsion.

Proposition A.42. *We have*

(i) *The torsion of a closed* SU(3)*-structure vanishes except for $\tau_0 = 1$ and $\hat{\tau}_2$.*

(ii) *The torsion of a coclosed* SU(3)*-structure vanishes except for $\tau_0 = 1$, $\hat{\tau}_0$ and τ_2 and τ_3. Moreover, we have $d\hat{\tau}_0 = -3\pi_6(d\tau_3)$ and $2d\hat{\tau}_0 = Jd^*\tau_2$.*

(iii) *A nearly Kähler structure vanishes except for $\tau_0 = 1$.*

Proof. Given an SU(3)-structure (ω, ρ) on M^6, the metric cone $(M \times \mathbb{R}_+, dt^2 + t^2 g_M)$ carries a G_2-structure given by

$$\varphi = r^2 dr \wedge \omega + r^3\rho \qquad\qquad * \varphi = \psi = -r^3 dr \wedge \hat{\rho} + r^4\frac{\omega^2}{2} \ .$$

Thus, if the SU(3)-structure is closed, we have

$$0 = d\varphi = r^2 dr \wedge (3\rho - d\omega) + r^3 d\rho$$

which implies

$$d\omega = 3\rho \qquad\qquad d\rho = 0 \ .$$

By Proposition A.40, the claim follows. Similarly, if the SU(3)-structure is coclosed,

$$0 = d\psi = r^3 dr \wedge (d\hat{\rho} + 4\frac{\omega^2}{2}) + r^4 d\frac{\omega^2}{2}$$

which implies

$$d\hat{\rho} = -2\omega^2 \qquad d\omega^2 = \omega \wedge d\omega = 0 \,.$$

Substituting in Proposition A.40, we get the first part of the claim. Differentiating the first and second equations on Proposition A.40 and using Lemma A.38 completes the claim.

Finally, combining conditions (i) and (ii), the last claim follows by Proposition A.19. □

Remark A.43. *The coclosed* SU(3) *condition in terms of the torsion appears to be overly complicated. One would expect* $\hat{\tau}_0 = C \in \mathbb{R}$ *and so* $\tau_2 = - * d\tau_3$*. However, we have not been able to prove this.*

Remark A.44. *Closed* SU(3)*-structures have been studied by physicists in the context of string theory, under the name of LT-structures, in [LT05]. They are a subclass of half-flat* SU(3)*-structures (cf. [MS13]).*

From Propositon A.6, we get

Lemma A.45 ([BV07, Thm 3.4])**.** *Let* (M, ω, ρ) *be an* SU(3)*-structure. The scalar curvature of the associated metric is given by*

$$s_g = 30(\tau_0^2 + \widehat{\tau_0}^2) + 2d^*(\tau_1 + \widehat{\tau_1}) - |\tau_1|^2 + 4\langle \tau_1, \widehat{\tau_1}\rangle - \frac{1}{2}\left(|\tau_2|^2 + |\widehat{\tau_2}|^2 + |\tau_3|^2\right) \,.$$

We have an explicit formula for the linearisation of Hitchin's duality map from Section 6 in terms of irreducible representations. We collect the result here as it is useful for computations in the next section:

Proposition A.46 ([Hit00] Section 3.3)**.** *Given* (ρ, σ) *defining an* SU(3)*-structure, consider* $\chi = \chi_1 + \chi_6 + \chi_8 \in \Omega^4$ *and* $\gamma = \gamma_{1\oplus 1} + \gamma_6 + \gamma_{12} \in \Omega^3$*. Then*

(i) The derivative of the Hitchin dual map at σ *in the direction of* χ *is*

$$\frac{d}{dt}\widehat{(\sigma + t\chi)}\Big|_{t=0} := \mathcal{K}(\chi) = \frac{1}{2} * \chi_1 + *\chi_6 - *\chi_8 \,.$$

(ii) The derivative of the Hitchin dual map at ρ *in the direction of* γ *is*

$$\frac{d}{dt}\widehat{(\rho + t\gamma)}\Big|_{t=0} := \mathcal{I}(\gamma) = *\gamma_{1\oplus 1} + *\gamma_6 - *\gamma_{12} \,.$$

As a straightforward corollary, we get

Lemma A.47. *Let* $\mathcal{I} : \Omega^3 \to \Omega^3$ *and* $\mathcal{K} : \Omega^4 \to \Omega^2$ *be the maps defined in Proposition A.46. For any* $X \in \Omega^1$*, we have* $\mathcal{L}_X\widehat{\rho} = \mathcal{I}\mathcal{L}_X\rho$ *and* $\mathcal{L}_X\widehat{\sigma} = \mathcal{K}\mathcal{L}_X\sigma$*.*

Nearly Kähler identities

We now restrict ourselves to the case where the SU(3)-structure is nearly Kähler. Most of the identities that follow are well-known and have been compiled here for convenience. The main reference is [Fos17]. The curl operator introduced in Equation 1 in the nearly Kähler case reads

$$\begin{aligned} \mathrm{curl} : \Omega^1 &\to \Omega^1 \\ X &\mapsto - * (dX \wedge \rho) \,. \end{aligned}$$

Proposition A.48. *Let $X \in \Omega^1$. We have*

(i) $d^*X = -*(dJX \wedge \frac{\omega^2}{2})$*;*

(ii) $dX = \frac{1}{3}d^*(JX)\omega - \frac{1}{2}J\mathrm{curl}(X) \lrcorner \rho + \pi_8(dX)$*;*

(iii) $\mathrm{curl}(X) = J\mathrm{curl}(JX) + 4X$*;*

(iv) $d^*(\mathrm{curl}(X)) = 0$ *and* $d^*(J\mathrm{curl}(JX)) = 4d^*X$*;*

(v) $\mathrm{curl}(df) = 0$*;*

(vi) $\mathrm{curl}(\mathrm{curl}(X)) = d^*dX + 3\mathrm{curl}(X) + Jdd^*(JX)$*.*

Proof. To get (i), differentiate the identity $*Y = JY \wedge \frac{\omega^2}{2}$. Now, the first term in (ii) follows from wedging by ω^2 and using (i). The term $\pi_6(dX)$ follows Lemma A.58.
The identity in (iii) follows from differentiating the identity $X \wedge \rho = JX \wedge \hat{\rho}$. We get

$$\mathrm{curl}(X) = -*(dX \wedge \rho) = -*d(JX \wedge \hat{\rho}) = -*(d(JX) \wedge \hat{\rho}) - 4*\left(JX \wedge \frac{\omega^2}{2}\right) = J\mathrm{curl}(JX) + 4X \ ,$$

where $*(d(JX) \wedge \hat{\rho}) = -J\mathrm{curl}(JX)$ by Lemma A.39. The identities in (iv) follow from acting by d^* in the definition of curl and the previous identity. Similarly, item (v) follows from the definition. Finally,

$$\mathrm{curl}(\mathrm{curl}(X)) = *(\rho \wedge d*(\rho \wedge dX)) = -*d**(\rho \wedge *(\rho \wedge dX) = d^**(\rho \wedge *(\rho \wedge dX) \ .$$

Using (vi) in Lemma A.39, we get

$$\begin{aligned}\mathrm{curl}(\mathrm{curl}(X)) =& d^*dX - *d(dX \wedge \omega) - d^*[d^*(JX)\omega] \\ =& d^*dX - 3*(dX \wedge \rho) + *\left(dd^*(JX) \wedge \frac{\omega^2}{2}\right) \\ =& d^*dX + 3\mathrm{curl}(X) + Jdd^*(JX) \ .\end{aligned}$$

□

Similarly, for 2-forms, we get

Proposition A.49. *Let $\beta = f\omega + X \lrcorner \rho + \beta_0$ for $f \in C^\infty(M)$, $X \in \Omega^1$ and $\beta_0 \in \Omega^2_8$. We get the following identities*

(i) $d(f\omega) = 3f\rho + df \wedge \omega$*,*

(ii) $d^*(f\omega) = Jdf$*,*

(iii) $d(X \lrcorner \rho) = -(\frac{1}{2}\mathrm{curl}(X) + X) \wedge \omega - \frac{1}{2}(d^*X)\rho - \frac{1}{2}d^*(JX)\hat{\rho} + \gamma_0$*, with* $\gamma_0 \in \Omega^3_{12}$*,*

(iv) $d^*(X \lrcorner \rho) = -\mathrm{curl}(JX)$*,*

(v) $d\beta_0 = \frac{1}{2}Jd^*\beta_0 \wedge \omega + \gamma_0'$*, for some* $\gamma_0' \in \Omega^3_{12}$*.*

Proof. Property (i) follows directly from the structure equations. For (ii), we get

$$d^*(f\omega) = - * d\Big(f\frac{\omega^2}{2}\Big) = - * (df \wedge \frac{\omega^2}{2}) = Jdf \ .$$

For (iii), we can obtain the $\pi_{1\oplus 1}$ by wedging with $\hat\rho$ and ρ respectively:

$$d(X \lrcorner \rho) \wedge \rho = d(X \lrcorner \rho \wedge \rho) = -2d * (JX) \ ,$$

$$d(X \lrcorner \rho) \wedge \hat\rho = d(X \lrcorner \rho \wedge \hat\rho) = 2d * (X) \ .$$

The claim now follows from the fact that $\rho \wedge \hat\rho = 4\,\mathrm{vol}$. Similarly, for the π_6 term, we consider

$$d(X \lrcorner \rho) \wedge \omega = d(X \lrcorner \rho \wedge \omega) - 3X \lrcorner \rho \wedge \rho = -\left(dJX \wedge \rho + 3X \wedge \omega^2\right) \ ,$$

From Lemmas A.38 and A.39, we get

$$-\pi_6\left(d(X \lrcorner \rho)\right) = \frac{1}{2}J\mathrm{curl}(JX) + 3X = \frac{1}{2}\mathrm{curl}(X) + X \ ,$$

where the last equality follows from Lemma A.48. For (iv), we need to use Lemma A.38 and the definition of curl:

$$d^*(X \lrcorner \rho) = *d(JX \wedge \rho) = -\mathrm{curl}(JX) \ .$$

Finally, for (v), we can differentiate the identity $(\beta_0 \wedge \omega) = - * \beta_0$. The statement follows from Lemma A.39. □

Finally, for 3-forms,

Proposition A.50. *For every $f, g \in C^\infty(M)$, $X \in \Omega^1$ and $\gamma_0 \in \Omega^3_{12}$, we get*

(i) $d(f\rho + g\hat\rho) = -4g\frac{\omega^2}{2} + (Jdf + dg) \wedge \hat\rho$;

(ii) $d^*(f\rho + g\hat\rho) = 4f\omega + (Jdg - df) \lrcorner \rho$;

(iii) $d(X \wedge \omega) = \frac{2}{3}(d^*JX)\frac{\omega^2}{2} - \Big(\frac{1}{2}J\mathrm{curl}(X) + 3JX\Big) \wedge \hat\rho + \pi_8(dX) \wedge \omega$

(iv) $d^*(X \wedge \omega) = \frac{2}{3}(d^*X)\omega - (\frac{1}{2}\mathrm{curl}(X) + X) \lrcorner \rho + \pi_8(dJX) \wedge \omega$,

(v) $\pi_6(d^*\gamma_0) = -J\pi_6(*d\gamma_0)$.

Proof. Properties (i) and (ii) follow from direct computation and the use of the structure equations. For (iii), we use (ii) from Proposition A.48. Similarly, (iv) follows from (iii) by using $*(X \wedge \omega) = JX \wedge \omega$. Let us prove (v). Using integration by parts and Lemma A.47

$$\begin{aligned} 2\int_M \langle X, \pi_6(*d\gamma_0)\rangle &= \int_M \langle X \lrcorner \rho, *d\gamma_0\rangle = \int_M d\gamma_0 \wedge (JX \lrcorner \rho) = \int_M \gamma_0 \wedge \mathcal{L}_{JX}\rho \\ &= -\int_M \gamma_0 \wedge *\mathcal{L}_{JX}\hat\rho = -\int_M \langle \gamma_0, d(JX \lrcorner \hat\rho)\rangle = -\int_M \langle d^*\gamma_0, *(JX \lrcorner \rho)\rangle \\ &= -2\int_M \langle JX, \pi_6(d^*\gamma_0)\rangle \ . \end{aligned}$$

Since X was arbitrary, we get $\pi_6(d^*\gamma_0) = -J\pi_6(*d\gamma_0)$. Notice that from the first to the second line, we used that $\mathcal{K}$ acts as (-1) on Ω^3_{12}. □

Moreover, using Lemma A.47 once more, we get

Lemma A.51. *For every $X \in \Omega^1$, we have*

$$d(X \lrcorner \rho) = (\mathrm{curl}(X) + 2X) \wedge \omega - d^* X \rho - d^*(JX)\hat{\rho} + d^*(JX \wedge \hat{\rho}) \ .$$

Proof. Expanding the right and left-hand sides of $\mathcal{L}_X \hat{\rho} = \mathcal{I}\mathcal{L}_X \rho$ from Lemma A.47 into irreducible parts, we get the desired claim. □

Combining the previous results, we have the following interesting corollary:

Corollary A.52. *For $\beta_0 \in \Omega^2_{8,coclosed}$ we have $d\beta_0 \in \Omega^3_{12}$ and for $\gamma_0 \in \Omega^3_{12,coclosed}$ we have $d\gamma_0 \in \Omega^4_8$.*

Finally, we have the following characterisation of Killing fields on a nearly Kähler manifold.

Lemma A.53. *Let (M, g, ω, ρ) be a nearly Kähler manifold and consider the spaces $\mathcal{E}_\lambda = \{\alpha \in \mathcal{C} | \ \mathrm{curl}(\alpha) = \lambda\alpha\}$. Assume that (M, g, ω, ρ) is not locally isometric to the round sphere (S^6, g_{round}). Then, $\mathfrak{isom}(M, g) \cong \mathfrak{aut}(M, \omega, \rho) \cong \mathcal{E}_{-2}$.*

Proof. As argued in [MNS05, Corollary 3.2], any Killing field must preserve the almost complex structure and thus the corresponding SU(3)-structure. The statement follows from enforcing $\mathcal{L}_X \rho = d(X \lrcorner \rho) = 0$ and Proposition A.49. □

In the case of the round sphere (S^6, g_{round}), we have

$$\mathcal{E}_{-2} \cong \mathfrak{g}_2 \qquad \qquad \mathcal{E}_5 \subseteq \mathfrak{g}_2^\perp \ ,$$

where the complement is taken as a subspace of $\mathfrak{so}(7)$. Proving the equality in the latter case is equivalent to proving $\mathfrak{isom}(S^6, g_{round}) \subset \mathcal{C}$. Since the round sphere is a symmetric space, this claim could be easily verified via representation theory (cf. Theorem 7.20). However, we have not carried out this computation.

Dirac operator and Hodge decomposition

The purpose of this section is to obtain a Hodge-type decomposition of 2-forms and 3-forms on nearly Kähler manifolds, which will be key in studying the second variations of the Hitchin functionals. Such decomposition is obtained by studying a Dirac-type operator and its mapping properties. The main decomposition result is due to Verbitsky [Ver11], although we present the proof given by Foscolo in [Fos17].

Recall that the choice of an SU(3)-structure is equivalent to the choice of a spin structure, together with the choice of a unit spinor, as discussed above. By the work of Bär[Bär93], from the point of view of spin geometry, the nearly Kähler condition can be rephrased as the unit spinor Φ satisfying the real Killing spinor condition:

$$\nabla_X \Phi = \frac{1}{2} X \cdot \Phi \ , \tag{134}$$

where $\cdot$ denotes Clifford multiplication and ∇ is the connection induced by the Levi-Civita connection on the spinor bundle. Clifford multiplication by the volume form vol yields a second Killing spinor, since $X \cdot \mathrm{vol} \cdot \Phi = -\mathrm{vol} \cdot X \cdot \Phi$, so

$$\nabla_X(\mathrm{vol} \cdot \Phi) = -\frac{1}{2} X \cdot (\mathrm{vol} \cdot \Phi) \ .$$

In terms of SU(3) representations, we can identify the real spinor bundle $\not{S}$ with $\Lambda^0 \oplus \Lambda^0 \oplus \Lambda^1$, where the isomorphism follows is given by

$$(f, g, X) \mapsto f\Phi + g \,\mathrm{vol} \cdot \Phi + X \cdot \Phi \ .$$

Let us compute the Dirac operator $\not{D}$ under this isomorphism. Since Φ satisfies (134), $\not{D}\Phi = -3\Phi$ and $\not{D}(\mathrm{vol} \cdot \Phi) = 3 \,\mathrm{vol} \cdot \Phi$. Thus,

$$\not{D}(f\Phi + g \,\mathrm{vol} \cdot \Psi) = -3f\Phi + 3g \,\mathrm{vol} \cdot \Phi + (\nabla f - J\nabla g) \cdot \Phi \ ,$$

since $JX \cdot \Phi = \mathrm{vol} \cdot X \cdot \Phi = -X \cdot \mathrm{vol} \cdot \varphi$. Similarly, we have

$$\not{D}(X \cdot \Phi) = \sum_{i=1}^{6} e_i \cdot \nabla_{e_i} X \cdot \Phi - X \cdot \Phi - X \cdot \not{D}\Phi = dX \cdot \Phi + (d^*X)\Phi + 2X \cdot \Phi \ .$$

To complete this computation, we need to understand the Clifford action of 2-forms.

Lemma A.54. *For any 2-form $\beta = f\omega + Y \lrcorner \rho + \beta_0$, we have*

$$\beta \cdot \Phi = 3f\Phi + 2JY \cdot \Phi \ .$$

Proof. First, we have that $\beta_0 \cdot \Phi = 0$. Now, we can write $\omega = \sum_{i=1}^{3} e_i \wedge Je_i$, with $\{e_i, Je_i\}_{i=1,2,3}$ an SU(3)-adapted orthonormal frame. Thus

$$\omega \cdot \Phi = \sum_{i=1}^{3} (e_i \wedge Je_i) \cdot \Phi = \sum_{i=1}^{3} e_i \cdot Je_i \cdot \Phi = -\sum_{i=1}^{3} e_i \cdot e_i \cdot \mathrm{vol} \cdot \Phi = 3 \,\mathrm{vol} \cdot \varphi \ .$$

Now, using Lemmas 1 and 2 from [CH16], we get

$$(Y \lrcorner \rho) \cdot \Phi = (JY \lrcorner \hat{\rho}) = -\frac{1}{2}(JY \cdot \hat{\rho} + \hat{\rho} \cdot JY) \cdot \Phi = 2JY \cdot \Phi \ . \qquad \square$$

Thus, using that $dX = \frac{1}{3} d^*(JX)\omega - \frac{1}{2} J\mathrm{curl}(X) \lrcorner \rho + \pi_8(dX)$, and collecting the computations above, we get

$$\begin{aligned} \not{D}(f\Phi + g \,\mathrm{vol} \cdot \Phi + X \cdot \Psi) = \ & (d^*X - 3f)\Phi + (d^*(JX) + 3g) \,\mathrm{vol} \cdot \Phi \\ & + (\nabla f - J\nabla g + \mathrm{curl}(X) + 2X) \cdot \Phi \ . \end{aligned} \tag{135}$$

Consider the operators

$$\begin{aligned} & D^+ : \Omega^2_{1\oplus 6} \oplus \Omega^4_1 \to \Omega^3_{1\oplus 1\oplus 6} \\ & \left(f\omega + X \lrcorner \rho, g\frac{\omega^2}{2} \right) \mapsto \pi_{1\oplus 1\oplus 6} \left[d(f\omega - X \lrcorner \rho) + d^* \left(\frac{g\omega^2}{2} \right) \right] \ . \end{aligned}$$

and $D^- : \Omega^3_{1\oplus1\oplus6} \to \Omega^4_{1\oplus6} \oplus \Omega^2_1$ given by $D^-(\sigma) = (\pi_{1\oplus6}(d\sigma), \pi_1(d^*\sigma))$. By Propositions A.49 and A.50, both these operators can be identified with the operator

$$D : \Omega^0 \oplus \Omega^0 \oplus \Omega^1 \to \Omega^0 \oplus \Omega^0 \oplus \Omega^1$$
$$(f, g, X) \mapsto \left(d^*X + 6f, d^*(JX) - 6g, \frac{1}{2}\mathrm{curl}(X) + X + df + Jdg\right)$$

by choosing appropriate identifications of $\Omega^2_{1\oplus6} \oplus \Omega^4_1$ and $\Omega^3_{1\oplus1\oplus6}$ with $\Omega^0 \oplus \Omega^0 \oplus \Omega^1$. The results we are interested in follow from the mapping properties of D. First, we have

Proposition A.55. *The operator D is an elliptic self-adjoint operator.*

This follows since, by Equation (135), D and $\not{D}$ coincide up to a self-adjoint term of order zero.

Proposition A.56. *Let (M^6, ω, ρ) be a complete nearly Kähler 6–manifold that is not isometric to the round 6–sphere. Then* $\ker(D) \cong \mathfrak{aut}(M, g, \omega, \rho) \cong \{X \in \Omega^1 | \ \mathrm{curl}(X) = -2X\}$.

Proof. Let $(f, g, X) \in \ker(D)$. Then

$$\frac{1}{2}\mathrm{curl}(X) + df - Jdg + X = 0 \tag{136}$$
$$d^*X + 6f = 0 \tag{137}$$
$$d^*(JX) - 6g = 0 \tag{138}$$

Acting by $d^* \circ J$ and d^* on (136) and using (137) and (138), we get $\Delta g + 18g = 0$ and $\Delta f - 6f = 0$. By Obata's theorem, we get $f = g = 0$, since M^6 is not isometric to the round sphere. The remaining equation is $\mathrm{curl}(X) + 2X = 0$, and the claim follows from Lemma A.53. □

Remark A.57. *For the round 6–sphere, the kernel of D consists of elements of the form $(f, 0, X - \nabla f)$, where X satisfies $\mathcal{L}_X\omega = 0 = \mathcal{L}_X\rho$ and f satisfies $\Delta f = 6f$.*

As in the nearly parallel G_2 manifold case, the Hodge decomposition for the elliptic operator yields:

Theorem A.58 ([Fos17] Proposition 3.22)**.** *Let (M^6, ω, ρ) be a nearly Kähler manifold that is not isometric to the round* 6*-sphere, and denote by* $\mathfrak{aut} = \mathfrak{aut}(M, g, \omega, \rho)$ *the set of infinitesimal automorphisms of the nearly Kähler structure. The following holds:*

(i) $\Omega^3 = \{X \wedge \omega | \ X \in \mathfrak{aut}\} \oplus d\Omega^2_{1\oplus6} \oplus d^*\Omega^4_1 \oplus \Omega^3_{12}$.

(ii) There is an L^2-orthogonal decomposition $\Omega^3_{exact} = d\Omega^2_{1\oplus6} \oplus \Omega^3_{12,exact}$.

(iii) $\Omega^4 = \{X \wedge \hat{\rho} | \ X \in \mathfrak{aut}\} \oplus d\Omega^3_{1\oplus1\oplus6} \oplus \Omega^4_8$.

(iv) For every $\chi \in \Omega^4$, there exists unique $X \in \mathfrak{aut}$, $Y \in \mathfrak{aut}^\perp$, $f \in \Omega^0$ and $\chi_0 \in \Omega^4_8$ such that

$$\chi = (X \wedge \hat{\rho}) + d(JY \wedge \omega + f\hat{\rho}) + \chi_0 \ ,$$

where $\mathfrak{aut}^\perp$ is the space L^2 complement to $\mathfrak{aut}$.

(v) There is an L^2-orthogonal decomposition $\Omega^4_{exact} = d\Omega^3_{1\oplus6} \oplus \Omega^4_{8,exact}$.

Proof. The first (resp. third) statement follows from the identification of D^+ (resp. D^-) with D and Proposition A.56. Item (ii) follows from (i), since for $X \in \mathfrak{aut}$,

$$0 = *\mathcal{L}_X\omega^2 = 2d^*(X \wedge \omega).$$

Therefore, by standard Hodge theory, the space $\{X \wedge \omega \mid X \in \mathfrak{aut}\} \oplus d^*\Omega^4_1$ is L^2-orthogonal to exact forms, and pointwise orthogonal to Ω^3_{12}. The orthogonality follows from Proposition A.50: if $\gamma_0 \in \Omega^3_{12}$ is closed, then $d^*\gamma_0 \in \Omega^2_8$.

For (iv), observe that every 4-form χ can be uniquely written as

$$\chi = X \wedge \hat{\rho} + d(JY \wedge \omega - g\rho + f\hat{\rho}) + \sigma_0,$$

with $X \in \mathfrak{aut}$, $Y \in \mathfrak{aut}^\perp$. This decomposition determines (X, Y, f, g, χ_0) uniquely, up to prescribing $d^*(JY \wedge \omega - g\rho + f\hat{\rho}) \wedge \omega^2$. Now, for every pair (f, Y'), we can set $g' = \frac{1}{6}d^*(JY)$, so that every solution (f, g, Y) to $D(f, g, Y) = (f', g', Y')$ satisfies $g = 0$.

Finally, (v) follows from (iii) and (iv) by the same argument as above. □

We conclude by proving

Proposition A.59. *Harmonic* 2*-forms are of type* 8*, and harmonic* 3*-forms are of type* 12*.*

Proof. If M^6 is diffeomorphic to S^6, then there are no non-trivial p-forms for $1 \leq p \leq 5$. Hence, we may assume we are under the hypotheses of Theorem A.58. By part (iv) of the theorem, any closed and coclosed 2-form β can be written as

$$\beta = X \lrcorner \rho + d^*\gamma + \beta_0,$$

with $X \in \mathfrak{aut}$, $\gamma \in \Omega^3_{1\oplus 6}$, and $\beta_0 \in \Omega^2_8$. Since β is coclosed, we have

$$d^*\beta_0 = -d^*(X \lrcorner \rho) = 6JX.$$

Now, we have

$$6\|X\|^2 = \langle d^*\beta_0, JX\rangle = \langle \beta_0, d(JX)\rangle = -3\langle \beta_0, X \lrcorner \rho\rangle = 0,$$

since $X \in \mathfrak{aut}$, and hence $X = 0$, so $d^*\beta_0 = 0$. By Proposition A.49, this implies $d\beta_0 \in \Omega^3_{12}$. Therefore,

$$0 = \langle d\beta, \gamma\rangle = \langle \beta, d^*\gamma\rangle = \langle \beta_0, d^*\gamma\rangle + \|d^*\gamma\|^2.$$

Analogously, by Theorem A.58 (i), any closed and coclosed 3-form γ can be written as

$$\gamma = X \wedge \omega + d\beta + d^*(f\omega^2) + \gamma_0,$$

with $X \in \mathfrak{aut}$, $\beta \in \Omega^2_{1\oplus 6}$, and $\gamma_0 \in \Omega^3_{12}$. First, observe that $X \wedge \omega$ is L^2-orthogonal to $d^*(f\omega^2)$:

$$\langle X \wedge \omega, d^*(f\omega^2)\rangle = \langle d(X \wedge \omega), f\omega^2\rangle = \langle dX \wedge \omega - 3X \wedge \rho, f\omega^2\rangle = 0.$$

Since γ is closed, we have:

$$0 = \langle \gamma, d^*(f\omega^2)\rangle = \|d^*(f\omega^2)\|^2 + \langle d\gamma_0, f\omega^2\rangle = \|d^*(f\omega^2)\|^2 .$$

Again, by closedness, $d\gamma_0 = -d(X \wedge \omega)$. Using $X \in \mathfrak{aut}$ and Proposition A.49, we have that $\pi_6(d\gamma_0) = 4X \wedge \rho$. Thus,

$$4\|X\|^2 = \langle d\gamma_0, X \wedge \rho\rangle = \langle \gamma_0, d^*(X \wedge \rho)\rangle = \langle \gamma_0, d(X \lrcorner \rho)\rangle = 0,$$

since $X \in \mathfrak{aut}$, and so $X = 0$ and $d\gamma_0 = 0$. Again by Proposition A.49, this implies $d^*\gamma_0 \in \Omega^2_8$. Finally, using this and the fact that γ is closed:

$$0 = \langle \gamma, d\beta\rangle = \langle d^*\gamma_0, d\beta\rangle + \|d\beta\|^2 = \|d\beta\|^2,$$

so $d\beta = 0$, as required. □

A.4 $\mathrm{U}(k) \times 1$-structures

We recall some well-known results on $\mathrm{U}(k)\times 1$-structures and Sasakian manifolds. Most of these results are classic and have been collected for convenience. A general reference for this material is the book of Boyer and Galicki [BG08].

Definition A.60. *An almost contact structure or $U(k) \times 1$-structure on a manifold M^{2k+1} is a reduction of its frame bundle to an $\mathrm{U}(k)$-principal bundle. A manifold equipped with a choice of frame reduction is called an almost contact manifold*[6]*.*

Equivalently, we may describe the $U(k) \times \mathbf{1}$-structure M^{2k+1} in terms of its invariant tensors.

Lemma A.61 ([BG08, Proposition 6.3.2])**.** *An almost contact structure on M^{2k+1} is equivalent to a triple (g, R, Φ), where g is a metric, R is a nowhere vanishing vector field, and Φ is an endomorphism on TM. They satisfy the relations*

$$\Phi^2 = -1 + R \otimes \eta \qquad \Phi^* = -\Phi\ ,$$

where $\eta = g(R, \cdot)$ is the dual 1-form to R and Φ^* is the adjoint map to Φ with respect to Φ.

The condition that R, called the Reeb field, has no zeroes means that M^{2k+1} is equipped with a one-dimensional foliation carrying a transverse $U(n)$-structure in the sense of Molino [Mol88], so we have a short exact sequence of bundles

$$0 \to \langle R\rangle \to TM \to H \to 0\ .$$

We refer to H as the horizontal subbundle of the almost contact structure. By taking the wedge of the previous short exact sequence, we have

$$0 \to \Lambda^k H \to \Lambda^k M \to \eta \wedge \Lambda^{k-1} H \to 0\ .$$

We abbreviate $\Lambda^k H$ by Λ^k_h, and refer to its sections as horizontal forms. As in the Kähler case, since $\Phi^* = -\Phi$, we have an associated 2-form $\omega := g(\ \cdot, \Phi\ \cdot)$, with maximal rank. The reduction of the structure group to $U(k)\times 1$ leads to a decomposition of $\Lambda^*(T^*M)$ into irreducible representations of $U(k)$, as usual.

[6] The term almost contact can be slightly ambiguous. In the literature, the definition above corresponds to strict almost contact metric structures (cf. [BG08]).

Lemma A.62. *Let $(M^{2k+1}, g, \eta, \Phi)$ be an almost contact manifold. Then for each k*

$$\Lambda^k = \eta \wedge \Lambda_h^{k-1} \oplus \Lambda_h^k \, .$$

Moreover, for each k we have the usual (p,q)-decomposition

$$\Lambda_h^k \otimes \mathbb{C} \cong \bigoplus_{p+q=k} \Lambda^{p,q}$$

from almost complex geometry.

As in the previous case, we have some identities involving these decompositions and the Hodge star, and they follow from the standard identities for (p,q)-forms.

As in the previous cases, one could study the intrinsic torsion of a $U(k) \times 1$-structure and relate it to the covariant derivative of its defining tensors (g, R, Φ). We omit this discussion and focus exclusively on the classes that interest us.

Definition A.63. *Let (Σ^{2k+1}, g) be a closed Riemannian manifold, and consider its metric cone $(\Sigma \times \mathbb{R}_+, dr^2 + r^2 g)$. We say (Σ, g) is*

(i) Sasaki if the induced metric cone has holonomy contained in $\mathrm{U}(k+1)$*;*

(ii) Sasaki-Einstein if the induced metric cone has holonomy contained in $\mathrm{SU}(k+1)$*;*

(iii) and 3-Sasaki if the induced metric cone has holonomy contained in $\mathrm{Sp}\left(\frac{k+1}{2}\right)$*.*

In particular, the chain of group inclusions $\mathrm{Sp}\left(\frac{k+1}{2}\right) \subseteq \mathrm{SU}(k+1) \subseteq \mathrm{U}(k+1)$ implies that any 3-Sasaki manifold is Sasaki-Einstein, and that both are Sasaki. Similarly, if $n = 4$, the inclusion $\mathrm{SU}(4) \subseteq \mathrm{Spin}(7)$ implies that every Sasaki-Einstein 7-manifold carries a nearly G_2-structure.

A detailed discussion between the relations of 3-Sasaki and Sasaki-Einstein and the induced G_2-geometries can be found in [AS12, Sect. 4].

Finally, since Calabi-Yau manifolds are Ricci-flat, we have, as the name suggests, the following

Lemma A.64. *A Sasaki manifold M^{2k+1} is Sasaki-Einstein if and only if it is Einstein with scalar curvature $2k(2k+1)$.*

Let us examine the structure of these manifolds in some more detail. Since the cone has holonomy included in $\mathrm{U}(n)$, it carries certain parallel tensors that induce a geometric structure on the link by restriction. In particular, if we denote by I the complex structure on the cone, we can consider the vector field $R \coloneqq I(r\partial_r)$. Such a vector field is constant in r, and so, it induces a nowhere-vanishing vector field on M, known as the Reeb vector field. Moreover, since I is parallel and $r\partial_r$ is a constant length Killing field, so is the Reeb field R.

Therefore, the flow induced by the Reeb field integrates to a smooth path in $\mathrm{Isom}_0(M, g)$, which we denote by Φ_t. By the Myers-Steenrod theorem, $\mathrm{Isom}_0(M, g)$ is a compact Lie group, so the compactification of the family Φ_t is a torus of a certain rank r, which allows us to distinguish two main classes of Sasaki manifolds.

If $r \geq 2$, we say that the Sasaki structure is irregular. If $r = 1$, it means that $\Phi_T = \mathrm{Id}$ for some $T > 0$, so the orbits of the Reeb flow are closed. By the structure theorem of Molino on

Riemannian foliations, almost all orbits have the same length and the orbit space $\mathcal{X}$ is an orbifold of dimension $2n$ with cyclic singularities. These are called quasi-regular Sasaki manifolds. If there are no orbifold singularities, i.e. the S^1-action is free, the Sasaki structure is said to be regular.

Going back to the structures on M, the 1-form dual to the Reeb field, denoted by η, satisfies $d\eta = 2\omega$, where ω is the pullback to M of the Kähler form on the cone.

Finally, in the Sasaki-Einstein case, we can also pull back the holomorphic volume Ω on the cone. Denote it by $\widetilde{\Omega}$. We have

Lemma A.65. *The k-form $\widetilde{\Omega}$ satisfies the equation $\mathcal{L}_R\widetilde{\Omega} = i(k+1)\widetilde{\Omega}$.*

Proof. The holomorphic volume form on the cone is a homogeneous form given by

$$\Omega = (dr + ir\eta) \wedge r^k\widetilde{\Omega} \ ,$$

where $\widetilde{\Omega} \in \Omega^k_h$. Now, since Ω is holomorphic, it is closed. Differentiating, we get

$$\begin{aligned} d\Omega &= (idr \wedge \eta + 2ir\omega) \wedge r^k\widetilde{\Omega} - (dr + ir\eta) \wedge r^{k-1}\left(kdr \wedge \widetilde{\Omega} + rd\widetilde{\Omega}\right) \\ &= i(k+1)r^k dr \wedge \eta \wedge \widetilde{\Omega} - r^k dr \wedge d\widetilde{\Omega} \ . \end{aligned}$$

Since $\widetilde{\Omega}$ is horizontal, the claim follows. □

Let us conclude by saying a few words on the Killing fields of a Sasaki-Einstein manifold. First, recall the notion of a foliate vector field:

Definition A.66. *A horizontal field $X \in \Gamma(H)$ is called foliate if $\mathcal{L}_R X = [R, X] \in \langle R\rangle$.*

Lemma A.67. *Let $X = f\eta + \overline{X}$ be the dual of a foliate Killing field, with $\overline{X} \in \Omega^1_h$.*

(i) If $\mathrm{curl}(X) = 2kX$*, then $f = C \in \mathbb{R}$ and $\overline{X} = 0$.*

(ii) If $\mathrm{curl}(X) = -2X$*, then $\overline{X}$ is a Killing field for the transverse metric g_T.*

(iii) Conversely, given $\overline{X}$ a Killing field for the transverse metric g_T, there exists X a foliate Killing field solving $\mathrm{curl}(X) = -2X$*.*

(iv) If X satisfies $\mathrm{curl}(X) = -2X$*, then X preserves the Sasaki structure.*

Proof. The foliate condition on X is equivalent to $\mathcal{L}_R\overline{X} = R\lrcorner d\overline{X} = 0$, so

$$\mathrm{curl}(X) = *\left[\left(df \wedge \eta + 2f\omega + d\overline{X}\right) \wedge \frac{\omega^{k-1}}{(k-1)!}\right] = -Jdf + (2kf + d^*(J\overline{X}))\eta \ .$$

Then $\mathrm{curl}(X) = AX$ becomes the system

$$-Jdf = A\overline{X} \qquad (2kf + d^*(J\overline{X})) = Af \ . \tag{139}$$

Using the Kähler identities, we have

$$\Delta\overline{X} = -Jdd^*J\overline{X} = A(A-2k)\overline{X} \ .$$

If $A = 2k$, $\overline{X}$ is basic harmonic and therefore zero by Bochner's argument. From the first equation in (139), we get that f is a constant.

Now, if $A = -2$, $\overline{X}$ is a basic coclosed 1-form solving $\Delta\overline{X} = 4(k+1)\overline{X}$. Since $\mathrm{Ric}_T = 2(k+1)$, it follows that $\overline{X}$ is a Killing field for g_T.

Conversely, given a vector field $\overline{X}$ preserving the horizontal component of the metric g, we need to find a basic function f such that $df = -2J\overline{X}$. By Hodge theory, we need $[J\overline{X}]_b \in H^1_b(M)$ in basic cohomology to vanish. But $H^1_b(M) = 0$ by Bochner's argument (cf. [BG08, Theorem 8.1.8]).

Finally, if a Killing vector field preserves the transverse metric g_T, it must preserve the Reeb field η. Using the relation $d\eta = 2g(\,\cdot\,, \Phi\,\cdot)$, it must also preserver Φ. □

Conversely, suppose X is an infinitesimal automorphism of the Sasaki structure. In that case, X is always a foliate vector field since for $X \in \mathfrak{aut}(M, g, \eta, \Phi, \rho)$ we must have $\mathcal{L}_X R = [X, R] = 0$. Thus, we have

$$\mathfrak{aut}(M, g, \eta, \Phi) \cong \{X \in \Omega^1 |\ \mathrm{curl}(X) = -2X\} \oplus \eta\ . \tag{140}$$

As usual, we consider the eigenspaces $\mathcal{E}_\lambda = \{X \in \Omega^1 |\ \mathrm{curl}(X) = \lambda X\}$.

From Lemma A.65 and Proposition A.10, we get the desired characterisation of $\mathfrak{aut}(M, g, \eta, \Phi)$; and from [BG08, Thm. 8.1.18 & Thm. 13.4.4], we have a comparison between the Lie algebras $\mathfrak{aut}(M, g, \eta, \Phi)$ and $\mathfrak{isom}(M, g)$ depending on the holonomy of the cone. Putting this together, we have the following:

Proposition A.68. *Let $(\Sigma^{2k+1}, g, \eta, \Phi)$ be a Sasaki-Einstein manifold. Then $\mathfrak{aut}(M, g, \eta, \Phi, \rho) \cong \mathcal{E}_{-2}$ and either*

(i) (Σ^{2k+1}, g) has constant sectional curvature, so it is covered by (S^{2k+1}, g_{round}),

(ii) (Σ^{2k+1}, g) is a (strict) 3-Sasaki manifold and $\dim \mathcal{E}_{2k} = 3$, or

(iii) (Σ^{2k+1}, g) is a (strict) Sasaki-Einstein manifold and $\mathcal{E}_{2k} = \langle\eta\rangle$.

In the round sphere case (S^{2k+1}, g_{round}), we have

$$\mathcal{E}_{-2} \cong \mathfrak{su}(k+1) \qquad\qquad \mathcal{E}_{2k} \subseteq \mathfrak{su}(k+1)^\perp\ , \tag{141}$$

where the complement is taken as a subspace of $\mathfrak{so}(2n+2)$, so $\dim \mathcal{E}_{2k} \leq k^2 + k + 1$.

Proving the equality in this case is equivalent to proving $\mathfrak{isom}(S^{2k+1}, g_{round}) \subset \mathcal{C}$. Since the round sphere is a symmetric space, this claim could be easily verified via representation theory (cf. Theorem 7.20). However, we have not carried out this computation.

Finally, in the 7-dimensional case, one may compare the two spaces of infinitesimal automorphisms through their curl operators. In the Sasaki-Einstein case, the associated 4-form is given by $\psi = -\eta \wedge \hat{\rho} + \frac{\omega^2}{2}$, and so

$$\mathrm{curl}_\psi(\eta) = *\left[\left(-\eta \wedge \hat{\rho} + \frac{\omega^2}{2}\right) \wedge 2\omega\right] = 6\eta\ .$$

Since $\mathfrak{aut}(M, g, \eta, \rho) \subseteq \mathfrak{aut}(M, g, \varphi) \subseteq \mathfrak{isom}(M, g)$, we see that the former inclusion is an equality.

Somewhat surprisingly, in the 3-Sasaki case, one of the Reeb additional Reeb fields induces an automorphism of the nearly parallel G_2, so $\mathcal{E}_6^\psi = \{X \in \Omega^1 | \operatorname{curl}_\psi(X) = 6X\}$ has dimension two. The proof follows from [AS12, Thm. 4.2]. Combining this with Lemma A.29, we get the following relation between the holonomy of a nearly parallel G_2-cone and the dimensions of $\mathcal{E}_6^\psi$ and $\mathcal{E}_6^{\omega^2/2}$, assuming Equation (141) holds.

$\mathrm{Hol}(C(\Sigma^7, g))$	Spin(7)	SU(4)	Sp(2)	$\{1\}$
$\dim \mathcal{E}_6^\psi$	0	1	2	7
$\dim \mathcal{E}_6^{\omega^2/2}$	-	1	3	≤ 13

Table 6: Multiplicity of the eigenvalue 6 for the operators $\operatorname{curl}_\psi(X) = *(\psi \wedge dX)$ and $\operatorname{curl}_{\omega^2/2}(X) = *\left(\frac{\omega^2}{2} \wedge dX\right)$ depending on the holonomy of the metric cone for (Σ^7, g).

B Non parabolicity of the nearly Kähler Laplacian flow

We investigate the nearly Kähler gradient flow introduced in Section 7.2:

$$\begin{cases} \partial_t \sigma = \Delta_\sigma \sigma - 12\sigma + \mathcal{L}_{V(\sigma)}\sigma \\ d\sigma = 0 \\ \sigma(0) = \sigma_0 \ , \end{cases}$$

with the Hitchin dual of σ_0 in $\mathcal{U} = \left\{\omega \in \Omega^2(\Sigma) \mid d\omega \text{ is stable, } \omega \text{ is stable and positive, } \omega^2 \text{ is exact}\right\}$.

We show that this flow is not strictly parabolic, even after using a DeTurck-type trick. Therefore, one cannot guarantee the short-time existence and uniqueness of solutions to the flow using standard techniques. In particular, the symbol of the nearly Kähler Laplacian flow (70) resembles the G_2 Laplacian coflow, introduced by Karigiannis, McKay, and Tsui in [KMT12] (cf. [Gri13]).

We begin by constructing suitable DeTurck vector fields, following the exposition of [BX11]. We then compute the symbol of the flow, modified by a suitable DeTurck field.

B.1 The DeTurck vector fields

We use the same recipe that DeTurck used for the Ricci flow, or Bryant and Xu [BX11] for the G_2 Laplacian flow. Let M be a manifold and g a metric, ∇ its Levi-Civita connection and ∇^0 a fixed torsion-free connection (e.g. the Levi-Civita of a background metric). The difference

$$T = \nabla - \nabla^0$$

is a well-defined section of $\mathrm{Sym}^2 TM^* \otimes TM$. Identifying TM with TM^* via the metric, and using the decomposition $\mathrm{Sym}^2 TM = TM \oplus \mathrm{Sym}^2_0 TM$, T is a section of $TM \oplus TM \otimes \mathrm{Sym}^2_0 TM$. We construct two vector fields from T, one from the first term of the decomposition, labelled V_1, and the other by contracting TM and $\mathrm{Sym}^2_0 TM$, labelled V_2. Therefore, whenever we have a G-structure on TM with $G \subseteq \mathrm{SO}(n)$, there is at least a two-dimensional family of vector fields associated with it, called the DeTurck field.

The linearisation of these vector fields depends exclusively on the variation of the metric g. Let g_t be a family of metrics with $g_0 = g$, and $h = \partial_t g_t$ at $t = 0$. By the Koszul formula,

$$g\left(\frac{\partial T}{\partial h}(X, Y), Z\right) = \frac{1}{2}\left[(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) - (\nabla_Z h)(X, Y)\right] .$$

Using the trace decomposition for h, $h = fg + h_0$, we get (cf. [BX11, Sect. 2.2])

$$V_{1*}(h) = \mathrm{grad}(f) \qquad\qquad V_{2*}(h) = \mathrm{div}(h_0) .$$

In our case of interest, the SU(3)-structure induces the further decomposition $\mathrm{Sym}^2_0 \cong \mathrm{Sym}^2_+ \oplus \mathrm{Sym}^2_-$ into traceless J-invariant and J-anti-invariant symmetric maps. Thus, we obtain a 3-dimensional family of suitable DeTurck vector fields. We only consider the trace and the J-invariant vector fields for order reasons.

Fix (ρ, σ) an SU(3)-structure. Using the isomorphisms from Lemma A.37, $\Lambda^4_1 \cong \mathbb{R}$ and $\mathrm{Sym}^2_+ \cong \Lambda^4_8$, it follows that there exists a universal constant A such that a variation of σ, $\delta\sigma = f\sigma + X \wedge \hat{\rho} + \chi_0$, the induced variation of the metric is given by

$$\delta g = \frac{1}{2} fg + A\iota(\chi_0) ,$$

where $\iota : \Omega^4_8 \to \mathrm{Sym}^2_+$ is the inverse map to the endomorphism action $S \mapsto S_*(\frac{\omega^2}{2})$. We need to compute $\mathrm{div}\left(\iota^{-1}(\chi_0)\right)$. We have the following lemma.

Lemma B.1. *Let $\chi \in \Lambda^4_8$. There is a universal constant B for which*

$$d\chi = B * \mathrm{div}\left[\iota(\chi_0)\right] + l.o.t ,$$

where $l.o.t$ is some 5-form depending smoothly on χ and the torsion of the SU(3)*-structure.*

Proof. Assume M carries a torsion-free SU(3)-structure (i.e. Calabi-Yau). Consider the diagram

$$\begin{array}{ccc} \mathrm{Sym}^2_+ TM \otimes \Lambda^1 & \xrightarrow{\iota \otimes 1} & \Lambda^4_8 \\ \Big\downarrow c & & \Big\downarrow * \circ \mathrm{Alt} \\ \Lambda^1 & \cdots\cdots\cdots & \Lambda^1 \end{array}$$

where Alt denotes skewsymmetrization and c denotes contraction by the metric. By Schur's lemma, there exists a universal constant B that completes the square, that is

$$* \mathrm{Alt}(\chi \otimes \alpha) = Bc\left(\iota^{-1}(\chi) \otimes \alpha\right) .$$

Now, we have $d(\chi) = \mathrm{Alt} \circ \nabla(\chi)$ and since ∇ preserves the Calabi-Yau structure, it commutes with the map ι, proving the statement for the torsion-free case. The torsion of the SU(3)-structure will modify the identity involving only zeroth-order terms. □

Lemma B.2. *Let (ρ, σ) be an* SU(3)*-structure. Then the DeTurck procedure outlined above allows us to construct two vector fields, $W_1(\rho, \sigma)$ and $W_2(\rho, \sigma)$, depending smoothly on the* SU(3)*-structure, whose linearisation along a variation $\delta\sigma = \chi = f\sigma + X \wedge \hat{\rho} + \chi_0$ is given by*

$$W_{1*}(\chi) = df \qquad\qquad W_{2*}(\chi) = *d\chi_0 + l.o.t\ .$$

We have rescaled our vector fields to eliminate all the constants and lighten the notation. Since they are universal, there are no ambiguities in us doing so. If we restrict ourselves to SU(3)-structures where the 4-form $\sigma = \frac{\omega^2}{2}$ is closed, we can further rewrite our DeTurck fields.

Proposition B.3. *Let (ρ, σ) be a* SU(3)*-structure such that $d\sigma = 0$. Then, the DeTurck fields can be chosen, so that*

$$V_{1*}(\chi) = df \qquad\qquad V_{2*}(\chi) = \mathrm{curl}(X) + l.o.t\ .$$

Proof. We need to prove that a linear combination of W_{1*} and W_{2*} is equal to V_{2*}, up to zeroth order terms. Linearising the condition $d\sigma = 0$, we have

$$\begin{aligned} d\chi &= df \wedge \sigma - dJX \wedge \rho + d\chi_0 \\ &= df \wedge \sigma - \mathrm{curl}(X) \wedge \sigma + d\chi_0 + l.o.t = 0\ , \end{aligned}$$

by Lemma A.48. Take $V_2 = W_2 + W_1$. □

B.2 The nearly Kähler Laplacian flow

Let us study the parabolicity of the nearly Kähler Laplacian flow (142), modified by the DeTurck term:

$$\begin{cases} \partial_t\sigma = \Delta_\sigma\sigma - 12\sigma + \mathcal{L}_{V(\sigma)}\sigma \\ d\sigma = 0 \\ \sigma(0) = \sigma_0\ , \end{cases} \tag{142}$$

for $V(\sigma) = 3V_1(\sigma) + 2V_2(\sigma)$, with V_i given in Proposition B.3.

We compute the linearisation of $P = \Delta_\sigma\sigma - 12\sigma + \mathcal{L}_{V(\sigma)}\sigma$ along $\chi = f\sigma + X \wedge \hat{\rho} + \chi_0 \in \Omega^4_{closed}$. Since σ is closed, $\Delta_\sigma\sigma = -d * d * \sigma$ and $\mathcal{L}_V\sigma = d(V \lrcorner \sigma)$ and so

$$D_\sigma P(\chi) = -d * d\mathcal{K}\chi + d(V_*(\chi) \lrcorner \sigma) = -d * d(2f\omega + X \lrcorner \rho - *\chi_0) + d\left((3V_{1*} - 2V_{2*}) \lrcorner \sigma\right)\ .$$

Similarly, we can compute $\Delta_\sigma\chi = dd^*\chi = -d * d(f\omega + X \lrcorner \rho + *\chi_0)$. By Lemma A.51, we have

$$\begin{aligned} D_\sigma P(\chi) + \Delta_\sigma\chi &= -d * d(3f\omega + 2X \lrcorner \rho) + d\left((3V_{1*} - 2V_{2*}) \lrcorner \sigma\right) \\ &= -d * \left[3df \wedge \omega + 2\left(\mathrm{curl}(X) \wedge \omega - d^*X\rho - d^*(JX)\hat{\rho} + d^*(JX \wedge \hat{\rho})\right]\right. \\ &\quad + d\left(3Jdf - 2\mathrm{curl}(X)\right) \wedge \omega + l.o.t. \\ &= -d\left[(3Jdf + 2J\mathrm{curl}(X)) \wedge \omega - 2d^*X\hat{\rho} + 2d^*(JX)\rho\right] + d\left(3Jdf - 2J\mathrm{curl}(X)\right) \wedge \omega + l.o.t. \\ &= 2\left[dd^*X - Jdd^*(JX)\right] \wedge \hat{\rho} + l.o.t\ . \end{aligned}$$

In particular, we have proved

Proposition B.4. *The linearization of the operator $P = \Delta_\sigma \sigma + \mathcal{L}_V(\sigma)\sigma$ along a closed 4-form $\chi = f\sigma + X \wedge \hat{\rho} + \chi_0$ is given by*

$$D_\sigma P(\chi) = -\Delta_\sigma \chi + 2(dd^* X - Jdd^* JX) \wedge \hat{\rho} + dF(\chi)$$

where $F(\chi)$ is a 3-form-valued algebraic function of χ that depends on the torsion of the SU(3)*-structure. In particular, its principal symbol in the direction ξ satisfies*

$$\langle \mathrm{S}_\xi(D_\sigma P)(\chi), \chi \rangle = -|\xi|^2 |\chi|^2 + 4\left(\langle \xi, X \rangle^2 + \langle \xi, JX \rangle^2\right) ,$$

which is not coercive, so the flow is not parabolic.

Proof. Only the symbol computation remains. Since we know that $\mathrm{S}_\xi(d) = \xi \wedge$ and $\mathrm{S}_\xi(d^*) = \xi \lrcorner$, the computation follows from the identity $\langle X \wedge \hat{\rho}, Y \wedge \hat{\rho} \rangle = 2\langle X, Y \rangle$. □

The term $dd^* X - Jdd^*(JX)$ cannot be reabsorbed by an additional term of the shape $\mathcal{L}_{W(\sigma)}\sigma$ for a different choice of field $W(\sigma)$. Indeed, the linearized operator for $W(\sigma) \lrcorner \sigma$ along $\chi = f\sigma + X \wedge \hat{\rho} + \chi_0$ will be a linear combination of $\mathrm{curl}(X)$, $\mathrm{curl}(JX)$, df and Jdf, plus lower order terms.

One could attempt to modify the flow further to make it elliptic, following Grigorian's construction of the modified G_2 Laplacian coflow [Gri13]. The idea is to construct second-order operators depending on σ, whose linearisation cancels out the terms $dd^* X$ and $Jdd^*(JX)$. In that direction, we have a first partial result.

Recall that $\tau_0(\sigma) = \frac{1}{3} * (d\omega \wedge \hat{\rho})$ is the 1-dimensional part of the torsion of σ, and it satisfies

$$\frac{1}{3}\sigma \wedge \omega = \frac{\tau_0^{-2}}{4} d\omega \wedge \widehat{d\omega} . \tag{143}$$

Lemma B.5. *The first order variation along $\chi = f\sigma + X \wedge \hat{\rho} + \chi_0$ of τ_0 is given by*

$$\partial_\chi \tau_0 = -\frac{1}{2} d^* X + l.o.t .$$

Proof. We differentiate Equation (143) with respect to χ:

$$\begin{aligned}(\delta_\chi \tau_0)\rho \wedge \hat{\rho} &= d(\delta_\chi \omega) \wedge \hat{\rho} + l.o.t. = d(\delta_\chi \omega \wedge \hat{\rho}) + l.o.t \\ &= d(X \lrcorner \rho \wedge \hat{\rho}) + l.o.t = -\frac{1}{2}(d^* X)\rho \wedge \hat{\rho} + l.o.t .\end{aligned}$$ □

We can introduce a first modification to the flow to remove one of the positive terms in the symbol.

Corollary B.6. *For $C \in \mathbb{R}$, consider the flow for $\sigma \in \Omega^4(M)$*

$$\begin{cases} \partial_t \sigma = \Delta_\sigma \sigma - 12\sigma + \mathcal{L}_{V(\sigma)}\sigma + d\left[(4\tau_0 + C)\hat{\rho}\right] \\ d\sigma = 0 \\ \sigma(0) = \sigma_0 , \end{cases} \tag{144}$$

with $V(\sigma)$ as before and $\hat{\rho}$ the associated 3-form as usual. The principal symbol of this flow satisfies

$$\langle \mathrm{S}_\xi(D_\sigma P)(\chi), \xi \rangle = -|\xi|^2 \chi + 4\langle \xi, JX \rangle^2 .$$

The question arises of whether we can further modify the flow (144) to obtain a parabolic flow.

C The Einstein–Hilbert action

Recall that Einstein metrics, solving the differential equation $\mathrm{Ric}_g = \lambda g$, are critical points of the Einstein–Hilbert action:

$$\begin{aligned} \mathcal{S} : \mathrm{Met}(M^n) &\to \mathbb{R} \\ g &\mapsto \frac{1}{n-1}\int_M s_g - \lambda(n-2)\,\mathrm{dvol}_g\ , \end{aligned} \tag{145}$$

where $\mathrm{Met}(M^n)$ is the space of metrics on M^n for $n \geq 2$, and s_g is the scalar curvature of the metric g.

Since nearly Kähler and nearly parallel G_2 manifolds are the links of Ricci-flat cones, they are Einstein for $\lambda = n-1$. In particular, they are critical points of the Einstein–Hilbert action. We investigate the relation between the second variation of the Hitchin functionals and the Einstein–Hilbert functional.

For the remainder of the section, assume (M^n, g) is not isometric to the round sphere. At a point $g \in \mathrm{Met}(M)$, we identify the tangent space of $\mathrm{Met}(M)$ with symmetric 2-tensors $\Gamma\left(\mathrm{Sym}^2(T^*M)\right)$. As in the case of the Hitchin functionals, the functional $\mathcal{S}$ is diffeomorphism invariant. Thus, it is convenient to study variations orthogonal to the diffeomorphism orbit. We have an L^2-orthogonal decomposition:

$$\Gamma\left(\mathrm{Sym}^2(T^*M)\right) = \mathbb{R}g\ \oplus \mathcal{C}_0^\infty(M)g\ \oplus \Gamma(TM) \oplus TT;$$

where the first and second terms correspond to constant rescalings and infinitesimal conformal deformations of the metric, respectively. The identification of the component $\Gamma(TM)$ in $\mathrm{Sym}^2(TM)$ is given by the map $X \mapsto \mathcal{L}_X g$ and corresponds to the orbit of the diffeomorphism group. The term TT corresponds to the traceless and transverse symmetric 2-tensors:

$$TT(M,g) = \left\{ h \in \Gamma(\mathrm{Sym}^2(T^*M)|\ \mathrm{tr}(h) = 0,\ \ \mathrm{div}(h) = -\sum_i e_i \lrcorner \nabla_{e_i} h = 0 \right\}\ .$$

By Ebin's slice theorem, this formal complement to the orbit of the diffeomorphism group is the tangent space to a genuine slice of the diffeomorphism orbit in a given conformal class.

Theorem C.1. *[Koi79, Thm 2.4 & Thm. 2.5] Let (M^n, g) be an Einstein metric with constant λ. Then, when restricted to conformal variations, the second variation is given by*

$$\delta^2\mathcal{S}_g(f,f') = \frac{n-2}{2}\int_M \left\langle \Delta f - \frac{n\lambda}{n-1} f, f' \right\rangle \mathrm{dvol}\ . \tag{146}$$

When restricted to tt-tensors, it is given by

$$\delta^2\mathcal{S}_g(h,h') = -\frac{1}{n-1}\int_M \langle \Delta_L h - 2\lambda h, h'\rangle\,\mathrm{dvol} = -\frac{1}{n-1}\left\langle \mathcal{Q}(h), h'\right\rangle_{L^2}\ . \tag{147}$$

for $h, h' \in TT(M,g)$.

If $\lambda = n-1$, the operator $(\Delta - n)f$ is strictly positive for $f \in \mathcal{C}_0^\infty(M)$, by Obata's theorem [Oba62]. The term $\Delta_L h$ is the Lichnerowicz Laplacian

$$\Delta_L = \nabla^*\nabla + q(\mathcal{R})\ ,$$

where $q(\mathcal{R}) = \sum_{i<j}(e_i \wedge e_j)_* \left(\mathcal{R}(e_i, e_j)\right)_*$ is the standard curvature endomorphism induced by the Riemannian curvature tensor $\mathcal{R}$. One defines the co-index of an Einstein metric as the maximal subspace along which $\mathcal{S}_g|_{TT}$ is positive definite. Since the operator $\Delta_L h - 2\lambda h$ is a strongly elliptic operator, the co-index is guaranteed to be finite.

Let us study the Einstein co-index of nearly Kähler and nearly parallel G_2-structures.

Nearly Kähler manifolds

We consider the case where (M, g) is a nearly Kähler manifold. Since its metric cone is Ricci-flat, the metric g is Einstein with $\lambda = 5$. By Lemma A.37, we have an isomorphism

$$\begin{aligned}\Phi : \mathrm{Sym}^2_0 T^*M &\to \Omega^2_8 \oplus \Omega^3_{12} \\ h = (h^+, h^-) &\mapsto \left(h^+_*(\omega), h^-_*(\rho)\right) ,\end{aligned}$$

with $h^\pm = 1/2\,(h \pm JhJ)$ the J-commuting and J-anti-commuting parts of a traceless symmetric 2-tensor. Thus, $\overline{\Delta} = \Phi \circ \Delta_L \circ \Phi^{-1}$ is a Laplacian-type operator on $\Omega^2_8 \oplus \Omega^3_{12}$.

The key result, due to Moroianu and Semmelmann [MS11, Section 5] (cf. [Sch22]), allows us to transform the eigenvalue problem for $\mathcal{S}_g|_{TT}$ to an eigenvalue problem for the Laplacian on forms:

Proposition C.2 ([Sch22, Lemma 3.1])**.** *Let (M^6, ω, ρ) be a nearly Kähler manifold, not isometric to the round sphere. For $\lambda < 16$, the operator $\mathcal{Q}(h) = \lambda h$ for $h \in TT(M)$ is identified via the map Φ to*

$$\begin{aligned}\Delta\beta + 4\beta + d^*\gamma &= \lambda\beta \\ \Delta\gamma + 6\gamma + 4d\beta &= \lambda\gamma ,\end{aligned}$$

with $(\beta, \gamma) \in \Omega_{TT} = \Omega^2_{8,coclosed} \times \Omega^3_{12,closed}$.

Now, the operator above commutes with the Laplacian acting on $\Omega^2_{8,coclosed}$, so they admit a common basis of eigenvectors. It is a linear algebra problem to compare the eigenvalues of the two operators, which Schwahn carried out in [Sch22]:

Proposition C.3 ([Sch22, Lemma 3.2])**.** *Let (M^6, ω, ρ) be a nearly Kähler manifold, not isometric to the round sphere. Consider the eigenspaces*

$$\mathcal{E}(\lambda) = \left\{\beta \in \Omega^2_8 \mid d^*\beta = 0\ ,\ \Delta\beta = \lambda\beta\right\} .$$

The Einstein index of (M, g) is given by

$$\mathrm{Ind}^{EH} = b^2(M) + b^3(M) + 3\sum_{\lambda\in(0,2)} \dim \mathcal{E}(\lambda) + 2\sum_{\lambda\in(2,6)} \dim \mathcal{E}(\lambda) + \sum_{\lambda\in(6,12)} \dim \mathcal{E}(\lambda) . \quad (148)$$

By comparing this formula with Equation (74), we immediately have

Corollary C.4. *The Einstein co-index is bounded below by the Hitchin index.*

Nearly parallel G_2 manifolds

We now consider the case where (M, g) is a nearly parallel G_2 manifold. Since its metric cone is Ricci-flat, the metric g is Einstein with $\lambda = 6$. By Lemma A.16 we have an isomorphism $\Phi : \Gamma\left(\mathrm{Sym}^2_0\right) \to \Omega^3_{27}$ given by $S \mapsto S_*\varphi$. The Laplacian comparison formula needed in this case is due to Alexandrov and Semmelmann.

Proposition C.5 ([AS12, Prop. 6.1])**.** *Under the map Φ, the operator*

$$\mathcal{Q}(h) = \Delta_L h - 12h$$

on $h \in TT(M)$ is identified with

$$\widehat{\mathcal{Q}} = \Delta\gamma + 2 * d\gamma - 8\gamma \tag{149}$$

acting on $\Omega_{TT} = \{\gamma \in \Omega^3_{27} |\ \pi_7(d\gamma) = 0\}$.

The proof strategy is the same as that of nearly Kähler structures. Let us study the eigenvalue problem for $\widehat{\mathcal{Q}}$.

Proposition C.6. *Let (M^7, g, φ) be a nearly parallel G_2 manifold. Consider the eigenspaces*

$$\mathcal{E}(\lambda) = \left\{\gamma \in \Omega^3_{27} |\ * d\gamma = \lambda\gamma\right\} \qquad \mathcal{F}(\lambda) = \left\{\gamma \in \Omega^3_{27} |\ dd^*\gamma = \lambda\gamma\right\}\ .$$

The Einstein index of (M, g) is given by

$$\mathrm{Ind}^{EH} = b^3(M) + \sum_{\lambda \in (-4,0)\cup(0,2)} \dim\, \mathcal{E}(\lambda) + \sum_{\lambda \in (0,8)} \dim\, \mathcal{F}(\lambda)\ . \tag{150}$$

Proof. The operator $\widehat{\mathcal{Q}}$ commutes with the self-dual operator $*d$ on Ω^3_{27}. Thus, we can find a common base of eigenforms. Let $\mu \in \mathbb{R}$ and consider the spaces $\mathcal{E}(\mu)$ and $\mathcal{F}(\mu)$ defined above. If $\gamma \in \mathcal{E}(\mu)$ for $\mu \neq 0$, we have $*d\gamma = \mu\gamma$, and substituting in Equation (149), we have that γ is an eigenform of $\widehat{\mathcal{Q}}$ with eigenvalue $\lambda = \mu^2 + 2\mu - 8$. If $\mu = 0$, γ is closed, and Equation (149) reduces to $\Delta\gamma = dd^*\gamma = (\lambda + 8)\gamma$, which concludes the proof of Equation (150). □

Remark C.7. *The purely topological bound* $\mathrm{Ind}^{EH} \geq b^3(M)$ *appeared in [SWW22].*

By comparing this formula with Corollary 8.16, we immediately have

Corollary C.8. *The Einstein co-index is bounded below by the Hitchin index.*

D Taylor expansions for cohomogeneity one nearly Kähler metrics

Lemma D.1 ([FH17]). *The first few terms of the Taylor expansion of* Ψ_a *are*

$$\begin{aligned}
\lambda(t) =& \frac{3}{2}t - \frac{2a^2+3}{12a^2}t^3 + O(t^5)\ ,\\
\mu(t) =& \sqrt{3}at + \frac{\sqrt{3}}{9a}(3-7a^2)t^3 + O(t^5)\ ,\\
u_0(t) =& a^2 - 3a^2t^2 + O(t^4)\ ,\\
u_1(t) =& a^2 - \frac{3}{2}(2a^2-1)t^2 + O(t^4)\ ,\\
u_2(t) =& -\frac{3\sqrt{3}}{2}at^2 + \frac{\sqrt{3}(16a^2-3)}{12a}t^4 + O(t^6)\ ,\\
v_0(t) =& 3a^2t^2 - \left(\frac{1}{4} + \frac{14}{3}a^2\right)t^4 + O(t^6)\ ,\\
v_1(t) =& 3a^2t^2 + \left(2 - \frac{14}{3}a^2\right)t^4 + O(t^6)\ ,\\
v_2(t) =& \frac{3\sqrt{3}}{2}at^2 - \frac{\sqrt{3}(34a^2-3)}{12a}t^4 + O(t^6)\ ,\\
w_0(t) =& \frac{\sqrt{3}}{3}at^{-1} - \frac{\sqrt{3}}{54a}\left(64a^2-39\right)t + O(t^3)\ ,\\
w_1(t) =& \frac{\sqrt{3}}{3}at^{-1} - \frac{2\sqrt{3}}{27a}\left(16a^2-3\right)t + O(t^3)\ ,\\
w_2(t) =& \frac{1}{2}t + \frac{9-76a^2}{54a^2}t^3 + O(t^5)\ .
\end{aligned}$$

Lemma D.2 ([FH17]). *The first few terms of the Taylor expansion of* Ψ_b *are*

$$\begin{aligned}
\lambda(t) =& b - \frac{9}{10}\frac{b^2-1}{b}t^2 + O(t^4)\ ,\\
\mu(t) =& 2bt + \frac{1}{10b}t^3 + O(t^5)\ ,\\
u_0(t) =& 2b^2t - \frac{1}{5}(17b^2+3)t^3 + O(t^5)\ ,\\
u_1(t) =& 2bt - \frac{23b^2-3}{5b}t^3 + O(t^5)\ ,\\
u_2(t) =& -2b^2t + \frac{1}{5}(17b^2-12)t^3 + O(t^5)\ ,\\
v_0(t) =& -\frac{2}{3}b^3 + 4b^3t^2 + O(t^4)\ ,\\
v_1(t) =& 4b^2t^2 + \frac{2}{5}t^4 + O(t^6)\ ,\\
v_2(t) =& \frac{2}{3}b^3 - b(4b^2-3)t^2 + O(t^4)\ ,\\
w_0(t) =& \frac{b}{3}t^{-1} - \frac{16b^2-29}{15b}t + O(t^3)\ ,\\
w_1(t) =& t + O(t^3)\ ,\\
w_2(t) =& -\frac{b}{3}t^{-1} + \frac{32b^2-13}{30b}t + O(t^3)\ .
\end{aligned}$$

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