EXACT SOLUTIONS FOR VORTEX EQUILIBRIA BY CONFORMAL MAPPING

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ABSTRACT

Conformal mapping is used to find exact, closed-form solutions for three classes of vortex sheet rotating equilibria. The first involves multi-sheet equilibria of the Protas-Sakajo class: N-fold symmetric equilibria consisting of multiple sheets stemming from a common origin. Conformal mapping of the exterior of the vortex structure to the exterior of the unit disk enables the solution construction using Fourier series. The solutions describe both the stream function field and the circulation density along the sheets and are found for $N = 2, 3$, and 4.

The approach is effective in reproducing equilibria of a second class due to O'Neil: a single, straight sheet in the presence of one or more point vortices. Finally, the method is used to construct new equilibrium families sharing features of both Protas-Sakajo and O'Neil classes. That is, a $N = 4$ Protas-Sakajo equilibria together with four point vortices located unit distance from the origin either (i) off each sheet tip, or (ii), on the bisector of the sheets. Members of each family are determined by a parameter γ measuring the total circulation of the sheets. For given γ, equilbria properties are determined numerical solution of a nonlinear algebraic equation. In case (ii), a non-rotating stationary equilibria is found.

1 Introduction

The search for equilibrium vortex structures in two-dimensional flow of an inviscid fluid has a rich history, especially in the case where the vorticity is confined to singular points, a problem which continues to generate interest, e.g. [1, 2, 3]. The next level of sophistication in which the vorticity is nonzero along two-dimensional curves – vortex sheets – has yielded far fewer exact equilibrium solutions owing to the dual challenge of finding both the vorticity distribution (or, equivalently, the circulation density) along the sheet and the shape of the sheet itself.

Recently [4] constructed a family of equilibria rotating with constant angular velocity consisting of an even number $N = 2, 4, \cdots$, of straight vortex sheets emanating from a common center of rotation and with endpoints at the vertices of a regular N-polygon. The mathematical problem is formulated using the Birkoff-Rott equation with Riemann-Hilbert methods used to derive an integral formula for the circulation density for $N = 2, 4, \cdots$. The integral formula is evaluated exactly for the case $N = 2$ recovering the classic single-sheet equilibrium solution [5, 6], and for the case $N = 4$ where they find a closed-form expression for the circulation density of a four-sheet equilibria. For $N > 4$, an integral formula for the circulation density along a sheet is derived and a numerical method is presented for its evaluation.

Another relatively recent approach to finding exact rotating vortex sheet equilibria is that used by [7] requiring the presence of both point vortices and a straight sheet. Complex analytic methods are used to formulate the problem as a quadratic differential from which, for given vortex positions satisfying certain symmetry conditions, the length and endpoints of the sheet and the circulation density along it are determined.

This work proposes an alternative approach to finding equilibria of both the Protas-Sakajo class, including the previously unexplored case of an odd number of sheets stemming from the origin, and the O'Neil class. The emphasis is on describing and demonstrating the method by producing previously known solutions, as well as some new solutions. The method, described in §2, conformally maps the exterior of the vortex sheet structure in the rotating frame to the exterior of the unit circle in a mathematical plane where the stream function is solved for using Fourier series subject to the boundary condition on the unit circle. The series is then summed explicitly to obtain closed-form solutions. Sections 3 and 4 illustrate the method by obtaining previously known Protas-Sakajo solutions for the circulation density in the cases $N = 2$ and $N = 4$. A new solution for the $N = 3$ case is presented in §5. Explicit, closed-form expressions for the stream function field are obtained in all these cases. The same approach is applied in §6 to construct rotating equilibria of the O'Neil class. Specifically, exact solutions for the stream function and circulation density along a single sheet are found for two examples: a single sheet and a point vortex where the vortex lies on the symmetry line of the sheet and a single sheet of arbitrary circulation and two point vortices located on the symmetry line off each tip of the sheet. The exact solutions for the along-sheet circulation densities take a different form to that of [7], but numerical evidence is presented which suggests they are the equivalent. New exact solution families for hybrid four-fold symmetric sheet and point vortex equilibria is obtained in §7. Remarks on generalisation, possible extensions and further applications of the method are given in §8.

2 General formulation and conformal mapping

Suppose a multi-sheet vortex equilibria comprising of N-'spokes', $N = 2, 3, \cdots$, rotates with angular velocity Ω. The spokes, or sheets, have the same length L and are equally spaced about a common hub coinciding with the centre of rotation $z = 0$, so that the structure has N-fold symmetry. One of the spokes is always taken to lie along $z = x + i0$, $x \in [0, L]$. In §§2–5 the problem is non-dimensionalised so that $\Omega = 1$ and $L = 1$.

Let the unbounded region exterior to the N-sheet structure be D_N, $N = 1, 2, 3, \cdots$. The conformal map from the exterior of unit ζ-circle to D_N can be obtained by first mapping to the upper half of the w-plane using $w = \tan(N^{-1}\cos^{-1}(z^{N/2}))$ [8] (page 79), and then using the Möbius map $\zeta = (1 - iw)/(1 + iw)$ to map to the exterior of the ζ-circle giving

$$\zeta = \exp\left(-\frac{2i}{N}\cos^{-1}(z^{N/2})\right). \tag{1}$$

The inverse of (1) is

$$z = \left[\cos\left(\frac{iN}{2}\log\zeta\right)\right]^{2/N}. \tag{2}$$

In a stationary frame, the stream function $\psi(x, y)$ is a real-valued harmonic function outside the sheet. Here the convention $u = -\psi_y$ and $v = \psi_x$ is used, so that the vorticity is $\nabla^2\psi$. In a frame rotating with angular velocity $\Omega = 1$ the spokes coincide with streamlines along which, without loss of generality, $\psi = 0$ ensuring the normal fluid velocity on the sheets vanishes. In this frame, the stream function satisfies Poisson's equation $\nabla^2\psi = -2$, and can be expressed in general as

$$\psi = -\frac{1}{2}|z|^2 + \alpha\log|\zeta| + a_0 + \mathrm{Re}\sum_{n=1}^{\infty} a_n\zeta^{-Nn}, \tag{3}$$

where α and a_0 are real, and the series representation preserves the N-fold symmetry. The first term on the RHS of (3) is the contribution of the background rotation, and accounts for the inhomogeneous term in Poisson's equation. Note that α is a measure of the total circulation of the vortex structure. Note also that ζ^{-Nn} is an analytic function in D_N so its real part is harmonic. The harmonic terms satisfy Laplace's equation which is conformally invariant.

For given N, the task is to find the unknown coefficients α and a_n, $n = 0, 1, 2\cdots$ subject to $\psi = 0$ on the sheets ∂D_N, and a Kutta-type condition that the velocity field in D_N is bounded at the tips of the sheets i.e. $|\nabla\psi| < \infty$ as $z \to \exp(2\pi in/N)$, $n = 0, 1, \cdots, N - 1$. This condition need only be enforced at one of the tips, since symmetry ensures it will then be satisfied at all tips. Symmetry also ensures that the average of the tangential velocities from either side of the sheet vanish as required for equilibrium. The resulting vortex equilibrium has a well-defined flow field everywhere including in the corner regions near the origin [4].

Once (3) is determined, the circulation density can be calculated by realising that along a spoke it is twice the tangential velocity jump across the spoke. Without loss of generality, consider the spoke along the real axis so that the circulation density is, by the anti-symmetry of the velocity field ψ_y across the sheet,

$$\rho(x) = -2u(x, 0^+) = 2\psi_y(x, 0^+), \quad x \in [0, 1], \tag{4}$$

where the sign is such that a positive sheet circulation locally induces an anticlockwise flow in a fixed reference frame.

The following results for the operators $\partial/\partial x$, $\partial/\partial y$ and their application to Re$f(\zeta)$, where $f(\zeta))$ is an analytic function are useful in the following sections:

$$\partial_x = \zeta_z\partial_\zeta + \overline{\zeta}_{\overline{z}}\partial_{\overline{\zeta}}, \quad \partial_y = i\left(\zeta_z\partial_\zeta - \overline{\zeta}_{\overline{z}}\partial_{\overline{\zeta}}\right), \tag{5}$$

so that

$$\partial_x \mathrm{Re}(f(\zeta)) = \mathrm{Re}(\zeta_z f_\zeta), \quad \partial_y \mathrm{Re}(f(\zeta)) = -\mathrm{Im}(\zeta_z f_\zeta). \tag{6}$$

In particular, the following results for constant a and $k \neq 0$ are used later

$$\partial_x \log|\zeta - a| = \mathrm{Re}\left(\frac{\zeta_z}{\zeta - a}\right), \quad \partial_y \log|\zeta - a| = -\mathrm{Im}\left(\frac{\zeta_z}{\zeta - a}\right), \tag{7}$$

and

$$\partial_x \mathrm{Re}\,\zeta^k = k\mathrm{Re}\left(\zeta_z\zeta^{k-1}\right), \quad \partial_y \mathrm{Re}\,\zeta^k = -k\mathrm{Im}\left(\zeta_z\zeta^{k-1}\right). \tag{8}$$

3 Single sheet $N = 2$

Let $N = 2$, so that the two spokes of the equilibria form one continuous straight sheet along $x \in [-1, 1]$, $y = 0$, corresponding to the well known case of a finite, straight sheet rotating equilibrium–see e.g. [5, 6, 9].

The maps to and from the exterior of the slit $[-1, 1]$ and the exterior of the unit ζ-circle are given by (1) and (2) with $N = 2$, reducing to the well-known maps

$$\zeta = z + \sqrt{z^2 - 1}, \quad z = \frac{1}{2}(\zeta + \zeta^{-1}). \tag{9}$$

On the sheet ∂D_2, $\zeta = \exp(i\theta)$, so that, from (9), $|z|^2 = \cos^2\theta = (\cos 2\theta + 1)/2$. Thus on ∂D_2, with $N = 2$ and realising $\log|\zeta| = 0$, (3) gives the Fourier series problem

$$a_0 + \mathrm{Re}\sum_{n=1}^{\infty} a_n\zeta^{-2n} = \frac{1}{4}(\cos 2\theta + 1), \tag{10}$$

from which by inspection $a_0 = 1/4$, $a_1 = 1/4$ and all other coefficients vanish. Hence the stream function is

$$\psi = -\frac{1}{2}|z|^2 + \alpha\log|\zeta| + \frac{1}{4} + \frac{1}{4}\mathrm{Re}\,\zeta^{-2}. \tag{11}$$

To determine α, the derivative ψ_y is evaluated on $y = 0$ with $x \to 1^-$ and α chosen to ensure the derivative remains bounded. In this $N = 2$ case, the analysis determining α can be carried directly in the z-plane after rewriting (11) purely in terms of z using (9). But here it is chosen to perform the calculation in the ζ-plane since this approach is more convenient in the $N = 3$ and 4 cases that follow.

Differentiating (9) gives

$$\zeta_z = \frac{i\zeta}{\sqrt{1 - z^2}}, \tag{12}$$

and using (7) it follows $\partial_y \log|\zeta| = -1/\sqrt{1 - x^2}$ on $z = x + 0i$, $|x| \leq 1$, and is singular at the tips as $x \to \pm 1$.

Differentiating the Re ζ^{-2} term in (11) using (8) and (12) gives

$$\partial_y \mathrm{Re}\,\zeta^{-2} = 2\mathrm{Im}\left(\zeta_z\zeta^{-3}\right) = 2\mathrm{Im}\left(\frac{i}{\zeta^2\sqrt{1 - z^2}}\right). \tag{13}$$

From (9) the limit $z = x \to 1^-$, corresponds to $\zeta \to 1$ and so (13) gives the singular behaviour $\partial_y \mathrm{Re}\,\zeta^{-2} \to 2/\sqrt{1 - x^2}$ as $x \to \pm 1$. Eliminating the two singular terms in (11) then gives $\alpha = 1/2$, so completing the derivation of the stream function. By explicit differentiation of (11) with respect to x, and using (7) and (8), it can be shown that $\psi_x = 0$ as $x \to 1^+$. That is, the sheet tips are stagnation points since both velocity components vanish. Figure 1 presents a plot of the streamlines. Note the total magnitude of the circulation of the sheet is $2\pi\alpha = \pi$. This argument does not require the branch of the square root to be specified, only that it is done consistently in the two singular terms.

Finally, to determine the circulation density $\rho(x)$ along the sheet requires that the tangential velocity ψ_y on one side of the sheet be calculated (which amounts to choosing a particular branch of any roots involved). On $y = 0$, $|x| \leq 1$ using (7) and (13) on (11) gives

$$\psi_y = -\frac{1}{2\sqrt{1 - x^2}} + \mathrm{Re}\left(\frac{1}{2\zeta^2\sqrt{1 - x^2}}\right). \tag{14}$$

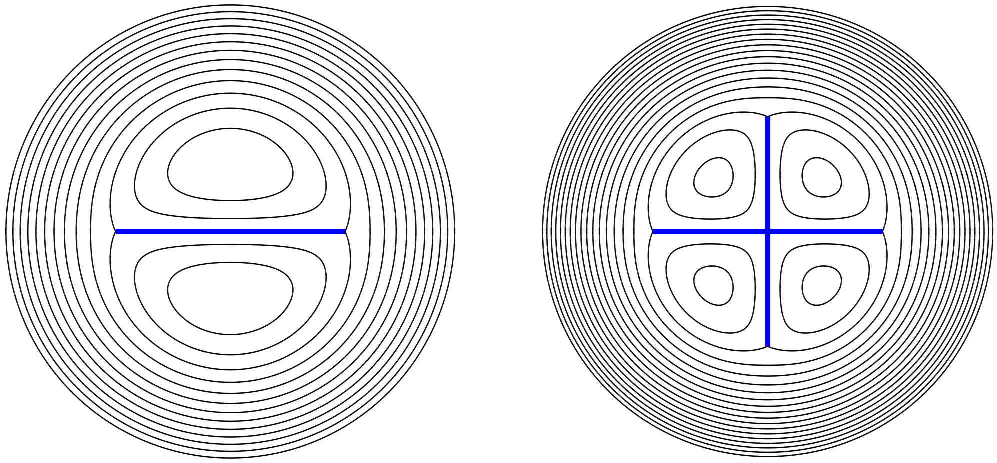

Figure 1: Streamlines (black) for the $N = 2$ and $N = 4$ equilibria. The vortex sheets (thick blue lines) have unit length from their centre of symmetry.

Using (9) (with $z = x$) in (14) and simplifying gives $\psi_y = -\sqrt{1-x^2}$, and so from (4) $\rho(x) = 2\psi_y = -2\sqrt{1-x^2}$ as in e.g. [9]. The negative sign outside the square root in $\rho(x)$ is unimportant: here the sheet circulation density must be everywhere positive so that the negative square root is chosen for $0 \le x \le 1$ implying $\rho(x) \ge 0$ leading to a self-driven anti-clockwise rotation of the sheet which precisely counteracts the imposed clockwise (negative) background flow represented by the $-|z|^2/2$ term in the stream function (11).

4 The $N = 4$ equilibria

The four spokes are arranged such that their tips are located at ± 1 and $\pm i$. The derivation in this section recovers the exact solution for the circulation density $\rho(x) \sim x \cosh^{-1}(1/\sqrt{1-x^4})$ found in [4], while additionally finding the stream function for the flow in D_4.

From (1) and (2), the maps between D_4 and the exterior of the unit ζ-circle are

$$\zeta = \exp\left(-\frac{i}{2}\cos^{-1}\left(z^2\right)\right), \quad z^2 = \cos(2i\log\zeta). \tag{15}$$

Analogous to the $N = 2$ case, it is necessary to find a Fourier series representation of $|z|^2/2$ on ∂D_4. From (15), $|z|^2 = |\cos 2\theta|$ which has 2π-periodic Fourier series

$$|\cos 2\theta| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos 4n\theta. \tag{16}$$

The stream function satisfying $\psi = 0$ on ∂D_4 is then

$$\psi = -\frac{1}{2}|z|^2 + \alpha\log|\zeta| + \frac{1}{\pi} + \frac{2}{\pi}\mathrm{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\zeta^{-4n}. \tag{17}$$

By expressing as partial fractions, the series on the RHS of (17) can be summed explicitly giving

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\zeta^{-4n} = -1 + \left(\zeta^2+\zeta^{-2}\right)\tan^{-1}\left(\zeta^{-2}\right), \tag{18}$$

and so

$$\psi = -\frac{1}{2}|z|^2 + \alpha\log|\zeta| + \frac{1}{\pi}\text{Re}\left[\left(\zeta^2+\zeta^{-2}\right)\tan^{-1}\left(\zeta^{-2}\right)\right]. \tag{19}$$

It is possible to express the stream function (19) purely in terms of z (and its conjugate), but the subsequent analysis is most easily done in its present 'hybrid' form involving both z and ζ variables.

To determine α and $\rho(x)$ the velocity field ψ_y along the sheet is calculated, using the operator (5) and the general derivative results (7) and (8). From (15), the result $\zeta_z/\zeta = iz/\sqrt{1-z^4}$ is helpful in what follows. Differentiating (19) and evaluating it on the sheet aligned with the positive real axis gives

$$\begin{aligned}\psi_y|_{y=0,x\in[0,1]} &= -\frac{\alpha x}{\sqrt{1-x^4}} - \frac{1}{\pi}\text{Im}\left[\zeta_z\left((2\zeta-2\zeta^{-3})\tan^{-1}(\zeta^{-2}) + (\zeta^2+\zeta^{-2})\frac{(-2\zeta^{-3})}{1+\zeta^{-4}}\right)\right],\\ &= -\frac{\alpha x}{\sqrt{1-x^4}} - \frac{2}{\pi}\text{Im}\left[\frac{\zeta_z}{\zeta}\left((\zeta^2-\zeta^{-2})\tan^{-1}(\zeta^{-2})-1\right)\right],\\ &= -\frac{\alpha x}{\sqrt{1-x^4}} - \frac{2}{\pi}\text{Im}\left[\frac{ix}{\sqrt{1-x^4}}\left((\zeta^2-\zeta^{-2})\tan^{-1}(\zeta^{-2})-1\right)\right],\end{aligned} \tag{20}$$

where $\zeta^2 = \exp[-i\cos^{-1}(x^2)]$ in (20). Since $\zeta \to 1$ as $z \to 1^-$, the singularity in ψ_y at the sheet tip $z = 1$ implied by (20) can be eliminated by the choice $\alpha = 2/\pi$. Equation (19) now completely determines ψ in D_4. Figure 1 shows a plot of the streamlines. The total magnitude of circulation of the 4-spoke equilibria is $2\pi\alpha = 4$.

The circulation density on $y = 0$, $x \in [0, 1]$, is then, from (20),

$$\rho(x) = 2\psi_y = -\frac{4}{\pi}\text{Im}\left[\frac{ix}{\sqrt{1-x^4}}\left(\zeta^2-\zeta^{-2}\right)\tan^{-1}(\zeta^{-2})\right]. \tag{21}$$

Now $\zeta^2 - \zeta^{-2} = -2i\sin(\cos^{-1}x^2) = -2i\sqrt{1-x^4}$ and substituting into (21), gives

$$\begin{aligned}\rho(x) &= -\frac{8}{\pi}x\,\text{Im}\left[\tan^{-1}(\zeta^{-2})\right],\\ &= -\frac{8}{\pi}x\,\text{Im}\left[\tan^{-1}(x^2+i\sqrt{1-x^4})\right],\\ &= -\frac{4}{\pi}x\coth^{-1}(1/\sqrt{1-x^4}).\end{aligned} \tag{22}$$

Aside from the unimportant negative sign in (22) which has been previously remarked on in §3, the result (22) differs from [4]'s equation (28b) by a factor $2/\pi$; the factor π in the denominator should appear in their result, and the factor two is needed for the choice $\Omega = 1$. Note that in (22), it is possible to express $\coth^{-1}(1/\sqrt{1-x^4}) = \tanh^{-1}(\sqrt{1-x^4})$, but the former is chosen to make the comparison of (22) to [4] obvious.

5 The $N = 3$ equilibria

This equilibria has been found numerically in [10]. Its analytical description is given here following the procedure used in the $N = 2$ and $N = 4$ cases.

From (1) and (2), the maps between D_3 and the exterior of the unit ζ-circle are

$$\zeta = \exp\left(-\frac{2i}{3}\cos^{-1}\left(z^{3/2}\right)\right), \quad z = \left(\cos\left(\frac{3i}{2}\log\zeta\right)\right)^{2/3}. \tag{23}$$

The Fourier series representation on ∂D_3 of $|z|^2 = |\cos(3\theta/2)|^{4/3}$ can be found using e.g. *Mathematica* and is

$$|\cos(3\theta/2)|^{4/3} = \frac{3\Gamma(7/6)}{\sqrt{\pi}\Gamma(2/3)}\left(\frac{1}{2} - \frac{2}{5}\sum_{n=1}^{\infty}(-1)^n\frac{(1/3)_{n-1}}{(8/3)_{n-1}}\cos 3n\theta\right), \tag{24}$$

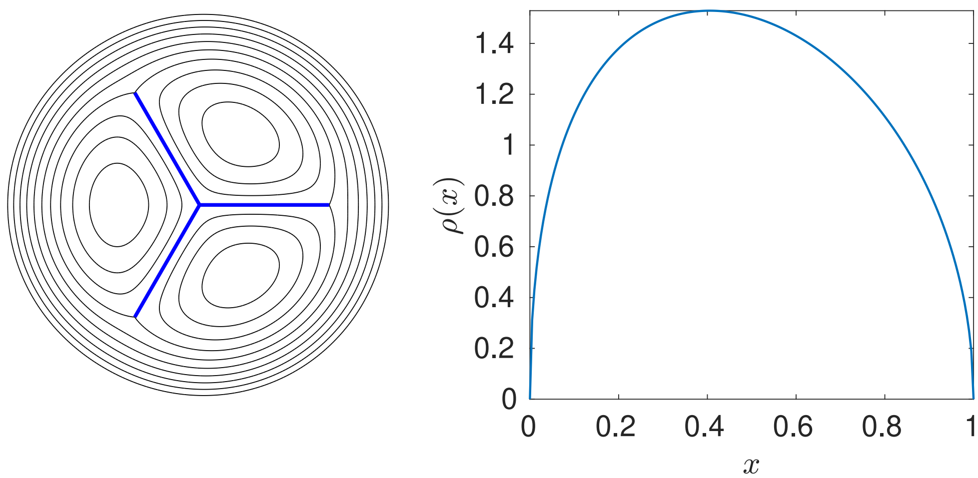

Figure 2: Streamlines (black) and unit length vortex sheets (thick blue lines) for the $N = 3$ equilibria (left) and corresponding circulation density $\rho(x)$ along the sheet $y = 0$, $x \in [0, 1]$ (right).

where $\Gamma()$ is the Gamma function and $(x)_n = x(x+1)\cdots(x+n-1)$, where $n = 1, 2, \cdots$, and $(x)_0 = 1$, is the Pochhammer symbol.

Using (3) and (24), the stream function is

$$\psi = -\frac{1}{2}|z|^2 + \alpha \log|\zeta| + \frac{3\Gamma(7/6)}{4\sqrt{\pi}\Gamma(2/3)} - \frac{3\Gamma(7/6)}{5\sqrt{\pi}\Gamma(2/3)}\mathrm{Re}\sum_{n=1}^{\infty}(-1)^n\frac{(1/3)_{n-1}}{(8/3)_{n-1}}\zeta^{-3n},$$
$$= -\frac{1}{2}|z|^2 + \alpha \log|\zeta| + \frac{3\Gamma(7/6)}{4\sqrt{\pi}\Gamma(2/3)} + \frac{3\Gamma(7/6)}{5\sqrt{\pi}\Gamma(2/3)}\mathrm{Re}\left[\frac{1}{\zeta^3}F\left(\frac{1}{3}, 1; \frac{8}{3}; -\frac{1}{\zeta^3}\right)\right], \quad (25)$$

where the sum has been expressed in terms of the hypergeometric function F [11].

As before, α is determined by examining the behaviour of singular terms of ψ_y as $z = x \to 1^-$. Using (7) and (8) in the limit that the tip $z = x = 1$ is approached along the sheet

$$\partial_y \alpha \log|\zeta| \to \frac{-\alpha}{\sqrt{1-x^3}}, \quad \text{and} \quad \partial_y \mathrm{Re}\,\zeta^{-3n} \to \frac{3n}{\sqrt{1-x^3}}. \quad (26)$$

Using (25) and (26), in order to remove the singularity it is required

$$\alpha = -\frac{9\Gamma(7/6)}{5\sqrt{\pi}\Gamma(2/3)}\sum_{n=1}^{\infty} n(-1)^n\frac{(1/3)_{n-1}}{(8/3)_{n-1}},$$
$$= \frac{3\Gamma(7/6)}{2\sqrt{\pi}\Gamma(2/3)}, \quad (27)$$

where the infinite sum on the RHS of the first line of (27) has been explicitly evaluated (e.g. using *Mathematica*) giving $-5/6$. The total magnitude of the circulation of the 3-spoke vortex is $2\pi\alpha = 3\sqrt{\pi}\Gamma(7/6)/\Gamma(2/3) \approx 3.64298$.

The circulation density can be found from $\rho(x) = 2\psi_y$ on $y = 0$, $x \in [0, 1]$ by applying the results (6), (7) and (8) to the series form of (25) to get

$$\rho(x) = -\frac{3\Gamma(7/6)}{\sqrt{\pi}\Gamma(2/3)}\frac{\sqrt{x}}{\sqrt{1-x^3}}\left(1 + \frac{6}{5}\mathrm{Re}\sum_{n=1}^{\infty} n(-1)^n\frac{(1/3)_{n-1}}{(8/3)_{n-1}}\exp\left(2in\cos^{-1}(x^{3/2})\right)\right),$$
$$= -\frac{3\Gamma(7/6)}{\sqrt{\pi}\Gamma(2/3)}\frac{\sqrt{x}}{\sqrt{1-x^3}}\left(1 - \frac{6}{5}\mathrm{Re}\left[\exp(2i\cos^{-1}(x^{3/2}))\,F\left(\frac{1}{3}, 2; \frac{8}{3}; -\exp(2i\cos^{-1}(x^{3/2}))\right)\right]\right). \quad (28)$$

Figure 2 show the streamlines and circulation density for the $N = 3$ equilibria, where the sign of the roots in (28) is chosen so that $\rho(x) \geq 0$ for $x \in [0, 1]$. Note that the circulation density vanishes at the origin as well as at the tip of the spoke. This property was also found in [4] for all even $N \geq 4$.

6 Rotating equilibria of the O'Neil class

6.1 Sheet plus one vortex

Consider now a single straight sheet together with a point vortex rotating steadily about the origin with constant angular velocity $\Omega > 0$. In the co-rotating frame the sheet occupies $x \in [c-r, c+r]$, $y = 0$, where c is the center of the sheet having length $2r > 0$. In this and following sections the problem is non-dimensionalised so that the point vortex has circulation $\Gamma = 2\pi$ and is located unit distance from the origin, here at $z = 1$. Note that $c + r < 1$ so that the entire sheet lies to the left of the point vortex. The task is to find Ω, r and c.

The conformal map from the exterior of the sheet to the exterior of the unit ζ-disc and its inverse is

$$\zeta = \frac{z - c + \sqrt{(z-c)^2 - r^2}}{r}, \quad z = c + \frac{r}{2}(\zeta + \zeta^{-1}). \tag{29}$$

The stream function is now given by

$$\psi = -\frac{\Omega}{2}|z|^2 + \alpha \log|\zeta| + \mathrm{Re}\sum_{n=0}^{\infty} a_n \zeta^{-n} + \log|\zeta - \zeta_0| - \log|\zeta - \zeta_0^{-1}|, \tag{30}$$

where the contributions from the point vortex and its image in the unit ζ-circle have been included where $\zeta_0 = (1 - c + \sqrt{(1-c)^2 - r^2})/r > 1$ is the corresponding point vortex location in the ζ-plane under the map (29). Inclusion of the $-\log|\zeta - \zeta_0^{-1}|$, the vortex image in the unit circle, implies that the logarithmic terms alone satisfy the requirement that $|\zeta| = 1$ is a streamline. The total circulation of the sheet is equivalent to the net circulation associated with the logarithmic terms inside the unit ζ-disk at $\zeta = 0, 1/\zeta_0$, and is $2\pi(\alpha - 1)$.

To find the coefficients, $a_0, a_1, \cdots$, the boundary condition $\psi = 0$ on $\zeta = \exp(i\theta)$ is used. Noting that on the sheet $\log|(\zeta - \zeta_0)/(\zeta - \zeta_0^{-1})| = \log|\zeta_0|$, (29) gives

$$\mathrm{Re}\sum_{n=0}^{\infty} a_n e^{-in\theta} = -\log|\zeta_0| + \frac{\Omega}{2}\left(c^2 + \frac{r^2}{2} + 2rc\cos\theta + \frac{r^2}{2}\cos 2\theta\right), \tag{31}$$

and so by inspection $a_0 = -\log|\zeta_0| + \Omega/2(c^2 + r^2/2)$, $a_1 = \Omega rc$, $a_2 = \Omega r^2/4$ and all other coefficients vanish. The stream function is now

$$\psi = -\frac{\Omega}{2}|z|^2 + \alpha\log|\zeta| + \frac{\Omega}{2}(c^2 + \frac{r^2}{2}) + \Omega rc\mathrm{Re}\,\zeta^{-1} + \frac{\Omega r^2}{4}\mathrm{Re}\,\zeta^{-2} + \log\left|\frac{\zeta - \zeta_0}{\zeta\zeta_0 - 1}\right|. \tag{32}$$

In this problem there is no symmetry in the real direction and so Kutta conditions must be enforced at both tips of the sheet. The derivative ψ_y is evaluated on $y = 0$ with $x \to c \pm r$, $\zeta \to \pm 1$ and α chosen to ensure the derivative remains bounded. From (29)

$$\zeta_z = \frac{\zeta}{\sqrt{(z-c)^2 - r^2}}, \tag{33}$$

so that

$$\left.\frac{\zeta_z}{\zeta}\right|_{z=x\in[c-r,c+r]} = -\frac{i}{\sqrt{r^2 - (x-c)^2}}, \tag{34}$$

is pure imaginary. Note that (34) has a different sign outside the square root compared to (12). This is not important and is a consequence of differentiating (29) instead of (1). It can be reconciled by choosing the appropriate signs of the square root. Using (33) and (34) in (7) and (8), the derivative of (32) evaluated at the tips is

$$\psi_y|_{z=x\to c\pm r} = \frac{1}{\sqrt{r^2 - (x-c)^2}}\left(\alpha + \frac{1}{1 \mp \zeta_0} - \frac{1}{1 \mp \zeta_0^{-1}} \mp \Omega rc - \frac{\Omega r^2}{2}\right). \tag{35}$$

For (35) to be bounded at the tips it follows

$$\alpha = \frac{1}{1 - \zeta_0^{-1}} - \frac{1}{1 - \zeta_0} + \Omega rc + \frac{\Omega r^2}{2}, \tag{36}$$

and

$$\alpha = \frac{1}{1 + \zeta_0^{-1}} - \frac{1}{1 + \zeta_0} - \Omega rc + \frac{\Omega r^2}{2}. \tag{37}$$

Equations (36) and (37) are combined to give

$$\Omega = -\frac{1}{c\sqrt{(1-c)^2 - r^2}}. \tag{38}$$

The choice $\alpha = 2$ is now made so that the sheet has the same total circulation as the point vortex.

For equilibrium it is required that the velocity, (u, v), at the point vortex location $z = 1$ to be zero. By symmetry $u = 0$ at $z = 1$ so it is sufficient to demand $v = \psi_x = 0$ here. Since the non-self induced vortex velocity is being sought at $z = 1$ via conformal mapping methods it is necessary to take into account the Routh correction term, see e.g. [6]. In particular the non-self induced velocity of the vortex is

$$u - iv = \frac{d}{dz}\left(F(z) + i\log(z-1)\right), \tag{39}$$

where F is the complex potential in the z-plane. Equation (39) can be written as

$$u - iv = \frac{d\zeta}{dz}\frac{d}{d\zeta}\left(F(f(\zeta)) + i\log(\zeta-\zeta_0)\right) + \frac{d\zeta}{dz}\frac{d}{d\zeta}\left(i\log\left(\frac{z-1}{\zeta-\zeta_0}\right)\right), \tag{40}$$

where $z = f(\zeta)$ is the conformal map from the ζ- to the z-plane and is given by (29).

Standard manipulation involving Taylor series expansion [6] near $z = 1$, $\zeta = \zeta_0$ then gives

$$v|_{z=1} = -\text{Im}\left[\frac{d\zeta}{dz}\frac{d}{d\zeta}\left(F(f(\zeta)) + i\log(\zeta-\zeta_0)\right)|_{\zeta_0}\right] - \text{Re}\left[\frac{f''(\zeta_0)}{2f'(\zeta_0)^2}\right], \tag{41}$$

where the first term on the RHS of (41) is given by $\psi_x(f(\zeta))|_{\zeta_0}$ and excludes the singular term at $\zeta = \zeta_0$, and can be found from (32). The second term in (41) is the Routh correction term. Hence, the velocity at the point vortex is given by

$$v|_{z=1} = \psi_x|_{z=1,\zeta=\zeta_0} - \partial_x \log|\zeta-\zeta_0|_{\zeta=\zeta_0} - \text{Re}\left[\frac{f''(\zeta_0)}{2f'(\zeta_0)^2}\right]. \tag{42}$$

To evaluate ψ_x at $z = 1$ and $\zeta = \zeta_0$, (32) is used with (7)-(8). The Routh correction term is evaluated using (33) and (29). Hence, (42) becomes

$$v|_{z=1} = -\Omega + \frac{1}{\sqrt{(1-c)^2 - r^2}}\left(2 - \frac{\zeta_0^2}{\zeta_0^2 - 1} - \frac{\Omega rc}{\zeta_0} - \frac{\Omega r^2}{2\zeta_0^2} - \frac{1}{\zeta_0^2 - 1}\right), \tag{43}$$

where $f''(\zeta_0)f'(\zeta_0)^{-2}/2 = \left((1-c)^2 - r^2\right)^{-1/2}(\zeta_0^2 - 1)^{-1}$ has been used. Setting (43) equal to zero and along with (37), (38) and $\zeta_0 = (1 - c + \sqrt{(1-c)^2 - r^2})/r$ the three nonlinear algebraic equations are solved numerically using *Mathematica* giving $\Omega \approx 0.589, c \approx -1.11$, and $r \approx 1.45$ to three significant figures. This agrees with the alternative analytical approach of [7] and also the numerical computation by [10]. Figure 3 shows the streamlines.

An expression for the circulation density $\rho(x) = 2\psi_y$ is obtained by finding the derivative ψ_y on $z = x \in [c-r, c+r]$ from (32) giving

$$\rho(x) = \frac{2}{\sqrt{r^2 - (x-c)^2}}\left(2 + \frac{\sqrt{(1-c)^2 - r^2}}{x-1} - \Omega\left(x^2 - xc - \frac{r^2}{2}\right)\right), \tag{44}$$

where the sign of the roots are chosen to give $\rho(x) \geq 0$ for $x \in [c-r, c+r]$. Figure 3 gives a comparison between (44) and that obtained by O'Neil's method (an appropriately scaled version which is given in [10]), namely

$$\rho_{\text{O'Neil}}(x) = 2\text{Re}\left(4\Omega - \Omega^2 x^2 - \frac{1}{(x-1)^2}\right)^{1/2}. \tag{45}$$

The comparison in figure 3 convincingly demonstrates the equivalence of (44) and (45), although this assertion remains to be proved.

6.2 Sheet plus two vortices

Now consider the case where a single straight sheet and two point vortices rotate about the origin with constant angular velocity $\Omega > 0$. This arrangement is such that in the co-rotating frame the sheet lies on $x \in [-r, r]$ with $0 \leq r < 1$ analogous to the $N = 2$ case of §3, with point vortices at $z = \pm 1$. Each point vortex has circulation $\Gamma = 2\pi$ and the

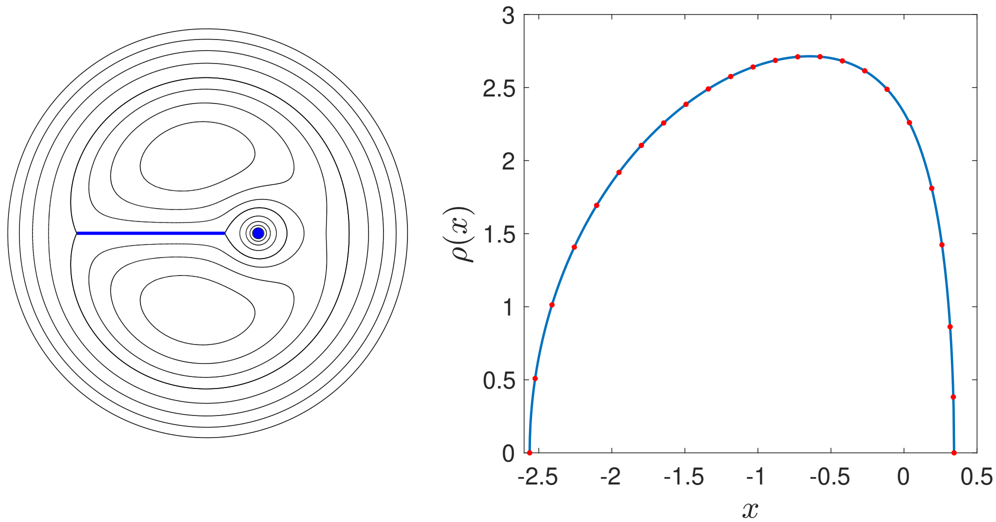

Figure 3: Streamlines (black) for the single vortex sheet and point vortex (blue) (left) and corresponding circulation density $\rho(x)$ along the sheet $y = 0$, $x \in [c - r, c + r]$ (blue) and points showing the solution given by [7] (orange) (right).

total circulation of the sheet is left as a given parameter $2\pi\gamma$ and the task, for given γ, is to determine the unknowns Ω and r which in turn enable explicit expressions for the stream function and the circulation density of the sheet to be found.

The conformal maps between the exteriors of the sheet and the unit ζ-disk are

$$\zeta = \frac{z + \sqrt{z^2 - r^2}}{r}, \quad z = \frac{r}{2}(\zeta + \zeta^{-1}). \tag{46}$$

In the rotating frame where the sheet and vortices are stationary, the stream function is

$$\psi = -\frac{\Omega}{2}|z|^2 + (\gamma + 2)\log|\zeta| + \text{Re}\sum_{n=0}^{\infty} a_n \zeta^{-n} + \log\left|\frac{\zeta^2 - \zeta_0^2}{\zeta^2 - \zeta_0^{-2}}\right|, \tag{47}$$

where $\zeta_0 = (1 + \sqrt{1 - r^2})/r > 1$. Applying $\psi = 0$ on $\zeta = \exp(i\theta)$ gives

$$\text{Re}\sum_{n=0}^{\infty} a_n e^{-in\theta} = -2\log|\zeta_0| + \frac{\Omega}{2}\left(\frac{r^2}{2} + \frac{r^2}{2}\cos 2\theta\right), \tag{48}$$

where it can be seen $a_0 = -2\log|\zeta_0| + \Omega r^2/4$, $a_2 = \Omega r^2/4$ and all other coefficients a_n vanish. By symmetry, the Kutta condition $\psi_y = 0$ need only be applied at one of the sheet tips $z = r$. Using (7), (8), (46) and (47) gives after some simplification

$$\frac{\Omega r^2}{2} = 2 + \gamma - \frac{2}{\sqrt{1 - r^2}}. \tag{49}$$

The velocity of the point vortex at $z = 1$ can be found using the same approach as in §6.1 that accounts for the Routh correction. Upon setting the velocity at either of the point vortices to zero (it vanishes at the other by symmetry) gives

$$\Omega = \frac{1}{\sqrt{1 - r^2}}\left(\frac{5}{2} + \gamma - \frac{\Omega r^2}{2\zeta_0^2} - \frac{2\zeta_0^2}{\zeta_0^2 - \zeta_0^{-2}} - \frac{1}{\zeta_0^2 - 1}\right). \tag{50}$$

Equations (49) and (50) can be simplified further using the expression for ζ_0 in terms of r to give $\Omega = \Omega(r)$ as an explicit function of r together with a nonlinear algebraic equation for r given γ:

$$\Omega = \frac{1 + r^2}{2\,(1 - r^2)^{3/2}}, \tag{51}$$

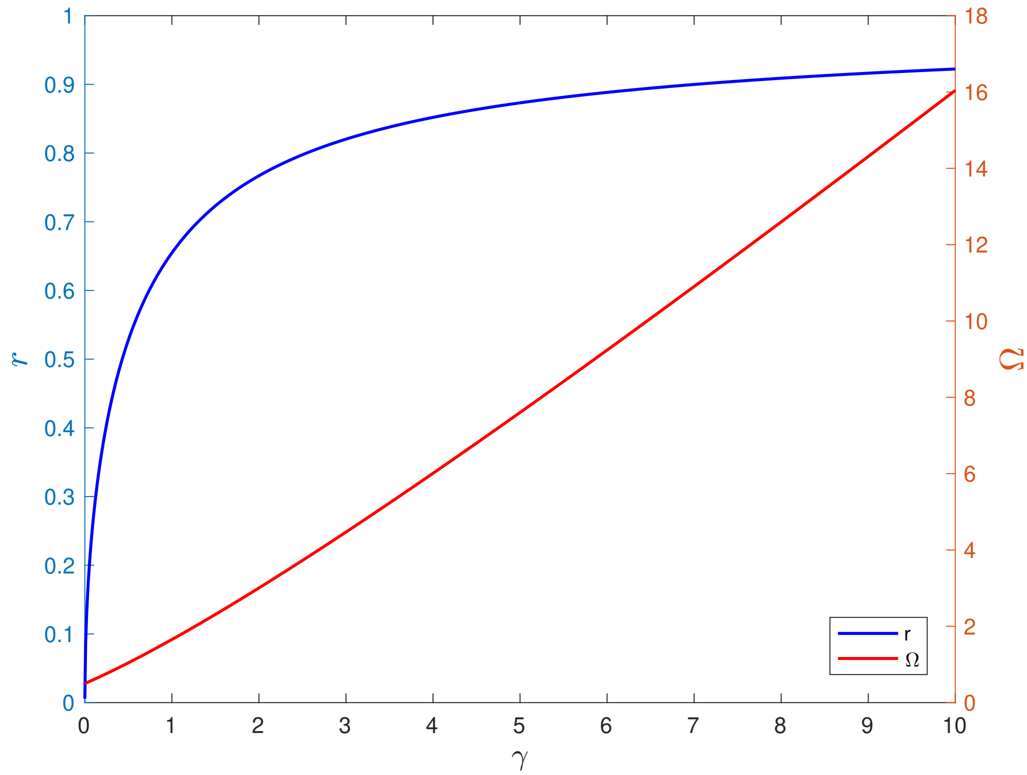

Figure 4: Plot showing the radius of the sheet, r, (blue, left axis) and the angular velocity of the system, Ω, (red, right axis) as a function of the total circulation of the sheet, γ.

$$\gamma + 2 = \frac{r^4 - 7r^2 + 8}{4\left(1 - r^2\right)^{3/2}}. \tag{52}$$

Note that (51) shows that there are no equilibria with $\Omega < 1/2$, unlike point vortices where a central vortex with opposite signed circulation can serve to slow the rotation of the satellite vortices at $z = \pm 1$. The minimum $\Omega = 1/2$ occurs in the limit $r \to 0$ which (52) implies $\gamma \to 0$ i.e. a pair of point vortices located at $z = \pm 1$. More generally, for given γ Figure 4 shows the dependence of Ω and r as γ increases. Figure 5 shows two example streamline plots of the equilibria for $\gamma = 2$ for which (52) gives $r \approx 0.767$ and (51) gives $\Omega \approx 3.00$, and for $\gamma = 0.5$ having corresponding $r \approx 0.528$ and $\Omega \approx 1.04$.

Evaluating $\rho(x) = 2\psi_y|_{z=x\in[-r,r]}$ using (46), (47) and (7) - (8) and simplifying using (51) and (52) gives the circulation density

$$\rho(x) = \frac{\sqrt{r^2 - x^2}}{\sqrt{1 - r^2}}\left(\frac{4}{1 - x^2} + \frac{1 + r^2}{1 - r^2}\right), \tag{53}$$

where the square roots are chosen to be positive for $x \in [-r, r]$.

As in §6.1, the expression (53) can be compared graphically to that obtained by [7]. Although not shown, the agreement is similarly good (for a given γ) as that shown in figure 3 for the one sheet and one vortex case.

7 New four-sheet equilibria with point vortices

7.1 Point vortices off the tips of the sheets

The conformal mapping method is used in this section to construct a new four-sheet equilibria of the type in §4 with the addition of four equal strength point vortices located off each tip. The sheets have length $0 \le r < 1$ and the point

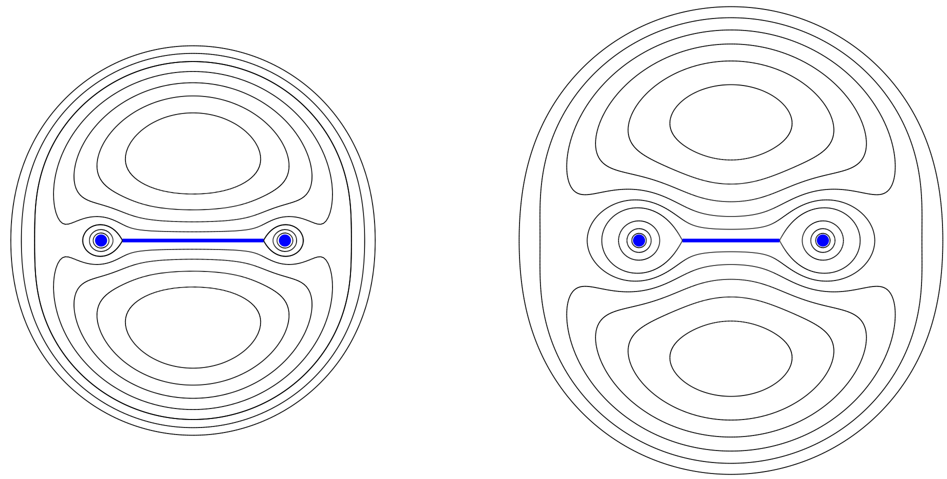

Figure 5: Streamlines (black) for a single vortex sheet with point vortices at $z = \pm 1$ (blue) when $\gamma = 2$, $r \approx 0.767$ and $\Omega \approx 3.00$ (left) and $\gamma = 0.5$, $r \approx 0.528$ and $\Omega \approx 1.04$ (right).

vortices are located at $z = \pm 1, \pm i$ and have circulation 2π. Following the notation of §6.2, the total circulation of the sheets is specified by the parameter γ.

The maps to and from the exterior of the sheets in the z-plane and the unit ζ-circle are the same as (15) scaled by r:

$$\zeta = \exp\left(-\frac{i}{2}\cos^{-1}\left(\frac{z^2}{r^2}\right)\right), \quad z^2 = r^2\cos(2i\log\zeta). \tag{54}$$

The map (54) locates the vortices at $\pm\zeta_0, \pm i\zeta_0$ in the ζ-plane, where

$$\zeta_0 = \frac{1}{r}\sqrt{1+\sqrt{1-r^4}} > 1. \tag{55}$$

Accounting for the images of the vortices in the boundary of the unit disk, the stream function has the form

$$\psi = -\frac{\Omega}{2}|z|^2 + (\gamma+4)\log|\zeta| + \mathrm{Re}\sum_{n=0}^{\infty} a_n\zeta^{-4n} + \log\left|\frac{\zeta^4-\zeta_0^4}{\zeta^4-\zeta_0^{-4}}\right|, \tag{56}$$

and the task is to find Ω, r, a_n for given γ. As in §4 application of the condition $\psi = 0$ on $\zeta = \exp(i\theta)$ determines a_n and summing gives

$$\psi = -\frac{\Omega}{2}|z|^2 + (\gamma+4)\log|\zeta| + \frac{\Omega r^2}{\pi}\mathrm{Re}\left[\left(\zeta^2+\zeta^{-2}\right)\tan^{-1}\left(\zeta^{-2}\right)\right] + \log\left|\frac{\zeta^4-\zeta_0^4}{\zeta_0^4\zeta^4-1}\right|. \tag{57}$$

Proceeding as in §6.2, the unknowns Ω and r are found by enforcing the Kutta condition at the sheet tips and demanding the velocity field vanish at the point vortices. By symmetry, it is sufficient to apply these conditions at $z = r$ and $z = 1$ respectively.

Differentiating (54) gives

$$\frac{\zeta_z}{\zeta} = \frac{iz}{\sqrt{r^4-z^4}}, \tag{58}$$

which is singular as $z \to r$. The Kutta condition removes the singularity in the velocity field $-\psi_y$ at $z = r$ by requiring, using (6) on (57),

$$\begin{aligned}\Omega &= \frac{\pi}{2r^2}\left[\gamma + 4 + 4\left(\frac{1+\zeta_0^4}{1-\zeta_0^4}\right)\right], \\ &= \frac{\pi}{2r^2}\left[\gamma + 4 - 4\frac{1}{\sqrt{1-r^4}}\right],\end{aligned} \tag{59}$$

where (55) has been used to simplify the last term.

Next, the non-self induced velocity $v = \hat{\psi}_x$ where $\hat{\psi}$ is the streamfunction with the singular term at the vortex at $z = 1$, or equivalently $\zeta = \zeta_0$, removed, is calculated giving

$$\hat{\psi}_x\big|_{z=1} = -\Omega + \frac{1}{\sqrt{1-r^4}}\left[4 + \gamma + \frac{3}{2} - \frac{4\zeta_0^4}{\zeta_0^4 - \zeta_0^{-4}} + \frac{2\Omega r^2}{\pi}\left\{(\zeta_0^2 - \zeta_0^{-2})\tan^{-1}\left(\zeta_0^{-2}\right) - 1\right\}\right] + \text{R.C.}, \tag{60}$$

where the Routh correction is

$$\begin{aligned}\text{R.C.} &= -\frac{z_{\zeta\zeta}}{2z_\zeta^2}\bigg|_{\zeta=\zeta_0}, \\ &= \frac{1}{2}\left[\frac{1}{\sqrt{1-r^4}} - \frac{1+r^4}{1-r^4}\right].\end{aligned} \tag{61}$$

Setting (60) to zero and substituting (61) gives after some simplification

$$\Omega = \frac{1}{\sqrt{1-r^4}}\left[4 + \gamma + \frac{r^4 - 5}{2\sqrt{1-r^4}} + \frac{2\Omega r^2}{\pi}\left\{\frac{2\sqrt{1-r^4}}{r^2}\tan^{-1}\left(\zeta_0^{-2}\right) - 1\right\}\right]. \tag{62}$$

Combining (59) and (62) gives an equation for Ω purely in terms of r

$$\Omega = \frac{r^4 + 3}{2(1-r^4)\left[1 - \frac{2}{\pi}\sin^{-1}(r^2)\right]}, \tag{63}$$

where, for given γ, r can be found by numerical solution of the nonlinear equation

$$\frac{r^2(r^4+3)}{\pi(1-r^4)\left[1 - \frac{2}{\pi}\sin^{-1}(r^2)\right]} = 4 + \gamma - \frac{4}{\sqrt{1-r^4}}. \tag{64}$$

In deriving (63) and (64), (55) followed by the simplification $2\tan^{-1}(\zeta_0^{-2}) = \sin^{-1}(r^2)$ has been used.

When $\gamma = 0$ (64) has solution $r = 0$, and by (63) $\Omega = 3/2$, corresponding to the expected angular velocity about the origin of an ensemble of four identical point vortices of circulation 2π located at the vertices of the unit square with no vortex sheets present. In the limit $r \to 1$, from (64) and (63) both Ω and γ become unbounded (so that the vorticity of the sheets dominates that of the point vortices) such that $\gamma/\Omega \to 2/\pi$. This is consistent with the result in §4 for the four-sheeted structure with no point vortices which has $\Omega = 1$ and total circulation of the sheets $\alpha = 2/\pi$. Figure 6 shows the behaviour of Ω and r as a function of $\gamma > 0$, where *MATLAB's fsolve* has been used to solve the nonlinear algebraic equation (64) for r.

Evaluating $\rho(x) = 2\psi_y|_{z=x\in[0,r]}$ using (54), (57) and (7) - (8) and simplifying using (59) gives the circulation density

$$\rho(x) = \frac{8x}{\sqrt{r^4 - x^4}}\left[\frac{1}{\sqrt{1-r^4}} - \frac{\sqrt{1-r^4}}{1-x^4} + \frac{\Omega}{2\pi}\sqrt{r^4-x^4}\tanh^{-1}\sqrt{1 - x^4/r^4}\right], \tag{65}$$

where all square roots are taken to be positive so that $\rho(x) \geq 0$ for $x \in [0, r]$. In the limit $\Omega \to \infty$ when the sheets dominate the point vortices, (65) (with $r = 1$) is consistent with (22).

Figure 7 shows an example of the streamlines for the choice $\gamma = 1$ which in turn gives $\Omega \approx 2.53$ and $r \approx 0.630$. The circulation density along a sheet given by (65) is also shown for this choice.

7.2 Point vortices located on the bisectors of the sheets

Consider the same four-sheeted structure, but now with four identical point vortices of circulation 2π located midway between each pair of sheets at unit distance from the origin i.e. $\pm\exp(\pm i\pi/4)$. Using the map (54) (with now no restriction on the sheet length except that it is $r \geq 0$) the images of the vortices in the ζ-plane are at $\pm\beta\exp(\pm i\pi/4)$, where

$$\beta = \frac{\sqrt{1+\sqrt{1+r^4}}}{r} > 1. \tag{66}$$

Analogous to (57), the stream function satisfying $\psi = 0$ on $|\zeta| = 1$ is

$$\psi = -\frac{\Omega}{2}|z|^2 + (\gamma+4)\log|\zeta| + \frac{\Omega r^2}{\pi}\text{Re}\left[\left(\zeta^2 + \zeta^{-2}\right)\tan^{-1}\left(\zeta^{-2}\right)\right] + \log\left|\frac{\zeta^4 + \beta^4}{\beta^4\zeta^4 + 1}\right|. \tag{67}$$

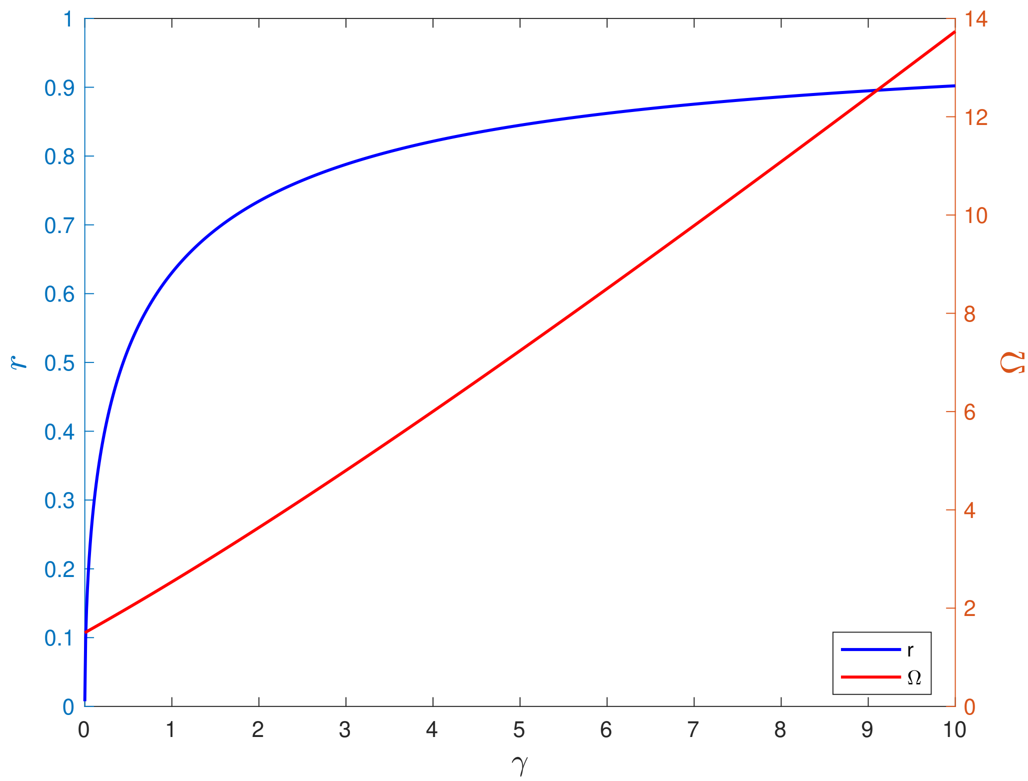

Figure 6: Plot showing the radius r (blue, left axis) for the §7.1 $N = 4$ sheet and the angular velocity of the system, Ω, (red, right axis) as a function of the total circulation of the sheet, γ.

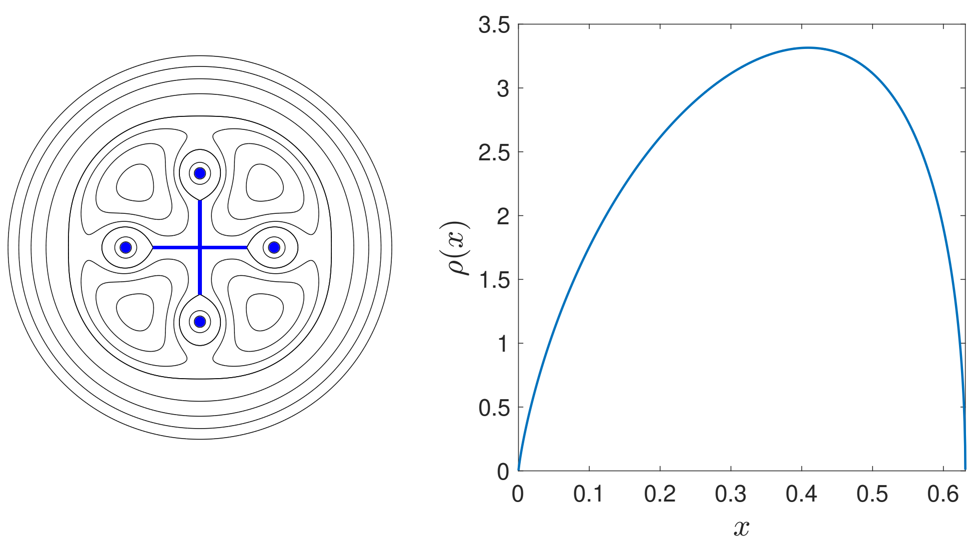

Figure 7: Streamlines (black) for the §7.1 $N = 4$ sheet (blue) and four point vortices (blue) when $\gamma = 1$, $\Omega \approx 2.53$ and $r \approx 0.630$ (left) and corresponding circulation density $\rho(x)$ along the sheet $y = 0$, $x \in [0, r]$ (right).

The Kutta condition at $z = r$ gives

$$\Omega = \frac{\pi}{2r^2}\left[\gamma + 4 + 4\left(\frac{1-\beta^4}{1+\beta^4}\right)\right], \\ = \frac{\pi}{2r^2}\left[\gamma + 4 - 4\frac{1}{\sqrt{1+r^4}}\right], \tag{68}$$

where (66) has been used to simplify the last term.

Following the same steps as in §7.1, the stationarity condition is, owing to symmetry, applied only to the vortex at $z = \exp(i\pi/4)$, and is equivalent to insisting $\partial_\rho \hat{\psi} + \text{R.C.} = 0$ at that point, where ρ is the radial coordinate in the z-plane, and, as in §7.1, $\hat{\psi}$ is the streamfunction with the singular term removed and R.C. is the Routh correction. It can be shown that

$$\partial_\rho \text{Re} f(\zeta)\big|_{\zeta=\beta\exp(i\pi/4)} = \frac{1}{\sqrt{1+r^4}}\text{Re}\left(\zeta f_\zeta\right)\big|_{\zeta=\beta\exp(i\pi/4)}. \tag{69}$$

Applying the operator ∂_ρ to (67), using (69), accounting for the Routh correction and setting the result to zero gives

$$\Omega = \frac{1}{\sqrt{1+r^4}}\left[4 + \gamma - \frac{2\Omega r^2}{\pi}\right] - \frac{(r^4+5)}{2(1+r^4)} + \frac{4\Omega}{\pi}\coth^{-1}\left(\frac{1+\sqrt{1+r^4}}{r^2}\right). \tag{70}$$

Equations (68) and (70) combine to give an equation for Ω purely in terms of r

$$\Omega = \frac{3 - r^4}{2(1+r^4)\left[1 - \frac{2}{\pi}\sinh^{-1}(r^2)\right]}, \tag{71}$$

where, for given γ, r can be found by numerical solution of the nonlinear equation

$$\frac{r^2(3-r^4)}{\pi(1+r^4)\left[1 - \frac{2}{\pi}\sinh^{-1}(r^2)\right]} = 4 + \gamma - \frac{4}{\sqrt{1+r^4}}. \tag{72}$$

In writing (71) and (72) the simplification $2\coth^{-1}[(1+\sqrt{1+r^4})/r^2] = \sinh^{-1}(r^2)$ has been used.

In contrast to §7.1, solutions (71) and (72) exist for both positive and negative γ. Moreover, for some choices of γ multiple solutions for r are possible, leading to multiple equilibria e.g. $\gamma = 0$ has solutions $(r, \Omega) = (0, 3/2)$ or $\approx (0.783, 1.51)$, among others. A thorough investigation of such multiple equilibria is not pursued here. Note the $(0, 3/2)$ solution coincides with the case when there are no sheets and the vortices located at the vertices of the unit square rotate with, as expected, angular velocity $3/2$.

Since solutions for $\gamma < 0$ are now available, so that the multi-sheet structure has total circulation opposite in sign to the point vortices, it follows that there is the possibility of a completely stationary equilibrium with $\Omega = 0$. This possibility is realised with the choice $\Omega = 0$, $\gamma = -2$ and $r = \sqrt[4]{3}$ which solves the system (71) and (72). Note that O'Neil [12], using a different method, gives a related example of a stationary structure involving three radial sheets with three point vortices located on each of the bisectors of the sheets (see Fig. 11 of [12]).

Evaluating $\rho(x) = 2\psi_y|_{z=x\in[0,r]}$ using (54), (67) and (6) - (8) and simplifying using (68) gives the circulation density

$$\rho(x) = \frac{8x}{\sqrt{r^4 - x^4}}\left[\frac{1}{\sqrt{1+r^4}} - \frac{\sqrt{1+r^4}}{1+x^4} + \frac{\Omega}{2\pi}\sqrt{r^4-x^4}\tanh^{-1}\sqrt{1-x^4/r^4}\right], \tag{73}$$

where all square roots are taken to be positive.

Figure 8 shows examples of streamline plots and the corresponding circulation density $\rho(x)$ for two different choices of γ. The top row show the case of the non-rotating stationary when $\gamma = -2$. Note that $\rho(x)$ is everywhere non-positive on the sheet. The second example has zero net sheet circulation $\gamma = 0$ and so $\rho(x)$ changes sign accordingly along the sheet. Note the existence of a streamline which intersects the sheet: it does so at the point at which $\rho(x) = 0$ and so is a stagnation point on the sheet.

8 Remarks

Attempts at applying the mapping method to find equilibria of the Protas-Sakajo class for $N \geq 5$ failed because of the lack of a sufficiently simple Fourier series representation of $|z|^2 = |\cos(N\theta/2)|^{4/N}$ on the unit ζ circle. But the method,

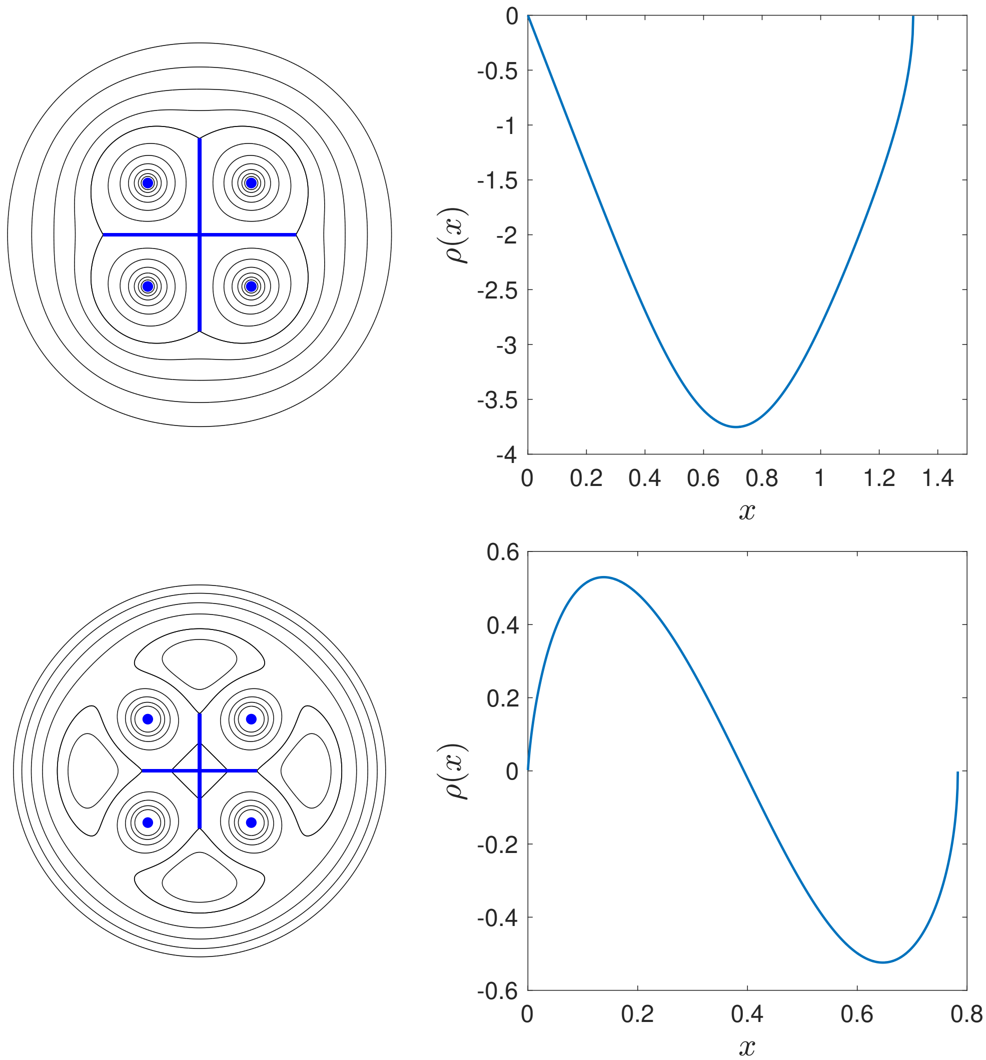

Figure 8: Streamlines (black) for the §7.2 $N = 4$ sheet (blue) and four point vortices (blue) when $\gamma = -2$, $\Omega = 0$ and $r = \sqrt[4]{3}$ (top left) and $\gamma = 0$, $\Omega \approx 1.51$ and $r \approx 0.783$ (bottom left) with corresponding circulation density $\rho(x)$ along the sheet $y = 0$, $x \in [0, r]$ (right) .

which requires seeking a harmonic function subject to known behaviour on a slit, or slits, and having logarithmic behaviour at infinity, suggests the problem can be tackled accurately and efficiently using numerical methods of the type developed over recent years by Trefethen, Costa, Baddoo, and others e.g. [13, 14, 15]. These methods effectively use least square methods to compute the Fourier coefficients. Such a numerical approach to computing vortex equilibria including those involving sheets and some of the examples considered in this work is explored in [10]. In fact, as demonstrated in [10], this numerical procedure is also able to accommodate the presence of point vortices, enabling equilibria of the O'Neil class to be similarly computed.

Further examples of O'Neil-type equilibria involving a single rectilinear sheet along the real axis with an arbitrary number of point vortices can, in principle, be constructed exactly using the present conformal mapping method. However, it is likely that, in order to satisfy the Kutta condition at the tips, the point vortices need to be arranged symmetrically with respect to the sheet i.e. in pairs located at z_i and $\overline{z_i}$. Indeed [7] gives such an example with two vortices located at $z = (1 \pm i)/2$. Two or more non-collinear, but straight, sheets are also potentially amenable to analysis by the conformal mapping method using methods of mapping multiply connected slit domains to the circular domains e.g. [16].

In addition to the examples presented in this work, the conformal mapping is capable of calculating other families of equilibria. For example, exact solutions of the type found in §7 for a three-sheet structure with three point vortices either off the tips or midway between the sheets ought to be possible to derive. The procedure is the same as that for the four-sheet structure, though there is the complexity of dealing with hypergeometric functions. Another possibility is combining a multi-sheet structure with any number of symmetrically placed point vortices. A natural starting point in this class would be to consider the equilibria for a four-sheet structure with eight point vortices located off the tips and on the bisectors of the sheets. Note in this case the stationary vortex condition must be applied at two vortices e.g. one on the positive real axis and another on the ray $\theta = \pi/4$. This implies the need for an additional unknown, namely, fixing the vortices off the tips to be unit distance from the origin but permitting those between the sheets to be at a different distance which is to be determined. Evidence for the need for such differing distances is in [10] who find a numerical solution for a single sheet along the real axis $\approx [-0.687, 0.687]$ with point vortices located at $z = \pm 1$ and $z \approx \pm 0.901i$.

A more general possibility is to use the conformal mapping method to construct equilibria involving curved sheets. O'Neil [17] approximates sheets as a distribution of a finite number of point vortices and so constructs equilibria suggestive of curved sheets. In separate work, O'Neil [12] uses a complex function based method to find exact solutions for translating dipole-like vorticity distribution involving a curved sheet and a point vortex. Finding exact solutions of this class by conformal mapping is challenging since it is a free boundary problem with the shape of the curved sheet needing to be calculated as part of the solution. Conformal mapping along with tools such as, for example, the Schwarz function or series approximation of the conformal maps have been successfully employed in other free boundary vortex equilibria problems e.g. [18], [10]. For this purpose the fact that the Joukowsky map (9) opens up arbitrary slits (i.e. curved sheets) and not just straight sheets is likely of use.

Translating equilibria can also be considered using the methods used here. As pointed out in [9], a finite rectilinear sheet is able to propagate steadily in the direction normal to itself, provided the existence of velocity singularities at the sheet tips is deemed acceptable. It seems possible that more general classes of translating sheet equilibria exist, including those for which the Kutta condition holds at the sheet tips. Mapping methods of the type presented here may be a way of finding these equilibria.

Finally it is noted that the problem of finding equilibria involving both sheets and vortices is essentially a variant of the Föppl problem in which the equilibrium location of point vortices near a body in a background flow is sought. The classic Föppl problem involves a circular cylinder immersed in a uniform stream e.g. [6]. Here the body (fixed in the rotating frame) is the sheet structure and uniform rotation provides the background flow. In the general Föppl problem the use of Hamiltonian vortex methods, and in particular the transformation properties of the vortex Hamiltonian under conformal mapping (which essentially accounts for the Routh correction), enables the consideration of non-circular bodies e.g. [19, 20]. This suggests Hamiltonian methods may also be of use in finding new families of vortex sheet equilibria when point vortices are present.

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