Arch. Rational Mech. Anal. (2025) 249:45
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-025-02114-y

Check for updates

Damping Versus Oscillations for a Gravitational Vlasov–Poisson System

M. Hadžić, G. Rein, M. Schrecker & C. Straub

Communicated by N. Masmoudi

Abstract

We consider a family of isolated inhomogeneous steady states of the gravitational Vlasov–Poisson system with a point mass at the centre. These are parametrised by the polytropic index $k > 1/2$, so that the phase space density of the steady state is C^1 at the vacuum boundary if and only if $k > 1$. We prove the following sharp dichotomy result: if $k > 1$, the linear perturbations Landau damp and if $1/2 < k \leq 1$ they do not. The above dichotomy is a new phenomenon and highlights the importance of steady state regularity at the vacuum boundary in the discussion of the long-time behaviour of the perturbations. Our proof of (nonquantitative) gravitational relaxation around steady states with $k > 1$ is the first such result for the gravitational Vlasov–Poisson system. The key novelty of this work is the proof that no embedded eigenvalues exist in the essential spectrum of the linearised system.

Contents

1. Introduction

The problem of the relaxation of stellar systems is a central question in the study of the dynamics of galaxies. It was explored in the 1960s in the pioneering works of Lynden-Bell [48,49], who was the first to point out an intimate connection between galaxy relaxation and the validity of so-called gravitational Landau damping. Landau damping originally referred to a well-known equilibration mechanism for the linearised electrostatic Vlasov–Poisson system around spatially homogeneous steady states discovered in 1946 [41]. In the gravitational case, the term Landau damping was used in [48] (see also [12] for an exhaustive list of references to the physics literature) to refer to the decay of macroscopic quantities of the linearised perturbations about a given steady state.

To study the stability around isolated and localised self-gravitating galaxies, one is forced to consider spatially inhomogeneous densities and this considerably complicates the stability analysis. There is a continuum of steady states of the gravitational Vlasov–Poisson (VP) system whose infinite-dimensional character is related to the invariance of the VP-system under the action of measure preserving diffeomorphisms. Moreover, the relevant steady states are compactly supported in both the space and the velocity variable, which means that particles are trapped in a finite region of phase-space, and this can act as an obstruction to decay.

In this work we construct a family of steady states for which we show that the question of relaxation depends strongly on the regularity of the equilibrium at the vacuum boundary. If the steady state is below a certain regularity threshold we prove that the linearised operator has pure oscillations in its spectrum and no damping occurs. If, by contrast, the steady state is above the threshold, there is no pure point spectrum and one can prove non-quantitative decay results using the RAGE theorem. This dichotomy is a striking feature of the gravitational dynamics, and we believe the methods developed in this paper to have a wide range of applicability.

The key mathematical novelty of the paper is the proof of absence of embedded eigenvalues in the spectrum of the linearised operator around sufficiently regular steady galaxies, see Sect. 4. Our method is new and exploits in a crucial way the underlying Hamiltonian geometry of the problem.

To focus on the main ideas, we consider the radial gravitational Vlasov–Poisson system including a fixed central potential generated by a point mass of size $M > 0$. The presence of the latter can be thought of as a Newtonian model for a central black hole, a feature found in many real-world galaxies. In addition, we assume that all the particles have angular momentum of fixed modulus.[1] This symmetry reduction removes several technical difficulties and allows us to focus on the key new ideas. The system reads as

$$\partial_t f + w\,\partial_r f - \left(U' + \frac{M}{r^2} - \frac{L}{r^3}\right)\partial_w f = 0, \tag{1.1}$$

[1] The situation of steady states with fixed modulus of angular momentum is discussed in [62, Sc. 3.1] and in the plasma case see also [55].

$$U' = \frac{4\pi}{r^2}\int_0^r s^2\rho(t,s)\,\mathrm{d}s, \qquad \lim_{r\to\infty} U(t,r) = 0, \tag{1.2}$$

$$\rho(t,r) = \frac{\pi}{r^2}\int_{\mathbb{R}} f(t,r,w)\,\mathrm{d}w. \tag{1.3}$$

Here $f(t,r,w) \geq 0$ is the phase-space number density, a function of time $t \in \mathbb{R}$, radial position $r > 0$, and radial velocity $w \in \mathbb{R}$, $U(t,r)$ is the gravitational potential induced by the stars of the galaxy, and $\rho(t,r) \geq 0$ their macroscopic mass density. The system (1.1)–(1.3) is the radial VP-system for an ensemble of particles all of which have angular momentum with the same squared modulus $L > 0$.

We consider a class of steady states to (1.1)–(1.3) of the form

$$f^{k,\varepsilon}(r,w) = \varphi(E(r,w)) = \varepsilon\,\tilde{\varphi}(E(r,w)), \qquad \tilde{\varphi}(E) = (E_0 - E)_+^k, \tag{1.4}$$

where $(\ldots)_+$ denotes the positive part of the argument, $\varepsilon > 0$ is a size-parameter, and $k > \frac{1}{2}$ the polytropic exponent. Here

$$E(r,w) = \frac{1}{2}w^2 + \Psi(r), \tag{1.5}$$

$$\Psi(r) = U(r) - \frac{M}{r} + \frac{L}{2r^2} \tag{1.6}$$

are the particle energy and the effective potential, respectively, while the cut-off energy $E_0 < 0$ is implicitly determined through the equation satisfied by the steady state. The gravitational potential U is induced by $f^{k,\varepsilon}$ through (1.2)–(1.3). For completeness of exposition, the existence of such steady states with finite radius and finite mass is shown in Sect. 3; this is actually easier than in the situation without a central point mass, cf. [56]. More precisely, fix a $k > \frac{1}{2}$. Then, for any $\varepsilon > 0$, there exists a whole 1-parameter family of steady states of the form (1.4) parametrised by the parameter

$$\kappa := E_0 - U(0) < 0, \tag{1.7}$$

which has the meaning of a relative gravitational potential at the origin. The resulting phase-space support is compact and the associated macroscopic density $\rho(r)$ is of size $\mathcal{O}_{\varepsilon\to 0}(\varepsilon)$, supported on a compact spherical shell $[R_{\min}, R_{\max}]$ of thickness $\mathcal{O}_{\varepsilon\to 0}(1)$ with a delta distribution of mass M centred at the origin, see Fig. 1. The parameter κ determines the inner vacuum radius $R_{\min} > 0$ as well as the (finite) limit of the outer vacuum radius $R_{\max}$ as $\varepsilon \to 0$. We shall suppress the dependence on κ and fix it to any value satisfying the *single gap condition*

$$-2^{-\frac{2}{3}}\frac{M^2}{2L} < \kappa < 0. \tag{1.8}$$

As shown in Corollary 3.11, condition (1.8) ensures that the essential spectrum of the linearised operator is simply connected for the relevant equilibria. The pivotal question is the dependence of the stability behaviour of the steady states $f^{k,\varepsilon}$ on the parameters k and ε.

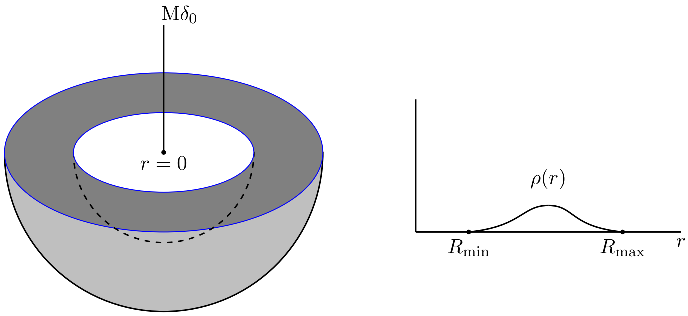

Fig. 1. Schematic depiction of the lower hemisphere of the spherical shell (on the left) and the macroscopic density distribution $\rho(r)$ (on the right)

We linearise the system (1.1)–(1.3) around a fixed steady state $f^{k,\varepsilon}$. If we denote the linear perturbation by F, a straightforward calculation gives the linearisation

$$\partial_t F + \tilde{\mathcal{L}} F = 0, \tag{1.9}$$

where

$$\tilde{\mathcal{L}} F := \mathcal{T}\left(F + \left|\varphi'(E)\right| U_F\right), \tag{1.10}$$

the *transport operator* $\mathcal{T}$ is given by

$$\mathcal{T} := w\,\partial_r - \Psi'(r)\,\partial_w, \tag{1.11}$$

and U_F solves the radial Poisson equation

$$U_F'(r) = \frac{4\pi}{r^2}\int_0^r s^2\,\rho_F(s)\,\mathrm{d}s = \frac{4\pi^2}{r^2}\int_0^r\int_{\mathbb{R}} F(s,w)\,\mathrm{d}w\,\mathrm{d}s, \qquad \lim_{r\to\infty} U_F(r) = 0. \tag{1.12}$$

Alternatively, one can apply the classical Antonov trick [1] and split (1.9) into separate equations for the even and odd in w parts $f_\pm(r,w) = \frac{1}{2}(F(r,w) \pm F(r,-w))$ of the perturbation F. The linear evolution is then fully described by the following second order system for f_-:

$$\partial_t^2 f_- + \mathcal{L} f_- = 0. \tag{1.13}$$

The *linearised operator* (also referred to as the *Antonov operator*) takes the form

$$\mathcal{L} := -\mathcal{T}^2 - \mathcal{R}, \tag{1.14}$$

where the *gravitational response operator* $\mathcal{R}$ is given by

$$\mathcal{R} g := 4\pi^2\left|\varphi'(E)\right|\frac{w}{r^2}\int_{\mathbb{R}} \tilde{w}\,g(r,\tilde{w})\mathrm{d}\tilde{w}. \tag{1.15}$$

Functional-analytic properties of the operators $\tilde{\mathcal{L}}$ and $\mathcal{L}$ are discussed in Sect. 3.4. We shall mostly work with the second order formulation (1.13), although the analysis can be carried out analogously in the first order formulation (1.9). The natural Hilbert space for our analysis is the weighted L^2-space

$$H := \{f : \Omega \to \mathbb{R} \mid f \text{ measurable and } \|f\|_H < \infty\},$$

where $\|\cdot\|_H$ is induced by the inner product

$$\langle f, g\rangle_H := \int_\Omega \frac{1}{|\varphi'(E)|} f(r,w)\,g(r,w)\,\mathrm{d}(r,w) \tag{1.16}$$

and $\Omega = \{f^{k,\varepsilon} > 0\}$ is the interior of the steady state support. Note that the integrand in (1.16) is well-defined, since

$$\varphi'(E(r,w)) < 0, \qquad (r,w) \in \Omega. \tag{1.17}$$

Since $\mathcal{L}$ only covers the evolution of the odd-in-w part of the linear perturbation, we further define the subspace of H consisting of odd-in-w functions as

$$\mathcal{H} := \{f \in H \mid f \text{ is odd in } w \text{ a.e. on } \Omega\}.$$

We shall see in Sect. 3.4 that $\mathcal{L}$ is self-adjoint on $\mathcal{H}$ when defined on its domain $\mathrm{D}(\mathcal{L})$.

The monotonicity condition (1.17) is known as the Antonov linearised stability criterion. For the case without a central point mass it was shown in the physics literature [19,38] that it implies the spectral stability, which is equivalent to the non-negativity of the quadratic form $\langle \mathcal{L}h, h\rangle_H$ on $\mathrm{D}(\mathcal{L})$. This result can also be thought of as the analogue of the Penrose stability criterion for plasmas [54]. Moreover, by a simple modification of the arguments in [26,42,43,63] one can prove that the steady states under consideration are nonlinearly orbitally stable in our symmetry class, which is essentially due to the energy subcritical nature of the problem. By contrast, nothing is known about the asymptotic-in-time behaviour of solutions close to such steady states and, unlike the classical Landau damping for plasmas, it is *a priori* unclear whether any form of damping occurs for the linearised dynamics (1.13).

To provide a meaningful formulation of Landau damping, we must consider initial data in the complement of the kernel of the operator $\mathcal{L}$. Viewed as an operator on $\mathcal{H}$ the kernel of $\mathcal{L}$ is trivial; see Lemma 3.10.

Definition 1.1. (Nonquantitative Landau damping) For $k > \frac{1}{2}$ and $\varepsilon > 0$, let $f^{k,\varepsilon}$ denote the steady state of the Vlasov–Poisson system (1.1)–(1.3) of the form (1.4). We say that the linearised Vlasov–Poisson equation (1.13) *Landau damps* if, for any initial data $f_0 \in \mathrm{D}(\mathcal{L}) \subset \mathcal{H}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\nabla U_{\mathcal{T}f(t,\cdot)}\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}t = 0, \tag{1.18}$$

where $\mathbb{R}_+ \ni t \to f(t,\cdot) \in \mathcal{H}$ is the unique solution to (1.13) with initial data $f(0,\cdot) = f_0$.

Definition 1.1 connects to the first-order dynamics as follows: if $t \mapsto F(t,\cdot)$ solves (1.9), then $\partial_t U_F = U_{\partial_t F} = U_{\partial_t f_+} = -U_{\mathcal{T}f_-} = -U_{\mathcal{T}f}$, where we recall f_+ is the even part of F, $f_- = f$ is the odd part. It follows that (1.18) is equivalent to the claim

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\nabla \partial_t U_{F(t,\cdot)}\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}t = 0.$$

Formula (1.18) implies a very weak form of decay of the macroscopic quantity $\|\nabla U_{\mathcal{T}f(t,\cdot)}\|^2_{L^2(\mathbb{R}^3)}$, without a rate. We chose to define Landau damping via (1.18) for specificity, but one could in principle consider other macroscopic quantities. We now state our main theorem.

Theorem 1.2. (Oscillation vs. relaxation) *For $k > \frac{1}{2}$ and $\varepsilon > 0$, let $f^{k,\varepsilon}$ denote the steady state of the Vlasov–Poisson system* (1.1)–(1.3) *of the form* (1.4)*. Then the following dichotomy holds:*

(a) For any $\frac{1}{2} < k \leq 1$ there exists an $\varepsilon_0 = \varepsilon_0(k) > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, the system (1.13) does not *damp. More precisely, there exists at least one strictly positive eigenvalue of $\mathcal{L}$.*

(b) For any $k > 1$ there exists an $\varepsilon_0 = \varepsilon_0(k) > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ the system (1.13) does *Landau damp in the sense of Definition 1.1. In particular, the point spectrum of $\mathcal{L}$ is empty.*

Our aim in the present paper is not to compute the rate of decay in the damped case ($k > 1$ and $0 < \varepsilon \ll 1$), but instead to focus on the dichotomy stated in Theorem 1.2. An important consequence of the theorem is that the gravitational relaxation is sensitive to the regularity of the underlying steady state. Note that the steady states $f^{k,\varepsilon}$ are always C^∞ in the interior of their phase-space support and $C^{\lfloor k\rfloor, k-\lfloor k\rfloor}$ up to and including the vacuum boundary $\{E = E_0\}$. Therefore the regularity limitation stems from the boundary behaviour.[2]

The polytropes defined through (1.4) are very commonly studied in the gravitational kinetic theory [12]. However, a simple examination of the proof shows that it is only the regularity of $f^{k,\varepsilon}$ near the phase-space vacuum boundary that discriminates between Landau damping and oscillations. We may therefore use more general ansatz functions $\tilde{\varphi}(E)$ whose Taylor expansion near the vacuum reads as $\tilde{\varphi}(E) \approx (E_0 - E)^k + o_{E\to E_0}\big((E_0 - E)^k\big)$; here k plays the same role as in Theorem 1.2. For example, linearised perturbations of the King model $\varphi_{\text{King}}(E) = \varepsilon(e^{E_0-E} - 1)_+$ with $0 < \varepsilon \ll 1$ do not damp.

A further direct consequence of the proof of the main theorem is that our methods can be used to give a criterion for the absence of embedded eigenvalues for general radial steady states, i.e., with and without the central point mass. The proof shows that a sufficient condition for the absence of embedded eigenvalues is for some explicitly computable constant to be sufficiently small, cf. Remark 4.6. Introducing the smallness parameter $0 < \varepsilon \ll 1$ gives a natural class of steady states where the constant is indeed small enough. In addition, the smallness assumption allows us to rigorously verify many of the structural properties of the steady states, most notably the monotonicity of the period function. From numerical simulations such properties are known to be true also when no small parameter is present.

The proof further shows that, in general, embedded eigenvalues can only exist at low frequencies. For steady states $f^{k,1}(r,w) = (E_0 - E)^k$ with $k > 1$ and $\varepsilon = 1$, there exists an integer m_0 depending only on the steady state such that

[2] We note that the steady states $f^{k,\varepsilon}$ fall in the regularity class for which one can prove local-in-time well-posedness.

there are no embedded eigenvalues larger than $\frac{4\pi^2 m_0^2}{T_{\min}^2}$, cf. Corollary 4.7; here, $T_{\min}$ denotes the minimal period occurring in the steady state, see (3.11). For this result no "smallness" of the steady state is imposed, and it also holds in the case without a point mass, see Remark 4.8. In the setting of generic radial equilibria of the form $f_0(r, w, L) = \varphi(E, L)$ we point the reader to the recent work by one of the authors [64], where the methods of this work have been extended to show nonquantitative damping.

The RAGE theorem was used to show nonquantitative damping around certain steady states of the 2D Euler equations by Lin and Zeng, see, e.g., [46, Thm. 11.7]. Our set-up is manifestly based on the second order formulation (1.13). However we can equivalently work in the first order formulation (1.9). Following the strategy of [46] we can restrict the dynamics to the invariant subspace of so-called *linearly dynamically accessible* perturbations $\mathrm{im}(\mathcal{T})$ and exhibit weak decay of $\|\nabla U_F\|_{L^2(\mathbb{R}^3)}$ for the data in the orthogonal complement of the kernel of $\tilde{\mathcal{L}}$. The subject of quantitative inviscid damping and the nonlinear stability around (typically) shear flow solutions of 2D Euler has been a very active area in the past decade, following the nonlinear stability result of Bedrossian and Masmoudi [8]. Without attempting to give an exhaustive overview, we refer the reader to the introductions of the recent articles [35,51], the review article [7], and the lecture [33] for an exhaustive list of references.

Theorem 1.2 is the first result which shows that Landau damping occurs around compactly supported, inhomogeneous equilibria of the gravitational Vlasov–Poisson system. The stated dichotomy between relaxation and oscillation, as well as the sharp transition threshold $k = 1$ are, to our knowledge, new. This situation is reminiscent of the well-known fact in the spectral theory of Schrödinger operators $-\Delta + V$ where the smallness of the potential V (in the right sense) helps to exclude bound states in dimension $d = 3$ and cannot exclude them when $d = 1$. In this analogy, the polytropic index k, which measures the regularity of the steady state at the vacuum boundary, plays the role of the dimension d. The mechanism that leads to this regularity threshold is in particular very different from the result of Lin and Zeng [45] which states that Landau damping in the plasma case around smooth space-homogeneous equilibria does not occur if the perturbations are not sufficiently smooth. The obstruction to damping in [45] comes from the existence of arbitrarily close nontrivial BGK waves which can only exist in a function class of sufficiently low regularity.

The possibility of oscillatory linear behaviour and the contrast to gravitational damping have been discussed in the physics literature [12, Ch. 5], see also [3,4, 47,52,66]. The observation that the smoothness of the perturbed steady state is relevant for the nonlinear damping is made in the numerical work of Ramming and Rein [57], where radial steady states without a central point mass are considered. We also point out the influential work of Kalnajs [36,37] where a formal approach is developed to study the decay of macroscopic quantities for linear perturbations using action-angle variables, see the discussion in [12, Sc. 5.3.2].

The question of gravitational relaxation was investigated in the pioneering work of Lynden-Bell [48,49], see also [12,50], who recognised that there exists a phase-mixing mechanism which could explain damping around stationary galaxies. By definition, phase mixing refers to a process according to which macroscopic quantities, like the spatial density or gravitational potential associated to the solutions of the pure transport problem

$$\partial_t^2 f - \mathcal{T}^2 f = 0, \tag{1.19}$$

decay in time. This mechanism was informally described by Lynden-Bell [48] and relies on the crucial monotonicity assumption $T'(E) \neq 0$ on $\bar{I}$, where $T(E)$ is the particle period function and $\bar{I}$ is the action interval of the steady state, see (2.1)–(2.2). Intuitively, this monotonicity condition allows the particles to explore the phase space very efficiently and therefore creates a mixing effect. In practice one can use arguments *à la* Riemann–Lebesgue lemma [13,62] or vector field commutators [14,53] to obtain decay. However, equation (1.19) is not the linearised dynamics around the steady state, and Theorem 1.2 shows that no mixing occurs when $\frac{1}{2} < k \leq 1$ despite the fact that the pure transport part does mix irrespective of how small the gravitational response term $\mathcal{R}$ is. More precisely, we show that the collective response of the gravitational system as measured by the operator-valued potential $\mathcal{R}$ can create nontrivial pure point spectrum. Furthermore, the fact that there is some form of mixing in the regime $k > 1$ as implied by Theorem 1.2 is highly nontrivial and involves a careful analysis of the response operator $\mathcal{R}$. More recently, decay results for the pure transport dynamics in the 3-D case and in the presence of a large point mass potential were shown in [15,28]. The former work deals with data nontrivially supported near the elliptic points. The latter work deals only with data supported away from such elliptic trapping, but contains some nonlinear applications to the VP-flow near the vacuum. We also mention a recent result [65], wherein the author shows pointwise nonquantitative decay-in-time of the gravitational force field for data in the absolutely continuous subspace, around plane-symmetric equilibria of the gravitational VP-system.

In the plasma case, nonlinear Landau damping around spatially homogeneous steady states was rigorously shown in the celebrated work of Mouhot and Villani [54], see also [9,23]. The results of [54] also apply to gravitational interactions (applying the Jeans swindle, see [12,39]), but such steady states do not represent isolated solutions of the Vlasov–Poisson system, see also the related work [6]. For plasma dynamics, linear damping around homogeneous equilibria in the whole space was recently analysed in [11,29], see also [22]. For a recent nonlinear result see [34], for the so-called screened case see [10,30,32], and for the case of massless electrons see [21]. Far less is known about damping around spatially inhomogeneous steady states. The Guo and Lin [25] constructed examples of stable BGK waves (that do not contain trapped particles) with a non-empty and with an empty point spectrum. The first Landau damping result for a class of BGK waves with a trapping region was shown by Després [18]. For a recent overview of known results about Landau damping, see [5].

The plan for this paper is as follows: the basic properties of the steady states and the linearised operator are explained in Sect. 3. In Sect. 4 we prove that there are no embedded eigenvalues when $k > 1$ and ε is sufficiently small, see Theorem 4.5. In Sect. 5 we derive the criterion for the existence of eigenvalues outside the essential spectrum, see Proposition 5.3. We then use it to show that such eigenvalues exist when $\frac{1}{2} < k \leq 1$ and do not when $k > 1$, see Theorems 5.5 and 5.4 respectively. Theorem 1.2 is finally proved in Sect. 6. In Appendix A we provide many key results about the underlying family of steady states, most notably various uniform-in-ε bounds for the period function $T(E)$ and its derivatives, as they play a crucial role in our analysis. Before we enter into the detailed proofs, in Sect. 2 we give a short overview of the general strategy which we employ.

2. An Overview of the Proof

The starting point for our analysis is a reformulation of (1.13) in action-angle variables [12,50]. We denote the minimal particle energy of the steady state by $E_{\min}$. Letting

$$I :=]E_{\min}, E_0[\tag{2.1}$$

be the "action" interval, we associate to any $E \in I$ two unique radii $r_-(E) < r_+(E)$ such that $\Psi(r_\pm(E)) = E$. Particles are trapped inside the potential well defined by the effective potential Ψ, and at any fixed energy level $E \in I$, they oscillate periodically between their turning points $r_-(E)$ and $r_+(E)$. The period $T(E)$ of this motion is given by the formula

$$T(E) := 2\int_{r_-(E)}^{r_+(E)} \frac{\mathrm{d}r}{\sqrt{2E - 2\Psi(r)}}, \qquad E \in I. \tag{2.2}$$

The angle θ parametrises this radial motion, suitably normalised by the period function. More precisely, for $(r, w) \in \Omega$ with $w \geq 0$ and $E = E(r, w)$ given by (1.5), the angle is defined as

$$\theta(r, w) = \frac{1}{T(E)} \int_{r_-(E)}^{r} \frac{\mathrm{d}s}{\sqrt{2E - 2\Psi(s)}} \in [0, \frac{1}{2}]. \tag{2.3}$$

Letting $\theta(r, w) = 1 - \theta(r, -w)$ for $w < 0$ leads to the one-to-one change of variables $(r, w) \mapsto (\theta, E)$, where Ω, i.e. the interior of the support of the steady state in phase space, is mapped onto the cylinder

$$\mathbb{S}^1 \times I.$$

Here $\mathbb{S}^1$ is the 1-dimensional torus, i.e. $\mathbb{S}^1 := [0, 1]$, where 0 and 1 are identified. In action-angle variables (θ, E), the transport operator $\mathcal{T}$ is now given by the simple formula

$$\mathcal{T} = \frac{1}{T(E)} \partial_\theta.$$

This allows us to explicitly determine the essential spectrum of $-\mathcal{T}^2$ in terms of the period function (2.2). Moreover, the gravitational response operator $\mathcal{R}$ does not affect the essential spectrum and we obtain that the operator $\mathcal{L}$ has essential spectrum of the form $[\frac{4\pi^2}{T_{\max}^2}, \infty[$ for $0 < \varepsilon \ll 1$, where $T_{\max} < \infty$ is the maximum of the period function T over $\bar{I}$, cf. Corollary 3.11. Proving these statements mainly relies on a frequency analysis in the angle variable θ. For $f \in L^2(\mathbb{S}^1)$ we let

$$\hat{f}(\ell) := \int_{\mathbb{S}^1} f(\theta)\, e^{-2\pi i \ell\theta}\, \mathrm{d}\theta, \;\; \ell \in \mathbb{Z}; \tag{2.4}$$

Fourier transformations always refer to the variable θ, also for functions of several variables.

Absence of embedded eigenvalues (Sect. 4). The hardest part of the proof of Theorem 1.2 is to show that there are no eigenvalues of $\mathcal{L}$ embedded in the essential spectrum when $k > 1$, see Theorem 4.5. If we assume, by contradiction, that there exists an eigenvalue of $\mathcal{L}$ of the form $\frac{4\pi^2 m^2}{T(E_m)^2}$ for some $(m, E_m) \in \mathbb{N} \times \bar{I}$, then $\pm\frac{2\pi i m}{T(E_m)}$ is an eigenvalue of $\tilde{\mathcal{L}}$, i.e. there exists an f such that $\tilde{\mathcal{L}} f = \frac{2\pi i m}{T(E_m)} f$. We move to action-angle variables and pass to the Fourier representation

$$f(\theta, E) = \sum_{\ell\in\mathbb{Z}} \hat{f}(\ell, E) e^{2\pi i \ell\theta}, \;\; U_f(\theta, E) = \sum_{\ell\in\mathbb{Z}} \widehat{U_f}(\ell, E) e^{2\pi i\ell\theta}$$

of the unknowns, where a simple calculation then shows that for almost every $E \in I$,

$$\hat{f}(\ell, E) = -T_m \frac{|\varphi'(E)| \widehat{U_f}(\ell, E)}{T_m - \frac{m}{\ell} T(E)}, \;\; E \in I, \; \ell \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}, \tag{2.5}$$

where $T_m := T(E_m)$, see Lemma 4.1.

Gravitational field via the Plancherel identity. The key idea is to use the Poisson equation (1.12) to express $\|\nabla U_f\|^2_{L^2(\mathbb{R}^3)}$ as $-16\pi^3 \int f\, U_f T(E)\, \mathrm{d}(\theta, E)$. By the Plancherel identity and (2.5) we then conclude that

$$\frac{1}{16\pi^3} \int_{\mathbb{R}^3} |\nabla U_f|^2\, \mathrm{d}x = T_m \sum_{\ell\neq 0} \int_I \frac{T(E)|\varphi'(E)|}{T_m - \frac{m}{\ell} T(E)} \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E. \tag{2.6}$$

Recall that by (1.4), $|\varphi'(E)| = \mathcal{O}(\varepsilon)$, so the way to reach a contradiction is to show that the right-hand side of (2.6) is bounded by $C\varepsilon \int |\nabla U_f|^2\, \mathrm{d}x$ and then use the smallness of ε to absorb $\|\nabla U_f\|_{L^2(\mathbb{R}^3)}$ into the left-hand side. The fundamental difficulty in doing so is the small denominators appearing inside the integral on the right-hand side of (2.6). Clearly there can exist frequency-energy pairs (ℓ, E_ℓ) such that $T_m - \frac{m}{\ell} T(E_\ell) = 0$.

Log-singularity. The idea is to rewrite such a possible singularity $\frac{1}{T_m - \frac{m}{\ell} T(E)}$ as $-\frac{\ell}{m T'(E)} \partial_E \left(\log(T_m - \frac{m}{\ell} T(E))\right)$ in the region where the argument of the logarithm is positive. Note that we are using the property $T' \neq 0$ on $\bar{I}$ in a fundamental way. Our idea is simple; for any frequency ℓ we integrate by parts in E to offload the E-derivative onto the gravitational potential $\widehat{U_f}(\ell, E)$ so that we schematically deal with terms of the form

$$\varepsilon \sum_{\ell\in\mathbb{Z}^*} \int_I g(E) \left|\log\left(T_m - \frac{m}{\ell} T(E)\right)\right| |\widehat{U_f}(\ell, E)|\, |\partial_E \widehat{U_f}(\ell, E)|\, \mathrm{d}E, \tag{2.7}$$

where g is some "well-behaved" weight. The log-singularity is very mild and the hope is that the integration in E will control it. The small factor of ε is there due to the assumption $|\varphi'(E)| \lesssim \varepsilon$.[3] The first big issue is that the integration-by-parts produces boundary terms, and they must either vanish or have to show up with the correct sign. This is a serious issue, and we must carefully analyse the frequency-energy pairs (ℓ, E) that produce small and vanishing denominators, see Lemma 4.3. The introduction of the above log-singularity is necessary only at frequencies for which the contributions from the right-hand side of (2.5) are positive. It is a structural feature of the problem that precisely in this range all the boundary terms are either of *good* sign or *vanish* due to the regularity and require no further estimates. For the vanishing boundary terms, we crucially use the regularity assumption $k > 1$ which implies $\varphi'(E_0) = 0$.

The second key issue is that the minimal point of the effective potential Ψ, corresponding to the radius r_* and energy $E_{\min}$, is a critical point with a strictly positive second derivative. This property, as shown in Lemma 3.5, implies that for any $\theta \in \mathbb{S}^1$ the map $E \mapsto r(\theta, E)$ is merely $C^{0,\frac{1}{2}}$ at $E = E_{\min}$ and, in particular,

$$|\partial_E r(\theta, E)| \lesssim (E - E_{\min})^{-\frac{1}{2}}, \quad (\theta, E) \in \mathbb{S}^1 \times I,$$

which creates singular powers of $E - E_{\min}$ when we try to compare $|\partial_E \widehat{U_f}(\ell, E)|$ to $\partial_r U_f$. This is intimately related to the particle trapping at the space-time cylinder $\{r = r_*\}$. We get around this by introducing positive powers of $E - E_{\min}$ as weights to "de-singularise" $\partial_E \widehat{U_f}(\ell, E)$ and compensate with negative powers of $E - E_{\min}$ hitting the mild log-singularity, so that we can close the estimates via Cauchy–Schwarz, see Step 2 of the proof of Theorem 4.5. The proof shows that the elliptic character of the Poisson equation as manifested through the energy-like identity (2.6) gives the winning strategy, as it permits us to estimate the function $\widehat{U_f}(\ell, E)$ by the derivatives of U_f.

Existence vs. absence of eigenvalues in the principal gap (Sect. 5). Existence of positive eigenvalues of $\mathcal{L}$ below the bottom of the essential spectrum parallels the classical quantum-mechanical problem of finding bound states below the absolutely continuous part of the spectrum of a Schrödinger operator. A classical strategy to study bound states is the Briman-Schwinger principle [44, Sc. 4.3.1], a version of which was pioneered by Mathur [52] for the Vlasov–Poisson system in a different context. In [27,40] the authors independently derived a criterion for the existence of eigenvalues in the *principal gap*

$$\mathcal{G} :=]0, \min \sigma_{\mathrm{ess}}(\mathcal{L})[=]0, \frac{4\pi^2}{T_{\max}^2}[. \tag{2.8}$$

The work [27] additionally gave examples of steady states where such a criterion can be verified. We apply a slightly different version of the principle developed in

[3] It turns out that the factor ε in (2.7) can be refined by an additional factor of $\frac{1}{\sqrt{m}}$, independent of ε, so that at high frequencies $m \gg 1$, smallness can be enforced without any smallness assumption on the microscopic equation of state φ, see Corollary 4.7.

[27] to obtain a necessary and sufficient condition for the existence of eigenvalues in the principal gap $]0, \min \sigma_{\mathrm{ess}}(\mathcal{L})[$; see Proposition 5.3. If $k > 1$ this criterion is used in Theorem 5.4 to show that there are no eigenvalues in the principal gap and if $\frac{1}{2} < k \leq 1$, we use it to prove the opposite, namely that there are oscillatory eigenvalues in the gap and therefore no damping occurs. Both of these proofs are again performed in the $0 < \varepsilon \ll 1$ regime in order to control steady state quantities like the period function T.

The RAGE theorem and the proof of the main result (Sect. 6). To complete the proof of Theorem 1.2 we observe that, by the above, the operator $\mathcal{L}$ has empty point spectrum on $\mathcal{H}$ when $k > 1$ and $\varepsilon > 0$ is sufficiently small. We rephrase the linear dynamics $\partial_t^2 f + \mathcal{L} f = 0$ as a first order system and then apply the RAGE theorem [60] to show the nonquantitative decay statement (1.18). To make this work, we only need to show that the operator $f \mapsto \left|\varphi'(E)\right| U_{\mathcal{T} f}$ is compact on a suitable function space, which again works by virtue of the smoothing properties of the solution operator to the Poisson equation (1.2), see Sect. 6.

Properties of the steady states and the period function $T(E)$ (Appendix A). One of the key analytical tools in our analysis are good uniform-in-ε estimates for steady states $f^{k,\varepsilon}$ with fixed $k > \frac{1}{2}$ and $0 < \varepsilon \ll 1$. Most notably, we show that, as $\varepsilon \to 0$, the period function T converges in C^2 to the explicitly known period function T^0 generated by the single point mass:

$$T^0(E) = \frac{\pi}{\sqrt{2}} \frac{M}{(-E)^{\frac{3}{2}}}.$$

In this way we deduce that $T = T(E)$ is strictly increasing in E for $0 < \varepsilon \ll 1$, which is a key ingredient in our analysis. In general, (monotonicity) properties of period functions are important in the analysis of the linearised Vlasov–Poisson system, cf. [27,40], as well as in the general context of Hamiltonian systems, cf. [16,17]. Further uniform-in-ε bounds on T up to its second derivative ensure that various constants appearing in the proof of Theorem 4.5 are ε-independent.

3. Steady States and Linearisation

3.1. Existence of Steady States

Lemma 3.1. *Fix the parameter $\kappa < 0$ so that the single-gap condition (1.8) holds. Then for any $k > \frac{1}{2}$ and $\varepsilon > 0$ there exists a steady state $f^{k,\varepsilon}$ of the system (1.1)–(1.3) defined by (1.4). The steady state is compactly supported in phase space, more precisely,*

$$\operatorname{supp}(f^{k,\varepsilon}) \subset [R^0_{\min}, R^0_{\max}] \times \left[-\frac{\sqrt{2M}}{\sqrt{R^0_{\min}}}, \frac{\sqrt{2M}}{\sqrt{R^0_{\min}}} \right], \tag{3.1}$$

where $0 < R^0_{\min} < R^0_{\max} < \infty$ are given by

$$R^0_{\min} := \frac{-M + \sqrt{M^2 + 2\kappa L}}{2\kappa}, \qquad R^0_{\max} := \frac{-M - \sqrt{M^2 + 2\kappa L}}{2\kappa}. \tag{3.2}$$

The total mass of the steady state is positive and finite, i.e.,

$$0 < M_{\mathrm{s}} := 4\pi \int_0^\infty r^2 \rho(r)\,\mathrm{d}r < \infty, \tag{3.3}$$

where ρ is the spatial density associated to $f^{k,\varepsilon}$.

The proof follows the strategy of [24,56]; we give the details in Appendix A.1.

An important quantity associated to the steady state is the effective potential Ψ defined in (1.6) whose properties we analyse next.

Lemma 3.2. *(a) There exists a unique radius $r_* > 0$ such that*

$$\min_{]0,\infty[} \Psi = \Psi(r_*) =: E_{\min} < 0. \tag{3.4}$$

This radius is given as the unique zero of Ψ' on $]0,\infty[$ and it holds that $\Psi' < 0$ on $]0, r_*[$ and $\Psi' > 0$ on $]r_*, \infty[$.

(b) Let

$$\mathbb{A} :=]E_{\min}, 0[\tag{3.5}$$

denote the set of all admissible particle energies. *Then, for any $E \in \mathbb{A}$ there exist two unique radii $r_\pm(E)$ satisfying*

$$0 < r_-(E) < r_* < r_+(E) < \infty$$

and

$$\Psi\left(r_\pm(E)\right) = E. \tag{3.6}$$

Proof. The assertions follow from the asymptotic behavior of Ψ and Ψ' at $r = 0, \infty$, and the fact that $r^3\Psi'$ is strictly increasing. □

In particular, since (A.5) implies that $\rho(r) > 0$ is equivalent to $\Psi(r) < E_0$ for $r > 0$, we conclude that

$$\mathrm{supp}\,(\rho) = [r_-(E_0), r_+(E_0)] =: [R_{\min}, R_{\max}] \subset [R^0_{\min}, R^0_{\max}]. \tag{3.7}$$

The steady state has the following regularity properties:

Lemma 3.3. *It holds that $U \in C^3([0,\infty[)$ and $\rho \in C^1([0,\infty[)$. In addition, $U, \rho \in C^\infty([0,\infty[\setminus\{R_{\min}, R_{\max}\})$.*

Proof. The continuous differentiability of ρ on $[0,\infty[$ follows by (A.5) since $E_0 - U = y \in C^1([0,\infty[)$ and $g \in C^1(\mathbb{R})$. Twice differentiating (A.1) then yields $U \in C^3([0,\infty[)$. Moreover, observe that $g \in C^\infty(\mathbb{R}\setminus\{0\})$ and that $E_0 - \Psi(r) = 0$ is equivalent to $r \in \{R_{\min}, R_{\max}\}$ by Lemma 3.2. Thus, we conclude that U and ρ are indeed infinitely differentiable on $[0,\infty[\setminus\{R_{\min}, R_{\max}\}$ by iterating the above argument. □

We note that a larger polytropic exponent k leads to higher regularity of U and ρ.

3.2. Particle Motions and the Period Function

Let $f^{k,\varepsilon}$ be a steady state as given by Lemma 3.1 with associated effective potential Ψ defined in (1.6). Because the particle energy is of the form $E(r, w) = \frac{1}{2}w^2 + \Psi(r)$, the characteristic flow of the steady state is governed by the system

$$\dot{r} = w, \qquad \dot{w} = -\Psi'(r). \tag{3.8}$$

Due to the structure of the effective potential established in Lemma 3.2, the behaviour of solutions of this system is similar to the three-dimensional case [27, p. 624f.]: The particle energy E is conserved along solutions of (3.8) and every solution with negative energy $E < 0$ is trapped, global in time, and either constant (with energy $E = E_{\min}$) or time-periodic with the period function $T(E)$ given by (2.2).

For $E \in \mathbb{A}$ let $(R, W)(\cdot, E)\colon \mathbb{R} \to]0, \infty[\times\mathbb{R}$ denote the global solution of (3.8) satisfying the initial condition

$$R(0, E) = r_-(E), \qquad W(0, E) = 0.$$

We further define that

$$r(\theta, E) := R(\theta\, T(E), E), \quad w(\theta, E) := W(\theta\, T(E), E), \qquad E \in \mathbb{A},\ \theta \in \mathbb{S}^1, \tag{3.9}$$

and note that $(r, w)(\cdot, E)$ is periodic with period 1 for $E \in \mathbb{A}$. The period function and the characteristics enjoy the following regularity properties:

Lemma 3.4. *It holds that* $(R, W) \in C^2(\mathbb{R} \times \mathbb{A})$ *and* $T \in C^2(\mathbb{A})$.

Proof. Since $\Psi \in C^3(]0, \infty[)$ by Lemma 3.3, the implicit function theorem implies that $r_\pm \in C^3(\mathbb{A})$. We thus conclude the claimed regularity of (R, W) by basic ODE theory.

Lebesgue's dominated convergence theorem yields that T is continuous on $\mathbb{A}$, cf. [27, Lemma B.7]. Because the period function is given as the solution of $W(T(E), E) = 0$ with $\dot{W}(T(E), E) > 0$ for $E \in \mathbb{A}$, applying the implicit function theorem similarly to [40, Theorem 3.6 et seq.] then implies that $T \in C^2(\mathbb{A})$. $\square$

A fundamental ingredient in our analysis is the use of action-angle variables introduced in (2.1)–(2.3). For functions $f\colon \Omega \setminus \{(r_*, 0)\} \to \mathbb{R}$, we write that

$$f(\theta, E) = f((r, w)(\theta, E))$$

for $(\theta, E) \in \mathbb{S}^1 \times I$. Note that integrals change via

$$\mathrm{d}w\, \mathrm{d}r = T(E)\, \mathrm{d}\theta\, \mathrm{d}E. \tag{3.10}$$

Action-angle coordinates are not defined at $(r, w) = (r_*, 0) \in \Omega$ since the characteristic system (3.8) possesses a stationary solution associated to the minimal energy $E_{\min}$ there (this corresponds to the so-called elliptic point of the Hamiltonian). The next result controls the behaviour of the action-angle coordinates at this singularity. Before we proceed, we introduce the abbreviations

$$T_{\min} := \inf_I T, \qquad T_{\max} := \sup_I T, \tag{3.11}$$

and also let $T'_{\min} := \inf_I T'$, $T'_{\max} := \sup_I T'$, $T''_{\min} := \inf_I T''$, and $T''_{\max} := \sup_I T''$. We later verify that each of these values is finite, cf. Remark A.18.

Lemma 3.5. *Let $r\colon \mathbb{S}^1 \times \mathbb{A} \to]0, \infty[$ be defined as in (3.9). Then $r \in C^2(\mathbb{S}^1 \times \mathbb{A})$ and there exists a constant $C > 0$ such that*

$$|r(\theta, E) - r_*| + |\partial_\theta r(\theta, E)| \le C\sqrt{E - E_{\min}} \tag{3.12}$$

as well as

$$|\partial_E r(\theta, E)| \le \frac{C}{\sqrt{E - E_{\min}}}, \qquad (\theta, E) \in \mathbb{S}^1 \times I.$$

The constant C is bounded in terms of $T_{\max}$, $T'_{\max}$, $|I|$, $\|\Psi''\|_{L^\infty([R_{\min}, R_{\max}])}$, $\|\Psi'''\|_{L^\infty([R_{\min}, R_{\max}])}$, and $\Psi''(r_*)^{-1}$.

Proof. The claimed regularity of r follows by Lemma 3.4. For $E_{\min} \le E < 0$ let $z = z(\cdot, E)\colon \mathbb{R} \to \mathbb{R}$ be the unique global solution of

$$\ddot{z} = -\Psi''(R(\cdot, E))\, z, \qquad z(0) = 1,\ \dot{z}(0) = 0, \tag{3.13}$$

where we set $R(\cdot, E_{\min}) \equiv r_*$. Grönwall's inequality implies that there exists a constant $C > 0$ as described in the statement of the lemma such that $|z(s, E)| \le C$ for $s \in [0, T_{\max}],\ E \in I$.

Furthermore, basic ODE theory yields

$$\partial_E R(s, E) = \partial_E r_-(E)\, z(s, E), \qquad s \in \mathbb{R},\ E \in I.$$

Because $\partial_E r(\theta, E) = \dot{R}(\theta\, T(E), E)\, \theta\, T'(E) + \partial_E R(\theta\, T(E), E)$ for $(\theta, E) \in \mathbb{S}^1 \times I$ and

$$|\dot{R}(s, E)| = \sqrt{2E - 2\Psi(R(s, E))} \le \sqrt{2}\sqrt{E - E_{\min}} \le \frac{\sqrt{2}\,|I|}{\sqrt{E - E_{\min}}} \tag{3.14}$$

for $(s, E) \in \mathbb{R} \times I$, it remains to show that

$$|\partial_E r_-(E)| \le \frac{C}{\sqrt{E - E_{\min}}}, \qquad E \in I, \tag{3.15}$$

for some constant $C > 0$ as specified in the statement of the lemma. In particular, (3.12) follows by (3.14). In order to establish (3.15), first observe that

$$\partial_E r_-(E) = \frac{1}{\Psi'(r_-(E))} \tag{3.16}$$

for $E \in I$ by the implicit function theorem. Moreover, the radial Poisson equation (A.1) yields

$$\Psi''(r) = -\frac{2\Psi'(r)}{r} + \frac{L}{r^4} + 4\pi\rho(r), \qquad r > 0. \tag{3.17}$$

In particular,

$$\alpha := \Psi''(r_*) > 0 = \Psi'(r_*) \tag{3.18}$$

by Lemma 3.2. This implies that in a small neighbourhood of $E = E_{\min}$ the denominator in (3.17) behaves to the leading order like $r_-(E) - r_*$, which then easily yields (3.15) using standard continuity arguments and the mean value theorem. □

3.3. Limiting Behaviour of Small Steady States

For fixed $k > \frac{1}{2}$ and κ satisfying (1.8) we study the behaviour of the steady state family $f^{k,\varepsilon} = \varphi(E) = \varepsilon\,\tilde{\varphi}(E)$ given by Lemma 3.1 as $\varepsilon \to 0$. In this section, we always add a superscript ε to steady state quantities to make the ε-dependencies more visible.

The limiting case $\varepsilon = 0$ corresponds to $U^0 \equiv 0$. Hence, the associated effective potential is of the form

$$\Psi^0(r) := -\frac{M}{r} + \frac{L}{2r^2}, \qquad r > 0. \tag{3.19}$$

The structure of this function is similar as in the case $\varepsilon > 0$ described in Lemma 3.2, with

$$\min_{]0,\infty[} \Psi^0 = \Psi^0(r_*^0) = E_{\min}^0 = -\frac{M^2}{2L}, \qquad r_*^0 = \frac{L}{M}, \tag{3.20}$$

and

$$r_\pm^0(E) = \frac{-M \mp \sqrt{M^2 + 2EL}}{2E}$$

for $E \in \mathbb{A}^0$, where

$$\mathbb{A}^0 :=]E_{\min}^0, 0[. \tag{3.21}$$

Accordingly, the period function takes on the form

$$T^0(E) := 2\int_{r_-^0(E)}^{r_+^0(E)} \frac{\mathrm{d}r}{\sqrt{2E - 2\Psi^0(r)}} = \frac{\pi}{\sqrt{2}} \frac{M}{(-E)^{\frac{3}{2}}} \tag{3.22}$$

for $E \in \mathbb{A}^0$; the latter identity is due to a straight-forward calculation.

Lemma 3.6. *The following assertions hold.*

(a) $E_{\min}^\varepsilon \to E_{\min}^0$ *and* $E_0^\varepsilon \to \kappa$ *as* $\varepsilon \to 0$*; recall* (3.4) *and* (3.20).

(b) $T^{\varepsilon}_{\min} \to T^0_{\min}$ *and* $T^{\varepsilon}_{\max} \to T^0_{\max}$ *as* $\varepsilon \to 0$, *where the limiting action interval (compare* (2.1)*) is*

$$I^0 :=]E^0_{\min}, \kappa[\tag{3.23}$$

and $T^0_{\min}$, $T^0_{\max}$ *are defined similar to* (3.11)*; recall* (2.2)*, and* (3.22)*. Moreover, there exist* $c, C > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $0 \le \varepsilon < \varepsilon_0$ *and* $j \in \{0, 1, 2\}$ *there holds*

$$c \le (T^{\varepsilon})^{(j)}(E) \le C, \qquad E \in I^{\varepsilon}. \tag{3.24}$$

In particular, T^{ε} *is strictly increasing on* I^{ε} *for* $0 \le \varepsilon < \varepsilon_0$.

Proof. The proof of these convergences is rather technical and postponed to Appendix A.2. Part (a) is shown in Lemmas A.1 and A.2, part (b) is proven in Lemmas A.9, A.14 and A.17. □

3.4. Linearisation

In order to analyse the linearised operator $\mathcal{L}$ given by (1.14) with methods from functional analysis, we first define the transport operator $\mathcal{T}$ in a weak sense, based on [61, Def. 2.1]:

For a function $f \in H$ the transport term $\mathcal{T} f$ *exists weakly* if there exists some $\mu \in H$ such that for every test function $g \in C^1_c(\Omega)$,

$$\langle f, \mathcal{T} g \rangle_H = -\langle \mu, g \rangle_H,$$

where $\mathcal{T} g$ is given by (1.11). In this case, $\mathcal{T} f := \mu$ *weakly*. The domain $\mathrm{D}(\mathcal{T})$ of $\mathcal{T}$ is the subspace of H where $\mathcal{T}$ exists weakly, while the domain of the squared transport operator is defined as

$$\mathrm{D}(\mathcal{T}^2) := \{f \in H \mid f \in \mathrm{D}(\mathcal{T}),\ \mathcal{T} f \in \mathrm{D}(\mathcal{T})\}.$$

We collect the following properties of the transport operator and its square as in [27], see also [24, Prop. 5.1] and [61] (further properties of $\mathcal{T}$ can be derived as in these papers):

Lemma 3.7. (Properties of $\mathcal{T}$ and $\mathcal{T}^2$)

(a) $\mathcal{T} \colon \mathrm{D}(\mathcal{T}) \to H$ *is skew-adjoint as a densely defined operator on* H*, i.e.,* $\mathcal{T}^* = -\mathcal{T}$*, and* $\mathcal{T}^2 \colon \mathrm{D}(\mathcal{T}^2) \to H$ *is self-adjoint.*

(b) The domains of $\mathcal{T}$ *and* $\mathcal{T}^2$ *can be characterised in action-angle coordinates as follows:*

$$\mathrm{D}(\mathcal{T}^m) = \Big\{ f \in H \mid f(\cdot, E) \in H^m_\theta \text{ for a.e. } E \in I \\ \text{and } \sum_{j=1}^{m} \int_I \frac{T(E)^{1-2j}}{|\varphi'(E)|} \int_{\mathbb{S}^1} |\partial_\theta^j f(\theta, E)|^2 \,\mathrm{d}\theta \,\mathrm{d}E < \infty \Big\}$$

for $m \in \{1, 2\}$*, where*

$$H^1_\theta := \{y \in H^1(]0, 1[) \mid y(0) = y(1)\}, \qquad H^2_\theta := \{y \in H^1_\theta \mid \dot{y} \in H^1_\theta\}. \tag{3.25}$$

In addition, for $f \in \mathrm{D}(\mathcal{T}^m)$ *with* $m \in \{1, 2\}$ *and a.e.* $(\theta, E) \in \mathbb{S}^1 \times I$,

$$(\mathcal{T}^m f)(\theta, E) = \left(\frac{1}{T(E)}\right)^m (\partial_\theta^m f)(\theta, E). \tag{3.26}$$

(c) The kernel of $\mathcal{T}$ *consists of functions only depending on* E, *i.e.,*

$$\ker(\mathcal{T}) = \{f \in H \mid \exists g : \mathbb{R} \to \mathbb{R} \text{ s.t. } f(r, w) = g(E(r, w)) \text{ a.e. on } \Omega\}. \tag{3.27}$$

(d) $\mathcal{T}$ *reverses* w*-parity and the restricted operator* $\mathcal{T}^2\big|_{\mathcal{H}} : \mathrm{D}(\mathcal{T}^2) \cap \mathcal{H} \to \mathcal{H}$ *is self-adjoint.*

(e) The spectrum and the essential spectrum of $-\mathcal{T}^2$ *are of the form*

$$\sigma(-\mathcal{T}^2) = \sigma_{\text{ess}}(-\mathcal{T}^2) = \overline{\left(\frac{2\pi \mathbb{N}_0}{T(I)}\right)^2},$$

$$\sigma(-\mathcal{T}^2\big|_{\mathcal{H}}) = \sigma_{\text{ess}}(-\mathcal{T}^2\big|_{\mathcal{H}}) = \overline{\left(\frac{2\pi \mathbb{N}}{T(I)}\right)^2}.$$

Proof. The skew-adjointness of $\mathcal{T}$ can be shown as in [61, Thm. 2.2], which then yields (a) using von Neumann's theorem [59, Thm. X.25]. Part (b) follows similarly to [27, Lemma 5.2 and Cor. 5.4]. The identity (3.26) then implies (c), while (d) is evident from parity considerations. Part (e) is due the observation that $\mathbb{S}^1 \times I \ni (\theta, E) \mapsto \sin(2\pi j\theta)\, \delta_{E^*}(E)$ defines an eigendistribution for $-\mathcal{T}^2$ or $-\mathcal{T}^2\big|_{\mathcal{H}}$; $j \in \mathbb{N}_0$ or $j \in \mathbb{N}$, respectively. The claimed structures of the spectra follow by applying Weyl's criterion [31, Thm. 7.2] similarly to [27, Thm. 5.7]. □

We next analyse the response operator $\mathcal{R}$ defined in (1.15).

Lemma 3.8. (Properties of $\mathcal{R}$) *The linear operator* $\mathcal{R} : H \to H$ *is bounded, symmetric, and non-negative (in the sense of quadratic forms, i.e.,* $\langle \mathcal{R}f, f\rangle_H \geq 0$ *for* $f \in H$*). The operator*

$$\sqrt{\mathcal{R}} : H \to H,\ \sqrt{\mathcal{R}} f(r, w) := 2\pi^{\frac{3}{2}} \left|\varphi'(E)\right| \frac{w}{r^2\sqrt{\rho(r)}} \int_{\mathbb{R}} \tilde{w}\, f(r, \tilde{w})\, \mathrm{d}\tilde{w} \tag{3.28}$$

is bounded, symmetric, non-negative, and $\sqrt{\mathcal{R}}\sqrt{\mathcal{R}} = \mathcal{R}$ *on* H*. Moreover,* $\sqrt{\mathcal{R}} f \in \mathcal{H}$ *and* $\mathcal{R}f \in \mathcal{H}$ *for* $f \in H$.

Proof. The claimed statements regarding $\mathcal{R}$ follow as in [27, Lemma 4.3]. The properties of $\sqrt{\mathcal{R}}$ can be derived similarly using the important identity

$$\int_{\mathbb{R}} w^2 \left|\varphi'(E)\right| \mathrm{d}w = \frac{r^2}{\pi}\, \rho(r), \qquad r > 0. \tag{3.29}$$

□

The response operator has a natural connection to the gravitational potential of the linear perturbation. Similarly to [27, Sc. A.1], we thus analyse the properties of such potentials defined by (1.12).

Lemma 3.9. *Let $g \in \mathrm{D}(\mathcal{T})$ and $f := \mathcal{T} g \in \mathrm{im}(\mathcal{T})$. Then $U_f \in H^2 \cap C^1([0,\infty[)$ with*

$$\|U_f\|_{H^2} + \|U_f\|_{L^\infty} + \|U_f'\|_{L^\infty} \le C\|f\|_H \tag{3.30}$$

for some constant $C > 0$ which can be estimated by ε and k. Furthermore,

$$U_f'(r) = \frac{4\pi^2}{r^2}\int_{\mathbb{R}} w\, g(r,w)\,\mathrm{d}w, \qquad r > 0, \tag{3.31}$$

$\operatorname{supp}(U_f') \subset [R_{\min}, R_{\max}]$, *and $U_f(|\cdot|) \in H^2 \cap C^1(\mathbb{R}^3)$. In action-angle coordinates, $U_f \in C^1(\mathbb{S}^1 \times \mathbb{A})$ with*

$$|\partial_E U_f| \le \frac{C}{\sqrt{E - E_{\min}}}\,|\partial_r U_f| \tag{3.32}$$

on $\mathbb{S}^1 \times I$ for $C > 0$ as in Lemma 3.5. Moreover, there exists a constant $C > 0$ such that for any $\ell \in \mathbb{Z}^*$,

$$\int_I \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E \le \frac{C}{\ell^2}\int |\nabla U_f|^2\,\mathrm{d}x. \tag{3.33}$$

Here $U_f(\theta, E) = U_f(r(\theta, E))$ for $(\theta, E) \in \mathbb{S}^1 \times \mathbb{A}$. For $j \in \mathbb{Z}$ it holds that $\hat{U}_f(j,\cdot) \in C^1(\mathbb{A})$ with $\partial_E \hat{U}_f(j,\cdot) = \widehat{\partial_E U_f}(j,\cdot)$ on $\mathbb{A}$.

Proof. First observe that

$$\begin{aligned}\|\rho_f\|^2_{L^2(]0,\infty[)} &= \pi^2 \int_0^\infty \frac{1}{r^4}\left(\int_{\mathbb{R}} f(r,w)\,\mathrm{d}w\right)^2 \mathrm{d}r \\ &\le C\int_0^\infty \left(\int_{\mathbb{R}} |\varphi'(E)|\,\mathrm{d}w\right)\left(\int_{\mathbb{R}} \frac{1}{|\varphi'(E)|}\, f(r,w)^2\,\mathrm{d}w\right)\mathrm{d}r \le C\|f\|_H^2\end{aligned}$$

since $\operatorname{supp}(\rho_f) \subset [R^0_{\min}, R^0_{\max}]$. In the last step we used the estimate

$$\int_{\mathbb{R}} |\varphi'(E)|\,\mathrm{d}w = C\,(E_0 - \Psi(r))_+^{k-\frac{1}{2}} \le C,$$

which follows by a calculation similar to (A.3). Hence the compact support of ρ_f implies that $\rho_f \in L^1 \cap L^2(]0,\infty[)$. By Lemma 3.7,

$$\frac{1}{\pi}\int_0^\infty r^2 \rho_f(r)\,\mathrm{d}r = \langle |\varphi'(E)|, \mathcal{T} g\rangle_H = -\langle \mathcal{T}|\varphi'(E)|, g\rangle_H = 0.$$

In particular, $U_f'(r) = 0$ for $r \in [0,\infty[\setminus[R^0_{\min}, R^0_{\max}]$, and $U_f(r) = 0$ for $r \ge R^0_{\max}$. Thus, $U_f \in C^1([0,\infty[)$ and $U_f, U_f' \in L^1 \cap L^\infty([0,\infty[)$ with $\|U_f\|_\infty + \|U_f'\|_\infty \le C\|\rho_f\|_2$ by (1.12). Together with the radial Poisson equation we conclude $U_f \in H^2([0,\infty[)$ and the estimate (3.30). The identity (3.31) follows via integration by parts together with a suitable approximation argument similar to [27, Eqn. (A.2)]. This regularity and Lemma 3.4 further imply that $U_f \in C^1(\mathbb{S}^1 \times \mathbb{A})$. The estimate (3.32) hence follows by Lemma 3.5.

To show (3.33) we use the assumption $\ell \neq 0$ to rewrite $\widehat{U_f}(\ell, E) = \frac{1}{2\pi i \ell} \widehat{\partial_\theta U_f}(\ell, E)$. Therefore, using $\partial_\theta U_f = \partial_r U_f \partial_\theta r$,

$$\int \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E \leq \frac{C}{\ell^2} \int_{\mathbb{S}^1 \times I} |\partial_\theta r|^2 |\partial_r U_f|^2 \,\mathrm{d}(\theta, E) \leq \frac{C}{\ell^2} \int |\nabla U_f|^2 \,\mathrm{d}x, \tag{3.34}$$

where we have used (3.12) in the last line. □

The identity (3.31) implies that for $f \in \mathrm{D}(\mathcal{T})$ and a.e. $(r, w) \in \Omega$,

$$\mathcal{R} f(r, w) = \left|\varphi'(E)\right| \, w \, U'_{\mathcal{T} f}(r), \qquad \sqrt{\mathcal{R}} f(r, w) = \left|\varphi'(E)\right| \, w \, \frac{U'_{\mathcal{T} f}(r)}{\sqrt{2\pi\rho(r)}}. \tag{3.35}$$

The natural domain of definition for the linearised operator $\mathcal{L} = -\mathcal{T}^2 - \mathcal{R}$ is

$$\mathrm{D}(\mathcal{L}) {:=} \mathrm{D}(\mathcal{T}^2) \cap \mathcal{H};$$

recall that $\mathcal{L}$ governs the dynamics of the odd-in-w part of the linearised perturbation. We obtain the following properties of this operator:

Lemma 3.10. (Properties of $\mathcal{L}$)

(a) The operator $\mathcal{L}\colon \mathrm{D}(\mathcal{L}) \to \mathcal{H}$ is self-adjoint as a densely defined operator on $\mathcal{H}$.

(b) The operators $\sqrt{\mathcal{R}}$ and $\mathcal{R}$ are relatively $(-\mathcal{T}^2)$-compact [31, Def. 14.1] and

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma_{\mathrm{ess}}(-\mathcal{T}^2\big|_{\mathcal{H}}) = \overline{\left(\frac{2\pi\mathbb{N}}{T(I)}\right)^2}. \tag{3.36}$$

(c) There exists $c > 0$ such that for all $f \in \mathrm{D}(\mathcal{L})$,

$$\langle \mathcal{L} f, f \rangle_H \geq c \left(\|f\|_H^2 + \|\mathcal{T} f\|_H^2 \right). \tag{3.37}$$

In particular, the kernel of $\mathcal{L}$ is trivial and $\sigma(\mathcal{L}) \subset]0, \infty[$.

An estimate of the form (3.37) is typically called an *Antonov coercivity bound*.

Proof. The self-adjointness of $\mathcal{L}$ is due to the Kato-Rellich theorem [59, Thm. X.12] and Lemmas 3.7 and 3.8. For part (b) it suffices to show that

$$\sqrt{\mathcal{R}}\colon \left(\mathrm{D}(\mathcal{L}),\ \|\mathcal{T}^2 \cdot\|_H + \|\cdot\|_H\right) \to \mathcal{H}$$

is compact, cf. [20, III Ex. 2.18.(1)]. This can be achieved similarly to [27, Thm. 5.9] using Lemma 3.7 (e), the identity (3.35), the bounds from Lemma 3.9, the compact embedding $H^2([0, R_{\max}]) \Subset H^1([0, R_{\max}])$, and (3.29).

For the last part we first recall the classical [2] Antonov coercivity bound

$$\langle \mathcal{L} f, f \rangle_H \geq \int_\Omega \frac{1}{|\varphi'(E)|} \frac{m(r)}{r^3} \, |f(r, w)|^2 \,\mathrm{d}(r, w) \tag{3.38}$$

for $f \in C_c^2(\Omega)$ odd in w, which can be derived as in [26, Lemma 1.1] or [42, (4.6)]. Extending the estimate (3.38) to $f \in \mathrm{D}(\mathcal{L})$ via a standard approximation argument [61, Prop. 2] implies $\sigma(\mathcal{L}) \subset [0, \infty[$ and $\ker(\mathcal{L}) = \{0\}$, cf. [27, Cor. 7.2 & 7.3]. In order to establish the coercivity bound (3.37), we then proceed as in [27, Prop. 7.4] and deduce that

$$\tilde{\lambda} := \inf_{\substack{f \in \mathrm{D}(\mathcal{T}) \\ f \notin \ker(\mathcal{T})}} \frac{\langle \mathcal{L} f, f \rangle_H}{\|\mathcal{T} f\|_H^2} = \inf_{\substack{f \in \mathrm{D}(\mathcal{T}) \\ f \notin \ker(\mathcal{T})}} \left(1 - \frac{\int_0^\infty r^2 U'_{\mathcal{T} f}(r)^2 \,\mathrm{d}r}{4\pi^2 \|\mathcal{T} f\|_H^2} \right) > 0$$

using Lemmas 3.9 and 3.7. Combining the latter estimate with Lemma 3.7 (e) similar to [27, Thm. 7.5] then concludes the proof of part (c). □

Corollary 3.11. (Single gap structure) *There exists an $\varepsilon_0 = \varepsilon_0(k) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the linearised operator associated to $f^{k,\varepsilon}$ satisfies*

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = [\frac{4\pi^2}{(T_{\max})^2}, \infty[.$$

Proof. The single gap condition (1.8) is equivalent to $\frac{T^0_{\max}}{T^0_{\min}} > 2$ by (A.15), which by Lemma 3.6 and (3.36) implies the claim. □

4. Absence of Embedded Eigenvalues

For fixed $k > \frac{1}{2}$ we consider the steady states $f^{k,\varepsilon}$ constructed in Lemma 3.1 with $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ be such that the statement of Corollary 3.11 and the uniform estimates from Lemma 3.6 (b) hold. Further ε-independent bounds on, e.g., $R_{\min}$, $R_{\max}$, E_0, and $E_{\min}$ for $0 < \varepsilon < \varepsilon_0$ follow by Lemmas 3.1 and 3.6 after suitably shrinking $\varepsilon_0 > 0$.

The central statement of this section is Theorem 4.5, which states that under the (regularity) assumption $k > 1$ there are no embedded eigenvalues of $\mathcal{L}$, i.e. no eigenvalues inside $\sigma_{\mathrm{ess}}(\mathcal{L})$ given by Lemma 3.10. We shall prove this by contradiction. To that end, we first make a simple observation relating the eigenvalues of $\mathcal{L}$ to those of $\tilde{\mathcal{L}}$; the latter operator is obviously well-defined on the domain $\mathrm{D}(\tilde{\mathcal{L}}) := \mathrm{D}(\mathcal{T})$, recall (1.10).

Lemma 4.1. *Assume that the operator $\mathcal{L} \colon \mathrm{D}(\mathcal{L}) \to \mathcal{H}$ has an embedded eigenvalue $\frac{4\pi^2 m^2}{T(E_m)^2}$ for some $m \in \mathbb{N}$ and $E_m \in \bar{I}$ with an eigenfunction $h \in \mathrm{D}(\mathcal{L})$. Then the function $f = h + \frac{T(E_m)}{2\pi i m} \mathcal{T} h$ enjoys the regularity $f \in \mathrm{D}(\mathcal{T})$ and satisfies the identity*

$$\hat{f}(\ell, E) = -T_m \frac{|\varphi'(E)| \widehat{U_f}(\ell, E)}{T_m - \frac{m}{\ell} T(E)}, \quad \textit{for a.e. } E \in I,\ \ell \in \mathbb{Z}^*, \tag{4.1}$$

where we have introduced the shorthand

$$T_m := T(E_m). \tag{4.2}$$

In addition, the statements of Lemma 3.9 apply to U_f, and $\nabla U_f \not\equiv 0$.

Here we employ the convention that a complex-valued function lies in some by definition real-valued function space like $\mathrm{D}(\mathcal{T})$, if its real and imaginary parts do.

Proof. Assume that λ^2 with $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{L}$ with an associated eigenfunction $h \in \mathrm{D}(\mathcal{L}) = \mathrm{D}(\mathcal{T}^2) \cap \mathcal{H}$. By Lemma 3.10 we have $\lambda \neq 0$. Using (3.35) and $U_h = 0$ (as h is odd in w), it is then easy to check that the pair of functions $f = h \pm \frac{1}{i\lambda}\mathcal{T}h$ are eigenfunctions of the operator $\tilde{\mathcal{L}}$ associated to eigenvalues $\pm i\lambda$. Observe here that $h \in \mathrm{D}(\mathcal{L})$ implies $h \pm \frac{1}{i\lambda}\mathcal{T}h \in \mathrm{D}(\mathcal{T}) = \mathrm{D}(\tilde{\mathcal{L}})$. Therefore, $\frac{2\pi i m}{T_m}$ is an eigenvalue of $\tilde{\mathcal{L}}$, and using action-angle coordinates and (3.26) yields the identity

$$\frac{1}{T(E)}\partial_\theta \left(f + |\varphi'(E)|U_f\right) = \frac{2\pi i m}{T_m} f, \quad \text{a.e. on } \mathbb{S}^1 \times I, \tag{4.3}$$

where we recall (4.2). Since $U_f = \frac{T_m}{2\pi i m} U_{\mathcal{T}h} \in C^1(\mathbb{S}^1 \times \mathbb{A})$ by Lemma 3.9, we may apply the Fourier transform w.r.t. $\theta \in \mathbb{S}^1$ to (4.3) to obtain the relation

$$\frac{\ell}{T(E)}\left(\hat{f}(\ell, E) + |\varphi'(E)|\widehat{U_f}(\ell, E)\right) = \frac{m}{T_m}\hat{f}(\ell, E), \quad \text{for a.e. } E \in I,\ \ell \in \mathbb{Z}, \tag{4.4}$$

where we recall (2.4). It is convenient to rewrite (4.4) in the following form

$$\left(T_m - \frac{m}{\ell}T(E)\right)\hat{f}(\ell, E) = -T_m|\varphi'(E)|\widehat{U_f}(\ell, E), \quad \text{for a.e. } E \in I,\ \ell \in \mathbb{Z}^*. \tag{4.5}$$

The strict monotonicity of $I \ni E \mapsto T(E)$ implies that for any given $\ell \in \mathbb{Z}^*$, there exists at most one energy $E_\ell \in \bar{I}$ such that $T_m - \frac{m}{\ell}T(E_\ell) = 0$. We hence conclude (4.1). Lastly, assume that $\nabla U_f \equiv 0$. Then $U_f \equiv 0$ since it decays to 0 as $r \to \infty$ and thus $f \equiv 0$ a.e. by (4.5). By definition of f it follows that $\mathcal{T}h = -\frac{2\pi i m}{T_m}h$ which is impossible since $h \neq 0$ is odd in w and $\mathcal{T}$ reverses w-parity. □

Remark 4.2. If $T_m - \frac{m}{\ell}T(E_\ell) = 0$ for $E_\ell = E_{\min} \in \bar{I}$, it follows by (4.5) that $\widehat{U_f}(\ell, E_{\min}) = 0$; we always extend T smoothly on $\bar{I}$ using Remark A.18. However, by Lemma 3.9, $\widehat{U_f}(\ell, \cdot)$ is only $C^{0,\frac{1}{2}}$ at $E = E_{\min}$, and therefore

$$\frac{\widehat{U_f}(\ell, E)}{T_m - \frac{m}{\ell}T(E)} \approx (E - E_{\min})^{-\frac{1}{2}} \text{ as } E \to E_{\min}.$$

In particular, the relation (4.1) does not make sense pointwise at $E = E_{\min}$, but it does weakly, or more precisely in $L^{2-\nu}(I)$ for any $0 < \nu \leq 1$.

The previous lemma suggests that the frequency-energy pairs where the denominator on the right-hand side of (4.1) vanishes play a distinguished role in the study of embedded eigenvalues. The next lemma provides simple quantitative bounds on the range of frequencies that are nearly resonant.

Lemma 4.3. (δ-resonant set) *Let $(m, E_m) \in \mathbb{N} \times \bar{I}$ be such that $\frac{4\pi^2 m^2}{T_m^2}$ is an eigenvalue of $\mathcal{L}$. Let $0 < \delta < \frac{1}{2}T_{\min}$ be given. Consider the* δ-resonant set

$$L_\delta^m := \left\{ \ell \in \mathbb{Z}^* \,\middle|\, \exists E \in \bar{I} \;\; \textit{such that} \;\; |T_m - \frac{m}{\ell} T(E)| < \delta \right\}.$$

Then $L_\delta^m \subset \mathbb{N}$ and there exists a constant $C_{res} = C(T_{\max}, T_{\min}) > 0$ such that

$$\left|\frac{\ell}{m}\right| + \left|\frac{m}{\ell}\right| \leq C_{res}, \quad \ell \in L_\delta^m. \tag{4.6}$$

Proof. If there exists an $E \in \bar{I}$ such that $-\delta < T_m - \frac{m}{\ell}T(E) < \delta$ then clearly $\frac{m}{\ell} > 0$ (since $\delta < \frac{1}{2}T_{\min}$) and

$$\frac{1}{2}\frac{T_{\min}}{T_{\max}} < \frac{T_{\min} - \delta}{T_{\max}} \leq \frac{T_m - \delta}{T(E)} < \frac{m}{\ell} < \frac{T_m + \delta}{T(E)} \leq \frac{T_{\max} + \delta}{T_{\min}} < \frac{3}{2}\frac{T_{\max}}{T_{\min}},$$

which implies the claim. □

Decomposition of the δ-resonant set. For $(m, E_m) \in \mathbb{N} \times \bar{I}$ and $\delta < \frac{1}{2}T_{\min}$ fixed, we decompose L_δ^m into three disjoint sets

$$L_\delta^m = \mathcal{R}_m \cup \mathcal{P}_m \cup \mathcal{N}_m,$$

where

$$\begin{aligned} \mathcal{R}_m &:= \left\{ \ell \in L_\delta^m \,\middle|\, \exists E \in \bar{I} \;\; \text{such that} \;\; T_m - \frac{m}{\ell}T(E) = 0 \right\}, \\ \mathcal{P}_m &:= \left\{ \ell \in L_\delta^m \,\middle|\, T_m - \frac{m}{\ell}T(E) > 0 \;\; \text{for all} \;\; E \in \bar{I} \right\}, \\ \mathcal{N}_m &:= \left\{ \ell \in L_\delta^m \,\middle|\, T_m - \frac{m}{\ell}T(E) < 0 \;\; \text{for all} \;\; E \in \bar{I} \right\}; \end{aligned} \tag{4.7}$$

recall that T is continuous on $\bar{I}$. We call the frequencies $\ell \in \mathcal{R}_m$, $\ell \neq m$, *resonant frequencies*. For any such frequency there exists an energy value $E_\ell \in \bar{I}$ at which the equation (4.5) degenerates, and by the monotonicity of $I \ni E \mapsto T(E)$ this energy value is unique. In particular,

$$T_m - \frac{m}{\ell}T(E) \begin{cases} < 0, & E \in]E_\ell, E_0], \\ \geq 0, & E \in [E_{\min}, E_\ell]. \end{cases} \tag{4.8}$$

An important piece of notation for the proof of Theorem 4.5 is given in the following definition:

Definition 4.4. (The function $p_{m,\ell}$) Let $\delta < \frac{1}{2}T_{\min}$. For any pair (m, ℓ) with $m \in \mathbb{N}$ and $\ell \in L_\delta^m$ let

$$p_{m,\ell}(t) := -\frac{\ell}{m}\log\left(T_m - \frac{m}{\ell}t\right) + C_p, \qquad \text{for } t \in [T_{\min}, T_{\max}] \text{ with } t < \frac{\ell}{m}T_m. \tag{4.9}$$

Here $C_p > 0$ is chosen independent of m, ℓ, and ε so that $p_{m,\ell} \geq 0$ on its domain of definition; this is possible by Lemma 4.3. Obviously, $p_{m,\ell}$ is an antiderivative of the map $t \mapsto \frac{1}{T_m - \frac{m}{\ell}t}$, and $p_{m,\ell}(t) \to \infty$ as $T_m - \frac{m}{\ell}t \searrow 0$.

Theorem 4.5. *Let $k > 1$. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the operator $\mathcal{L}$ has no embedded eigenvalues.*

Proof. By way of contradiction, we assume that there is an eigenvalue in the essential spectrum of $\mathcal{L}$, which by Lemma 3.10 means that it is of the form $\frac{4\pi^2 m^2}{T(E_m)^2}$ for some $m \in \mathbb{N}$ and $E_m \in \bar{I}$. Let $f \in \mathrm{D}(\mathcal{T})$ be a function as in Lemma 4.1, i.e., the relation (4.1) holds, the statements of Lemma 3.9 apply to U_f, and $\nabla U_f \not\equiv 0$. Throughout the proof, we keep track of the dependence of constants on the steady state, and hence on ε, and on the frequency m.

Step 1. An energy-type identity. We multiply (4.1) by the complex conjugate of $-\widehat{U_f}(\ell, E)$, sum over $\ell \in \mathbb{Z}^*$, and integrate against $T(E)\,\mathrm{d}E$. By Parseval's theorem, the left-hand side equals

$$-\int_I \int_{\mathbb{S}^1} f\, U_f\, T(E)\,\mathrm{d}\theta\,\mathrm{d}E = -\iint_\Omega f\, U_f\,\mathrm{d}(r, w) = \frac{1}{16\pi^3}\int_{\mathbb{R}^3} |\nabla U_f|^2\,\mathrm{d}x;$$

observe that $\hat{f}(0, \cdot) = 0$ by (4.4). As a result, we obtain the identity

$$\frac{1}{16\pi^3}\int |\nabla U_f|^2\,\mathrm{d}x = T_m \sum_{\ell \neq 0} \int_I \frac{T(E)|\varphi'(E)|}{T_m - \frac{m}{\ell}T(E)} \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E. \tag{4.10}$$

Now we let $\delta = \frac{1}{4}T_{\min}$ so that the conclusions of Lemma 4.3 apply. It is clear that there exists a constant $C = C(T_{\min}, T_{\max}) > 0$ such that

$$\frac{1}{\left|T_m - \frac{m}{\ell}T(E)\right|} \leq C\left|\frac{\ell}{m}\right|, \qquad m \in \mathbb{Z}^*,\ E \in I,\ \ell \in \mathbb{Z}^* \setminus L_\delta^m. \tag{4.11}$$

By (4.10),

$$\begin{aligned}\frac{1}{16\pi^3}\int |\nabla U_f|^2\,\mathrm{d}x &= T_m \sum_{\substack{\ell \neq 0 \\ \ell \in (L_\delta^m)^c}} \int_I \frac{T(E)|\varphi'(E)|}{T_m - \frac{m}{\ell}T(E)} \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E \\ &\quad + T_m \left(\sum_{\ell \in \mathcal{R}_m} + \sum_{\ell \in \mathcal{P}_m} + \sum_{\ell \in \mathcal{N}_m}\right) \int_I \frac{T(E)|\varphi'(E)|}{T_m - \frac{m}{\ell}T(E)} \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E.\end{aligned}$$

We bound the first term on the right-hand side using (3.33), (4.11), and the bound $|T_m - \frac{m}{\ell}T(E)|^{\frac{1}{2}} \geq \delta^{\frac{1}{2}}$ for $\ell \in (L_\delta^m)^c$ to get

$$\begin{aligned}&T_m \sum_{\substack{\ell \neq 0 \\ \ell \in (L_\delta^m)^c}} \int_I \frac{T(E)|\varphi'(E)|}{T_m - \frac{m}{\ell}T(E)} \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E \\ &\leq \frac{C\|\varphi'\|_{L^\infty(I)}}{\delta^{\frac{1}{2}}} \sum_{\substack{\ell \neq 0 \\ \ell \in (L_\delta^m)^c}} \frac{|\ell|^{\frac{1}{2}}}{|m|^{\frac{1}{2}}} \int_I \left|\widehat{U_f}(\ell, E)\right|^2 \mathrm{d}E \\ &\leq \frac{C\|\varphi'\|_{L^\infty(I)}}{\delta^{\frac{1}{2}} m^{\frac{1}{2}}} \|\nabla U_f\|_{L^2}^2 \sum_{\substack{\ell \neq 0 \\ \ell \in (L_\delta^m)^c}} |\ell|^{-\frac{3}{2}} \leq \frac{C\|\varphi'\|_{L^\infty(I)}}{\delta^{\frac{1}{2}} m^{\frac{1}{2}}} \|\nabla U_f\|_{L^2}^2.\end{aligned}$$

We thus rearrange the above identity and use (4.8) to obtain

$$\begin{aligned}\frac{1}{16\pi^3}\int |\nabla U_f|^2\,\mathrm{d}x &\le \frac{C\|\varphi'\|_{L^\infty(I)}}{\delta^{\frac12}m^{\frac12}}\|\nabla U_f\|_{L^2}^2\\ &+ T_m\sum_{\ell\in\mathcal{P}_m}\int_I \frac{T(E)|\varphi'(E)|}{|T_m-\frac{m}{\ell}T(E)|}\left|\widehat{U_f}(\ell,E)\right|^2\mathrm{d}E\\ &+ T_m\sum_{\ell\in\mathcal{R}_m}\int_{E_{\min}}^{E_\ell}\frac{T(E)|\varphi'(E)|}{|T_m-\frac{m}{\ell}T(E)|}\left|\widehat{U_f}(\ell,E)\right|^2\mathrm{d}E. \end{aligned}\tag{4.12}$$

By Definition 4.4, $\frac{1}{T'(E)}\partial_E\left(p_{m,\ell}(T(E))\right)=\frac{1}{|T_m-\frac{m}{\ell}T(E)|}$ for $\ell\in\mathcal{P}_m$ and $E\in I$ or $\ell\in\mathcal{R}_m$ and $E<E_\ell$. We use this to rewrite the integrals above and then integrate by parts in E. For $\ell\in\mathcal{P}_m$ this results in

$$\begin{aligned}&\int_I \frac{T(E)|\varphi'(E)|}{T'(E)}\partial_E\left(p_{m,\ell}(T(E))\right)\left|\widehat{U_f}(\ell,E)\right|^2\mathrm{d}E\\ &= A_\ell+B_\ell-\frac{T(E)|\varphi'(E)|}{T'(E)}p_{m,\ell}(T(E))|\widehat{U_f}(\ell,E)|^2\Big|_{E=E_{\min}}\le A_\ell+B_\ell,\end{aligned}\tag{4.13}$$

where for $\ell\in\mathcal{P}_m$,

$$A_\ell := -2\int_I \frac{T(E)|\varphi'(E)|}{T'(E)}p_{m,\ell}(T(E))\,\mathrm{Re}\left(\partial_E\widehat{U_f}(\ell,E)\,\overline{\widehat{U_f}(\ell,E)}\right)\mathrm{d}E,\tag{4.14}$$

$$B_\ell := -\int_I \partial_E\left(\frac{T(E)|\varphi'(E)|}{T'(E)}\right)p_{m,\ell}(T(E))\left|\widehat{U_f}(\ell,E)\right|^2\mathrm{d}E.\tag{4.15}$$

In (4.13), we used $k>1$ to conclude that $\varphi'(E_0)=0$ and the regularity of $\widehat{U}_f(\ell,\cdot)$ from Lemma 3.9 to infer that the boundary term at $E=E_0$ vanishes. Analogously, for $\ell\in\mathcal{R}_m$ we have

$$\begin{aligned}&\int_{E_{\min}}^{E_\ell}\frac{T(E)|\varphi'(E)|}{T'(E)}\partial_E\left(p_{m,\ell}(T(E))\right)\left|\widehat{U_f}(\ell,E)\right|^2\mathrm{d}E\\ &= A_\ell+B_\ell-\frac{T(E)|\varphi'(E)|}{T'(E)}p_{m,\ell}(T(E))|\widehat{U_f}(\ell,E)|^2\Big|_{E=E_{\min}}\le A_\ell+B_\ell,\end{aligned}\tag{4.16}$$

where for $\ell\in\mathcal{R}_m$

$$A_\ell := -2\int_{E_{\min}}^{E_\ell}\frac{T(E)|\varphi'(E)|}{T'(E)}p_{m,\ell}(T(E))\,\mathrm{Re}\left(\partial_E\widehat{U_f}(\ell,E)\,\overline{\widehat{U_f}(\ell,E)}\right)\mathrm{d}E,\tag{4.17}$$

$$B_\ell := -\int_{E_{\min}}^{E_\ell}\partial_E\left(\frac{T(E)|\varphi'(E)|}{T'(E)}\right)p_{m,\ell}(T(E))\left|\widehat{U_f}(\ell,E)\right|^2\mathrm{d}E.\tag{4.18}$$

In order to see that the boundary term at $E=E_\ell$ in (4.16) vanishes, first note that we may assume $E_\ell>E_{\min}$. If, in addition, $E_\ell<E_0$, we have $\widehat{U}_f(\ell,E_\ell)=0$ by (4.5)

and thus obtain $|\widehat{U}_f(\ell,E)|^2 p_{m,\ell}(T(E)) \to 0$ as $E \nearrow E_\ell$ using the regularities of $\widehat{U}_f$ and T. Otherwise, $E_\ell = E_0$ and the boundary term vanishes because $k > 1$.

Step 2. Estimates for A_ℓ, $\ell \in \mathcal{P}_m \cup \mathcal{R}_m$. The main challenge in our estimates is that the term $\partial_E \widehat{U_f}(\ell,E)$ at $E = E_{\min}$ inherits the singular behaviour $(E - E_{\min})^{-\frac{1}{2}}$ and for any given $\ell \in \mathbb{Z}$ the function $E \mapsto \partial_E \widehat{U_f}(\ell,\cdot)$ just fails to be in $L^2(I)$. To go around this we shall introduce powers of

$$\delta E := E - E_{\min}$$

as weights in our estimates. For any $\ell \in \mathcal{P}_m$ we first rewrite $\widehat{U_f}(\ell,E)$ as $\frac{1}{2\pi i\ell}\widehat{\partial_\theta U_f}$. Using (4.14) and Cauchy–Schwarz

$$\begin{aligned}
|A_\ell|^2 &\le \frac{1}{\pi^2\ell^2}\left|\int_I \frac{T(E)|\varphi'(E)|}{T'(E)} p_{m,\ell}(T(E))\partial_E \widehat{U_f}(\ell,E)(\delta E)^{\frac{1}{2}} \overline{\widehat{\partial_\theta U_f}(\ell,E)}(\delta E)^{-\frac{1}{2}}\,\mathrm{d}E\right|^2 \\
&\le \frac{C}{m^2}\int_I \left|\partial_E \widehat{U_f}(\ell,E)\right|^2 T(E)\,\delta E\,|\varphi'(E)|\,\mathrm{d}E \\
&\int_I \frac{p_{m,\ell}^2(T(E))}{T'(E)^2}\left|\widehat{\partial_\theta U_f}(\ell,E)\right|^2 \frac{T(E)}{\delta E}|\varphi'(E)|\,\mathrm{d}E,
\end{aligned}$$

where we have used (4.6). Applying Cauchy's inequality and summing over $\ell \in \mathcal{P}_m$ yields

$$\begin{aligned}
\sum_{\ell\in\mathcal{P}_m}|A_\ell| &\le Cm^{-\frac{1}{2}}\sum_{\ell\in\mathcal{P}_m}\int_I \left|\partial_E \widehat{U_f}(\ell,E)\right|^2 T(E)\,\delta E\,|\varphi'(E)|\,\mathrm{d}E \\
&+Cm^{-\frac{3}{2}}\sum_{\ell\in\mathcal{P}_m}\int_I \frac{p_{m,\ell}^2(T(E))}{T'(E)^2}\left|\widehat{\partial_\theta U_f}(\ell,E)\right|^2 \frac{T(E)}{\delta E}|\varphi'(E)|\,\mathrm{d}E. \quad (4.19)
\end{aligned}$$

By the same arguments as above we conclude that

$$\begin{aligned}
\sum_{\ell\in\mathcal{R}_m}|A_\ell| &\le Cm^{-\frac{1}{2}}\sum_{\ell\in\mathcal{R}_m}\int_{E_{\min}}^{E_\ell} \left|\partial_E \widehat{U_f}(\ell,E)\right|^2 T(E)\,\delta E\,|\varphi'(E)|\,\mathrm{d}E \\
&+Cm^{-\frac{3}{2}}\sum_{\ell\in\mathcal{R}_m}\int_{E_{\min}}^{E_\ell} \frac{p_{m,\ell}^2(T(E))}{T'(E)^2}\left|\widehat{\partial_\theta U_f}(\ell,E)\right|^2 \frac{T(E)}{\delta E}|\varphi'(E)|\,\mathrm{d}E.
\end{aligned} \quad (4.20)$$

The first sums on the right-hand sides of (4.19)–(4.20) respectively combine to give

$$\begin{aligned}
&\frac{C}{m^{\frac{1}{2}}}\sum_{\ell\in\mathcal{P}_m}\int_I \left|\partial_E \widehat{U_f}(\ell,E)\right|^2 T(E)\,\delta E\,|\varphi'|\,\mathrm{d}E \\
&+\frac{C}{m^{\frac{1}{2}}}\sum_{\ell\in\mathcal{R}_m}\int_{E_{\min}}^{E_\ell} \left|\partial_E \widehat{U_f}(\ell,E)\right|^2 T(E)\,\delta E\,|\varphi'|\,\mathrm{d}E \\
&\le \frac{C}{m^{\frac{1}{2}}}\sum_{\ell\in\mathbb{Z}}\int_I \left|\partial_E \widehat{U_f}(\ell,E)\right|^2 T(E)\,\delta E\,|\varphi'|\,\mathrm{d}E
\end{aligned}$$

$$= \frac{C}{m^{\frac{1}{2}}} \int_{\mathbb{S}^1 \times I} \left|\partial_E U_f\right|^2 T(E)\, \delta E\, |\varphi'|\, \mathrm{d}(\theta, E), \tag{4.21}$$

where we have used the Plancherel identity in the last line. We use (3.32) and change variables $\theta \mapsto r$, keeping in mind that $\frac{\partial r}{\partial \theta} = T(E)\sqrt{2E - 2\Psi(r)}$, to obtain

$$\begin{aligned}
&\frac{C}{m^{\frac{1}{2}}} \int_{\mathbb{S}^1 \times I} \left|\partial_E U_f(\theta, E)\right|^2 T(E)\, \delta E\, |\varphi'|\, \mathrm{d}(\theta, E) \\
&\quad \leq \frac{C}{m^{\frac{1}{2}}} \int_{\mathbb{S}^1 \times I} \left|\partial_r U_f(\theta, E)\right|^2 T(E)\, |\varphi'|\, \mathrm{d}(\theta, E) \\
&\quad = \frac{C}{m^{\frac{1}{2}}} \|\varphi'\|_{L^\infty(I)} \int_{R_{\min}}^{R_{\max}} \left|\partial_r U_f(r)\right|^2 \int_{\Psi(r)}^{E_0} \frac{\mathrm{d}E}{\sqrt{E - \Psi(r)}}\, \mathrm{d}r \\
&\quad = \frac{C}{m^{\frac{1}{2}}} \|\varphi'\|_{L^\infty(I)} \int_{R_{\min}}^{R_{\max}} \left|\partial_r U_f(r)\right|^2 (E_0 - \Psi(r))^{\frac{1}{2}}\, \mathrm{d}r \leq \frac{c_1}{m^{\frac{1}{2}}} \|\varphi'\|_{L^\infty(I)} \|\nabla U_f\|_2^2.
\end{aligned} \tag{4.22}$$

In this estimate only the general assumption $k > 1$ and the uniform-in-ε bounds on $R_{\min}$, $R_{\max}$, E_0, and $E_{\min}$ have been used; constants denoted by C never depend on ε, m, or ℓ.

It remains to estimate the second sums on the right-hand sides of (4.19) and (4.20) respectively. We start with the resonant contribution from (4.20). We recall that $\partial_\theta U_f = \partial_r U_f \frac{\partial r}{\partial \theta}$, change variables $\theta \mapsto r$, and apply the estimates from Lemma 3.6 to find that

$$\begin{aligned}
&\sum_{\ell \in \mathcal{R}_m} \int_{E_{\min}}^{E_\ell} \frac{p_{m,\ell}^2(T(E))}{T'(E)^2} \left|\widehat{\partial_\theta U_f}(\ell, E)\right|^2 \frac{T(E)}{\delta E} |\varphi'(E)|\, \mathrm{d}E \\
&\quad \leq \sum_{\ell \in \mathcal{R}_m} \int_{E_{\min}}^{E_\ell} \int_{\mathbb{S}^1} \left|\partial_\theta U_f(\theta, E)\right|^2 \frac{p_{m,\ell}^2(T(E))}{T'(E)^2} \frac{T(E)}{\delta E} |\varphi'(E)|\, \mathrm{d}\theta\, \mathrm{d}E \\
&\quad \leq \sum_{\ell \in \mathcal{R}_m} \int_{r_-(E_\ell)}^{r_+(E_\ell)} \left|\partial_r U_f\right|^2 \int_{\Psi(r)}^{E_\ell} \frac{p_{m,\ell}^2(T(E))}{T'(E)^2} \frac{T^2(E)}{\delta E} \sqrt{2E - 2\Psi(r)}\, |\varphi'(E)|\, \mathrm{d}E\, \mathrm{d}r \\
&\quad \leq C \sum_{\ell \in \mathcal{R}_m} \int_{R_{\min}}^{R_{\max}} \left|\partial_r U_f\right|^2 \mathrm{d}r \int_{E_{\min}}^{E_\ell} p_{m,\ell}^2(T(E))(\delta E)^{-\frac{1}{2}} |\varphi'(E)|\, \mathrm{d}E \\
&\quad \leq C \|\nabla U_f\|_2^2 \sum_{\ell \in \mathcal{R}_m} I_{\ell m},
\end{aligned} \tag{4.23}$$

where

$$I_{\ell m} := \int_{E_{\min}}^{E_\ell} p_{m,\ell}^2(T(E))(\delta E)^{-\frac{1}{2}} |\varphi'(E)|\, \mathrm{d}E.$$

In order to estimate the energy integrals $I_{\ell m}$ accordingly, we first note that for $\ell \in \mathcal{R}_m$ and $E \in [E_{\min}, E_\ell[$,

$$T_m - \frac{m}{\ell} T(E) = \frac{m}{\ell}(T(E_\ell) - T(E)) \geq C(E_\ell - E), \tag{4.24}$$

where we used Lemmas 3.6 and 4.3. For some $\alpha > 0$ sufficiently small we estimate $p_{m,\ell}(T(E))$ against $C(T_m - \frac{m}{\ell}T(E))^{-\alpha}$ and apply the standard integral identity

$$\int_a^b (s-a)^{\alpha}\ (b-s)^{\beta}\,\mathrm{d}s = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\ (b-a)^{\alpha+\beta+1}, \\ \alpha, \beta > -1,\ a \leq b,$$

to find that

$$\begin{aligned} I_{\ell m} &\leq C\|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)} \int_{E_{\min}}^{E_\ell} (E_0-E)^{k-1}\frac{(E_\ell - E)^{-2\alpha}}{\sqrt{E-E_{\min}}}\,\mathrm{d}E \\ &\leq C\|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)}(E_0-E_{\min})^{k-\frac{1}{2}-2\alpha} \leq C\|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)}, \end{aligned}$$

where the constant $C > 0$ is independent of ε and m; recall that $k > 1$. It then follows from (4.23) that there exists a constant $c_2 > 0$ such that

$$\begin{aligned} &\sum_{\ell\in\mathcal{R}_m}\int_{E_{\min}}^{E_\ell}\frac{p_{m,\ell}^2(T(E))}{T'(E)^2}\left|\widehat{\partial_\theta U_f}(\ell,E)\right|^2\frac{T(E)}{\delta E}|\varphi'|\,\mathrm{d}E \\ &\quad \leq c_2 m\|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)}\|\nabla U_f\|_2^2, \end{aligned} \tag{4.25}$$

where we recall that $|\mathcal{R}_m| \leq Cm$ by Lemma 4.3. By a completely analogous argument

$$\begin{aligned} &\sum_{\ell\in\mathcal{P}_m}\int_I\frac{p_{m,\ell}^2(T(E))}{T'(E)^2}\left|\widehat{\partial_\theta U_f}(\ell,E)\right|^2\frac{T(E)}{\delta E}|\varphi'(E)|\,\mathrm{d}E \\ &\leq C\sum_{\ell\in\mathcal{P}_m}\int_{R_{\min}}^{R_{\max}}\left|\partial_r U_f\right|^2\mathrm{d}r\ I_{\ell m} \leq c_3 m\|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)}\|\nabla U_f\|_{L^2(\mathbb{R}^3)}^2 \end{aligned} \tag{4.26}$$

for some ε, m, ℓ-independent constant $c_3 > 0$. The difference to the estimate (4.23) is that for $\ell \in \mathcal{P}_m$ the energy integrals $I_{\ell m}$ extend over the whole energy interval I, and (4.24) is replaced by

$$T_m - \frac{m}{\ell}T(E) > \frac{m}{\ell}(T(E_0) - T(E)) \geq C(E_0 - E).$$

Hence

$$I_{\ell m} \leq C\|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)}(E_0-E_{\min})^{k-\frac{1}{2}-2\alpha}\int_0^1(1-s)^{k-1-2\alpha}s^{-\frac{1}{2}}\,\mathrm{d}s,$$

which is again uniformly bounded in the same way. In conclusion, from (4.19)–(4.22) and (4.25)–(4.26) we conclude

$$\sum_{\ell\in\mathcal{P}_m\cup\mathcal{R}_m}|A_\ell| \leq c_4 m^{-\frac{1}{2}}\,\|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)}\,\|\nabla U_f\|_{L^2(\mathbb{R}^3)}^2 \tag{4.27}$$

for some ε, m, ℓ-independent constant $c_4 > 0$; notice that $\|\varphi'\|_{L^\infty(I)} \leq C \|\frac{\varphi'}{(E_0-\cdot)^{k-1}}\|_{L^\infty(I)}$.

Step 3. Estimates for B_ℓ, $\ell \in \mathcal{P}_m \cup \mathcal{R}_m$, see (4.15) *and* (4.18). These estimates are analogous to the bounds (4.25) and (4.26), and we obtain

$$\sum_{\ell \in \mathcal{P}_m \cup \mathcal{R}_m} |B_\ell| \leq c_5\, m^{-1} \left(\| \frac{\varphi'}{(E_0 - \cdot)^{k-1}} \|_{L^\infty(I)} + \| \frac{\varphi''}{(E_0 - \cdot)^{k-2}} \|_{L^\infty(I)} \right) \|\nabla U_f\|^2_{L^2(\mathbb{R}^3)} \tag{4.28}$$

for some ε, m, ℓ-independent constant $c_5 > 0$. Here, we again rely on the assumption $k > 1$ to guarantee the integrability of φ'' near $E = E^0$ and use the uniform bounds on T, T', and T''.

Step 4. Conclusion. We use (4.27), (4.28), and (4.12) to get

$$\frac{1}{16\pi^3} \|\nabla U_f\|_2^2 \leq C m^{-\frac{1}{2}} \left(\| \frac{\varphi'}{(E_0 - \cdot)^{k-1}} \|_{L^\infty(I)} + \| \frac{\varphi''}{(E_0 - \cdot)^{k-2}} \|_{L^\infty(I)} \right) \|\nabla U_f\|_2^2 \tag{4.29}$$

$$\leq C_{\text{final}}\, m^{-\frac{1}{2}} \varepsilon\, \|\nabla U_f\|_2^2, \tag{4.30}$$

where the ε in (4.30) appears due to the polytropic choice of the steady state (1.4). With $\varepsilon_0 > 0$ small enough this gives the contradiction for $0 < \varepsilon < \varepsilon_0$, recall $\nabla U_f \not\equiv 0$ by Lemma 4.1. □

Remark 4.6. The final constant $C_{\text{final}} > 0$ on the right-hand side of (4.30) depends on M, L, k, κ, $R_{\min}$, $R_{\max}$, $T_{\min}$, $T_{\max}$, $T'_{\min}$, $T'_{\max}$, and $T''_{\max}$ in an explicitly computable way. The proof shows that as long as $C_{\text{final}}\, \varepsilon$ is smaller than $\frac{1}{16\pi^3}$, no embedded eigenvalues occur.

We have carefully tracked the occurrence of the small factor $0 < \varepsilon \ll 1$ in the proof of Theorem 4.5, which appears only in (4.30). Due to the presence of the factor $m^{-\frac{1}{2}}$ in (4.29), it follows trivially that at high frequencies m there cannot be any embedded eigenvalues, even if we *do not* impose any smallness.

Corollary 4.7. (No embedded eigenvalues at large frequencies for large steady states) *Consider the family of steady states* (1.4) *with $\varepsilon = 1$, i.e.,*

$$f^{k,1}(r, w) = \varphi(E) = (E_0 - E)_+^k, \qquad k > 1. \tag{4.31}$$

Assume further that $T'_{\min} > 0$. Then there exists an integer $m_0 = m_0(k) > 0$ such that the operator $\mathcal{L}$ has no embedded eigenvalues larger than $\frac{4\pi^2 m_0^2}{T_{\min}^2}$.

Proof. By (4.31) it is clear that

$$M_\varphi := \| \frac{\varphi'}{(E_0 - \cdot)^{k-1}} \|_{L^\infty(I)} + \| \frac{\varphi''}{(E_0 - \cdot)^{k-2}} \|_{L^\infty(I)} < \infty.$$

Thus, the claim follows directly from (4.29) upon choosing m sufficiently large. □

Remark 4.8. Under the strict monotonicity assumption on the period function $E \mapsto T(E)$ (which is expected to hold generically), the above corollary applies to a broad class of steady states satisfying the assumption $M_\varphi < \infty$. This clearly also includes isotropic steady states without the central point mass at the origin.

5. Principal Gap Analysis

Throughout this section let $f^{k,\varepsilon}$ be a steady state given by Lemma 3.1 with fixed $k > \frac{1}{2}$ and $\varepsilon > 0$.

5.1. A Birman–Schwinger Principle

In order to characterise the presence of eigenvalues of the linearised operator $\mathcal{L} = -\mathcal{T}^2 - \mathcal{R}\colon \mathrm{D}(\mathcal{L}) \to \mathcal{H}$ in the principal gap $\mathcal{G}$ defined in (2.8), we provide a criterion similar to [27, Sc. 8], see also [40] and [24, Sc. 6].

Lemma 5.1. (A Birman–Schwinger principle, cf. [27, Lemmas 8.1–8.3]) *For* $\lambda \in \mathcal{G}$ *let*

$$Q_\lambda := \sqrt{\mathcal{R}} \left(-\mathcal{T}^2 - \lambda\right)^{-1} \sqrt{\mathcal{R}} : \mathcal{H} \to \mathcal{H}. \tag{5.1}$$

We refer to Q_λ *as the* Birman–Schwinger operator *associated to* $\mathcal{L}$. *This operator is linear, bounded, symmetric, non-negative, and compact. Furthermore, the linearised operator* $\mathcal{L}$ *possessing an eigenvalue in the principal gap* $\mathcal{G}$ *is equivalent to the existence of* $\lambda \in \mathcal{G}$ *such that* Q_λ *has an eigenvalue greater or equal than* 1.

Proof. The properties of Q_λ for $\lambda \in \mathcal{G}$ follow by the properties of $-\mathcal{T}^2$ and $\sqrt{\mathcal{R}}$ derived in Lemmas 3.7 and 3.8. In particular, Q_λ being compact is due to $\sqrt{\mathcal{R}}$ being relatively $(-\mathcal{T}^2)$-compact, cf. Lemma 3.10 (b).

In order to relate the spectra of Q_λ and $\mathcal{L}$ to each other, we consider the operators

$$\mathcal{L}_\mu := -\mathcal{T}^2 - \frac{1}{\mu}\mathcal{R}\colon \mathrm{D}(\mathcal{L}) \to \mathcal{H}, \qquad \mu > 0.$$

Similar to Lemma 3.10, the operators $\mathcal{L}_\mu$ are self-adjoint with $\sigma_{\mathrm{ess}}(\mathcal{L}_\mu) = \sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma(-\mathcal{T}^2\big|_{\mathcal{H}})$. Furthermore, for $\lambda \in \mathcal{G}$ and $\mu \geq 1$ there holds

$$\lambda \text{ is an eigenvalue of } \mathcal{L}_\mu \quad \Leftrightarrow \quad \mu \text{ is an eigenvalue of } Q_\lambda. \tag{5.2}$$

This equivalency is due to the following two observations: If $f \in \mathrm{D}(\mathcal{L}) \setminus \{0\}$ solves $\mathcal{L}_\mu f = \lambda f$, then $g := \sqrt{\mathcal{R}} f \in \mathcal{H} \setminus \{0\}$ satisfies $Q_\lambda g = \mu g$. Conversely, if $g \in \mathcal{H} \setminus \{0\}$ solves $Q_\lambda g = \mu g$, then $f := (-\mathcal{T}^2 - \lambda)^{-1} \sqrt{\mathcal{R}} g \in \mathrm{D}(\mathcal{L}) \setminus \{0\}$ defines a solution of $\mathcal{L}_\mu f = \lambda f$. Next, we deduce that

$$\mathcal{L} = \mathcal{L}_1 \text{ has an eigenvalue in } \mathcal{G} \quad \Leftrightarrow \quad \exists \mu \geq 1\colon \mathcal{L}_\mu \text{ has an eigenvalue in } \mathcal{G} \tag{5.3}$$

by the non-negativity of $\mathcal{R}$ (cf. Lemma 3.8) and the positivity of $\mathcal{L}$ with $\sigma_{\mathrm{ess}}(\mathcal{L}) \cap \mathcal{G} = \emptyset$ (cf. Lemma 3.10) together with the min-max principle for operators [31, Prop. 5.12].

Combining (5.2) and (5.3) then concludes the proof. □

We note that Q_λ slightly differs from the respective operator defined in [27, Eq. (8.1)]. The benefit of the definition (5.1) is that Q_λ is symmetric, which is not the case in [27].

When searching for eigenfunctions of Q_λ for $\lambda \in \mathcal{G}$ associated to non-zero eigenvalues, we may restrict ourselves to the space

$$\begin{aligned} \operatorname{im}(Q_\lambda) \subset \operatorname{im}(\sqrt{\mathcal{R}}) &\subset \{f \in H \mid \exists F = F(r)\colon f(r,w) = \left|\varphi'(E)\right| \, w \, F(r)\} \\ &= \left\{\Omega \ni (r,w) \mapsto \left|\varphi'(E)\right| \frac{w}{r\sqrt{\rho(r)}} F(r) \mid F \in L^2([R_{\min}, R_{\max}])\right\}. \end{aligned} \tag{5.4}$$

This leads to the following operator which was first introduced by Mathur [52].

Lemma 5.2. (The Mathur operator, cf. [27, Def. 8.5, Prop. 8.6, and Lemma 8.8]) *For $F \in L^2([R_{\min}, R_{\max}])$ let $f \in \mathcal{H}$ be defined via*

$$f(r,w) = \left|\varphi'(E)\right| \frac{w}{r\sqrt{\rho(r)}} F(r) \quad \text{for a.e. } (r,w) \in \Omega.$$

Due to (5.4), for any $\lambda \in \mathcal{G}$ there exists a unique $G \in L^2([R_{\min}, R_{\max}])$ such that

$$Q_\lambda f(r,w) = \left|\varphi'(E)\right| \frac{w}{r\sqrt{\rho(r)}} G(r) \quad \text{for a.e. } (r,w) \in \Omega.$$

The resulting mapping

$$\mathcal{M}_\lambda\colon L^2([R_{\min}, R_{\max}]) \to L^2([R_{\min}, R_{\max}]), \ \mathcal{M}_\lambda F := G$$

is the Mathur operator. *This operator is linear, bounded, symmetric, non-negative, and a compact Hilbert-Schmidt operator [58, Thm. VI.22 et seq.]. We have the representation*

$$(\mathcal{M}_\lambda F)(r) = \int_{R_{\min}}^{R_{\max}} K_\lambda(r,s)\, F(s)\, \mathrm{d}s, \quad F \in L^2([R_{\min}, R_{\max}]), \ r \in [R_{\min}, R_{\max}], \tag{5.5}$$

with integral kernel $K_\lambda \in C([R_{\min}, R_{\max}]^2)$ given by

$$K_\lambda(r,s) := \frac{16\pi^{\frac{3}{2}}}{rs} \sum_{j=1}^{\infty} \int_{I(r)\cap I(s)} \frac{\left|\varphi'(E)\right|}{T(E)} \frac{\sin(2\pi j\,\theta(r,E))\,\sin(2\pi j\,\theta(s,E))}{\frac{4\pi^2}{T(E)^2} j^2 - \lambda} \,\mathrm{d}E \tag{5.6}$$

for $r, s \in [R_{\min}, R_{\max}]$, where θ is defined in (2.3) and

$$I(r) := \{E \in I \mid r_-(E) < r < r_+(E)\}, \qquad r > 0.$$

Proof. The operator $\mathcal{M}_\lambda$ being bounded, symmetric, non-negative, and compact follows by the respective properties of the Birman–Schwinger operator Q_λ together with the identity (3.29).

In order to verify that the Mathur operator is a Hilbert-Schmidt operator, the key observation is that using action-angle variables (cf. Sect. 3.2 and Lemma 3.7 (b)) yields

$$
\begin{aligned}
&(-\mathcal{T}^2-\lambda)^{-1}g(\theta,E)\\
&=\frac{4}{T(E)}\sum_{j=1}^{\infty}\int_{r_-(E)}^{r_+(E)}\frac{g(\theta(r,E),E)\,\sin(2\pi j\,\theta(r,E))}{\sqrt{2E-2\Psi(r)}}\mathrm{d}r\;\frac{\sin(2\pi j\theta)}{\frac{4\pi^2}{T(E)^2}j^2-\lambda}
\end{aligned}
$$

for $\lambda\in\mathcal{G}$ and $g\in\mathcal{H}$. Inserting the definition of $\sqrt{\mathcal{R}}$ from Lemma 3.8, it is then straight-forward to verify that the identity (5.5) holds. The continuity of the kernel K_λ follows by the dominated convergence theorem. $\square$

Due to (5.4), the Mathur operator still contains all the relevant information of the spectrum of the Birman–Schwinger operator. More precisely, for $\lambda\in\mathcal{G}$, any $\mu>0$ is an eigenvalue of $Q_\lambda\colon\mathcal{H}\to\mathcal{H}$ if and only if it is an eigenvalue of $\mathcal{M}_\lambda\colon L^2([R_{\min},R_{\max}])\to L^2([R_{\min},R_{\max}])$; this is similar to [27, Lemma 8.10]. In addition, the properties of the Mathur operator derived above together with [31, Prop. 5.12] and [58, Thm. VI.6] imply

$$
\sup(\sigma(\mathcal{M}_\lambda))=\max(\sigma(\mathcal{M}_\lambda))=\|\mathcal{M}_\lambda\|
$$

for $\lambda\in\mathcal{G}$, where $\|\cdot\|$ denotes the operator norm on $L^2([R_{\min},R_{\max}])$ given by

$$
\begin{aligned}
M_\lambda:=\|\mathcal{M}_\lambda\|&=\sup\{\|\mathcal{M}_\lambda F\|_2\mid F\in L^2([R_{\min},R_{\max}]),\ \|F\|_2=1\}\\
&=\sup\{\langle\mathcal{M}_\lambda F,F\rangle_2\mid F\in L^2([R_{\min},R_{\max}]),\ \|F\|_2=1\},\qquad\lambda\in\mathcal{G}.
\end{aligned}
\tag{5.7}
$$

Overall, we arrive at the following criterion for the presence of eigenvalues of $\mathcal{L}$ in the principal gap $\mathcal{G}$ defined in (2.8).

Proposition 5.3. (cf. [27, Thm. 8.11]) *The linearised operator $\mathcal{L}$ possesses an eigenvalue in the principal gap $\mathcal{G}$ if and only if there exists a $\lambda\in\mathcal{G}$ such that $M_\lambda\geq 1$.*

5.2. Absence of Eigenvalues in the Principal Gap for $k>1$

We now prove the absence of eigenvalues in the principal gap $\mathcal{G}$ defined (2.8) using the Birman–Schwinger principle derived above.

Theorem 5.4. *Assume that the polytropic exponent satisfies $k>1$. Then there exists $\varepsilon_0>0$ such that the linearised operator $\mathcal{L}$ associated to the equilibrium $f^{k,\varepsilon}$ has no eigenvalues in the principal gap $\mathcal{G}$ for any $0<\varepsilon<\varepsilon_0$.*

Proof. In order to apply Proposition 5.3, let $\lambda \in \mathcal{G}$ and $F \in L^2([R_{\min}, R_{\max}])$ with $\|F\|_2 = 1$. Using the representation (5.5) of the Mathur operator $\mathcal{M}_\lambda$ yields

$$\begin{aligned}\langle \mathcal{M}_\lambda F, F\rangle_2 = 16\pi^{\frac{3}{2}} \sum_{j=1}^{\infty} \int_I \frac{\left|\varphi'(E)\right|}{T(E)} \frac{1}{\frac{4\pi^2}{T(E)^2} j^2 - \lambda} \\ \times \left(\int_{r_-(E)}^{r_+(E)} \sin(2\pi j \theta(r, E)) \frac{F(r)}{r} \mathrm{d}r \right)^2 \mathrm{d}E.\end{aligned} \tag{5.8}$$

We next apply the Cauchy–Schwarz inequality together with the bounds on $R_{\min}$ and $R_{\max}$ from (3.7) to estimate the radial integral; the constant $C > 0$ changes from line to line but is always uniform in $\varepsilon \in]0, \varepsilon_0[$. In addition, we use the bound $\lambda < \frac{4\pi^2}{T_{\max}^2}$ and thus arrive at

$$M_\lambda \le C \sum_{j=1}^{\infty} \int_I \frac{\left|\varphi'(E)\right|}{T(E)} \frac{1}{\frac{4\pi^2}{T(E)^2} j^2 - \frac{4\pi^2}{T_{\max}^2}} \mathrm{d}E. \tag{5.9}$$

For the first summand, recall that there exists $\varepsilon_0 > 0$ such that $T = T^\varepsilon$ is increasing on $I = I^\varepsilon$ for $0 < \varepsilon < \varepsilon_0$ by Lemma 3.6 (b). Together with the uniform bounds from the latter lemma and the mean value theorem (cf. Lemma 3.4 for the regularity of T) we deduce

$$\frac{4\pi^2}{T(E)^2} - \frac{4\pi^2}{T_{\max}^2} = \frac{4\pi^2}{T(E)^2} - \frac{4\pi^2}{T(E_0)^2} \ge \frac{1}{C} (E_0 - E)$$

for $E \in I$. Therefore,

$$\int_I \frac{\left|\varphi'(E)\right|}{T(E)} \frac{1}{\frac{4\pi^2}{T(E)^2} - \frac{4\pi^2}{T_{\max}^2}} \mathrm{d}E \le C \int_I \frac{\left|\varphi'(E)\right|}{E_0 - E} \mathrm{d}E = Ck\varepsilon \int_I (E_0 - E)^{k-2} \mathrm{d}E \le C\varepsilon \tag{5.10}$$

because $k > 1$; recall that $I = I^\varepsilon$ is uniformly bounded as $\varepsilon \to 0$ by Lemma 3.6 (a).

In order to bound the remaining summands on the right hand side of (5.9), observe that $\frac{4\pi^2}{T(E)^2} j^2 - \frac{4\pi^2}{T_{\max}^2} \ge \frac{j^2}{C}$ for $j \ge 2$ and $E \in I$. Thus,

$$\sum_{j=2}^{\infty} \int_I \frac{\left|\varphi'(E)\right|}{T(E)} \frac{1}{\frac{4\pi^2}{T(E)^2} j^2 - \frac{4\pi^2}{T_{\max}^2}} \mathrm{d}E \le C \int_I \left|\varphi'(E)\right| \mathrm{d}E \le C\varepsilon. \tag{5.11}$$

Inserting (5.10) and (5.11) into (5.9) implies $M_\lambda \le C\varepsilon$. Applying the Birman–Schwinger–Mathur criterion from Proposition 5.3 then concludes the proof. □

5.3. Existence of Pure Oscillations for $\frac{1}{2} < k \leq 1$

We now apply Proposition 5.3 to prove the existence of pure oscillations à la [27, Thm. 8.13].

Theorem 5.5. *Assume that the polytropic exponent satisfies $\frac{1}{2} < k \leq 1$. Then there exists $\varepsilon_0 > 0$ such that the linearised operator $\mathcal{L}$ associated to the equilibrium $f^{k,\varepsilon}$ possesses an eigenvalue in the principal gap $\mathcal{G}$ for any $0 < \varepsilon < \varepsilon_0$.*

Proof. For $\lambda \in \mathcal{G}$ and $F \in L^2([R_{\min}, R_{\max}])$ we rewrite $\langle \mathcal{M}_\lambda F, F\rangle_2$ as in (5.8) to deduce

$$\langle \mathcal{M}_\lambda F, F\rangle_2 \geq 16\pi^{\frac{3}{2}} \int_I \frac{|\varphi'(E)|}{T(E)} \frac{1}{\frac{4\pi^2}{T(E)^2} - \lambda} \left(\int_{r_-(E)}^{r_+(E)} \sin(2\pi\theta(r, E)) \frac{F(r)}{r} \,\mathrm{d}r \right)^2 \mathrm{d}E.$$

Now choose $\eta > 0$ and a non-empty set $S \subset]R_{\min}, R_{\max}[$ such that for all $E \in]E_0 - \eta, E_0[$ it holds that $S \subset]r_-(E), r_+(E)[$ and $\sin(2\pi\theta(r, E)) \geq \frac{1}{2}$ for $r \in S$; this is possible since $r_\pm$ are smooth and $\theta(\cdot, E)\colon]r_-(E), r_+(E)[\to]0, \frac{1}{2}[$ is one-to-one for $E \in I$. Setting $F := \mathbb{1}_S$ leads to

$$\begin{aligned} \limsup_{\lambda \to \frac{4\pi^2}{T_{\max}^2}} M_\lambda &\geq \limsup_{\lambda \to \frac{4\pi^2}{T_{\max}^2}} \frac{\langle \mathcal{M}_\lambda F, F\rangle_2}{\|F\|_2^2} \geq \frac{4\pi^{\frac{3}{2}} |S|}{R_{\min}^2} \limsup_{\lambda \to \frac{4\pi^2}{T_{\max}^2}} \int_{E_0-\eta}^{E_0} \frac{|\varphi'(E)|}{T(E)} \frac{1}{\frac{4\pi^2}{T(E)^2} - \lambda} \,\mathrm{d}E \\ &= C\,|S| \int_{E_0-\eta}^{E_0} \frac{|\varphi'(E)|}{T(E)} \frac{1}{\frac{4\pi^2}{T(E)^2} - \frac{4\pi^2}{T_{\max}^2}} \,\mathrm{d}E \end{aligned}$$

for some constant $C = C^\varepsilon > 0$. Using the bounds from Lemma 3.6 (b) implies

$$\frac{4\pi^2}{T(E)^2} - \frac{4\pi^2}{T_{\max}^2} = \frac{4\pi^2}{T(E)^2} - \frac{4\pi^2}{T(E_0)^2} \leq C\,(T(E_0) - T(E)) \leq C(E_0 - E)$$

for $E \in I$. Hence,

$$\limsup_{\lambda \to \frac{4\pi^2}{T_{\max}^2}} M_\lambda \geq C\,|S| \int_{E_0-\eta}^{E_0} \frac{|\varphi'(E)|}{E_0 - E} \,\mathrm{d}E = C\,|S|\,\varepsilon k \int_{E_0-\eta}^{E_0} (E_0 - E)^{k-2} \,\mathrm{d}E.$$

Because $k \leq 1$, the integral in the latter expression is infinite. Applying Proposition 5.3 then concludes the proof. □

6. Proof of the Main Theorem

We can now complete the proof of Theorem 1.2. Part (a) is the content of Theorem 5.5. To prove part (b) we first observe that since $\sigma(\mathcal{L}) \subset]0, \infty[= \mathcal{G} \cup \sigma_{\mathrm{ess}}(\mathcal{L})$ by Corollary 3.11, Theorems 4.5 and 5.4 imply that there are no eigenvalues in the spectrum and therefore the pure point spectrum is empty. It remains to show the damping formula (1.18). To that end we view the linear evolution (1.13) as a first order system of the form

$$\partial_t \psi = A\psi, \quad \psi = \begin{pmatrix} f \\ \partial_t f \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\mathcal{L} & 0 \end{pmatrix}. \tag{6.1}$$

Following [20, Sc. VI.3], we consider this system on the Hilbert space $\mathcal{X} := (\mathrm{D}(\mathcal{T}) \cap \mathcal{H}) \times \mathcal{H}$ with

$$\langle (f, g), (F, G) \rangle_{\mathcal{X}} := \langle \mathcal{T} f, \mathcal{T} F \rangle_H - \frac{1}{4\pi^2} \int_0^\infty r^2\, U'_{\mathcal{T} f}(r)\, U'_{\mathcal{T} F}(r)\, \mathrm{d}r + \langle g, G \rangle_H \tag{6.2}$$

for $(f, g), (F, G) \in \mathcal{X}$. Here we recall (1.16). If additionally $f \in \mathrm{D}(\mathcal{L})$, the above expression can be rewritten as

$$\langle (f, g), (F, G) \rangle_{\mathcal{X}} = \langle \mathcal{L} f, F \rangle_H + \langle g, G \rangle_H$$

using (3.35). Hence, extending Antonov's coercivity bound from Lemma 3.10 (c) onto $\mathrm{D}(\mathcal{T}) \cap \mathcal{H}$ via a standard approximation argument [61, Prop. 2] shows that (6.2) indeed defines an inner product on $\mathcal{X}$.

The natural domain of definition for the operator A is $\mathrm{D}(A) := \mathrm{D}(\mathcal{L}) \times (\mathrm{D}(\mathcal{T}) \cap \mathcal{H})$, which is a dense subset of $\mathcal{X}$. Moreover, since $\mathcal{L} \colon \mathrm{D}(\mathcal{L}) \to \mathcal{H}$ is self-adjoint and invertible by Lemma 3.10, it is straight-forward to verify that $A \colon \mathrm{D}(A) \to \mathcal{X}$ is skew-adjoint, i.e., $A^* = -A$. By Stone's theorem [20, Thm. II.3.24], A thus generates a unitary C^0-group and the system (6.1) is well-posed, i.e., any initial datum $(f_0, g_0) \in \mathrm{D}(A)$ launches a unique, global solution of the form $\mathbb{R} \ni t \mapsto e^{tA}(f_0, g_0) \in \mathrm{D}(A)$, cf. [20, Thm. II.6.7]. The analogous statement clearly carries over to the second order equation (1.13).

Consider the operator

$$K \colon \mathcal{X} \to \mathcal{X}, \quad K \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} 0 \\ \left|\varphi'(E)\right| U_{\mathcal{T} f} \end{pmatrix},$$

which is bounded by Lemma 3.9. Moreover, for any bounded sequence $(f_n, g_n) \subset \mathcal{X}$ we obtain that $(\mathcal{T} f_n) \subset H$ is bounded by Lemma 3.10 (c). Similar to Lemma 3.10 (b), we thus conclude that K is compact by applying Lemma 3.9. Thus, since the point spectrum of A is empty by the above discussion, the RAGE theorem [60, Thm. XI.115] implies

$$0 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \| K e^{tA} (f_0, g_0) \|_{\mathcal{X}}^2 \, \mathrm{d}t = \lim_{T \to \infty} \frac{1}{T} \int_0^T \| |\varphi'(E)| U_{\mathcal{T} f(t)} \|_H^2 \, \mathrm{d}t. \tag{6.3}$$

Furthermore,

$$\begin{aligned} \frac{1}{16\pi^3} \| \nabla U_{\mathcal{T} f(t)} \|_{L^2}^2 &= - \int_\Omega U_{\mathcal{T} f(t)}(r)\, \mathcal{T} f(t, r, w)\, \mathrm{d}(r, w) \\ &\le \| |\varphi'(E)| U_{\mathcal{T} f(t)} \|_H \, \| \mathcal{T} f(t) \|_H \end{aligned}$$

by Cauchy–Schwarz and

$$\| \mathcal{T} f(t) \|_H^2 \le C \langle \mathcal{L} f(t), f(t) \rangle_H \le C \| (f(t), \partial_t f(t)) \|_{\mathcal{X}}^2 = C \| (f_0, g_0) \|_{\mathcal{X}}^2 \tag{6.4}$$

due to Lemma 3.10 (c) and $(e^{tA})_{t\in\mathbb{R}}$ being unitary. Therefore,

$$\frac{1}{T}\int_0^T \|\nabla U_{\mathcal{T}f(t)}\|_{L^2}^2\,\mathrm{d}t \le \frac{C}{T}\int_0^T \||\varphi'(E)|U_{\mathcal{T}f(t)}\|_H\,\|\mathcal{T}f(t)\|_H\,\mathrm{d}t$$
$$\le C\left(\frac{1}{T}\int_0^T \||\varphi'(E)|U_{\mathcal{T}f(t)}\|_H^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T \|\mathcal{T}f(t)\|_H^2\,\mathrm{d}t\right)^{\frac{1}{2}}$$

for $T>0$ and (1.18) follows by (6.3) and (6.4).

Acknowledgements. The authors thank Alexander Pushnitski for many stimulating discussions. M. Hadžić's research is supported by the EPSRC Early Career Fellowship EP/S02218X/1. M. Schrecker's research is supported by the EPSRC Post-doctoral Research Fellowship EP/W001888/1.

Data Availability Statement All data generated or analysed during this study are included in this published article.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

A. Steady state theory

A.1. Proof of Lemma 3.1

The ansatz (1.4) indeed yields a stationary solution of the system (1.1)–(1.3) provided that U is the potential associated to $f^{k,\varepsilon}$, i.e.

$$U'(r)=\frac{4\pi}{r^2}\int_0^r s^2\rho(s)\,\mathrm{d}s,\quad r>0,\qquad \lim_{r\to\infty}U(r)=0, \tag{A.1}$$

where ρ is induced by $f^{k,\varepsilon}$ via (1.3), i.e,

$$\rho(r)=\frac{\pi}{r^2}\int_{\mathbb{R}} f^{k,\varepsilon}(r,w)\,\mathrm{d}w. \tag{A.2}$$

In order to get a closed system for U, we insert the ansatz (1.4) into (A.2) and obtain

$$\begin{aligned}\rho(r) &= \frac{\sqrt{2}\pi}{r^2}\,\varepsilon\int_{\Psi(r)}^{E_0}(E_0-E)_+^k\,(E-\Psi(r))^{-\frac{1}{2}}\,\mathrm{d}E \\ &= \sqrt{2}\pi^{\frac{3}{2}}\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}\,\frac{1}{r^2}\,\varepsilon\,(E_0-\Psi(r))_+^{k+\frac{1}{2}} =: \frac{c_k}{r^2}\,\varepsilon\,(E_0-\Psi(r))_+^{k+\frac{1}{2}} \end{aligned} \tag{A.3}$$

for $r>0$, where the effective potential Ψ is given by (1.6). Hence, defining

$$g(z) := c_k\, z_+^{k+\frac{1}{2}}, \qquad z\in\mathbb{R}, \tag{A.4}$$

yields that

$$\rho(r) = \frac{\varepsilon}{r^2}\,g(E_0-\Psi(r)) = \frac{\varepsilon}{r^2}\,g\left(E_0-U(r)+\frac{M}{r}-\frac{L}{2r^2}\right). \tag{A.5}$$

Observe that $g\in C^1(\mathbb{R})\cap C^\infty(\mathbb{R}\setminus\{0\})$ since $k>\frac{1}{2}$. Inspired by [56], we now consider the quantity

$$y := E_0 - U$$

instead of U. Then y solves

$$y'(r) = -\frac{4\pi c_k}{r^2}\,\varepsilon\int_0^r g\left(y(s)+\frac{M}{s}-\frac{L}{2s^2}\right)\mathrm{d}s, \qquad r>0. \tag{A.6}$$

We equip this equation with the initial condition

$$y(0)=\kappa \tag{A.7}$$

for prescribed κ satisfying the single gap condition (1.8). It is straight-forward to verify that there exists a unique solution $y\in C^1([0,\infty[)$ of (A.6)–(A.7), cf. [56]. This solution possesses a vacuum region at the centre, more precisely, $\rho(r)=0$ and $y(r)=\kappa$ for $0\le r\le R_{\min}$ with $R_{\min}=R^0_{\min}$ given by (3.2) and $R_{\min}>0$ is the maximal radius with this property. Furthermore, inserting $y\le\kappa<0$ into (A.5) yields $\rho(r)=0$ for $r\ge R^0_{\max}$ where $R^0_{\max}$ is given by (3.2). Hence, the limit $y_\infty := \lim_{r\to\infty} y(r)\in]-\infty,0[$ exists. Then, setting $E_0 := y_\infty < 0$ and $U := E_0 - y$ yields a solution of (A.1). The estimate for the w-part of the support in (3.1) follows by

$$\frac{1}{2}\,w^2 \le E_0 - E(r,w) = y(r) + \frac{M}{r} - \frac{L}{2r^2} \le \frac{M}{R_{\min}}, \quad (r,w)\in\operatorname{supp}(f^{k,\varepsilon}),$$

where we have used the bounds $y\le 0$, $r\ge R_{\min}$, and $-\frac{L}{2r^2}<0$ on the galaxy support. □

A.2. *Convergence of the steady state family*

The aim of this section is to prove Lemma 3.6. For fixed $k>\frac{1}{2}$ and κ as above, this requires a detailed understanding of the behaviour of the steady states $f^{k,\varepsilon}$ given by Lemma 3.1 as $\varepsilon\to 0$. We hence add a superscript ε to all steady state quantities to make the ε-dependencies more visible, i.e., ρ^ε is the stationary mass density, U^ε is the stationary potential, and the associated local mass function is given by

$$m^\varepsilon(r) := 4\pi\int_0^r s^2\rho^\varepsilon(s)\,\mathrm{d}s, \qquad r\ge 0. \tag{A.8}$$

The first result is similar to [24, Lemma 3.3] and forms the basis for all further convergence results. Recall Lemma 3.3 for the regularity properties of ρ^ε and U^ε.

Lemma A.1. *As $\varepsilon \to 0$ it holds that ρ^ε, $(\rho^\varepsilon)' \to 0$ uniformly on $[0,\infty[$, $M_{\mathrm{S}}^\varepsilon \to 0$, $E_0^\varepsilon \to \kappa$, and*

$$(U^\varepsilon)^{(j)} \to 0 \text{ uniformly on } [0,\infty[, \quad j \in \{0,1,2,3\}. \tag{A.9}$$

Proof. Inserting the uniform bound of the radial support given by (3.7) and the estimate $y^\varepsilon \leq y^\varepsilon(0) = \kappa$ into (A.5) yields the uniform convergence $\rho^\varepsilon \to 0$, which immediately implies that $M_{\mathrm{S}}^\varepsilon \to 0$; recall (3.3). Together with $(y^\varepsilon)'(r) = -\frac{m^\varepsilon(r)}{r^2}$ for $r > 0$ by (A.6) this also leads to $(y^\varepsilon)' \to 0$ uniformly. After integration, we then deduce that $y^\varepsilon \to \kappa$ uniformly on $[0,\infty[$ and $E_0^\varepsilon = y_\infty^\varepsilon \to \kappa$. In addition, after differentiating (A.5) we obtain the uniform convergence $(\rho^\varepsilon)' \to 0$. Combining all these limits then also yields the uniform convergence of the second and third derivative of U^ε after further differentiating (A.1). □

We next prove that r_*^ε and $r_\pm^\varepsilon$ converge to r_*^0 and $r_\pm^0$, respectively, as $\varepsilon \to 0$; recall Lemma 3.2 and Section 3.3 for the definitions of these radii. We start with the minimising radius r_*^ε and the associated minimal energy value $E_{\mathrm{min}}^\varepsilon$.

Lemma A.2. *It holds that $r_*^\varepsilon \to r_*^0$ and $E_{\mathrm{min}}^\varepsilon \to E_{\mathrm{min}}^0$ as $\varepsilon \to 0$.*

Proof. The radius r_*^ε is given as the unique zero of the increasing function

$$\xi^\varepsilon\colon]0,\infty[\to \mathbb{R},\ \xi^\varepsilon(r) := M + m^\varepsilon(r) - \frac{L}{r}$$

for $\varepsilon \geq 0$; in the case $\varepsilon = 0$ we have $m^0 \equiv 0$. Because $m^\varepsilon \to 0$ uniformly on $[0,\infty[$ as $\varepsilon \to 0$ by Lemma A.1, we obtain that r_*^ε indeed tends to r_*^0 as $\varepsilon \to 0$.
Together with the uniform convergence $U^\varepsilon \to 0$ from Lemma A.1 we then deduce that $E_{\mathrm{min}}^\varepsilon = \Psi^\varepsilon(r_*^\varepsilon) \to \Psi^0(r_*^0) = E_{\mathrm{min}}^0$ as $\varepsilon \to 0$. □

The next step is to show analogous properties for the radii $r_\pm^\varepsilon(E)$ as well. In addition, we also analyse the behaviour of these radii for $(\varepsilon, E) \to (0, E_{\mathrm{min}}^0)$, i.e. in the near circular regime. For this sake, let

$$\mathcal{A} := \{(\varepsilon, E) \mid \varepsilon \geq 0,\ E \in \mathbb{A}^\varepsilon\} \tag{A.10}$$

denote the set of all *admissible* (ε, E)*-pairs*; recall that $\mathbb{A}^\varepsilon =]E_{\mathrm{min}}^\varepsilon, 0[$ by (3.5) and (3.21).

Lemma A.3. *(a) The mappings $\mathcal{A} \ni (\varepsilon, E) \mapsto r_\pm^\varepsilon(E)$ are continuous at $\varepsilon = 0$ locally uniformly in E. More precisely, for any $\delta > 0$ and $E_1 < 0$ there exists some $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$, $E_{\mathrm{min}}^\varepsilon < E < E_1$, and $E_{\mathrm{min}}^0 < E^* < E_1$ with $|E - E^*| < \varepsilon_0$ it holds that $|r_\pm^\varepsilon(E) - r_\pm^0(E^*)| < \delta$.*

(b) The radii $r_\pm^\varepsilon(E)$ converge to r_*^0 as $E \to E_{\mathrm{min}}^\varepsilon$ and $\varepsilon \to 0$. More precisely, for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$ and $E_{\mathrm{min}}^\varepsilon < E < E_{\mathrm{min}}^\varepsilon + \eta < 0$ it holds that $|r_\pm^\varepsilon(E) - r_*^0| < \delta$.

Proof. The radius $r_-^\varepsilon(E)$ is the unique zero of the decreasing function

$$\xi_E^\varepsilon\colon]0, r_*^\varepsilon[\to \mathbb{R},\ \xi_E^\varepsilon(r) := \Psi^\varepsilon(r) - E$$

for $\varepsilon \geq 0$ and $E \in \mathbb{A}^\varepsilon$. We continuously extend r_-^0 by setting $r_-^0(E_{\mathrm{min}}^0) := r_*^0$ and let $\delta > 0$ and $E_1 \in]E_{\mathrm{min}}^0, 0[$ be arbitrary. By taking $\delta > 0$ sufficiently small, we ensure that there exist $E \in]E_{\mathrm{min}}^0, E_1[$ with $r_-^0(E) + \delta \leq r_*^0$. We then observe that, as $\delta > 0$, for such E, $\xi_E^0(r_-^0(E) + \delta)$ is uniformly negative, that is,

$$\zeta_- := \sup\{\xi_E^0(r_-^0(E) + \delta) \mid E_{\mathrm{min}}^0 < E \leq E_1,\ r_-^0(E) + \delta \leq r_*^0\} < 0.$$

Now let ε, E^*, and E be as specified in the statement of part (a) of the lemma for some $\varepsilon_0 > 0$ which we define below. If $r_-^0(E^*)+2\delta \geq r_*^\varepsilon$, then $r_-^\varepsilon(E) \leq r_*^\varepsilon \leq r_-^0(E^*)+2\delta$. Otherwise, i.e., $r_-^0(E^*)+2\delta < r_*^\varepsilon$, we also have $r_-^0(E^*)+\delta < r_*^0$ after choosing $\varepsilon_0 > 0$ sufficiently small by Lemma A.2. Due to $\zeta_- < 0$ and the uniform bound $r_-^0(E^*)+\delta \geq r_-^0(E_1)$, we then obtain that $\xi_E^\varepsilon(r_-^0(E^*)+\delta) < 0$ after potentially shrinking $\varepsilon_0 > 0$ again according to Lemma A.1. This implies that $r_-^\varepsilon(E) \leq r_-^0(E^*)+\delta$. Showing that $r_-^\varepsilon(E) \geq r_-^0(E^*)-\delta$ works similarly. An analogous proof yields the respective estimates for r_+ as well, which concludes the proof of part (a).

Part (b) then follows by combining part (a) with the convergence of $E_{\min}^\varepsilon$ from Lemma A.2.
□

In particular, Lemmas A.1 and A.3 imply the convergence of the radial support of the steady state, i.e.

$$R_{\min}^\varepsilon = R_{\min}^0, \qquad R_{\max}^\varepsilon \to R_{\max}^0 \text{ as } \varepsilon \to 0, \tag{A.11}$$

recall (3.2) and (3.7).

We now consider the period function $T^\varepsilon : \mathbb{A}^\varepsilon \to]0,\infty[$ for $\varepsilon \geq 0$ defined in (2.2) and (3.22). The aim is to establish the uniform convergence of T^ε on the energy support $\overline{I}^\varepsilon$ of the steady state as $\varepsilon \to 0$; recall (2.1) and (3.23). The main difficulty in this task is that the set I^ε changes in ε, in particular, the minimal energy value $E_{\min}^\varepsilon$ depends on ε. We thus first consider the case of this "near circular regime", i.e., the region where E gets close to the minimal energy value $E_{\min}^\varepsilon$. It turns out that T^ε is essentially determined by $(\Psi^\varepsilon)''(r_*^\varepsilon)$ in this regime, which is why we start by establishing the following auxiliary result; recall Lemma 3.3 for the regularity of the effective potential Ψ^ε for $\varepsilon > 0$.

Lemma A.4. *Let $j \in \mathbb{N}_0$. Then $(\Psi^\varepsilon)^{(j)}(s)$ converges to $(\Psi^0)^{(j)}(r_*^0)$ as $E \to E_{\min}^\varepsilon$ and $\varepsilon \to 0$ uniformly in $s \in [r_-^\varepsilon(E), r_+^\varepsilon(E)]$. More precisely, for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$ and $E_{\min}^\varepsilon < E < E_{\min}^\varepsilon + \eta < 0$ it holds that $|(\Psi^\varepsilon)^{(j)}(s) - (\Psi^0)^{(j)}(r_*^0)| < \delta$ for $s \in [r_-^\varepsilon(E), r_+^\varepsilon(E)]$.*

Proof. Applying Lemmas A.1–A.3 and (A.11) implies that there exist $\tilde\delta, \varepsilon_0, \eta > 0$ such that

$$[r_-^\varepsilon(E), r_+^\varepsilon(E)] \subset [R_{\min}^\varepsilon + 2\tilde\delta, R_{\max}^\varepsilon - 2\tilde\delta] \subset [R_{\min}^0 + \tilde\delta, R_{\max}^0 - \tilde\delta]$$

for $E_{\min}^\varepsilon < E < E_{\min}^\varepsilon + \eta$ and $0 \leq \varepsilon < \varepsilon_0$. In particular, $(\Psi^\varepsilon)^{(j)}$ exists on $[r_-^\varepsilon(E), r_+^\varepsilon(E)]$ by Lemma 3.3; observe that $\Psi^0 \in C^\infty(]0,\infty[)$ by (3.19). If $j \leq 3$, the statement then follows by Lemma A.1. In the case of a larger j one has to iterate the arguments of Lemma A.1, i.e., further differentiate (A.1) and (A.5), to deduce that $(U^\varepsilon)^{(j)} \to 0$ uniformly on $[R_{\min}^0 + \tilde\delta, R_{\max}^0 - \tilde\delta]$ as $\varepsilon \to 0$. □

We then obtain the following behaviour of $T^\varepsilon(E)$ as $\varepsilon \to 0$ for E in the near circular regime:

Lemma A.5. *The period function $T^\varepsilon(E)$ converges to $T^0(E_{\min}^0)$ as $E \to E_{\min}^\varepsilon$ and $\varepsilon \to 0$. More precisely, for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$ and $E_{\min}^\varepsilon < E < E_{\min}^\varepsilon + \eta < 0$ it holds that $|T^\varepsilon(E) - T^0(E_{\min}^0)| < \delta$.*

Here, $T^0(E_{\min}^0)$ denotes the continuous extension of T^0 onto $E_{\min}^0$. Due to (3.19), (3.20), and (3.22), this value is given by

$$T^0(E_{\min}^0) = 2\pi\,\frac{L^{\frac{3}{2}}}{M^2} = \frac{2\pi}{\sqrt{(\Psi^0)''(r_*^0)}}. \tag{A.12}$$

Proof. For any $\varepsilon \geq 0$ and $E \in \mathbb{A}^\varepsilon$ we obtain that

$$T^\varepsilon(E) = 2\int_{r_-^\varepsilon(E)}^{r_*^\varepsilon} \frac{\mathrm{d}r}{\sqrt{2E - 2\Psi^\varepsilon(r)}} + 2\int_{r_*^\varepsilon}^{r_+^\varepsilon(E)} \frac{\mathrm{d}r}{\sqrt{2E - 2\Psi^\varepsilon(r)}}$$
$$= \int_{E_{\min}^\varepsilon}^{E} \frac{\sqrt{2}}{\sqrt{(E-\tilde{\eta})\,(\Psi^\varepsilon)'(r_-^\varepsilon(\tilde{\eta}))^2}}\,\mathrm{d}\tilde{\eta} + \int_{E_{\min}^\varepsilon}^{E} \frac{\sqrt{2}}{\sqrt{(E-\tilde{\eta})\,(\Psi^\varepsilon)'(r_+^\varepsilon(\tilde{\eta}))^2}}\,\mathrm{d}\tilde{\eta}$$

by changing variables via $\tilde{\eta} = \Psi^\varepsilon(r)$, $r = r_\pm^\varepsilon(\tilde{\eta})$ in both integrals. We focus on the first integral in the last line of the above calculation; the arguments for the second integral are similar.

Due to the extended mean value theorem, for every $\tilde{\eta} \in]E_{\min}^\varepsilon, E[$ there exists some $s \in]r_-^\varepsilon(\tilde{\eta}), r_*^\varepsilon[\subset]r_-^\varepsilon(E), r_*^\varepsilon[$ such that

$$\frac{(\Psi^\varepsilon)'(r_-^\varepsilon(\tilde{\eta}))^2}{\tilde{\eta} - E_{\min}^\varepsilon} = \frac{(\Psi^\varepsilon)'(r_-^\varepsilon(\tilde{\eta}))^2}{\Psi^\varepsilon(r_-^\varepsilon(\tilde{\eta})) - \Psi^\varepsilon(r_*^\varepsilon)} = 2\,(\Psi^\varepsilon)''(s);$$

recall that $(\Psi^\varepsilon)'(r_*^\varepsilon) = 0$. Hence, the integrand of the integral under consideration can be rewritten as

$$\frac{\sqrt{2}}{\sqrt{(E-\tilde{\eta})\,(\Psi^\varepsilon)'(r_-^\varepsilon(\tilde{\eta}))^2}} = \frac{1}{\sqrt{(E-\tilde{\eta})\,(\tilde{\eta} - E_{\min}^\varepsilon)\,(\Psi^\varepsilon)''(s)}}. \tag{A.13}$$

Now Lemma A.4 implies that for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$ and $E_{\min}^\varepsilon < E < E_{\min}^\varepsilon + \eta$ it holds that $|(\Psi^\varepsilon)''(s)^{-\frac{1}{2}} - (\Psi^0)''(r_*^0)^{-\frac{1}{2}}| < \frac{\delta}{2\pi}$ for $s \in [r_-^\varepsilon(E), r_+^\varepsilon(E)]$; note that $(\Psi^0)''(r_*^0) = \frac{M^4}{L^3} > 0$. Together with (A.12) we thus conclude the following estimate for the integral under consideration for $0 \leq \varepsilon < \varepsilon_0$ and $E_{\min}^\varepsilon < E < E_{\min}^\varepsilon + \eta$:

$$\left|\int_{E_{\min}^\varepsilon}^{E} \frac{\sqrt{2}}{\sqrt{(E-\tilde{\eta})\,(\Psi^\varepsilon)'(r_-^\varepsilon(\tilde{\eta}))^2}}\,\mathrm{d}\tilde{\eta} - \frac{1}{2}T^0(E_{\min}^0)\right|$$
$$= \left|\int_{E_{\min}^\varepsilon}^{E} \frac{\sqrt{2}}{\sqrt{(E-\tilde{\eta})\,(\Psi^\varepsilon)'(r_-^\varepsilon(\tilde{\eta}))^2}}\,\mathrm{d}\tilde{\eta} - \frac{1}{\sqrt{(\Psi^0)''(r_*^0)}}\int_{E_{\min}^\varepsilon}^{E} \frac{\mathrm{d}\tilde{\eta}}{\sqrt{(E-\tilde{\eta})\,(\tilde{\eta} - E_{\min}^\varepsilon)}}\right|$$
$$\leq \frac{\delta}{2\pi}\int_{E_{\min}^\varepsilon}^{E} \frac{\mathrm{d}\tilde{\eta}}{\sqrt{(E-\tilde{\eta})(\tilde{\eta} - E_{\min}^\varepsilon)}} = \frac{\delta}{2}.$$

□

In order to establish the desired uniform convergence of T^ε on the energy support as $\varepsilon \to 0$, we next verify the pointwise convergence of $T^\varepsilon(E)$. This is based on the following concavity estimate which originates from [42, Lemma 2.1 (iii)] (the proof only uses the elliptic equation (A.1)):

Lemma A.6. *For any $\varepsilon \geq 0$, $E \in \mathbb{A}^\varepsilon$, and $r \in [r_-^\varepsilon(E), r_+^\varepsilon(E)]$ the following concavity estimate holds:*

$$E - \Psi^\varepsilon(r) \geq L\,\frac{(r_+^\varepsilon(E) - r)\,(r - r_-^\varepsilon(E))}{2r^2\,r_-^\varepsilon(E)\,r_+^\varepsilon(E)}. \tag{A.14}$$

The continuity of T^ε at $\varepsilon = 0$ now follows similar to [27, Lemma B.7].

Lemma A.7. *The mapping $\mathcal{A} \ni (\varepsilon, E) \mapsto T^\varepsilon(E)$ is continuous at $\varepsilon = 0$; recall (A.10). More precisely, for any $\delta > 0$ and $E^* \in \mathbb{A}^0$ there exists $\varepsilon_0 > 0$ such that for all $0 \le \varepsilon < \varepsilon_0$ and $E \in \mathbb{A}^\varepsilon$ with $|E^* - E| < \varepsilon_0$ it holds that $|T^\varepsilon(E) - T^0(E^*)| < \delta$.*

Proof. For any $\varepsilon \ge 0$ and $E \in \mathbb{A}^\varepsilon$ the affine change of variables $r = r_-^\varepsilon(E) + (r_+^\varepsilon(E) - r_-^\varepsilon(E))\, s$ leads to

$$T^\varepsilon(E) = \sqrt{2} \int_0^1 \frac{r_+^\varepsilon(E) - r_-^\varepsilon(E)}{\sqrt{E - \Psi^\varepsilon\left(r_-^\varepsilon(E) + (r_+^\varepsilon(E) - r_-^\varepsilon(E))\, s\right)}}\, \mathrm{d}s.$$

Lemmas A.1 and A.3 imply the pointwise convergence of the integrand in the integral above as $(\varepsilon, E) \to (0, E^*)$, the concavity estimate from Lemma A.6 shows that the integrand is bounded by an integrable, E-independent function. Hence, Lebesgue's dominated convergence theorem implies the desired continuity statement. □

Combining Lemmas A.5 and A.7 with standard continuity arguments yields the main result of the present section.

Lemma A.8. *The period function $T^\varepsilon(E)$ converges to $T^0(E^*)$ as $\varepsilon \to 0$ and $E \to E^*$ locally uniformly. More precisely, for any $\delta > 0$ and $E_1 < 0$ there exists $\varepsilon_0 > 0$ such that for all $0 \le \varepsilon < \varepsilon_0$ as well as $E^0_{\min} < E^* < E_1$ and $E^\varepsilon_{\min} < E < E_1$ with $|E^* - E| < \varepsilon_0$ it holds that $|T^\varepsilon(E) - T^0(E^*)| < \delta$.*

We now conclude the convergence of the minimal and maximal value of the period function on the steady state support. Recall that $I^\varepsilon =]E^\varepsilon_{\min}, E^\varepsilon_0[$ for $\varepsilon \ge 0$ by (2.1) and (3.23) as well as the definitions of $T^\varepsilon_{\min}$ and $T^\varepsilon_{\max}$ in (3.11).

Lemma A.9. *It holds that $\lim_{\varepsilon\to 0} T^\varepsilon_{\min} = T^0_{\min}$ and $\lim_{\varepsilon\to 0} T^\varepsilon_{\max} = T^0_{\max}$, with*

$$0 < T^0_{\min} = T^0(E^0_{\min}) = 2\pi \frac{L^{\frac{3}{2}}}{M^2} < \frac{\pi}{\sqrt{2}} \frac{M}{(-\kappa)^{\frac{3}{2}}} = T^0(\kappa) = T^0_{\max} < \infty \tag{A.15}$$

due to (1.8), (3.22), and (A.12).

Proof. Combine Lemma A.8 with the limit results $E^\varepsilon_{\min} \to E^0_{\min}$ and $E^\varepsilon_0 \to \kappa$ as $\varepsilon \to 0$ established in Lemmas A.1 and A.2. □

We later show that T^ε is increasing on I^ε for $0 \le \varepsilon \ll 1$, which implies that $T^\varepsilon_{\min}$ and $T^\varepsilon_{\max}$ are attained on the boundary of I^ε.

We now establish results similar to Lemmas A.8 and A.9 for the first and second order derivatives of the period function; recall the regularity of T^ε shown in Lemma 3.3. We first derive suitable representations of these derivatives. Proceeding as in Lemma 3.3 leads to a relation between $\partial_E T^\varepsilon(E)$ and $(\partial_E W^\varepsilon)(T^\varepsilon(E), E)$, cf. [40, Lemma A.12]. However, as $\partial_E W^\varepsilon$ is only implicitly known as the solution of a suitable ODE, this quantity is rather hard to control; in particular in the vicinity of the minimum of the potential well $E_{\min}$.

Instead, we proceed as suggested in [27, Sc. B.3] and derive a suitable integral expression for $\partial_E T^\varepsilon$.

Lemma A.10. *For $\varepsilon \ge 0$ and $E \in \mathbb{A}^\varepsilon$ it holds that*

$$(T^\varepsilon)'(E) = \frac{1}{E - E^\varepsilon_{\min}} \int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} \frac{G_0^\varepsilon(r)}{\sqrt{2E - 2\Psi^\varepsilon(r)}}\, \mathrm{d}r, \tag{A.16}$$

where the continuous function $G_0^\varepsilon \colon]0, \infty[\to \mathbb{R}$ is defined by

$$G_0^\varepsilon(r) := \begin{cases} \frac{(\Psi^\varepsilon)'(r)^2 - 2(\Psi^\varepsilon(r) - E^\varepsilon_{\min})\, (\Psi^\varepsilon)''(r)}{(\Psi^\varepsilon)'(r)^2}, & r \ne r_*^\varepsilon, \\ 0, & r = r_*^\varepsilon. \end{cases} \tag{A.17}$$

Proof. Taylor expanding (A.17) in the limit $r \to r_*^\varepsilon$ yields

$$G_0^\varepsilon(r) = \frac{-\frac{1}{3}\,(r-r_*^\varepsilon)^3\,(\Psi^\varepsilon)''(r_*^\varepsilon)\,(\Psi^\varepsilon)'''(r_*^\varepsilon) + o((r-r_*^\varepsilon)^3)}{(r-r_*^\varepsilon)^2\,(\Psi^\varepsilon)''(r_*^\varepsilon)^2 + o((r-r_*^\varepsilon)^2)} \to 0 \quad \text{as } r \to r_*^\varepsilon \tag{A.18}$$

by (3.18). Thus, G_0^ε is indeed continuous and, in particular, the integral on the right-hand side of (A.16) is well-defined. A calculation similar to [17, Thm. 2.1] then yields the identity (A.16). □

It is again crucial to understand the behaviour of $(T^\varepsilon)'(E)$ in the near circular regime, i.e. when E gets close to the minimal energy value $E_{\min}^\varepsilon$. One difficulty in this task is the factor in front of the integral on the right-hand side of (A.16). It is thus convenient to rewrite the integral expression (A.16) as

$$(T^\varepsilon)'(E) = -\frac{1}{E-E_{\min}^\varepsilon}\int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} \frac{G_0^\varepsilon(r)}{(\Psi^\varepsilon)'(r)}\,\partial_r\left[\sqrt{2E-2\Psi^\varepsilon(r)}\right]\mathrm{d}r \tag{A.19}$$

for $\varepsilon \geq 0$ and $E \in \mathbb{A}^\varepsilon$, with the intention to integrate by parts in (A.19). For this sake we introduce the function

$$G_1^\varepsilon\colon\,]0,\infty[\to\mathbb{R}, \qquad G_1^\varepsilon(r) := \begin{cases} \frac{G_0^\varepsilon(r)}{(\Psi^\varepsilon)'(r)}, & \text{if } r \neq r_*^\varepsilon, \\ -\frac{1}{3}\,\frac{(\Psi^\varepsilon)'''(r_*^\varepsilon)}{(\Psi^\varepsilon)''(r_*^\varepsilon)^2}, & \text{if } r = r_*^\varepsilon, \end{cases} \tag{A.20}$$

with G_0^ε defined in Lemma A.10; recall that $(\Psi^\varepsilon)''(r_*^\varepsilon) > 0$ by (3.18). A Taylor expansion similar to (A.18) yields that G_1^ε is continuous on $]0,\infty[$. In fact, recalling from (A.18) that G_0^ε and $(\Psi^\varepsilon)'$ are both smooth and $\mathcal{O}(r-r_*^\varepsilon)$ as $r \to r_*^\varepsilon$, and that $(\Psi^\varepsilon)'$ vanishes only at r_*^ε, we deduce that G_1^ε is smooth on $]R_{\min}^\varepsilon, R_{\max}^\varepsilon[$ (recall the regularities established in Lemma 3.3) and that its derivatives admit explicit representation in terms of derivatives of Ψ^ε.

For $\varepsilon \geq 0$ and $E \in \mathbb{A}^\varepsilon$ we now continue the calculation (A.19) and integrate by parts:

$$(T^\varepsilon)'(E) = \frac{1}{E-E_{\min}^\varepsilon}\int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} (G_1^\varepsilon)'(r)\,\sqrt{2E-2\Psi^\varepsilon(r)}\,\mathrm{d}r. \tag{A.21}$$

Let us hence analyse the behaviour of integrals of the latter form in the near circular regime:

Lemma A.11. *Let $]0,\infty[\ni r \mapsto F(r)$ be continuous. For fixed $\varepsilon \geq 0$ it holds that*

$$\lim_{E\searrow E_{\min}^\varepsilon}\int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} F(r)\,\sqrt{2E-2\Psi^\varepsilon(r)}\,\mathrm{d}r = 0. \tag{A.22}$$

Moreover, it holds that

$$\partial_E\left[\int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} F(r)\,\sqrt{2E-2\Psi^\varepsilon(r)}\,\mathrm{d}r\right] = \int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} \frac{F(r)}{\sqrt{2E-2\Psi^\varepsilon(r)}}\,\mathrm{d}r, \qquad E \in \mathbb{A}^\varepsilon. \tag{A.23}$$

Proof. First observe that Lemma 3.2 implies that $r_\pm^\varepsilon(E) \to r_*^\varepsilon$ as $E \searrow E_{\min}^\varepsilon$; the proof is similar to the one of Lemma A.3. Then (A.22) is obvious. The derivative relation (A.23) is straight-forward to verify using Lebesgue's dominated convergence theorem. □

Due to the mean value theorem together with (A.21), (A.22), and (A.23), we conclude that for any $E \in \mathbb{A}^\varepsilon$ there exists some $\tilde{E} \in]E^\varepsilon_{\text{min}}, E[$ such that

$$(T^\varepsilon)'(E) = \int_{r^\varepsilon_-(\tilde{E})}^{r^\varepsilon_+(\tilde{E})} \frac{(G^\varepsilon_1)'(r)}{\sqrt{2\tilde{E} - 2\Psi^\varepsilon(r)}} \,\mathrm{d}r. \tag{A.24}$$

We hence arrive at the following limiting behaviour of $(T^\varepsilon)'$ in the near circular regime:

Lemma A.12. *The derivative of the period function $(T^\varepsilon)'(E)$ converges to $(T^0)'(E^0_{\text{min}})$ as $E \to E^\varepsilon_{\text{min}}$ and $\varepsilon \to 0$. More precisely, for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$ and $E^\varepsilon_{\text{min}} < E < E^\varepsilon_{\text{min}} + \eta < 0$ it holds that $|(T^\varepsilon)'(E) - (T^0)'(E^0_{\text{min}})| < \delta$.*
Here, $(T^0)'(E^0_{\text{min}})$ denotes the continuous extension of $(T^0)'$ onto E^0_{min}. Due to (3.19), (3.20), and (3.22), this value is given by

$$(T^0)'(E^0_{\text{min}}) = 6\pi \frac{L^{\frac{5}{2}}}{M^4} = \pi \frac{(G^0_1)'(r^0_*)}{\sqrt{(\Psi^0)''(r^0_*)}}. \tag{A.25}$$

Proof. We proceed as in the proof of Lemma A.5 and change variables via $\tilde{\eta} = \Psi^\varepsilon(r)$ in (A.24) to deduce that for any $\varepsilon \geq 0$ and $E \in \mathbb{A}^\varepsilon$ there exists some $\tilde{E} \in]E^\varepsilon_{\text{min}}, E[$ such that

$$(T^\varepsilon)'(E) = \int_{E^\varepsilon_{\text{min}}}^{\tilde{E}} \frac{(G^\varepsilon_1)'(r^\varepsilon_-(\tilde{\eta}))}{\sqrt{2(\tilde{E} - \tilde{\eta})\,(\Psi^\varepsilon)'(r^\varepsilon_-(\tilde{\eta}))^2}} \,\mathrm{d}\tilde{\eta} + \int_{E^\varepsilon_{\text{min}}}^{\tilde{E}} \frac{(G^\varepsilon_1)'(r^\varepsilon_+(\tilde{\eta}))}{\sqrt{2(\tilde{E} - \tilde{\eta})\,(\Psi^\varepsilon)'(r^\varepsilon_+(\tilde{\eta}))^2}} \,\mathrm{d}\tilde{\eta}. \tag{A.26}$$

Lemma A.4 implies that for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $0 \leq \varepsilon < \varepsilon_0$ and $E^\varepsilon_{\text{min}} < E < E^\varepsilon_{\text{min}} + \eta$ it holds that $|(G^\varepsilon_1)'(\tilde{s})\,(\Psi^\varepsilon)''(s)^{-\frac{1}{2}} - (G^0_1)'(\tilde{s})\,(\Psi^0)''(r^0_*)^{-\frac{1}{2}}| < \frac{\delta}{2\pi}$ for $s, \tilde{s} \in [r^\varepsilon_-(E), r^\varepsilon_+(E)]$; recall that $(G^\varepsilon_1)'$ admits explicit representation in terms only of derivatives of Ψ^ε derived from (A.20). For $0 \leq \varepsilon < \varepsilon_0$ and $E^\varepsilon_{\text{min}} < E < E^\varepsilon_{\text{min}} + \eta$ we thus conclude the following estimate for the first integral in (A.26) after rewriting the integrand with the extended mean value theorem similar to (A.13):

$$\begin{aligned}
&\Big| \int_{E^\varepsilon_{\text{min}}}^{\tilde{E}} \frac{(G^\varepsilon_1)'(r^\varepsilon_-(\tilde{\eta}))}{\sqrt{2(\tilde{E} - \tilde{\eta})\,(\Psi^\varepsilon)'(r^\varepsilon_-(\tilde{\eta}))^2}} \mathrm{d}\tilde{\eta} - \frac{1}{2}(T^0)'(E^0_{\text{min}}) \Big| \\
&= \Big| \int_{E^\varepsilon_{\text{min}}}^{\tilde{E}} \frac{(G^\varepsilon_1)'(r^\varepsilon_-(\tilde{\eta}))}{\sqrt{2(\tilde{E} - \tilde{\eta})\,(\Psi^\varepsilon)'(r^\varepsilon_-(\tilde{\eta}))^2}} \mathrm{d}\tilde{\eta} \\
&\quad - \frac{(G^0_1)'(r^0_*)}{2\sqrt{(\Psi^0)''(r^0_*)}} \int_{E^\varepsilon_{\text{min}}}^{\tilde{E}} \frac{\mathrm{d}\tilde{\eta}}{\sqrt{(\tilde{E} - \tilde{\eta})\,(\tilde{\eta} - E^\varepsilon_{\text{min}})}} \Big| \\
&\leq \frac{\delta}{2\pi} \int_{E^\varepsilon_{\text{min}}}^{\tilde{E}} \frac{\mathrm{d}\tilde{\eta}}{\sqrt{(\tilde{E} - \tilde{\eta})(\tilde{\eta} - E^\varepsilon_{\text{min}})}} = \frac{\delta}{2}.
\end{aligned}$$

Similar arguments also apply to the second integral in (A.26). □

The next step is again to verify a suitable pointwise convergence of $(T^\varepsilon)'$ as $\varepsilon \to 0$.

Lemma A.13. *The mapping* $\mathcal{A} \ni (\varepsilon, E) \mapsto (T^\varepsilon)'(E)$ *is continuous at* $\varepsilon = 0$*; recall* (A.10). *More precisely, for any* $\delta > 0$ *and* $E^* \in \mathbb{A}^0$ *there exists* $\varepsilon_0 > 0$ *such that for all* $0 \leq \varepsilon < \varepsilon_0$ *and* $E \in \mathbb{A}^\varepsilon$ *with* $|E^* - E| < \varepsilon_0$ *it holds that* $|(T^\varepsilon)'(E) - (T^0)'(E^*)| < \delta$.

Proof. First observe that $G_0^\varepsilon \to G_0^0$ as $\varepsilon \to 0$ locally uniformly by Lemmas A.1 and A.2; recall the definition of G_0^ε in (A.17). Then the claimed continuity follows similarly to Lemma A.7 using the concavity estimate (A.14) and Lebesgue's dominated convergence theorem applied to the representation (A.16) of $(T^\varepsilon)'(E)$; also note that $E_{\min}^\varepsilon \to E_{\min}^0$ by Lemma A.2. □

We then arrive at the desired convergence results for the derivative of the period function.

Lemma A.14. *It holds that* $\lim_{\varepsilon\to 0} T'^{\varepsilon}_{\min} = T'^{0}_{\min}$ *and* $\lim_{\varepsilon\to 0} T'^{\varepsilon}_{\max} = T'^{0}_{\max}$, *with*

$$0 < T'^{0}_{\min} = (T^0)'(E^0_{\min}) = 6\pi\,\frac{L^{\frac{5}{2}}}{M^4} < \frac{3\pi}{2\sqrt{2}}\,\frac{M}{(-\kappa)^{\frac{5}{2}}} = (T^0)'(\kappa) = T'^{0}_{\max} < \infty. \tag{A.27}$$

Proof. Combining Lemmas A.12 and A.13 yields that $(T^\varepsilon)'(E)$ converges to $(T^0)'(E^*)$ as $\varepsilon \to 0$ and $E \to E^*$ locally uniformly in E^*; see Lemma A.8 for similar arguments. Because $E_{\min}^\varepsilon \to E_{\min}^0$ and $E_0^\varepsilon \to \kappa$ as $\varepsilon \to 0$ by Lemmas A.1 and A.2 we conclude the desired convergence results. □

The next step is to establish a similar result for the second derivative of the period function; recall that $T^\varepsilon \in C^2(\mathbb{A}^\varepsilon)$ by Lemma 3.4. Again differentiating (A.21) using (A.23) and rearranging the integrand yields

$$(T^\varepsilon)''(E) = \frac{1}{(E - E^\varepsilon_{\min})^2}\int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} (G_1^\varepsilon)'(r)\left(\frac{\Psi^\varepsilon(r) - E^\varepsilon_{\min}}{\sqrt{2E - 2\Psi^\varepsilon(r)}} - \frac{1}{2}\sqrt{2E - 2\Psi^\varepsilon(r)}\right)\mathrm{d}r \tag{A.28}$$

for $E \in \mathbb{A}^\varepsilon$. We integrate by parts to rewrite the first summand and arrive at

$$\begin{aligned}(T^\varepsilon)''(E) \\ = \frac{1}{(E - E^\varepsilon_{\min})^2}\int_{r_-^\varepsilon(E)}^{r_+^\varepsilon(E)} \partial_r\left[(G_1^\varepsilon)'(r)\frac{\Psi^\varepsilon(r) - E^\varepsilon_{\min}}{(\Psi^\varepsilon)'(r)} - \frac{1}{2}G_1^\varepsilon(r)\right]\sqrt{2E - 2\Psi^\varepsilon(r)}\mathrm{d}r\end{aligned} \tag{A.29}$$

for $E \in I^\varepsilon$. This is possible because the function

$$G_2^\varepsilon\colon]R^\varepsilon_{\min}, R^\varepsilon_{\max}[\to \mathbb{R},\ G_2^\varepsilon(r) := \begin{cases} \partial_r\left[(G_1^\varepsilon)'(r)\frac{\Psi^\varepsilon(r) - E^\varepsilon_{\min}}{(\Psi^\varepsilon)'(r)} - \frac{1}{2}G_1^\varepsilon(r)\right], & \text{if } r \neq r_*^\varepsilon, \\ 0, & \text{if } r = r_*^\varepsilon, \end{cases}$$

is continuous by Taylor's theorem; recall Lemma 3.3 and that G_1^ε defined in (A.20) is continuously differentiable.
Applying the extended mean value theorem to (A.29) and using (A.23) yields that for any $E \in I^\varepsilon$ there exists $\tilde{E} \in]E^\varepsilon_{\min}, E[$ such that

$$(T^\varepsilon)''(E) = \frac{1}{2(\tilde{E} - E^\varepsilon_{\min})}\int_{r_-^\varepsilon(\tilde{E})}^{r_+^\varepsilon(\tilde{E})} \frac{G_2^\varepsilon(r)}{\sqrt{2\tilde{E} - 2\Psi^\varepsilon(r)}}\,\mathrm{d}r. \tag{A.30}$$

In order to eliminate the factor $(\tilde{E} - E^\varepsilon_{\min})^{-1}$ we integrate by parts again. Observing that G_2^ε is again $\mathcal{O}(r - r_*^\varepsilon)$ as $r \to r_*^\varepsilon$ and smooth on $]R^\varepsilon_{\min}, R^\varepsilon_{\max}[$, we deduce that the function $G_3^\varepsilon \colon]R^\varepsilon_{\min}, R^\varepsilon_{\max}[\to \mathbb{R}$ defined by

$$G_3^\varepsilon(r) := \begin{cases} \frac{G_2^\varepsilon(r)}{(\Psi^\varepsilon)'(r)}, & \text{if } r \neq r_*^\varepsilon, \\ -\frac{1}{10} \frac{(\Psi^\varepsilon)^{(5)}(r_*^\varepsilon)}{(\Psi^\varepsilon)''(r_*^\varepsilon)^3} + \frac{1}{2} \frac{(\Psi^\varepsilon)'''(r_*^\varepsilon)\,(\Psi^\varepsilon)^{(4)}(r_*^\varepsilon)}{(\Psi^\varepsilon)''(r_*^\varepsilon)^4} - \frac{4}{9} \frac{(\Psi^\varepsilon)'''(r_*^\varepsilon)^3}{(\Psi^\varepsilon)''(r_*^\varepsilon)^5}, & \text{if } r = r_*^\varepsilon, \end{cases}$$

is continuously differentiable with derivative that is explicitly computable in terms of derivatives of Ψ^ε. Here the value of $G_3^\varepsilon(r_*^\varepsilon)$ follows from Taylor expansion in G_2^ε. Hence,

$$(T^\varepsilon)''(E) = \frac{1}{2(\tilde{E} - E^\varepsilon_{\min})} \int_{r_-^\varepsilon(\tilde{E})}^{r_+^\varepsilon(\tilde{E})} (G_3^\varepsilon)'(r) \sqrt{2\tilde{E} - 2\Psi^\varepsilon(r)}\, \mathrm{d}r \tag{A.31}$$

for $E \in I^\varepsilon$ and $\tilde{E} \in]E^\varepsilon_{\min}, E[$ as in (A.30). The mean value theorem and (A.23) imply the existence of $\bar{E} \in]E^\varepsilon_{\min}, \tilde{E}[\subset]E^\varepsilon_{\min}, E[$ such that

$$(T^\varepsilon)''(E) = \frac{1}{2} \int_{r_-^\varepsilon(\bar{E})}^{r_+^\varepsilon(\bar{E})} \frac{(G_3^\varepsilon)'(r)}{\sqrt{2\bar{E} - 2\Psi^\varepsilon(r)}} \mathrm{d}r. \tag{A.32}$$

Because this identity is similar to (A.24), we deduce the following behaviour of $(T^\varepsilon)''(E)$ in the near circular regime.

Lemma A.15. *The second order derivative of the period function $(T^\varepsilon)''(E)$ converges to $(T^0)''(E^0_{\min})$ as $E \to E^\varepsilon_{\min}$ and $\varepsilon \to 0$. More precisely, for any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that for all $0 \le \varepsilon < \varepsilon_0$ and $E^\varepsilon_{\min} < E < E^\varepsilon_{\min} + \eta < 0$ it holds that $|(T^\varepsilon)''(E) - (T^0)''(E^0_{\min})| < \delta$.*
Here, $(T^0)''(E^0_{\min})$ denotes the continuous extension of $(T^0)''$ onto $E^0_{\min}$ given by

$$(T^0)''(E^0_{\min}) = 30\pi \frac{L^{\frac{7}{2}}}{M^6} = \frac{\pi}{2} \frac{(G_3^0)'(r_*^0)}{\sqrt{(\Psi^0)''(r_*^0)}}. \tag{A.33}$$

Proof. The statement can be proven similarly to Lemma A.12 by using (A.32) and the properties of G_3^ε derived above. □

The next step is again to verify a suitable pointwise convergence of $(T^\varepsilon)''$ as $\varepsilon \to 0$.

Lemma A.16. *The mapping $\mathcal{A} \ni (\varepsilon, E) \mapsto (T^\varepsilon)''(E)$ is continuous at $\varepsilon = 0$. More precisely, for any $\delta > 0$ and $E^* \in \mathbb{A}^0$ there exists $\varepsilon_0 > 0$ such that for all $0 \le \varepsilon < \varepsilon_0$ and $E \in \mathbb{A}^\varepsilon$ with $|E^* - E| < \varepsilon_0$ it holds that $|(T^\varepsilon)''(E) - (T^0)''(E^*)| < \delta$.*

Proof. First observe that $(G_1^\varepsilon)' \to (G_1^0)'$ as $\varepsilon \to 0$ locally uniformly by Lemmas A.1 and A.2; recall that $(G_1^\varepsilon)'$ admits explicit representation in terms of derivatives of Ψ^ε. Then the claimed continuity follows similarly to Lemmas A.7 and A.13 using the concavity estimate (A.14) and Lebesgue's dominated convergence theorem applied to the representation (A.28) of $(T^\varepsilon)''(E)$. □

We then arrive at the desired convergence results for the second order derivative of the period function.

Lemma A.17. *It holds that* $\lim_{\varepsilon\to 0} T''^{\varepsilon}_{\min} = T''^{0}_{\min}$ *and* $\lim_{\varepsilon\to 0} T''^{\varepsilon}_{\max} = T''^{0}_{\max}$, *with*

$$0 < T''^{0}_{\min} = (T^0)''(E^0_{\min}) = 30\pi\,\frac{L^{\frac{7}{2}}}{M^6} < \frac{15\pi}{4\sqrt{2}}\,\frac{M}{(-\kappa)^{\frac{5}{2}}} = (T^0)''(\kappa) = T''^{0}_{\max} < \infty. \tag{A.34}$$

Proof. The proof is based on Lemmas A.15 and A.16 and proceeds similarly as the proof of Lemma A.14. □

Remark A.18. Similar arguments as to those in the proofs of Lemmas A.5, A.12, and A.15 imply that

$$T^{\varepsilon}(E) \to \frac{2\pi}{\sqrt{(\Psi^{\varepsilon})''(r^{\varepsilon}_*)}}, \qquad (T^{\varepsilon})'(E) \to \pi\,\frac{(G^{\varepsilon}_1)'(r^{\varepsilon}_*)}{\sqrt{(\Psi^{\varepsilon})''(r^{\varepsilon}_*)}},$$
$$(T^{\varepsilon})''(E) \to \frac{\pi}{2}\,\frac{(G^{\varepsilon}_3)'(r^{\varepsilon}_*)}{\sqrt{(\Psi^{\varepsilon})''(r^{\varepsilon}_*)}}$$

as $E \searrow E^{\varepsilon}_{\min}$ for fixed $\varepsilon \geq 0$, where $(G^{\varepsilon}_1)'(r^{\varepsilon}_*)$ and $(G^{\varepsilon}_3)'(r^{\varepsilon}_*)$ are explicitly computable in terms of derivatives of Ψ^{ε} at r^{ε}_*. Together with Lemma 3.4 and (3.18) we hence conclude that $T^{\varepsilon}_{\max}, T'^{\varepsilon}_{\max}, T''^{\varepsilon}_{\max} < \infty$ as well as $T^{\varepsilon}_{\min} > 0$ for any $\varepsilon \geq 0$.

References

1. Antonov, V.A.: Remarks on the problems of stability in stellar dynamics. *Soviet Astronom. AJ* **4**, 859–867, 1960
2. Antonov, V.A.: Solution of the Problem of Stability of a Stellar System with Emden's Density Law and a Spherical Distribution of Velocities. Vestnik Leningradskogo Universiteta, Leningrad (1962)
3. Barré, J., Yamaguchi, Y.Y.: On algebraic damping close to inhomogeneous Vlasov equilibria in multi-dimensional spaces. *J. Phys. A* **46**, 225501, 2013
4. Barré, J., Olivetti, A., Yamaguchi, Y.Y.: Landau damping and inhomogeneous reference states. *C. R. Physique* **16**, 723–728, 2015
5. Bedrossian, J.: A brief introduction to the mathematics of Landau damping. Preprint, arXiv:2211.13707 2022
6. Bedrossian, J.: Nonlinear echoes and Landau damping with insufficient regularity. *Tunis. J. Math.* **3**, 121–205, 2021
7. Bedrossian, J., Germain, P., Masmoudi, N.: Stability of the Couette flow at high Reynolds numbers in two dimensions and three dimensions. *Bull. Am. Math. Soc. (N.S.)* **56**, 373–414, 2019
8. Bedrossian, J., Masmoudi, N.: Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publ. Math. Inst. Hautes Études Sci.* **122**, 195–300, 2015
9. Bedrossian, J., Masmoudi, N., Mouhot, C.: Landau damping: paraproducts and Gevrey regularity. *Ann. PDE* **2**, 71, 2016
10. Bedrossian, J., Masmoudi, N., Mouhot, C.: Landau damping in finite regularity for unconfined systems with screened interactions. *Commun. Pure Appl. Math.* **71**, 537–576, 2018
11. Bedrossian, J., Masmoudi, N., Mouhot, C.: Linearized wave-damping structure of Vlasov–Poisson in $\mathbb{R}^3$. *SIAM J. Math. Anal.* **54**, 4379–4406, 2022
12. Binney, J., Tremaine, S.: Galactic Dynamics (second edition), Princeton Series in Astrophysics, vol. 4. Princeton University Press, Princeton (2008)

13. Bruè, E., Coti Zelati, M., Marconi, E.: Enhanced dissipation for two-dimensional Hamiltonian flows. Preprint, arXiv:2211.14057 2022
14. Chaturvedi, S., Luk, J.: Phase mixing for solutions to 1D transport equation in a confining potential. *Kinet. Relat. Models* **15**, 403–416, 2022
15. Chaturvedi, S., Luk, J.: Linear and nonlinear phase mixing for the gravitational Vlasov-Poisson system under an external Kepler potential. Preprint, arXiv:2409.14626
16. Chicone, C.: The monotonicity of the period function for planar Hamiltonian vector fields. *J. Differ. Equ.* **69**, 310–321, 1987
17. Chow, S.-N., Wang, D.: On the monotonicity of the period function of some second order equations. *Časopis Pěst. Mat.* **111**, 14–25, 1986
18. Després, B.: Scattering structure and Landau damping for linearized Vlasov equations with inhomogeneous Boltzmannian states. *Ann. Henri Poincaré* **20**, 2767–2818, 2019
19. Doremus, J.-P., Feix, M.R., Baumann, G.: Stability of encounterless spherical stellar systems. *Phys. Rev. Lett.* **26**, 725–728, 1971
20. Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics 194. Springer, New York (2000)
21. Gagnebin, A., Iacobelli, M.: Landau damping on the torus for the Vlasov–Poisson system with massless electrons. Preprint, arXiv:2209.04676 2022
22. Glassey, R., Schaeffer, J.: Time decay for solutions to the linearized Vlasov equation. *Transp. Theory Stat. Phys.* **23**, 411–453, 1994
23. Grenier, E., Nguyen, T.T., Rodnianski, I.: Landau damping for analytic and Gevrey data. *Math. Res. Lett.* **28**, 1679–1702, 2021
24. Günther, S., Rein, G., Straub, C.: A Birman–Schwinger principle in general relativity: linearly stable shells of collisionless matter surrounding a black hole. Preprint, arXiv:2204.10620 2022
25. Guo, Y., Lin, Z.: The existence of stable BGK waves. *Commun. Math. Phys.* **352**, 1121–1152, 2017
26. Guo, Y., Rein, G.: A non-variational approach to nonlinear stability in stellar dynamics applied to the King model. *Commun. Math. Phys.* **271**, 489–509, 2007
27. Hadžić, M., Rein, G., Straub, C.: On the Existence of Linearly Oscillating Galaxies. *Arch. Ration. Mech. Anal.* **243**, 611–696, 2022
28. Hadžić, M., Rein, G., Schrecker, M., Straub, C.: Quantitative phase mixing for Hamiltonians with trapping. Preprint, arXiv:2405.17153 2024
29. Han-Kwan, D., Nguyen, T.T., Rousset, F.: On the Linearized Vlasov–Poisson System on the Whole Space Around Stable Homogeneous Equilibria. *Commun. Math. Phys.* **387**, 1405–1440, 2021
30. Han-Kwan, D., Nguyen, T.T., Rousset, F.: Asymptotic stability of equilibria for screened Vlasov–Poisson systems via pointwise dispersive estimates. *Ann. PDE* **7**, 37, 2021
31. Hislop, P.D., Sigal, I.M.: Introduction to Spectral Theory, Applied Mathematical Sciences 113. Springer, New York (1996)
32. Huang L., Nguyen, Q.-H., Xu, Y.: Sharp estimates for screened Vlasov–Poisson system around Penrose-stable equilibria in $\mathbb{R}^d$, $d \geq 3$. Preprint arXiv:2205.10261 2022
33. Ionescu, A.: On the global stability of shear flows and vortices. *Video Lecture at ICM 2022*, https://www.youtube.com/watch?v=tO81EKR7n0s
34. Ionescu, A., Pausader, B., Wang, X., Widmayer, K.: Nonlinear Landau damping for the Vlasov-Poisson system in $\mathbb{R}^3$: the Poisson equilibrium. Preprint, arXiv:2205.04540 2022
35. Ionescu, A.D., Iyer, S., Jia, H.: Linear inviscid damping and vorticity depletion for non-monotonic shear flows. Preprint, arXiv:2301.00288 2023
36. Kalnajs, A.J.: Dynamics of flat galaxies. *I. Astrophys. J.* **166**, 275–293, 1971
37. Kalnajs, A.J.: Dynamics of flat galaxies. IV. The integral equation for normal modes in matrix form. *Astrophys. J.* **212**, 637–644, 1977
38. Kandrup, H.E., Sygnet, J.F.: A simple proof of dynamical stability for a class of spherical clusters. *Astrophys. J.* **298**, 27–33, 1985

39. Kiessling, M.K.-H.: The "Jeans swindle": a true story–mathematically speaking. *Adv. in Appl. Math.* **31**, 132–149, 2003
40. Kunze, M.: A Birman–Schwinger Principle in Galactic Dynamics, Progress in Mathematical Physics, vol. 77. Birkhäuser, Basel (2021)
41. Landau, L.: On the vibrations of the electronic plasma. *Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz.* **16**, 574–586, 1946
42. Lemou, M., Méhats, F., Raphaël, P.: A new variational approach to the stability of gravitational systems. *Commun. Math. Phys.* **302**, 161–224, 2011
43. Lemou, M., Méhats, F., Raphaël, P.: Orbital stability of spherical galactic models. *Invent. Math.* **187**, 145–194, 2012
44. Lieb, E., Seiringer, R.: The Stability of Matter in Quantum Mechanics. Cambridge University Press, Cambridge (2010)
45. Lin, Z., Zeng, C.: Small BGK waves and nonlinear Landau damping. *Commun. Math. Phys.* **306**, 291–331, 2011
46. Lin, Z., Zeng, C.: Instability, index theorem, and exponential trichotomy for linear Hamiltonian PDEs. *Mem. Am. Math. Soc.* **275**, 136, 2022
47. Louis, P.D.: Discrete oscillation modes and damped stationary density waves in one-dimensional collisionless systems. *Mon. Not. R. Astron. Soc.* **258**, 552–570, 1992
48. Lynden-Bell, D.: The stability and vibrations of a gas of stars. *Mon. Not. R. Astron. Soc.* **124**, 279–296, 1962
49. Lynden-Bell, D.: Statistical mechanics of violent relaxation in stellar systems. *Mon. Not. R. Astron. Soc.* **136**, 101–121, 1967
50. Lynden-Bell, D.: Lectures on Stellar Dynamics, in Galactic dynamics and N-Body Simulations. Lecture Notes in Physics, vol. 433, pp. 3–31. Springer, Berlin (1994)
51. Masmoudi, N., Zhao, W., Nonlinear inviscid damping for a class of monotone shear flows in finite channel. Preprint, arXiv:2001.08564 2020
52. Mathur, S.D.: Existence of oscillation modes in collisionless gravitating systems. *Mon. Not. R. Astron. Soc.* **243**, 529–536, 1990
53. Moreno, M., Rioseco, P., Van Den Bosch, H.: Mixing in anharmonic potential well. *J. Math. Phys.* **63**, 071502, 2022
54. Mouhot, C., Villani, C.: On Landau damping. *Acta Math.* **207**, 29–201, 2011
55. Pausader, B., Widmayer, K.: Stability of a point charge for the Vlasov–Poisson system: the radial case. *Commun. Math. Phys.* **385**, 1741–1769, 2021
56. Ramming, T., Rein, G.: Spherically symmetric equilibria for self-gravitating kinetic or fluid models in the non-relativistic and relativistic case–A simple proof for finite extension. *SIAM J. Math. Anal.* **45**, 900–914, 2013
57. Ramming, T., Rein, G.: Oscillating solutions of the Vlasov–Poisson system—a numerical investigation. *Phys. D* **365**, 72–79, 2018
58. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. I. Functional analysis, 2nd edn. Academic Press, Inc, New York (1980)
59. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, II. Fourier Analysis Self-Adjointness. Academic Press, New York (1975)
60. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, III. Scattering Theory. Academic Press, New York (1979)
61. Rein, G., Straub, C.: On the transport operators arising from linearizing the Vlasov–Poisson or Einstein–Vlasov system about isotropic steady states. *Kinet. Relat. Models* **13**, 933–949, 2020
62. Rioseco, P., Sarbach, O.: Phase space mixing in an external gravitational central potential. *Class. Quant. Gravity* **37**, 195027, 2020
63. Schulze, A.: Existence and stability of static shells for the Vlasov–Poisson system with a fixed central point mass. *Math. Proc. Camb. Philos. Soc.* **146**, 489–511, 2009
64. Straub, C., *Pulsating Galaxies*, Doctoral thesis, Universität Bayreuth 2024
65. Weder, R., Galaxy dynamics, gravitational Vlasov–Poisson system, Landau damping, and scattering theory. Preprint, arXiv:2501.04175, 2025

66. Weinberg, M.D.: Weakly damped modes in star clusters and galaxies. *Astrophys. J.* **421**, 481–490, 1994

M. Hadžić
University College London,
London
UK.
e-mail: m.hadzic@ucl.ac.uk

and

G. Rein & C. Straub
University of Bayreuth,
Bayreuth
Germany.
e-mail: gerhard.rein@uni-bayreuth.de
e-mail: christopher.straub@uni-bayreuth.de

and

M. Schrecker
University of Bath,
Bath
UK.
e-mail: mris21@bath.ac.uk

(*Received July 29, 2024 / Accepted May 19, 2025*)
Published online July 17, 2025