The construction of 'reasoning' in a KS3 textbook

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Analysis of a textbook scheme for Key Stage 3 characterises the ways students are expected to engage with mathematical reasoning. Key findings include the lack of a clear distinction between problem-solving and reasoning and strong differences in both the types and amounts of reasoning expected of students perceived to need support or depth. We discuss relationships between the discourse of reasoning in the textbook and the discourses found in research and in the official curriculum.

Keywords: mathematical reasoning; textbook; recontextualisation

Introduction

The development of mathematical reasoning is one of the aims of the National Curriculum for England as well as being a feature of many curricula worldwide. Yet reviews of its use in research (Hjelte et al., 2020), curriculum documents (Reid, 2022) and among teachers (Herbert et al., 2015) suggest that diverse meanings are ascribed to the term. Our project seeks to understand how mathematical reasoning is construed in discourses occurring at different points in the curriculum chain, tracking how these discourses are recontextualised as they move from policy into practice. By comparing the curriculum chains in Sweden and in England, we hope to contribute to a general understanding of processes of recontextualisation.

In the project so far we have analysed the characterisations of reasoning in three key documents arising from the mathematics education research community that have played significant roles in influencing curriculum reforms in many countries (Morgan et al., 2024). We have also analysed how reasoning is characterised in the official discourse of the National Curriculum in England and its Non-Statutory Guidance as well as in dictionaries of 'everyday' language. This analysis has considered not only how reasoning is defined but also how it is exemplified in descriptions of classroom activities and tasks for students. Our current focus is on how textbooks construe mathematical reasoning and the messages they convey for students and teachers about student engagement in reasoning activity.

As a discourse moves from one site to another, it is transformed according to the functions of the new site. The transformed discourse is constructed by a recontextualisation that "selectively appropriates, relocates, refocuses and relates other discourses to constitute its own order" (Bernstein, 2000, p. 33). Thus, while the discourse on mathematical reasoning produced by researchers has influenced its inclusion in curricula, and curricular requirements have influenced the production of textbooks and, in turn, the practices of teachers, we should not expect reasoning to be identical in all these contexts. Our wider project seeks to characterise and understand the transformations that take place as discourse on mathematical reasoning moves across these sites. After presenting our findings from analysis of one KS3 scheme, we will discuss some of our initial observations of recontextualisation, relating our findings to the discourses from the research field and from the official curriculum.

Background

Previous research on conceptualisations of reasoning has resulted in categorisation of a range of types or meanings, including, in the research field: generalising, conjecturing, identifying patterns, comparing, classifying, validating, justifying, proving, formal proving (e.g. Jeannotte & Kieran, 2017) and, among teachers: thinking, communicating thinking, problem solving, validating thinking, conjecturing, validating conjectures, and connecting aspects of mathematics (e.g. Herbert et al., 2015). In our earlier study (Morgan et al., 2024), we found 39 different types of reasoning across three research-based frameworks for reform, including five frequent types common to all three: argument, (logical) chain, explanation, justification and proof.

Analyses of reasoning in mathematics textbooks tend to consider specific aspects of reasoning. For example, Lithner (2008) distinguishes *creative* reasoning, involving problems where a solution method needs to be constructed, from *imitative* reasoning. Other studies use the *reasoning-and-proving* analytic framework (Stylianides, 2008) focusing on two main parts: making mathematical generalisations and providing support for mathematical claims. Research from these two perspectives yields some similar results across cultural contexts. A substantial majority of textbook tasks demand none of the types of mathematical reasoning (or only *imitative* reasoning) (Bieda et al., 2014; Jäder et al., 2020; Zhang & Qi, 2019). Reasoning tasks are commonly placed at the end of sections or chapters (Bieda et al., 2014; Jäder et al., 2020) calling into question whether all students will encounter them. Tasks involving proving are very infrequent (Jäder et al., 2020; Zhang & Qi, 2019) Furthermore, Jäder et al. (2020) found that more than 60% of the tasks labelled as problem solving or reasoning involve only *imitative* reasoning. Our study focuses on tasks that are labelled as reasoning, to analyse how the textbooks construe mathematical reasoning.

The data set

The textbook scheme we have chosen for analysis is *Maths Progress KS3*, published by Pearson and widely used in secondary schools in England. We focus on the materials for Year 7. Within these materials, tasks for students are presented across several texts:

- Core Textbook, intended for use with all students
- Support Book, designed to "*Provide extra scaffolding and support*"
- Depth Book, designed to "*Deepen students' understanding*"
- Teacher Notes, providing guidance on some of the items in the student books.

While schools and teachers may differ in the ways they use these texts, it seems likely that all students will have access to the *Core* materials (though they may not use all the items) and different groups of students will also be provided with some or all the items in the *Support* or *Depth* materials. Our sample consists of materials from all these texts across three Units from different areas of the curriculum: *Expressions, functions and formulae* (Algebra); *Ratio and Proportion* (Ratio); *Lines and Angles* (Angles).

An important feature of this scheme is that some student items are labelled as "Reasoning" or "Problem-Solving" (in the *Depth* materials "R", "P-S" or in some cases "P-S/R"). The implementation of reasoning in student items is thus made explicit. Our data set consists of items labelled "Reasoning", "R" or "P-S/R" in each of the books, together with *Teacher Notes* associated with any of these items. It also includes additional items for which the *Teacher Notes* use the term *reason**, referring to an intended student activity. These items and notes give explicit messages to students and/or teachers about the nature of mathematical reasoning. Many of the items (numbered in the student texts) involve more than one task (making a separate demand for an action or an answer). Our unit of analysis is the task.

Analytic approach

The analysis focuses on the types of reasoning and the forms of student engagement expected. This matches the approach taken in analysing the reform frameworks and the official curriculum documents. Our analysis of reform frameworks identified terms used to identify types of reasoning or products of reasoning (Morgan et al., 2024). In the current analysis of teaching materials, we use these terms to characterise the types of reasoning expected for student tasks explicitly identified as involving reasoning.

Tasks also vary in the ways students are expected to engage with these types of reasoning. As in our analysis of frameworks, we define three forms of engagement, drawing on Bernstein's (2000) theory of pedagogic discourse. According to Bernstein, successful acquisition of the specialised discourse of a school subject such as mathematics involves *recognising* the discourse by distinguishing between what is and is not to be considered mathematical and knowing the rules for *realising* the discourse by producing legitimate mathematical texts. Moreover, acquisition is more likely to occur when the learner is aware of the *evaluation* criteria for legitimating mathematical text. Students may engage with a type of reasoning in one of these ways:

- *recognition*, defined as processes of observing or understanding;
- *realisation*, defined as constructive processes by which the student produces something communicable to others (e.g. an argument, a proof, a conjecture);
- *evaluation*, defined as processes of judging, investigating, comparing.

Each reasoning task is thus assigned a *task type* comprising the *form of engagement* and the *reasoning type* involved (e.g. *realise + generalisation*).

The extract shown in Figure 1 is an item presented in the National Curriculum Non-Statutory Guidance as an example of good practice, providing an opportunity for reasoning (DfE & NCETM, 2021, p. 126). The item is considered together with guidance about its use.

Figure 1: Example of an item involving reasoning tasks and guidance for its use

> ***Example 3****: A number has been rounded to a number of significant figures, with the result of 76 500.*
>
> *a) Kayla says that it has been rounded to three significant figures. Lakshmi says that it has been rounded to four significant figures.*
>
> *Who is correct? Why?*
>
> *b) What might the original number have been before rounding?*
>
> Example 3 offers the opportunity … Students should be given the opportunity to reason fully regarding the zero digits in the answer and which of these may be significant.

Table 1: Analysis of Figure 1 item

Task	Task type	Comment
Who is correct?	Evaluate solution	Judgement about solutions offered by named others
Why?	Realise justification of evaluation	Produce a text justifying the evaluation
What might the original number have been before rounding?	Recognise structure	A correct response may be taken as evidence of understanding when a zero may be significant (an object of reasoning mentioned in the guidance).

To illustrate our analysis, we identify three tasks within this item that expect responses from students , categorising these tasks as shown in Table 1.

Findings

The number of reasoning tasks in each of the student texts is summarised in Table 2. Overall, the three topic areas do not differ widely in the number of reasoning tasks compared to the total number of items in the unit. However, in the Algebra unit, a high proportion of these are in the Depth materials. It is notable that the Support materials provide few additional reasoning tasks while the Depth materials provide many.

Table 2: Distribution of reasoning tasks in the sample units

	Core tasks (*items*)	Support tasks (*items*)	Depth tasks (*items*)	Total tasks (*items*)
Angles	52 (*134*)	4 (*51*)	19 (*90*)	75 (*275*)
Algebra	22 (*148*)	4 (*50*)	73 (*79*)	99 (*277*)
Ratio	49 (*180*)	11 (*34*)	34 (*100*)	94 (*314*)
Total	123 (*462*)	19 (*135*)	126 (*269*)	268 (*866*)

The sample units contained 42 distinct reasoning task types. The most common of these (frequency ≥ 10) are shown in Table 3. The high frequency of *realise solution* tasks in all three units reflects the complexity of the relationship between reasoning and problem solving. A substantial majority (50/73) of this task type occurs in the Depth materials and of these 39 are within items labelled as P-S/R, not distinguishing explicitly between problem-solving and reasoning.

Table 3: Frequent reasoning task types by topic

Task type		Angles	Algebra	Ratio	All units
evaluate	method	7	1	7	15
evaluate	solution	4	4	5	13
realise	explanation of method	1	4	9	14
realise	generalisation	6	12	0	18
realise	justification of evaluation	8	8	10	26
realise	logical chain	5	1	4	10
realise	solution	21	25	27	73
recognise	relationship	0	4	6	10

Table 4: Frequent reasoning task types by level of materials (omitting ‘realise solution’ task types)

Task type		Core	Support	Depth	Total
evaluate	method	10	0	5	15
evaluate	solution	7	3	3	13
realise	explanation of method	11	0	3	14
realise	generalisation	6	0	12	18
realise	justification of evaluation	14	4	8	26
realise	logical chain	10	0	0	10
recognise	relationship	2	0	8	10

Table 4 compares patterns of occurrence of the most frequent task types (omitting *realise solution*, discussed above) between the Core, Support and Depth materials. Differences between these texts may indicate varying expectations and experiences of students perceived to be low or high achievers, though further investigation would be needed to clarify how teachers actually make use of these materials with different groups of students. As noted above, the Support materials provide few additional reasoning tasks; the type of task is also limited, involving only *evaluate solution* and *realise justification of evaluation*. The Depth materials offer not only many additional opportunities for reasoning but also a wider range of task types, with a particularly high representation of *realise generalisation* (mainly in the Algebra unit) and *recognise relationship*.

Discussion of recontextualization

Our discussion focusses on how the construction of reasoning in this textbook scheme varies from the discourses of research-based reform frameworks and of the National Curriculum. The first theme we address is the relationship between reasoning and problem solving. The reform frameworks and the National Curriculum define these as distinct, though in some cases recognising that they may be intertwined in practice. The label P-S/R on items in our data set reflects such intertwining. We frequently found *realise solution* tasks within these items but they also occurred within items labelled simply as Reasoning, indicating that the messages to students and teachers do not make a clear distinction between Problem-Solving and Reasoning. This suggests that the recontextualisation into textbooks moves the conceptualisation of mathematical reasoning towards problem solving and away from the generalisation and proving aspects emphasised in research and curriculum discourses.

After *realise solution*, the most frequent task type is *realise justification of evaluation*. This task type is exemplified in the official Non-Statutory Guidance (one such example is shown in Figure 1), asking students first to evaluate given solutions and then to justify their evaluation. Evaluation of solutions, methods and arguments is also exemplified in the reform frameworks, as is the demand to justify. The frequency of *realise justification of evaluation* suggests that this form of reasoning moves seamlessly into the textbook. However, it often appears without an initial *realise evaluation* task, instead informing students that the given solution is correct/incorrect and only asking them to explain why.

We found variation in the distribution of reasoning tasks across the Core, Support and Depth materials. The limited variety and quantity of reasoning tasks in Support materials suggests that "scaffolding and support" is not seen to involve engagement with reasoning. In contrast, students steered towards the Depth materials are provided with opportunities to engage with a wide range of reasoning tasks. Neither research-based frameworks nor the National Curriculum suggest that different groups of student should engage in different amounts or types of reasoning. On the other hand, previous research has found reasoning tasks placed at the end of chapters, suggesting a form of differentiation (Bieda et al., 2014; Jäder et al., 2020). As the discourse moves closer to classroom practice we seem to observe assumptions of difference in students' learning and different expectations of engagement with mathematical reasoning.

Next steps

As we proceed with the project we will need to develop our analytic method for tasks not explicitly identified as reasoning, both within this scheme and in others, including extending to KS2 and KS4. We also intend to investigate teachers' discourses and practices of reasoning in order not only to determine how the discourses of research, curriculum and textbooks are recontextualised into the classroom but also to consider how classroom practices and discourses may influence the formation of discourses elsewhere in the curriculum chain. Some work has already been done with teachers in

Sweden and it will be interesting to compare how reasoning is constructed through the curriculum chains in two countries with differing educational cultures.

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