Chapter 3

Character formula for conjugacy classes in a coset

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Let G be a finite group and $N \triangleleft G$ a normal subgroup with G/N abelian. We show how the conjugacy classes of G in a given coset qN relate to the irreducible characters of G that are not identically 0 on qN. We describe several consequences. In particular, we deduce that when G/N is cyclic generated by q, the number of irreducible characters of N that extend to G is the number of conjugacy classes of G in qN.

Let G be a finite group and $N \triangleleft G$ a normal subgroup with $Q = G/N$ abelian. The character group $\widehat{Q}$ acts on the set of irreducible characters $\operatorname{Irr} G$ by tensoring, and it is well known that (see e.g. [5, Theorem 1.3])

$$\#(\text{conjugacy classes of } G \text{ inside } N) = \#(\widehat{Q}\text{-orbits on } \operatorname{Irr} G).$$

In this note, we give a simple representation-theoretic interpretation of conjugacy classes in other cosets of N, and discuss some corollaries. We write $[g]$ for the conjugacy class of $g \in G$, and $[\rho]$ for the $\widehat{Q}$-orbit of $\rho \in \operatorname{Irr} G$.

Theorem 1. *Let $N \triangleleft G$ be finite groups with $Q = G/N$ abelian, and $q \in G$. Consider*

$$\begin{aligned} J_q &= \text{ set of conjugacy classes of } G \text{ inside } qN \\ R_q &= \text{ set of } \widehat{Q}\text{-orbits } [\rho] \text{ on } \operatorname{Irr} G \text{ with } \rho \text{ not identically 0 on } qN. \end{aligned}$$

Then $\#J_q = \#R_q$, and the following matrix is unitary:

$$M_q = \Big(\sqrt{\tfrac{\#[g]\#[\rho]}{\#G}}\, \rho(g)\Big)_{[\rho]\in R_q,\, [g]\in J_q}.$$

Here we pick any representative ρ for each orbit in R_q.

Through the article, a character χ of a group G containing a normal subgroup $N \triangleleft G$ in its kernel is sometimes seen as a character of G/N and vice versa. (See e.g. [4, Lemma 2.22] and the discussion after that.) When Q is abelian, recall that $\hat{Q}$ is a group under $\otimes$, and $\hat{Q} \cong Q$ non-canonically (see [4, Problem 2.7]).

2020 Mathematics Subject Classification. Primary 20C15; Secondary 20E45, 20C25, 20K35.
Keywords. finite groups, conjugacy classes, characters, quotient group.

Proof of Theorem 1. Consider the class functions $\pi_q = \frac{1}{\#\widehat{Q}} \sum_{\chi\in\widehat{Q}} \overline{\chi(q)}\chi$ in the character ring of G. We prove the theorem in six steps.

(i) *Claim.* $\pi_q(g) = \begin{cases} 0 & \textit{if } g\notin qN \\ 1 & \textit{if } g\in qN \end{cases}$, *and hence* $(\pi_q\otimes\rho)(g) = \begin{cases} 0 & \textit{if } g\notin qN \\ \rho(g) & \textit{if } g\in qN \end{cases}$.

Indeed,

$$\pi_q(g) = \frac{1}{\#\widehat{Q}} \sum_{\chi\in\widehat{Q}} \overline{\chi(q)}\chi(gN) = \begin{cases} 0 & \text{if } g\notin qN \\ 1 & \text{if } g\in qN, \end{cases}$$

by column orthogonality in the character table for Q.

(ii) *Claim. If* $q\notin\bigcap_{\chi\in\mathrm{Stab}(\rho)}\ker\chi$, *then* $\pi_q\otimes\rho=0$. *Otherwise,* $\langle\pi_q\otimes\rho,\pi_q\otimes\rho\rangle = \frac{1}{\#[\rho]}$.

Let S be a set of representatives for $\widehat{Q}/\,\mathrm{Stab}(\rho)$. Every $\chi\in\widehat{Q}$ can be written uniquely as $\chi_1\chi_2$ with $\chi_1\in\mathrm{Stab}\,\rho$ and $\chi_2\in S$. Then

$$\pi_q\otimes\rho = \frac{1}{\#\widehat{Q}} \sum_{\chi\in\widehat{Q}} \overline{\chi(q)}(\chi\otimes\rho) = \sum_{\chi_2\in S}(\chi_2\otimes\rho)\overline{\chi_2(q)} \sum_{\chi_1\in\mathrm{Stab}(\rho)} \overline{\chi_1(q)}.$$

If $q\notin\bigcap_{\chi\in\mathrm{Stab}(\rho)}\ker\chi$, then the inner sum is 0 by column orthogonality for q and the identity element in the character table of $\mathrm{Stab}(\rho)$. Otherwise, it is $\#\,\mathrm{Stab}(\rho)$, so

$$\pi_q\otimes\rho = \frac{1}{\#[\rho]} \sum_{\chi_2\in S} \overline{\chi_2(q)}(\chi_2\otimes\rho). \qquad (\dagger)$$

In that case, the characters $\chi_2\otimes\rho$ are all distinct, hence orthonormal, and

$$\langle\pi_q\otimes\rho,\pi_q\otimes\rho\rangle = \frac{1}{(\#[\rho])^2} \sum_{\chi_2\in S} \chi_2(q)\overline{\chi_2(q)} = \frac{1}{(\#[\rho])^2} \sum_{\chi_2\in S} 1 = \frac{1}{\#[\rho]}.$$

(iii) *Claim.* $[\rho]\in R_q \iff q\in\bigcap_{\chi\in\mathrm{Stab}(\rho)}\ker\chi$.

Suppose $[\rho]\in R_q$, so $\rho\not\equiv 0$ on qN. If $\chi\in\mathrm{Stab}\,\rho$, then $\chi\otimes\rho=\rho$, and in particular $\chi(q)=1$ as χ is constant on qN. Therefore, $q\in\bigcap_{\chi\in\mathrm{Stab}(\rho)}\ker\chi$. Conversely, if q lies in this intersection, then $\langle\pi_q\otimes\rho,\pi_q\otimes\rho\rangle\neq 0$ by (ii). As $\pi_q\otimes\rho$ is zero outside qN by (i), we must have $\rho\neq 0$ on qN. In other words $[\rho]\in R_q$.

(iv) *Claim. Choose a set of representatives U of orbits of $\widehat{Q}$ on* Irr G. *Then*

$$\left\{\sqrt{\#[\rho]}(\pi_q\otimes\rho)\;\middle|\;\rho\in U, q\in\bigcap_{\chi\in\mathrm{Stab}(\rho)}\ker\chi\right\}$$

is an orthonormal basis of class functions for G.

From (†) it is clear that $\pi_q\otimes\rho$ and $\pi_{q'}\otimes\rho'$ are orthogonal whenever $\rho\neq\rho'$, as $[\rho]$ and $[\rho']$ are disjoint. From (i) it follows that they are orthogonal when $q\neq q'$ as well, and (ii) shows orthonormality.

Next, for abelian groups $B < A$, we have

$$\bigcap_{g \in B} \left\{ \psi \in \hat{A} \;\middle|\; \psi(g) = 1 \right\} = \left\{ \psi \in \hat{A} \;\middle|\; B \subset \ker \psi \right\} = \widehat{A/B}.$$

Applying this to $B = \operatorname{Stab} \rho$, $A = \hat{Q}$ and $\hat{A} = \hat{\hat{Q}} = Q$ we find that

$$\bigcap_{\chi \in \operatorname{Stab}(\rho)} \ker \chi = \bigcap_{\chi \in \operatorname{Stab} \rho} \left\{ q \in Q \;\middle|\; \chi(q) = 1 \right\} = (\hat{Q} / \operatorname{Stab} \rho)\widehat{\ }.$$

In particular, for each $\rho \in U$, the left-hand side is a group of order $\#Q / \# \operatorname{Stab} \rho = \#[\rho]$. Thus our set of class functions has cardinality

$$\sum_{\rho \in U} \#[\rho] = \# \operatorname{Irr} G,$$

and is therefore a basis.

(v) *Claim.* $\#J_q = \#R_q$.

The class functions $\sqrt{\#[\rho]}(\pi_q \otimes \rho)$ with $q \in \bigcap_{\chi \in \operatorname{Stab}(\rho)} \ker \chi$ are 0 outside qN and are the only such functions from the basis in (iv). So they are a basis of class functions that are zero outside qN, and hence their number is the number of conjugacy classes in qN.

(vi) *Claim.* M_q *is unitary.*

Choose a set of representatives U_q of $\widehat{Q}$-orbits in R_q. By (i), for $\rho, \rho' \in U_q$, we have

$$\begin{aligned} &\sum_{g \in J_q} \sqrt{\tfrac{\#[g]\#[\rho]}{\#G}}\, \rho(g) \sqrt{\tfrac{\#[g]\#[\rho']}{\#G}}\, \overline{\rho'(g)} \\ &\quad = \frac{1}{\#G} \sum_{g \in J_q} \left(\#[g] \sqrt{\#[\rho]} (\pi_q \otimes \rho)(g) \cdot \sqrt{\#[\rho']}\, \overline{(\pi_q \otimes \rho')(g)} \right). \end{aligned}$$

By (i), $\pi_q \otimes \rho$ is 0 outside qN, so this is just

$$\left\langle \sqrt{\#[\rho]} \pi_q \otimes \rho, \sqrt{\#[\rho']} \pi_q \otimes \rho' \right\rangle,$$

which is 1 if $\rho = \rho'$ and 0 otherwise, by (iv). So the rows of M_q are orthonormal. As M_q is a square matrix by (v), it is unitary. ∎

Example 2. Consider $G = F_5 = C_5 \rtimes C_4$ of order 20, and $N = C_5 \lhd G$. Pick $h \in N$ and $q \in G$ of order 5 and 4, respectively. The character table of G is given by

	1	h	q	q^2	q^3
order	1	5	4	2	4
size	1	4	5	5	5
ρ_1	1	1	1	1	1
ρ_2	1	1	-1	1	-1
ρ_3	1	1	$-i$	-1	i
ρ_4	1	1	i	-1	$-i$
ρ_5	4	-1	0	0	0

For the trivial coset N we have

$$J_N = \{[1],[h]\}, \qquad R_N = \{[\rho_4],[\rho_5]\}.$$

Thus, $\#J_N = \#R_N$, and indeed

$$M_N = \begin{pmatrix} \sqrt{\frac{4}{20}}\cdot 1 & \sqrt{\frac{4\cdot 4}{20}}\cdot 1 \\ \sqrt{\frac{1}{20}}\cdot 4 & \sqrt{\frac{4}{20}}\cdot(-1) \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

is unitary. All other cosets have

$$J_{q^jN} = \{[q^j]\}, \quad R_{q^jN} = \{[\rho_1]\} \quad \text{and} \quad M_{q^jN} = (1).$$

Remark 3. Generally, Clifford–Fischer theory links cosets of N, even when G/N is not abelian, to certain projective representations of G; see [3] and the references in [1]. It is quite possible that Theorem 1 is a special case, although usually in Clifford–Fischer theory to deduce results for ordinary (rather than projective) characters, one requires either that $G = N \rtimes Q$ or that every character of N extends to its inertia group. Theorem 1 does not need those assumptions (which fail e.g. when $N = C_2$, $G = Q_8$), and it is probably easier to prove from first principles in any case.

Remark 4. When $N = G$, the unitary matrix $M_G = \left(\sqrt{\frac{\#[g]}{\#G}}\rho(g)\right)$ is the usual modified character table.

Lemma 5. *In the setting of Theorem 1,*

(1) $\#\operatorname{Stab}_{\widehat{Q}}(\rho) = \langle \operatorname{Res}_N \rho, \operatorname{Res}_N \rho\rangle$.

(2) $[\rho] \in R_q \iff q \in \bigcap_{\chi\in \operatorname{Stab}(\rho)} \ker \chi$.

(3) $R_q \subset R_{q^k}$ *and* $\#J_q \le \#J_{q^k}$ *for all* $q \in G$ *and* $k \ge 1$.

(4) *If* G/N *is cyclic generated by* $q \in G$, *then*

$$[\rho] \in R_q \iff \operatorname{Stab}_{\widehat{Q}}(\rho) = \{\mathbf{1}\} \iff \operatorname{Res}_N \rho \textit{ is irreducible.}$$

Proof. (1) Since Q is abelian, $\sum_{\psi\in\widehat{Q}}\psi$ is the character of the regular representation of Q, in other words $\mathrm{Ind}_N^G\mathbf{1}$ as a character of G. Recall that $\rho\otimes\mathrm{Ind}_N^G\mathbf{1}=\mathrm{Ind}_N^G((\mathrm{Res}_N\rho)\otimes\mathbf{1})$, see e.g. [4, Problem 5.3]. So, by Frobenius reciprocity,

$$\#\,\mathrm{Stab}_{\widehat{Q}}(\rho)=\langle\rho,\rho\otimes\sum_{\psi\in\widehat{Q}}\psi\rangle=\langle\rho,\rho\otimes\mathrm{Ind}_N^G\mathbf{1}\rangle=\langle\rho,\mathrm{Ind}_N^G((\mathrm{Res}_N\rho)\otimes\mathbf{1})\rangle$$
$$=\langle\mathrm{Res}_N\rho,\mathrm{Res}_N\rho\rangle.$$

(2) This is Claim (iii) in the proof of Theorem 1.

(3) Immediate from (2) and the equality $\#R_q=\#J_q$.

(4) The first equivalence follows from (2), noting that $q\notin\ker\chi$ for any $\mathbf{1}\neq\chi\in\widehat{Q}$. The second equivalence follows from (1). ■

Corollary 6. *Let $N\lhd G$ with G/N cyclic generated by $q\in G$. The following are equal:*

- *The number of conjugacy classes of G inside qN.*
- *The number of $\widehat{Q}$-orbits $[\rho]$ on $\mathrm{Irr}\,G$ with ρ not identically 0 on qN.*
- *The number of $\widehat{Q}$-orbits on $\mathrm{Irr}\,G$ of length $(G:N)$.*
- *$\frac{1}{(G:N)}$ times the number of $\rho\in\mathrm{Irr}\,G$ whose restriction to N is irreducible.*
- *The number of $\tau\in\mathrm{Irr}\,N$ that extend to a character of G.*

Proof. The first equivalence follows from Theorem 1 ($\#R_q=\#J_q$). The second and third are the two equivalences in Lemma 5 (2). For the last one, observe that $\tau\in\mathrm{Irr}\,N$ extends to a character of G if and only if $\tau=\mathrm{Res}\,\rho$ for some $\rho\in\mathrm{Irr}\,G$. Suppose $\tau=\mathrm{Res}\,\rho$. Then

$$\mathrm{Res}_N\rho'=\tau\;\Rightarrow\;1=\langle\mathrm{Res}_N\rho',\tau\rangle=\langle\rho',\mathrm{Ind}_N^G\mathrm{Res}\,\rho\rangle=\langle\rho',\rho\otimes\mathrm{Ind}_N^G\mathbf{1}_N\rangle\;\Rightarrow\;\rho'\in[\rho].$$

Conversely, if $\rho'\in[\rho]$, then clearly $\tau=\mathrm{Res}\,\rho'$. In other words, the characters that restrict to τ are exactly those in $[\rho]$, so there are $(G:N)$ of them. The last equivalence now follows. ■

Corollary 7. *Let $N\lhd G$ with G/N cyclic. Then N has a non-trivial irreducible character that extends to G if and only if G has no conjugacy classes of size $\#N$.*

Proof. Let $q\in G$ generate G/N. Then G has a conjugacy class of size $\#N$ if and only if qN is such a class, by Lemma 5 (3). Equivalently, by Corollary 6, only $\mathbf{1}_N$ extends to a character of G. ■

Example 8. For $p>2$, there are exactly p irreducible representations of $N=\mathrm{SL}_2(\mathbb{F}_p)$ that extend to $G=\mathrm{GL}_2(\mathbb{F}_p)$. Indeed, fix a generator $qN\in G/N\cong\mathbb{F}_p^\times$, a primitive root mod p. Note that if the determinant of a matrix in $\mathrm{GL}_2(\mathbb{F}_p)$ generates $\mathbb{F}_p^\times$, then the element is automatically semisimple, provided $p > 2$. Thus there are p conjugacy classes in the coset qN, characterised by their trace, and hence, by Corollary 6, exactly p irreducible representations that extend to $\mathrm{GL}_2(\mathbb{F}_p)$.

We end with an 'inversion formula', which reconstructs a character Θ on G from its values on a sufficient number of cosets. This was our original motivation in the number-theoretic setting when G is a local Galois group and $N \triangleleft G$ its inertia subgroup. In that case, this formula explicitly reconstructs a representation of G from characteristic polynomials of Frobenius over sufficiently many intermediate fields. We refer the reader to [2] (Chapter 4 of the current volume) for the applications of the formula.

Corollary 9 (Inversion formula). *Suppose $N \triangleleft G$ with $Q = G/N$ cyclic, generated by $q \in G$. Let U be a set of representatives of orbits of $\widehat{Q}$ on* Irr G*, and denote $m_\rho = \langle \mathrm{Res}_N\, \rho, \mathrm{Res}_N\, \rho\rangle$ for $\rho \in U$. Let Θ be a character of G.*

(i) *Θ can be written as*

$$\Theta = \sum_{\rho\in U} \Psi_\rho \otimes \rho \qquad \text{for some character } \Psi_\rho \text{ of } G \text{ with } N \subset \ker \Psi_\rho.$$

(ii) *The eigenvalues of the matrix associated to $\Psi_\rho(q)$ are well defined up to multiplication by m_ρth roots of 1.*

(iii) *For every $\rho \in U$ and every $d \geq 0$,*

$$\Psi_\rho(q^{dm_\rho}) = \frac{1}{\#N \cdot m_\rho} \sum_{g\in q^{dm_\rho}N} \overline{\rho(g)}\, \Theta(g).$$

(iv) *$\Psi_\rho \otimes \rho$ is uniquely determined by* (iii). *Concretely, suppose* $\dim \Psi_\rho \leq B$. *There is a unique $0 \leq n \leq B$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^\times$ such that $\sum_k \lambda_k^d = \Psi_\rho(q^{dm_\rho})$ for $d = 1, \ldots, B$. Then $\Psi_\rho(q) = \sqrt[m_\rho]{\lambda_1} + \ldots + \sqrt[m_\rho]{\lambda_n}$ for some choice of the roots. The character $\Psi_\rho \otimes \rho$ is independent of this choice.*

Proof. (i) Clear.

(ii) Since the decomposition into irreducibles is unique, the terms $\Psi_\rho \otimes \rho$ are uniquely determined by Θ. It remains to show that for $\psi, \psi' \in \widehat{Q}$,

$$\rho \otimes \psi \cong \rho \otimes \psi' \quad \Longleftrightarrow \quad (\psi/\psi')^{m_\rho} = \mathbf{1}.$$

By Lemma 5 (1), we have $m_\rho = \#\mathrm{Stab}_{\widehat{Q}}(\rho)$. Because $\widehat{Q}$ is cyclic, $\mathrm{Stab}_{\widehat{Q}}(\rho)$ is exactly the subgroup of characters of order m_ρ, as required.

(iii) Fix $\rho \in U$ and a multiple $j = dm_\rho$. Let $U_{q^j} \subset U$ be the subset of characters that are not identically 0 on q^jN. Define the matrix M_{q^j} as in the theorem with these representatives for R_{q^j}. For $g \in J_{q^j}$, we have

$$\begin{aligned}\Theta(g) &= \sum_{\rho\in U} \Psi_\rho(g)\rho(g) = \sum_{\rho\in U} \Psi_\rho(q^j)\rho(g) = \sum_{\rho\in U_{q^j}} \Psi_\rho(q^j)\rho(g) \\ &= \sqrt{\tfrac{\#G}{\#[g]}} \sum_{\rho\in U_{q^j}} M_{q^j,g,\rho} \frac{\Psi_\rho(q^j)}{\sqrt{\#[\rho]}}.\end{aligned}$$

In other words, we have a matrix equation

$$\Big(\sqrt{\tfrac{\#[g]}{\#G}}\,\Theta(g)\Big)_{g\in J_{q^j}} = M_{q^j} \cdot \Big(\frac{\Psi_\rho(q^j)}{\sqrt{\#[\rho]}}\Big)_{\rho\in U_{q^j}}.$$

As M_{q^j} is unitary, we can rewrite it as

$$\Big(\frac{\Psi_\rho(q^j)}{\sqrt{\#[\rho]}}\Big)_{\rho\in U_{q^j}} = \overline{M^t_{q^j}} \cdot \Big(\sqrt{\tfrac{\#[g]}{\#G}}\,\Theta(g)\Big)_{g\in J_{q^j}}.$$

Using the fact that $m_\rho = \#\operatorname{Stab}_{\widehat{Q}}(\rho)$ proved in (ii), and the orbit-stabiliser equality $m_\rho[\rho] = (G:N)$, we get

$$\begin{aligned}\Psi_\rho(q^j) &= \sqrt{\#[\rho]} \sum_{g\in J_{q^j}} \overline{M_{q^j,g,\rho}} \sqrt{\tfrac{\#[g]}{\#G}}\,\Theta(g) = \sum_{g\in J_{q^j}} \tfrac{\#[\rho]\#[g]}{\#G}\overline{\rho(g)}\Theta(g) \\ &= \sum_{g\in J_{q^j}} \frac{\#G}{\#N m_\rho}\frac{\#[g]}{\#G}\overline{\rho(g)}\Theta(g) = \frac{1}{\#N m_\rho}\sum_{g\in q^j N}\overline{\rho(g)}\Theta(g),\end{aligned}$$

as claimed.

(iv) Let $\mu_1,\dots,\mu_n$ be the eigenvalues of the matrix associated to $\Psi_\rho(q)$, so that $\Psi_\rho(q^{dm_\rho}) = \sum_k \mu_k^{dm_\rho}$ for any $d \geq 0$. Generally, the Vandermonde system of equations $\sum_{k=1}^B \nu_k^d = a_d$ for $d = 1,\dots,B$ has a unique (unordered) solution $\nu_1,\dots,\nu_B$. Thus, the unique solution to $\sum_k \lambda_k^d = \Psi_\rho(q^{dm_\rho})$ is $\mu_1^{m_\rho},\dots,\mu_n^{m_\rho},0,\dots,0$. The formula for $\Psi(\rho)$ follows, and independence of the choice is proved in (ii). ∎

Acknowledgements. We would like to thank the Warwick Mathematics Institute and King's College London, where parts of this research were carried out, Geoffrey Robinson for helpful discussions, and the referee for helpful comments.

Funding. This research was partially is supported by EPSRC grants EP/M016838/1 and EP/M016846/1 'Arithmetic of hyperelliptic curves'. The second author was supported by a Royal Society University Research Fellowship.

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