Detecting Blow-ups via Mirror Laurent Polynomials

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I, Hannah Tillmann-Morris, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The classification of Fano varieties up to deformation is a longstanding open problem. The Fanosearch program is an approach to Fano classification which uses mirror symmetry to translate the geometric classification question into a combinatorial problem. Under mirror symmetry, deformation classes of n-dimensional Fano varieties conjecturally correspond to mutation classes of rigid maximally mutable Laurent polynomials in n variables. In this thesis, we use this correspondence to better understand the birational classification of Fano varieties, by asking the question:

> Is there a combinatorial condition on pairs of Laurent polynomials that is equivalent to their mirror Fano varieties being related by a blow-up?

We introduce a new method of constructing a Fano mirror to a given Laurent polynomial, using constructions from the Gross–Siebert program. Our new construction is more complicated than previous approaches, but is more conceptual and applies in significantly greater generality – in particular, it does not rely on a construction of the Fano as a complete intersection inside a toric variety. In the case that two given Laurent polynomials satisfy a particular combinatorial relationship, both mirror schemes produced by our method can be related by a birational map.

We prove that the mirrors to the two Laurent polynomials $f = x + y + 1/xy$ and $g = x + y + xy + 1/xy$ produced by our new construction are related by a birational morphism $X_g \to X_f$. We show moreover that the restriction of this morphism to the general fibres of the families X_g and X_f gives the blow-up of the projective plane $\mathbb{P}^2$ in a single point.

Research Impact Statement

Fano varieties are fundamental building blocks in algebraic geometry, both in the birational classification of algebraic varieties and as a rich source of explicit examples and constructions. After bounding the complexity of the singularities allowed, there are finitely many deformation classes of Fano variety in each dimension, and their classification – finding a 'Periodic Table of shapes' – is a longstanding open problem.

On the other hand, a set of ideas coming from string theory, called Mirror Symmetry, has had a remarkable impact on mathematics. Mirror Symmetry expresses an equivalence between type IIA and type IIB string theory: in mathematical terms, it implies that the enumerative geometry of a space X is equivalent to the complex geometry of a different space $\check{X}$ called the mirror to X.

The main idea behind the Fanosearch program is to use Mirror Symmetry to understand the Fano classification problem. Mirror symmetry induces a conjectural correspondence between deformation classes of Fano varieties and mutation classes of certain Laurent polynomials. In this thesis I explore how combinatorial changes to the Laurent polynomials result in birational transformations of their Fano mirrors. This is an important step towards understanding the mirror correspondence, both in terms of proving it and in terms of using it to understand the classification of Fano varieties – i.e., 'find the groups in the Periodic Table'.

The Fanosearch approach to mirror symmetry has a wealth of examples, but there are few systematic results about the correspondence. Conversely, the mirror constructions from the Gross–Siebert program apply in much greater generality, but their output is complicated to describe, and as such few examples have been computed. This thesis links the two approaches to mirror symmetry, providing new methods for describing the relationships between Gross–Siebert mirrors to a pair of general Laurent polynomials, without needing the full description of their coordinate rings.

As well as deepening our conceptual understanding of mirror symmetry, this new approach has implications for the Fano classification problem. Once the results have been adapted to higher dimensions, they can be applied to find combinatorial criteria on pairs of Laurent polynomials in more than two variables which imply that their mirror Fanos are related by a blow-up map. It is then a manageable computational task to search for chains of Laurent polynomials linked by the criteria, mapping out mirror Fano varieties and birational morphisms between them.

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Introduction

0.1 Mirror symmetry and Fano varieties

Fano varieties are fundamental building blocks in algebraic geometry, both in the birational classification of algebraic varieties [8,32,39,45,46], and as a rich source of explicit examples and constructions. If we bound the complexity of the singularities allowed, then there are finitely many Fano varieties in each dimension up to deformation [7,39,40]. Many explicit constructions have been given, but the classification of (deformation classes of) Fano varieties is a longstanding open problem. Smooth Fano varieties have been classified up to dimension three. In dimension one, the only Fano variety is projective space $\mathbb{P}^1$. In dimension two, there are the ten del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-up of $\mathbb{P}^2$ in up to 8 points [17]. The smooth three-dimensional Fano varieties were classified by Fano [19], Iskovskikh [36–38] and Mori–Mukai [47]: there are 105 deformation families. For smooth Fano varieties in dimensions four and higher, very little is known.

In the setting of the Minimal Model Program, a more natural question is not the classification of *smooth* Fano varieties, but rather the classification of Fano varieties with $\mathbb{Q}$-factorial terminal singularities – called $\mathbb{Q}$-Fano varieties – up to $\mathbb{Q}$-Gorenstein deformation. Despite the fundamental importance of this problem, once again very little is known – even in dimension three.

The main idea behind the Fanosearch program [12,13] is to use Mirror Symmetry, a set of ideas coming from string theory, to understand the Fano classification problem. Mirror Symmetry expresses an equivalence between type IIA and type IIB string theory: that the enumerative geometry of a space X is equivalent to the complex geometry of a different space $\check{X}$ called the mirror to X. In our context, enumerative invariants of a Fano variety X determine and are determined by the complex geometry of its mirror $\check{X}$, which is expected to be a *cluster variety*. One can equivalently think of an n-dimensional cluster variety, plus a distinguished function on it, as being given by a collection of Laurent polynomials $f \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Roughly speaking, even though classifying Fano varieties is a hard problem in geometry, classifying the Laurent polynomials that give 'mirror partners' to Fano varieties turns out to be an easier problem in combinatorics. Mirror Symmetry, at least in the form that we need it, is still conjectural, but the Fanosearch approach potentially both reveals the classification of Fano varieties and opens the way to proving it. The mirror partners to $\mathbb{Q}$-Fano varieties are called *maximally mutable Laurent polynomials* (MMLPs).

Conjecture 0.1.1 (see [15])**.** *There is a one-to-one correspondence between n-dimensional $\mathbb{Q}$-Fano varieties with terminal locally toric rigid singularities (up to deformation) and rigid maximally mutable Laurent polynomials in n variables (up to an equivalence relation called mutation).*

Let us say more about what it means for a $\mathbb{Q}$-Fano variety X to correspond to a Laurent polynomial f under Mirror Symmetry [12]. The *regularised quantum period* of X is a power series

$$\hat{G}_X := 1 + \sum_{k=2}^{\infty} k! c_k t^k$$

with coefficients c_k that are genus-zero Gromov–Witten invariants [6, 41, 42, 44]. These can be thought of intuitively as counts of rational curves of a given degree k which pass though a fixed generic point and satisfy certain constraints on their complex structure. We say a Laurent polynomial f is *mirror* to a Fano variety X if the classical period

$$\pi_f = \sum_{k=0}^{\infty} \mathrm{coeff}_1(f^k) t^k$$

and the regularised quantum period $\hat{G}_X$ are equal as power series. Here $\mathrm{coeff}_1(f^k)$ denotes the coefficient of the constant term of f^k. This numerical relationship between power series is expected to arise from a geometric relationship between X and f: if X is mirror to f then there is expected to be a one-parameter degeneration with general fibre X and special fibre the toric variety X_Σ, where Σ is the spanning fan of the Newton polytope of f.

There are expected to be tens of thousands of deformation classes of both smooth Fano fourfolds and $\mathbb{Q}$-Fano threefolds. To *understand* the classification problem, therefore, we will need to regard the many deformation classes as forming a much smaller number of families – namely the families of varieties related by birational equivalence studied in the Minimal Model Program. A fundamental example of such a birational equivalence is the blow-up $\mathrm{Bl}_Z : Y \to X$ of X in a subvariety Z. This leads to the motivating question of this thesis:

> Is there a combinatorial condition on pairs f, g of maximally mutable Laurent polynomials that is equivalent to the mirror Fano varieties X_f and X_g being related by a blow-up?

A positive answer to this question would make the Fano classification much easier to understand and work with.[1] It would also open up new ways to prove the mirror correspondence between Fano varieties and Laurent polynomials.

Evidence for a positive answer comes from toric geometry. For a toric variety X with fan Σ, the Givental/Hori–Vafa construction gives a Laurent polynomial f that is mirror to X, such that the Newton polytope of f is the convex hull of the primitive generators of the rays of Σ [20,34]. Applying this to toric varieties X and Y such that there is a toric blow-up $Y \to X$ reveals a natural relationship between the corresponding Laurent polynomials f and g. Even for X non-toric, the toric variety X_{Σ_f} defined by the spanning fan Σ_f of the Newton polytope of a mirror Laurent polynomial f is expected to occur as the special fiber of a degeneration of X.

The Givental/Hori–Vafa construction also produces Laurent polynomial mirrors to complete intersections inside toric Fano varieties. In these cases, the toric degenerations X to X_Σ can be described in terms of explicit equations via the Przyjalkowski method [48] as described in [11]. For certain pairs of Fano complete intersections, with mirror Laurent polynomials f and g, these equations can be chosen compatibly to give a morphism of families $\mathfrak{X}_g \to \mathfrak{X}_f$, which restricts to a blowup $X_g \to X_f$ on the general fibre, and on the central fibre is given by the toric morphism $X_{\Sigma_g} \to X_{\Sigma_f}$ induced by the the map of fans $\Sigma_g \to \Sigma_f$.

However, in order to prove that some combinatorial condition on a general pair of MMLPs f, g implies that their mirrors are related by a blowup $X_g \to X_f$, one

[1] By *the* mirror Fano variety to a Laurent polynomial f in the motivating question, we mean a general element of the deformation family of varieties X with $\hat{G}_X = \pi_f$.

requires a method of recovering Fano varieties from their mirror MMLPs that does not rely on a construction of the Fano as a complete intersection inside a toric variety. Thus we need to move beyond existing methods, such as Laurent inversion [14], which in any case sometimes fail to produce a $\mathbb{Q}$-Fano from its Laurent polynomial mirror.

0.2 The Gross–Siebert program

Mirror symmetry was originally conceived of as an equivalence between pairs of Calabi–Yau manifolds [9], rather than Fano manifolds and Landau–Ginzburg models. The famous conjecture of Strominger–Yau–Zaslow [49] states that a Calabi–Yau manifold X and its mirror partner $\check{X}$ admit dual special Lagrangian torus fibrations over the same base $X \to B \leftarrow \check{X}$. The Gross–Siebert program [5, 10, 21–27, 30, 31] is a new approach to Mirror Symmetry, born out of an attempt to use the affine structure on the base B of the dual fibrations [33] to give an intrinsic algebro-geometric construction of the mirror to a variety X. This bypasses any need to consider X as living inside an ambient toric variety. Gross and Siebert also show how to extend the SYZ approach from Calabi–Yau manifolds to Fano varieties, by passing from a Fano variety X to a *log Calabi–Yau pair* (X, D), where D is an anticanonical divisor.

The version of the Gross–Siebert mirror construction that we will use takes a log Calabi–Yau pair (X, D) and produces a formal family $\check{\mathfrak{X}} \to \mathfrak{S}$ whose fibres are mirror to (X, D). The construction passes through an affine manifold B that plays the role of the base of the SYZ fibration. Here B is a real n-dimensional manifold with a polyhedral decomposition – the dual intersection complex of the pair (X, D) – and carries an affine structure determined by the intersection numbers between components of D. Codimension-one strata of B correspond to curve classes in X, and we can use these classes to glue 'thickened' torus charts associated to each chamber of B across the codimension-one strata. The resulting space is an approximation to the mirror family that we seek, but we need to modify the gluing maps so as to produce a family with the correct general fibre[2]. Corrections to the gluing are made

[2] Another reason to correct the gluing maps is that we want the general fibre to carry a distinguished basis of functions, called *theta functions*, indexed by integral points of B. This is by analogy with mirror symmetry for toric varieties, where lattice points index sections of appropriate line bundles on the mirror.

order by order, using a process called *scattering*. This subdivides the chambers of B using extra codimension one polyhedra called walls, each equipped with a function that reparametrises the thickened torus chart on either side of the wall. A central insight of Gross and Siebert is that this extra combinatorial structure is determined by punctured log Gromov–Witten invariants of the pair (X, D) – the coefficients in the wall functions involve counts of log maps into X whose tropicalisation is supported on the wall. In this way, as expected from the duality between type IIA and type IIB string theory, the enumerative geometry of the pair (X, D) determines the complex geometry of the mirror.

The Gross–Siebert approach fits closely with the story of Mirror Symmetry for Fano varieties: when X is Fano variety and D is an anticanonical divisor, there is a distinguished regular function on the mirror family, namely the sum of the the theta functions corresponding to the primitive integral points on each ray of the polyhedral decomposition. This function, when restricted to a torus chart, gives a Laurent polynomial f that is mirror to X in the sense of Fanosearch: $\pi_f = \hat{G}_X$. All examples of the Gross–Siebert mirror construction for Fano varieties appearing in the literature to date have taken this perspective.

In contrast, in this thesis we will make use of the Gross–Siebert construction to 'cross the mirror' in the opposite direction. The first step here is to produce a log Calabi–Yau pair from a maximally mutable Laurent polynomial, as we will now describe.

0.3 From Laurent polynomials to log Calabi–Yau pairs

Starting with a Laurent polynomial f, one can generate a log Calabi–Yau pair (Y, D) as follows. The Newton polytope of f determines a polarized toric variety. Note this is the toric variety defined by the normal fan Σ', as opposed to the spanning fan Σ considered above. The integral points on Newt f represent a basis for the ring of sections of the polarizing line bundle, and this allows us to consider f as a regular function on $Y_{\Sigma'}$. Pairing f with the section corresponding to the origin yields a rational map $[f : 1] : Y_{\Sigma'} \dashrightarrow \mathbb{P}^1$ which is resolved by blowing up finitely many times on the toric boundary. Let $Y \to Y_{\Sigma'}$ be this resolution, and D be the strict transform of the toric boundary. The pair (Y, D) is log Calabi–Yau, and the general fibre of the family produced by the Gross–Siebert mirror construction is expected to be $X \setminus E$

where X is the Fano variety mirror to f and E is a smooth anticanonical divisor.

0.4 The main result

In this thesis we focus on producing and comparing the Gross–Siebert mirrors of the maximally mutable Laurent polynomials

$$f := x + y + \frac{1}{xy} \quad \text{and} \quad g := x + y + \frac{1}{xy} + xy, \tag{1}$$

which are mirror to the del Pezzo surfaces $\mathbb{P}^2$ and its blowup in a single point $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2$. (We know that f is a mirror to $\mathbb{P}^2$, and that g is a mirror to $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2$, because they are both toric varieties, but this will play no role in what follows.)

In Chapters 4 and 5, we apply Gross–Siebert mirror symmetry to the log Calabi–Yau pairs determined by f and g. This produces two families, $\check{\mathfrak{X}}_f \to \mathfrak{S}$ and $\check{\mathfrak{X}}_g \to \mathfrak{S}$, with general fibres isomorphic to $\mathbb{P}^2$ and $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2$ respectively. The main result of this thesis is the construction, directly from f, g and Gross–Siebert mirror symmetry, of a map of families

$$\begin{array}{ccc} \check{\mathfrak{X}}_g & \longrightarrow & \check{\mathfrak{X}}_f \\ & \searrow \quad \swarrow & \\ & \mathfrak{S} & \end{array}$$

which in the general fibre is the blowup of $\mathbb{P}^2$ in a point. See Theorem 5.2.2 for details.

0.5 Conjectures and future directions

In this thesis we prove the result for the pair of Laurent polynomials f, g given in (1) above. However, since Gross–Pandharipande–Siebert's 'tropical vertex' construction [24] of the Gross–Siebert mirror can be applied to any rigid maximally mutable Laurent polynomial $F \in \mathbb{C}[x^{\pm}, y^{\pm}]$, we expect that the result generalises to pairs of rigid MMLPs which satisfy certain combinatorial conditions.

Definition 0.5.1 (The perturbed scattering diagram $\mathfrak{D}_{\mathrm{pert}}(f, g)$)**.** Suppose that $f, g \in \mathbb{C}[x^{\pm}, y^{\pm}]$ are rigid maximally mutable Laurent polynomials such that

(i) $\operatorname{Newt} f \subset \operatorname{Newt} g$.

(ii) There is a unique edge $E_{\mathrm{pert}} \subset \operatorname{Newt} f$ contained in the interior of $\operatorname{Newt} g$.

For each edge $E \subset \operatorname{Newt} g$ that intersects E_{pert}, let E_f denote the segment of E which is an edge of $\operatorname{Newt} f$. Let l_E denote the lattice length of E, and let v_E denote the inward-pointing primitive normal vector E, where E is any edge of $\operatorname{Newt} f$ or $\operatorname{Newt} g$. Let h_E denote the height of E above the origin with respect to v_E. We define the *initial scattering diagram associated to f and g* as follows:

$$\begin{aligned}
\mathfrak{D}^0_{\text{pert}}(f,g) := &\left\{ \left(\mathbb{R}_{\geq 0} v_E, (1 + t_{E,i} z^{v_E})\right) \;\middle|\; \begin{array}{l} E \subset \operatorname{Newt} g \text{ s.t. } E \subset \operatorname{Newt} f, \\ i \in \{1, \dots, l_E/h_E\} \end{array} \right\} \\
\cup &\left\{ \begin{array}{l} \left(\mathbb{R}_{\geq 0} v_E, (1 + t_{E,i} z^{v_E})\right), \\ \left(v_{E_{\text{pert}}} + \mathbb{R}_{\geq 0} v_E, (1 + t_{E,j} z^{v_E})\right) \end{array} \;\middle|\; \begin{array}{l} E \subset \operatorname{Newt} g \text{ s.t. } E \cap E_{\text{pert}} \neq \emptyset, \\ i \in \{1, \dots, l_{E_f}/h_E\}, \\ j \in \{(l_{E_f}/h_E) + 1, \dots, l_E/h_E\} \end{array} \right\} \\
\cup &\left\{ \left(v_{E_{\text{pert}}} + \mathbb{R}_{\geq 0} v_E, (1 + t_{E,i} z^{v_E})\right) \;\middle|\; \begin{array}{l} E \subset \operatorname{Newt} g \text{ s.t. } E \subset \operatorname{Newt} g \setminus \operatorname{Newt} f, \\ i \in \{1, \dots, l_E/h_E\} \end{array} \right\}
\end{aligned}$$

We define the *perturbed scattering diagram associated to f and g* to be the result of scattering of the initial scattering diagram:

$$\mathfrak{D}_{\text{pert}}(f,g) := \operatorname{Scatter}\left(\mathfrak{D}^0_{\text{pert}}(f,g)\right).$$

This generalises the construction of $\mathfrak{D}_{\text{pert}}$ for the Laurent polynomials in (1), given in equation (4.3), whose initial scattering diagram is pictured in Figure 4.1.

Conjecture 0.5.2. *Let $f, g \in \mathbb{C}[x^{\pm}, y^{\pm}]$ be rigid maximally mutable Laurent polynomials with Newton polytopes that satisfy the following conditions:*

(i) $\operatorname{Newt} f \subset \operatorname{Newt} g$.

(ii) There is a unique edge $E_{\text{pert}} \subset \operatorname{Newt} f$ contained in the interior of $\operatorname{Newt} g$.

(iii) There is an open neighborhood $U \subset \mathbb{R}^2$ about the origin such that $U \cap \mathfrak{D}_{\text{pert}}(f,g)$ only contains lines passing through the origin.

Then there is an $\mathfrak{S}$-birational morphism $\pi : \check{\mathfrak{X}}_g \to \check{\mathfrak{X}}_f$ between the mirror families to g and f. If, moreover,

(iv) the complement $\operatorname{Newt} g \setminus \operatorname{Newt} f$ *is a triangle,*

then the morphism π of families over $\mathfrak{S}$ is, in the general fibre, a blowup in a single point.

This blowup π can be seen on the level of local charts on $\check{\mathfrak{X}}_f$ and $\check{\mathfrak{X}}_g$ as *formal* schemes over $\operatorname{Spf} \Bbbk[\![t]\!]$ – here explicit equations for the local charts $\mathfrak{U}_i = \operatorname{Spec} R_i$ are given by the affine geometry of B and the scattering determined by f and g. In order to construct the morphism π the *general fibres* of the families, however, it is necessary to algebraise the formal schemes. Once they are considered as schemes over $\operatorname{Spec} \Bbbk[\![t]\!]$, the rings R_i no longer define affine charts on $\check{\mathfrak{X}}_f$ and $\check{\mathfrak{X}}_g$. Therefore, in order to prove Conjecture 0.5.2, we need to either:

1. understand affine charts on $\check{\mathfrak{X}}_f$ and $\check{\mathfrak{X}}_g$ as schemes over $\operatorname{Spec} \Bbbk[\![t]\!]$, or

2. generalise the more abstract argument in Chapter 5 to this setting.

While the second of these options is likely to be easier, the first would be more satisfactory in terms of giving a direct link between the combinatorial relationship between f and g and the birational relationship between their mirrors. Moreover, the first option will generalise more easily to higher dimensions.

Looking further into the future, we hope to generalise Conjecture 0.5.2 to Laurent polynomials in more than two variables (i.e. mirrors of higher-dimensional Fano varieties). This will require developments in the theory of punctured log Gromov–Witten invariants. The mutation class of a rigid MMLP in more than two variables is not uniquely determined by its Newton polytope, so the statement of the conjecture needs to take the coefficients of the Laurent polynomials into account. Furthermore, the higher-dimensional log Calabi–Yau pair associated to a Laurent polynomial in more than two variables has a more complicated toric model than those considered in Argüz and Gross's HDTV theorem [5], so we need to extend their work to cover the case where the hyperplanes in the boundary of the toric model are non-reduced. We also need to establish higher-dimensional analogues of the results in Section 4.1.1 which allow us to control the scattering process.

Conjecture 0.5.2 can be interpreted in terms of the Gromov–Witten theory of the mirror Fano varieties as follows. The difference $g - f$ between the two Laurent polynomials in the conjecture contribute to the extra terms in the difference of their respective periods $\pi_g - \pi_f$. In terms of the Gromov–Witten theory of the mirrors X_f and X_g, these extra terms in the difference of the regularised quantum periods $\hat{G}_{X_g} - \hat{G}_{X_f}$ correspond to the extra curve classes that arise in the blow-up $X_g \to X_f$.

The criteria (i)-(iii) almost capture the entire classification of smooth Fano surfaces – so far we have been able to find pairs of Laurent polynomials mirror to the eight of the nine pairs smooth del Pezzo surfaces related by the blow-up in a point (all except the pair $dP_2 \to dP_1$). In [10, Section 7], Carl–Pumperla–Siebert give another set of constructions of the affine manifolds underlying the mirrors to the del Pezzo surfaces. For the five toric del Pezzo surfaces, it is clear to see which choices of rigid MMLPs relates relate the mirror construction of this thesis to the constructions in [10]. Carl–Pumperla–Siebert's constructions of the wall structures underlying mirrors to lower-degree del Pezzo surfaces [10, Construction 7.15] are given by adding more singularities to the affine manifold associated to the blow-up of $\mathbb{P}^2$ in three points. Studying how the toric degenerations associated to these constructions relate to the toric degenerations associated to rigid MMLPs will shed light on how [10, Construction 7.15] relates to our Conjecture 0.5.2, and may lead to natural generalisations.

0.6 Outline of the thesis

We will now describe the contents of the rest of the thesis in more detail.

Chapter 1

We begin with an introduction to wall structures, the combinatorial objects that underlie the mirror constructions of the Gross–Siebert program. In particular, we describe how to construct a scheme from the data of a finite wall structure on an affine manifold, following the conventions of [23]. Every wall structure that we consider as part of a mirror construction is, in fact, infinite, so we also need to handle infinite wall structures. In the final section of the chapter, we introduce the notion of a *compatible system of wall structures* – that is, a notion of infinite wall structure that is general enough to encompass every case we need – and describe how a compatible system of wall structures gives rise to a formal scheme. Grothendieck's Existence Theorem gives general conditions under which formal schemes can be algebraised, and we apply this to describe a class of compatible systems of wall structures that produce algebraisable formal schemes.

Chapter 2

In this chapter, we introduce two types of wall structure that we will use in the construction of the mirror to a log Calabi–Yau pair (Y, D). One of these is the *canonical wall structure* $\mathfrak{D}_{\text{can}}$ associated to (Y, D), as defined by Gross–Siebert in [29, 31]. The other is the *algorithmic wall structure* associated to (Y, D) in the presence of a toric model (see Definition 2.3.1). This wall structure arises from a scattering diagram, called the *HDTV scattering diagram* [5], in a way that generalises a construction of Gross–Hacking–Keel [21, Chapter 3]. We prove that the schemes constructed from each of these wall structures are isomorphic over a sublocus of the base, called the *Gross–Siebert locus*. Furthermore, we describe how to compactify the mirror families, by truncating the corresponding wall structures. We begin the chapter by defining scattering diagrams in two dimensions.

Chapter 3

Here we apply the theory we set up in Chapters 1 and 2 to construct what we call the *Gross–Siebert mirror* $\mathfrak{X}_F$ to a rigid maximally mutable Laurent polynomial F in two variables. In order to do this, we construct the log Calabi–Yau pair (Y_F, D_F) associated to F, and describe the two wall structures associated to (Y_F, D_F) in terms of F. Additionally, we give canonical truncations of the wall structures – these produce a compactification $\overline{\mathfrak{X}_F}$ of the formal mirror family $\mathfrak{X}_F$, which is necessary for it to be algebraisable. Throughout the chapter, we apply the construction to the Laurent polynomials in (1), comparing the results of each stage of the construction for f and g as we go.

Chapter 4

Here we relate the mirror families to f and g of (1) by constructing an *intermediate mirror family* $\overline{\mathfrak{X}_\sim}$ that admits morphisms to both $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$. The scheme $\overline{\mathfrak{X}_\sim}$ is constructed from the wall structure associated to a three-dimensional log Calabi–Yau pair $(\tilde{Y}, \tilde{D})$ that is the total space of a degeneration of (Y_g, D_g). The HDTV scattering diagram which determines this algorithmic wall structure turns out to be equivalent to the cone over $\mathfrak{D}_{\text{pert}}$ (see Definition 0.5.1).

In Section 4.1 we show that the scattering process in $\mathfrak{D}_{\text{pert}}$ is compatible with the HDTV scattering diagrams associated to f and g: we prove that condition (iii) from Conjecture 0.5.2 holds for $\mathfrak{D}_{\text{pert}}$, so that the HDTV diagram for f includes

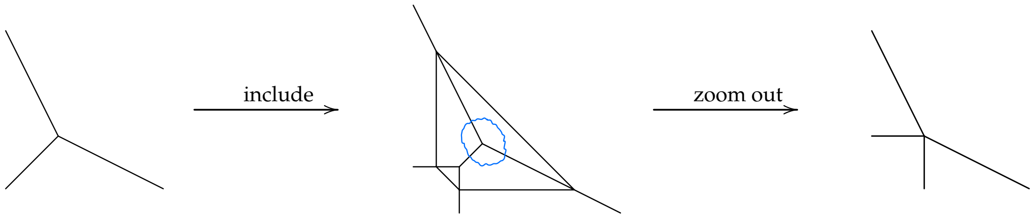

Figure 1: The relationship between the HDTV scattering diagram for f, the scattering diagram $\mathfrak{D}_{\text{pert}}$, and the HDTV scattering diagram for g

(after an appropriate change of variables on wall functions) into $\mathfrak{D}_{\text{pert}}$. Taking the asymptotic version of $\mathfrak{D}_{\text{pert}}$ ('zooming out') yields the HDTV diagram for g – see Figure 1.

In order to translate the relationships between the scattering diagrams to relationships between the mirror families, we need a wall structure $\mathfrak{D}_{f\leftrightarrow g}$ determined by $\mathfrak{D}_{\text{pert}}$. Since our goal is to construct morphisms between the *compactified* mirrors, we need $\mathfrak{D}_{f\leftrightarrow g}$ to be truncated in such a way that is compatible with the truncations of the wall structures producing the compactifications $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$. In Section 4.2 we construct this wall structure, $\mathfrak{D}_{f\leftrightarrow g}$, and use it to define the intermediate mirror family $\overline{\mathfrak{X}_\sim}$.

The three schemes $\overline{\mathfrak{X}_\sim}$, $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$ are families over different bases. In Section 4.3 we construct morphisms from the bases of $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$ to the base of $\overline{\mathfrak{X}_\sim}$. In order for these morphisms to be well defined, it is necessary to restrict further to a sublocus of the Gross–Siebert locus, which we call the *extended Gross–Siebert locus*. To finish the construction of the morphisms

$$\begin{array}{ccc} & \overline{\mathfrak{X}_\sim} & \\ \swarrow & & \searrow \\ \overline{\mathfrak{X}_f} & & \overline{\mathfrak{X}_g}, \end{array} \tag{2}$$

we also prove some general results about wall structures which are not written down in the literature.

Chapter 5

In the final chapter, we show that the morphisms in (2) are birational, and that the generic fibres of the three families are smooth. After restricting to a general one-parameter subscheme $\operatorname{Spec} \mathbb{k}[\![t]\!]$ of the base, we show that the generic fibre of

$\overline{\mathfrak{X}_f}$ is $\mathbb{P}^2_{\Bbbk((t))}$, and the generic fibres of $\overline{\mathfrak{X}_\sim}$ and $\overline{\mathfrak{X}_g}$ are both isomorphic to $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^2_{\Bbbk((t))}$.

Chapter 1

Constructions of the Gross–Siebert program I: schemes from wall structures

1.1 Overview

In this chapter we describe, in abstract terms, the construction of a scheme starting from the combinatorial data of a *wall structure* on an affine manifold, without reference to mirror symmetry. We follow the conventions of Gross–Hacking–Keel, giving what is essentially a summary of Chapters 1–4 of their paper [23].

Given an n-dimensional Calabi–Yau manifold $U := Y \setminus D$, the base B of the special Lagrangian torus fibration $f : U \to B$ of SYZ mirror symmetry can be described combinatorially, as an n-dimensional affine manifold with singularities with a polyhedral decomposition $\mathscr{P}$. This is the *polyhedral affine pseudomanifold* $(B, \mathscr{P})$ defined in Section 1.2.1. In Chapter 2 we will see that B is constructed from the pair (Y, D) by tropicalising the divisorial log structure on Y induced by D – for a toric variety Y with its toric boundary D, this just corresponds to taking the fan.

The mirror to U is expected to admit a dual torus fibration over B, by the SYZ conjecture. In the Gross–Siebert program the mirror is constructed from B, by considering cells of the polyhedral subdivision $\mathscr{P}$ of B to be the moment polytopes of a toric degeneration – the scheme $\mathfrak{X}$ is constructed as a smoothing over $\operatorname{Spec} \Bbbk[Q]/I$ of the union of toric varieties X_0 whose moment polytopes are given by $\mathscr{P}$, where Q is a monoid containing the effective curve classes of Y and $I \subset \Bbbk[Q]$ is an ideal. This degeneration is constructed locally by considering the upper convex hull of a Q-valued convex piecewise linear function on B which respects the polyhedral decomposition: the open affine chart on the interior of a maximal cell in the polyhedral decomposition of B gives a relative torus $\operatorname{Spec}(\Bbbk[Q]/I) \times \mathbb{G}_m^n$, and the kink $\kappa \in Q$ of the piecewise affine function over a codimension one cell of the polyhedral decomposition determines the smoothing of the transverse intersection between two varieties

$$V\left(YZ - t^{\kappa}\prod_{i}^{n-1} X_i^{m_i}\right) \subset \operatorname{Spec}\Bbbk[Q]/I \times \mathbb{A}^2_{Y,Z} \times (\mathbb{G}_m^{n-1})_{X_i}.$$

Here the numbers $m_i \in \mathbb{Z}$, as well as the gluings of these charts, are determined by the affine structure on B. The correct notion of convex piecewise affine function, and the naïve gluing construction given above, are defined in Section 1.2.2. The result is a flat scheme $\mathfrak{X}^\circ$ over $\operatorname{Spec}\Bbbk[Q]/I$, but this only gives a deformation of X_0 away from the codimension two strata of the toric varieties unless the pair (Y, D) is toric.

The mirror to U is an affine variety with its complex structure determined by counts of curves on U – there are global functions ϑ_{D_i} on U associated to the components of D, which are given on the local charts by expressions of the form

$$\sum_{\beta\in Q} n_\beta t^\beta z^{m_\beta} \tag{1.1}$$

where $m_\beta \in \Lambda_B$ is an integral tangent vector on B and n_β is, roughly speaking, a count of curves on Y intersecting D_i in a certain way. In order for these global functions to be well defined on $\mathfrak{X}^\circ$, some modifications to the gluing of the charts must be made. These modifications (the 'instanton corrections' in the symplectic heuristic) are given by a subdivision of $\mathscr{P}$ along codimension one sets $\mathfrak{d} \subset B$ which are decorated with functions $f_\mathfrak{d} \in (\Bbbk[Q]/I)[\Lambda_B]$ that determine birational transformations of the relative torus on either side of $\mathfrak{d}$:

$$z^m \longmapsto z^m f_\mathfrak{d}^{\langle m, n_\mathfrak{d}\rangle},$$

where $n_\mathfrak{d} \in \Lambda^\perp$ is the primitive normal vector to $\mathfrak{d}$. These modifications to $(B, \mathscr{P})$ are called *wall structures*; they are defined in Section 1.2.3.

Sections 1.3 and 1.4 are devoted to laying out the combinatorial conditions on a wall structure $\mathscr{S}$ necessary to define a scheme $\mathfrak{X}$ which is a flat deformation of

X_0, filling in the codimension two locus. In the case that the wall structure $\mathscr{S}$ is *consistent*, the integral tangent vectors $m \in \Lambda_B$ determine theta functions ϑ_m of the form (1.1), which form a $\Bbbk[Q]/I$-module basis of $\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$. In Section 1.5, a partial compactification $\mathfrak{X} \supset \mathfrak{X}^\circ$ is constructed via the algebra of theta functions associated to a consistent wall structure $\mathscr{S}$.

In the final section of this chapter, we introduce the notion of a *compatible system of wall structures* to explicitly link the abstract definition of finite wall structures, which define schemes over $\operatorname{Spec} \Bbbk[Q]/I$ for some ideal $I \subset \Bbbk[Q]$, with the notion of wall structure that actually appears in the Gross–Siebert mirror constructions. A compatible system is a series of wall structures which defines a formal scheme over $\operatorname{Spf} \widehat{\Bbbk[Q]}$. Under certain conditions, this formal scheme is algebraisable, and we can use the construction from Section 1.5.2 to define a scheme over $\operatorname{Spec} \widehat{\Bbbk[Q]}$.

1.2 Wall structures

1.2.1 The affine manifold

Below we define the affine manifold B with polyhedral decomposition $\mathscr{P}$, as given in Construction 1.1.1 of [23] which forms the basis of the setup for the construction of $\mathfrak{X}$.

Definition 1.2.1 (The polyhedral affine pseudomanifold $(B, \mathscr{P})$)**.** Let $\mathscr{P}$ be an integral affine polyhedral complex. That is, $\mathscr{P}$ is a category consisting of a set of integral polyhedra together with a set of integral affine maps $\omega \to \tau$ identifying ω with a face of τ. Assume that any proper face of $\tau \in \mathscr{P}$ occurs as the domain of an element of $\hom(\mathscr{P})$ with target τ. Define a topological space

$$B := \varinjlim_{\tau \in \mathscr{P}} \tau$$

and suppose that the pair $(B, \mathscr{P})$ satisfies the following conditions:

1. For each $\tau \in \mathscr{P}$ the map $\tau \to B$ is injective.

2. The intersection of any two cells $\tau \in \mathscr{P}$ is a cell of $\mathscr{P}$, where by abuse of notation a *cell* is considered to be both an element of $\mathscr{P}$ and a subset of B.

3. Every cell of $\mathscr{P}$ is contained in an n-dimensional cell, so B is of pure dimension n.

4. Every $(n-1)$-cell is contained in one or two n-cells, making B a manifold with boundary away from the cells of codimension 2.

5. If $\tau \in \mathscr{P}$ is a cell of codimension 2 or higher, then any point $x \in \mathrm{Int}\tau$ has a neighbourhood basis in B consisting of open sets V such that $V \setminus \tau$ is connected.

The *vertices*, *edges* and *maximal cells* of $\mathscr{P}$ are the 0-,1- and n-cells respectively. The *boundary* ∂B consists of the $(n-1)$-cells contained in only one maximal cell, $\rho \in \mathscr{P}_{\partial}^{[n-1]}$. An *interior* cell $\tau \in \mathscr{P}_{\mathrm{int}}$ is a cell not contained in the boundary.

We define the discriminant locus Δ, a codimension two subset of B. See [23, Construction 1.1.1] for a description of a choice of Δ. In this thesis however, we may simply assume that Δ is the union of the codimension two cells in $\mathscr{P}$. We endow $B_0 := B \setminus \Delta$ with an affine structure compatible with the given affine structure on the cells. It suffices to provide an integral $\mathscr{P}$-piecewise linear embedding ψ_ρ for every codimension one interior cell $\rho \in \mathscr{P}_{\mathrm{int}}^{[n-1]}$, with pair of adjacent maximal cells $\sigma, \sigma' \in \mathscr{P}$

$$\psi_\rho : \sigma \cup \sigma' \longrightarrow \mathbb{R}^n,$$

well-defined up to $\mathrm{GL}_n(\mathbb{Z})$.

We refer to the data of the pseudomanifold B, the discriminant locus Δ, the polyhedral decomposition $\mathscr{P}$ and the compatible integral affine structure on B_0 as the *polyhedral affine pseudomanifold*, and denote it by $(B, \mathscr{P})$.

Notation 1.2.2. We denote by Λ the sheaf of integral tangent vectors on B_0, and its stalk at a point $x \in B_0$ by Λ_x. For a cell $\tau \in \mathscr{P}$ we will denote the sheaf of integral tangent vectors to τ by Λ_τ. Suppose that x is an interior point of τ, and that τ is of codimension one in the cell $\sigma \supset \tau$. We denote by $\Lambda_{\sigma,\tau}$ the subgroup of Λ_x given by integral tangent vector which point into σ from x.

Construction 1.2.3 (The central fibre X_0)**.** For each $\tau \in \mathscr{P}$, let X_τ be the toric variety associated to the polyhedron τ. Each integral affine map $\omega \to \tau$ identifying ω with a face of τ induces a closed embedding of toric varieties $X_\omega \hookrightarrow X_\tau$. We define the variety

$$X_0 := \varinjlim_{\tau \in \mathscr{P}} X_\tau.$$

It is a union of n-dimensional toric varieties corresponding to the maximal cells of

$\mathscr{P}$, intersecting each other in their toric boundaries according to the integral affine maps of $\mathscr{P}$.

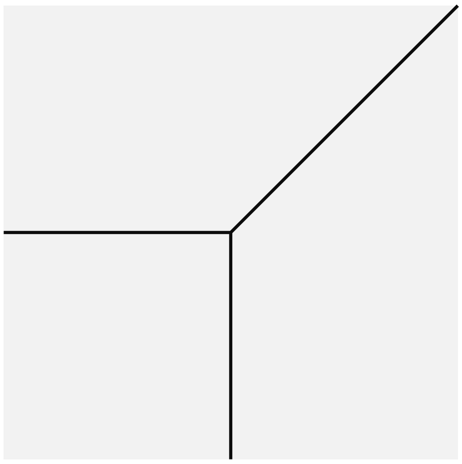

Figure 1.1: A polyhedral affine manifold $B \cong \mathbb{R}^2$, with a polyhedral complex $\mathscr{P}$ given by the fan Σ, subdividing $\mathbb{R}^2$ into three cells. The codimension one cells are the rays generated by $(-1,0)$, $(0,-1)$ and $(1,1) \in \Lambda \cong \mathbb{N}^2$. The toric variety associated to $(B, \mathscr{P})$ is the union of three copies of $\mathbb{A}^2$, intersecting each other along their coordinate hyperplanes.

1.2.2 The convex piecewise affine function

We define a multi-valued convex piecewise affine function on B_0.

Definition 1.2.4 (Toric monoid)**.** A finitely generated, integral, saturated monoid Q such that Q^{gp} is torsion-free is a *toric monoid*.

Definition 1.2.5 (Q^{gp}-valued multivalued piecewise affine function)**.** A Q^{gp}*-valued piecewise affine function* on an open set $U \subseteq B_0$ is a continuous map

$$U \longrightarrow Q^{\mathrm{gp}}_{\mathbb{R}}$$

which restricts to a $Q^{\mathrm{gp}}_{\mathbb{R}}$-valued integral affine function on each maximal cell of $\mathscr{P}$. Denote the sheaf of Q^{gp}-valued piecewise affine functions on B_0 by $\mathcal{PA}(B, Q^{\mathrm{gp}})$, and denote the sheaf of Q^{gp}-valued integral affine functions on B_0 by $\mathcal{A}ff(B, Q^{\mathrm{gp}})$. The sheaf of Q^{gp}*-valued multivalued piecewise affine (MPA-) functions* on B_0 is defined to be

$$\mathcal{MPA}(B, Q^{\mathrm{gp}}) := \mathcal{PA}(B, Q^{\mathrm{gp}}) / \mathcal{A}ff(B, Q^{\mathrm{gp}}).$$

Remark 1.2.6. To define a Q^{gp}-valued MPA function $\varphi \in \mathcal{MPA}(B, Q^{\mathrm{gp}})$, it suffices to define local representatives φ_ρ on the open neighbourhood around each interior codimension one cell $\rho \in \mathscr{P}$. These are Q^{gp}-valued piecewise affine functions

$$\varphi_\rho : \sigma \cup \sigma' \longrightarrow Q^{\mathrm{gp}}_{\mathbb{R}}$$

where σ and σ' are the maximal cells containing ρ. If a maximal cell σ contains interior codimension one cells ρ and ρ', the restriction of the difference $(\varphi_\rho - \varphi_{\rho'})|_\sigma$ is an integral affine function on σ.

Definition 1.2.7 (Kink of an MPA-function)**.** Suppose that $(B, \mathscr{P})$ is a polyhedral affine pseudomanifold, Q is a toric monoid, and that $\varphi \in \mathcal{MPA}(B, Q^{\mathrm{gp}})$. Let ρ be an interior codimension one cell in $\mathscr{P}$, contained in maximal cells σ and σ', and let $x \in \operatorname{Int}\rho$. Define $\delta \in \check{\Lambda}_x$ to be the quotient map

$$\delta : \Lambda_x \longrightarrow \Lambda_x / \Lambda_\rho \cong \mathbb{Z},$$

where the signs are fixed so that δ is non-negative on $\Lambda_{\sigma',\rho}$. Let φ_ρ be the unique representative of φ on $\sigma \cup \sigma'$ such that $\varphi_\rho|_\sigma$ is identically zero. Considering φ_ρ as a piecewise affine function on Λ_x, there exists $\kappa_\rho \in Q^{\mathrm{gp}}$ such that

$$\varphi_\rho|_{\sigma'} = \kappa_\rho \cdot \delta.$$

Note that the value κ_ρ is independent of the choice of orientation of σ and σ'. We call $\kappa_\rho \in Q^{\mathrm{gp}}$ the *kink* of φ along ρ.

Remark 1.2.8 (MPA-functions determined by their kinks across codimension one cells)**.** We see that a Q^{gp}-valued MPA-function $\varphi \in \mathcal{MPA}(B, Q^{\mathrm{gp}})$ is completely determined by the data of its kinks κ_ρ across every interior codimension one cell $\rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$.

Definition 1.2.9 (Convex MPA-function)**.** A Q^{gp}-valued MPA-function φ on B is *convex* if the kink $\kappa_\rho(\varphi)$ of φ along ρ is contained in Q for every interior codimension one cell $\rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$.

Example 1.2.10. Let $(B, \mathscr{P})$ be the polyhedral affine manifold $(\mathbb{R}^2, \Sigma)$ shown in Figure 1.1. Let $Q = \mathbb{N}\kappa \oplus \mathbb{N}a \oplus \mathbb{N}b \cong \mathbb{N}^3$. Then the convex MPA-function φ on $(B, \mathscr{P})$

with kink κ across each of the three codimension one cells in $\mathscr{P}$ has a *single*-valued representative given by

$$\varphi(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \mathbb{R}_{\geq 0}(0,-1) + \mathbb{R}_{\geq 0}(-1,0) \\ \kappa x & \text{if } (x,y) \in \mathbb{R}_{\geq 0}(0,-1) + \mathbb{R}_{\geq 0}(1,1) \\ \kappa y & \text{if } (x,y) \in \mathbb{R}_{\geq 0}(-1,0) + \mathbb{R}_{\geq 0}(1,1). \end{cases}$$

Construction 1.2.11 (The $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor $\mathbb{B}_\varphi \to B$)**.** Let $(B,\mathscr{P})$ be a polyhedral affine pseudomanifold, Q a toric monoid and φ a piecewise affine function on B. Let $\mathbb{B}_\varphi := B \times Q^{\mathrm{gp}}_{\mathbb{R}}$. We endow $\mathbb{B}_\varphi$ with the following integral affine structure. There is a polyhedral decomposition of $\mathbb{B}_\varphi$ coming from $\mathscr{P}$:

$$\Sigma_\varphi := \{\tau \times Q^{\mathrm{gp}}_{\mathbb{R}} \mid \tau \in \mathscr{P}\}$$

Let $U_\rho = \sigma \cup \sigma'$ with $\sigma, \sigma' \in \mathscr{P}_{\max}$ and $\rho = \sigma \cap \sigma' \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$. Putting together the affine charts ψ_ρ on U_ρ and the piecewise linear function φ_ρ we define integral affine charts on $\mathbb{B}_\varphi$

$$\begin{aligned} U_\rho \times Q^{\mathrm{gp}}_{\mathbb{R}} &\longrightarrow \mathbb{R}^n \times Q^{\mathrm{gp}}_{\mathbb{R}} \\ (x,q) &\longmapsto (\psi_\rho(x), q + \varphi_\rho(x)) \end{aligned}$$

Remark 1.2.12 (φ is a section of the torsor)**.** The projection map $\pi : \mathbb{B}_\varphi \longrightarrow B$ is integral affine and we have a section $\varphi : B \longrightarrow \mathbb{B}_\varphi$, making $\mathbb{B}_\varphi$ a $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor.

Definition 1.2.13 (The sheaf of monoids $\mathcal{P}^+$)**.** Let $(B,\mathscr{P})$ be a polyhedral affine pseudomanifold, Q a toric monoid, φ a Q-valued convex piecewise affine function on B, with $\mathbb{B}_\varphi$ the $Q^{\mathrm{gp}}_{\mathbb{R}}$-torsor constructed in Construction 1.2.11. Let $\Lambda_{\mathbb{B}_\varphi}$ be the system of integral tangent vectors on $\mathbb{B}_\varphi$ and define a locally constant sheaf of abelian groups on B_0,

$$\mathcal{P} := \varphi^* \Lambda_{\mathbb{B}_\varphi}$$

which has fibres $\mathbb{Z}^n \times Q^{\mathrm{gp}}_{\mathbb{R}}$. Define $\mathcal{P}^+ \subset \mathcal{P}$ to be the subsheaf with sections over an open set $U \subseteq B_0$ given by $p \in \mathcal{P}(U)$ such that, under the canonical identification

$$\Gamma(\operatorname{Int}\sigma, \mathcal{P}) = \Lambda_\sigma \times Q^{\mathrm{gp}},$$

we have that

$$\begin{aligned} p|_{\operatorname{Int}\sigma} &\in \Lambda_\sigma \times Q \quad \text{for any } \sigma \in \mathscr{P}_{\max}, \text{ and} \\ p|_{\operatorname{Int}\sigma} &\in \Lambda_{\sigma,\rho} \times Q \quad \text{for any } \sigma \in \mathscr{P}_{\max} \text{ such that } \sigma \cap \partial B = \rho \in \mathscr{P}^{[n-1]} \text{ with } \rho \subset U. \end{aligned}$$

Remark 1.2.14. The sheaf $\mathcal{P}$ fits into the canonical exact sequence

$$0 \longrightarrow \underline{Q}^{\mathrm{gp}} \longrightarrow \mathcal{P} \xrightarrow{\pi_*} \Lambda_B \longrightarrow 0.$$

where $\underline{Q}^{\mathrm{gp}}$ is the constant sheaf with stalk Q^{gp} on B_0, and π_* is the homomorphism induced by the affine projection $\pi : \mathbb{B}_\varphi \to B$.

Notation 1.2.15. Let $I \subset \Bbbk[Q]$ be an ideal and let $\mathfrak{m} := \sqrt{I}$ be the radical ideal of I. We assume that $\mathfrak{m}$ contains the kinks $\kappa_\varphi(\rho)$ of the MPA-function φ for all interior codimension-one cells $\rho \in \mathscr{P}$. In all the examples we consider in this thesis, $\mathfrak{m}$ is generated by a monoid ideal in Q – we will conflate the two notions of ideals in the monoid and ideals in the associated semigroup ring.

We now use the sheaf of monoids $\mathcal{P}^+$ to define a deformation of the union of toric varieties X_0 associated to $(B, \mathscr{P})$ over $\operatorname{Spec} \Bbbk[Q]/I$, where I is an ideal of $\Bbbk[Q]$ such that $\sqrt{I} = \mathfrak{m}$. The total space of the deformation will be described in Construction 1.2.16 with affine charts given by the stalks of

$$\Bbbk[\mathcal{P}^+]/\mathcal{I}$$

where $\mathcal{I}$ is the ideal sheaf $I \cdot \Bbbk[\mathcal{P}^+]$, and the gluing morphisms are induced by parallel transport.

First, let us describe the stalks $\mathcal{P}^+_x$ and the effects of parallel transport more explicitly. For $x \in \operatorname{Int}\sigma$, an interior point of a maximal cell $\sigma \in \mathscr{P}_{\max}$, the stalk of $\mathcal{P}^+$ at x is

$$\mathcal{P}^+_x = \Lambda_x \times Q,$$

and for $x \in \operatorname{Int}\rho$, $\rho \in \mathscr{P}_{\partial}^{[n-1]}$, the stalk of $\mathcal{P}^+$ at x is

$$\mathcal{P}_x^+ = \Lambda_{\sigma,\rho} \times Q.$$

When $x \in \operatorname{Int}\rho$ is an interior point of an interior codimension one cell $\rho \in \mathscr{P}_{\mathrm{int}}^{[n-1]}$, the stalk at x has the following description:

$$\mathcal{P}_x^+ = \left(\Lambda_\rho \oplus \mathbb{N}Z_+ \oplus \mathbb{N}Z_- \oplus Q\right)/(Z_+ + Z_- = \kappa_\rho),$$

where $\kappa_\rho \in Q$ is the kink of the convex piecewise affine function φ across ρ. By our assumptions ρ is the intersection of two maximal cells σ and σ'; fix an ordering of σ and σ' and a tangent vector $\xi \in \Lambda_x$ pointing into σ and generating $\Lambda_\sigma/\Lambda_\rho$. Then parallel transport in $\mathcal{P}$ along a path from $x \in \operatorname{Int}\rho$ to $y \in \operatorname{Int}\sigma$ induces the map

$$\begin{aligned} \mathrm{t}_{\sigma,\rho} : \mathcal{P}_x^+ &\longrightarrow \mathcal{P}_y^+ \\ (\lambda_\rho, aZ_+, bZ_-, q) &\longmapsto (\lambda_\rho + (a-b)\xi, q + b\kappa_\rho) \end{aligned}$$

and parallel transport along a path from x to $y' \in \operatorname{Int}\sigma'$ induces the map

$$\begin{aligned} \mathrm{t}_{\sigma',\rho} : \mathcal{P}_x^+ &\longrightarrow \mathcal{P}_{y'}^+ \\ (\lambda_\rho, aZ_+, bZ_-, q) &\longmapsto (\lambda_\rho + (a-b)\xi, q + a\kappa_\rho), \end{aligned}$$

whereas parallel transport along a path completely contained in a cell τ simply induces the identity map

$$\mathcal{P}_x^+ \longrightarrow \mathcal{P}_y^+.$$

Construction 1.2.16 (The uncorrected Mumford degeneration)**.** The rings given by the stalks of $\mathbb{k}[\mathcal{P}^+]/\mathcal{I}$, together with the isomorphisms and localisation homomorphisms induced by parallel transport, form a category of $\mathbb{k}[Q]/I$-algebras. Taking Spec of this category defines a category of affine schemes and open embeddings. There exists (see Section 2.3 in [23]) a separated scheme over $\operatorname{Spec}\mathbb{k}[Q]/I$ which is isomorphic to the colimit of this category of schemes. Since parallel transport along a path contained in the interior of a cell $\tau \in \mathscr{P}$ induces the identity map on stalks of $\mathcal{P}^+$, this scheme admits a finite open cover

$$\left\{ \operatorname{Spec} R_\tau \,\middle|\, \tau \in \left(\mathscr{P}_{\max} \cup \mathscr{P}^{[n-1]} \right) \right\},$$

where R_τ is defined to be the stalk of $\Bbbk[\mathcal{P}^+]/\mathcal{I}$ at an interior point of τ. There are three types of rings R_τ: one type corresponding to each of the three cases

$$\tau \in \mathscr{P}_{\max}, \quad \tau \in \mathscr{P}^{[n-1]}_{\mathrm{int}} \quad \text{or} \quad \tau \in \mathscr{P}^{[n-1]}_{\partial}.$$

We describe R_τ more explicitly in each case: when τ is a maximal cell $\sigma \in \mathscr{P}_{\max}$, we have

$$R_\sigma := (\Bbbk[Q]/I)\,[\Lambda_\sigma]. \tag{1.2}$$

When τ is a boundary cell $\rho_\partial \in \mathscr{P}^{[n-1]}_{\partial}$,

$$R^\partial_\rho := (\Bbbk[Q]/I)\,[\Lambda_{\sigma,\rho}], \tag{1.3}$$

where σ is the unique maximal cell containing ρ. Parallel transport from an interior point of ρ_∂ to an interior point of σ induces the localisation homomorphism

$$\chi^\partial_{\sigma,\rho} : R^\partial_\rho \longrightarrow R_\sigma \tag{1.4}$$

canonically defined by the inclusion $\Lambda_{\sigma,\rho} \subseteq \Lambda_\sigma$. Finally, when τ is an interior cell of codimension one $\rho \in \mathscr{P}^{[n-1]}_{\mathrm{int}}$, we have

$$R_\rho := (\Bbbk[Q]/I)\,[\Lambda_\rho][Z_+, Z_-]/(Z_+ Z_- - z^{\kappa_\rho}) \tag{1.5}$$

where κ_ρ is the kink of the MPA-function φ across ρ. The maps induced by parallel transport from ρ into its adjacent chambers σ and σ', $\mathrm{t}_{\sigma,\rho}$ and $\mathrm{t}_{\sigma',\rho}$, define the localisation homomorphisms

$$\chi_{\sigma,\rho} : R_\rho \longrightarrow R_\sigma \quad \text{and} \quad \chi_{\sigma',\rho} : R_\rho \longrightarrow R_{\sigma'}. \tag{1.6}$$

Recall that we consider Z_+ to be pointing into σ and Z_- to be pointing into σ', so that $\chi_{\sigma,\rho}$ identifies R_σ with $(R_\rho)_{Z_+}$ and $\chi_{\sigma',\rho}$ identifies $R_{\sigma'}$ with $(R_\rho)_{Z_-}$. In other words, the maps are defined by the canonical inclusion of Λ_ρ into Λ_σ, together with

$$\chi_{\sigma,\rho} : Z_+ \longmapsto z^{\xi} \qquad\qquad \chi_{\sigma',\rho} : Z_+ \longmapsto z^{\xi} z^{\kappa_\rho}$$
$$Z_- \longmapsto z^{-\xi} z^{\kappa_\rho} \qquad\qquad Z_- \longmapsto z^{-\xi}$$

where $\xi \in \Lambda_x$ points into σ and generates $\Lambda_x/\Lambda_\rho \cong \mathbb{Z}$, for some point $x \in \operatorname{Int}\rho$. Here we use the identifications $\Lambda_{\sigma'} \cong \Lambda_x \cong \Lambda_\sigma$ given by the chart on B_0,

$$\psi_\rho : \sigma \cup \sigma' \longrightarrow \mathbb{R}^n.$$

Example 1.2.17. Let us consider the uncorrected Mumford degeneration of Construction 1.2.16 given by the piecewise linear function φ from Example 1.2.10. The central fibre X_0 is three copies of $\mathbb{A}^2$ glued along their coordinate hyperplanes, as described in the caption to Figure 1.1. Since φ only takes values in $\mathbb{N}\kappa \hookrightarrow Q$, we may restrict the family to one-parameter family over the base $\mathbb{A}^1 \cong \operatorname{Spec} \Bbbk[\mathbb{N}\kappa]$. Then the affine charts $\operatorname{Spec} R_\rho$, where ρ runs through the three codimension one cells, are smoothings of X_0 about the corresponding strata, away from the central intersection point of all three planes. However, since φ is single-valued in this case, the smoothing can be extended over the central point. The general fibre here is a copy of the algebraic torus $\mathbb{G}_m^2$.

Notation 1.2.18 (Monomials on B_0)**.** We write the monomial $z^p \in \Bbbk[\mathcal{P}_x^+]$ associated to $p \in \mathcal{P}_x^+$ as $t^q z^m$, where $q \in Q$ and $m := \pi_*(p) \in \Lambda_x$ is the *tangent vector* of the monomial, where π_* is the second map of the short exact sequence

$$0 \longrightarrow \underline{Q}^{\mathrm{gp}} \longrightarrow \mathcal{P} \xrightarrow{\pi_*} \Lambda_B \longrightarrow 0$$

defined in Remark 1.2.14.

Example 1.2.19. Let φ be the $\mathbb{N}^3$-valued MPA-function on $(\mathbb{R}^2, \Sigma)$ defined in Example 1.2.10. If we denote $\Bbbk[\Lambda] \cong \Bbbk[x^\pm, y^\pm]$, with $x = z^{(1,0)}$ and $y = z^{(0,1)}$, then we have

$$\Bbbk[\mathcal{P}_P^+] = \Bbbk[t^\kappa, t^a, t^b, x^\pm, y^\pm] \cong \Bbbk[\mathbb{N}^3 \oplus \mathbb{Z}^2]$$

for every interior point $P \in B_0$.

1.2.3 Wall structures

Definition 1.2.20 (Wall structure)**.** A *wall* on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise affine function φ is a codimension one rational polyhedral subset

$$\mathfrak{d} \subseteq \sigma \in \Sigma_{\max} \quad \text{such that} \quad \mathfrak{d} \not\subseteq \partial B,$$

equipped with a *wall function*

$$f_{\mathfrak{d}} = \sum_{\substack{p \in \mathcal{P}_x^+ \\ \pi_*(p) \in \Lambda_{\mathfrak{d}}}} c_p z^p \in \Bbbk[\mathcal{P}_x^+]$$

for some interior point $x \in \operatorname{Int} \mathfrak{d}$. By parallel transport inside σ we require that $p \in \mathcal{P}_y^+$ for all $y \in \mathfrak{d} \setminus \Delta$ when $c_p \neq 0$. We also require that[1]

$$f_{\mathfrak{d}} \equiv \begin{cases} 1 & \mod \mathfrak{m} \quad \text{when } \mathfrak{d} \text{ has codimension zero} \\ f_\rho & \mod \mathfrak{m} \quad \text{otherwise, for some } f_\rho \in \Bbbk[Q][\Lambda_\rho]. \end{cases}$$

Here the *codimension* $k \in \{0, 1\}$ of a wall $\mathfrak{d}$ is defined as the codimension of the minimal cell of $\mathscr{P}$ containing $\mathfrak{d}$. Walls of codimension one are also referred to as *slabs*, denoted $\mathfrak{b}$.

A *wall structure* on $(B, \mathscr{P})$ is a set $\mathscr{S}$ of walls which induces a rational polyhedral decomposition $\mathscr{P}_{\mathscr{S}}$ refining $\mathscr{P}$; the codimension one cells of $\mathscr{P}_{\mathscr{S}}$ are subsets of the elements of

$$|\mathscr{S}| := \bigcup_{\mathfrak{d} \in \mathscr{S}} \mathfrak{d} \cup \bigcup_{\rho \in \mathscr{P}^{[n-1]}} \rho.$$

A maximal cell $\mathfrak{u}$ of $\mathscr{P}_{\mathscr{S}}$ is called a *chamber* of the wall structure. A chamber $\mathfrak{u}$ with $\dim(\mathfrak{u} \cap \partial B) = n - 1$ is called a *boundary chamber* - all other chambers are called *interior chambers*. Two chambers are *adjacent* if $\dim(\mathfrak{u} \cap \mathfrak{u}') = n - 1$. A cell $\mathfrak{j} \in \mathscr{P}_{\mathscr{S}}$ of codimension two is called a *joint*. If a joint is contained in the boundary ∂B it is called a *boundary joint*, otherwise an *interior joint*. We define the *codimension* $k \in \{0, 1, 2\}$ of a joint $\mathfrak{j}$ to be the codimension of the smallest cell of $\mathscr{P}$ containing $\mathfrak{j}$.

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[1] In a more general setting, where the discriminant locus Δ is not required to lie inside the codimension two locus of $(B, \mathscr{P})$, there would be additional compatibility conditions on the choice of the functions f_ρ: see [23, (2.10)].

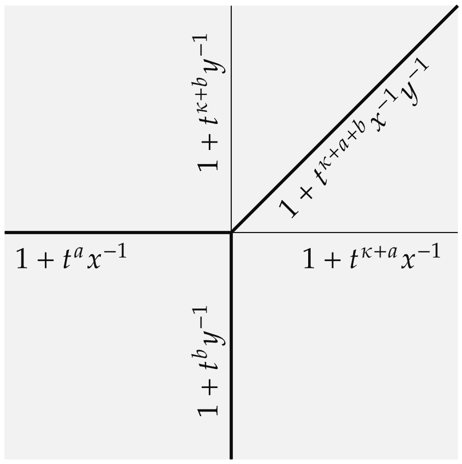

Figure 1.2: A wall structure $\mathscr{S}$ on the polyhedral affine pseudomanifold $(\mathbb{R}^2, \mathscr{P})$ equipped with piecewise linear function φ from Example 1.2.10, with each of the five walls labelled with their wall function, and the three slabs shown in bold.

Definition 1.2.21 (Equivalence of wall structures)**.** Given two wall structures $\mathscr{S}$ and $\mathscr{S}'$ on a polyhedral affine pseudomanifold with a piecewise affine function $(B, \mathscr{P}, \varphi)$, we define

$$f_x := \prod_{x \in \mathfrak{d} \in \mathscr{S}} f_{\mathfrak{d}} \quad \text{and} \quad f'_x := \prod_{x \in \mathfrak{d}' \in \mathscr{S}'} f_{\mathfrak{d}'}$$

for points $x \in B$ which are not contained in the discriminant locus Δ, nor any joint of $\mathscr{S}$ or $\mathscr{S}'$. Let I be a monoid ideal of Q such that $\sqrt{I} = \mathfrak{m}$. We say that the two wall structures are *equivalent (modulo I)* if

$$f_x \equiv f'_x \mod I$$

for all such points $x \in B$.

Remark 1.2.22 (Trivial modifications to $\mathscr{S}$)**.** Several trivial modifications can be made to a wall structure $\mathscr{S}$ without changing its equivalence class. In particular, we can make the following assumptions to simplify the construction of the family $\mathfrak{X}$.

1. $\mathscr{S} = \mathscr{P}^{[n-1]}_{\mathscr{S}}$. In other words, the interior of a wall does not intersect the interior of any other wall.
2. For any wall $\mathfrak{d} \in \mathscr{S}$, $\operatorname{Int} \mathfrak{d} \cap \Delta = \emptyset$.
3. For any chamber $\mathfrak{u}$ of $\mathscr{P}_{\mathscr{S}}$, there is at most one $\rho \in \mathscr{P}^{[n-1]}_{\partial}$ with $\dim(\mathfrak{u} \cap \rho) = n-1$.

1.3 Local construction of the family

In this section we describe the local charts and gluing data allowing construction of $\mathfrak{X}^\circ \subset \mathfrak{X}$, the family over $\operatorname{Spec} \Bbbk[Q]/I$ which defines a deformation of X_0 away from the codimension two locus.

1.3.1 Corrections given by the wall structure

Construction 1.3.1 (Local charts on the degeneration)**.** Given a wall structure $\mathscr{S}$ on a polyhedral affine pseudomanifold $(B, \mathscr{P})$, and an $\mathfrak{m}$-primary ideal $I \subset \Bbbk[Q]$, we define the following three types of affine scheme $\operatorname{Spec} R$ over $\operatorname{Spec} \Bbbk[Q]/I$ associated to $\mathscr{S}$. First, for any chamber $\mathfrak{u}$ define the ring

$$R_{\mathfrak{u}} := R_\sigma = (\Bbbk[Q]/I)[\Lambda_\sigma] \tag{1.7}$$

where $\sigma \in \mathscr{P}_{\max}$ is the unique maximal cell containing $\mathfrak{u}$. Second, if $\mathfrak{u}$ is a boundary chamber, we define the subring

$$R^\partial_{\mathfrak{u}} := R^\partial_\rho \subseteq R_{\mathfrak{u}} \tag{1.8}$$

where ρ is the unique codimension one boundary cell contained in $\mathfrak{u}$. The third ring is a deformation of R_ρ from Construction 1.2.16 associated to any slab $\mathfrak{b} \subseteq \rho \in \mathscr{P}^{[n-1]}$ of the wall structure:

$$R_{\mathfrak{b}} := (\Bbbk[Q]/I)[\Lambda_\rho][Z_+, Z_-]/(Z_+Z_- - f_{\mathfrak{b}} z^{\kappa_\rho}), \tag{1.9}$$

where we have that $R_{\mathfrak{b}}/\mathfrak{m} = R_\rho$. We also define gluing morphisms between the affine charts $\operatorname{Spec} R_{\mathfrak{u}}$, $\operatorname{Spec} R^\partial_{\mathfrak{u}}$ and $\operatorname{Spec} R_{\mathfrak{b}}$. First, we note that the rings come with localisation homomorphisms as in Construction 1.2.16:

$$\chi_{\mathfrak{b},\mathfrak{u}} : R_{\mathfrak{b}} \longrightarrow R_{\mathfrak{u}} \quad \text{and} \quad \chi^\partial_{\mathfrak{u}} : R^\partial_{\mathfrak{u}} \longrightarrow R_{\mathfrak{u}}. \tag{1.10}$$

The first of these localisation homomorphisms $\chi_{\mathfrak{b},\mathfrak{u}}$ is defined by the inclusion $\Lambda_\rho \subseteq \Lambda_\sigma$ and

$$Z_+ \longmapsto z^\xi, \quad Z_- \longmapsto z^{-\xi} f_{\mathfrak{b}} z^{\kappa_\rho}$$

for some choice of generator ξ of $\Lambda_\sigma/\Lambda_\rho$; the second localisation morphism $\chi^\partial_{\mathfrak{u}}$ is just equal to $\chi^\partial_{\sigma,\rho}$ (1.4), induced by the inclusion $\Lambda_{\sigma,\rho} \subseteq \Lambda_\sigma$. We now introduce additional non-trivial isomorphisms of R_σ coming from the codimension zero walls. For a codimension zero wall $\mathfrak{d}$ separating the chamber $\mathfrak{u}$ and $\mathfrak{u}'$, we define the *wall-crossing automorphism*

$$\theta_{\mathfrak{d}} : R_{\mathfrak{u}} \longrightarrow R_{\mathfrak{u}'}, \quad z^p \longmapsto f_{\mathfrak{d}}^{\langle n_{\mathfrak{d}}, \pi_*(p)\rangle} z^p, \tag{1.11}$$

where $n_{\mathfrak{d}}$ is defined to be a generator of $\Lambda_{\mathfrak{d}}^\perp \subseteq \check{\Lambda}_x$, for some $x \in \operatorname{Int}\mathfrak{d}$, such that $\langle n_{\mathfrak{d}}, m\rangle \geq 0$ when $m \in \Lambda_x$ points into $\mathfrak{u}$. Note that this map does indeed define an isomorphism of rings when we consider $f_{\mathfrak{d}}$ to be invertible by reduction modulo I.

Definition 1.3.2 (Consistency in codimension zero)**.** Let $\mathscr{S}$ be a wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ, and suppose that $\mathfrak{j}$ is a codimension zero joint of $\mathscr{S}$. Let $\mathfrak{d}_1, \dots, \mathfrak{d}_r$ be the walls containing $\mathfrak{j}$, taken in cyclic order, with each $\mathfrak{d}_i$ contained in the chambers $\mathfrak{u}_i$ and $\mathfrak{u}_{i+1}$ (where the indices of the chambers are taken modulo r). Since the codimension of $\mathfrak{j}$ is zero, each chamber $\mathfrak{u}_i$ is contained in the unique maximal cell $\sigma \in \mathscr{P}_{\max}$ containing $\mathfrak{j}$. Therefore, there is an automorphism of R_σ,

$$\theta_{\mathfrak{d}_i} : R_{\mathfrak{u}_i} \longrightarrow R_{\mathfrak{u}_{i+1}},$$

as defined in (1.11) for each $\mathfrak{d}_i$. We say that $\mathfrak{j}$ is *consistent* if

$$\theta := \theta_{\mathfrak{d}_r} \circ \cdots \circ \theta_{\mathfrak{d}_1} = \mathrm{Id}$$

as an automorphism of R_σ. The wall structure $\mathscr{S}$ is *consistent in codimension zero* if every codimension zero joint of $\mathscr{S}$ is consistent.

Definition 1.3.3 (Consistency in codimension one)**.** Let $\mathscr{S}$ be a wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ, and suppose that $\mathfrak{j}$ is an interior joint of $\mathscr{S}$ of codimension one. Let ρ be the unique codimension one cell of $\mathscr{P}$ containing $\mathfrak{j}$, and σ, σ' the unique maximal cells containing ρ. Since $\mathfrak{j} \not\subseteq \partial B$ there are unique slabs $\mathfrak{b}_1, \mathfrak{b}_2 \subset \rho$ with $\mathfrak{j} = \mathfrak{b}_1 \cap \mathfrak{b}_2$. Denote the codimension zero walls containing $\mathfrak{j}$ by $\mathfrak{d}_1, \dots, \mathfrak{d}_r \subset \sigma$ and $\mathfrak{d}'_1, \dots, \mathfrak{d}'_s \subset \sigma'$, such that

$$\mathfrak{b}_1, \mathfrak{d}_1, \ldots, \mathfrak{d}_r, \mathfrak{b}_2, \mathfrak{d}'_1, \ldots, \mathfrak{d}'_s$$

is a cyclic ordering of all the walls containing $\mathfrak{j}$. There are localisation homomorphisms

$$\chi_{\mathfrak{b}_i,\sigma} : R_{\mathfrak{b}_i} \longrightarrow R_\sigma, \quad \chi_{\mathfrak{b}_i,\sigma'} : R_{\mathfrak{b}_i} \longrightarrow R_{\sigma'}$$

defined as in (1.10), and the compositions of wall crossings on either side of ρ

$$\theta := \theta_{\mathfrak{d}_r} \circ \cdots \circ \theta_{\mathfrak{d}_1} \quad \text{and} \quad \theta' := \theta_{\mathfrak{d}'_1} \circ \cdots \circ \theta_{\mathfrak{d}'_s}$$

give automorphisms of R_σ and $R_{\sigma'}$ respectively. We say that $\mathfrak{j}$ is *consistent* if

$$(\theta \times \theta')\left((\chi_{\mathfrak{b}_1,\sigma}, \chi_{\mathfrak{b}_1,\sigma'})(R_{\mathfrak{b}_1})\right) = (\chi_{\mathfrak{b}_2,\sigma}, \chi_{\mathfrak{b}_2,\sigma'})(R_{\mathfrak{b}_2}).$$

In this case there is a well-defined isomorphism

$$\theta_{\mathfrak{j}} : R_{\mathfrak{b}_1} \longrightarrow R_{\mathfrak{b}_2} \tag{1.12}$$

which is induced by $\theta \times \theta'$, since the map

$$(\chi_{\mathfrak{b}_i,\sigma}, \chi_{\mathfrak{b}_i,\sigma'}) : \ R_{\mathfrak{b}_i} \longrightarrow R_\sigma \times R_{\sigma'}$$

is injective. We say the wall structure $\mathscr{S}$ is *consistent in codimension one* if every codimension one joint is consistent.

Proposition 1.3.4 (The local construction of $\mathfrak{X}^\circ$). *Let $\mathscr{S}$ be a wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ. If $\mathscr{S}$ is consistent in codimensions zero and one, then there exists a unique scheme $\mathfrak{X}^\circ$ flat over* Spec $(\Bbbk[Q]/I)$ *with open embeddings of the affine schemes*

Spec $R_\mathfrak{u}$, *defined (1.7) for every chamber* $\mathfrak{u}$,

Spec $R^\partial_\mathfrak{u}$, *defined (1.8) for every boundary chamber* $\mathfrak{u}$,

Spec $R_\mathfrak{b}$, *defined (1.9) for every slab* $\mathfrak{b}$, *that are compatible with the isomorphisms*

$$\begin{aligned} &\theta_{\mathfrak{b}} : \operatorname{Spec} R_{\mathfrak{u}'} \longrightarrow \operatorname{Spec} R_{\mathfrak{u}} && \textit{and} \\ &\theta^{\partial}_{\mathfrak{b}} : \operatorname{Spec} R^{\partial}_{\mathfrak{u}'} \longrightarrow \operatorname{Spec} R^{\partial}_{\mathfrak{u}}, && \textit{defined (1.11) for every codimension zero wall } \mathfrak{b}, \\ &\theta_{\mathfrak{j}} : \operatorname{Spec} R_{\mathfrak{b}_2} \longrightarrow \operatorname{Spec} R_{\mathfrak{b}_1}, && \textit{defined (1.12) for every codimension one joint } \mathfrak{j}, \end{aligned}$$

and compatible with the open embeddings

$$\begin{aligned} &\operatorname{Spec} R_{\mathfrak{u}} \hookrightarrow \operatorname{Spec} R_{\mathfrak{b}}, && \textit{defined (1.10) for every slab } \mathfrak{b} \subset \mathfrak{u}, \\ &\operatorname{Spec} R_{\mathfrak{u}} \hookrightarrow \operatorname{Spec} R^{\partial}_{\mathfrak{u}}, && \textit{defined (1.8) for every boundary chamber } \mathfrak{u}. \end{aligned}$$

Proof. This is Proposition 2.4.1 in [23]. □

Proposition 1.3.5 (The central fibre of $\mathfrak{X}^\circ$ is X_0)**.** *The reduction of $\mathfrak{X}^\circ$ modulo $\mathfrak{m}$ is canonically isomorphic to the complement of the codimension two strata in X_0.*

Proof. This is Proposition 2.4.4 in [23]. □

1.4 The algebra of theta functions

The goal of this section is to define a canonical set of global functions ϑ_m on $\mathfrak{X}^\circ$ indexed by the integral tangent vectors on B. These 'theta functions' are given locally by sums of monomials coming from piecewise linear paths on B which interact with the wall structure $\mathscr{S}$, called *broken lines*.

The expression for ϑ_m on the chart $\operatorname{Spec} R_{\mathfrak{u}}$ is a sum of monomials associated to broken lines with endpoint in the chamber $\mathfrak{u}$. If the expression for ϑ_m on different charts $\operatorname{Spec} R_{\mathfrak{u}'}$ is compatible with the gluing maps between charts, then ϑ_m is well-defined as a global function. In this case, we say that the wall structure is *consistent*. Here we define broken lines and notion of consistency of a wall structure more precisely.

1.4.1 Broken lines

The theta functions will be indexed by the *asymptotic monomials* of B.

Definition 1.4.1 (Asymptotic monomials on $(B, \mathscr{P})$)**.** Let $(B, \mathscr{P})$ be a polyhedral affine pseudomanifold equipped with a piecewise affine function φ. Suppose that

$\tau \in \mathscr{P}$ and $x \in \operatorname{Int}\tau$. An *asymptotic monomial on* τ is a monomial

$$z^m \in \Bbbk[\mathcal{P}_x^+]$$

where $m \in \Lambda_\tau$ such that

$$\tau + \mathbb{R}_{\geq 0} m \subseteq \tau.$$

By abuse of notation we also refer to m as the asymptotic monomial. An *asymptotic monomial on* $(B, \mathscr{P})$ is an asymptotic monomial on any $\tau \in \mathscr{P}$.

We would like to be able to propagate monomials across all of B, but parallel transport of a monomial across a slab is not necessarily well defined. However, we can define a notion of parallel transport across slabs for monomials that point away from the slab. We therefore consider monomials $m \in \Lambda_B$ to be propagating in direction $-m$.

Definition 1.4.2 (Change of chambers morphisms)**.** Suppose $\mathfrak{u}$ and $\mathfrak{u}'$ are two adjacent chambers. If both chambers are contained in the same cell $\sigma \in \mathscr{P}_{\max}$, they are separated by a codimension zero wall. In this case, we define the *change of chambers morphism* $\theta_{\mathfrak{u}',\mathfrak{u}}$ to be the isomorphism between $R_{\mathfrak{u}}$ and $R_{\mathfrak{u}'}$ given by the wall-crossing automorphism (1.11) defined in Construction 1.3.1.

$$\theta_{\mathfrak{u},\mathfrak{u}'} := \theta_{\mathfrak{b}} : R_{\mathfrak{u}} \longrightarrow R_{\mathfrak{u}'} \tag{1.13}$$

When $\mathfrak{u}$ and $\mathfrak{u}'$ are separated by a slab $\mathfrak{b}$, there is no well-defined ring homomorphism from $R_{\mathfrak{u}}$ to $R_{\mathfrak{u}'}$ which factors through $R_{\mathfrak{b}}$. There is, however, a subring of $R_{\mathfrak{u}}$,

$$R_{\mathfrak{u}}^{\mathfrak{b}} := (\Bbbk[Q]/I)\,[\Lambda_\rho][\chi_{\mathfrak{b},\sigma}(Z_+)] \subseteq R_{\mathfrak{u}},$$

on which the localisation homomorphism $\chi_{\mathfrak{b},\sigma}$ is invertible. We define the change of chambers morphism $\theta_{\mathfrak{u}',\mathfrak{u}}$ in this case to be

$$\theta_{\mathfrak{u},\mathfrak{u}'} = \theta_{\mathfrak{b}} : R_{\mathfrak{u}}^{\mathfrak{b}} \longrightarrow R_{\mathfrak{u}'} \tag{1.14}$$

which is the ring homomorphism given by

$$\chi_{\mathfrak{b},\sigma}(Z_+) \longmapsto \chi_{\mathfrak{b},\sigma'}(Z_-)^{-1} \cdot f_{\mathfrak{b}} \cdot z^{\kappa_\rho}$$

and the canonical inclusion $\Lambda_\rho \subseteq \Lambda_{\sigma'}$.

Definition 1.4.3 (Broken lines for $\mathscr{S}$ on $(B, \mathscr{P})$)**.** Let $\mathscr{S}$ be a wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ. Consider a proper continuous map with image disjoint from any joints of $\mathscr{S}$

$$\beta : (-\infty, 0] \longrightarrow B_0$$

with image disjoint from the joints of $\mathscr{S}$, such that there exists a sequence

$$-\infty = t_0 < t_1 < \cdots < t_r = 0$$

corresponding to points on the path with $\beta(t_i) \in |\mathscr{S}|$ for $i \in \{1, \ldots, r-1\}$, such that $\beta|_{(t_i,t_{i+1})}$ is a non-constant affine map with image in $B \setminus |\mathscr{S}|$. To each domain of linearity (t_{i-1}, t_i) assign a monomial $a_i z^{m_i} \in \Bbbk[\mathcal{P}^+_{\beta(t)}]$ satisfying the following conditions:

1. $a_i \in \Bbbk[Q]/I$ and $-m_i = \beta'(t)$ for $t \in (t_{i-1}, t_i)$.
2. $a_1 = 1$ and z^{m_1} is an asymptotic monomial of $(B, \mathscr{P})$.
3. For all $i \in \{1, \ldots, r-1\}$ the monomial $a_{i+1} t^{m_{i+1}}$ is a summand of the expression for the image of $a_i z^{m_i}$ under the change of chambers morphism at $\beta(t_i)$,

$$\theta_{\mathfrak{u}_{i+1},\mathfrak{u}_i}(a_i z^{m_i}) = \sum_j a_j z^{m_j}.$$

We say the data of the map β together with the monomials $\{a_i z^{m_i} \,|\, 1 \le i \le r\}$ is a *(normalized) broken line for $\mathscr{S}$ with asymptotic monomial z^{m_1} and endpoint $\beta(0)$.*

Remark 1.4.4. There are finitely many (normalized) broken lines with asymptotic monomial z^m and fixed endpoint $p \in B$.

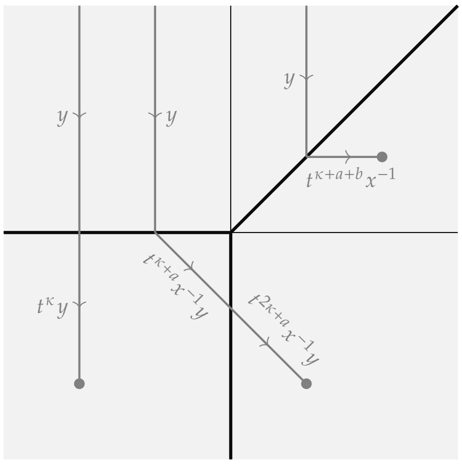

Figure 1.3: Three broken lines with asymptotic monomial y on the wall structure $\mathscr{S}$ from Figure 1.2, with the image of each domain of linearity labelled with the monomial it carries.

Definition 1.4.5 (Theta function)**.** We define a *theta function* for an asymptotic monomial m on $(B, \mathscr{P})$ and a general point p contained in a chamber $\mathfrak{u}$ of a wall structure $\mathscr{S}$ to be the sum

$$\vartheta_m(p) := \sum_{\beta} a_\beta z^{m_\beta} \in R_{\mathfrak{u}},$$

where the sum runs over all broken lines for $\mathscr{S}$ with asymptotic monomial m and end point p, and we define the associated monomial by

$$a_\beta z^{m_\beta} := a_r z^{m_r}. \tag{1.15}$$

Definition 1.4.6 (Consistency in codimension two)**.** Let $\mathscr{S}$ be a wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ, and suppose that $\mathfrak{j}$ be a joint of codimension two. We define a new polyhedral affine pseudomanifold $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$, where

$$\mathscr{P}_{\mathfrak{j}} := \left\{ (\Lambda_{\tau,\mathfrak{j}})_{\mathbb{R}} \mid \mathfrak{j} \subseteq \tau \in \mathscr{P} \right\},$$

where the inclusions of cells $(\Lambda_{\tau,\mathfrak{j}})_{\mathbb{R}} \hookrightarrow (\Lambda_{\tau',\mathfrak{j}})_{\mathbb{R}}$ are induced by the inclusions $\tau \hookrightarrow \tau' \in \mathscr{P}$, and the affine structure on $B_{\mathfrak{j}}$ is the one induced by the affine structure on

B. We define the wall structure on $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$ induced by $\mathscr{S}$,

$$\mathscr{S}_{\mathfrak{j}} := \left\{ \left((\Lambda_{\mathfrak{d},\mathfrak{j}})_{\mathbb{R}}, f_{\mathfrak{d}} \right) \,\middle|\, (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathscr{S} \text{ and } \mathfrak{j} \subseteq \mathfrak{d} \right\}.$$

Then $\mathscr{S}_{\mathfrak{j}}$ is the local model for $\mathscr{S}$ near $\mathfrak{j}$, all of whose chambers are cones containing the codimension two joint $\Lambda_{\mathfrak{j},\mathbb{R}}$, which is the only joint of $\mathscr{S}_{\mathfrak{j}}$. We say that $\mathfrak{j}$ is *consistent* if for any asymptotic monomial m of $(B_{\mathfrak{j}}, \mathscr{P}_{\mathfrak{j}})$, the theta functions $\vartheta_m(p)$

1. do not depend on the choice of $p \in \mathfrak{u} \in (\mathscr{P}_{\mathfrak{j}})_{\mathscr{S}_{\mathfrak{j}}}$, and
2. are compatible with the change of chambers morphisms for adjacent chambers $\mathfrak{u}, \mathfrak{u}' \in (\mathscr{P}_{\mathfrak{j}})_{\mathscr{S}_{\mathfrak{j}}}$. That is,

$$\theta_{\mathfrak{u}',\mathfrak{u}}(\vartheta_m(p)) = \vartheta_m(p') \quad \in R_{\mathfrak{u}'}$$

where $p \in \mathfrak{u}$, $p' \in \mathfrak{u}'$, and $\theta_{\mathfrak{u}',\mathfrak{u}}$ is defined in (1.13) and (1.14).

Definition 1.4.7 (Consistency)**.** A wall structure $\mathscr{S}$ is *consistent* if every joint $\mathfrak{j} \in \mathscr{P}_{\mathscr{S}}$ is consistent.

Example 1.4.8. The wall structure pictured in Figure 1.2 is consistent.

Theorem 1.4.9 (Consistency implies counts of broken lines are global functions on $\mathfrak{X}^\circ$)**.** *Let $\mathscr{S}$ be a consistent wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ. Let $\mathfrak{X}^\circ$ be the corresponding flat scheme over $\Bbbk[Q]/I$, as defined in Proposition 1.3.4.*

Then for each asymptotic monomial m (Definition 1.4.1) there exists a function $\vartheta_m \in \Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$ which restricts on $R_{\mathfrak{u}}$ to the theta function

$$\vartheta_m(p) := \sum_{\beta} a_\beta z^{m_\beta}$$

for each chamber $\mathfrak{u}$, where the sum runs over all broken lines for $\mathscr{S}$ with asymptotic monomial m and ending at a general point $p \in \mathfrak{u}$. Moreover, the ϑ_m form a $\Bbbk[Q]/I$-module basis of $\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$:

$$\Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ}) = R(\mathscr{S}) := \bigoplus_m (\Bbbk[Q]/I) \cdot \vartheta_m \tag{1.16}$$

Proof. This is Theorem 3.3.1 in [23]. □

Lemma 1.4.10 (Structure constants of the algebra of theta functions)**.** *Let $\mathscr{S}$ be a consistent wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ. Let $\mathfrak{X}^\circ$ be the corresponding flat scheme over $\Bbbk[Q]/I$, as defined in Proposition 1.3.4.*

For any two asymptotic monomials m_1, m_2, the product of the corresponding theta functions has expansion

$$\vartheta_{m_1}\vartheta_{m_2} = \sum_m \alpha_m(m_1, m_2) \cdot \vartheta_m, \tag{1.17}$$

where the coefficients are of the form

$$\alpha_m(m_1, m_2) = \sum_{(\beta_1,\beta_2)\in T_m(m_1,m_2)} a_{\beta_1} a_{\beta_2}. \tag{1.18}$$

Here a_{β_i} is the monomial in $\Bbbk[Q]/I$ associated to β_i by (1.15), and $T_m(m_1, m_1)$ is the set of pairs of broken lines (β_1, β_2) such that

1. *β_i has asymptotic monomial m_i.*

2. *β_1 and β_2 have the same endpoint $p \in \mathfrak{u}$, a general point for both asymptotic monomials m_1 and m_2.*

3. *The pair of broken lines satisfy the balancing condition $m_{\beta_1} + m_{\beta_2} = m \in \Lambda_{\mathfrak{u}}$, where $m_{\beta_i} \in \Lambda_{\mathfrak{u}}$ is the tangent vector associated to β_i by (1.15).*

Proof. This is Theorem 3.5.1 in [23]. □

1.4.2 Convexity

Consistency of a boundary joint is implied by the following notion of *convexity*. Checking convexity is simpler than checking consistency, and we will make use of this in Chapters 3 and 4.

Definition 1.4.11 (Convexity)**.** Suppose that $\mathfrak{j} \subset \partial B$ is a boundary joint of a wall structure $\mathscr{S}$ on a polyhedral affine pseudomanifold $(B, \mathscr{P})$. Let $\mathscr{S}_\mathfrak{j}$ be the local model for $\mathscr{S}$ near $\mathfrak{j}$, as defined in Definition 1.4.6. Since $\mathfrak{j}$ is a polyhedral subset of B of codimension two, there is an affine submersion

$$\pi : B_\mathfrak{j} \longrightarrow \mathbb{R}^2$$

that contracts $\mathfrak{j}$ to the origin. By extending the base ring $\Bbbk[Q]$ to $\Bbbk[Q \oplus \Lambda_{\mathfrak{j}}]$, we may consider wall functions $f_{\mathfrak{d}}$ for walls of $\mathscr{S}_{\mathfrak{j}}$ as wall functions for $\pi(\mathfrak{d})$, and therefore define a wall structure $\pi(\mathscr{S}_{\mathfrak{j}})$ on $\pi(B_{\mathfrak{j}})$. Broken lines $\beta \subset B_{\mathfrak{j}}$ with asymptotic monomial not in $\Lambda_{\mathfrak{j}}$ and endpoint p correspond one to one with broken lines in $\pi(B_{\mathfrak{j}})$ with endpoint $\pi(p)$.

The image $\pi(B_{\mathfrak{j}})$ will be a cone in $\mathbb{R}^2$. Let $m_0 \in \Lambda_{\pi(\partial B_{\mathfrak{j}})}$ be the primitive vector pointing away from the origin along one edge of the cone, and consider a broken line $\beta \subset \pi(B_{\mathfrak{j}})$ with asymptotic monomial m_0. It will cross the walls contained in the interior of the cone $\pi(B_{\mathfrak{j}})$ in succession; assume that, as β crosses $\pi(\mathfrak{d}) \subset \operatorname{Int} \pi(B_{\mathfrak{j}})$, it bends maximally away from the boundary $\pi(\partial B_{\mathfrak{j}})$. Let $a_\beta z^{m_\beta}$ be the monomial associated to the last possible domain of linearity of β – after it has bent across maximally many of the interior walls of $\pi(B_{\mathfrak{j}})$. If

$$\mathbb{R}_{\geq 0} m_\beta \not\subseteq \operatorname{Int} \pi(B_{\mathfrak{j}})$$

unless $a_\beta = 0 \in \Bbbk[Q]/I$, we say that the wall structure $\mathscr{S}$ is *convex* at the joint $\mathfrak{j}$.

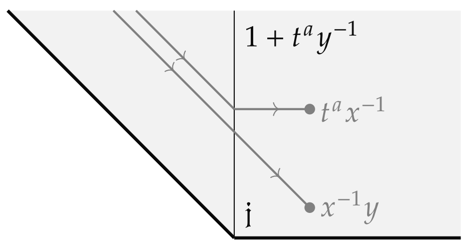

Figure 1.4: A neighbourhood of a convex boundary joint $\mathfrak{j}$ in a two-dimensional wall structure, and two broken lines with asymptotic monomial $x^{-1}y \in \Bbbk[x^{-1}y, x] \cong \Bbbk[\Lambda_{\mathfrak{j}}]$. The single interior wall containing $\mathfrak{j}$ is labelled with its wall function, and the two broken lines are labelled with the monomials associated to the last domain of linearity. The upper broken line is bent maximally away from the boundary.

Proposition 1.4.12 (Convexity implies consistency). *If $\mathscr{S}$ is convex at a boundary joint $\mathfrak{j}$, then $\mathfrak{j}$ is consistent.*

Proof. This is [23, Proposition 3.2.5]. □

1.5 The global construction of the family

When the wall structure $\mathscr{S}$ is conical, the scheme $\mathfrak{X}$ is affine and can be constructed as Spec of the algebra of theta functions. When the wall structure is not conical, we use it to construct a line bundle $\mathfrak{L}^\circ$ on $\mathfrak{X}^\circ$, which restricts on X_0 to the ample line bundle given by the polytopes. This line bundle is constructed as the inverse of the line bundle $(\mathfrak{L}^\circ)^{-1}$, whose total space is constructed as an affine scheme over $\mathfrak{X}^\circ$, by truncating the cone over $\mathscr{S}$. Then the partial compactification of $\mathfrak{X}^\circ$ is constructed by taking Proj of the graded algebra of theta functions on $(\mathfrak{L}^\circ)^{-1}$.

1.5.1 The affine case

Theorem 1.5.1 (Open embedding of affine $\mathfrak{X}^\circ \to \operatorname{Spec} R$)**.** *Suppose that $\mathscr{S}$ is a conical consistent wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ - that is, every cell $\tau \in \mathscr{P}_{\mathscr{S}}$ is a cone. Then the theta functions ϑ_m freely generate $R(\mathscr{S}) := \Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ})$ as a $\Bbbk[Q]/I$-module, and the induced canonical morphism*

$$\mathfrak{X}^\circ \longrightarrow \mathfrak{X} := \operatorname{Spec} R(\mathscr{S})$$

is an open embedding restricting to $X_0^\circ \longrightarrow X_0$ modulo $\mathfrak{m}$.

Proof. This is Proposition 3.4.2 in [23]. □

1.5.2 The projective case

Definition 1.5.2 (Cone over a polyhedron)**.** Let $\sigma \subseteq \mathbb{R}^n$ be an integral polyhedron. The *cone over* σ is defined to be

$$\mathbf{C}\sigma := \overline{\mathbb{R}_{\geq 0} \cdot (\sigma \times \{1\})} \subseteq \mathbb{R}^n \times \mathbb{R}.$$

Remark 1.5.3. If the polyhedron σ is unbounded, then $\sigma = \sigma_0 + \sigma_\infty$ where σ_0 is bounded and σ_∞ is a cone, which is uniquely determined for σ. In this case

$$\mathbf{C}\sigma = \mathbb{R}_{\geq 0} \cdot (\sigma_0 \times \{1\}) + \sigma_\infty \times \{0\}$$

where the two Minkowski summands only intersect in the origin $O \in \mathbb{R}^{n+1}$. We call σ_∞ the *asymptotic cone* or *tail cone* of σ.

Definition 1.5.4 (Cone over a polyhedral affine manifold)**.** The *cone over the polyhedral*

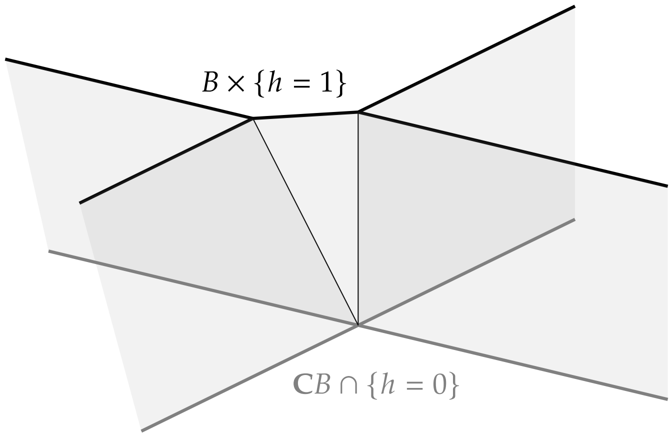

Figure 1.5: The segment $\{0 \leq h \leq 1\}$ of the cone over a two-dimensional polyhedral affine manifold B with four maximal cells and one bounded cell of dimension one. Here the codimension one cells of $\mathbf{C}B$ are filled in grey, with the codimension one cells of $B \times \{1\}$ shown in black, and the codimension one cells of $\mathbf{C}B \cap \{h = 0\}$ shown in gray.

affine pseudomanifold $(B, \mathscr{P})$ is the topological space

$$\mathbf{C}B := \varinjlim_{\tau \in \mathscr{P}} \mathbf{C}\tau$$

with polyhedral decomposition $\mathbf{C}\mathscr{P} := \{\mathbf{C}\tau \mid \tau \in \mathscr{P}\}$ and affine structure on $\mathbf{C}B_0 \subseteq \mathbf{C}B \setminus \mathbf{C}\Delta$ defined by the charts

$$\begin{aligned} \mathbf{C}\psi : \mathbf{C}U \setminus \{O\} &\longrightarrow \mathbb{R}^{n+1} \\ (x, h) &\longmapsto (h \cdot \psi(x), h) \end{aligned}$$

associated to each chart $\psi : U \to \mathbb{R}^n$ on B_0.

Definition 1.5.5 (Cone over a wall structure)**.** Let $\mathscr{S}$ be a wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ. We define the cone over $\mathscr{S}$ as follows. First, we note that φ induces a piecewise linear function on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ with kinks given by

$$\kappa_{\mathbf{C}\rho}(\mathbf{C}\varphi) := \kappa_\rho(\varphi).$$

Given a wall $(\mathfrak{d}, f_\mathfrak{d}) \in \mathscr{S}$, let $a \in \mathbb{N} \setminus \{0\}$ be the index of the image of $\Lambda_{\mathbf{C}\mathfrak{d}}$ in $\mathbb{Z}$ under the map

$$\Lambda_{\mathbf{C}\mathfrak{d}} \longrightarrow \mathbb{Z}$$

induced by projection to the height of the cone. Since $f_{\mathfrak{d}} \in 1 + \mathfrak{m}$, where $\mathfrak{m}$ is a nilpotent ideal in $\Bbbk[Q]/I$, there exists a unique element $(f_{\mathfrak{d}})^{1/a} \in 1 + \mathfrak{m}$ such that $\left((f_{\mathfrak{d}})^{1/a}\right)^a = f_{\mathfrak{d}}$. We define the *cone over* $\mathscr{S}$ as

$$\mathbf{C}\mathscr{S} := \left\{ \left(\mathbf{C}\mathfrak{d}, f_{\mathfrak{d}}^{1/a}\right) \,\middle|\, (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathscr{S} \right\}.$$

Lemma 1.5.6 (Consistency of cone over wall structure)**.** *Let $\mathscr{S}$ be a wall structure on polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise affine function φ. If $\mathscr{S}$ is consistent, then $\mathbf{C}\mathscr{S}$ is a consistent wall structure on $(\mathbf{C}B, \mathbf{C}\mathscr{P})$ equipped with the function $\mathbf{C}\varphi$.*

Proof. This is Proposition 4.2.6 in [23]. □

Theorem 1.5.7 (Open embedding $\mathfrak{X}^\circ \to \operatorname{Proj} S$)**.** *Let $\mathscr{S}$ be a consistent wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with a piecewise linear function φ. Let $\mathfrak{X}^\circ$ and $\mathfrak{Y}^\circ$ be the flat $\Bbbk[Q]/I$-schemes associated to $\mathscr{S}$ and $\mathbf{C}\mathscr{S}$ respectively by Proposition 1.3.4, and let*

$$R(\mathscr{S}) = \Gamma(\mathfrak{X}^\circ, \mathcal{O}_{\mathfrak{X}^\circ}), \quad R(\mathbf{C}\mathscr{S}) = \Gamma(\mathfrak{Y}^\circ, \mathcal{O}_{\mathfrak{Y}^\circ})$$

be the $\Bbbk[Q]/I$-algebras constructed in Theorem 1.4.9. The algebras have canonical $\Bbbk[Q]/I$-module bases of sections ϑ_m for asymptotic monomials on B and $\mathbf{C}B$ respectively.

Then the following holds.

1. *The ring $R(\mathbf{C}\mathscr{S})$ is a $\mathbb{Z}$-graded $R(\mathscr{S})$-algebra, with degree zero part $R(\mathbf{C}\mathscr{S})_0 = R(\mathscr{S})$.*

2. *There is a canonical embedding*

$$\mathfrak{X}^\circ \longrightarrow \mathfrak{X} := \operatorname{Proj} R(\mathbf{C}\mathscr{S})$$

of $\mathfrak{X}^\circ$ as an open dense subscheme in $\mathfrak{X}$. Moreover, $\mathfrak{X}$ is the unique flat extension of X_0 from $\Bbbk[Q]/\mathfrak{m}$ to $\Bbbk[Q]/I$ which contains $\mathfrak{X}^\circ$ as an open subscheme and is proper over $\operatorname{Spec} R(\mathscr{S})$.

3. *$R(\mathbf{C}\mathscr{S}) = \bigoplus_{d\in\mathbb{N}} \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(d))$, the homogeneous coordinate ring of $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(1))$.*

Proof. This is Theorem 4.3.2 in [23]. □

1.6 Compatible systems of wall structures

Definition 1.6.1 (Compatible system of wall structures)**.** Let $(B, \mathscr{P})$ be a polyhedral affine pseudomanifold equipped with a Q-valued MPA-function φ. Let $\mathfrak{D}$ be a set of wall structures $\mathscr{S}_I$ on $(B, \mathscr{P}, \varphi)$, indexed by $\mathfrak{m}$-primary ideals $I \subset \Bbbk[Q]$. We will call $\mathfrak{D}$ a *compatible system of wall structures* if we have

$$\mathscr{S}_I \equiv \mathscr{S}_{I'} \mod J$$

for any $\mathfrak{m}$-primary ideal J containing I and I'. If each wall structure $\mathscr{S}_I$ is consistent modulo I we say that $\mathfrak{D}$ is a compatible system of *consistent* wall structures.

A compatible system of wall structures on $(B, \mathscr{P})$ is morally an infinite collection of walls that form polyhedral subdivisions of $\mathscr{P}$ modulo each $\mathfrak{m}$-primary ideal I – that is, there are only finitely many walls $\mathfrak{d}$ with wall function $f_{\mathfrak{d}}$ nontrivial modulo I. The reason we do not simply define wall structures with wall functions $f_{\mathfrak{d}}$ as elements of $\widehat{\Bbbk[Q]}[\Lambda_x]$ is that taking the completion with respect to $\mathfrak{m}$ does not commute with the localisation maps induced by parallel transport on B. In the rest of this thesis we will, however, abuse notation by letting $\mathfrak{D}/I := \mathscr{S}_I$ for a compatible system of wall structures $\mathfrak{D} = \{\mathscr{S}_I \mid \sqrt{I} = \mathfrak{m}\}$.

Construction 1.6.2 (The formal family)**.** Let $(B, \mathscr{P})$ be a polyhedral affine pseudomanifold equipped with a Q-valued MPA-function φ, and let $I \subset \Bbbk[Q]$ be an $\mathfrak{m}$-primary ideal. Let $\mathfrak{D}$ be a compatible system of consistent wall structures on $(B, \mathscr{P}, \varphi)$. Let $\mathfrak{X}_I := \operatorname{Proj} R(\mathbf{C}\mathscr{S}_I)$ be the flat scheme over $\operatorname{Spec} \Bbbk[Q]/I$ associated to $\mathscr{S}_I$ via Theorem 1.5.7. For any ideal $J \subset I$ we have

$$\mathscr{S}_I \equiv \mathscr{S}_J \mod I,$$

so in particular $\mathfrak{X}_I$ is the pullback of $\mathfrak{X}_J$ via $\operatorname{Spec} \Bbbk[Q]/I \to \operatorname{Spec} \Bbbk[Q]/J$. We can therefore define a formal scheme

$$\hat{\mathfrak{X}} := \operatorname{colim}_{\sqrt{I}=\mathfrak{m}} \mathfrak{X}_I,$$

which is a formal flat family over

$$\operatorname{Spf}\widehat{\Bbbk[Q]} := \operatorname{colim}_{\sqrt{I}=\mathfrak{m}} \operatorname{Spec}\Bbbk[Q]/I.$$

Here $\widehat{\Bbbk[Q]}$ is the completion of $\Bbbk[Q]$ with respect to the maximal ideal $\mathfrak{m}$.

Lemma 1.6.3 (Algebraising the family)**.** *If B is bounded, then the formal family*

$$\hat{\mathfrak{X}} \longrightarrow \operatorname{Spf}\widehat{\Bbbk[Q]}$$

is algebraisable. That is, there exists a family

$$\mathfrak{X} \longrightarrow \operatorname{Spec}\widehat{\Bbbk[Q]}$$

such that $\mathfrak{X} \times_{\operatorname{Spec}\widehat{\Bbbk[Q]}} \operatorname{Spec}\Bbbk[Q]/I$ is isomorphic to $\mathfrak{X}_I$ for any $\mathfrak{m}$-primary ideal I.

Proof. The construction of $\mathfrak{X}_I$ in the the proof of [23, Theorem 4.3.2] involves an invertible sheaf whose restriction to $\mathfrak{X}_{\mathfrak{m}}$ is ample. By the Grothendieck Existence Theorem [18, III 5.4.5] it follows that $\hat{\mathfrak{X}}$ is algebraisable (see [35]). □

Remark 1.6.4. Alternatively, we can consider the algebraisation of $\hat{\mathfrak{X}}$ (when B is bounded) to be given by taking Proj of the $\widehat{\Bbbk[Q]}$-algebra

$$R(\mathbf{C}\mathfrak{D}) := \bigoplus_{m\in\mathbf{C}B(\mathbb{Z})} \widehat{\Bbbk[Q]} \cdot \vartheta_m, \tag{1.19}$$

where the theta functions are now formal variables whose product is given, modulo each $\mathfrak{m}$-primary ideal I, by the formula in Lemma 1.4.10. This is only a well-defined algebra because for each pair $m_1, m_2 \in \mathbf{C}B(\mathbb{Z})$, there are only finitely many asymptotic monomials $m \in \mathbf{C}B(\mathbb{Z})$ such that $\alpha_m(m_1, m_2) \neq 0$. That is, the expression for $\vartheta_{m_1}\vartheta_{m_2}$ is polynomial in the ϑ_m's.

Chapter 2

Constructions of the Gross–Siebert program II: sources of wall structures

In this Chapter we introduce two main constructions of a compatible system of wall structures $\mathfrak{D}$ associated to a log Calabi–Yau pair (Y, D): we define the canonical wall structure $\mathfrak{D}_{\text{can}}$ in Section 2.2 and the *algorithmic wall structure* $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$ in Section 2.3. We show that the formal schemes produced by these two compatible systems wall structures are in fact the same when restricted to a sublocus of the base, called the *Gross–Siebert locus*. In Section 2.4 we introduce an auxiliary construction of a wall structure, which enables us to further compactify the mirror family $\mathfrak{X}$.

In Section 2.1, we introduce the concept of *scattering diagrams* in two dimensions. This is closely related to the notion of wall structure, discussed in Section 1.2.3, except that:

1. We work on $\mathbb{R}^2$ rather than a polyhedral affine pseudomanifold. In particular, there is no polyhedral decomposition involved and there are no singularities in the affine structure. As a result, the notion of consistency, which in this context we call compatibility, is equivalent to consistency in codimension zero.

2. Wall functions are regarded as living in an $\mathfrak{m}$-adic completion of $\Bbbk[Q]$, rather than in $\Bbbk[Q]/I$ for some $\mathfrak{m}$-primary ideal I. Scattering diagrams can contain infinite numbers of rays and lines (the analogues of walls) as long as there only finitely many modulo I for every $\mathfrak{m}$-primary ideal I.

These technical differences, although small, are important. The conventions adopted when defining scattering diagrams allow us to make streamlined arguments about an algorithmic *scattering process*, in which an initial scattering diagram is iteratively refined to produce a compatible scattering diagram. The conventions when defining wall structures, on the other hand, are better adapted to the construction of mirror families by gluing local charts.

As we will see below, any scattering diagram determines a compatible system of wall structures on $M_{\mathbb{R}}$. Furthermore, a key result of the Gross–Siebert program is that the canonical wall structure for a log Calabi–Yau pair – which gives rise to the mirror family – can be obtained from a scattering diagram (see Remark 2.1.2 below). Although the canonical wall structure can be infinite and complicated, the initial scattering diagram involved is finite and straightforward to describe.

2.1 Scattering diagrams

In this section we define scattering diagrams on $M_{\mathbb{R}}$ where $M \cong \mathbb{Z}^2$. There is also a notion of scattering diagram in higher dimensions [28], and an analogue of the Kontsevich–Soibelman Lemma, Theorem 2.1.8, is known to hold for those higher-dimensional scattering diagrams that are formed of *widgets* – see [5, Theorem 5.6]. However, since the only higher dimensional scattering diagrams appearing in this thesis are cones over two dimensional scattering diagrams, it isn't necessary to define higher dimensional scattering here. See [5] for a detailed discussion.

Definition 2.1.1 (Scattering diagram)**.** Let $M = \mathbb{Z}^2$ and Q be a monoid with map $r : Q \to M$. Let $\mathfrak{m}_Q = Q \setminus Q^\times$, and $\widehat{\Bbbk[Q]}$ denote the completion of $\Bbbk[Q]$ with respect to the monomial ideal $\mathfrak{m}_Q$. A *ray* or a *line* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ such that

- $\mathfrak{d} \subset M_{\mathbb{R}}$ is given by $\mathfrak{d} = p + \mathbb{R}_{\geq 0} m$ if $\mathfrak{d}$ is a ray and $\mathfrak{d} = p + \mathbb{R}m$ if $\mathfrak{d}$ is a line, where $p \in M_{\mathbb{R}}$ and $m \in M_{\mathbb{R}} \setminus \{0\}$.
- $f_{\mathfrak{d}} \in \widehat{\Bbbk[Q]}$.
- $f_{\mathfrak{d}} \equiv 1 \bmod \mathfrak{m}_Q$.
- $f_{\mathfrak{d}} = 1 + \sum_q c_q z^q$ where the sum is over either $q \in Q$ such that $r(q) \in \mathbb{R}_{\geq 0} m$, or $q \in Q$ such that $-r(q) \in \mathbb{R}_{\geq 0} m$.

  If $\mathfrak{d}$ is a ray we say that $\mathfrak{d}$ is *incoming* if $r(q) \in \mathbb{R}_{\geq 0} m$ for all q with $c_q \neq 0$, and that $\mathfrak{d}$ is *outgoing* ray if $-r(q) \in \mathbb{R}_{\geq 0} m$ for all q with $c_q \neq 0$.

A *scattering diagram* for the data $r : Q \to M$ is a set of rays and lines such that for every power $k > 0$, there are only a finite number of $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$ such that $f_{\mathfrak{d}} \not\equiv 1$

mod $\mathfrak{m}_Q^k$. We denote by Sing($\mathfrak{D}$) the set of codimension 2 cells in the the polyhedral decomposition of $M_\mathbb{R}$ induced by the scattering diagram $\mathfrak{D}$.

Remark 2.1.2. A scattering diagram $\mathfrak{D}$ for the data $r : Q \to M$ induces, for each $\mathfrak{m}_Q$-primary ideal I, a wall structure $\mathscr{S}_I$ on the polyhedral affine manifold $M_\mathbb{R}$. Here the Q-valued MPA-function φ is zero, walls are obtained from rays or lines $(\mathfrak{d}, f_\mathfrak{d})$ by replacing $f_\mathfrak{d}$ by $f_\mathfrak{d} \mod I$, and we discard any walls such that $f_\mathfrak{d} \equiv 1 \mod I$. This leaves only finitely many walls in $\mathscr{S}_I$. By abuse of notation, we define the compatible system of wall structures

$$\mathfrak{D} = \left\{ \mathscr{S}_I \,\middle|\, \sqrt{I} = \mathfrak{m}_Q \right\},$$

and refer to both rays and lines as *walls* of $\mathfrak{D}$.

We say a ray or line $\mathfrak{d}$ *passes through* a point $p \in M_\mathbb{R}$ if p lies in the interior of the support of $\mathfrak{d}$. If $\mathfrak{d}$ is an outgoing ray and p is the boundary point of the support of $\mathfrak{d}$, we say that $\mathfrak{d}$ *emanates from* p.

Remark 2.1.3. Definition 2.1.1 generalises to higher dimensions as one would expect: the walls $\mathfrak{d}$ are polyhedral subsets of $M_\mathbb{R}$ of codimension one (here $M \cong \mathbb{Z}^n$), and every monomial z^q appearing with non-zero coefficient in the associated wall function $f_\mathfrak{d} \in \widehat{\Bbbk[Q]}$ has $r(q) \in \Lambda_\mathfrak{d} \subset M$. The definition of a general scattering diagram can be found in [28] or [5].

Definition 2.1.4 (Path ordered product)**.** Let $\gamma : [0, 1] \to M_\mathbb{R} \setminus \mathrm{Sing}(\mathfrak{D})$ be a path whose endpoints are not contained in $\mathfrak{D}$ and such that all intersections with any ray or line of $\mathfrak{D}$ are transverse. We define the *path ordered product*

$$\theta_{\gamma,\mathfrak{D}} : \widehat{\Bbbk[Q]} \longrightarrow \widehat{\Bbbk[Q]}$$

of $\mathfrak{D}$ along γ as follows. For each $k > 0$ the set

$$\{0 < t_1 \leq t_2 \leq \cdots \leq t_n < 1\} := \{t \in [0, 1] \mid \gamma(t) \in \mathfrak{d} \in \mathfrak{D} \text{ such that } f_\mathfrak{d} \not\equiv 1 \bmod \mathfrak{m}_Q^k\}$$

is finite as $\mathfrak{D}$ is a scattering diagram. For each t_i such that $\gamma(t_i) \in \mathfrak{d}$, define

$$\theta^k_{\mathfrak{d},n_\mathfrak{d}} : \Bbbk[Q]/\mathfrak{m}_Q^k \longrightarrow \Bbbk[Q]/\mathfrak{m}_Q^k$$

by

$$z^q \longmapsto z^q f_{\mathfrak{d}}^{\langle n_{\mathfrak{d}}, r(q)\rangle}$$

where $n_{\mathfrak{d}} \in N = M^{\vee}$ is primitive, annihilates the tangent space to $\mathfrak{d}$, and is chosen with the sign convention

$$\langle n_{\mathfrak{d}}, \gamma'(t_i)\rangle < 0.$$

We define

$$\theta^k_{\gamma,\mathfrak{D}} = \theta^k_{\mathfrak{d}_s,n_s} \circ \cdots \circ \theta^k_{\mathfrak{d}_1,n_1}.$$

The product does not depend on any choices, since $t_i = t_{i+1}$ implies that $\mathfrak{d}_i \cap \mathfrak{d}_{i+1}$ must be a one-dimensional cell in $M_{\mathbb{R}}$, and one can see easily $\theta^k_{\mathfrak{d}_i,n_i}$ commutes with $\theta^k_{\mathfrak{d}_{i+1},n_{i+1}}$. Therefore we can define the γ-ordered product

$$\theta_{\gamma,\mathfrak{D}} = \lim_{k\to\infty} \theta^k_{\gamma,\mathfrak{D}}.$$

Remark 2.1.5. Consider the following automorphism of $\widehat{\Bbbk[Q]}$

$$\exp\left(\log(f_{\mathfrak{d}})\,\partial_{n_{\mathfrak{d}}}\right),$$

where $n_{\mathfrak{d}}$ is a primitive normal vector to $\mathfrak{d}$. Here $\partial_n \in \mathrm{Hom}(M, \widehat{\Bbbk[Q]}) = \widehat{\Bbbk[Q]} \otimes N$ is the log derivation defined by $n \in N = M^{\vee}$:

$$\partial_n(z^q) = \langle n, r(q)\rangle z^q$$

and $\exp(\theta)$ is an automorphism of $\widehat{\Bbbk[Q]}$ for any $\theta \in \mathfrak{m}_Q\left(\widehat{\Bbbk[Q]} \otimes N\right)$, defined by

$$\exp(\theta)(g) = \mathrm{Id}(g) + \sum_{i=1}^{\infty} \frac{\theta^i(g)}{i!}.$$

We note that

$$\theta^k_{\mathfrak{d},n_{\mathfrak{d}}} \equiv \exp\left(\log(f_{\mathfrak{d}})\,\partial_{n_{\mathfrak{d}}}\right) \mod \mathfrak{m}_Q^k$$

when $n_{\mathfrak{d}}$ is chosen with respect to γ with the same sign convention as in the definition

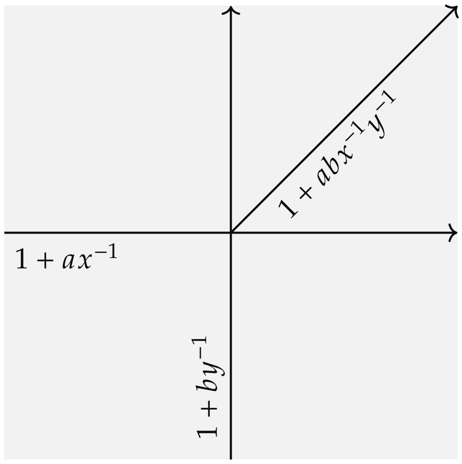

Figure 2.1: A consistent scattering diagram for the data $r : Q = \mathbb{N}^2 \oplus M \to M$ with two lines and one ray. Here r is the canonical projection and $\Bbbk[Q]$ is denoted $\Bbbk[a, b, x^{\pm}, y^{\pm}]$.

of $\theta^k_{\gamma,\mathfrak{d}}$, and so path ordered products can always be written in the form

$$\theta_{\gamma,\mathfrak{D}} \equiv \exp\left(\sum_i c_i z^{q_i} \partial_{n_i}\right) \quad \text{modulo } \mathfrak{m}_Q^k$$

where $n_i \in r(q_i)^\perp$ and $c_i \in k$. The set of automorphisms of $\widehat{\Bbbk[Q]}$ with such an expression form a group, introduced in [28] and discussed in [24] as the *tropical vertex group*.

Definition 2.1.6. A scattering diagram $\mathfrak{D}$ is *compatible* if

$$\theta_{\gamma,\mathfrak{D}} = \mathrm{Id}$$

for all loops γ for which $\theta_{\gamma,\mathfrak{D}}$ is defined.

Remark 2.1.7. If a scattering diagram $\mathfrak{D}$ is compatible, then the induced wall structure $\mathscr{S}$ from Remark 2.1.2 is consistent. Indeed, every joint of $\mathscr{S}$ is of codimension zero, so the conditions for consistency of $\mathscr{S}$ are equivalent to the conditions for compatibility of $\mathfrak{D}$.

Theorem 2.1.8 (Kontsevich–Soibelman [43])**.** *Let $\mathfrak{D}$ be a scattering diagram. There exists a compatible scattering diagram* Scatter($\mathfrak{D}$) *containing $\mathfrak{D}$ such that* Scatter($\mathfrak{D}$) $\setminus$ $\mathfrak{D}$ *consists only of rays.*

Proof. By definition, $\mathfrak{D}$ is compatible modulo $\mathfrak{m}_Q$. Construction 4.1.2 gives an algorithm for constructing a scattering diagram $\mathrm{Scatter}^k(\mathfrak{D})$ which is compatible modulo $\mathfrak{m}_Q^{k+1}$ and contains $\mathrm{Scatter}^{k-1}(\mathfrak{D})$, the scattering diagram compatible modulo $\mathfrak{m}_Q^k$ containing $\mathfrak{D}$. The algorithm only adds outgoing rays to $\mathrm{Scatter}^{k-1}(\mathfrak{D})$ to obtain $\mathrm{Scatter}^k(\mathfrak{D})$. □

Definition 2.1.9. Two scattering diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ for the same data $r : Q \to M$ are *equivalent* if

$$\theta_{\gamma,\mathfrak{D}} = \theta_{\gamma,\mathfrak{D}'}$$

for every curve γ for which both sides are defined.

Remark 2.1.10. $\mathrm{Scatter}(\mathfrak{D})$ is unique up to equivalence of scattering diagrams. We refer to the application of the functor Scatter to a scattering diagram $\mathfrak{D}$ as *scattering* and the application of $\mathrm{Scatter}^k$ as *scattering to order k*.

The following Lemma is proved in Section 4.1.1.

Lemma 2.1.11. *Scattering is functorial up to equivalence in the following sense.*

Suppose Q and P are two monoids equipped with maps $r : Q \to M$ and $s : P \to M$, and $\varphi : Q \to P$ is a morphism of monoids such that there exists an automorphism $\alpha \in GL_2(\mathbb{Z})$ of $M_\mathbb{R}$ such that $s \circ \varphi = \alpha \circ r$. Then define

$$\varphi(\mathfrak{D}) := \{(\alpha(\mathfrak{d}), \hat{\varphi}(f_\mathfrak{d})) \mid (\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}\}$$

where $\hat{\varphi} : \widehat{\Bbbk[Q]} \longrightarrow \widehat{\Bbbk[P]}$ is the ring homomorphism induced by φ. Then $\mathrm{Scatter}(\varphi(\mathfrak{D}))$ *is equivalent to* $\varphi(\mathrm{Scatter}(\mathfrak{D}))$.

2.2 The canonical wall structure

In this section we define the canonical wall structure associated to a log Calabi–Yau pair (Y, D) in terms of its *punctured invariants* (see [2] for a definition). We follow the notations and conventions of [31], simplifying where possible to restrict the definitions and constructions to a smaller set of log Calabi–Yau pairs which contains all the examples of this thesis.

Definition 2.2.1 (Log Calabi–Yau pair)**.** Let Y be a non-singular variety and $D \subset Y$ a simple normal crossings divisor. The divisor D induces a divisorial log structure on Y which is log smooth over the log point $\operatorname{Spec}\Bbbk$. We say the pair (Y, D) is *log Calabi–Yau* if the logarithmic canonical class $K_Y + D$ is numerically equivalent to an effective $\mathbb{Q}$-divisor supported on D.

Remark 2.2.2. In general (see [31]) the polyhedral affine pseudomanifold underlying the canonical wall structure depends on the presentation

$$K_Y + D \equiv_{\mathbb{Q}} \sum_j a_j D_j,$$

where $a_j \geq 0$ and the D_j are the irreducible components of D. In this thesis, we will further assume that $a_j = 0$ for all j. This will allow us to simplify the definition of the polyhedral affine pseudomanifold.

Remark 2.2.3. In addition, we assume that the stratum

$$D_{j_1} \cap \cdots \cap D_{j_k}$$

is connected or empty for every $j_1, \ldots, j_k$. This is already true for the examples considered in this thesis, but can also be achieved in general via a log étale birational modification of Y [31]. We further assume that there exists at least one zero-dimensional stratum.

The tropicalisation of the log scheme given by (Y, D) is defined in [1]. When we make the above assumptions, it has the following description.

Construction 2.2.4 (Tropicalisation of the log Calabi–Yau pair)**.** Given a log Calabi–Yau pair (Y, D) satisfying the above assumptions, the tropicalisation $\Sigma(Y)$ is given by the dual intersection complex of $D = D_1 + \ldots + D_r$. More precisely, it is the polyhedral cone complex in $\operatorname{Div}_D(Y)^*_{\mathbb{R}}$

$$\Sigma(Y) := \left\{ \sum_{j \in J} \mathbb{R}_{\geq 0} D_j^* \,\middle|\, J \subseteq \{1, \ldots, r\}, \bigcap_{j \in J} D_j \neq \varnothing \right\}.$$

Construction 2.2.5 (The integral affine structure)**.** We now let $\mathscr{P} = \Sigma(Y)$ and define an integral affine structure on $B = \bigcup_{\sigma \in \mathscr{P}} \sigma$ as follows. For each codimension one cone $\rho \in \Sigma(Y)$ such that $\rho = \sigma \cap \sigma'$ for $\sigma, \sigma' \in \Sigma(Y)_{\max}$, we define a chart ψ_ρ :

$\sigma \cup \sigma' \longrightarrow \mathbb{R}^n$. Say that

$$\rho = \sum_{k=1}^{n-1} \mathbb{R}_{\geq 0} D^*_{j_k}, \qquad \sigma = \sum_{k=1}^{n} \mathbb{R}_{\geq 0} D^*_{j_k}, \qquad \sigma' = \sum_{k=1}^{n-1} \mathbb{R}_{\geq 0} D^*_{j_k} + \mathbb{R}_{\geq 0} D^*_{j'_n}$$

and pick two bases $e_1, \ldots, e_{n-1}, e_n$ and $e_1, \ldots, e_{n-1}, e'_n$ for $\mathbb{R}^n$ such that

$$e_n + e'_n = -\sum_{k=1}^{n-1} (D_{j_k} \cdot Y_\rho) e_k,$$

where Y_ρ is the curve defined by the stratum of the intersection complex given by ρ. Then define the chart ψ_ρ by

$$\begin{aligned} D^*_{j_k} &\longmapsto e_k \quad \forall k \in \{1, \ldots, n\} \\ \text{and} \qquad D^*_{j'_n} &\longmapsto e'_n, \end{aligned}$$

and note that it is well-defined up to an element of $\mathrm{GL}_n(\mathbb{Z})$.

Remark 2.2.6. Under our assumptions, $\Sigma(Y)$ contains an n-dimensional cone, and therefore the tropicalisation of (Y, D) defines a polyhedral affine pseudomanifold in the sense of Definition 1.2.1. See Proposition 1.3 in [31].

Construction 2.2.7 (The monoid Q)**.** We fix a finitely generated monoid $Q \subseteq N_1(Y)$ such that

1. Q contains the classes of all stable maps to Y,
2. Q is saturated, and
3. $Q^\times = N_1(Y)_{\mathrm{tors}}$.

In the examples considered in this thesis, $N_1(Y)$ will be torsion-free, and so the third condition implies that the monoid Q is sharp. This set of conditions on the choice of monoid Q is a simplification of the conditions given in [30, Basic Setup 1.6] – in this section we only describe the canonical wall structure in the *absolute case*.

Construction 2.2.8 (The multivalued piecewise affine function)**.** We define $\varphi \in \mathcal{MPA}(B, Q^{\mathrm{gp}})$ to be the unique MPA-function such that, for every codimension one cell $\rho \in \Sigma(Y)$, the kink across ρ is

$$\kappa_\rho = [Y_\rho] \in N_1(Y).$$

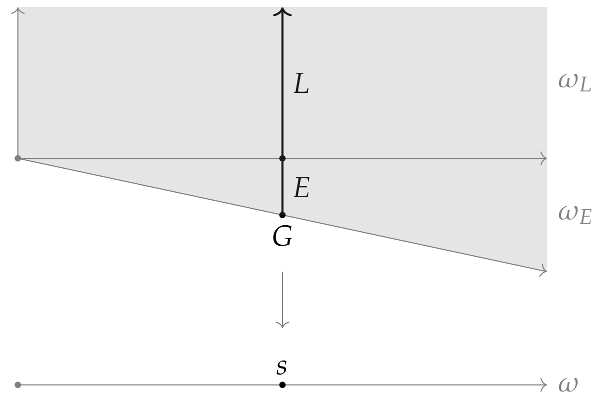

Figure 2.2: A family of tropical curves $\Gamma(G) \longrightarrow \omega = \mathbb{R}_{\geq 0}$. The graph G over each interior point has two vertices, an edge E and a leg L.

Definition 2.2.9 (Tropical type)**.** Recall that the tropicalisation of a log stable curve is a graph G, which has a vertex for each irreducible component, an edge between two vertices for each point of intersection of irreducible components, and a leg (an edge with a single vertex) for each punctured or marked point. A function $g : V(G) \to \mathbb{N}$ assigns a genus to each component of the log stable curve.

Let ω be a rational polyhedral cone and C be a family of curves over $W = \operatorname{Spec} \Bbbk[S_\omega]$, where S_ω is the semigroup consisting of the integral points of $\omega^\vee$. The tropicalisation of $C \to W$ is a morphism of cone complexes $\Gamma(G) \to \omega$ which is a family of graphs with G being the fibre over each interior point of ω. For each vertex $v \in V(G)$ there is a copy of ω, $\omega_v \in \Gamma(G)$, and for each edge or leg $E \in E(G) \cup L(G)$ there is a cone $\omega_E \subset \omega \times \mathbb{R}_{\geq 0}$. See Figure 2.2 for an example.

Now consider the tropicalisation of a punctured log map with domain $C \to W = \operatorname{Spec} \Bbbk[Q]$ and target Y. By functoriality of tropicalisation this is a family of tropical maps: a morphism of cone complexes

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{h} & \Sigma(Y) \\ \downarrow & & \\ \Sigma(W) = \omega = Q^\vee_{\mathbb{R}_{\geq 0}}. & & \end{array}$$

The *type* of this family of tropical maps is the data $\boldsymbol{\tau} = (G, \sigma, u, \beta)$, where

- G is the fibre over an interior point of ω,
- $\sigma : V(G) \cup E(G) \cup L(G) \longrightarrow \Sigma(Y)$ sends x to the minimal cone containing $h(\omega_x)$,
- $u : E(G) \cup L(G) \longrightarrow \Lambda_B$ sends E to its *contact order*, the image $h((0,1)) \in \Lambda_{\sigma(E)}$ of $(0,1) \in \Lambda_{\omega_E} = \Lambda_\omega \oplus \mathbb{Z}$,
- $\beta : V(G) \longrightarrow NE(Y)$ assigns a curve class to each vertex – the class of the image of the associated component of C.

The *dimension* of a type $\boldsymbol{\tau}$ is the dimension of the base cone ω. We say that $\boldsymbol{\tau}$ is *balanced* if for each vertex $v \in V(G)$ with $\sigma(v) \in \mathscr{P}$ a cone of codimension zero or one, we have

$$\sum_i u(E_i) = 0, \tag{2.1}$$

where the sum runs over all edges and legs incident to v.

Remark 2.2.10. Note that u records the contact order of the points associated to each edge or leg of G. See [2, Definition 2.17] for the definition of the contact order of a punctured map at a point.

Definition 2.2.11 (Wall type)**.** A *wall type* is a balanced tropical type $\boldsymbol{\tau}$ of dimension $\dim Y - 2$, with only vertices of genus 0 and a single leg L_{out} whose associated cone $\omega_L = \tau_{\text{out}}$ has codimension one image $h(\tau_{\text{out}}) \subset \Sigma(Y)$.

Construction 2.2.12 (The canonical wall structure $\mathfrak{D}_{\text{can}}$)**.** Denote by $\mathfrak{D}_{\text{can}}$ the set of pairs

$$(\mathfrak{d}, f_{\mathfrak{d}}) := \left(h(\tau_{\text{out}}), \exp(k_\tau N_{\boldsymbol{\tau}} t^\beta z^{-u})\right), \tag{2.2}$$

where $N_{\boldsymbol{\tau}} := \deg[\mathcal{M}_{\boldsymbol{\tau}}(Y)]^{\text{virt}}$ counts families of punctured maps to Y whose tropicalisation is of wall type $\boldsymbol{\tau}$. The wall $\mathfrak{d}$ itself is swept out by the image of the leg of the tropicalisation, and the exponent $-u \in \Lambda_B$ is the contact order of the single leg $u(L_{\text{out}})$. The term $k_{\boldsymbol{\tau}}$ is a positive integer defined as

$$k_{\boldsymbol{\tau}} := |\Lambda_{h(\tau_{\text{out}})} / h_*(\Lambda_{\tau_{\text{out}}})|$$

where h_* is the morphism $\Lambda_{\tau_{\text{out}}} \to \Lambda_\sigma$ induced by $h|_{\tau_{\text{out}}}$. The curve class β ranges over values in Q.

Remark 2.2.13. Strictly speaking, the collection of pairs $(\mathfrak{d}, f_\mathfrak{d})$ defined in (2.2) do not form a wall structure in the sense of Definition 1.2.20; the functions $f_\mathfrak{d}$ live in the *completion* of $\mathbb{k}[\mathcal{P}_x^+]$ with respect to the maximal ideal $\mathfrak{m} = Q \setminus Q^\times$, and there are infinitely many polyhedral subsets of the form $h(\tau_{\mathrm{out}})$.

However, if I is an $\mathfrak{m}$-primary ideal then there are only finitely many such functions $f_\mathfrak{d}$ which are non-trivial modulo I. Thus $\mathfrak{D}_{\mathrm{can}}$ defines a compatible system of wall structures in the sense of Definition 1.6.1, similarly to a scattering diagram (see Remark 2.1.2). We abuse notation by referring as $\mathfrak{D}_{\mathrm{can}}$ to both the compatible system of wall structures and wall structure obtained by truncating $f_\mathfrak{d}$ modulo I.

2.3 Algorithmic construction of the canonical wall structure

We now describe a construction of the canonical wall structure via scattering, following Argüz–Gross [5], in the setting where our log Calabi–Yau pair (Y, D) has a toric model.

Definition 2.3.1 (Toric model associated to a log Calabi–Yau pair)**.** Consider a log Calabi–Yau pair (Y, D) where Y is obtained by a blow-up

$$\mathrm{Bl}_H : Y \longrightarrow Y_\Sigma$$

of a smooth toric variety Y_Σ associated to a complete fan Σ in $\mathbb{R}^n$, where the centre of the blow-up H is a union of general smooth hypersurfaces in the toric boundary,

$$H = \bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} H_{ij}$$

and $H_{ij} \subset D_i$ meets each toric stratum of D_i transversally. Here the D_i are distinct toric divisor components in Y_Σ. The divisor $D \subset Y$ is taken to be the strict transform of the toric boundary $D_\Sigma \subset Y_\Sigma$. We call the data (Y_Σ, H) a *toric model* for (Y, D).

Remark 2.3.2. The term *toric model* is sometimes used to refer to the toric model of a log birational modification $(\tilde{Y}, \tilde{D})$ of (Y, D). The logarithmic Gromov–Witten theories of $(\tilde{Y}, \tilde{D})$ and (Y, D) coincide [3], so their canonical wall structures are equivalent. When (Y, D) is a Looijenga pair, there exists a log birational modification admitting a toric model. In this thesis, we only refer to toric models of a given pair (Y, D) in the stricter sense of Definition 2.3.1.

In the rest of this chapter and the next, it will be useful to fix the following notation.

Notation 2.3.3. Suppose that (Y, D) is a log Calabi–Yau pair with a toric model (Y_Σ, H). The existence of a toric model means that (Y, D) satisfies the assumptions in Remarks 2.2.2 and 2.2.3. Constructions 2.2.4 and 2.2.5 therefore give us a polyhedral affine pseudomanifold $(B, \mathscr{P})$. The log Calabi–Yau pair (Y_Σ, D_Σ) also satisfies the assumptions – the associated polyhedral affine manifold is $(M_\mathbb{R}, \Sigma)$, where $M \cong \mathbb{Z}^n$ denotes the lattice containing Σ.

Let $D_i \subset Y_\Sigma$ be the toric divisor component associated to the ray $\rho_i \in \Sigma$ as above, and more generally let $Y_\tau \subset Y_\Sigma$ be the toric stratum associated to the cell $\tau \in \Sigma$. Denote by $\overline{D}_i$ and $\overline{Y}_\tau$ the proper transforms of D_i and Y_τ under the blow-up $\mathrm{Bl}_H : Y \to Y_\Sigma$. Then $D \subset Y$ has the following expression as the sum of its irreducible components:

$$D = \sum_{i=1}^{r} \overline{D}_i,$$

where $r \geq s$. Moreover, the dual intersection complexes of D and D_Σ are the same. Therefore, there is a canonical piecewise linear map of polyhedral affine pseudomanifolds

$$\nu : B \to M_\mathbb{R} \tag{2.3}$$

given by the piecewise linear identification of $\mathscr{P}$ with Σ. For any cell $\tau \in \Sigma$, denote its pre-image by

$$\overline{\tau} := \nu^{-1}(\tau) \in \mathscr{P}. \tag{2.4}$$

The stratum $Y_{\overline{\tau}} \subset Y$ corresponding to $\overline{\tau} \in \mathscr{P}$ is equal to $\overline{Y}_\tau$.

Construction 2.3.4 (The HDTV scattering diagram)**.** Given a log Calabi–Yau (Y, D) with a toric model (Y_Σ, H) as above, we construct a scattering diagram as follows. Fix a monoid

$$P := \left\langle (m_i, e_{ij}) \,\middle|\, 1 \leq i \leq s,\ 1 \leq j \leq s_i \right\rangle \subset M \oplus \bigoplus_{i=1}^{s} \mathbb{N}^{s_i},$$

where m_i is the primitive generator of the ray ρ_i corresponding to the component D_i of the toric boundary of Y_Σ that contains H_{ij}, and e_{ij} is the generator corresponding to the (ij)-th standard basis element of $\bigoplus_i \mathbb{N}^{s_i}$. For each H_{ij}, fix a polynomial

$$f_{ij} := 1 + t_{ij} z^{m_i},$$

where $t_{ij} := z^{e_{ij}}$, and define a scattering diagram

$$\mathfrak{D}_{ij} := \left\{ \left(\rho, f_{ij}^{Y_\rho \cdot H_{ij}} \right) \middle| \rho \in \Sigma^{[n-1]} \text{ such that } \rho_i \subset \rho \right\},$$

where the intersection product $Y_\rho \cdot H_{ij}$ is evaluated on D_i. Now define

$$\mathfrak{D}_{(Y_\Sigma, H)} := \text{Scatter}\left(\bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} \mathfrak{D}_{ij} \right).$$

The compatible scattering diagram $\mathfrak{D}_{(Y_\Sigma,H)}$ exists by Theorem 5.6 in [5].

Construction 2.3.5 (The HDTV scattering diagram determines the canonical wall structure)**.** In [5, Section 6], Argüz and Gross construct a wall structure $\Upsilon(\mathfrak{D}_{(Y_\Sigma,H)})$ on $(B, \mathscr{P})$ from the data of the toric model. On the support of the scattering diagrams/wall structures, their map Υ is given by the canonical piecewise linear map $\nu^{-1} : M_{\mathbb{R}} \to B$. The key result of the paper [5, Theorem 6.1] is that $\Upsilon(\mathfrak{D}_{(Y_\Sigma,H)})$ is equivalent to the canonical wall structure $\mathfrak{D}_{\text{can}}$ associated to (Y, D).

Construction 2.3.6 (The affine mirror family)**.** Given a log Calabi–Yau pair (Y, D) with toric model (Y_Σ, H), the canonical wall structure $\mathfrak{D}_{\text{can}}$ is a conical wall structure. Therefore Theorem 1.5.1 applies and mirror family can be constructed by taking Spec of the algebra of theta functions associated to $\mathfrak{D}_{\text{can}}$:

$$\mathfrak{X}(\mathfrak{D}_{\text{can}}/I) := \operatorname{Spec} R(\mathfrak{D}_{\text{can}}/I).$$

We can then define the *canonical mirror family* to (Y, D) via Construction 1.6.2:

$$\mathfrak{X}(\mathfrak{D}_{\text{can}}) := \operatorname{colim}_{\sqrt{I}=\mathfrak{m}_Q} \mathfrak{X}(\mathfrak{D}_{\text{can}}/I). \tag{2.5}$$

This is a formal scheme over $\operatorname{Spf} \widehat{\Bbbk[Q]}$, where Q is the monoid from Construction 2.2.7.

One of the advantages of working with $\mathfrak{D}_{(Y_\Sigma,H)}$ rather than $\mathfrak{D}_{\text{can}}$ is of course that the initial scattering diagram determining $\mathfrak{D}_{(Y_\Sigma,H)}$ is finite and simple to describe. However, another useful property of $\mathfrak{D}_{(Y_\Sigma,H)}$ is its support $M_\mathbb{R}$, which is a flat affine manifold that is independent of the geometry of the log Calabi–Yau pair, rather than the affine pseudomanifold B, whose singularities are determined by intersection numbers on (Y, D). In Chapter 4 we will need to relate the wall structures associated to different log Calabi–Yau pairs (Y_f, D_f) and (Y_g, D_g) in order to construct morphisms between their mirror families. Relating the two wall structures is much easier if the same affine manifold supports both. Instead of working with the canonical wall structure, therefore, for each pair (Y, D) we will work with a different wall structure $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$ that is supported on $(M_\mathbb{R}, \Sigma)$.

The rest of this section will be devoted to defining the wall structure $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$, and proving that, after restricting to a sublocus in the base of the formal family, the associated formal scheme $\mathfrak{X}(\alpha\mathfrak{D}_{(Y_\Sigma,H)})$ is isomorphic to the canonical mirror family. This generalises the approach of Gross, Hacking and Keel, who prove consistency of $\mathfrak{D}_{\mathrm{can}}$ for two-dimensional log Calabi–Yau pairs in [21] by showing that $\mathfrak{X}(\mathfrak{D}_{\mathrm{can}})$ and $\mathfrak{X}(\alpha\mathfrak{D}_{(Y_\Sigma,H)})$ are isomorphic over the *Gross–Siebert locus*, which we define below.

Definition 2.3.7 (The Gross–Siebert locus)**.** Suppose that (Y, D) is a log Calabi–Yau pair with a toric model (Y_Σ, H). We define a monoid

$$Q^{\mathrm{gs}} := \mathrm{Bl}^*_H(NE(Y_\Sigma)) \oplus E, \tag{2.6}$$

where $NE(Y_\Sigma) = NE(Y_\Sigma)_\mathbb{R} \cap N_1(Y)$ is the monoid of integral points in the Mori cone and $E \subset N_1(Y)$ is the lattice generated by the classes of the exceptional curves of the blow-up $Y \longrightarrow Y_\Sigma$. The maximal ideal $\mathfrak{m}^{\mathrm{gs}} \subset Q^{\mathrm{gs}}$ is now equal to $Q^{\mathrm{gs}} \setminus E$, and we denote the completion of the associated ring with respect to $\mathfrak{m}^{\mathrm{gs}}$ by

$$R^{\mathrm{gs}} := \widehat{\Bbbk[Q^{\mathrm{gs}}]}. \tag{2.7}$$

We use the term *Gross–Siebert locus* to refer to either of the schemes $\operatorname{Spec} R^{\mathrm{gs}}$ or $\operatorname{Spec} R^{\mathrm{gs}}/J$, where J is an $\mathfrak{m}^{\mathrm{gs}}$-primary ideal, and also use the same term to refer to the formal scheme $\operatorname{Spf} R^{\mathrm{gs}}$.

Remark 2.3.8 (The Gross–Siebert locus as a subscheme of the base)**.** Let Q be a sharp monoid associated to (Y, D) as in Construction 2.2.7. The monoid $NE(Y_\Sigma)$

is a convex cone generated by the classes of the one-dimensional toric strata, and E is generated by the classes of the exceptional curves over a general point in each component of H, so the monoid Q^{gs} is finitely generated. Moreover, Q^{gs} contains $NE(Y)$, and so we may assume, by shrinking Q if necessary, that

(i) $Q \subset Q^{\mathrm{gs}}$, and

(ii) $E \cap Q$ is a face of Q.

It follows from (i) and (ii) that $Q \oplus E = Q^{\mathrm{gs}}$. If $I \subset Q$ is an $\mathfrak{m}$-primary ideal, where $\mathfrak{m}$ is the maximal ideal of Q, then $I \oplus E$ is an $\mathfrak{m}^{\mathrm{gs}}$-primary ideal in Q^{gs}. Thus we may think of the Gross–Siebert locus $\operatorname{Spec} R^{\mathrm{gs}}/(I \oplus E)$ as a subscheme of $\operatorname{Spec} R^{\#}/I$, where

$$R^{\#} := \widehat{\Bbbk[Q]} \tag{2.8}$$

is the completion of $\Bbbk[Q]$ with respect to $\mathfrak{m}$. More concretely, there is a diagram of rings

$$\begin{array}{ccc} R^{\mathrm{gs}}/(I \oplus E) & & \\ \uparrow & & \\ R'/(I \oplus E \cap Q) & \longrightarrow & R^{\#}/I, \end{array} \tag{2.9}$$

where R' is the completion of $\Bbbk[Q]$ with respect to $\mathfrak{m}^{\mathrm{gs}} \cap Q$. In order to view the family over the Gross–Siebert locus as a subfamily of the canonical mirror family $\mathfrak{X}(\mathfrak{D}_{\mathrm{can}})$, we will use the fact (see Lemma 2.3.10 below) that $\mathfrak{X}(\mathfrak{D}_{\mathrm{can}})$ is the pullback of a formal family over R' via the horizontal map in the diagram above.

Construction 2.3.9 (The algorithmic wall structure $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$). Suppose that (Y, D) is a log Calabi–Yau pair with a toric model (Y_Σ, H), and suppose that Q is a monoid for (Y, D) as in Construction 2.2.7, and P is the monoid defined for the toric model as in Construction 2.3.4. There is a *single-valued* piecewise linear function

$$\psi_\Sigma : M_{\mathbb{R}} \longrightarrow N_1(Y_\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}, \tag{2.10}$$

such that the kink across $\rho \in \Sigma^{[n-1]}$ is the curve class represented by the toric stratum $Y_\rho \subset Y_\Sigma$ (see [5, Lemma 2.9]). Therefore, there is a Q^{gp}-valued single-valued piecewise linear function ψ on $M_{\mathbb{R}}$, defined by

$$\psi := \mathrm{Bl}_H^* \circ \psi_\Sigma : M_{\mathbb{R}} \longrightarrow Q_{\mathbb{R}}^{\mathrm{gp}}.$$

This function has kink across each codimension one cell of Σ given by

$$\kappa_\rho(\psi) = \mathrm{Bl}_H^*[Y_\rho].$$

Since ψ is a single-valued function on an affine manifold with no singularities, it induces the constant sheaf $\mathcal{P}_\psi$ on M with fibre $M \oplus Q^{\mathrm{gp}}$, as in Definition 1.2.13, and we can define a subsheaf of monoids

$$\mathcal{P}_\psi^+ := \{(m, \psi(m) + q) \mid m \in M,\ q \in Q^{\mathrm{gs}}\}$$

with respect to the monoid Q^{gs} defining the Gross–Siebert locus in Definition 2.3.7 above. We define a morphism of monoids

$$\alpha : P \longrightarrow \mathcal{P}_\psi^+, \quad (m_i, e_{ij}) \longmapsto (m_i, \psi(m_i) - E_{ij})$$

which clearly restricts to the identity on M, and thus induces a consistent scattering diagram

$$\alpha\mathfrak{D}_{(Y_\Sigma,H)} := \left\{(\mathfrak{d}, \alpha_*(f_\mathfrak{d})) \,\middle|\, (\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_{(Y_\Sigma,H)}\right\}$$

for the data $\mathcal{P}^+ \to M$, via Lemma 2.1.11. Moreover, the scattering diagram $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$ gives a system of compatible wall structures on the polyhedral affine manifold $(M_{\mathbb{R}}, \Sigma)$ equipped with the convex piecewise-linear function ψ, with respect to $\mathfrak{m}^{\mathrm{gs}}$.

Lemma 2.3.10. *Both $\mathfrak{D}_{\mathrm{can}}$ and $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$ are compatible systems of wall structures with respect to the ideal $\mathfrak{m}^{\mathrm{gs}} \subset Q^{\mathrm{gs}}$. That is, for each $\mathfrak{m}^{\mathrm{gs}}$-primary ideal J, $\mathfrak{D}_{\mathrm{can}}/J$ and $\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J$ are wall structures.*

For two-dimensional log Calabi–Yau pairs, this lemma is a consequence of [21, Lemma 3.16 and Theorem 3.33]. Below we prove the lemma for HDTV scattering diagrams $\mathfrak{D}_{(Y_\Sigma,H)}$ satisfying the following assumption, which covers every case treated in this thesis.

Assumption 2.3.11. Let $\mathfrak{D}_{(Y_\Sigma,H)}$ be the HDTV scattering diagram associated to a toric model (Y_Σ, H) for a log Calabi–Yau pair. We assume that there exists an $\mathfrak{m}_P$-primary ideal $I \subset P$, and that for every maximal cone $\sigma \in \Sigma$ there exists a convex cone $\tau \subset M$ such that

(i) $\tau \cap \sigma = 0$, and

(ii) if $a_p z^p \in \Bbbk[P]$ is a monomial summand of a wall function $f_\mathfrak{d}$ for some wall $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_{(Y_\Sigma,H)}$ such that $\mathfrak{d} \subset \sigma$, then $r(p) \in M$ is contained in τ unless z^p is trivial modulo I.

Note that all HDTV scattering diagrams of dimension two automatically satisfy this assumption, as every wall in $\mathfrak{D}_{(Y_\Sigma,H)} \setminus \bigcup \mathfrak{D}_{ij}$ is an outgoing ray emanating from the origin. For the three-dimensional HDTV scattering diagram $\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$ constructed in Section 4.2.1, Assumption 2.3.11 holds as a consequence of Claims 4.1.13 and 4.1.14 – for more details see the proof of Lemma 4.3.5 in the case $\mathfrak{D} = \tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$.

Proof of Lemma 2.3.10 under Assumption 2.3.11. Fix an $\mathfrak{m}^{\mathrm{gs}}$-primary ideal $J \subset Q^{\mathrm{gs}}$. We prove that $\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J$ is a wall structure. Since the collection of walls $\Psi\mathfrak{D}_{\mathrm{can}}$ defined in Construction 2.3.15 is a wall structure modulo J if and only if $\mathfrak{D}_{\mathrm{can}}/J$ is a wall structure, it then follows from Lemma 2.3.16 that $\mathfrak{D}_{\mathrm{can}}/J$ is also a wall structure. It suffices to show that

(i) For every wall $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_{(Y_\Sigma,H)}$, the infinite sum of monomials $\alpha_*(f_\mathfrak{d})$ is a polynomial modulo J.

(ii) There are finitely many walls $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_{(Y_\Sigma,H)}$ such that $\alpha_*(f_\mathfrak{d}) \not\equiv 1 \mod J$.

Since there are finitely many maximal cells $\sigma \in \Sigma$, it suffices to prove these statements after replacing $\mathfrak{D}_{(Y_\Sigma,H)}$ with $\mathfrak{D}_{(Y_\Sigma,H)} \cap \sigma$. For statement (i), it suffices to show that there are only finitely many monomials $t^\beta z^m \in \Bbbk[Q^{\mathrm{gs}} \oplus M]$ that appear with nonzero coefficient in a wall function $\alpha_*(f_\mathfrak{d})$ for some $\mathfrak{d} \in \mathfrak{D}_{(Y_\Sigma,H)} \cap \sigma$ and are nontrivial modulo J. Here α_* is determined on σ by the representative for ψ which is zero on σ. If $t^\beta z^m$ appears in $\alpha_*(f_\mathfrak{d})$ then

$$t^\beta z^m = \alpha_*\left(\prod_{i=1}^{s}\prod_{j=1}^{s_i}\left(t_{ij}z^{m_i}\right)^{a_{ij}}\right)$$

for some collection $a_{ij} \in \mathbb{N}$. Let $I_\sigma := \{i \in [s] \mid \rho_i \subset \sigma\}$, where $[s] = \{1, \dots, s\}$. We have $\alpha_*(t_{ij}) = t^{-E_{ij}} \notin \mathfrak{m}^{\mathrm{gs}}$ for all $i \in I_\sigma$. Since every kink of ψ is contained in $\mathfrak{m}^{\mathrm{gs}}$, we have $\alpha_*(t_{ij}) \in \mathfrak{m}^{\mathrm{gs}}$ for all $i \in [s] \setminus I_\sigma$. There exists an $N \in \mathbb{N}$ such that $(\mathfrak{m}^{\mathrm{gs}})^N \subset J$, so

$$\sum_{i \in [s] \setminus I_\sigma} \sum_{j=1}^{s_i} a_{ij} \leq N,$$

which means that the area containing

$$\sum_{i \in [s] \setminus I_\sigma} \sum_{j=1}^{s_i} a_{ij} m_i \tag{2.11}$$

is bounded. Unless $t^\beta z^m$ is the image of one of the finitely many monomial appearing in $\mathfrak{D}_{(Y_\Sigma, H)}/I$, we must have $m \in \tau$ by Assumption 2.3.11. Here τ is a convex cone that depends only on σ, and so

$$\sum_{i \in I_\sigma} \sum_{j=1}^{s_i} a_{ij}$$

is also bounded, otherwise m cannot be contained in τ. □

Construction 2.3.12 (The algorithmic family $\mathfrak{X}(\alpha \mathfrak{D}_{(Y_\Sigma, H)})$)**.** The finite wall structure $\alpha \mathfrak{D}_{(Y_\Sigma, H)}/J$ is conical for all $\mathfrak{m}^{\mathrm{gs}}$-primary ideals J, and so by Theorem 1.5.1 it defines an affine scheme

$$\mathfrak{X}(\alpha \mathfrak{D}_{(Y_\Sigma, H)}/J) := \operatorname{Spec} R(\alpha \mathfrak{D}_{(Y_\Sigma, H)}/J)$$

over $\operatorname{Spec} R^{\mathrm{gs}}/J$. Via Construction 1.6.2, we can associate to the system of wall structures $\alpha \mathfrak{D}_{(Y_\Sigma, H)}$ a formal family

$$\mathfrak{X}\left(\alpha \mathfrak{D}_{(Y_\Sigma, H)}\right) := \operatorname{colim}_{\sqrt{J} = \mathfrak{m}^{\mathrm{gs}}} \mathfrak{X}(\alpha \mathfrak{D}_{(Y_\Sigma, H)}/J)$$

over the Gross–Siebert locus $\operatorname{Spf} R^{\mathrm{gs}}$.

Remark 2.3.13 (The canonical mirror family over the Gross–Siebert locus)**.** Lemma 2.3.10 above implies that the canonical family $\mathfrak{X}(\mathfrak{D}_{\mathrm{can}})$ over $\operatorname{Spf} R^\#$ is actually the pullback of a formal family over $\operatorname{Spf} R'$ via the horizontal morphisms in (2.9). That is,

$$\mathfrak{X}(\mathfrak{D}_{\mathrm{can}}/(J \cap Q)) := \operatorname{Spec} R(\mathfrak{D}_{\mathrm{can}}/(J \cap Q))$$

is a scheme over $\operatorname{Spec} R'/(J \cap Q)$ for all $\mathfrak{m}^{\mathrm{gs}}$-primary ideals $J \subset Q^{\mathrm{gs}}$, and

$$\mathfrak{X}(\mathfrak{D}_{\mathrm{can}}/I) = \mathfrak{X}\big(\mathfrak{D}_{\mathrm{can}}/(I \oplus E \cap Q)\big) \times_{\operatorname{Spec} R'/(I \oplus E \cap Q)} \operatorname{Spec} R^{\#}/I$$

for all $\mathfrak{m}$-primary ideals I. The canonical family over the Gross–Siebert locus $\operatorname{Spf} R^{\mathrm{gs}}$ can then be viewed as a subfamily of $\mathfrak{X}(\mathfrak{D}_{\mathrm{can}})$ via the vertical maps in (2.9):

$$\begin{aligned}\mathfrak{X}(\mathfrak{D}_{\mathrm{can}}/J) &:= \operatorname{Spec} R(\mathfrak{D}_{\mathrm{can}}/J)\\ &= \mathfrak{X}(\mathfrak{D}_{\mathrm{can}}/(J \cap Q)) \times_{\operatorname{Spec} R'/(J\cap Q)} \operatorname{Spec} R^{\mathrm{gs}}/J\end{aligned}$$

for all $\mathfrak{m}^{\mathrm{gs}}$-primary ideals J, and

$$\mathfrak{X}^{\mathrm{gs}}(\mathfrak{D}_{\mathrm{can}}) := \operatorname{colim}_{\sqrt{J}=\mathfrak{m}^{\mathrm{gs}}} \mathfrak{X}(\mathfrak{D}_{\mathrm{can}}/J).$$

Theorem 2.3.14 (The mirror families are isomorphic over the Gross–Siebert locus)**.** *Let (Y, D) be a log Calabi–Yau pair with toric model (Y_Σ, H). Then the schemes $\mathfrak{X}^{\mathrm{gs}}(\mathfrak{D}_{\mathrm{can}}/J)$ and $\mathfrak{X}(\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J)$ are isomorphic over the Gross–Siebert locus $\operatorname{Spec} R^{\mathrm{gs}}/J$ for all $\mathfrak{m}^{\mathrm{gs}}$-primary ideals J.*

Proof. In Construction 2.3.15 we construct a map Ψ of the canonical wall structure $\mathfrak{D}_{\mathrm{can}}$, restricted to the Gross–Siebert locus. By Lemma 2.3.16, $\Psi\mathfrak{D}_{\mathrm{can}}$ is a compatible system of wall structures on $(M_{\mathbb{R}}, \Sigma, \psi)$ which is equivalent to $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$. In particular, $\Psi\mathfrak{D}_{\mathrm{can}}$ is consistent and $R(\Psi\mathfrak{D}_{\mathrm{can}}/J) = R(\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J)$.

By Lemma 2.3.17, the map of wall structures Ψ induces isomorphisms of families over $\operatorname{Spec} R^{\mathrm{gs}}/J$ between $\mathfrak{X}^\circ(\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J)$ and $\mathfrak{X}^\circ(\mathfrak{D}_{\mathrm{can}}/J)$, where $\mathfrak{X}^\circ$ denotes the family constructed by gluing local charts in Proposition 1.3.4. We therefore have an isomorphism of (R^{gs}/J)-algebras

$$\Gamma\left(\mathfrak{X}^\circ(\mathfrak{D}_{\mathrm{can}}/J), \mathcal{O}_{\mathfrak{X}^\circ(\mathfrak{D}_{\mathrm{can}}/J)}\right) \longrightarrow \Gamma\left(\mathfrak{X}^\circ(\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J), \mathcal{O}_{\mathfrak{X}^\circ(\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J)}\right) \tag{2.12}$$

induced by Ψ.

By Theorem 1.5.1, these are freely generated by the theta functions ϑ_m, indexed by asymptotic monomials on $(B, \mathscr{P})$ and $(M_{\mathbb{R}}, \Sigma)$ respectively. However, since both polyhedral affine pseudomanifolds are conical, the asymptotic monomials are in bijection with the integral points. Therefore, the piecewise linear map $\nu : B \rightarrow$

$M_{\mathbb{R}}$ (2.3) induces a bijection between asymptotic monomials of $(B, \mathscr{P})$ and $(M_{\mathbb{R}}, \Sigma)$ given by

$$B(\mathbb{Z}) \ni m \longmapsto \nu(m) \in M.$$

In fact, the isomorphisms (2.12) are given by

$$\vartheta_m \longmapsto \vartheta_{\nu(m)}.$$

by Corollary 2.3.19. These isomorphisms therefore define an isomorphism of the completions of algebras with respect to $\mathfrak{m}^{\mathrm{gs}}$. □

Construction 2.3.15 (The wall structure $\Psi\mathfrak{D}_{\mathrm{can}}$)**.** Let (Y, D) be a log Calabi–Yau pair with toric model (Y_Σ, H). Then $\nu : B \longrightarrow M_{\mathbb{R}}$ (2.3) induces a map on the support of $\mathfrak{D}_{\mathrm{can}}$ – for any wall $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_{\mathrm{can}}$, we simply define

$$\Psi(\mathfrak{d}) := \nu(\mathfrak{d}). \tag{2.13}$$

By [5, Theorem 6.1], we can assume that the wall $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_{\mathrm{can}}$ is the image of some wall $(\mathfrak{d}', f_{\mathfrak{d}'}) \in \mathfrak{D}_{(Y_\Sigma, H)}$ under the map Υ from Construction 2.3.5. The definition of the wall function $f_{\nu(\mathfrak{d})}$ depends on whether $(\mathfrak{d}', f_{\mathfrak{d}'})$ is an incoming or outgoing wall of $\mathfrak{D}_{(Y_\Sigma, H)}$.

The restriction of ν to a maximal cell $\overline{\sigma} \in \mathscr{P}$ is a linear map; moreover, it is a linear isomorphism of smooth cones, since Y_Σ is a smooth toric variety. Thus ν induces a canonical isomorphism of integral tangent spaces

$$\nu_* : \Lambda_{\overline{\sigma}} \longrightarrow M. \tag{2.14}$$

Recall the 'canonical' MPA-function φ on $(B, \mathscr{P})$ defined in Construction 2.2.8 – we now consider φ to be taking values in $Q^{\mathrm{gs}} \supset Q$. Let $\mathcal{P}^+_\varphi$ be the sheaf of monoids on B associated to φ via Definition 1.2.13, and let $\mathcal{P}^+_\psi$ be the sheaf on $M_{\mathbb{R}}$ defined in Construction 2.3.12. Following the notation of Definition 1.2.13, we fix a representative φ_0 such that $\varphi_0|_{\overline{\sigma}}$ is the zero function. This gives the canonical identification

$$\left(\mathcal{P}^+_\varphi\right)_{\overline{x}} \cong \Lambda_{\overline{\sigma}} \times Q^{\mathrm{gs}}.$$

for any interior point $\overline{x} \in \operatorname{Int} \overline{\sigma}$, along with an induced isomorphism of monoids

$$\mu_\sigma : \left(\mathcal{P}^+_\varphi\right)_{\overline{x}} \xrightarrow{\sim} \left(\mathcal{P}^+_\psi\right)_x \tag{2.15}$$

given by

$$(m, p) \longmapsto (\nu_*(m),\ \psi|_\sigma(\nu_*(m)) + p).$$

Denoting the induced isomorphism of algebras by $(\mu_\sigma)_*$, we define

$$\Psi(f_{\mathfrak{d}}) := (\mu_\sigma)_*(f_{\mathfrak{d}}) \tag{2.16}$$

for all walls

$$(\mathfrak{d}, f_{\mathfrak{d}}) \in \Upsilon\left(\mathfrak{D}_{(Y_\Sigma,H)} \setminus \left(\bigcup_{i=1}^{s}\bigcup_{j=1}^{s_i} \mathfrak{D}_{ij}\right)\right) \quad \text{such that} \quad \mathfrak{d} \subset \sigma.$$

Now suppose that $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_{\text{can}}$ is the image of an incoming wall of $\mathfrak{D}_{(Y_\Sigma,H)}$, so $(\mathfrak{d}, f_{\mathfrak{d}}) \in \Upsilon\left(\mathfrak{D}_{ij}\right)$ for some $\mathfrak{D}_{ij}$. Then (by [5, (6.2)]) $(\mathfrak{d}, f_{\mathfrak{d}})$ takes the form $(\overline{\rho}, f_{\overline{\rho}})$, where $\rho \in \Sigma$ is a codimension one cell containing the ray $\rho_i \in \Sigma$, and

$$f_{\overline{\rho}} = \prod_{k=1}^{Y_\rho \cdot H_{ij}} \left(1 + t^{E_{ij}} z^{-m_i}\right).$$

Here $m_i \in \Lambda_{\overline{\rho}_i}$ positively generates $\overline{\rho}_i$, and E_{ij} is the class of an exceptional curve of the blow-up $\mathrm{Bl}_H : Y \longrightarrow Y_\Sigma$ over H_{ij}. We then define

$$\Psi(f_\rho) := (\mu_\sigma)_*\left(\prod_{k=1}^{Y_\rho \cdot H_{ij}} \left(1 + t^{-E_{ij}} z^{m_i}\right)\right), \tag{2.17}$$

for some $\sigma \in \mathscr{P}_{\max}$ containing ρ. Note that the direction of the wall has been flipped before applying $(\mu_\sigma)_*$ – it is only possible to invert E_{ij} when working over the Gross–Siebert locus. We define

$$\Psi\mathfrak{D}_{\text{can}} := \left\{(\nu(\mathfrak{d}), \Psi(f_{\mathfrak{d}})) \mid (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}_{\text{can}}\right\}.$$

Lemma 2.3.16 (Equivalence of $\Psi\mathfrak{D}_{\text{can}}$ and $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$)**.** *Let (Y, D) be a log Calabi–Yau* *pair with toric model* (Y_Σ, H)*. Then* $\Psi\mathfrak{D}_{\mathrm{can}}$ *is a system of consistent wall structures on* $(M_\mathbb{R}, \Sigma)$ *equipped with the* Q^{gs}*-valued piecewise linear function* ψ*. Moreover,* $\Psi\mathfrak{D}_{\mathrm{can}}$ *and* $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$ *are equivalent.*

Proof. Following through the definitions of the maps α, Υ and Ψ, one can check that

$$\alpha((\mathfrak{d}, f_\mathfrak{d})) = \Psi \circ \Upsilon((\mathfrak{d}, f_\mathfrak{d}))$$

for every wall $(\mathfrak{d}.f_\mathfrak{d}) \in \mathfrak{D}_{(Y_\Sigma,H)}$. It is clear that $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$ defines a system of wall structures on $(M_\mathbb{R}, \Sigma)$ equipped with the convex piecewise linear function ψ, and by functoriality of scattering (Lemma 2.1.11), $\alpha\mathfrak{D}_{(Y_\Sigma,H)}$ is a compatible scattering diagram on $M_\mathbb{R}$. Therefore, it defines a system of consistent wall structures. □

Lemma 2.3.17 (Ψ induces an isomorphism of local mirror families over the Gross–Siebert locus)**.** *Let* (Y, D) *be a log Calabi–Yau pair with toric model* (Y_Σ, H)*. Let* $J \subset R^{\mathrm{gs}}$ *be an* $\mathfrak{m}^{\mathrm{gs}}$*-primary ideal. Let*

$$\mathfrak{X}^\circ(\mathfrak{D}_{\mathrm{can}}/J) \longrightarrow \operatorname{Spec} R^{\mathrm{gs}}/J \quad \textit{and} \quad \mathfrak{X}^\circ(\Psi\mathfrak{D}_{\mathrm{can}}/J) \longrightarrow \operatorname{Spec} R^{\mathrm{gs}}/J$$

be the two families constructed from the wall structures $\mathfrak{D}_{\mathrm{can}}$ *and* $\Psi\mathfrak{D}_{\mathrm{can}}$ *by gluing local charts, as defined in Proposition 1.3.4. Then the map* Ψ *defined in Construction 2.3.15 induces an isomorphism*

$$\mathfrak{X}^\circ(\mathfrak{D}_{\mathrm{can}}/J) \xrightarrow{\sim} \mathfrak{X}^\circ(\alpha\mathfrak{D}_{(Y_\Sigma,H)}/J)$$

of schemes over $\operatorname{Spec} R^{\mathrm{gs}}/J$.

Proof. The map $\nu : B \to M_\mathbb{R}$ (2.3) induces a piecewise linear identification of the polyhedral subdivisions induced by $\mathfrak{D}_{\mathrm{can}}$ and $\Psi\mathfrak{D}_{\mathrm{can}}$ which, following the conventions of Notation 2.3.3, is given by

$$\mathscr{P}_{\mathfrak{D}_{\mathrm{can}}} \ni \overline{\tau} := \nu^{-1}(\tau) \longmapsto \tau \in \mathscr{P}_{\mathfrak{D}_{(Y_\Sigma,H)}}.$$

It is therefore enough to show that there are are isomorphisms of the rings defined in Construction 1.3.1

$$\mu_\mathfrak{b} : R_{\overline{\mathfrak{b}}} \longrightarrow R_\mathfrak{b} \quad \text{and} \quad \mu_\mathfrak{u} : R_{\overline{\mathfrak{u}}} \longrightarrow R_\mathfrak{u} \tag{2.18}$$

for every slab $\mathfrak{b} \in \mathscr{P}_{\mathfrak{D}_{(Y_\Sigma,H)}}$ and every chamber $\mathfrak{u} \in \mathscr{P}_{\mathfrak{D}_{(Y_\Sigma,H)}}$, such that $\mu_\mathfrak{b}$ and $\mu_\mathfrak{u}$ are compatible with the localisation homomorphisms (1.10)

$$\chi_{\overline{\mathfrak{b}},\overline{\mathfrak{u}}} : R_{\overline{\mathfrak{b}}} \longrightarrow R_{\overline{\mathfrak{u}}} \quad \text{and} \quad \chi_{\mathfrak{b},\mathfrak{u}} : R_\mathfrak{b} \longrightarrow R_\mathfrak{u}, \tag{2.19}$$

and compatible with the wall crossing automorphisms (1.11) across walls of codimension zero

$$\theta_{\overline{\mathfrak{d}}} : R_{\overline{\mathfrak{u}}} \longrightarrow R_{\overline{\mathfrak{u}}'} \quad \text{and} \quad \theta_\mathfrak{d} : R_\mathfrak{u} \longrightarrow R_{\mathfrak{u}'}. \tag{2.20}$$

Let $\sigma \in \Sigma$ be the maximal cell containing the chamber $\mathfrak{u} \in \mathscr{P}_{\Psi\mathfrak{D}_{\text{can}}}$. By definition,

$$R_{\overline{\mathfrak{u}}} = \Bbbk\left[\left(\mathcal{P}^+_\varphi\right)_{\overline{x}}\right]/I \quad \text{and} \quad R_\mathfrak{u} = \Bbbk\left[\left(\mathcal{P}^+_\psi\right)_x\right]/I,$$

and so the isomorphism of monoids μ_σ (2.15) induces an isomorphism

$$\mu_\mathfrak{u} := (\mu_\sigma)_* : R_{\overline{\mathfrak{u}}} \to R_\mathfrak{u}.$$

Moreover, for any pair of adjacent chambers $\mathfrak{u}, \mathfrak{u}' \subset \sigma$ separated by a codimension zero wall $\mathfrak{d} \subset \sigma$, we have that

$$f_\mathfrak{d} = \Psi(f_{\overline{\mathfrak{d}}}) = (\mu_\sigma)_*(f_{\overline{\mathfrak{d}}})$$

by definition of Ψ, and we also know that

$$\langle n_{\overline{\mathfrak{d}}}, m\rangle = \langle n_\mathfrak{d}, \nu_*(m)\rangle,$$

precisely because ν_* is an isomorphism of lattices. Thus $(\mu_\sigma)_*$ commutes with wall-crossing, and so

$$\mu_{\mathfrak{u}'} \circ \theta_{\overline{\mathfrak{d}}} = \theta_\mathfrak{d} \circ \mu_\mathfrak{u} \tag{2.21}$$

as required.

Now suppose that $\mathfrak{b}$ is a slab contained in $\rho \in \Sigma^{[n-1]}$. Since Y_Σ is a smooth toric variety, the fan Σ is a simplicial cone complex. We can thus assume that ρ is spanned by rays $\rho_1, \ldots, \rho_{n-1} \in \Sigma$, and is contained in the two maximal cells $\sigma_+ = \rho + \rho_+$ and $\sigma_- = \rho + \rho_-$, where ρ_+ and ρ_- are rays in Σ. To see the isomorphism $\mu_\mathfrak{b}$ induced by

Ψ, we write down the following presentations of $R_{\overline{\mathfrak{b}}}$ and $R_{\mathfrak{b}}$.

Let the primitive vectors $m_i, m_+, m_- \in M$ be positive generators of the rays $\rho_i, \rho_+, \rho_- \in \Sigma$ respectively, and denote the primitive tangent vectors positively generating $\overline{\rho}_i, \overline{\rho}_+, \overline{\rho}_-$ by $\overline{m}_i, \overline{m}_+, \overline{m}_-$. Fixing a single-valued representative for φ on $\overline{\sigma}_+ \cup \overline{\sigma}_-$, we define

$$\begin{aligned} \overline{X}_i &:= z^{(\overline{m}_i, \varphi(\overline{m}_i))}, \quad \overline{X}_+ := z^{(\overline{m}_+, \varphi(\overline{m}_+))}, \quad \overline{X}_- := z^{(\overline{m}_-, \varphi(\overline{m}_-))} \\ X_i &:= z^{(m_i, \psi(m_i))}, \quad X_+ := z^{(m_+, \psi(m_+))}, \quad X_- := z^{(m_-, \psi(m_-))}. \end{aligned} \tag{2.22}$$

These are generators of $\mathbb{R}_{\overline{\mathfrak{b}}}$ and $R_{\mathfrak{b}}$ as R^{gs}/J-algebras. By blowing up Y in the curve $\overline{Y}_\rho$ if necessary, we may assume that only one of the divisors D_i contains components of H. Let $l \in \{1, \ldots, n-1\}$ be the index of this divisor. We use the fact that $\mathfrak{D}_{\mathrm{can}} \equiv \Upsilon(\mathfrak{D}_{(Y_\Sigma, H)})$ in order to factorise $f_{\overline{x}}$ for $\overline{x} \in \mathrm{Int}\,\overline{\mathfrak{b}}$ as

$$f_{\overline{x}} = g_{\overline{\mathfrak{b}}} \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} \left(1 + t^{E_{lj}} z^{-\overline{m}_l}\right)\right)$$

where $g_{\overline{\mathfrak{b}}}$ is the image of outgoing walls of $\mathfrak{D}_{(Y_\Sigma, H)}$ under Υ. Then we have

$$R_{\overline{\mathfrak{b}}} = \frac{(R^{\mathrm{gs}}/J)\,[\overline{X}_1^{\pm}, \ldots, \overline{X}_{n-1}^{\pm}][\overline{X}_+, \overline{X}_-]}{\left(\overline{X}_+\overline{X}_- - t^{\overline{Y}_\rho} \left(\prod_{i=1}^{n-1} \overline{X}_i^{\left(-\overline{D}_i \cdot \overline{Y}_\rho\right)}\right) g_{\overline{\mathfrak{b}}} \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} \left(1 + t^{E_{lj}} \overline{X}_l^{-1}\right)\right)\right)}, \tag{2.23}$$

where $g_{\overline{\mathfrak{b}}} \in (R^{\mathrm{gs}}/J)[\overline{X}_1^{\pm}, \ldots, \overline{X}_{n-1}^{\pm}]$. Note that the presentation of $R_{\overline{\mathfrak{b}}}$ is independent of the choice of single valued-representative of φ. Using the equivalence of $\Psi\mathfrak{D}_{\mathrm{can}}$ and $\Psi \circ \Upsilon(\mathfrak{D}_{(Y_\Sigma, H)})$, we can write

$$f_x \equiv \Psi(f_{\overline{x}}) \equiv (\mu_{\sigma_+})_*(g_{\overline{\mathfrak{b}}}) \cdot \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} \left(1 + t^{-E_{lj}} z^{m_l}\right)\right) \mod J,$$

and so we have

$$R_{\mathfrak{b}} = \frac{(R^{\mathrm{gs}}/J)[X_1^{\pm}, \ldots, X_{n-1}^{\pm}][X_+, X_-]}{\left(X_+X_- - t^{\mathrm{Bl}^*(Y_\rho)} \left(\prod_{i=1}^{n-1} X_i^{\left(-D_i \cdot Y_\rho\right)}\right) g_{\mathfrak{b}} \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} \left(1 + t^{-E_{lj}} X_l\right)\right)\right)}, \tag{2.24}$$

where $g_{\mathfrak{b}} = (\mu_{\sigma_+})_*(g_{\overline{\mathfrak{b}}})$. With these presentations, the isomorphism $\mu_{\mathfrak{b}}$ is given by

$$\overline{X}_i \mapsto X_i, \quad ,\overline{X}_+ \mapsto X_+, \quad \overline{X}_- \mapsto X_-. \tag{2.25}$$

and the identity on R^{gs}/J. Indeed, using the identities

$$\overline{Y}_\rho = \mathrm{Bl}^*(Y_\rho) - \sum_{j=1}^{s_l} \sum_{k=1}^{Y_\rho \cdot H_{lj}} E_{lj}$$

and

$$\begin{aligned}
\overline{D}_i \cdot \overline{Y}_\rho &= \overline{D}_i \cdot \left(\mathrm{Bl}^*(Y_\rho) - \sum_{j=1}^{s_l} \sum_{k=1}^{Y_\rho \cdot H_{lj}} E_{lj} \right) \\
&= \mathrm{Bl}_* \left(\overline{D}_i \right) \cdot Y_\rho - \overline{D}_i \cdot \left(\sum_{j=1}^{s_l} \sum_{k=1}^{Y_\rho \cdot H_{lj}} E_{lj} \right) \\
&= \begin{cases} D_i \cdot Y_\rho - \sum_{j=1}^{s_l} Y_\rho \cdot H_{lj} & \text{if } i = l \\ D_i \cdot Y_\rho & \text{if } i \neq l, \end{cases}
\end{aligned}$$

we can rewrite the expression in (2.23) via

$$\begin{aligned}
& t^{\overline{Y}_\rho} \left(\prod_{i=1}^{n-1} \overline{X}_i^{\left(-\overline{D}_i \cdot \overline{Y}_\rho\right)} \right) g_{\overline{\mathfrak{b}}} \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} \left(1 + t^{E_{lj}} \overline{X}_l^{-1} \right) \right) \\
&= t^{Y_\rho} \left(\prod_{i=1}^{n-1} \overline{X}_i^{\left(-D_i \cdot Y_\rho\right)} \right) \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} t^{-E_{lj}} \overline{X}_l \right) g_{\overline{\mathfrak{b}}} \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} \left(1 + t^{E_{lj}} \overline{X}_l^{-1} \right) \right) \\
&= t^{Y_\rho} \left(\prod_{i=1}^{n-1} \overline{X}_i^{\left(-D_i \cdot Y_\rho\right)} \right) g_{\overline{\mathfrak{b}}} \left(\prod_{j=1}^{s_l} \prod_{k=1}^{Y_\rho \cdot H_{lj}} \left(1 + t^{-E_{lj}} \overline{X}_l \right) \right).
\end{aligned}$$

Finally, we note that

$$(\mu_{\sigma_+})_*(\overline{X}_+) = X_+, \quad (\mu_{\sigma_-})_*(\overline{X}_-) = X_-, \quad \text{and } (\mu_{\sigma_+})_*(\overline{X}_i) = (\mu_{\sigma_-})_*(\overline{X}_i) = X_i,$$

since the two representatives of φ involved in the definition of μ_{σ_+} and μ_{σ_-} agree on the codimension on cell $\overline{\rho}$. Therefore we have

$$\chi_{\mathfrak{b},\mathfrak{u}_+} \circ \mu_{\mathfrak{b}} = \mu_{\mathfrak{u}_+} \circ \chi_{\overline{\mathfrak{b}},\overline{\mathfrak{u}}_+} \quad \text{and} \quad \chi_{\mathfrak{b},\mathfrak{u}_-} \circ \mu_{\mathfrak{b}} = \mu_{\mathfrak{u}_-} \circ \chi_{\overline{\mathfrak{b}},\overline{\mathfrak{u}}_-} \tag{2.26}$$

as required. □

Lemma 2.3.18. *The piecewise linear function $\nu : B \longrightarrow M_{\mathbb{R}}$ (2.3) induces a bijection of broken lines:*

$$\left\{ \begin{array}{c} \text{broken lines on } \mathfrak{D}_{\mathrm{can}} \\ \text{with endpoint } p \text{ and} \\ \text{asymptotic monomial } m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{broken lines on } \alpha\mathfrak{D}_{(Y_\Sigma,H)} \\ \text{with endpoint } \nu(p) \text{ and} \\ \text{asymptotic monomial } \nu(m) \end{array} \right\}$$

$$\text{given by} \quad \beta \longmapsto \nu \circ \beta.$$

Corollary 2.3.19. *If $p \in B$ is a general point with $\nu(p)$ contained in the chamber $\mathfrak{u} \in \alpha\mathfrak{D}_{(Y_\Sigma,H)}$, then*

$$\mu_{\mathfrak{u}}(\vartheta_m(p)) = \vartheta_{\nu(m)}(\nu(p)).$$

Proof of Lemma 2.3.18. It is clear that $\nu \circ \beta$ must have endpoint $\nu(p)$. Since $\mathscr{P}$ and Σ are both conical, the asymptotic monomials are in bijection with integral points of B and $M_{\mathbb{R}}$ respectively. It is clear that the last domain of linearity of $\nu \circ \beta$ must be parallel to $\nu(m)$.

Suppose that $x \in (\infty, t_1)$ is a point in the last domain of linearity, and let $\mathfrak{u} \subset \sigma$ be the chamber of $\mathfrak{D}_{(Y_\Sigma,H)}$ containing $(\nu \circ \beta)(x)$. To consider $z^{\nu(m)}$ as a monomial in $R_{\mathfrak{u}}$, one takes the trivialisation of $\mathcal{P}^+_\psi$ corresponding to the choice of single valued representative of ψ which is identically zero on σ. Then we have

$$\mu_{\mathfrak{u}}(z^m) = z^{\nu(m)}.$$

It remains to show that bending of a broken line across each wall $\mathfrak{d} \in \alpha\mathfrak{D}_{(Y_\Sigma,H)}$ is the same as bending across $\overline{\mathfrak{d}} \in \mathfrak{D}_{\mathrm{can}}$. In other words, we need to show that change of chambers morphisms $\theta_{\mathfrak{u},\mathfrak{u}'}$ and $\theta_{\overline{\mathfrak{u}},\overline{\mathfrak{u}'}}$ of Definition 1.4.2 commute with the local isomorphisms of rings $\mu_{\mathfrak{u}}$ and $\mu_{\mathfrak{u}'}$ defined in the proof of Lemma 2.3.17 above. For two chambers separated by a codimension zero wall $\mathfrak{d}$, this follows directly from the compatibility of $\mu_{\mathfrak{u}}$ with $\theta_{\mathfrak{d}}$ and $\theta_{\overline{\mathfrak{d}}}$ (2.21). For two chambers $\mathfrak{u}_+$ and $\mathfrak{u}_-$ separated by a slab $\mathfrak{b}$, we note that $R^{\mathfrak{b}}_{\mathfrak{u}_+}$ embeds into $R_{\mathfrak{b}}$ as

$$R^{\mathfrak{b}}_{\mathfrak{u}_+} \cong (R^{\mathrm{gs}}/J)[X_1^{\pm},\dots,X_{n-1}^{\pm}][X_+] \subset R_{\mathfrak{b}}$$

in the notation of (2.24), on which the restriction of $\chi_{\mathfrak{b},\mathfrak{u}_-}$ is equal to $\theta_{\mathfrak{u}_+,\mathfrak{u}_-}$. The restriction of $\mu_{\mathfrak{b}}$ to

$$R^{\overline{\mathfrak{b}}}_{\overline{\mathfrak{u}}_+} \cong (R^{\mathrm{gs}}/J)[\overline{X}_1^{\pm},\dots,\overline{X}_{n-1}^{\pm}][\overline{X}_+] \subset R_{\overline{\mathfrak{b}}}$$

is equal to $\mu_{\mathfrak{u}_+}$. Therefore compatibility of the change of chambers morphisms with local isomorphisms

$$\mu_{\mathfrak{u}_-} \circ \theta_{\overline{\mathfrak{u}}_+,\overline{\mathfrak{u}}_-} = \theta_{\mathfrak{u}_+,\mathfrak{u}_-} \circ \mu_{\mathfrak{u}_+}$$

follows from (2.26). □

2.4 Compactifying the mirror family

Remark 2.4.1 (Structure constants in terms of punctured Gromov–Witten invariants)**.** In Construction 2.2.12 the wall functions in the canonical wall structure $\mathfrak{D}_{\mathrm{can}}$ associated to a log Calabi–Yau pair (Y,D) are defined in terms of punctured Gromov–Witten invariants. The structure constants (1.17) of the algebra of theta functions $R(\mathfrak{D}_{\mathrm{can}})$ associated to $\mathfrak{D}_{\mathrm{can}}$ can also be expressed in terms of these invariants. By [31, Theorem 6.1] we can write

$$\alpha_m(m_1,m_2) = \sum_{A\in Q} N^A_{m_1 m_2 m} t^A \;\in \Bbbk[Q]/I \tag{2.27}$$

where the numbers $N^A_{m_1 m_2 m}$ are certain punctured Gromov–Witten invariants defined in [30, Definition 3.21], counting punctured maps into (Y,D) with curve class A and contact orders m_1, m_2 and $-m$.

Construction 2.4.2 (Adding a boundary to a conical wall structure)**.** Suppose that $\mathfrak{D}$ is a consistent, *conical* wall structure on $(B,\mathscr{P})$. By Theorem 1.5.1 we have an affine $\Bbbk[Q]/I$-family $\mathfrak{X}(\mathfrak{D}) := \operatorname{Spec} R(\mathfrak{D})$ containing the locally constructed family $\mathfrak{X}^\circ(\mathfrak{D})$ from Proposition 1.3.4, where $R(\mathfrak{D})$ is the algebra of theta functions on $\mathfrak{D}$. Our goal is to construct a compactification of $\mathfrak{X}(\mathfrak{D})$.

It is clear that $\mathfrak{X}^\circ(\mathfrak{D})$ can be partially compactified by truncating (some of) the conical chambers $\mathfrak{u} \in \mathscr{P}_{\mathfrak{D}_{\mathrm{can}}}$, turning them into (partially) bounded chambers which we will denote by $\overline{\mathfrak{u}}$. In other words, one "adds a boundary" to the underlying conical affine pseudomanifold $(B, \mathscr{P})$ to obtain a wall structure $\overline{\mathfrak{D}}$ on a (partially) bounded affine pseudomanifold $(\overline{B}, \overline{\mathscr{P}})$. The open embedding

$$\mathfrak{X}^\circ(\mathfrak{D}) \hookrightarrow \mathfrak{X}^\circ(\overline{\mathfrak{D}})$$

is given by the natural isomorphisms of affine charts

$$\operatorname{Spec} R_{\mathfrak{u}} \xrightarrow{\sim} \operatorname{Spec} R_{\overline{\mathfrak{u}}}.$$

The boundary $\mathfrak{X}^\circ(\overline{\mathfrak{D}}) \setminus \mathfrak{X}^\circ(\mathfrak{D})$ is given by the zero set of the ideal

$$\langle z^m \mid m \notin \Lambda_{\mathfrak{b}} \rangle \subset R^{\partial}_{\overline{\mathfrak{u}}} = (\Bbbk[Q]/I)\,[\Lambda_{\overline{\mathfrak{u}},\mathfrak{b}}],$$

in the boundary chart $\operatorname{Spec} R^{\partial}_{\overline{\mathfrak{u}}}$ (1.8) – here $\mathfrak{b}$ is the boundary slab contained in $\overline{\mathfrak{u}}$.

Proposition 1.4.12 implies that the boundary joints of $\overline{\mathfrak{D}}$ are consistent if their are convex in the sense of Definition 1.4.11. All interior joints of $\overline{\mathfrak{D}}$ are joints of $\mathfrak{D}$, and so adding any boundary to B inducing only convex boundary joints in $\overline{\mathfrak{D}}$ will result in a consistent (partially) bounded wall structure $\overline{\mathfrak{D}_{\mathrm{can}}}$.

Let us assume that $\overline{\mathfrak{D}}$ is consistent. As long as the boundary is a rational polyhedral set, the cone over $(\overline{B}, \overline{\mathscr{P}})$ will be an integral affine pseudomanifold. In this case $\mathbf{C}\overline{\mathfrak{D}}$ is a consistent, conical wall structure, and its associated algebra of theta functions $R(\mathbf{C}\overline{\mathfrak{D}})$ has a natural grading, given by the height of asymptotic monomial $m \in \mathbf{C}B(\mathbb{Z})$ associated to each theta function ϑ_m. By Theorem 1.5.7 there is a partial compactification

$$\mathfrak{X}^\circ(\overline{\mathfrak{D}}) \hookrightarrow \mathfrak{X}(\overline{\mathfrak{D}}) := \operatorname{Proj} R(\mathbf{C}\overline{\mathfrak{D}}).$$

In fact,

Claim 2.4.3. $\mathfrak{X}(\overline{\mathfrak{D}})$ *is a partial compactification of* $\mathfrak{X}(\mathfrak{D})$.

Proof. Indeed, $\mathfrak{D}$ is conical, so the origin must be contained in $\overline{\mathfrak{D}}$, and no broken line on $\mathbf{C}\overline{\mathfrak{D}}$ with asymptotic monomial $(0,k) \in \mathbf{C}\overline{B}(\mathbb{Z})$, where $k \in \mathbb{N}$, may cross a wall in

$\mathbf{C}\overline{\mathfrak{D}}$ before it reaches its endpoint. Therefore we must have

$$\vartheta_{(0,k)} \cdot \vartheta_{(m,l)} = \vartheta_{(m,k+l)}$$

for any asymptotic monomial $(m,l) \in \mathbf{C}\overline{B}(\mathbb{Z})$, and so there is an isomorphism

$$R(\mathbf{C}\overline{\mathfrak{D}})_{\vartheta_{(0,1)}} \cong R(\mathfrak{D})$$

given by $\vartheta_{(m,l)}/(\vartheta_{(0,1)})^l \longmapsto \vartheta_m$. □

Since all the walls of $\mathfrak{D}_{\mathrm{can}}$ are outgoing, Proposition 1.4.12 implies that a truncation $\overline{\mathfrak{D}_{\mathrm{can}}}$ is consistent if the boundary is locally convex. We will see that good truncations of $\mathfrak{D}_{\mathrm{can}}$ can be obtained from nef divisors in Y which are supported on D. Construction 2.4.5 below is from [30, Construction 1.20] and was motivated by [22, Section 8.5]. It requires the following lemma – recall that $D \subset Y$ is an snc divisor with connected strata.

Lemma 2.4.4. *[30, Corollary 1.14] Suppose that $f : (C, x_1, \ldots, x_n) \to Y$ is a punctured map with contact orders $u_1, \ldots, u_n$, where u_i is an integral tangent vector to a cone $\sigma_i \in \Sigma(Y)$. Then for any divisor D' supported on D, we have*

$$\deg f^*\mathcal{O}_Y(D') = \sum_i \langle u_i, D' \rangle.$$

Construction 2.4.5 (The Rees construction)**.** Suppose that D' is a nef divisor supported on D. Then for a punctured map with three marked points with contact orders m_1, m_2 and $-m$ and total curve class A, we have

$$0 \le A \cdot D' = \langle m_1, D' \rangle + \langle m_2, D' \rangle - \langle m, D' \rangle.$$

Therefore we have

$$\langle m, D' \rangle \le \langle m_1, D' \rangle + \langle m_2, D' \rangle \tag{2.28}$$

for any theta function ϑ_m appearing in the expansion of $\vartheta_{m_1} \cdot \vartheta_{m_2}$, which gives rise to a filtered ring structure on $R(\mathfrak{D}_{\mathrm{can}})$. We can thus take the graded Rees algebra

$$\tilde{R}_{D'}(\mathfrak{D}_{\mathrm{can}}) = \bigoplus_{k\in N} \bigoplus_{\substack{m\in B(\mathbb{Z}) \\ \langle m,D'\rangle\leq k}} (\Bbbk[Q]/I)\cdot \vartheta_m X^k \subseteq R(\mathfrak{D}_{\mathrm{can}})[X],$$

where the grading is given by the power of X.

Remark 2.4.6 (The truncation $\overline{\mathfrak{D}_{\mathrm{can}}}$ induced by D')**.** The Rees construction for a choice of nef divisor D' supported on D is equivalent to Construction 2.4.2 in the following sense. Recall that since $\Sigma(Y)$ is conical, the asymptotic monomials are in bijection with integral points on B. Thus the function $\langle -, D'\rangle$ induces a $\Sigma(Y)$-piecewise linear function

$$\psi_{D'} : B \longrightarrow \mathbb{R},$$

where we consider each integral point $m \in \sigma \in \Sigma(Y)$ to be a positive tangent vector on σ. If we define our truncated affine pseudomanifold to be

$$\overline{B} := \{b \in B \,|\, \psi_{D'}(b) \leq 1\}\,,$$

with corresponding wall structure $\overline{\mathfrak{D}_{\mathrm{can}}} := \mathfrak{D}_{\mathrm{can}} \cap \overline{B}$, then there is an isomorphism of graded $\Bbbk[Q]/I$-algebras

$$\tilde{R}_{D'}(\mathfrak{D}_{\mathrm{can}}) \cong R(\mathbf{C}\overline{\mathfrak{D}_{\mathrm{can}}}),$$

given by $\vartheta_m X^k \longmapsto \vartheta_{(m,k)}$. Note that consistency at the boundary joints of $\mathfrak{D}_{\mathrm{can}}$ is implied by (2.28), and $\mathbf{C}\overline{B}$ is an integral polyhedral affine pseudomanifold if $h_i \in \mathbb{Q}$ for all $i = 1,\ldots,r$.

Chapter 3

Mirrors to f and g

We now apply all of this construction in our context: the goal of this chapter is the construction of a mirror scheme to a given rigid maximally mutable Laurent polynomial F. When F is a Laurent polynomial in two variables, we show that this mirror is isomorphic to the scheme constructed from an algorithmic wall structure as in Construction 2.3.9. This will enable the comparison of mirrors to different rigid MMLPs in Chapter 4, via comparison of the affine geometry of the initial scattering diagrams. The methods and constructions in this chapter all apply to a general rigid MMLP F in two variables. We discuss their application to

$$f := x + y + \frac{1}{xy} \quad \text{and} \quad g := x + y + \frac{1}{xy} + xy$$

in examples, and explain where the method can be applied to rigid MMLPs in more variables in remarks.

The first step is to pass from a Laurent polynomial F in two variables to a log Calabi–Yau pair (Y, D).

3.1 From Laurent polynomials to log Calabi–Yau pairs

Construction 3.1.1 (A toric variety associated to F, and a line bundle on it)**.** Let $F \in \Bbbk[x^{\pm}, y^{\pm}]$ be a Laurent polynomial, and let $P := \operatorname{Newt} F$ be its Newton polytope. Denote by $\Sigma[P]$ the normal fan to P, and let $Y_{\Sigma[P]}$ denote the corresponding toric variety. The Newton polytope P defines a very ample line bundle on $Y_{\Sigma[P]}$, with divisor $\sum_i h_i D_i$ supported on the toric boundary. Here $h_i > 0$ is the lattice height of the origin above the facet of P corresponding to the toric divisor D_i.

Construction 3.1.2 (A rational map $Y_{\Sigma[P]} \dashrightarrow \mathbb{P}^1$)**.** Observe that both F and the unit monomial define sections of the line bundle in Construction 3.1.1. Therefore, there is a rational map $Y_{\Sigma[P]} \dashrightarrow \mathbb{P}^1$ defined by $[1 : F]$.

Construction 3.1.3 (The log Calabi–Yau pair associated to F)**.** The toric variety $Y_{\Sigma[P]}$ is in general singular, and the rational map $Y_{\Sigma[P]} \dashrightarrow \mathbb{P}^1$ has basepoints. Let Y be the variety obtained from $Y_{\Sigma[P]}$ by taking the minimal resolution of singularities and resolving basepoints, and let D be the proper transform of the toric boundary under the resolution of singularities, and the strict transform under the resolution of basepoints. Then D is anticanonical, and the pair (Y, D) is log Calabi–Yau. We say that (Y, D) is the log Calabi–Yau pair associated to F.

Remark 3.1.4. Since we are working in two dimensions, the resolution of $Y_{\Sigma[P]}$ in Construction 3.1.3 is unique. In higher dimensions, one would need to make a choice of resolution of the ambient space, and any two minimal choices would be related by flops. However, since these flops are log birational transformations – they only affect the boundary of the pair D without affecting $U = Y \setminus D$ – the choice of minimal resolution does not affect the associated canonical wall structure. One may therefore apply the three constructions above to a rigid MMLP in any number of variables and obtain a log Calabi–Yau pair.

Henceforth we restrict our attention to Laurent polynomials F which are *rigid maximally mutable* [15, Definitions 2.5 and 2.6]. This is the class of Laurent polynomials which provides mirrors to smooth del Pezzo surfaces [4], and which, in higher dimensions, are expected to provide mirrors to n-dimensional Fano varieties with terminal locally toric qG-rigid singularities that admit a qG-degeneration with reduced fibres to a normal toric variety [15, Conjecture 5.1].

Example 3.1.5. The rigid maximally mutable Laurent polynomial

$$F = \frac{(1+x)^4(1+y)^2}{x^2y} - 12$$

is a mirror to the del Pezzo surface dP_2. It has Newton polytope shown in Figure 3.1. The lattice points in Figure 3.1 are labelled with the coefficient of the corresponding monomial in F. Note the binomial coefficients along the edges.

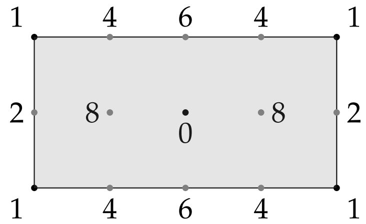

Figure 3.1: Newton polytope for a Laurent polynomial mirror to dP_2.

Lemma 3.1.6 below, which is a consequence of Propositions 3.7 and 3.9 in [15], allows us to describe the log Calabi–Yau pair (Y, D) associated to a rigid maximally mutable Laurent polynomial F more explicitly.

Lemma 3.1.6. *Suppose that $F \in \Bbbk[x^{\pm}, y^{\pm}]$ is a rigid maximally mutable Laurent polynomial. Then F has binomial coefficients along each edge E of* Newt F*, and the lattice length l_E of each edge is divisible by its height over the origin h_E. Moreover, the rational map $Y_\Sigma \dashrightarrow \mathbb{P}^1$ associated to F via Construction 3.1.2 is resolved by blowing up $n_E := l_E/h_E$ times in a point on the toric boundary component of Y_Σ corresponding to each edge E.*

Proof. Let $P = \operatorname{Newt} F$ be the Newton polytope of a Laurent polynomial $F \in \mathbb{C}[x, y]$. Proposition 3.7 in [15] associates to P a set of Laurent polynomials with Newton polytope P, zero constant term, and fixed mutation graph $\mathcal{G}_P$. By [15, Proposition 3.9], F is maximally mutable if and only if it coincides with the general Laurent polynomial f_P associated to P by [15, Proposition 3.7].

But by definition of rigidity, F is the unique Laurent polynomial with Newton polytope P, zero constant term and mutation graph $\mathcal{G}_P$. Therefore there are no free parameters in the coefficients of the general f_P. By [15, Proposition 3.7], the free parameters in the construction of f_P are in bijection with any *choice of residual points* (see [15, Definition 3.6]) of P. Since there are no free parameters, any choice of residual points must be empty. Therefore there are no R-cones in the spanning fan of P – that is, the length of any edge of P is divisible by its height over the origin.

Given an edge E of P, fix a primitive vector v_E in the direction of E and denote by w_E the primitive inward-pointing normal vector to E. Let l_E be the length of E, let h_E be its height over the origin, and define $n_E := l_E/h_E$. Let $C_E \cong \mathbb{P}^1$ be the toric boundary component of the (desingularisation of) Y_Σ corresponding to E. By [15, Proposition 3.7], $F = f_P$ is *mutable with respect to* the pair $(w_E, (1 + x^{v_E})^{n_E})$

(see [15, Definition 1.6]). This means that, in toric local coordinates (z, w), where z is a local coordinate along C_E, the rational map $[1 : F]$ takes the form

$$\left[w^{h_E} g(z) : \sum_{i=1}^{h_E} w^{(h_E - i)} (1+z)^{(i \cdot n_E)} g_i(z,w) + O(w^{h_E}) \right],$$

where $g_i(z, w) \in \Bbbk[w_E^{\pm}]$. In these coordinates C_E is given by the locus $w = 0$. The basepoints of the rational map $Y_\Sigma \dashrightarrow \mathbb{P}^1$ occur where F and the unit monomial simultaneously vanish – that is, at the points on the toric boundary where F vanishes. From the expression above, we see that there is a unique zero of F on C_E at $z = -1$, and that the rational map $[1 : F]$ is resolved by blowing up n_E times successively at $z = -1$ on C_E. □

This analysis also makes clear that the resulting variety Y is deformation equivalent to the blow-up of the desingularisation of $Y_{\Sigma[P]}$ in n_E distinct general points on each component C_E of the toric boundary. One can construct this deformation by taking a deformation of F which has n_E roots of multiplicity h_E on the toric boundary component C_E.

Remark 3.1.7. For a rigid MMLP in more than two variables, the base locus of the map $[1 : F]$ will be a subvariety of codimension one in the toric boundary of $Y_{\Sigma[P]}$. It is not guaranteed to be irreducible or reduced.

Example 3.1.8. In this thesis we will consider the two Laurent polynomials

$$f := x + y + \frac{1}{xy} \quad \text{and} \quad g := x + y + xy + \frac{1}{xy},$$

which are mirrors to the del Pezzo surfaces $\mathbb{P}^2$ and dP_8, the blow-up of $\mathbb{P}^2$ in a point, respectively. We describe their associated log Calabi–Yau pairs (Y_f, D_f) and (Y_g, D_g) via Construction 3.1.3 explicitly as follows.

The two Newton polytopes in the lattice $N \cong \mathbb{Z}^2$ are shown in Figures 3.2 and 3.3 respectively, with the origin coloured in black.

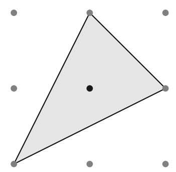

Figure 3.2: Newton polytope for $f = x + y + \frac{1}{xy}$

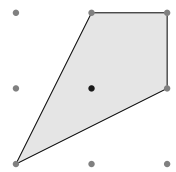

Figure 3.3: Newton polytope for $g = x + y + xy + \frac{1}{xy}$

Since each edge of $\mathrm{Newt}\, f$ and $\mathrm{Newt}\, g$ has lattice length one, the rational map to $\mathbb{P}^1$ is resolved by blowing up three and four distinct points respectively – one on each component of the respective toric boundaries of $Y_{\Sigma[\mathrm{Newt}\, f]}$ and $Y_{\Sigma[\mathrm{Newt}\, g]}$.

The toric variety $Y_{\Sigma[\mathrm{Newt}\, f]}$ is isomorphic to $\mathbb{P}^2/\mu_3$ – each vertex of $\mathrm{Newt}\, f$ corresponds to one of the three singular points. Each of these points corresponds to a singular cone in $\Sigma[\mathrm{Newt}\, f]$, and is resolved by performing the toric blow-ups determined by adding rays to $\Sigma[\mathrm{Newt}\, f]$ in the interior of said cone, decomposing it into three smooth cones. The fan Σ_f defining the desingularisation of $Y_{\Sigma[\mathrm{Newt}\, f]}$ is shown in Figure 3.4. It is a refinement of $\Sigma[\mathrm{Newt}\, f]$ – the rays in $\Sigma_f \setminus \Sigma[\mathrm{Newt}\, f]$, which define the blow-ups desingularising $Y_{\Sigma[\mathrm{Newt}\, f]}$, are shown in grey. The divisor $D_f \subset Y_f$ has nine components, one corresponding to each ray in Σ_f.

The toric variety $Y_{\Sigma[\mathrm{Newt}\, g]}$ has three singular points, one corresponding to each vertex of $\mathrm{Newt}\, g$ apart from the top right vertex $(1, 1)$. The fan defining the desingularisation of $Y_{\Sigma[\mathrm{Newt}\, g]}$ is shown in Figure 3.5. The divisor $D_g \subset Y_g$ is made up of eight irreducible components. Both Newton polytopes are reflexive, and correspond to the anticanonical divisor on the desingularised toric varieties Y_{Σ_f} and Y_{Σ_g}.

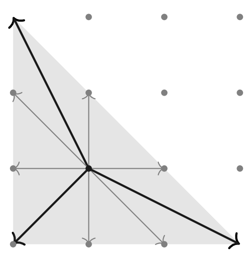

Figure 3.4: The fan Σ_f, the rays defining $\Sigma[\mathrm{Newt}\, f]$ shown in black

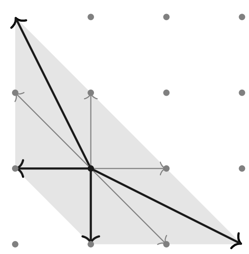

Figure 3.5: The fan Σ_g, the rays defining $\Sigma[\mathrm{Newt}\, g]$ shown in black

3.2 The canonical wall structure associated to a Laurent polynomial

Now consider a log Calabi–Yau pair (Y, D) obtained from a rigid maximally mutable Laurent polynomial F as in Construction 3.1.3 above. The canonical wall structure $\mathfrak{D}_{\mathrm{can}}$ associated to (Y, D) has the following description.

Let Σ be the fan associated to the toric desingularisation of $Y_{\Sigma[P]}$ from Construction 3.1.1. Note that $P = \mathrm{Newt}\, F$ defines a line bundle on Y_Σ with divisor

$$D_{\mathrm{Newt}\, f} := \sum_{\rho \in \Sigma^{[1]}} h_\rho D_\rho, \tag{3.1}$$

where D_ρ is the toric boundary component of Y_Σ corresponding to the ray ρ. Here h_ρ is the height above the origin of the edge of P corresponding to D_ρ for any ray $\rho \in \Sigma \cap \Sigma$; and for rays $\rho \in \Sigma \setminus \Sigma[P]$ not corresponding to an edge of P, we define $h_\rho \in \mathbb{N}$ to be the height above the origin of the unique affine hyperplane $H \subset \mathbb{R}^2$ normal to ρ such that H intersects the boundary of P and ρ points from H into P. Now considering F and the unit monomial to be a sections of this line bundle, let

$$\mathrm{Bl} : Y \longrightarrow Y_\Sigma \tag{3.2}$$

denote the blow-up resolving the basepoints of $[1 : F]$ as a rational map $Y_\Sigma \dashrightarrow \mathbb{P}^1$.

The tropicalisation of the pair (Y, D), defined in Construction 2.2.4, can be identified with Σ as a polyhedral complex. The affine structure around each ray $\rho \in \Sigma$ is defined by the self-intersection of the component of D associated to ρ: see Construction 2.2.5. This affine structure has a unique singularity, at the unique dimension zero cell of Σ. The Q-valued MPA function defined in Construction 2.2.8 has kink across ρ given by the curve class of the divisor component corresponding to ρ. All the walls in $\mathfrak{D}_{\mathrm{can}}$ are rays emanating from the vertex in Σ, refining the fan Σ.

Example 3.2.1. Consider the Laurent polynomials f and g as in our running example (Example 3.1.8). Every component of both D_f and D_g has self-intersection -2, so the polyhedral affine pseudomanifolds B_f and B_g have the universal covers shown in Figures 3.6 and 3.7. Each universal cover consists of the open upper half plane with a polyhedral decomposition given by rays generated by $(i, 1)$ for all $i \in \mathbb{Z}$. The deck transformations shift $(i, 1)$ to $(i + 9n, 1)$ and $(i + 8n, 1)$ respectively.

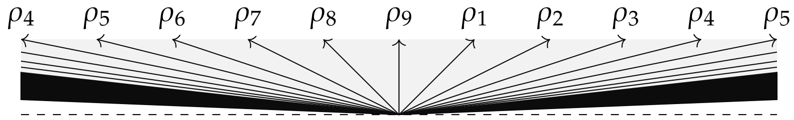

Figure 3.6: The universal cover of B_f, with each ray labelled by the ray $\rho_i \in \mathscr{P}_f := \Sigma(Y_f)$ it covers

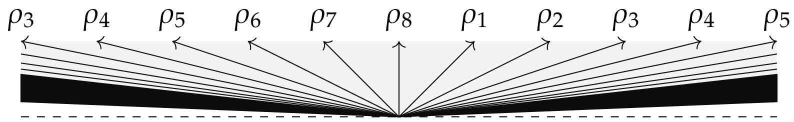

Figure 3.7: The universal cover of B_g with each ray labelled by the ray $\rho_i \in \mathscr{P}_g := \Sigma(Y_g)$ it covers

Recall that our log Calabi–Yau pair (Y, D) is obtained from the desingularisation Y_Σ by a sequence of infinitely-near (non-toric) blow-ups on boundary components of Y_Σ. Lemma 3.9 in [21] shows that the canonical wall structure is unchanged if we replace each chain of n infinitely-near blow-ups at p with the blow-up in n distinct general points of the boundary component that contains p. Note that the argument in [21, Lemma 3.9] is phrased in terms of relative Gromov–Witten invariants, whereas our canonical scattering diagram is defined in terms of punctured Gromov–Witten invariants. In this setting, these invariants coincide.

3.3 The algorithmic wall structure

By the discussion above, we can consider the canonical wall structure $\mathfrak{D}_{\mathrm{can}}(F)$ associated to a maximally mutable Laurent polynomial F to be the canonical wall structure associated to a log Calabi–Yau pair with a toric model as in the set-up of Construction 2.3.4. Here the toric model for the log Calabi–Yau pair (Y, D) is given by (Y_Σ, H_F), where Y_Σ is the desingularisation of the toric variety $Y_{\Sigma[P]}$ associated to F, and the centre of the blow-up

$$H_F := \bigcup_i \bigcup_{j=1}^{n_i} H_{ij}$$

is a union of $n_i = l_i/h_i$ general points on each component D_i of the toric boundary of Y_Σ, where l_i is the length of the corresponding edge of $\operatorname{Newt} F$ and h_i is its height over the origin – if D_i does not correspond to an edge of $\operatorname{Newt} F$ then $n_i = 0$. Thus $\mathfrak{D}_{\text{can}}(F)$ can be defined algorithmically using Constructions 2.3.4 and 2.3.5.

The HDTV scattering diagram from Construction 2.3.4 is simply the result of scattering of an initial scattering diagram which we will denote by $\mathfrak{D}_{\text{init}}(F)$:

$$\mathfrak{D}_{(Y_\Sigma,H_F)} = \operatorname{Scatter}\left(\mathfrak{D}_{\text{init}}(F)\right).$$

This initial scattering diagram simply consists of n_i incoming rays

$$\mathfrak{d}_{ij} := (\rho_i, 1 + t_{ij}z^{m_i}), \quad j = 1, \ldots, n_i$$

along each ray $\rho_i \in \Sigma[P] \cap \Sigma$ corresponding to an edge of $\operatorname{Newt} F$, where $m_i \in M$ is the primitive generator of ρ_i.

We denote the canonical mirror family associated to f by

$$\mathfrak{X}_f^{\text{can}} := \mathfrak{X}(\mathfrak{D}_{\text{can}}).$$

This is a formal scheme over $\operatorname{Spf} R^{\#}$, but when considered as a formal scheme over the Gross–Siebert locus $\operatorname{Spf} R^{\text{gs}}$, it is isomorphic to

$$\mathfrak{X}_f := \mathfrak{X}(\alpha\mathfrak{D}_{(Y_\Sigma,H_f)}) \tag{3.3}$$

by Theorem 2.3.14.

Example 3.3.1 (The initial scattering diagrams associated to f and g)**.** Let f and g be the two Laurent polynomials from our running example (see Examples 3.1.8 and 3.2.1). The initial scattering diagrams for the algorithmic construction of the canonical wall structures associated to f and g are pictured below. Henceforth we will denote the canonical scattering diagrams on $(B_f, \mathscr{P}_f)$ and $(B_g, \mathscr{P}_g)$ by $\mathfrak{D}_{\text{can}}(f)$ and $\mathfrak{D}_{\text{can}}(g)$, and the algorithmic scattering diagrams on $M_\mathbb{R}$ by

$$\mathfrak{D}_{(\Sigma_f,H_f)} = \operatorname{Scatter}(\mathfrak{D}_{\text{init}}(f)) \quad \text{and} \quad \mathfrak{D}_{(\Sigma_g,H_g)} = \operatorname{Scatter}(\mathfrak{D}_{\text{init}}(g)).$$

They are scattering diagrams for the data $Q^{abc} := M \oplus \mathbb{N}^3$ and $Q^{abcd} := M \oplus \mathbb{N}^4$ respectively, where we denote the associated rings by $\Bbbk[x^{\pm}, y^{\pm}][a,b,c]$ and $\Bbbk[x^{\pm}, y^{\pm}][a,b,c,d]$. We will denote the monoids defining the Gross–Siebert locus in each case by

$$Q_f^{\mathrm{gs}} := \mathrm{Bl}^*_{H_f}\left(NE\left(Y_{\Sigma_f}\right)\right) \oplus E_{abc} \quad \text{and} \quad Q_g^{\mathrm{gs}} := \mathrm{Bl}^*_{H_g}\left(NE\left(Y_{\Sigma_g}\right)\right) \oplus E_{abcd}, \tag{3.4}$$

where the exceptional curves generating the groups E_{abc} and E_{abcd} are indexed by a, b, c and d.

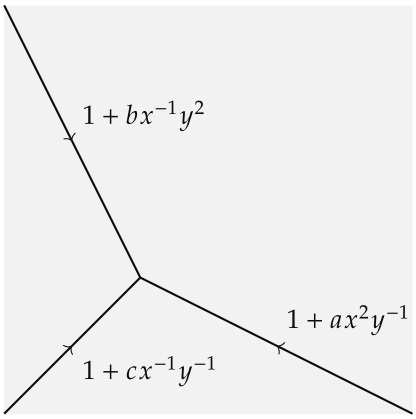

Figure 3.8: The initial scattering diagram $\mathfrak{D}_{\mathrm{init}}(f)$

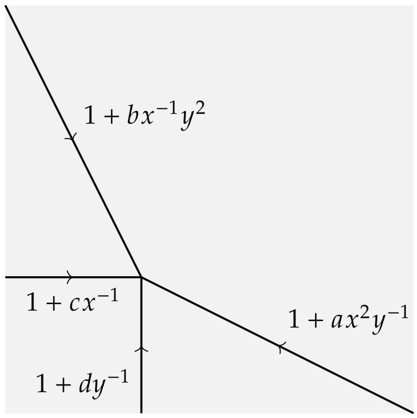

Figure 3.9: The initial scattering diagram $\mathfrak{D}_{\mathrm{init}}(g)$

Remark 3.3.2. In general, the log Calabi–Yau pair associated to a rigid MMLP F in more than two variables will not have a toric model in the sense of Definition 2.3.1, and proving it is deformation equivalent to a pair with a toric model is more complicated. However, the HDTV construction can be applied if the map $[1 : F]$ is resolved by a single blow-up in the base locus H, which takes the form described in Definition 2.3.1.

3.4 Compactifying the mirrors

Since the log Calabi–Yau pair (Y, D) constructed from a rigid maximally mutable Laurent polynomial F via Construction 3.1.3 has a toric model (Y_Σ, H_F), both the canonical scattering diagram $\mathfrak{D}_{\mathrm{can}}$ and the algorithmic scattering diagram $\mathfrak{D}_{(Y_\Sigma, H_F)}$ are conical wall structures. There is a canonical choice of nef divisor supported on D_F, in order to compactify the affine mirror families $\mathfrak{X}(\mathfrak{D}_{\mathrm{can}})$ and $\mathfrak{X}(\mathfrak{D}_{(Y_\Sigma, H_F)})$ by applying Constructions 2.4.5:

Denote by $\overline{D}_{\operatorname{Newt} F} \subset Y$ the proper transform of the divisor (3.1) of Y_Σ defined by P under the non-toric blow-up (3.2).

Lemma 3.4.1. *If f is rigid maximally mutable, then $\overline{D}_{\operatorname{Newt} F} \subset Y$ is nef divisor supported on D.*

Proof. We can write

$$\overline{D}_{\operatorname{Newt} F} = \sum_{\rho \in \tilde{\Sigma}[1]} h_\rho \overline{D}_\rho,$$

where $\overline{D}_\rho$ is the proper transform of the toric boundary stratum $D_\rho \subset Y_\Sigma$, and h_ρ is defined as asbove in (3.1). Clearly $\overline{D}_{\operatorname{Newt} F} \cdot E \geq 0$ for any of the exceptional divisors of the blow-up (3.2). We also claim that $\overline{D}_{\operatorname{Newt} F} \cdot \overline{D}_\rho = 0$ for all $\rho \in \Sigma$. The degree of the restriction of the line bundle (3.1) to the toric boundary stratum $D_\rho \subset Y_\Sigma$ is equal to the length l_ρ of the corresponding edge of P ($l_\rho = 0$ when there is no such edge). Since Y_Σ is a toric surface, we have

$$D_\rho \cdot D_{\rho'} = \begin{cases} \left(D_\rho\right)^2 & \text{if } \rho = \rho' \\ 1 & \text{if } \rho \text{ and } \rho' \text{ are adjacent rays in } \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

(See [16, Theorem 10.4.4].) Thus we have

$$l_\rho = D_\rho \cdot D_{\operatorname{Newt} F} = h_\rho (D_\rho)^2 + h_{\rho_+} + h_{\rho_-} \tag{3.5}$$

where ρ_+ and ρ_- are the rays adjacent to ρ in Σ. Now $\overline{D}_\rho$ is the proper transform of D_ρ blown up in l_ρ / h_ρ general points, so

$$\overline{D}_\rho \cdot \overline{D}_{\rho'} = \begin{cases} (\overline{D}_\rho)^2 = (D_\rho)^2 - l_\rho / h_\rho & \text{if } \rho = \rho' \\ 1 & \text{if } \rho \text{ and } \rho' \text{ are adjacent rays in } \Sigma \\ 0 & \text{otherwise,} \end{cases}$$

and, putting this together with (3.5), we have

$$\overline{D}_{\operatorname{Newt} F} \cdot \overline{D}_\rho = h_\rho (\overline{D}_\rho)^2 + h_{\rho_+} + h_{\rho_-} = 0.$$

Any curve class on Y is a positive linear combination of proper transforms of toric curves in Y_Σ and exceptional curves of the blow-up (3.2). □

We may therefore compactify $\mathfrak{X}_F^{\mathrm{can}}$ by applying the Rees construction (Construction 2.4.5) with $\overline{D}_{\mathrm{Newt}\,F}$. Let $\psi_{\mathrm{Newt}\,F}$ be the piecewise linear function on $(B_F, \Sigma(Y))$ defined by $\overline{D}_{\mathrm{Newt}\,F}$ – it is an integral function defined by its values on the rays of $\Sigma(Y)$:

$$\psi_{\mathrm{Newt}\,F}(m_\rho) = h_\rho$$

where m_ρ is the primitive generator of ρ, and h_ρ is defined for $D_{\mathrm{Newt}\,F}$ as in (3.1). Since h_ρ is strictly positive for all rays ρ, the level set of $\psi_{\mathrm{Newt}\,F}$ forms a bounded polyhedron. Denote this polyhedron on B by

$$\overline{B}_F := \{m \in B \mid \psi_{\mathrm{Newt}\,F}(m) \leq 1\}\,.$$

By Remark 2.4.6, the compactification of $\mathfrak{X}_F^{\mathrm{can}}$ given by the Rees construction may equivalently be constructed by applying Construction 2.4.2 with the bounded wall structure $\overline{\mathfrak{D}_{\mathrm{can}}}$ given by the truncation of $\mathfrak{D}_{\mathrm{can}}$ by $\overline{B}_F$. On $M_\mathbb{R}$, the analogous bounded polyhedron given by the $\psi_{\mathrm{Newt}\,F} \circ \nu^{-1}$ is exactly the polar dual polytope to the Newton polytope of F,

$$(\mathrm{Newt}\,F)^\vee := \{m \in M_\mathbb{R} \mid \langle m, p\rangle \geq -1 \quad \forall p \in P = \mathrm{Newt}\,F\}\,.$$

We denote the corresponding bounded wall structure by $\overline{\mathfrak{D}_{(Y_\Sigma, H_F)}}$.

Construction 3.4.2 (The compactified mirror to F)**.** The piecewise linear identification (2.3) between $(B_F, \mathscr{P})$ and $(M_\mathbb{R}, \Sigma)$ preserves the bounded subsets:

$$\nu\left(\overline{B}_F\right) = (\mathrm{Newt}\,F)^\vee.$$

Therefore, broken lines on $\mathbf{C}\overline{B}_F$ correspond one-to-one with broken lines on $\mathbf{C}(\mathrm{Newt}\,F)^\vee$ as in Lemma 2.3.18, and so the algebras of theta functions $R(\mathbf{C}\overline{\mathfrak{D}_{\mathrm{can}}}/J)$ and $R(\mathbf{C}\overline{\mathfrak{D}_{(Y_\Sigma,H_F)}}/J)$ are isomorphic over R^{gs}/J for all $\mathfrak{m}^{\mathrm{gs}}$-primary ideals J. Moreover, the underlying affine pseudomanifolds are bounded, so the algebras of theta functions $R(\mathbf{C}\overline{\mathfrak{D}_{\mathrm{can}}}) \cong R(\mathbf{C}\overline{\mathfrak{D}_{(Y_\Sigma,H_F)}})$ are in fact well-defined as graded R^{gs}-algebras.

By Lemma 1.6.3 therefore, we have a family over the Gross–Siebert locus Spec R^{gs},

$$\overline{\mathfrak{X}_F} := \operatorname{Proj} R(\mathbf{C}\alpha\overline{\mathfrak{D}_{(Y_\Sigma,H_F)}}) \cong \operatorname{Proj} R(\mathbf{C}\overline{\mathfrak{D}_{\mathrm{can}}}) \tag{3.6}$$

which is the algebraisation of a compactification of $\mathfrak{X}_F$.

Example 3.4.3 (Compactifying the mirrors to f and g)**.** Let f and g be the Laurent polynomials from our running example (see Examples 3.1.8, 3.2.1 and 3.3.1). In these cases, the Newton polytopes are both reflexive, so $\overline{D}_{\operatorname{Newt} f} = D_f$ and $\overline{D}_{\operatorname{Newt} g} = D_g$. The preimage of the level set

$$\{m \in B_f \mid \psi_{\operatorname{Newt} f}(m) = 1\} \subset B_f$$

in the universal cover $\tilde{B}_f$ (pictured in Figure 3.6) is simply the affine line

$$\{y = 1\} \subset \mathbb{R}^2.$$

Since every ray $(\mathfrak{d}, f_{\mathfrak{d}}) \in \overline{\mathfrak{D}_{\mathrm{can}}}(f)$ is outgoing, any broken line propagating along the boundary of $\overline{B}_f$ which crosses $\mathfrak{d}$ either bends upwards (towards the boundary) or doesn't bend at all. It is therefore easy to see consistency at the new boundary joints, as convexity of $\mathfrak{D}_{\mathrm{can}}(f)$ follows from convexity of the bounded polyhedron $\overline{B}_f$. The universal cover of the boundary defined by the level set where $\psi_{\operatorname{Newt} g}(m) = 1$ looks exactly the same on $\tilde{B}_g$.

Since the Newton polytopes of f and g are both reflexive, their dual polyhedra are lattice polygons. Figures 3.10 and 3.11 below show the initial scattering diagrams defining the bounded algorithmic scattering diagrams $\overline{\mathfrak{D}_{(\Sigma_f,H_f)}}$ and $\overline{\mathfrak{D}_{(\Sigma_g,H_g)}}$ on the polyhedral affine pseudomanifolds $\big((\operatorname{Newt} f)^\vee, \Sigma_f\big)$ and $\big((\operatorname{Newt} g)^\vee, \Sigma_g\big)$ respectively.

The compactified mirrors to f and g are the two families over the respective Gross–Siebert loci

$$\overline{\mathfrak{X}_f} := \operatorname{Proj} R(\mathbf{C}\alpha\overline{\mathfrak{D}_{(\Sigma_f,H_f)}}) \longrightarrow \operatorname{Spec} R_f^{\mathrm{gs}}$$
$$\text{and} \quad \overline{\mathfrak{X}_g} := \operatorname{Proj} R(\mathbf{C}\alpha\overline{\mathfrak{D}_{(\Sigma_g,H_g)}}) \longrightarrow \operatorname{Spec} R_g^{\mathrm{gs}}.$$

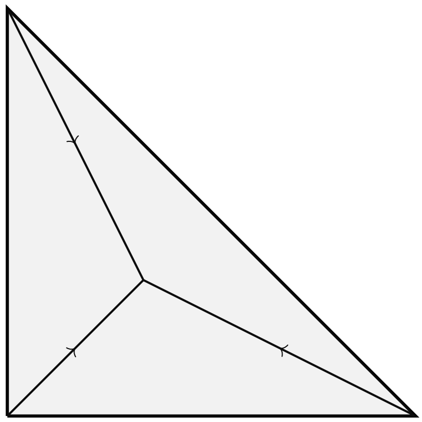

Figure 3.10: The bounded initial scattering diagram $\overline{\mathfrak{D}_{\text{init}}}(f)$

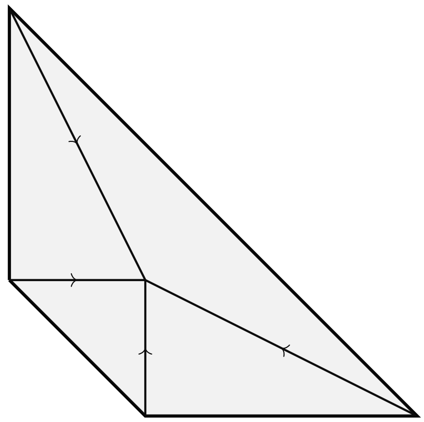

Figure 3.11: The bounded initial scattering diagram $\overline{\mathfrak{D}_{\text{init}}}(g)$

Chapter 4

The intermediate mirror family

The overall goal of this chapter is to construct a third scheme $\overline{\mathfrak{X}_\sim}$ which admits morphisms to both $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$, after some appropriate changes of the bases. In the next chapter we will show that the restriction of the morphism $\overline{\mathfrak{X}_\sim} \to \overline{\mathfrak{X}_g}$ to the general fibre of $\overline{\mathfrak{X}_\sim}$ is an isomorphism, and that the restriction of the morphism $\overline{\mathfrak{X}_\sim} \to \overline{\mathfrak{X}_f}$ to the general fibre is the blow-up in a point on the boundary $\overline{\mathfrak{X}_f} \setminus \mathfrak{X}_f$.

The general strategy is as follows. In Section 4.1 we define a third scattering diagram $\mathfrak{D}_{\text{pert}}$ on $\mathbb{R}^2$, constructed by perturbing the incoming rays in $\mathfrak{D}_{\text{init}}(g)$. The diagram $\mathfrak{D}_{\text{pert}}$ provides a combinatorial link between the two HDTV scattering diagrams associated to f and g; the main result of Section 4.1, Theorem 4.1.1, boils down to saying that $\mathfrak{D}_{(\Sigma_f,H_f)}$ can in some sense be included into $\mathfrak{D}_{(\Sigma_g,H_g)}$.

The next step is turning the scattering diagram $\mathfrak{D}_{\text{pert}}$ into a wall structure that can be used to construct the scheme $\overline{\mathfrak{X}_\sim}$. In Section 4.2.1 we construct a three-dimensional wall structure $\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$ and an associated family of surfaces $\mathfrak{X}_\sim$. This wall structure converts the combinatorial relationship between the HDTV scattering diagrams $\mathfrak{D}_{(\Sigma_f,H_f)}$ and $\mathfrak{D}_{(\Sigma_g,H_g)}$, given by $\mathfrak{D}_{\text{pert}}$, into a relationship between the affine mirrors $\mathfrak{X}_f$ and $\mathfrak{X}_g$. We describe this relationship on the level of the local constructions of the formal schemes: we obtain an inclusion $\mathfrak{X}_f^\circ \hookrightarrow \mathfrak{X}_g$. In Section 4.2.2 we construct a four-dimensional wall structure $\mathfrak{D}_{f\leftrightarrow g}$, and use it to define a partial compactification $\overline{\mathfrak{X}_\sim}$ of $\mathfrak{X}_\sim$. Extending the ideas of Theorem 1.5.7, we use the bi-conical structure of $\mathfrak{D}_{f\leftrightarrow g}$ to produce two auxiliary schemes $\overline{\mathfrak{X}_{0,\bullet}}$ and $\overline{\mathfrak{X}_{\bullet,0}}$

from $\mathfrak{D}_{f\leftrightarrow g}$ such that there are natural morphisms

$$\begin{array}{ccc} & \overline{\mathfrak{X}_{\sim}} & \\ \swarrow & & \searrow \\ \overline{\mathfrak{X}_{\bullet,0}} & & \overline{\mathfrak{X}_{0,\bullet}}. \end{array} \tag{4.1}$$

In Section 4.3 we define morphisms from the base of $\overline{\mathfrak{X}_{\sim}}$ to the bases of $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$, such that $\overline{\mathfrak{X}_{\bullet,0}}$ can be identified with the pullback of $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_{0,\bullet}}$ can be identified with the pullback of $\overline{\mathfrak{X}_g}$. That is, the diagram (4.1) extends to a diagram

$$\begin{array}{ccccccccc} & & & & \overline{\mathfrak{X}_{\sim}} & & & & \\ & & & \swarrow & \downarrow & \searrow & & & \\ \overline{\mathfrak{X}_f} & \longleftarrow & \overline{\mathfrak{X}_{0,\bullet}} & & & & \overline{\mathfrak{X}_{\bullet,0}} & \longrightarrow & \overline{\mathfrak{X}_g} \\ & \searrow & & \searrow & \downarrow & \swarrow & & \swarrow & \\ & & \operatorname{Spec} R^{\mathrm{egs}}_f & \longleftarrow & \operatorname{Spec} \tilde{R}^{\mathrm{egs}} & \longrightarrow & \operatorname{Spec} R^{\mathrm{gs}}_g & & \end{array} \tag{4.2}$$

The morphisms of the bases are not actually well defined until we restrict to a further sublocus inside the Gross–Siebert locus. This extends the notion of the Gross–Siebert locus by further extending the monoid Q^{gs}, so we call it the *extended Gross–Siebert locus*. The central fibres of the families over the extended Gross–Siebert locus is not longer toric, but in Section 4.3.2 we show that the schemes themselves are still well-defined.

4.1 The perturbed scattering diagram

Recall that in Example 3.3.1 we defined $Q^{abcd} = \mathbb{N}^4 \oplus M$, where $\Bbbk[Q]$ is denoted by $\Bbbk[a, b, c, d, x^{\pm 1}, y^{\pm 1}]$. Define another scattering diagram for the same data $Q^{abcd} \to M$ as

$$\mathfrak{D}_{\mathrm{pert}} := \mathrm{Scatter}(\mathfrak{D}^0_{\mathrm{pert}}), \tag{4.3}$$

where

$$\mathfrak{D}^0_{\mathrm{pert}} := (\{\mathfrak{d}_a, \mathfrak{d}_b, \mathfrak{d}_c, \mathfrak{d}_d\}),$$

and

$$(\mathfrak{d}_a, f_a) := \left(\mathbb{R}_{\geq 0}(2,-1),\ 1 + ax^2y^{-1}\right),$$
$$(\mathfrak{d}_b, f_b) := \left(\mathbb{R}_{\geq 0}(-1,2),\ 1 + bx^{-1}y^2\right),$$
$$(\mathfrak{d}_c, f_c) := \left((-1,-1) + \mathbb{R}_{\geq 0}(-1,0),\ 1 + cx^{-1}\right),$$
$$(\mathfrak{d}_d, f_d) := \left((-1,-1) + \mathbb{R}_{\geq 0}(0,-1),\ 1 + dy^{-1}\right).$$

The initial scattering diagram $\mathfrak{D}^0_{\text{pert}}$ is shown in Figure 4.1 below. It is equivalent to a perturbed version of $\mathfrak{D}_{\text{init}}(g)$, with the bottom left two incoming rays translated by $(-1,-1)$.

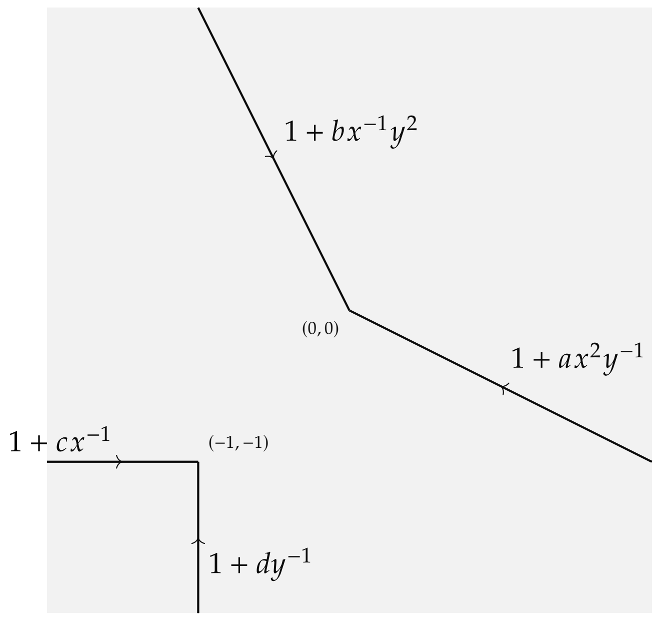

Figure 4.1: The initial scattering diagram $\mathfrak{D}^0_{\text{pert}}$

Theorem 4.1.1. *There exists an open neighbourhood $U \subset M_{\mathbb{R}}$ about the origin such that*

$$\mathfrak{D}_{\text{pert}} \cap U$$

consists of lines through the origin and rays emanating from the origin. Furthermore, it is equivalent to

$$\phi\left(\mathfrak{D}_{(\Sigma_f, H_f)} \cap U\right),$$

where ϕ is the morphism of monoids

$$\phi : Q^{abc} \longrightarrow Q^{abcd} \tag{4.4}$$

given by $a \mapsto a$, $b \mapsto b$, $c \mapsto cd$.

4.1.1 Principles of scattering

In this section we describe a series of facts and techniques for calculating $\mathrm{Scatter}(\mathfrak{D})$ given a two-dimensional scattering diagram $\mathfrak{D}$, which we make use of in the proof of Theorem 4.1.1.

We begin by describing the algorithm which takes as its input $\mathrm{Scatter}^{k-1}(\mathfrak{D})$, a scattering diagram which is compatible modulo $\mathfrak{m}_Q^k$, and outputs $\mathrm{Scatter}^k(\mathfrak{D})$, the unique (modulo $\mathfrak{m}_Q^{k+1}$) scattering diagram which contains $\mathrm{Scatter}^{k-1}(\mathfrak{D})$ and is compatible modulo $\mathfrak{m}_Q^{k+1}$. It forms the inductive step in the proof of the Kontsevich–Soibelman Lemma (Lemma 2.1.8). In the recreation below algorithm is slightly modified so that $\mathrm{Scatter}^k(\mathfrak{D})$ is obtained from $\mathrm{Scatter}^{k-1}(\mathfrak{D})$ by only adding rays $\mathfrak{d}$ such that $f_{\mathfrak{d}} \not\equiv 1 \bmod \mathfrak{m}_Q^{k+1}$.

Construction 4.1.2 (A scattering diagram $\mathfrak{D}_k$ compatible to order k and containing $\mathfrak{D}_{k-1}$)**.** Suppose $\mathfrak{D}_{k-1}$ is compatible to order $k-1$, that is

$$\theta_{\gamma,\mathfrak{D}_{k-1}} \equiv \mathrm{Id} \mod \mathfrak{m}_Q^k$$

for all loops γ for which $\theta_{\gamma,\mathfrak{D}_{k-1}}$ is defined. Let $\mathfrak{D}'_{k-1}$ be the finite subset of $\mathfrak{D}_{k-1}$ consisting of the walls $\mathfrak{d}$ in $\mathfrak{D}_{k-1}$ such that $f_{\mathfrak{d}} \not\equiv 1 \bmod \mathfrak{m}_Q^{k+1}$. Let γ_p be a closed simple loop around $p \in \mathrm{Sing}(\mathfrak{D}'_{k-1})$, small enough so that it contains no other points of $\mathrm{Sing}(\mathfrak{D}'_{k-1})$. By assumption we can write uniquely

$$\theta_{\gamma_p,\mathfrak{D}'_{k-1}} = \exp\left(\sum_{i=1}^{s} c_i z^{m_i} \partial_{n_i}\right)$$

where $m_i \in M \setminus 0$, $n_i \in m_i^{\perp}$ is primitive and $c_i \in \mathfrak{m}_Q^k$, and we know that

$$\theta_{\gamma_p,\mathfrak{D}'_{k-1}} \equiv \theta_{\gamma_p,\mathfrak{D}_{k-1}} \mod \mathfrak{m}_Q^{k+1}.$$

We let

$$\mathfrak{D}[p] = \left\{(p + \mathbb{R}_{\geq 0}(-m_i), 1 \pm c_i z^{m_i}) \mid i \in \{1, \dots, s\} \text{ and } c_i \equiv 0 \bmod \mathfrak{m}_Q^{k+1}\right\},$$

where the sign for each ray is chosen so that its contribution to $\theta_{\gamma_p, \mathfrak{D}'_{k-1}}$ is $\exp(-c_i z^{m_i} \partial_{n_i})$ modulo $\mathfrak{m}_Q^{k+1}$, giving

$$\theta_{\gamma_p, \mathfrak{D}_{k-1} \cup \mathfrak{D}[p]} \equiv \mathrm{Id} \mod \mathfrak{m}_Q^{k+1}.$$

The set $\mathrm{Sing}(\mathfrak{D}'_{k-1})$ is finite since $\mathfrak{D}'_{k-1}$ is a finite scattering diagram, and so we can define

$$\mathfrak{D}_k = \mathfrak{D}_{k-1} \cup \bigcup_{p \in \mathrm{Sing}(\mathfrak{D}'_{k-1})} \mathfrak{D}[p].$$

Any automorphism coming from $\mathfrak{D}[p]$ commutes modulo $\mathfrak{m}_Q^{k+1}$ with any automorphism coming from $\mathfrak{D}_{k-1}$, since $c_i \in \mathfrak{m}_Q^k$ for all rays

$$(p + \mathbb{R}_{\geq 0} m_i, 1 \pm c_i z^{m_i} \partial_{n_i}) \in \mathfrak{D}[p].$$

This means that

$$\theta_{\gamma, \mathfrak{D}_k} \equiv \mathrm{Id} \mod \mathfrak{m}_Q^{k+1}$$

for all loops γ for which $\theta_{\gamma, \mathfrak{D}_k}$ is defined, so $\mathfrak{D}_k$ is compatible to order k, and has been obtained from $\mathfrak{D}_{k-1}$ by adding only outgoing rays $\mathfrak{d}$ with nontrivial wall function modulo $\mathfrak{m}_Q^{k+1}$.

Remark 4.1.3. Note that if $f_\mathfrak{d} = 1$ then $\theta_{\gamma,\mathfrak{d}} = \mathrm{Id}$, and if two walls $\mathfrak{d}_1$ and $\mathfrak{d}_2$ share the same support, then the automorphisms $\theta_{\gamma,\mathfrak{d}_1}$ and $\theta_{\gamma,\mathfrak{d}_2}$ commute. We can therefore make trivial modifications to $\mathfrak{D}$ without leaving its equivalence class, like removing a wall $\mathfrak{d}$ with trivial wall function, or replacing a collection of walls $\mathfrak{d}_1, \dots, \mathfrak{d}_n$ with the same support with a single wall with wall function

$$f_\mathfrak{d} = \prod_{i=1}^{n} f_{\mathfrak{d}_i}.$$

We will also consider the notion of a line

$$(\mathfrak{l}, f_{\mathfrak{l}}) = (p + \mathbb{R}m_{\mathfrak{l}}, f_{\mathfrak{l}})$$

passing through a point p to be interchangeable with the notion of the union of an incoming ray to towards p and outgoing ray emanating from p,

$$(\mathfrak{l}_{\text{in}}, f_{\mathfrak{l}}) = (p + \mathbb{R}_{\geq 0}m_{\mathfrak{l}}, f_{\mathfrak{l}}) \quad \text{and} \quad (\mathfrak{l}_{\text{out}}, f_{\mathfrak{l}}) = (p - \mathbb{R}_{\geq 0}m_{\mathfrak{l}}, f_{\mathfrak{l}}).$$

We will say that $\mathfrak{l}_{\text{in}}$ and $\mathfrak{l}_{\text{out}}$ are the incoming and outgoing rays *contained in* $\mathfrak{l}$.

Proof of Lemma 2.1.11. Since Scatter($\varphi(\mathfrak{D})$) is the unique scattering diagram containing $\varphi(\mathfrak{D})$ which is compatible to all orders, and $\varphi(\text{Scatter}(\mathfrak{D}))$ clearly contains $\varphi(\mathfrak{D})$, it is enough to show that $\varphi(\text{Scatter}(\mathfrak{D}))$ is compatible to all orders. We claim that

$$\theta_{\alpha(\gamma),\alpha(\mathfrak{D})}(z^{\varphi(q)}) = \hat{\varphi}\left(\theta_{\gamma,\mathfrak{D}}(z^{q})\right).$$

The only thing to show is that

$$\langle n_{\alpha(\mathfrak{d})}, s \circ \varphi(q)\rangle = \langle n_{\mathfrak{d}}, r(q)\rangle.$$

This holds because α and α^{-1} are elements of $GL_2(\mathbb{Z})$, so must preserve primitivity. This means that $n_{\alpha(\mathfrak{d})} \in N$, defined to be the primitive element of N which annihilates $\varphi(m_{\mathfrak{d}}) = \alpha(m_{\mathfrak{d}})$ and is negative on $(\alpha \circ \gamma)'(t) = \alpha(\gamma'(t))$, precisely equals $n_{\mathfrak{d}} \circ \alpha^{-1}$. Thus

$$\begin{aligned}\langle n_{\alpha(\mathfrak{d})}, s \circ \varphi(q)\rangle &= \langle n_{\mathfrak{d}} \circ \alpha^{-1}, s \circ \varphi(q)\rangle \\ &= \langle n_{\mathfrak{d}} \circ \alpha^{-1}, \alpha \circ r(q)\rangle \\ &= \langle n_{\mathfrak{d}}, r(q)\rangle.\end{aligned}$$

□

Lemma 4.1.4. *Suppose that a scattering diagram $\mathfrak{D}$ for the data $Q = \mathbb{N}^k \oplus M$ consists of a set of rays all incoming to a single point*

$$(\mathfrak{d}_i, f_{\mathfrak{d}_i}) = (p + \mathbb{R}_{\geq 0}m_i,\ 1 + t_i z^{m_i}),$$

where $m_i \in M$ and t_i is a generator of the ring $\Bbbk[Q] = \Bbbk[t_1, \cdots, t_k, x^{\pm}, y^{\pm}]$.

Then $\mathrm{Scatter}^1(\mathfrak{D})$*, the scattering diagram containing $\mathfrak{D}$ and compatible modulo $\mathfrak{m}_Q^2$, is equivalent to the set of lines*

$$(\mathfrak{l}_i, f_{\mathfrak{l}_i}) = (p + \mathbb{R}m_i,\ f_{\mathfrak{d}_i}),$$

each of which is the continuation of the ray $\mathfrak{d}_i$. That is, each line $\mathfrak{l}_i \in \mathrm{Scatter}_1(\mathfrak{D})$ is the union $\mathfrak{d}_i \cup \mathfrak{d}'_i$ of a ray $\mathfrak{d}_i \in \mathfrak{D}$ and the outgoing ray $\mathfrak{d}'_i$ with the same wall function $f_{\mathfrak{d}'_i} = f_{\mathfrak{d}_i}$.

Proof. Consider the path ordered product $\theta_{\gamma,\mathfrak{D}}$ around the loop γ about the point p modulo $\mathfrak{m}_Q^2 = (t_1, \cdots, t_k)^2$

$$\theta_{\gamma,\mathfrak{D}} = \theta_{\mathfrak{d}_k,n_k} \circ \cdots \circ \theta_{\mathfrak{d}_1,n_1}.$$

We may consider $\log(f_i)$ to be a power series in $\Bbbk[M][\![t_i]\!]$. Its leading term is $t_i z^{m_i}$ and all other terms are contained in $\mathfrak{m}_Q^2$, so we have

$$\begin{aligned}\theta_{\gamma,\mathfrak{D}} &\equiv \exp(t_n z^{m_n}\partial_{n_k}) \circ \cdots \circ \exp(t_1 z^{m_1}\partial_{n_1}) \\ &\equiv \exp\left(\sum_{i=1}^{k} t_i z^{m_i}\partial_{n_i}\right) \qquad\qquad \mod \mathfrak{m}_Q^2.\end{aligned}$$

Following Construction 4.1.2 we add the rays

$$(\mathfrak{d}'_i, f_{\mathfrak{d}'_i}) = (p + \mathbb{R}_{\geq 0}(-m_i),\ 1 + t_i z^{m_i})$$

to $\mathfrak{D}$ in order to obtain $\mathrm{Scatter}_1(\mathfrak{D})$, as $\mathfrak{d}'_i$ then contributes

$$\theta_{\mathfrak{d}'_i,n'_i} \equiv \exp(t_i z^{m_i}\partial_{-n_i}) \equiv \exp(-t_i z^{m_i}\partial_{n_i}) \mod \mathfrak{m}_Q^2$$

to the path ordered product around γ. □

Lemma 4.1.5 (Scattering of two lines)**.** *Suppose that an initial scattering diagram $\mathfrak{D}$ consisting of two lines $\mathfrak{l}_1$ and $\mathfrak{l}_2$ meeting in a single point p. Then* $\mathrm{Scatter}(\mathfrak{D}) \setminus \mathfrak{D}$ *consists of outgoing rays contained in the cone spanned by the two outgoing rays $(\mathfrak{l}_{1out}, f_{\mathfrak{l}_1})$ and $(\mathfrak{l}_{2out}, f_{\mathfrak{l}_2})$ emanating from p.*

Proof. One can see from Construction 4.1.2 that any outgoing ray

$$(p + \mathbb{R}_{\geq 0}(-r(q_{\mathfrak{d}})),\ 1 + c_{\mathfrak{d}} z^{q_{\mathfrak{d}}})$$

added to obtain $\mathrm{Scatter}^k(\mathfrak{D})$ must have its exponent $q_{\mathfrak{d}}$ equal a non-negative linear combination of the exponents $q_j^i \in Q$ appearing in the wall functions

$$f_{\mathfrak{l}_i} = 1 + \sum_j c_j^i z^{q_j^i}.$$

The outgoing rays emanating from p which are contained in the two lines are described by

$$(\mathfrak{l}_{i\mathrm{out}}, f_{\mathfrak{l}_i}) = \left(p + \mathbb{R}_{\geq 0}(-r(q_1^i)),\ \sum_j c_j^i z^{q_i} \right)$$

with $r(q_j^i) \in \mathbb{R}_{\geq 0}(r(q_1^i))$ for all k. Thus $r(q_{\mathfrak{d}})$ is contained in the cone spanned by $r(q_1^1)$ and $r(q_1^2)$, so $\mathfrak{d}$ is contained in

$$p + \mathbb{R}_{\geq 0}(-r(q_1^1) + \mathbb{R}_{\geq 0}(-r(q_1^2).$$

□

Perturbation of a scattering diagram $\mathfrak{D}$ is a useful technique, given in Section 1.4 of [24], for calculating $\mathrm{Scatter}(\mathfrak{D})$ when $\mathrm{Sing}(\mathfrak{D})$ contains points with more than two walls passing through them. The technique relies on the concept of the *asymptotic scattering diagram*.

Definition 4.1.6. Given a scattering diagram $\mathfrak{D}$, the *asymptotic scattering diagram* $\mathcal{S}_{\mathrm{as}}$ is obtained from $\mathfrak{D}$ by replacing each ray $(p + \mathbb{R}_{\geq 0} m, f)$ with the ray $(\mathbb{R}_{\geq 0} m, f)$ emanating from the origin, and replacing each line $(p + \mathbb{R} m, f)$ with the line $(\mathbb{R} m, f)$ passing through the origin.

Remark 4.1.7. Let $\mathbf{C}\mathfrak{D}$ be the cone over the scattering diagram $\mathfrak{D}$, as defined in Definition 1.5.5. The slice of the cone at height zero,

$$\mathbf{C}\mathfrak{D}_0 := \mathbf{C}\mathfrak{D} \cap (M_{\mathbb{R}} \times \{0\})$$

is clearly equivalent to the asymptotic scattering diagram $\mathfrak{D}_{\mathrm{as}}$. Note that here we mean equivalence of *scattering diagrams* in the sense of Definition 2.1.9 – $\mathscr{S}_{\mathrm{as}}$ is not necessarily equivalent to $\mathbf{C}\mathfrak{D}_0$ in the sense of Definition 1.2.21, as the wall functions may have been modified slightly when taking the cone over $\mathfrak{D}$.

Remark 4.1.8. Compatibility of $\mathfrak{D}$ implies compatibility of $\mathscr{S}_{\mathrm{as}}$. Indeed, if γ is a sufficiently large simple loop about the origin containing all the points of $\mathrm{Sing}(\mathfrak{D})$, then

$$\theta_{\gamma,\mathfrak{D}} = \theta_{\gamma,\mathscr{S}_{\mathrm{as}}}.$$

Construction 4.1.9 (Scattering of more than two lines passing through a point)**.** Suppose that a scattering diagram consists of a collection of lines

$$\mathfrak{D} = \{\mathfrak{l}_1, \cdots, \mathfrak{l}_n\}$$

which all pass through $p \in M_{\mathbb{R}}$. The following method reduces the calculation of $\mathrm{Scatter}^k(\mathfrak{D})$ to the calculation of $\mathrm{Scatter}^k(\mathfrak{D}_q)$ for a finite set of scattering diagrams $\mathfrak{D}_q$, each of which consists of only two lines passing through a point. The point is that the scattering of only two lines meeting in a point is easier to control, using e.g. Lemma 4.1.5, or Lemmas 4.1.11 and 4.1.12 below.

For all $j \in \mathbb{N}$, the only point in $\mathrm{Sing}(\mathrm{Scatter}^j(\mathfrak{D}))$ is p. Thus $\mathrm{Scatter}^{k-1}(\mathfrak{D}) \setminus \mathfrak{D}$ contains only outgoing rays emanating from p by Construction 4.1.2. Perturb $\mathfrak{D}$ so that no more than two of the lines $\mathfrak{l}_i$ pass through any point in $M_{\mathbb{R}}$, and denote the resulting diagram $\tilde{\mathfrak{D}}$. Then $\tilde{\mathfrak{D}}_{\mathrm{as}}$ is equivalent to $\mathfrak{D} - p$, where

$$\mathfrak{D} - p := \{(\mathfrak{d} - p, f_{\mathfrak{d}}) \mid (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}\}.$$

Since $(\mathrm{Scatter}^k(\tilde{\mathfrak{D}}))_{\mathrm{as}}$ contains $\mathfrak{D} - p$ and is compatible modulo $\mathfrak{m}_Q^{k+1}$ by Remark 4.1.8, it must be equivalent to $\mathrm{Scatter}^k(\mathfrak{D} - p)$, which clearly equals $\mathrm{Scatter}^k(\mathfrak{D}) - p$. Therefore

$$\mathrm{Scatter}^k(\mathfrak{D}) \equiv \left(\mathrm{Scatter}^k(\tilde{\mathfrak{D}})\right)_{\mathrm{as}} + p \mod \mathfrak{m}_Q^{k+1}.$$

It remains to show that $\mathrm{Scatter}^k(\tilde{\mathfrak{D}})$ can be obtained only by calculating scattering between two lines at a time. Following Construction 4.1.2, $\mathrm{Scatter}^1(\tilde{\mathfrak{D}})$ is calculated by considering path ordered products around loops about each $\tilde{p} \in \mathrm{Sing}(\mathrm{Scatter}^1(\tilde{\mathfrak{D}}))$ separately, and adding rays emanating from the $\tilde{p}$.

Now let $k \geq 2$. We can assume that no more than two rays of $\mathrm{Scatter}^{k-1}(\tilde{\mathfrak{D}})$ pass through each point in

$$S_k := \mathrm{Sing}(\mathfrak{D}) \cup \mathrm{Sing}(\mathrm{Scatter}^1(\mathfrak{D})) \cup \cdots \cup \mathrm{Sing}(\mathrm{Scatter}^{k-1}(\mathfrak{D}),$$

since S_k is finite and the configuration of the lines of $\tilde{\mathfrak{D}}$ has been chosen generally. Let $\mathfrak{q}$ and $\mathfrak{p}$ be the two rays in $\mathrm{Scatter}^k(\tilde{\mathfrak{D}})$ to pass through the point $q \in \mathrm{Sing}(\mathrm{Scatter}^{k-1}(\tilde{\mathfrak{D}}))$. Define

$$\mathfrak{D}^k[q] := \left\{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathrm{Scatter}^k(\{\mathfrak{q}, \mathfrak{p}\}) \,\middle|\, \mathfrak{d} \text{ emanates from } q\right\} \setminus \mathrm{Scatter}^{k-1}(\tilde{\mathfrak{D}}).$$

Note that the only rays of $\mathrm{Scatter}^{k-1}(\tilde{\mathfrak{D}})$ to intersect a the point q other than $\mathfrak{q}$ and $\mathfrak{p}$ must be outgoing rays emanating from q. Since $\tilde{\mathfrak{D}}$ contains only lines, these outgoing rays emanating from q must have arisen as part of the scattering process, and so are contained in $\mathrm{Scatter}^{k-1}(\{\mathfrak{q}, \mathfrak{p}\})$. In particular,

$$\mathrm{Scatter}^k(\{\mathfrak{q}, \mathfrak{p}\}) = \mathrm{Scatter}^k\left(\left\{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathrm{Scatter}^{k-1}(\tilde{\mathfrak{D}}) \,\middle|\, q \in \mathfrak{d}\right\}\right),$$

and therefore

$$\mathrm{Scatter}^k(\tilde{\mathfrak{D}}) \equiv \mathrm{Scatter}^{k-1}(\tilde{\mathfrak{D}}) \cup \bigcup_{q \in \mathrm{Sing}\left(\mathrm{Scatter}^{k-1}(\tilde{\mathfrak{D}})\right)} \mathfrak{D}^k[q].$$

Remark 4.1.10 (More than two incoming rays to a point)**.** We can apply the method of Construction 4.1.9 to any initial scattering diagram $\mathfrak{D}$ which consists of only incoming rays and lines. By Lemma 4.1.4, $\mathrm{Scatter}^1(\mathfrak{D})$ will consist of only lines. Then one can apply a variant of the Construction, letting $\mathrm{Pert}(\mathrm{Scatter}^1(\mathfrak{D}))$ be a perturbation of $\mathrm{Scatter}^1(\mathfrak{D})$ such that

$$\mathrm{Scatter}^k\left(\mathrm{Pert}(\mathrm{Scatter}^1(\mathfrak{D}))\right)$$

can be calculated only by calculating scattering between two lines at a time. Then $\mathrm{Scatter}^k(\mathfrak{D})$ is not obtained by taking the asymptotic scattering diagram of $\mathrm{Scatter}^k\left(\mathrm{Pert}(\mathrm{Scatter}^1(\mathfrak{D}))\right)$, but a variant which is uniquely determined by the position of the lines in $\mathrm{Pert}(\mathrm{Scatter}^1(\mathfrak{D}))$.

In general, given a finite scattering diagram $\mathfrak{D}$, one needs to add infinitely many outgoing rays to obtain Scatter($\mathfrak{D}$). However, in the proof of Theorem 4.1.1 we will make use of the following case when Scatter($\mathfrak{D}$) is finite.

Lemma 4.1.11. *Let $Q = \mathbb{N}^2 \oplus M$. Suppose that an initial scattering diagram $\mathfrak{D}_{m_1,m_2}$ consists of two incoming rays*

$$(\mathfrak{d}_1, f_{\mathfrak{d}_1}) = (\mathbb{R}_{\geq 0} m_1, 1 + a z^{m_1}), \quad (\mathfrak{d}_2, f_{\mathfrak{d}_2}) = (\mathbb{R}_{\geq 0} m_2, 1 + b z^{m_2}),$$

where $a = z^{(1,0,0,0)}$, $b = z^{(0,1,0,0)}$, and $m_1, m_2 \in M$ form a $\mathbb{Z}$-basis of M. Then Scatter($\mathfrak{D}_{m_1,m_2}$) *is a finite scattering diagram consisting of two lines and an outgoing ray*

$$\begin{aligned}
(\mathfrak{d}_1, f_{\mathfrak{d}_1}) &= (\mathbb{R} m_1, 1 + a z^{m_1}),\\
(\mathfrak{d}_2, f_{\mathfrak{d}_2}) &= (\mathbb{R} m_2, 1 + b z^{m_2}),\\
(\mathfrak{d}_3, f_{\mathfrak{d}_3}) &= (\mathbb{R}_{\geq 0}(-m_1 - m_2), 1 + ab z^{m_1 + m_2}).
\end{aligned}$$

Proof. When $m_1 = e_1$ and $m_2 = e_2$, this example is discussed in detail in [24]. The condition

$$\theta_{\gamma, \mathrm{Scatter}(\mathfrak{D}_{e_1,e_2})} = \mathrm{Id}$$

for a loop γ about the origin can be easily checked by hand, as γ crosses a wall only five times. The scattering diagram $\mathfrak{D}_{m_1,m_2}$ can be obtained from $\mathfrak{D}_{e_1,e_2}$ by applying $\alpha \in GL_2(\mathbb{Z})$, the automorphism of M which sends $e_1 \mapsto m_1$, $e_2 \mapsto m_2$. The result then follows by Lemma 2.1.11. □

Lemma 4.1.12. *Let $Q = \mathbb{N}^2 \oplus M$ and denote the generators of Q by*

$$a = z^{(1,0,0,0)}, \ b = z^{(0,1,0,0)}, \ x = z^{(0,0,1,0)}, \ y = z^{(0,0,0,1)}.$$

Suppose the initial scattering diagram $\mathfrak{D}$ consists of two incoming rays

$$\begin{aligned}
(\mathfrak{d}_1, f_{\mathfrak{d}_1}) &= (\mathbb{R}_{\geq 0}(-n, m), 1 + a x^{-n} y^{m})\\
(\mathfrak{d}_2, f_{\mathfrak{d}_2}) &= \left(\mathbb{R}_{\geq 0}(0, -1), 1 + b y^{-1}\right),
\end{aligned}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then Scatter($\mathfrak{D}$) *consists of the two lines which are the* *continuation of the two initial rays, plus a collection of rays contained in the cone* $\mathbb{R}_{\geq 0}\langle (n,-m),(n,n-m)\rangle$.

In particular, if both n and m are non-negative, then $\mathrm{Scatter}(\mathfrak{D})$ *consists of the continuation of* $\mathfrak{d}_1$ *and* $\mathfrak{d}_2$*, plus a collection of outgoing rays contained in* $\mathbb{R}_{\geq 0}\langle (n,-m),(1,1)\rangle$.

Proof. By Lemma 4.1.4 one adds two outgoing rays

$$(\mathfrak{d}_1', f_{\mathfrak{d}_1'}) = (\mathbb{R}_{\geq 0}(n,-m), 1 + ax^{-n}y^m)$$
$$(\mathfrak{d}_2', f_{\mathfrak{d}_2'}) = \left(\mathbb{R}_{\geq 0}(0,1), 1 + by^{-1}\right)$$

when scattering $\mathfrak{D}$ to order 1, and by Lemma 4.1.5 all the additional outgoing rays to be added to obtain $\mathrm{Scatter}(\mathfrak{D})$ will be contained in the cone

$$\mathbb{R}_{\geq 0}\langle (n,-m),(0,1)\rangle.$$

Consider a loop γ which goes about the origin anticlockwise and starts to at a point just to the right of $\mathfrak{d}_2'$. We would like to show that to any order $k \geq 1$

$$\theta_{\gamma,\mathrm{Scatter}^k(\mathfrak{D})} \equiv \exp\left(\sum_{i=1}^{N_k} c_i z^{m_i}\partial_{n_i}\right) \quad \mathrm{mod}\ \mathfrak{m}_Q^{k+1}$$

where $-m_i$ lies in $\sigma := \mathbb{R}_{\geq 0}\langle (n,-m),(n,n-m)\rangle$ for all $1 \leq i \leq N_k$, so that scattering to order $k+1$ only involves adding finitely many outgoing rays in the specified cone. We claim it suffices to show that

$$\theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1} \circ \theta_{\mathfrak{d}_2',n_2'} = \exp\left(\sum_{j\in\mathbb{N}} c_j z^{m_j}\partial_{n_j}\right)$$

where $-m_j \in \sigma$ for all $j \in \mathbb{N}$. This is because after composing with $\theta_{\gamma,\mathfrak{d}}$ for any $\mathfrak{d} \subset \sigma$, the vectors m_i that appear in the expression

$$\theta_{\mathfrak{d},n_{\mathfrak{d}}} \circ \left(\theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1} \circ \theta_{\mathfrak{d}_2',n_2'}\right) = \exp\left(\sum_{i} c_i z^{m_i}\partial_{n_i}\right)$$

must be positive linear combinations of the m_j and $m_{\mathfrak{d}}$. Cones are closed under positive linear combination, and so $m_j \in -\sigma$ would imply $m_i \in -\sigma$. To check that $-m_j \in -\sigma$ for all $j \in \mathbb{N}$, we calculate the images of a basis $\{y^{-1}, x^{-1}\}$ for M]. Let

$p = \langle n_{\mathfrak{d}_1}, -e_2 \rangle$ and $q = \langle n_{\mathfrak{d}_1}, -e_1 \rangle$. Then p is negative, so

$$\begin{aligned}\theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1} \circ \theta_{\mathfrak{d}'_2,n'_2}(y^{-1}) &= \theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1}(y^{-1}) \\ &= \theta_{\mathfrak{d}_2,n_2}\left(y^{-1}(1+ax^{-n}y^m)^p\right) \\ &= y^{-1}\left(1+ax^{-n}y^m(1+by^{-1})^n\right)^p,\end{aligned}$$

and

$$\begin{aligned}\theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1} \circ \theta_{\mathfrak{d}'_2,n'_2}(x^{-1}) &= \theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1}\left(\frac{x^{-1}}{(1+by^{-1})}\right) \\ &= \theta_{\mathfrak{d}_2,n_2}\left(\frac{x^{-1}(1+ax^{-n}y^m)^{q-p}}{(1+ax^{-n}y^m)^{-p}+by^{-1}}\right) \\ &= \frac{x^{-1}(1+by^{-1})\left(1+ax^{-n}y^m(1+by^{-1})^n\right)^{q-p}}{\left(1+ax^{-n}y^m(1+by^{-1})^n\right)^{-p}+by^{-1}} \\ &= x^{-1}\left(\frac{\left(1+ax^{-n}y^m(1+by^{-1})^n\right)^{q-p}}{1+\sum_{i=1}^{-p}\binom{-p}{i}(ax^{-n}y^m)^i(1+by^{-1})^{i-1}}\right).\end{aligned}$$

We see that after multiplying out the brackets in the expressions above, all the monomials in $x^i y^j$ appearing in the resulting rational functions are equal to some z^m where $m \in -\sigma$. The monomials appearing in the Taylor expansions of these rational functions are positive linear combinations of these $x^i y^j$, and so

$$\begin{aligned}\theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1} \circ \theta_{\mathfrak{d}'_2,n'_2} : \quad y^{-1} &\longmapsto y^{-1}\left(1+\sum c_l z^{m_l}\right) \\ x^{-1} &\longmapsto x^{-1}\left(1+\sum c_r z^{m_r}\right)\end{aligned}$$

where $m_l, m_r \in -\sigma$. Therefore any m_j appearing in

$$\theta_{\mathfrak{d}_2,n_2} \circ \theta_{\mathfrak{d}_1,n_1} \circ \theta_{\mathfrak{d}'_2,n'_2} = \exp\left(\sum c_j z^{m_j} \partial_{n_j}\right)$$

must lie in $-\sigma$. □

4.1.2 Proof of Theorem 4.1.1

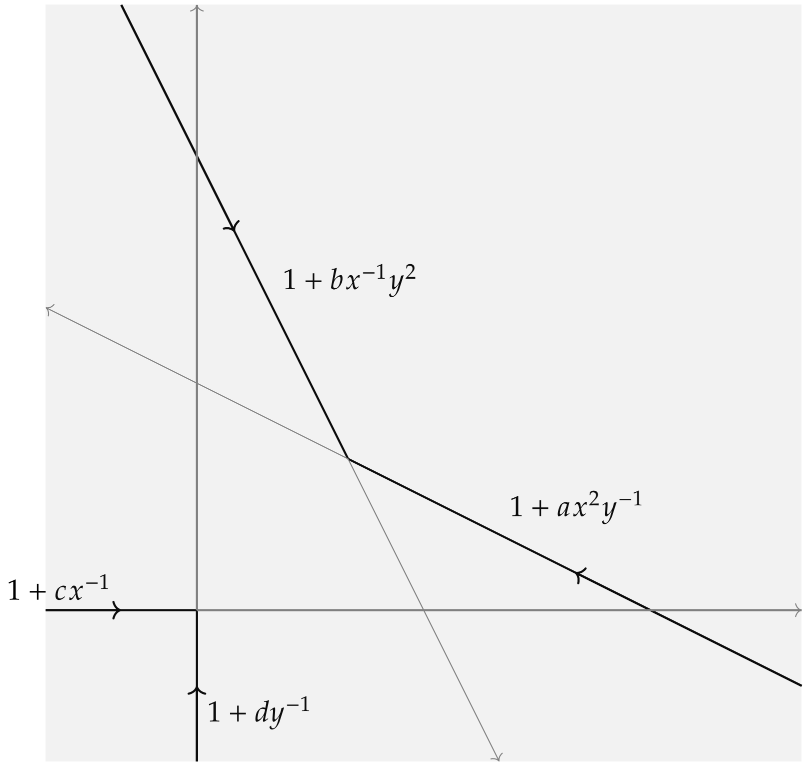

Figure 4.2: The compatible to order 1 scattering diagram containing $\mathfrak{D}^0_{\text{pert}}$, $\text{Scatter}^1(\mathfrak{D}^0_{\text{pert}})$. The rays in the complement of $\mathfrak{D}^0_{\text{pert}}$ shown in grey.

Proof of Theorem 4.1.1. We start by scattering $\mathfrak{D}^0_{\text{pert}}$ to order 2. By Lemma 4.1.4 we have

$$\text{Scatter}^1(\mathfrak{D}^0_{\text{pert}}) = \mathfrak{D}^0_{\text{pert}} \cup \left\{(\mathfrak{d}'_a, f_a), (\mathfrak{d}'_b, f_b), (\mathfrak{d}'_c, f_c), (\mathfrak{d}'_d, f_d)\right\}$$

where $\mathfrak{d}'_i$ is the outgoing ray extending $\mathfrak{d}_i$ to a line.

Following Construction 4.1.2, one must add outgoing rays emanating from the points in

$$\text{Sing}(\text{Scatter}^1(\mathfrak{D}^0_{\text{pert}})) = \left\{\mathfrak{d}_a \cap \mathfrak{d}_b,\ \mathfrak{d}_c \cap \mathfrak{d}_d,\ \mathfrak{d}_a \cap \mathfrak{d}'_c,\ \mathfrak{d}_b \cap \mathfrak{d}'_d,\ \mathfrak{d}'_c \cap \mathfrak{d}'_b,\ \mathfrak{d}'_d \cap \mathfrak{d}'_a\right\}.$$

in order to obtain $\text{Scatter}^2(\mathfrak{D}^0_{\text{pert}})$. By Lemma 4.1.11 we add the outgoing rays

$$\begin{aligned} (\mathfrak{d}_{cd}, f_{cd}) &:= ((-1,-1) + \mathbb{R}_{\geq 0}(1,1), 1 + cdx^{-1}y^{-1}), \\ (\mathfrak{d}_{ac}, f_{ac}) &:= ((2,-1) + \mathbb{R}_{\geq 0}(-1,1),\ 1 + acxy^{-1}) \\ \text{and} \quad (\mathfrak{d}_{bd}, f_{bd}) &:= ((-1,2) + \mathbb{R}_{\geq 0}(1,-1),\ 1 + bdx^{-1}y). \end{aligned}$$

to achieve compatibility around $\mathfrak{d}_c \cap \mathfrak{d}_d$, $\mathfrak{d}_a \cap \mathfrak{d}'_c$ and $\mathfrak{d}_b \cap \mathfrak{d}'_d$. By Lemma 4.1.5 the outgoing rays emanating from $\mathfrak{d}'_c \cap \mathfrak{d}'_b$ and $\mathfrak{d}'_d \cap \mathfrak{d}'_a$ which must be added to obtain $\text{Scatter}^2(\mathfrak{D}^0_{\text{pert}})$ point away from the origin. One also adds rays emanating from the origin to achieve compatibility to order 2 about $\mathfrak{d}_a \cap \mathfrak{d}_b$.

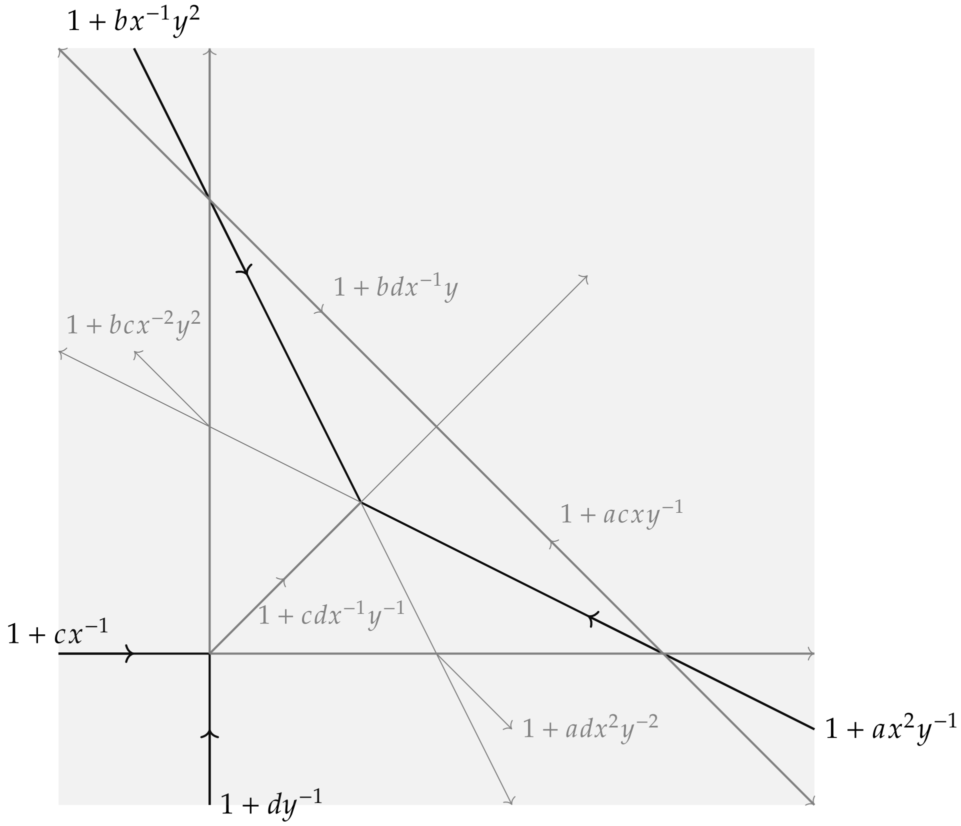

Figure 4.3: A subset of $\text{Scatter}^2(\mathfrak{D}^0_{\text{pert}})$, the scattering diagram containing $\mathfrak{D}^0_{\text{pert}}$ which is compatible to order 2. The rays in the complement of $\mathfrak{D}^0_{\text{pert}}$ shown in grey.

We now note that the collection of rays $\{\mathfrak{d}'_c, \mathfrak{d}'_d, \mathfrak{d}_{ac}, \mathfrak{d}_{bd}\}$ forms a triangle Δ around the origin with vertices $(-1,-1)$, $(-1,2)$ and $(2,-1)$. Denote by $\Delta^\circ \subset \Delta$ the interior of this triangle.

Claim 4.1.13. *The only walls in* Scatter $\mathfrak{D}^0_{\text{pert}}$ *which intersect* Δ° *are*

- *the two incoming rays to the origin* $\mathfrak{d}_a$ *and* $\mathfrak{d}_b$*,*
- *the ray* $\mathfrak{d}_{cd}$ *propagating through the origin, and*
- *outgoing rays emanating from the origin.*

Theorem 4.1.1 follows easily from Claim 4.1.13, as any outgoing ray emanating from the origin is clearly the result of scattering of the three incoming rays to the origin, $\mathfrak{d}_a$ $\mathfrak{d}_b$ and $\mathfrak{d}_{cd}$. Then

$$\mathrm{Scatter}^1(\mathfrak{D}^0_{\mathrm{pert}}) \cap \Delta^\circ \equiv \phi(\mathrm{Scatter}^1(\mathfrak{D}_{ABC})) \cap \Delta^\circ,$$

so the theorem follows by Lemma 2.1.11.

It remains to prove Claim 4.1.13: we show that any outgoing ray in $\mathrm{Scatter}(\mathfrak{D}^0_{\mathrm{pert}}) \setminus \mathrm{Scatter}^2(\mathfrak{D}^0_{\mathrm{pert}})$ points away from (or along the boundary of) Δ.

Consider the complement in $M_\mathbb{R}$ of the interior of the triangle and the four incoming rays of $\mathfrak{D}^0_{\mathrm{pert}}$. It comprises of four connected components:

$$M_\mathbb{R} \setminus \left(\Delta^\circ \cup \mathfrak{D}^0_{\mathrm{pert}}\right) = M_{ab} \sqcup M_{bc} \sqcup M_{cd} \sqcup M_{da},$$

where M_{ij} denotes the connected component that is bounded by the two incoming rays $\mathfrak{d}_i$ and $\mathfrak{d}_j$ in $\mathfrak{D}^0_{\mathrm{pert}}$. We prove the following claim by induction on the order of scattering.

Claim 4.1.14. *Any ray* $\mathfrak{d}$ *in* $\mathrm{Scatter}(\mathfrak{D}^0_{\mathrm{pert}})$ *intersecting* $M_\mathbb{R} \setminus \left(\Delta^\circ \cup \mathfrak{D}^0_{\mathrm{pert}}\right)$ *has wall function* $1 + c_\mathfrak{d} z^{q_\mathfrak{d}}$*, where* $c_\mathfrak{d} \in k$ *and* $q_\mathfrak{d} \in Q$ *such that*

$$-r(q_\mathfrak{d}) \in \begin{cases} \mathbb{R}(-1,1) + \mathbb{R}_{\geq 0}(1,1) & \text{if } \mathfrak{d} \cap M_{ab} \neq \varnothing \\ \mathbb{R}(0,1) + \mathbb{R}_{\geq 0}(-1,0) & \text{if } \mathfrak{d} \cap M_{bc} \neq \varnothing \\ \mathbb{R}(-1,1) + \mathbb{R}_{\geq 0}(-1,-1) & \text{if } \mathfrak{d} \cap M_{cd} \neq \varnothing \\ \mathbb{R}(1,0) + \mathbb{R}_{\geq 0}(0,-1) & \text{if } \mathfrak{d} \cap M_{da} \neq \varnothing \end{cases}$$

Assuming the statement of the claim holds for $\mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})$, we show it holds for $\mathrm{Scatter}^{k}(\mathfrak{D}^0_{\mathrm{pert}})$ one connected component at a time. Here we only treat M_{da} - the arguments for the other components are identical. First, note that all outgoing rays in

$$\mathrm{Scatter}^{k}(\mathfrak{D}^0_{\mathrm{pert}}) \setminus \mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})$$

which emanate from a point in Δ° must emanate from the origin, as the origin is the only point in

$$\mathrm{Sing}(\mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})) \cap \Delta^\circ.$$

Any ray emanating from the origin and intersecting M_{da} must lie in the cone

$$\mathbb{R}(1,0) + \mathbb{R}_{\geq 0}(0,-1)$$

as specified by Claim 4.1.14. Next, consider an outgoing ray $\mathfrak{d}$ in

$$\mathrm{Scatter}^{k}(\mathfrak{D}^0_{\mathrm{pert}}) \setminus \mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})$$

which emanates from a point $p \in M_{da} \cup \left(\bar{M}_{da} \cap \partial\Delta\right)$. We claim that $\mathfrak{d}$ satisfies the condition of Claim 4.1.14. We may assume by Construction 4.1.9 and Remark 4.1.10 that there are exactly two walls of $\mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})$ passing through p.

By Lemma 4.1.5, $\mathfrak{d}$ is contained in the cone spanned by the halves of these two walls which are equivalent to outgoing rays from p. But by our inductive assumption, the cone spanned by the two walls passing through to p is a subset of the cone

$$p + \mathbb{R}(1,0) + \mathbb{R}_{\geq 0}(0,-1),$$

so all rays in $\mathrm{Scatter}^{k}(\mathfrak{D}^0_{\mathrm{pert}}) \cap M_{da}$ coming from $\mathrm{Scatter}^{k-1}(\mathfrak{D})[p]$ satisfy Claim 4.1.14.

This tells us that any new ray intersecting $\mathrm{Scatter}^{k}(\mathfrak{D}^0_{\mathrm{pert}}) \cap M_{da}$ whose generator does not lie in the cone of the claim must emanate from some point p on $\mathfrak{d}_a$ or $\mathfrak{d}_d$. Suppose that

$$\mathfrak{d} \in \mathrm{Scatter}^{k}(\mathfrak{D}^0_{\mathrm{pert}}) \setminus \mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})$$

emanates from $p \in \mathfrak{d}_a \setminus \Delta^\circ$ and intersects M_{da}. Assuming there no more than two walls passing through p as before, we can again apply Lemma 4.1.5 to see that $\mathfrak{d}$ is contained in the cone spanned by $\mathfrak{d}_a$ and a ray $\mathfrak{d}' \in \mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})$ which intersects M_{ab}. By the inductive assumption, $\mathfrak{d}'$ has its generator $-r(q_{\mathfrak{d}'})$ contained in

$$\mathbb{R}(-1,1) + \mathbb{R}_{\geq 0}(1,1).$$

We can therefore apply the $GL_2(\mathbb{Z})$-transformation given by the matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

to $\left(\mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}}) \cap U_p\right)$ and use the functoriality of scattering (Lemma 2.1.11) to apply Lemma 4.1.12. It shows that $\mathfrak{d}$ lies in the cone

$$p + \mathbb{R}_{\geq 0}(-1,0) - \mathbb{R}_{\geq 0}\left(r(q_{\mathfrak{d}'})\right) \subset p + \mathbb{R}_{\geq 0}(-1,0) + \mathbb{R}_{\geq 0}(2,-1).$$

Thus $\mathfrak{d}$ satisfies the statement of Claim 4.1.14. A similar argument will work to show that any new ray emanating from $p \in \mathfrak{d}_d \setminus \Delta^\circ$ and intersecting M_{da} lies in the cone specified by the claim, although in this case Lemma 4.1.12 applies directly without needing to apply a $GL_2(\mathbb{Z})$-transformation and use functoriality of scattering. Thus all rays $\mathfrak{d}$ in

$$\mathrm{Scatter}^k(\mathfrak{D}^0_{\mathrm{pert}}) \setminus \mathrm{Scatter}^{k-1}(\mathfrak{D}^0_{\mathrm{pert}})$$

which intersect M_{da} have the form

$$(p + \mathbb{R}_{\geq 0}(-r(q_{\mathfrak{d}})), 1 + c_{\mathfrak{d}} z^{q_{\mathfrak{d}}})$$

where $-r(q_{\mathfrak{d}}) \in \mathbb{R}_{\geq 0}(-1,0) + \mathbb{R}_{\geq 0}(2,-1)$. Since the statement of Claim 4.1.14 clearly holds for $\mathrm{Scatter}^2(\mathfrak{D}^0_{\mathrm{pert}})$, this completes the proof of Theorem 4.1.1. □

Remark 4.1.15. The application of Lemma 4.1.12 to prove Claim 4.1.14 mirrors the statement that the boundary joints of $\alpha_f \overline{\mathfrak{D}_{(\Sigma_g, H_g)}}$ are convex – the product of scattering of a set of walls, considered as rays propagating in a particular direction, follows the possible trajectories of broken lines on a wall structure.

4.2 The intermediate wall structure

In order to use Theorem 4.1.1 to relate the mirrors to f and g, we construct a wall structure from the scattering diagram $\mathfrak{D}_{\mathrm{pert}}$. This wall structure will define a scheme which interpolates between $\mathfrak{X}_f$ and $\mathfrak{X}_g$. The wall structure turns out to be determined by a three-dimensional log Calabi–Yau pair $(\tilde{Y}, \tilde{D})$, equipped with a toric model that relates the toric models associated to f and g.

In Subsection 4.2.1 we construct this log Calabi–Yau pair and its Gross–Siebert mirror. The pair $(\tilde{Y}, \tilde{D})$ is in fact the total space of a degeneration of (Y_g, D_g) over $(\mathbb{A}^1, 0)$. As a *relative log scheme*, the pair $(\tilde{Y}, \tilde{D})$ therefore has a two-dimensional mirror family $\mathfrak{X}_\sim$, which can be constructed from the canonical wall structure associated to $(\tilde{Y}, \tilde{D})$. The statements about scattering diagrams proved in the previous section now translate to statements about the schemes defined by the resulting algorithmic wall structures:

1. The equivalence of scattering diagrams $(\mathfrak{D}_{\text{pert}})_{\text{as}} \equiv \mathfrak{D}_{(\Sigma_g, H_g)}$ implies that $\mathfrak{X}_\sim$ is a projective scheme over the affine mirror $\mathfrak{X}_g$ (after an appropriate change of base).

2. Theorem 4.1.1 implies that there is a natural inclusion of the locally-constructed mirrors $\mathfrak{X}_f^\circ \hookrightarrow \mathfrak{X}_\sim^\circ$ (after an appropriate change of base).

We leave the issue of changing the bases of our mirror families to Section 4.3.

Composing the two statements above is enough to see that the mirrors to f and g are birational: we have a morphism $\mathfrak{X}_f^\circ \hookrightarrow \mathfrak{X}_\sim^\circ \hookrightarrow \mathfrak{X}_\sim \to \mathfrak{X}_g$. In order to extend this birational map to a morphism

$$\overline{\mathfrak{X}_g} \longrightarrow \overline{\mathfrak{X}_f}$$

between the compactified mirrors, however, we need a wall structure with a truncation that interpolates between the cones over Newt g and Newt f. In Subsection 4.2.2 we construct this wall structure, called the *intermediate wall structure* $\mathfrak{D}_{f \leftrightarrow g}$, and use it to define a partial compactification $\overline{\mathfrak{X}_\sim}$ of $\mathfrak{X}_\sim$.

4.2.1 The mirror to $(\tilde{Y}, \tilde{D})$

We define a family of log Calabi–Yau pairs over $\mathbb{A}^1$ with general fibre isomorphic to (Y_g, D_g) as follows.

Construction 4.2.1 (The degeneration $(\tilde{Y}, \tilde{D})$ and its toric model)**.** Noting that the projection

$$Y_{\Sigma_g} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$

is given by the canonical morphism of fans

$$\Sigma_g \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0},$$

we define a toric blow-up

$$Y_{\tilde{\Sigma}} \longrightarrow Y_{\Sigma_g} \times \mathbb{A}^1$$

by letting $\tilde{\Sigma}$ be the refinement of the fan $\Sigma_g \times \mathbb{R}_{\geq 0}$ by the ray $\mathbb{R}_{\geq 0}(-1,-1,1)$. This is the blow-up of $Y_{\Sigma_g} \times \mathbb{A}^1$ in the torus-fixed point in $Y_{\Sigma_g} \times \{0\}$ corresponding to the top right corner of $\operatorname{Newt} g$. Let

$$\epsilon : Y_{\tilde{\Sigma}} \longrightarrow \mathbb{A}^1$$

be the composition of the blow-up map and the projection to $\mathbb{A}^1$. This defines a flat family over $\mathbb{A}^1$ with general fibre Y_{Σ_g} and central fibre consisting of two irreducible components: one equal to Y_{Σ_f} and one equal to $\mathbb{P}^2$.

If ρ is a ray in Σ_g then

$$(\rho, 0) := \rho \times \{0\} \subset M_{\mathbb{R}} \times \mathbb{R}_{\geq 0} \tag{4.5}$$

is a ray in $\tilde{\Sigma}$. If $\rho \neq \mathbb{R}_{\geq 0}(-1,0)$ or $\mathbb{R}_{\geq 0}(0,-1)$, then the toric divisor $D_{(\rho,0)} \subset Y_{\tilde{\Sigma}}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{A}^1$, where we identify $\mathbb{P}^1 \cong D_\rho$ as a toric boundary component of Y_{Σ_g}. When $\rho \in \Sigma_g$ is equal to $\mathbb{R}_{\geq 0}(-1,0)$ or $\mathbb{R}_{\geq 0}(0,-1)$, the toric divisor $D_{(\rho,0)}$ is isomorphic to $\mathrm{Bl}_{(0,0)}(\mathbb{P}^1 \times \mathbb{A}^1)$. We define the collection of hypersurfaces of the toric boundary

$$\tilde{H} = H_a \cup H_b \cup H_c \cup H_d \quad \subset D_{\tilde{\Sigma}}$$

as follows. Let $\rho_a := \mathbb{R}_{\geq 0}(2,-1)$ and $\rho_b := \mathbb{R}_{\geq 0}(-1,2)$, and let

$$H_i := \{p_i\} \times \mathbb{A}^1 \subset \mathbb{P}^1 \times \mathbb{A}^1 \cong D_{(\rho_i,0)}$$

for $i \in \{a,b\}$, where $p_i \in \mathbb{P}^1 \cong D_{\rho_i}$ is a general point. Let $\rho_c := \mathbb{R}_{\geq 0}(-1,0)$ and $\rho_d := \mathbb{R}_{\geq 0}(0,-1)$, and for $i \in \{c,d\}$ let $H_i \subset D_{(\rho_i,0)}$ be a curve isomorphic to $\mathbb{A}^1$, which intersects the exceptional divisor transversely and does not intersect the toric curves corresponding to codimension one strata in $\tilde{\Sigma} \cap (M_{\mathbb{R}} \times \{0\})$. We then define

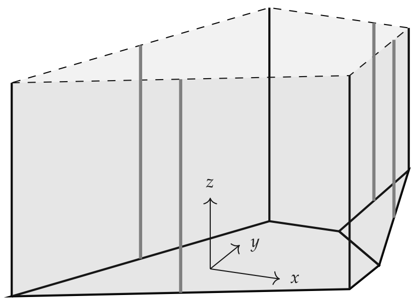

Figure 4.4: A schematic version P of the moment polytope of $Y_{\tilde{\Sigma}}$: the normal fan of P is a coarsening of $\tilde{\Sigma}$, obtained by forgetting the rays generated by $(1,0,0),(0,1,0),(1,-1,0),(-1,1,0)$. The four hyperplanes which form $\tilde{H}$ are shown as grey lines on the corresponding faces of P.

$(\tilde{Y},\tilde{D})$ to be the three-dimensional log Calabi–Yau pair with toric model $(Y_{\tilde{\Sigma}},\tilde{H})$. Note that

$$\epsilon^{-1}(t) \cong Y_g \quad \text{and} \quad \epsilon^{-1}(t) \cap \tilde{D} \cong D_g$$

for all $t \neq 0 \in \mathbb{A}^1$.

Lemma 4.2.2. *There is an equivalence of three-dimensional scattering diagrams between the HDTV scattering diagram for the toric model $(Y_{\tilde{\Sigma}},\tilde{H})$ and the cone over the perturbed scattering diagram:*

$$\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})} \equiv \mathbf{C}\mathfrak{D}_{\mathrm{pert}}.$$

Proof. Given a ray $\rho \in \Sigma_g$, we may define a two-cell of $\tilde{\Sigma}$ by

$$\tilde{\rho} := (\rho, 0) + \mathbb{R}_{\geq 0}(0,0,1), \tag{4.6}$$

where $(\rho,0)$ is the ray in $\tilde{\Sigma}$ defined in (4.5). Following Construction 2.3.4,

$$\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})} = \mathrm{Scatter}\left(\bigcup_{i=1}^{4}\mathfrak{D}_i\right),$$

where

$$\begin{aligned}
\mathfrak{D}_1 &= \left\{\left(\tilde{\rho}_a,\ 1 + az^{(2,-1,0)}\right)\right\},\\
\mathfrak{D}_2 &= \left\{\left(\tilde{\rho}_b,\ 1 + bz^{(-1,2,0)}\right)\right\},\\
\mathfrak{D}_3 &= \left\{\left((\rho_c, 0) + \mathbb{R}_{\geq 0}(-1,-1,1),\ 1 + cz^{(-1,0,0)}\right)\right\}\\
\text{and}\quad \mathfrak{D}_4 &= \left\{\left((\rho_d, 0) + \mathbb{R}_{\geq 0}(-1,-1,1),\ 1 + dz^{(0,-1,0)}\right)\right\}
\end{aligned}$$

It is easy to see that the slice of the initial scattering diagram at height 1 equals the initial scattering diagram defining the perturbed scattering diagram:

$$\left(\bigcup_{i=1}^{4} \mathfrak{D}_i\right) \cap M_{\mathbb{R}} \times \{1\} = \mathfrak{D}^0_{\mathrm{pert}}.$$

Taking the cone over $\mathfrak{D}^0_{\mathrm{pert}}$ does not change the wall functions – for each incoming ray $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}^0_{\mathrm{pert}}$, the map

$$\Lambda_{\mathbf{C}\mathfrak{d}} \longrightarrow \mathbb{Z}$$

induced by projection to the height of the cone is surjective, so we have

$$\mathbf{C}\mathfrak{D}^0_{\mathrm{pert}} = \left\{(\mathbf{C}\mathfrak{d}, f_{\mathfrak{d}}) \,\middle|\, (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}^0_{\mathrm{pert}}\right\}.$$

Therefore

$$\bigcup_{i=1}^{4} \mathfrak{D}_i = \mathbf{C}\mathfrak{D}^0_{\mathrm{pert}},$$

and so by uniqueness of scattering we have

$$\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})} = \mathrm{Scatter}\left(\bigcup_{i=1}^{4} \mathfrak{D}_i\right) = \mathrm{Scatter}\left(\mathbf{C}\mathfrak{D}^0_{\mathrm{pert}}\right) = \mathbf{C}\left(\mathrm{Scatter}\left(\mathfrak{D}^0_{\mathrm{pert}}\right)\right) = \mathbf{C}\mathfrak{D}_{\mathrm{pert}}.$$

□

Notation 4.2.3 (The Gross–Siebert locus and algorithmic wall structure associated to $(\tilde{Y}, \tilde{D})$)**.** Let $\tilde{Q}^{\mathrm{gs}}$ denote the monoid defining the Gross–Siebert locus for the log Calabi–Yau pair $(\tilde{Y}, \tilde{D})$:

$$\tilde{Q}^{\mathrm{gs}} := \mathrm{Bl}^*_{\tilde{H}}\left(NE\left(Y_{\tilde{\Sigma}}\right)\right) \oplus E_{abcd}, \tag{4.7}$$

where here E_{abcd} the subgroup of $N_1\left(\tilde{Y}\right)$ generated by the exceptional curves over the hyperplanes H_i. We carry through the notation for rings that we set up in Definition 2.3.7 and Remark 2.3.8 – in particular, we let $\tilde{R}^{\mathrm{gs}}$ denote the completion of $\Bbbk[\tilde{Q}^{\mathrm{gs}}]$ with respect to its maximal ideal $\tilde{\mathfrak{m}}^{\mathrm{gs}}$. We let $\tilde{\psi}$ denote the $\tilde{Q}^{\mathrm{gs}}$-valued piecewise linear function on the polyhedral affine manifold $(M_{\mathbb{R}}\times\mathbb{R}_{\geq 0}, \tilde{\Sigma})$ as defined in Construction 2.3.12. We denote the algorithmic wall structure associated to $(\tilde{Y}, \tilde{D})$ by $\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$.

Construction 4.2.4 (The open intermediate family $\mathfrak{X}_{\sim}$)**.** The restriction of the $\tilde{Q}^{\mathrm{gs}}$-valued piecewise linear function $\tilde{\psi}$ to the slice $M_{\mathbb{R}}\times\{1\}$ is a piecewise *affine* function on the polyhedral affine manifold

$$\left(M_{\mathbb{R}}, \tilde{\Sigma}\cap(M_{\mathbb{R}}\times\{1\})\right). \tag{4.8}$$

Applying the steps of Construction 2.3.12 to $\mathfrak{D}_{\mathrm{pert}}$ and the restriction of $\tilde{\psi}$ gives us a consistent wall structure $\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}$ on the polyhedral affine manifold (4.8) equipped with the piecewise affine function $\tilde{\psi}\big|_{M_{\mathbb{R}}\times\{1\}}$ such that

$$\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})} \equiv \mathbf{C}\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}. \tag{4.9}$$

Thus Theorem 1.5.7 applies to $\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$. In particular, the algebra of theta functions

$$\tilde{S} := R(\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}/J)$$

is a graded $\tilde{R}^{\mathrm{gs}}/J$-algebra for any $\tilde{\mathfrak{m}}^{\mathrm{gs}}$-primary ideal $J\subset\tilde{Q}^{\mathrm{gs}}$, so we can define a formal scheme

$$\mathfrak{X}_{\sim} := \operatorname{colim}_{\sqrt{J}=\tilde{\mathfrak{m}}^{\mathrm{gs}}}\operatorname{Proj}\tilde{S}$$

over $\operatorname{Spf}\tilde{R}^{\mathrm{gs}}$.

Remark 4.2.5 (The relative mirror construction)**.** The degeneration $\tilde{Y}\to\mathbb{A}^1$ satisfies the *relative log Calabi–Yau* case described in [31, Section 1.1]; there is a projective log smooth morphism $\epsilon:\tilde{Y}\to\mathbb{A}^1$, where the varieties are considered as log schemes via the divisorial log structures coming from $\tilde{D}$ and 0 respectively. The scheme $\mathfrak{X}_{\sim}$ produced by Construction 4.2.4 above is the canonical mirror to $(\tilde{Y},\tilde{D})$ as a relative log Calabi–Yau pair, restricted to the Gross–Siebert locus.

Construction 4.2.6 (The local construction and affine scheme related to $\mathfrak{X}_\sim$)**.** By the equivalence of wall structures (4.9) and Theorem 1.5.7, there is an open embedding

$$\mathfrak{X}^\circ\left(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}/J\right) \hookrightarrow \operatorname{Proj}\tilde{S}$$

over $\operatorname{Spec}\tilde{R}^{\mathrm{gs}}/J$. Moreover, there is a natural birational map

$$\operatorname{Proj}\tilde{S} \longrightarrow \operatorname{Spec}\tilde{S}_0,$$

over $\operatorname{Spec}\tilde{R}^{\mathrm{gs}}/J$, where $\tilde{S}_0$ is the zero-graded part of $\tilde{S}$. Recall the open neighbourhood of the origin $U \subset M_{\mathbb{R}}$ from Theorem 4.1.1. Let

$$\mathfrak{X}^\circ\left((\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}/J) \cap U\right)$$

be the family over $\operatorname{Spec}\tilde{R}^{\mathrm{gs}}/J$ defined by gluing local charts defined for all chambers and slabs intersecting U. Clearly this family is open and dense in $\mathfrak{X}^\circ\left(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}/J\right)$, and so we have an open embedding

$$\mathfrak{X}^\circ\left((\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}/J) \cap U\right) \hookrightarrow \mathfrak{X}^\circ\left(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}/J\right) \hookrightarrow \operatorname{Proj}\tilde{S}.$$

Remark 4.2.7 (Idea of the proof of the main theorem)**.** In Section 4.3 we will define a sublocus T of the Gross–Siebert locus such that, after changing the base appropriately, we have isomorphisms of formal schemes

$$\begin{aligned} \operatorname{Spf}\tilde{S}_0 \times_{\operatorname{Spf}\tilde{R}^{\mathrm{gs}}} T &\cong \mathfrak{X}_g \times_{\operatorname{Spf}R_g^{\mathrm{gs}}} T \\ \text{and} \quad \mathfrak{X}^\circ\left(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}} \cap U\right) \times_{\operatorname{Spf}\tilde{R}^{\mathrm{gs}}} T &\cong \mathfrak{X}_f^\circ \times_{\operatorname{Spf}R_f^{\mathrm{gs}}} T. \end{aligned}$$

It follows that $\mathfrak{X}_f$ and $\mathfrak{X}_g$ are birational; we have a T-birational map

$$\mathfrak{X}_f^\circ \times_{\operatorname{Spf}R_f^{\mathrm{gs}}} T \longrightarrow \mathfrak{X}_g \times_{\operatorname{Spf}R_g^{\mathrm{gs}}} T. \tag{4.10}$$

The rest of this section will be devoted to constructing a certain compactification $\overline{\mathfrak{X}_\sim}$ of $\mathfrak{X}_\sim$ which will enable us to extend (4.10) to a birational map between the compactifications of the mirrors to f and g,

$$\begin{array}{ccc} & \overline{\mathfrak{X}_\sim} & \\ \swarrow & & \searrow \\ \overline{\mathfrak{X}_f} & \dashrightarrow & \overline{\mathfrak{X}_g} \end{array} \tag{4.11}$$

In Chapter 5, analysis of the birational morphisms in (4.11) will show that $\overline{\mathfrak{X}}_g$ is the blow-up of $\mathfrak{X}_f$ in a point on the boundary $\overline{\mathfrak{X}_f} \setminus \mathfrak{X}_f$.

4.2.2 The intermediate wall structure

Here we will construct a four-dimensional bi-conical wall structure $\mathfrak{D}_{f\leftrightarrow g}$, which we will later use to construct the compactification $\overline{\mathfrak{X}_\sim}$.

Construction 4.2.8 (The intermediate wall structure $\mathfrak{D}_{f\leftrightarrow g}$)**.** Consider the cone over the algorithmic wall structure for $(\tilde{Y}, \tilde{D})$. This is a four-dimensional, conical wall structure

$$\mathbf{C}\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$$

on the polyhedral affine manifold $\mathbf{C}(M_{\mathbb{R}} \times \mathbb{R}_{\geq 0}, \tilde{\Sigma})$ equipped with piecewise linear function $\mathbf{C}\tilde{\psi}$. Since $\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$ is itself conical, the cone over it is equal to the product of $\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$ and $\mathbb{R}_{\geq 0}$ – that is,

$$\mathbf{C}(\mathfrak{d}, f_{\mathfrak{d}}) = (\mathfrak{d} \times \mathbb{R}_{\geq 0}, f_{\mathfrak{d}})$$

for all walls $(\mathfrak{d}, f_{\mathfrak{d}}) \in \tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$. In particular,

$$\mathbf{C}(M_{\mathbb{R}} \times \mathbb{R}_{\geq 0}, \tilde{\Sigma}) \cong \left(M_{\mathbb{R}} \times (\mathbb{R}_{\geq 0})^2,\ \tilde{\Sigma} \times \mathbb{R}_{\geq 0}\right)$$

and, denoting coordinates on $M_{\mathbb{R}} \times (\mathbb{R}_{\geq 0})^2$ by (x, y, t, s), we have

$$\mathbf{C}\tilde{\psi}(x, y, t, s) = \tilde{\psi}(x, y, t).$$

We construct $\mathfrak{D}_{f\leftrightarrow g}$ by restricting $\mathbf{C}\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$ to a conical polyhedral subset

$$B_{f\leftrightarrow g} \subset M_{\mathbb{R}} \times (\mathbb{R}_{\geq 0})^2,$$

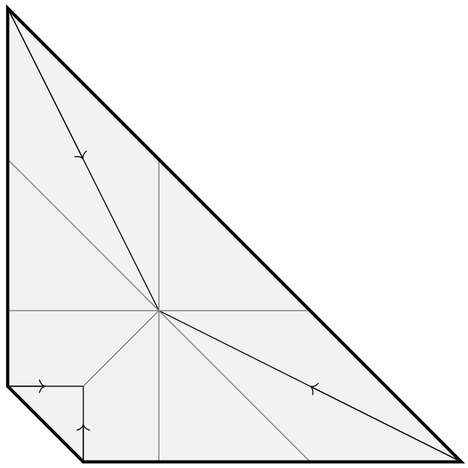

Figure 4.5: The cross section of $B_{f\leftrightarrow g}$ at height (1,1). The cross-sections of incoming walls in $\mathbf{C}\mathbf{C}\mathfrak{D}_{\text{pert}}$ are shown in black, and the cross-sections of the remaining codimension-one cells of $\tilde{\Sigma} \times \mathbb{R}_{\geq 0}$ are shown in grey.

adding in a boundary as follows. We define the two-dimensional slice of $B_{f\leftrightarrow g}$ at height (a, b) to be

$$B_{f\leftrightarrow g} \cap \{t = a, s = b\} := \text{Conv}\left\{\begin{matrix} a(-1,-1) + b(-1,0), \\ a(-1,-1) + b(0,-1), \\ (a+b)(2,-1), \\ (a+b)(-1,2) \end{matrix}\right\}. \tag{4.12}$$

Figure 4.5 shows the slice of $B_{f\leftrightarrow g}$ at height $(1, 1)$. The four-dimensional polytope $B_{f\leftrightarrow g}$ is then defined to be the convex hull of the $\mathbb{N}^2$-indexed set of these two-dimensional polyhedra. The intermediate wall structure $\mathfrak{D}_{f\leftrightarrow g}$ is simply defined as the restriction of $\mathbf{C}\tilde{\alpha}(\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})})$ to $(B_{f\leftrightarrow g}, \tilde{\Sigma} \times \mathbb{R}_{\geq 0})$.

Remark 4.2.9 ($\mathfrak{D}_{f\leftrightarrow g}$ is bi-conical)**.** First, let us describe the polyhedral affine manifold $(B_{f\leftrightarrow g}, \tilde{\Sigma} \times \mathbb{R}_{\geq 0})$ in more detail. The four-dimensional cone $B_{f\leftrightarrow g}$ has 6 facets: one given by $B_{f\leftrightarrow g} \cap \{t = 0\}$, one given by $B_{f\leftrightarrow g} \cap \{s = 0\}$ and the others given by the convex hulls, over $(a, b) \in \mathbb{N}^2$, of each of the four facets of $B_{f\leftrightarrow g} \cap \{t = a, s = b\}$. Moreover, the polyhedral affine manifold is a cone with respect to both the t-height and the s-height:

$$\begin{aligned}\left(B_{f\leftrightarrow g}, \tilde{\Sigma}\times\mathbb{R}_{\geq 0}\right) &= \mathbf{C}\left(B_{f\leftrightarrow g}\cap\{t=1\},\ \left(\tilde{\Sigma}\cap\{t=1\}\right)\times\mathbb{R}_{\geq 0}\right)\\ &= \mathbf{C}\left(B_{f\leftrightarrow g}\cap\{s=1\},\ \tilde{\Sigma}\times\{s=1\}\right)\end{aligned}$$

Since we defined $\mathfrak{D}_{f\leftrightarrow g}$ to be the restriction of $\mathbf{C}\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$, the cone over a conical wall structure, to $\left(B_{f\leftrightarrow g}, \tilde{\Sigma}\times\mathbb{R}_{\geq 0}\right)$, it is easy to see that

$$\mathfrak{D}_{f\leftrightarrow g} = \mathbf{C}\left(\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}\cap\left(B_{f\leftrightarrow g}\cap\{s=1\}\right)\right) = \mathbf{C}\left(\mathfrak{D}_{f\leftrightarrow g}\cap\{s=1\}\right). \tag{4.13}$$

By Lemma 4.2.2, we can write

$$\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})} = \mathbf{C}\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}},$$

where $\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}$ is a consistent wall structure on $(M_{\mathbb{R}}, \tilde{\Sigma}\cap\{t=1\})$ equipped with the (single-valued) piecewise *affine* function $\tilde{\psi}\big|_{\{t=1\}}$. Therefore, $\mathfrak{D}_{f\leftrightarrow g}$ is a cone with respect to the t-height as well:

$$\mathfrak{D}_{f\leftrightarrow g} = \mathbf{C}\left((\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}\times\mathbb{R}_{\geq 0})\ \cap\ (B_{f\leftrightarrow g}\cap\{t=1\})\right).$$

Lemma 4.2.10 ($\mathfrak{D}_{f\leftrightarrow g}$ is consistent)**.** *The wall structure $\mathfrak{D}_{f\leftrightarrow g}$ on the polyhdral affine manifold $(B_{f\leftrightarrow g}, \tilde{\Sigma}\times\mathbb{R}_{\geq 0})$, equipped with the piecewise linear function $\mathbf{C}\tilde{\psi}$, is consistent in the sense of Definition 1.4.7.*

Proof. By Lemma 1.5.6 and (4.13), it is sufficient to check that $\mathfrak{D}_{f\leftrightarrow g}\cap\{s=1\}$ is a consistent wall structure on the polyhedral affine manifold

$$(B_{f\leftrightarrow g}\cap\{s=1\}, \tilde{\Sigma}\times\{s=1\})$$

equipped with piecewise linear function $\tilde{\psi}$. Consistency of the interior joints of $\mathfrak{D}_{f\leftrightarrow g}\cap\{s=1\}$ follows from the consistency of $\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$. By Proposition 1.4.12, all that is left is to check convexity of the boundary joints. Since $B_{f\leftrightarrow g}\cap\{s=1\}$ is a convex polyhedron, any boundary joint not contained in an incoming wall is automatically convex. Let $\mathfrak{j}$ be a joint incident to one of the four incoming walls

$$\left(\mathfrak{d}_i \cap (B_{f\leftrightarrow g} \cap \{s=1\}), f_{\mathfrak{d}_i}\right)$$

where $i \in \{a,b,c,d\}$. Then, following the notation of Definition 1.4.11, we have

$$\pi\left((\mathfrak{D}_{f\leftrightarrow g} \cap \{s=1\})_{\mathfrak{j}}\right) \cong (\mathfrak{D}_{f\leftrightarrow g} \cap \{s=1, t=1\})_{(\mathfrak{j}\cap\{s=1,t=1\})}.$$

Convexity of a joint is independent of the piecewise linear function on the polyhedral affine manifold or the monoid in which it takes values, so convexity of a joint on $\mathfrak{D}_{f\leftrightarrow g} \cap \{s=1, t=1\}$ is equivalent to convexity of the same (i.e. set-theoretic equal) joint on $\mathfrak{D}_{\text{pert}} \cap (B_{f\leftrightarrow g} \cap \{s=1, t=1\})$. The convexity of $(\mathfrak{j} \cap \{s=1, t=1\})$ on $\mathfrak{D}_{\text{pert}} \cap (B_{f\leftrightarrow g} \cap \{s=1, t=1\})$ follows by Claim 4.1.14. □

Lemma 4.2.11 ($R(\mathfrak{D}_{f\leftrightarrow g})$ is bigraded)**.** *Let $J \subset \tilde{Q}^{\text{gs}}$ be a $\tilde{\mathfrak{m}}^{\text{gs}}$-primary ideal. The algebra of theta functions $R(\mathfrak{D}_{f\leftrightarrow g}/J)$ is bi-graded with respect to the t-height and s-height of the asymptotic monomials.*

Proof. In Remark 4.2.9, we showed that $\mathfrak{D}_{\text{pert}}$ is a conical wall structure with respect to both the t-height and the s-height. It follows that the theta functions on $\mathfrak{D}_{f\leftrightarrow g}$ are indexed by the integral points of $B_{f\leftrightarrow g}(\mathbb{Z})$, and that the ring of theta functions is graded with respect to both heights. □

Corollary 4.2.12. *The limit of the algebras of theta functions $R(\mathfrak{D}_{f\leftrightarrow g}/J)$ along the compatible system $\mathfrak{D}_{f\leftrightarrow g}$ is a bi-graded $\tilde{R}^{\text{gs}}$-algebra*

$$R(\mathfrak{D}_{f\leftrightarrow g}) := \bigoplus_{m\in B_{f\leftrightarrow g}(\mathbb{Z})} \tilde{R}^{\text{gs}} \cdot \vartheta_m$$

Construction 4.2.13 (The intermediate family)**.** Denote the bi-graded algebra of theta functions on $\mathfrak{D}_{f\leftrightarrow g}$ by $S_{\bullet,\bullet} := R(\mathfrak{D}_{f\leftrightarrow g})$, and the (a,b)-graded piece by $S_{a,b}$. We define the *intermediate family* to be

$$\overline{\mathfrak{X}_{\sim}} := \operatorname{Proj} \bigoplus_{k\in\mathbb{N}} S_{k,k}. \tag{4.14}$$

This is a scheme over Spec $\tilde{R}^{\mathrm{gs}}$, and it is the algebraisation of the formal scheme

$$\operatorname{colim}_{\sqrt{J}=\tilde{\mathfrak{m}}^{\mathrm{gs}}}\left(\overline{\mathfrak{X}_{\sim}}/J\right)$$

over Spf $\tilde{R}^{\mathrm{gs}}$, where for each $\tilde{\mathfrak{m}}^{\mathrm{gs}}$-primary ideal J, the scheme

$$\overline{\mathfrak{X}_{\sim}}/J := \operatorname{Proj}\bigoplus_{k\in\mathbb{N}} S_{k,k}/J$$

is the pullback of $\overline{\mathfrak{X}_{\sim}}$ via the map Spec $\tilde{R}^{\mathrm{gs}}/J \to$ Spec $\tilde{R}^{\mathrm{gs}}$.

Remark 4.2.14 (Local construction of the intermediate family)**.** In the next section (Lemma 4.3.15) we will prove that

$$\bigoplus_{k\in\mathbb{N}} S_{k,k} \cong R(\mathfrak{D}_{f\leftrightarrow g}\cap\{s=t\}).$$

It follows from this that $\overline{\mathfrak{X}_{\sim}}$ can be in some sense be locally constructed by charts defined by a truncation of the wall structure $\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}$, given by $B_{f\leftrightarrow g}\cap\{s=t=1\}$ (see Figure 4.5). More precisely, Theorem 1.5.7 says that there is an open embedding

$$\mathfrak{X}^{\circ}\left(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}/J\cap\left(B_{f\leftrightarrow g}\cap\{s=t=1\}\right)\right)\hookrightarrow\overline{\mathfrak{X}_{\sim}}/J$$

of schemes over Spec $\tilde{R}^{\mathrm{gs}}/J$ for any $\tilde{\mathfrak{m}}^{\mathrm{gs}}$-primary ideal J. We see that the formal scheme associated to $\overline{\mathfrak{X}_{\sim}}$ is a compactification of $\mathfrak{X}_{\sim}$, the open intermediate family $(\tilde{Y},\tilde{D})$ defined in Construction 4.2.4.

This concludes the discussion of the relationships between the open mirror family $\mathfrak{X}_{\sim}$ and the affine mirror families $\mathfrak{X}_f$ and $\mathfrak{X}_g$. In the rest of this chapter we will focus on the compactified mirrors, $\overline{\mathfrak{X}_{\sim}}$, $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$, without needing to pass to affine charts. We thus have no more need for the language of formal schemes until Chapter 5.

The following proposition puts the bi-conical structure of $\mathfrak{D}_{f\leftrightarrow g}$ to use to construct maps from $\overline{\mathfrak{X}_{\sim}}$ to two more families over Spec $\tilde{R}^{\mathrm{gs}}$.

Proposition 4.2.15 (Morphisms to Proj $S_{\bullet,0}$ and Proj $S_{0,\bullet}$)**.** *There are natural morphisms*

$$\begin{array}{ccc} & \overline{\mathfrak{X}_{\sim}} & \\ \swarrow & & \searrow \\ \operatorname{Proj} S_{\bullet,0} & & \operatorname{Proj} S_{0,\bullet}. \end{array} \tag{4.15}$$

of schemes over Spec $\tilde{R}^{\mathrm{gs}}$*.*

Proof. Let π_t and π_s be the canonical projections

$$\begin{array}{ccc} & \operatorname{Spec} S_{\bullet,\bullet} & \\ {}^{\pi_s}\swarrow & & \searrow^{\pi_t} \\ \operatorname{Spec} S_{\bullet,0} & & \operatorname{Spec} S_{0,\bullet}. \end{array}$$

The bi-graded $\tilde{R}^{\mathrm{gs}}$-algebra $S_{\bullet,\bullet}$ is finitely generated by $S_{1,0}$ and $S_{0,1}$, and the $\mathbb{G}_m^2$-action on Spec $S_{\bullet,\bullet}$ is the composition of the action of its first factor on $S_{0,\bullet}$ and the second factor on $S_{\bullet,0}$. Thus the projections π_t and π_s are $\mathbb{G}_m^2$-equivariant. The $\mathbb{G}_m^2$-invariant sections of

$$\mathcal{O}_{\operatorname{Spec} S_{\bullet,0}}(k), \quad \mathcal{O}_{\operatorname{Spec} S_{0,\bullet}}(k) \quad \text{and} \quad \left(\pi_s^*\mathcal{O}(1) \otimes \pi_t^*\mathcal{O}(1)\right)^{\otimes k},$$

with linearisations given by the bi-grading on $S_{\bullet,\bullet}$, are $S_{k,0}$, $S_{0,k}$, and $S_{k,k}$ respectively. Noting that

$$S_{+,+} = S_{\bullet,+} \cdot S_{+,\bullet},$$

we see that the respective images of the semistable locus of Spec $S_{\bullet,\bullet}$ under π_s and π_t are contained in the semistable loci of Spec $S_{\bullet,0}$ and Spec $S_{0,\bullet}$. Therefore, π_s and π_t induce maps of GIT quotients

$$\begin{array}{ccc} & \operatorname{Proj} \bigoplus_{k\in\mathbb{N}} S_{k,k} & \\ {}^{\pi_s}\swarrow & & \searrow^{\pi_t} \\ \operatorname{Proj} \bigoplus_{k\in\mathbb{N}} S_{k,0} & & \operatorname{Proj} \bigoplus_{k\in\mathbb{N}} S_{0,k} \end{array}$$

as in (4.15). ☐

4.3 Morphisms to the mirrors to f and g

In Section 4.3.1 we construct morphisms between the bases of the three families $\overline{\mathfrak{X}_\sim}$, $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$, by defining inclusions of the monoids associated to the corresponding log Calabi–Yau pairs. These morphisms are only defined after extending the monoids slightly – we call the corresponding sublocus of the base the *extended Gross–Siebert locus*.

The problem with this is that the kinks of the MPA-functions $\tilde{\psi}$ and ψ_f are no longer contained in the maximal ideals of the extended monoids. Thus the theory of Chapter 1 can no longer be applied to the wall structures $\mathfrak{D}_{f\leftrightarrow g}$ and $\alpha_f \overline{\mathfrak{D}_{(\Sigma_f,H_f)}}$ to obtain schemes over the extended Gross–Siebert locus. However, in Section 4.3.2 we prove that the two wall structures satisfy enough conditions in order to define schemes over the Gross–Siebert locus by taking Proj of the algebra of theta functions.

In Section 4.3.3 we introduce the notion of an asymptotic wall structure in order to prove that the auxiliary schemes $\overline{\mathfrak{X}_{0,\bullet}}$ and $\overline{\mathfrak{X}_{\bullet,0}}$ are pullbacks of the mirrors to f and g, as in (4.2). We also show that the intermediate family $\overline{\mathfrak{X}_\sim}$ can be constructed locally by gluing charts on a two-dimensional wall structure induced by $\mathfrak{D}_{\mathrm{pert}}$.

4.3.1 Changing the bases of the mirror families

Construction 4.3.1 (Inclusions of monoids)**.** We construct inclusions of the monoids (3.4) defining the Gross–Siebert locus for (Y_f, D_f) and (Y_g, D_g) into $\tilde{Q}^{\mathrm{gs}}$ as follows. The inclusion of Y_g into $\tilde{Y}$ as the general fibre of the family over $\mathbb{A}^1$ induces a morphism of monoids

$$\iota_g : Q_g^{\mathrm{gs}} \hookrightarrow \tilde{Q}^{\mathrm{gs}} \tag{4.16}$$

given by the natural inclusion

$$N_1\left(Y_g\right) \hookrightarrow N_1\left(\tilde{Y}\right).$$

Although is no natural inclusion of Y_f into $\tilde{Y}$, there *is* an embedding of the toric variety Y_{Σ_f} into $Y_{\tilde{\Sigma}}$ as one of the two components of the central fibre. We can therefore define a morphism of monoids

$$\iota_f : Q_f^{\mathrm{gs}} \hookrightarrow \tilde{Q}^{\mathrm{gs}} \oplus \left\langle -\tilde{E} \right\rangle, \tag{4.17}$$

where

$$\tilde{E} := \mathrm{Bl}^*_{\tilde{H}}[Y_{\mathbb{R}_{\geq 0}(-1,-1,1)+\mathbb{R}_{\geq 0}(0,0,1)}], \tag{4.18}$$

by letting ι_f send

$$\begin{aligned} \mathrm{Bl}^*_{H_f}[Y_\rho] &\longmapsto \begin{cases} \tilde{E} & \text{if } \rho = \mathbb{R}_{\geq 0}(-1,-1) \\ \mathrm{Bl}^*_{\tilde{H}}[Y_{(\rho,0)+\mathbb{R}_{\geq 0}(0,0,1)}] & \text{for all other rays } \rho \in \Sigma_f \end{cases} \\ -E_a &\longmapsto -E_a \\ -E_b &\longmapsto -E_b \\ -E_c &\longmapsto \tilde{E} - (E_c + E_d). \end{aligned}$$

We note that ι_f is well defined because the generators of the exceptional lattices E_{abc} and E_{abcd} are linearly independent, and the restriction to the other summand of Q_f^{gs} is

$$\iota_f\big|_{\mathrm{Bl}^*_{H_f}\left(NE\left(Y_{\Sigma_f}\right)\right)} = \mathrm{Bl}^*_{\tilde{H}} \circ \iota_* \circ \left(\mathrm{Bl}_{H_f *}\big|_{\mathrm{Bl}^*_{H_f}\left(NE\left(Y_{\Sigma_f}\right)\right)}\right)$$

where $\iota_* : NE\left(Y_{\Sigma_f}\right) \hookrightarrow NE\left(Y_{\tilde{\Sigma}}\right)$ is induced by the inclusion of toric varieties.

Note that it was necessary to extend the monoid $\tilde{Q}^{\mathrm{gs}}$ in order for the morphism ι_f to be well-defined. This motivates the definition of a further sublocus of the base of the family $\overline{\mathfrak{X}_{\sim}}$:

Definition 4.3.2 (The extended Gross–Siebert locus)**.** We define the monoids

$$\tilde{Q}^{\mathrm{egs}} := \tilde{Q}^{\mathrm{gs}} \oplus \langle -\tilde{E} \rangle \quad \text{and} \quad Q_f^{\mathrm{egs}} := Q_f^{\mathrm{gs}} \oplus \langle -E_f \rangle, \tag{4.19}$$

where $\tilde{E} \in \tilde{Q}^{\mathrm{gs}}$ is the curve class defined in (4.18) and

$$E_f := \mathrm{Bl}^*_{H_f}[Y_{\mathbb{R}_{\geq 0}(-1,-1)}] \in Q_f^{\mathrm{gs}}$$

is the preimage of $\tilde{E}$ under the map ι_f. If Q^{egs} denotes either of the monoids in (4.19), we denote its maximal ideal by $\mathfrak{m}^{\mathrm{egs}}$, and the completion of the associated semigroup ring by

$$R^{\mathrm{egs}} := \widehat{\Bbbk[Q^{\mathrm{egs}}]}.$$

We use the term *extended Gross–Siebert locus* to refer either of the schemes Spec R^{egs} or Spec R^{egs}/G, where G is an $\mathfrak{m}^{\mathrm{egs}}$-primary ideal. When necessary, we use the same term to refer to the formal scheme Spf R^{egs}.

Lemma 4.3.3. *The monoid Q^{egs} is a convex cone containing Q^{gs}, the monoid associated to the Gross–Siebert locus associated to the respective toric models for $(\tilde{Y}, \tilde{D})$ and (Y_f, D_f). The invertible elements of Q^{egs} are given by*

$$(Q^{\mathrm{egs}})^{\times} = E \oplus L,$$

where E is the lattice in $N_1(Y)$ generated by the classes of the blow-up $Y \longrightarrow Y_{\Sigma}$, and $L \cong \mathbb{Z}$ is the lattice generated by the respective curves $\tilde{E} \in N_1(\tilde{Y})$ or $E_f \in N_1(Y_f)$.

Proof. In case of the pair $(\tilde{Y}, \tilde{D})$, this is because $Y_{\mathbb{R}_{\geq 0}(-1,-1,1)+\mathbb{R}_{\geq 0}(0,0,1)}$ is an exceptional curve of the blow-up

$$Y_{\tilde{\Sigma}} \longrightarrow Y_{\Sigma_g} \times \mathbb{A}^1$$

defined by the refinement of fans $\tilde{\Sigma} \to \Sigma_g \times \mathbb{R}_{\geq 0}$. Thus $\tilde{E}$ generates an extremal ray of the cone $\mathrm{Bl}^*_{\tilde{H}}\left(NE\left(\tilde{Y}\right)\right)$. This ray must lie on the boundary of the cone $\tilde{Q}^{\mathrm{gs}}$, since the lattice E_{abcd} is linearly independent from $\mathrm{Bl}^*_{\tilde{H}}\left(NE\left(\tilde{Y}\right)\right)$. Therefore, inverting $\tilde{E}$ preserves the convexity of $\tilde{Q}^{\mathrm{gs}}$, and the lattice $E_{abcd} \oplus \langle \tilde{E} \rangle \simeq \mathbb{Z}^5$ is a face of Q^{egs}. In the case of (Y_f, D_f), convexity is preserved because $Y_{\mathbb{R}_{\geq 0}(-1,-1)}$ is an exceptional curve of the toric blow-up

$$Y_{\Sigma_f} \longrightarrow Y_{\Sigma_g}$$

determined by the refinement of fans $\Sigma_f \to \Sigma_g$, and so E_f generates an extremal ray of the convex cone $\mathrm{Bl}^*_{H_f}\left(NE\left(Y_f\right)\right)$. As in the former case, it follows that Q_f^{egs} is convex, and that the lattice $E_{abc} \oplus \langle E_f \rangle \simeq \mathbb{Z}^4$ is a face of Q_f^{egs}. □

Remark 4.3.4 (The extended Gross–Siebert locus as a subscheme of the base)**.** If $J \subset Q^{\mathrm{gs}}$ is an $\mathfrak{m}^{\mathrm{gs}}$-primary ideal, then $J \oplus L$ is an $\mathfrak{m}^{\mathrm{egs}}$-primary ideal in Q^{egs}. The diagram of rings (2.9) can thus be extended to a diagram

$$\begin{array}{ccccc} R^{\mathrm{egs}}/(I \oplus E \oplus L) & & & & \\ \uparrow & & & & \\ R'''/(I \oplus E \oplus L \cap Q^{\mathrm{gs}}) & \longrightarrow & R^{\mathrm{gs}}/(I \oplus E) & & \\ \uparrow & & \uparrow & & \\ R''/(I \oplus E \oplus L \cap Q) & \longrightarrow & R'/(I \oplus E \cap Q) & \longrightarrow & R^{\#}/I, \end{array} \tag{4.20}$$

where R'' is the completion of $\Bbbk[Q]$ with respect to the ideal $\mathfrak{m}^{\mathrm{egs}} \cap Q$, and R''' is the completion of $\Bbbk[Q^{\mathrm{gs}}]$ with respect to the ideal $\mathfrak{m}^{\mathrm{egs}} \cap Q^{\mathrm{gs}}$. Thus we may think of the extended Gross–Siebert locus $\operatorname{Spec}(R^{\mathrm{egs}}/(I \oplus E \oplus L))$ as a subscheme of the base $\operatorname{Spec} R^{\#}/I$. Moreover, this diagram of rings extends to a diagram of the completions

$$\begin{array}{ccccc} R^{\mathrm{egs}} & & & & \\ \uparrow & & & & \\ R''' & \hookrightarrow & R^{\mathrm{gs}} & & \\ \uparrow & & \uparrow & & \\ R'' & \hookrightarrow & R' & \hookrightarrow & R^{\#}. \end{array} \tag{4.21}$$

4.3.2 Wall structures over the extended Gross–Siebert locus

The intermediate wall structure $\mathfrak{D}_{f\leftrightarrow g}$ does not, strictly speaking, give a compatible of system wall structures for $\tilde{Q}^{\mathrm{egs}}$ in the sense of Definition 1.6.1, because the kinks of $\tilde{\psi}$ across three of the codimension one cells in $\tilde{\Sigma}$ are not contained in $\tilde{\mathfrak{m}}^{\mathrm{egs}}$. Similarly, the canonical and algorithmic wall structures associated to (Y_f, D_f) do not give systems of compatible wall structures for Q_f^{egs}, as the kink of ψ_f across the ray $\mathbb{R}_{\geq 0}(-1,-1) \in \Sigma_f$ is not contained in $\mathfrak{m}_f^{\mathrm{egs}}$.

However, we will see in Construction 4.3.9 that $\mathfrak{D}_{f\leftrightarrow g}$ and $\alpha_f \overline{\mathfrak{D}_{(\Sigma_f, H_f)}}$ still define schemes over the associated extended Gross–Siebert loci. The following two Lemmas establish the properties of $\mathfrak{D}_{\mathrm{can}}(f)$ and $\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}}, \tilde{H})}$ necessary to define families over the extended Gross–Siebert locus.

Lemma 4.3.5. *Let $\mathfrak{D}$ be one of the two compatible systems of wall structures $\mathfrak{D}_{\mathrm{can}}(f)$ and $\tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}}, \tilde{H})}$. Then*

(i) For every wall $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$, the wall function $f_{\mathfrak{d}} \in R^{\mathrm{gs}}$ is contained in R''', and so we can consider it to be an element of R^{egs}.

(ii) If $G \subset Q^{\mathrm{egs}}$ is an $\mathfrak{m}^{\mathrm{egs}}$-primary ideal, then $f_{\mathfrak{d}} \equiv 1 \mod G$ for all but finitely many walls $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$.

Proof in the case $\mathfrak{D} = \mathfrak{D}_{\mathrm{can}}(f)$. First we consider $\mathfrak{D} = \mathfrak{D}_{\mathrm{can}}(f)$. Since there are finitely many maximal cells $\sigma \in \mathscr{P}$, it is enough to show conditions (i) and (ii) hold after replacing $\mathfrak{D}$ with $\mathfrak{D} \cap \sigma$. To do this, we compare $\mathfrak{D} \cap \sigma$ with $\Psi(\mathfrak{D} \cap \sigma)$. For each wall $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \cap \sigma$, its image under Ψ is a wall function of the scattering diagram $\alpha_f \mathfrak{D}_{(\Sigma_f, H_f)}$. If the representative of the piecewise-linear function ψ_f on Σ_f is chosen to be zero on the maximal cell $\nu(\sigma)$, then a monomial $t^\beta z^m \in \Bbbk[Q \oplus \Lambda_\sigma] \subset \Bbbk[Q^{\mathrm{egs}} \oplus \Lambda_\sigma]$ is nontrivial modulo G if and only if $(\mu_\sigma)_*(t^\beta z^m) \in \Bbbk[Q \oplus M] \subset \Bbbk[Q^{\mathrm{egs}} \oplus M]$ is nontrivial modulo G. We will say that t^β *appears in* a wall function $f_{\mathfrak{d}}$ if $\alpha t^\beta z^m$ is a summand of the formal power series $f_{\mathfrak{d}}$ for some $\alpha \in \Bbbk$ and $m \in \Lambda_{\mathfrak{d}}$. Since $\Psi(f_{\mathfrak{d}})$ only differs from $(\mu_\sigma)_*(f_{\mathfrak{d}})$ by at worst a monomial factor contained in $\Bbbk[Q^{\mathrm{egs}} \oplus M] \setminus G$, conditions (i) and (ii) hold for $\mathfrak{D} \cap \sigma$ if and only they hold for $\Psi(\mathfrak{D} \cap \sigma)$. By the equivalence of scattering diagrams

$$\Psi \mathfrak{D}_{\mathrm{can}}(f) \equiv \alpha_f \mathfrak{D}_{(\Sigma_f, H_f)} \equiv \mathrm{Scatter}\left(\alpha_f \mathfrak{D}_{\mathrm{init}}(f)\right),$$

every monomial appearing as a summand in the formal power series $\Psi(f_{\mathfrak{d}})$ is a product of multiples of the three monomials

$$t^{\psi_f(2,-1)-E_a} z^{(2,-1)}, \quad t^{\psi_f(-1,2)-E_b} z^{(-1,2)} \quad \text{and} \quad t^{\psi_f(-1,-1)-E_c} z^{(-1,-1)} \in \Bbbk[Q \oplus M],$$

where ψ_f is chosen to be zero on $\nu(\sigma)$. Therefore any monomial $t^\beta \in \Bbbk[Q]$ appears at most finitely many times in finitely many wall functions in $\Psi(\mathfrak{D} \cap \sigma)$.

It remains to show that only finitely many monomials $t^\beta \in \Bbbk[Q^{\mathrm{egs}}] \setminus G$ appear in a wall function of $\mathfrak{D}$. Recall from (2.2) that wall functions in $\mathfrak{D}_{\mathrm{can}}$ are of the form

$$f_{\mathfrak{d}} = \exp\left(\sum k_\tau N_{\boldsymbol{\tau}} t^\beta z^{-u}\right), \tag{4.22}$$

where the sum is over all wall types $\boldsymbol{\tau}$ such that $h(\tau_{\mathrm{out}}) = \mathfrak{d}$. Since Y_f is two-dimensional, we may use the equivalent definition of the canonical wall structure given in [21], via relative invariants of blowups of (Y_f, D_f) rather than punctured invariants of (Y_f, D_f). Thus we may consider the sum to be over curve classes β, and consider N_τ to denote a relative invariant associated to β and k_τ. If β is a multiple of E_i for $i \in \{a, b, c\}$, then $\mathfrak{d}$ is the slab corresponding to the component of D_f which intersects E_i. Moreover, by [21, Lemma 3.15] the contribution to $f_\mathfrak{d}$ from all such terms is

$$\exp\left(\sum_{\beta=kE_i} k_\tau N_\tau t^\beta z^{-u}\right) = (1 + t^{E_i} z^{-m_i}),$$

where $m_i \in \Lambda_\mathfrak{d}$ is the primitive tangent vector generating $\mathfrak{d}$ as a cone. Therefore, it suffices to show that only finitely many monomials $t^\beta \in \Bbbk[Q^{\mathrm{egs}}] \setminus G$ such that β is *not* a multiple of E_a, E_b or E_c appear in the argument of exp in (4.22).

Every curve class β appearing in (4.22) must be contained in $NE(Y_f)$, as $\beta = \pi_*\phi_*[C]$ for some toric blowup $\pi : \tilde{Y}_f \to Y_f$ and relative stable map $\phi : C \to \tilde{Y}_f$. Recall that there is a chain of blowups

$$Y_f \xrightarrow{p} Y_{\Sigma_f} \xrightarrow{q} Y_{\Sigma_g}, \tag{4.23}$$

where $p = \mathrm{Bl}_{H_f}$ and q is the toric blowup determined by the refinement of fans $\Sigma_f \hookrightarrow \Sigma_g$. Note that the class of the exceptional divisor of q is $[Y_{\mathbb{R}_{\geq 0}(-1,-1)}] \in NE(Y_{\Sigma_f})$, and $p^*[Y_{\mathbb{R}_{\geq 0}(-1,-1)}] = E_f$. The complement of $\mathfrak{m}^{\mathrm{egs}}$ intersects $NE(Y_f)$ in a face, so there is an ample divisor H on Y_{Σ_g} such that

$$NE(Y_f) \setminus \mathfrak{m}^{\mathrm{egs}} = NE(Y_f) \cap (p^*q^*H)^\perp.$$

Since $\sqrt{I} = \mathfrak{m}^{\mathrm{egs}}$, there is a bound n such that $\beta \in NE(Y_f) \setminus I$ implies $\beta \cdot p^*q^*H < n$, and so if $t^\beta \in \Bbbk[Q^{\mathrm{egs}}] \setminus I$ appears in a wall function of $\mathfrak{D}$, there are finitely many possible values that $q_*p_*\beta$ can take. We claim that given a choice of $\alpha = q_*p_*\beta$, there are only finitely many possible values of β which appear in a wall function of $\mathfrak{D}$. We have

$$\beta = p^*q^*\alpha + \lambda_a E_a + \lambda_b E_b + \lambda_c E_c + \lambda_f E_f$$

for some collection of $\lambda_i \in \mathbb{Z}$. For every ray $\rho \in \Sigma_f$ we have $\beta \cdot \overline{Y_\rho} \geq 0$, where $\overline{Y_\rho}$ is the strict transform of $Y_\rho \subset Y_{\Sigma_f}$. Taking $\rho = \mathbb{R}_{\geq 0}(2,-1), \mathbb{R}_{\geq 0}(-1,2)$ or $\mathbb{R}_{\geq 0}(-1,0)$, this implies that λ_a, λ_b and λ_f are bounded below for fixed α. When $\rho = \mathbb{R}_{\geq 0}(-1,-1)$, then $\beta \cdot \overline{Y_\rho} = \lambda_c - \lambda_f$, so λ_c is bounded below by λ_f; in particular $\lambda_f > 0$ implies $\lambda_c > 0$. In the proof of [21, Corollary 3.16], Gross–Hacking–Keel show that if any of

$\lambda_a, \lambda_b, \lambda_c$ are positive, then $N_\tau = 0$ unless β is a multiple of E_a, E_b or E_c. Therefore each λ_i is also bounded above, and so there are only finitely many possible values for β given $\alpha = q_* p_* \beta$. □

When the dimension of the log Calabi–Yai pair is two, the walls of $\mathfrak{D}_{\text{can}}$ and the curve classes appearing in (2.2) are particularly easy to characterise. To prove Lemma 4.3.5 for the three-dimensional wall structure $\mathfrak{D} = \tilde{\alpha}(\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})})$, however, it is easier to use the results about the geometry of the scattering diagram $\mathfrak{D}_{\text{pert}}$ from Section 4.1.

Proof in the case $\mathfrak{D} = \tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$. There are ten maximal cells $\sigma \in \tilde{\Sigma}$; we show that conditions (i) and (ii) holds for each subset $\mathfrak{D} \cap \sigma$ separately. The wall functions live in $\widehat{\Bbbk[\mathcal{P}_{\tilde{\psi}}]}$, where the completion is taken with respect to the maximal ideal $\mathfrak{m}^{\text{gs}}$ associated to the Gross–Siebert locus. We identify $\widehat{\Bbbk[\mathcal{P}_{\tilde{\psi}}]}$ with $\Bbbk[\widehat{Q^{\text{gs}} \oplus M}]$ by choosing the representative of $\tilde{\psi}$ to be zero on σ, and consider the wall functions to be formal power series in $\Bbbk[\widehat{Q^{\text{gs}} \oplus M}]$. We will say that a monomial $t^\beta z^m \in \Bbbk[Q^{\text{egs}}]$ *appears in* the wall function $f_\mathfrak{d}$ if $\alpha t^\beta z^m$ is a summand of $f_\mathfrak{d}$ as a formal power series for some $\alpha \neq 0 \in \Bbbk$.

Lemma 4.3.5 follows if there are finitely many monomials $t^\beta z^m \in \Bbbk[Q^{\text{egs}}] \setminus G$ which appear in the function $f_\mathfrak{d}$ associated to any wall $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D} \cap \sigma$. All such monomials are positive linear combinations of the monomials in the incoming walls, taking the form

$$\prod_{i\in\{a,b,c,d\}} \left(t^{\tilde{\psi}(m_i)-E_i} z^{m_i}\right)^{\lambda_i} \tag{4.24}$$

for some collection of $\lambda_i \in \mathbb{N}$.

Suppose first that σ is not contained in $\mathbb{R}_{\geq 0}\langle(-1,0,0),(0,-1,0),(0,0,1)\rangle$. Then $m_i \in \sigma$ for at most one $j \in \{a,b,c,d\}$, and therefore at most one of the incoming monomials is not contained in $\mathfrak{m}^{\text{egs}}$. Thus the assumption that the monomial (4.24) is nontrivial modulo G bounds λ_i for all $i \neq j$. It follows that the area containing $\sum_{i\neq j} \lambda_i m_i$ is bounded. Moreover, we know by Claims 4.1.13 and 4.1.14 that there exists a (non-strictly) convex cone $\tau \subset M$ such that

1. $\sigma \cap \tau = \{0\}$ and
2. all monomials $t^\beta z^m$ appearing in $f_\mathfrak{d}$ for some $\mathfrak{d} \subset \sigma$ have $m \in \tau$, apart from the

(at most one) incoming ray contained in σ.

The second condition implies $\sum_i \lambda_i m_i \in \tau$. However, since $\sum_{i\neq j} \lambda_i m_i$ is bounded and $m_j \in \sigma$, the value of λ_j must be bounded.

Now suppose that $\sigma \subset \mathbb{R}_{\geq 0}\langle(-1,0,0),(0,-1,0),(0,0,1)\rangle$. This time only $\tilde{\psi}(m_a)$ and $\tilde{\psi}(m_b)$ are contained in $\mathfrak{m}^{\mathrm{egs}}$. In the case $\sigma = \mathbb{R}_{\geq 0}\langle(-1,0,0),(0,-1,0),(-1,-1,1)\rangle$, the same argument as above shows that all the λ_i are bounded. In the case $\sigma = \mathbb{R}_{\geq 0}\langle(0,-1,0),(-1,-1,1),(0,0,1)\rangle$, the cone $\tau = \mathbb{R}_{\geq 0}\langle(1,0,0),(-1,0,0),(0,1,0)\rangle$ contains all tangent vectors $m \in M$ for which a monomial $t^{\beta}z^{m}$ appears in $f_{\mathfrak{d}}$ for some $\mathfrak{d} \subset \sigma$, apart from the two monomials appearing in the functions af the two incoming walls contained in σ:

$$\begin{aligned} \tilde{\alpha}(\mathfrak{d}_d, f_{\mathfrak{d}_d}) &= \left(\mathbb{R}_{\geq 0}\langle(0,-1,0),(-1-1,1)\rangle,\ \tilde{\alpha}_*(1+dz^{(0,-1,0)})\right) \\ \text{and}\quad \tilde{\alpha}(\mathfrak{d}_{cd}, f_{\mathfrak{d}_{cd}}) &= \left(\mathbb{R}_{\geq 0}\langle(-1-1,1),(0,0,1)\rangle,\ \tilde{\alpha}_*(1+cdz^{(-1,-1,0)})\right). \end{aligned}$$

The same argument as in the previous cases implies that if the monomial (4.24) is nontrivial modulo G and $\sum_i \lambda_i m_i \in \mathbb{R}_{\geq 0}\langle(1,0,0),(0,1,0)\rangle \subset \tau$, then the λ_i are bounded. However, Claims 4.1.13 and 4.1.14 imply that any monomial $t^{\beta}z^{m}$ appearing in a wall function in σ with $m \in \mathbb{R}_{\geq 0}\langle(-1,0,0),(0,1,0)\rangle$ must be either

(a) a product of multiples of $\tilde{\alpha}_*(dz^{(0,-1,0)})$ and a monomial $t^{\beta'}z^{m'}$ appearing in $f_{\mathfrak{d}}$ for some $\mathfrak{d} \subset \mathbb{R}_{\geq 0}\langle(-1,0,0),(0,-1,0),(-1,-1,1)\rangle$,

(b) a product of multiples of $\tilde{\alpha}_*(cdz^{(-1,-1,0)})$ and a monomial $t^{\beta'}z^{m'}$ appearing in $f_{\mathfrak{d}}$ where $\mathfrak{d} = \mathbb{R}_{\geq 0}\langle(-1,-1,1),(-1,0,1)\rangle$, or

(c) a product of multiples of monomials of the form (a), (b), and monomial summands $t^{\beta'}z^{m'}$ of $f_{\mathfrak{d}}$ such that $\mathfrak{d} \subset \sigma$ and $m' \in \mathbb{R}_{\geq 0}\langle(1,0,0),(0,1,0)\rangle$.

It remains to show that there are only finitely monomials of the form (a) and (b) which are nontrivial modulo G.

There are finitely many nontrivial monomials modulo G which are summands of $f_{\mathfrak{d}}$ for some $\mathfrak{d} \subset \mathbb{R}_{\geq 0}\langle(-1,0,0),(0,-1,0),(-1,-1,1)\rangle$. Moreover, the only such monomials which are nontrivial modulo $\mathfrak{m}^{\mathrm{egs}}$ are $\tilde{\alpha}_*(cz^{(-1,0,0)})$ and $\tilde{\alpha}_*(dz^{(0,-1,0)})$. By Lemma 4.1.11 the scattering diagram $\mathrm{Scatter}(\tilde{\alpha}(\mathfrak{d}_c, f_{\mathfrak{d}_c}), \tilde{\alpha}(\mathfrak{d}_d, f_{\mathfrak{d}_d}))$ is finite, so when $m' = \tilde{\alpha}_*(cz^{(-1,0,0)})$ there are finitely many monomials of the form (a). When

$m' \neq \tilde{\alpha}_*(cz^{(-1,0,0)})$, then Claim 4.1.14 implies $m' \in \mathbb{R}_{\geq 0}\langle(-1,1,0),(0,1,0)\rangle$, and so $m \in \mathbb{R}_{\geq 0}\langle(0,1,0),(-1,-1,0)\rangle$ by Lemma 4.1.12. Since $\beta' \in \mathfrak{m}^{\mathrm{egs}}$ there are only finitely many monomials $t^{\beta}z^{m}$ of the form (a) which are nontrivial modulo G.

If $\mathfrak{d} = \mathbb{R}_{\geq 0}\langle(-1,-1,1),(-1,0,1)\rangle$ and $t^{\beta'}z^{m'}$ is a summand of $f_{\mathfrak{d}}$ which contributes to a monomial of them form (b), then we must have $m' \in \mathbb{R}_{\geq 0}(0,1,0)$. Since $\mathbb{R}_{\geq 0}(0,1,0)$ does not contain $(-1,0,0)$ or $(0,-1,0)$, there are finitely many monomials $t^{\beta'}z^{m'}$ which are nontrivial modulo I. By Lemma 2.1.11 and Lemma 4.1.12, any monomial $t^{\beta}z^{m}$ of the form (b) must have $m \in R_{\geq 0}\langle(-1,0,0),(0,1,0)\rangle$. Moreover, we must have $\beta' \in \mathfrak{m}^{\mathrm{egs}}$. Thus there are finitely monomials $t^{\beta}z^{m}$ of the form (b) which are nontrivial modulo G. Thus conditions (i) and (ii) hold for $\mathfrak{D}\cap\sigma$. A similar argument holds in the remaining case $\sigma = \mathbb{R}_{\geq 0}\langle(-1,0,0),(-1,-1,1),(0,0,1)\rangle$. □

Corollary 4.3.6. *The wall structure $\mathfrak{D} = \alpha_f\mathfrak{D}_{(\Sigma_f,H_f)}$ also satisfies conditions (i) and (ii) of Lemma 4.3.5.*

Proof. In the proof of Lemma 4.3.5 in the case $\mathfrak{D} = \mathfrak{D}_{\mathrm{can}}(f)$, we showed that for any $\sigma \in \mathscr{P}$, conditions (i) and (ii) hold for $\mathfrak{D}_{\mathrm{can}}(f)\cap\sigma$ if and only if they hold for $\Psi(\mathfrak{D}_{\mathrm{can}}(f)\cap\sigma) \equiv \alpha_f\mathfrak{D}_{(\Sigma_f,H_f)}\cap\Psi(\sigma)$. Thus $\alpha_f\mathfrak{D}_{(\Sigma_f,H_f)}$ satsifies conditions (i) and (ii). □

Lemma 4.3.7. *Let $\mathfrak{D}$, Q and Q^{egs} be as in Lemma 4.3.5. Then*

$$R^{\mathrm{egs}}(\mathfrak{D}) := \bigoplus_{m} R^{\mathrm{egs}}\cdot\vartheta_m \tag{4.25}$$

is well-defined as an R^{egs}-algebra.

Proof. $\mathfrak{D}$ is a compatible system of consistent wall structures for Q^{gs}, and satisfies conditions (i) and (ii) from Lemma 4.3.5. Therefore, given an ideal $G \subset Q^{\mathrm{egs}}$ such that $\sqrt{G} = \mathfrak{m}^{\mathrm{egs}}$, the definitions for consistency in codimensions zero and one are satisfied. Consistency in codimension two is satisfied as long as, for any asymptotic monomial m of $\mathfrak{D}$ and general point p for m, the theta function $\vartheta_m(p)$ is well defined as an element of $R^{\mathrm{gs}}/(G\cap Q^{\mathrm{gs}})$. That is, the sum over broken lines β with endpoint p and asymptotic monomial m

$$\sum_{\beta} a_\beta z^{m_\beta} \in \Bbbk[\widehat{Q^{\mathrm{gs}}\oplus\Lambda_p}]$$

is finite modulo G. By Corollary 4.3.6 and the fact that $R(\mathfrak{D}_{\text{can}})$ and $R(\mathfrak{D}_{(Y_\Sigma,H)})$ are isomorphic as R^{gs}-algebras, we may consider the broken lines to live on $\mathfrak{D} = \alpha_f \mathfrak{D}_{(\Sigma_f,H_f)}$ instead of on $\mathfrak{D}_{\text{can}}(f)$.

Let $a_{i-1}z^{m_{i-1}}$ and $a_i z^{m_i} \in \Bbbk[Q^{\text{gs}} \oplus \Lambda_{\beta(t_i)}]$ be the monomials carried by β on the domains of linearity either side of $t_i \in \mathbb{R}_{<0}$, where $\beta(t_i) \in \mathfrak{d} \in \mathfrak{D}$. If $a_i z^{m_i} \neq a_{i-1}z^{m_{i-1}}$ we say that β *interacts nontrivially with* $\mathfrak{d}$. We say that β *picks up the monomial* $(a_i z^{m_i})/(a_{i-1}z^{m_{i-1}})$ at t_i. If β changes direction at t_i it must interact nontrivially with $\mathfrak{d}$. If $\mathfrak{d}$ is a slab contained in the codimension one cell ρ and the kink of the piecewise linear function κ_ρ is non-zero, then β must interact nontrivially with $\mathfrak{d}$; the monomial β picks up at t_i must be divisible by z^{κ_ρ}.

Suppose for a contradiction that there are infinitely many broken lines β on $\mathfrak{D}$ with endpoint p and asymptotic monomial m such that $a_\beta z^{m_\beta} \in \Bbbk[Q^{\text{gs}} \oplus \Lambda_p]$ is nontrivial modulo I. This implies broken lines on $\mathfrak{D}$ can pick up arbitrarily many nontrivial monomials modulo $\mathfrak{m}^{\text{egs}}$ and only a bounded number of trivial monomials modulo $\mathfrak{m}^{\text{egs}}$. By Lemma 4.3.5, the wall structure $\mathfrak{D} \mod \mathfrak{m}^{\text{egs}}$ is finite. In fact, we have

$$\alpha_f \mathfrak{D}_{(\Sigma_f,H_f)} \equiv \alpha_f \mathfrak{D}_{\text{init}}(f) \mod \mathfrak{m}^{\text{egs}}. \tag{4.26}$$

However, the kinks of ψ_f across the two rays supporting the incoming walls $\mathfrak{d}_a$ and $\mathfrak{d}_b$ are contained in $\mathfrak{m}^{\text{egs}}$, so broken lines cannot interact trivially with these walls modulo I. Thus our assumption implies that a broken line β on $\alpha_f \mathfrak{D}_{(\Sigma_f,H_f)}$ can interact nontrivially with the incoming ray $\alpha_f(\mathfrak{d}_c)$ arbitrarily many times, without interacting nontrivially with any other wall. This is clearly impossible. When $\mathfrak{D} = \tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$ we have

$$\mathfrak{D} \equiv \begin{matrix} \tilde{\alpha}\{\mathfrak{d}_a, \mathfrak{d}_b\} \cup \\ \tilde{\alpha}\left(\text{Scatter}\{\mathfrak{d}_c, \mathfrak{d}_d\} \cap \mathbb{R}_{\geq 0}\langle(-1,0,0),(0,-1,0),(0,0,1)\rangle\right) \end{matrix} \mod \mathfrak{m}^{\text{egs}} \tag{4.27}$$

In this case, the kinks $\tilde{\psi}$ across the two incoming walls $\mathfrak{d}_a$ and $\mathfrak{d}_b$ are also contained in $\mathfrak{m}^{\text{egs}}$. Broken lines in $\mathfrak{D}$ cannot interact nontrivially arbitrarily many times with walls in $\tilde{\alpha}\,\text{Scatter}\{\mathfrak{d}_c, \mathfrak{d}_d\}$ without also interacting nontrivially with other walls arbitrarily many times. □

Corollary 4.3.8. *The algebra $R^{\text{egs}}(\mathfrak{D})$ is also a well-defined R^{egs}-algebra (4.25) when*

$\mathfrak{D} = \mathbf{C}\alpha_f\overline{\mathfrak{D}_{(\Sigma_f,H_f)}}$ *or* $\mathfrak{D} = \mathfrak{D}_{f\leftrightarrow g}$.

Proof. The two wall structures are truncations of cones over $\alpha_f\mathfrak{D}_{(\Sigma_f,H_f)}$ and $\tilde{\alpha}\mathfrak{D}_{Y_{(\tilde{\Sigma}},\tilde{H})}$. The wall functions of $\mathbf{C}\mathfrak{D}$ are $f_{\mathfrak{d}}^{1/a}$, where a is the index of the map $\Lambda_{\mathbf{C}\mathfrak{d}} \longrightarrow \mathbb{Z}$ induced by projection to the height. Therefore, $\mathbf{C}\alpha_f\overline{\mathfrak{D}_{(\Sigma_f,H_f)}}$ and $\mathfrak{D}_{f\leftrightarrow g}$ satisfy conditions (i) and (ii) of Lemma 4.3.5 if $f_{\mathfrak{d}}^{1/a}$ is well-defined as an element of R^{egs} for all $\mathfrak{d} \in \alpha_f\mathfrak{D}_{(\Sigma_f,H_f)}$ and $\mathfrak{d} \in \tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}},\tilde{H})}$. If $f_{\mathfrak{d}} \equiv 1 \mod \mathfrak{m}^{\mathrm{egs}}$ then $f_{\mathfrak{d}}^{1/a}$ is uniquely defined in R^{egs}. By the equations (4.26) and (4.27) from the proof of Lemma 4.3.7, we see that if $f_{\mathfrak{d}} \not\equiv 1 \mod \mathfrak{m}^{\mathrm{egs}}$ we have $a = 1$. If $\mathbf{C}\mathfrak{D}$ satisfies conditions (i) and (ii), and $R^{\mathrm{egs}}(\mathfrak{D})$ is a R^{egs}-algebra, then it follows that $R^{\mathrm{egs}}(\mathbf{C}\mathfrak{D})$ is a R^{egs}-algebra. □

Construction 4.3.9 (Families over the extended Gross–Siebert locus)**.** Let $\overline{\mathfrak{X}} \to \operatorname{Spec} R^{\mathrm{gs}}$ denote either of the families $\overline{\mathfrak{X}_{\sim}}$ or $\overline{\mathfrak{X}}_f$. Following Corollary 4.3.8 above, we may consider $\overline{\mathfrak{X}}$ to be the pullback of a scheme over $\operatorname{Spec} R'''$ via the upper horizontal morphism in (4.21) - we write

$$\overline{\mathfrak{X}} = \operatorname{Proj}\,(S \otimes_{R'''} R^{\mathrm{gs}})\,,$$

where S is the graded R'''-algebra given by the algebras of theta functions $\bigoplus_{k\in\mathbb{N}} S_{k,k}$ or $R(\mathbf{C}\alpha_f\overline{\mathfrak{D}_{(\Sigma_f,H_f)}})$ respectively. Then we may define the corresponding family over the extended Gross–Siebert locus

$$\overline{\mathfrak{X}^{\mathrm{egs}}} := \operatorname{Proj}\,(S \otimes_{R'''} R^{\mathrm{egs}})$$

as a scheme over $\operatorname{Spec} R^{\mathrm{egs}}$.

Remark 4.3.10. Not all the kinks of the piecewise linear function are contained in $\mathfrak{m}^{\mathrm{egs}}$, and not all wall functions $f \equiv 1 \mod \mathfrak{m}^{\mathrm{egs}}$, and so the central fibre of the family $\overline{\mathfrak{X}^{\mathrm{egs}}}$ is not a union of toric varieties. However, $\overline{\mathfrak{X}^{\mathrm{egs}}}$ is the pullback of a family $\overline{\mathfrak{X}''}$ over $\operatorname{Spec} R''$, and this family can be pulled back to a family $\overline{\mathfrak{X}''} \times_{\operatorname{Spec} R''} \operatorname{Spec} R^{\#}$ which *does* have a toric central fibre.

4.3.3 Asymptotic wall structures

Before we make use of the monoid maps $Q_f^{\mathrm{egs}} \hookrightarrow \tilde{Q}^{\mathrm{egs}}$ and $Q_g \hookrightarrow \tilde{Q}^{\mathrm{egs}}$ to relate the three wall structures $\mathfrak{D}_{f\leftrightarrow g}$, $\overline{\mathfrak{D}_{(\Sigma_f,H_f)}}$ and $\overline{\mathfrak{D}_{(\Sigma_g,H_g)}}$, we need to introduce the notion of an asymptotic wall structure.

Definition 4.3.11 (Asymptotic wall structure $\mathscr{S}_{\mathrm{as}}$)**.** Let $\mathscr{S}$ be a wall structure on a polyhedral affine pseudomanifold $(B, \mathscr{P})$ equipped with MPA-function φ. Define the *asymptotic wall structure* $\mathscr{S}_{\mathrm{as}}$ similarly to the asymptotic scattering diagram of Definition 4.1.6, by replacing each wall $(\mathfrak{d}, f_{\mathfrak{d}})$ by $(\mathfrak{d}_\infty, f_{\mathfrak{d}})$, where $\mathfrak{d}_\infty$ is the tail cone of $\mathfrak{d} = \mathfrak{d}_0 + \mathfrak{d}_\infty$. (See Remark 1.5.3 – here $\mathfrak{d}_0$ is a bounded set.)

Let $h : \mathbf{C}B \to \mathbb{R}_{\geq 0}$ be the global affine function defined by projection to the second factor – i.e. the height. Suppose that

$$(B_{\mathrm{as}}, \mathscr{P}_{\mathrm{as}}) := (\mathbf{C}B, \mathbf{C}\mathscr{P}) \cap h^{-1}(0)$$

is a polyhedral affine pseudomanifold of dimension $\dim B$ in the sense of Definition 1.2.1. Then

$$\varphi_{\mathrm{as}} := \mathbf{C}\varphi|_{h^{-1}(0)}$$

defines a convex MPA-function on $(B_{\mathrm{as}}, \mathscr{P}_{\mathrm{as}})$, and $\mathscr{S}_{\mathrm{as}}$ is a wall structure for the data $(B_{\mathrm{as}}, \mathscr{P}_{\mathrm{as}}, \varphi_{\mathrm{as}})$.

Lemma 4.3.12. *Given a consistent wall structure $\mathscr{S}$, the asymptotic wall structure $\mathscr{S}_{\mathrm{as}}$ is consistent if it is defined. Moreover, there is an isomorphism of $\Bbbk[Q]/I$-algebras*

$$R(\mathscr{S}) \cong R(\mathscr{S}_{\mathrm{as}}). \tag{4.28}$$

Proof. For every joint $\mathfrak{j}_{\mathrm{as}}$ of $\mathscr{S}_{\mathrm{as}}$ there is a simply connected subset of $\mathscr{S}$

$$\mathscr{S}_{\mathfrak{j}_{\mathrm{as}}} := \{\mathrm{Int}\, \tau \in \mathscr{P}_{\mathfrak{D}} \mid \mathrm{Int}\, \tau \in B_{\mathfrak{j}} \text{ where } \mathfrak{j}_\infty = \mathfrak{j}_{\mathrm{as}}\}$$

containing all the joints of $\mathscr{S}$ with tail cone equal to $\mathfrak{j}_{\mathrm{as}}$.

(Consistency in codimension zero.) Consistency of the codimension zero joints of $\mathscr{S}_{\mathrm{as}}$ follows from the codimension zero joints of $\mathscr{S}$, similarly to the compatibility of an asymptotic scattering diagram (see Remark 4.1.8). For any maximal cell $\sigma_{\mathrm{as}} \in \mathscr{P}_{\mathrm{as}}$, there is a unique maximal cell $\sigma \in \mathscr{P}$ with tail cone σ_∞ equal to σ_{as}. Moreover, $R_{\sigma_{\mathrm{as}}} \cong R_\sigma$, and if $\mathfrak{j}_{\mathrm{as}} \subset \sigma_{\mathrm{as}}$ is of codimension zero, then any joint $\mathfrak{j}$ of $\mathscr{S}$ with tail cone $\mathfrak{j}_\infty$ equal to $\mathfrak{j}_{\mathrm{as}}$ must have codimension zero itself, and so $\mathscr{S}_{\mathfrak{j}_{\mathrm{as}}}$ is contained in $\sigma \in \mathscr{P}$.

Given any wall $\mathfrak{d}_{\mathrm{as}} \in (\mathscr{S}_{\mathrm{as}})_{\mathfrak{j}_{\mathrm{as}}}$, the walls $\mathfrak{d} \in \mathfrak{D}$ with asymptotic cone containing

$\mathfrak{d}_{\mathrm{as}}$ are contained in $\mathscr{S}_{\mathfrak{j}_{\mathrm{as}}}$, and

$$f_{\mathfrak{d}_{\mathrm{as}}} = \prod_{\mathfrak{d}_\infty \supset \mathfrak{d}_{\mathrm{as}}} f_{\mathfrak{d}}.$$

In particular, all such walls are parallel, so their change of chambers morphisms commute, and $\theta_{\mathfrak{d}_{\mathrm{as}}}$ is equal to the composition of all $\theta_{\mathfrak{d}}$ such that $\mathfrak{d}$ is a wall in $\mathscr{S}$ with tail cone containing in $\mathfrak{d}_{\mathrm{as}}$. Since $\mathscr{S}$ is consistent in codimension zero, the composition of isomorphisms

$$\theta_\gamma = \theta_{\mathfrak{d}_r} \circ \cdots \circ \theta_{\mathfrak{d}_1} : R_\sigma \longrightarrow R_\sigma$$

determined by any closed loop contained in σ is equal to the identity. There is a closed loop γ in $\mathscr{S}_{\mathfrak{j}_{\mathrm{as}}}$ such that θ_γ is equal to $\theta_{\gamma'}$ where γ' is the simple closed loop in $(\mathscr{S}_{\mathrm{as}})_{\mathfrak{j}_{\mathrm{as}}}$ around $\mathfrak{j}_{\mathrm{as}}$. Therefore $\theta_{\gamma'}$ is the identity on $R_{\sigma_{\mathrm{as}}}$.

(Consistency in codimension one.) If $\mathfrak{j}_{\mathrm{as}}$ is a joint of codimension one, then all the joints contained in $\mathscr{S}_{\mathfrak{j}_{\mathrm{as}}}$ are joints of codimension zero or one in $\mathscr{S}$. Given a slab $\mathfrak{b}_{\mathrm{as}} \in \mathscr{S}_{\mathrm{as}}$, the change of chambers morphism

$$\theta_{\mathfrak{b}_{\mathrm{as}}} : R^{\mathfrak{b}_{\mathrm{as}}}_{\mathfrak{u}_{\mathrm{as}}} \longrightarrow R_{\mathfrak{u}'_{\mathrm{as}}}$$

is equal to the composition of all change of chambers morphisms $\theta_{\mathfrak{b}}$ or $\theta_{\mathfrak{d}}$ restricted to $R^{\mathfrak{b}}_{\mathfrak{u}}$, where $\mathfrak{b}$ and $\mathfrak{d}$ are all the slabs and walls with tail cone containing $\mathfrak{b}_{\mathrm{as}}$. Note that the order of the composition does not matter as the change of chambers morphisms commute, and here $R^{\mathfrak{b}}_{\mathfrak{u}}$ and $R^{\mathfrak{b}_{\mathrm{as}}}_{\mathfrak{u}_{\mathrm{as}}}$ are canonically identified because $\mathfrak{b}$ is necessarily parallel to $\mathfrak{b}_{\mathrm{as}}$. Consistency of $\mathfrak{j}_{\mathrm{as}}$ therefore follows from consistency of all the joints in $\mathscr{S}_{\mathfrak{j}_{\mathrm{as}}}$.

(Consistency in codimension two and isomorphism of algebras of theta functions.) If $\mathfrak{j}_{\mathrm{as}}$ is a codimension two joint of $\mathscr{S}_{\mathrm{as}}$, then we have

$$(\mathscr{S}_{\mathfrak{j}_{\mathrm{as}}})_{\mathrm{as}} \equiv (\mathscr{S}_{\mathrm{as}})_{\mathfrak{j}_{\mathrm{as}}} .$$

Therefore, consistency of $\mathfrak{j}_{\mathrm{as}}$ follows if we show that there is a bijective correspondence between broken lines on $\mathscr{S}$ and broken lines on $\mathscr{S}_{\mathrm{as}}$ with fixed asymptotic monomial and endpoint. Since asymptotic monomials on $\mathscr{S}_{\mathrm{as}}$ are asymptotic monomials on $\mathscr{S}$ by definition, the correspondence between broken lines would also imply that the identification of theta functions $\vartheta_m \mapsto \vartheta_m$ induces an isomorphism of the algebras $R(\mathscr{S}) \to R(\mathscr{S}_{\mathrm{as}})$.

For any chamber $\mathfrak{u}_{\mathrm{as}}$ of $\mathscr{S}_{\mathrm{as}}$, there exists a unique chamber $\mathfrak{u}$ of $\mathscr{S}$ with tail cone $\mathfrak{u}_\infty = \mathfrak{u}_{\mathrm{as}}$. Suppose that β_{as} is a broken line on $\mathscr{S}_{\mathrm{as}}$ with endpoint $p_{\mathrm{as}} \in \mathfrak{u}^{p_{\mathrm{as}}}$. By choosing p to be far enough away from the origin, it is easy to see how β_{as} corresponds to a broken line β on $\mathscr{S}$ with endpoint $p \in \mathfrak{u}_0 + p_{\mathrm{as}} \subset \mathfrak{u}^p$, where $\mathfrak{u}^p = \mathfrak{u}_0 + \mathfrak{u}^{p_{\mathrm{as}}}$. The broken line β only passes through chambers $\mathfrak{u}$ of $\mathscr{S}$ with tail cone equal to a chamber or wall of $\mathscr{S}_{\mathrm{as}}$. If $\mathfrak{u}_\infty = \mathfrak{d}_{\mathrm{as}} \in \mathscr{S}_{\mathrm{as}}$, then β enters an exits $\mathfrak{u}$ via walls $\mathfrak{d}$ and $\mathfrak{d}'$ whose tail cone contains in $\mathfrak{d}_{\mathrm{as}}$. We claim that there exists a choice of general point $p_\infty \in \mathfrak{u}^p$ such that no broken lines β contributing to $\vartheta_m(q)$ pass through any chamber $\mathfrak{u}$ of $\mathscr{S}$ without tail cone $\mathfrak{u}_\infty$ equal to a chamber or wall of $\mathscr{S}_{\mathrm{as}}$. Thus every broken line contributing to $\vartheta_m(p_\infty) \in \mathscr{S}$ corresponds to a broken line β_{as} on $\mathscr{S}_{\mathrm{as}}$, contributing to $\vartheta_m(p) \in R(\mathscr{S}_{\mathrm{as}})$.

Suppose that β is a broken line on $\mathscr{S}$ with endpoint $q \in \mathfrak{u}^p$, passing through a chamber $\mathfrak{u}$ with tail cone of codimension greater than one. Suppose that $\mathfrak{u}$ is adjacent to $\mathfrak{u}^p$, and β passes through the wall $\mathfrak{d} = \mathfrak{u} \cap \mathfrak{u}^p$. The final monomial $a_\beta z^{m_\beta}$ carried by β is a summand of $\theta_{\mathfrak{d}}(a_{r-1}t^{m_{r-1}})$, where $a_{r-1}t^{m_{r-1}}$ is the monomial carried by β in $\mathfrak{u}$. The endpoint q must be contained in $\mathfrak{d} + \mathbb{R}_{\geq 0} m_\beta$, and since $\mathfrak{d}$ has tail cone of codimension at least two, the set $\mathfrak{d} + \mathbb{R}_{\geq 0} m_\beta \subset \mathfrak{u}^p$ has tail cone of codimension at least one.

Now suppose that a broken line β with endpoint $q \in \mathfrak{u}^p$ passes through chambers with tail cone of codimension greater than one, but that the final such chamber $\mathfrak{u}$ is not adjacent to $\mathfrak{u}^p$. We may argue by induction that $q \in \tau + \mathbb{R}_{\geq 0} m_\beta$, where τ is a subset of a wall of $\mathfrak{u}^p$, with tail cone τ_∞ of codimension at least two. The tail cone of $\tau + \mathbb{R}_{\geq 0} m_\beta$ is also of codimension at least one.

Since there are finitely many broken line types in $\mathscr{S}$, there are finitely many possible subsets $\tau + \mathbb{R}_{\geq 0} m_\beta \subset \mathfrak{u}^p$ containing endpoints of broken lines which pass through chambers with tail cone of codimension greater than one. The complement of these subsets of $\mathfrak{u}^p$ is therefore a union of polyhedra with combined tail cone equal to $\mathfrak{u}^{p_{\mathrm{as}}}$, the tail cone of $\mathfrak{u}^p$. Thus we can choose p_∞ to be in this complement in $\mathfrak{u}^p$. □

Corollary 4.3.13. *Given a consistent wall structure $\mathscr{S}$, there is an isomorphism of* $\Bbbk[Q]/I$*-*

algebras

$$R(\mathbf{C}\mathscr{S})_0 \cong R(\mathscr{S}_{\mathrm{as}}), \tag{4.29}$$

where $R(\mathbf{C}\mathfrak{D})$ is the graded algebra of theta functions defined on $\mathscr{S}$, and $R(\mathscr{S}_{\mathrm{as}})$ is the algebra of theta functions defined on $\mathscr{S}_{\mathrm{as}}$.

Proof. By [23, Theorem 4.3.2], we know that $R(\mathbf{C}\mathscr{S})_0 \cong R(\mathscr{S})$, and by Lemma 4.3.12 we have $R(\mathscr{S}) \cong R(\mathscr{S}_{\mathrm{as}})$. □

Lemma 4.3.14. *The monoid maps ι_f and ι_g induce ring homomorphisms*

$$R_f^{\mathrm{egs}} \longrightarrow \tilde{R}^{\mathrm{egs}} \quad \textit{and} \quad R_g^{\mathrm{gs}} \longrightarrow \tilde{R}^{\mathrm{egs}}. \tag{4.30}$$

Moreover, we have

$$\mathrm{Proj}\left(S_{0,\bullet} \otimes_{\tilde{R}'''} \tilde{R}^{\mathrm{egs}}\right) \cong \overline{\mathfrak{X}_g} \times_{\iota_g} \mathrm{Spec}\, \tilde{R}^{\mathrm{egs}} \tag{4.31}$$

and

$$\mathrm{Proj}\left(S_{\bullet,0} \otimes_{\tilde{R}'''} \tilde{R}^{\mathrm{egs}}\right) \cong \overline{\mathfrak{X}_f}_{\mathrm{egs}} \times_{\iota_f} \mathrm{Spec}\, \tilde{R}^{\mathrm{egs}} \tag{4.32}$$

Proof. The monoid map ι_f and ι_g induce the ring homomorphisms (4.30) because

$$\iota_f^{-1}(\tilde{\mathfrak{m}}^{\mathrm{egs}}) = \mathfrak{m}_f^{\mathrm{egs}} \quad \text{and} \quad \iota_g^{-1}(\tilde{\mathfrak{m}}^{\mathrm{egs}}) = \mathfrak{m}_g^{\mathrm{gs}}.$$

In order to prove (4.31), we show that there is an isomorphism of $\tilde{R}^{\mathrm{egs}}$-algebras

$$R\left(\mathbf{C}\alpha_g \overline{\mathfrak{D}_{(\Sigma_g,H_g)}}\right) \otimes_{\iota_g} \tilde{R}^{\mathrm{egs}} \cong S_{0,\bullet} \otimes_{\tilde{R}'''} \tilde{R}^{\mathrm{egs}}. \tag{4.33}$$

Since $S_{0,\bullet} = R\left(\mathfrak{D}_{f\leftrightarrow g}\right)_0$, we use Corollary 4.3.13 and relate $\mathbf{C}\alpha_g \overline{\mathfrak{D}_{(\Sigma_g,H_g)}}$ to $\mathfrak{D}_{f\leftrightarrow g}$ in the following sense. By Remark 4.2.9, we can express $\mathfrak{D}_{f\leftrightarrow g}$ as a cone with respect to the t-height:

$$\mathfrak{D}_{f\leftrightarrow g} = \mathbf{C}\left(\left(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}} \times \mathbb{R}_{\geq 0}\right) \cap \left(B_{f\leftrightarrow g} \cap \{t=1\}\right) \right),$$

so we aim to show that

$$\left(\left(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}} \times \mathbb{R}_{\geq 0}\right) \cap \left(B_{f\leftrightarrow g} \cap \{t=1\}\right) \right)_{\mathrm{as}} = \iota_g \mathbf{C}\alpha_g \overline{\mathfrak{D}_{(\Sigma_g,H_g)}}. \tag{4.34}$$

This is done by simply tracing through the identifications – we start by noting that

$$B_{f\leftrightarrow g}\cap\{t=0\}=\mathbf{C}(\mathrm{Newt}\,g)^\vee \tag{4.35}$$

and

$$\left(\tilde{\Sigma}\times\mathbb{R}_{\geq 0}\right)\cap\{t=0\}=\mathbf{C}\Sigma_g. \tag{4.36}$$

To show (4.34), it therefore suffices to show that

$$\begin{aligned} &(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}}\times\mathbb{R}_{\geq 0})_{\mathrm{as}}\equiv\iota_g\mathbf{C}\alpha_g\mathfrak{D}_{(\Sigma_g,H_g)}\\ \iff\qquad &(\tilde{\alpha}\mathfrak{D}_{\mathrm{pert}})_{\mathrm{as}}\equiv\iota_g\circ\alpha_g\mathfrak{D}_{(\Sigma_g,H_g)},\end{aligned}$$

where the latter is an equivalence of wall structures on $(M_{\mathbb{R}},\Sigma_g)$ equipped with the $\tilde{Q}^{\mathrm{egs}}$-valued piecewise linear function $\tilde{\psi}|_{M_{\mathbb{R}}\times\{0\}}$. This is easy to see since

$$\tilde{\psi}|_{M_{\mathbb{R}}\times\{0\}}=\iota_g\circ\psi_g\quad\text{and}\quad(\mathfrak{D}_{\mathrm{pert}})_{\mathrm{as}}=\mathfrak{D}_{(\Sigma_g,H_g)},$$

so

$$\begin{aligned}\left(\tilde{\alpha}|_{\{t=1\}}\,\mathfrak{D}_{\mathrm{pert}}\right)_{\mathrm{as}}&=\tilde{\alpha}|_{\{t=0\}}\,(\mathfrak{D}_{\mathrm{pert}})_{\mathrm{as}}\\ &=\tilde{\alpha}|_{\{t=0\}}\,\mathfrak{D}_{(\Sigma_g,H_g)}\\ &=\iota_g\circ\alpha_g\mathfrak{D}_{(\Sigma_g,H_g)}.\end{aligned}$$

In order to prove (4.32), we show that there is an isomorphism of $\tilde{R}^{\mathrm{egs}}$-algebras

$$R\left(\mathbf{C}\alpha_f\overline{\mathfrak{D}_{(\Sigma_f,H_f)}}\right)\otimes_{\iota_f}\tilde{R}^{\mathrm{egs}}\cong S_{\bullet,0}\otimes_{R'''}R^{\mathrm{egs}}. \tag{4.37}$$

Following the same argument as for $\overline{\mathfrak{X}_g}$, we note that

$$B_{f\leftrightarrow g}\cap\{s=0\}=\mathbf{C}(\mathrm{Newt}\,f)^\vee \tag{4.38}$$

and

$$\left(\tilde{\Sigma}\times\mathbb{R}_{\geq 0}\right)\cap\{s=0\}=\tilde{\Sigma}.$$

However, we have

$$\tilde{\Sigma} \cap \mathbf{C}(\operatorname{Newt} f)^\vee \cong \left(\Sigma_f \times \mathbb{R}_{\geq 0}\right) \cap \mathbf{C}(\operatorname{Newt} f)^\vee,$$

and the restriction $\tilde{\psi}\big|_{\mathbf{C}(\operatorname{Newt} f)^\vee}$ can be linearly continued to give a $\tilde{Q}^{\mathrm{egs}}$-valued piecewise linear function on $\Sigma_f \times \mathbb{R}_{\geq 0}$, which we will denote by $\tilde{\psi}\big|_{(\Sigma_f \times \mathbb{R}_{\geq 0})}$. Since all of the codimension one cells of $\Sigma_f \times \mathbb{R}_{\geq 0}$ are vertical, the same MPA-function is also represented by

$$\tilde{\psi}\big|_{\Sigma_f} \times 0 := \left(\tilde{\psi}\big|_{(\Sigma_f \times \mathbb{R}_{\geq 0})}\Big|_{M_\mathbb{R} \times \{0\}} \times 0 \right) : \; M_\mathbb{R} \times \mathbb{R}_{\geq 0} \longrightarrow \tilde{Q}^{\mathrm{egs}}.$$

With this representative, we can identify

$$\tilde{\psi}\big|_{\Sigma_f} \times 0 = \iota_f \circ \left(\psi_f \times 0\right)$$

as piecewise linear functions on $(M_\mathbb{R} \times \mathbb{R}_{\geq 0}, \Sigma_f \times \mathbb{R}_{\geq 0})$, and make the identification

$$\tilde{\alpha} \circ (\phi \times 0) = \iota_f \circ (\alpha_f \times 0) : \; P_f \times \mathbb{R}_{\geq 0} \longrightarrow \mathcal{P}^+_{\tilde{\psi}|_{\Sigma_f \times \mathbb{R}_{\geq 0}}} \tag{4.39}$$

of functions of sheaves on $(M_\mathbb{R} \times \mathbb{R}_{\geq 0}, \Sigma_f \times \mathbb{R}_{\geq 0})$, where $\phi : P_f \to P_g$ is defined in (4.4). We can now argue that

$$\begin{aligned}
\left(\mathfrak{D}_{f \leftrightarrow g} \cap \{s = 1\}\right)_{\mathrm{as}} &= \mathfrak{D}_{f \leftrightarrow g} \cap \{s = 0\} \\
&= \tilde{\alpha}\mathfrak{D}_{(Y_{\tilde{\Sigma}}, \tilde{H})} \cap \mathbf{C}(\operatorname{Newt} f)^\vee && \text{by (4.38)} \\
&= \tilde{\alpha}\left(\mathfrak{D}_{(Y_{\tilde{\Sigma}}, \tilde{H})} \cap \mathbf{C}(\operatorname{Newt} f)^\vee\right) \\
&= \iota_f \circ (\alpha_f \times 0)\left(\mathbf{C}\mathfrak{D}_{(\Sigma_f, H_f)} \cap \mathbf{C}(\operatorname{Newt} f)^\vee\right) && \text{by (4.39)} \\
&= \iota_f \mathbf{C} \alpha_f \overline{\mathfrak{D}_{(\Sigma_f, H_f)}},
\end{aligned}$$

and so (4.37) holds by Remark 4.2.9 and Corollary 4.3.13. □

Lemma 4.3.15. *The wall structure $\mathfrak{D}_{f \leftrightarrow g} \cap \{s = t\}$ is equivalent to the cone over $\tilde{\alpha}\overline{\mathfrak{D}_{\mathrm{pert}}}$, where*

$$\overline{\mathfrak{D}_{\mathrm{pert}}} := \mathfrak{D}_{\mathrm{pert}} \cap \left(B_{f \leftrightarrow g} \cap \{s = t = 1\}\right). \tag{4.40}$$

Moreover, there is an isomorphism of graded $\Bbbk[\tilde{Q}]/\tilde{I}$-algebras

$$\bigoplus_{k\in\mathbb{N}} S_{k,k} \cong R\left(\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\}\right). \tag{4.41}$$

Proof. The first statement follows from the discussion in Remark 4.2.9. We show that broken lines on $\mathfrak{D}_{f\leftrightarrow g}$ which contribute to a theta function $\vartheta_{(m,k,k)} \in S_{k,k}$ correspond bijectively to broken lines on $\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\}$ which contribute to $\vartheta_{(m,k)} \in R(\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\})$. One direction of this bijection is clear – any broken line β contained in $\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\}$ is automatically a broken line in $\mathfrak{D}_{f\leftrightarrow g}$, and its asymptotic monomial must have degree (k, k).

Now suppose that β is a broken line on $\mathfrak{D}_{f\leftrightarrow g}$ with asymptotic monomial of degree (k, k). Since the monomials in the wall functions of $\mathfrak{D}_{f\leftrightarrow g}$ have degree $(0, 0)$, the monomial carried by β has constant degree, and thus the image of β is completely contained in a hyperplane

$$H := \{s - t = c\} \cap B_{f\leftrightarrow g}$$

for some constant $c \in \mathbb{R}$, parallel to $\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\}$. If $c < 0$, then H carries a wall structure $\mathfrak{D}_{f\leftrightarrow g} \cap H$ such that $\mathfrak{D}_{f\leftrightarrow g} \cap H \setminus \partial\mathfrak{D}_{f\leftrightarrow g}$ is equivalent to a subset of $\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\}$. Therefore, β determines a broken line on $\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\}$ which has endpoint in $\mathfrak{u}^{s=t}$. If $c > 0$, then H carries a wall structure $\mathfrak{D}_{f\leftrightarrow g} \cap H$ such that $(\mathfrak{D}_{f\leftrightarrow g} \cap H)_{\mathrm{as}} \equiv \mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\}$. Since β contributes to $\vartheta_{(m,k)} \in R(\mathfrak{D}_{f\leftrightarrow g} \cap H)$ and $R(\mathfrak{D}_{f\leftrightarrow g} \cap H) \cong R(\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\})$ by Lemma 4.3.12, β must correspond to a broken line contributing to $\vartheta_{(m,k)} \in R(\mathfrak{D}_{f\leftrightarrow g} \cap \{s = t\})$. □

Chapter 5

The morphisms are birational

In the previous chapter, we constructed three families over Spec $\widehat{\mathbb{k}[\tilde{Q}^{\mathrm{egs}}]}$ and maps between them. In this chapter, we will show that the generic fibres of these families are smooth, and that the maps between them are birational. For simplicity, we adopt uniform notation for the three families.

Notation 5.0.1. Recall the three monoids $\tilde{Q}^{\mathrm{gs}}$, Q_f^{gs} and Q_g^{gs} associated to the Gross–Siebert loci for the respective log Calabi–Yau pairs $(\tilde{Y}, \tilde{D})$, (Y_f, D_f) and (Y_g, D_g). We denote these monoids by Q^{gs} when the context is clear, and denote by Q^{egs} the monoid associated to the corresponding extended Gross–Siebert locus. (Here $Q_g^{\mathrm{egs}} := Q_g^{\mathrm{gs}}$.) We set

$$R^{\mathrm{gs}} := \widehat{\mathbb{k}[Q^{\mathrm{gs}}]} \quad \text{and} \quad R^{\mathrm{egs}} := \widehat{\mathbb{k}[Q^{\mathrm{egs}}]}, \tag{5.1}$$

and let $\mathfrak{D}$ denote one of the following three wall structures over R^{egs}:

$$\tilde{\alpha}\overline{\mathfrak{D}_{\mathrm{pert}}}, \qquad \alpha_f\overline{\mathfrak{D}_{(\Sigma_f,H_f)}}, \qquad \text{or} \qquad \alpha_g\overline{\mathfrak{D}_{(\Sigma_g,H_g)}}. \tag{5.2}$$

We write $\overline{\mathfrak{X}}$ for the family

$$\overline{\mathfrak{X}} := \operatorname{Proj} R(\mathbf{C}\mathfrak{D}) \longrightarrow \operatorname{Spec} R^{\mathrm{egs}} \tag{5.3}$$

where $R(\mathbf{C}\mathfrak{D})$ is the graded R^{egs}-algebra of theta functions for the three-dimensional wall structure $\mathbf{C}\mathfrak{D}$.

5.1 Smoothness of the generic fibre

Theorem 5.1.1. *The families* $\overline{\mathfrak{X}} \longrightarrow \operatorname{Spec} R^{\mathrm{egs}}$ *are smooth over the generic point.*

Proof. We will prove this via a local analysis of the two-dimensional wall structure $\mathfrak{D}$. Roughly speaking[1], neighbourhoods of joints in $\mathfrak{D}$ give rise to local charts on $\overline{\mathfrak{X}}$. The most difficult point in the analysis involves the joint at the origin: here we use results of Gross–Hacking–Keel [21], for which we need to work over a sharp monoid.

Fix a sharp monoid $Q^\#$ that satisfies the conditions of Construction 2.2.7 and is contained in Q^{gs}, and let

$$R^\# := \widehat{\Bbbk[Q^\#]}. \tag{5.4}$$

Let $\mathfrak{m}'$ denote the intersection of $Q^\#$ with the maximal ideal of Q^{gs}, let $\mathfrak{m}''$ denote the intersection of $Q^\#$ with the maximal ideal of Q^{egs}, and recall the diagram of rings (4.21).

Claim 5.1.2. *There exists a family* $\overline{\mathfrak{X}''} \to \operatorname{Spec} R''$ *such that* $\overline{\mathfrak{X}} \cong \overline{\mathfrak{X}''} \times_{\operatorname{Spec} R''} \operatorname{Spec} R^{\mathrm{egs}}$.

Proof. We need to show that the structure constants (1.18) in the R^{egs}-algebra of theta functions $R(\mathbf{C}\mathfrak{D})$ are actually contained in R''. The claim therefore follows if $a_\beta \in \Bbbk[Q^\#]$ for every broken line β on $\mathfrak{D}$. We only show this in the first case, but the argument in the other two cases is similar.

When $\mathfrak{D} = \tilde{\alpha}\overline{\mathfrak{D}_{\mathrm{pert}}}$, there are only five walls $\mathfrak{d} \in \mathfrak{D}$ whose wall function $f_{\mathfrak{d}}$ admits a monomial summand $t^A z^m$ such that $A \notin \tilde{Q}^\#$, namely the slabs arising from the five rays in $\mathfrak{D}_{\mathrm{pert}}$ which propagate towards the origin: $\mathfrak{d}_a, \mathfrak{d}_b, \mathfrak{d}_c, \mathfrak{d}_d, \mathfrak{d}_{cd} \in \operatorname{Scatter}^2(\mathfrak{D}^0_{\mathrm{pert}})$. The wall functions carried by these slabs $\mathfrak{b}_a$, $\mathfrak{b}_b$, $\mathfrak{b}_c$, $\mathfrak{b}_d$, $\mathfrak{b}_{cd}$ are of the form $f_{\mathfrak{b}_i} = (1 + t^{A_i} z^{m_i}) g_{\mathfrak{b}_i}$, where $m_i \in M$,

$$A_i = \begin{cases} -E_i & \text{if } i = a, b, c \text{ or } d \\ \tilde{E} - E_c - E_d & \text{if } i = cd, \end{cases}$$

and every monomial summand of $g_{\mathfrak{b}_i}$ is contained in $\Bbbk[Q^\# \oplus M]$. Suppose that a broken line β crosses one of these five slabs at $t_k \in (-\infty, 0)$. Then the monomial $(a_{k+1} z^{m_{k+1}})/(a_k z^{m_k}) \in \Bbbk[Q^{\mathrm{egs}} \oplus M]$ that β picks up at t_k is a summand of

$$(t^{\kappa_{\mathfrak{b}_i}} f_{\mathfrak{b}_i})^n$$

[1]This is not quite true, which is why we work with a formal family below, rather than arguing directly on $\overline{\mathfrak{X}}$.

for some $n \in \mathbb{N}$. Noting that $\kappa_{\mathfrak{d}_i} + A_i \in NE(\tilde{Y})$ for every i, we conclude that $a_{k+1}/a_k \in \Bbbk[Q^{\mathrm{egs}}]$ in fact lies in $\Bbbk[Q^{\#}]$. Then it follows by induction that $a_\beta \in \Bbbk[Q^{\#}]$. Therefore, we may consider the ring of theta functions on $\mathbf{C}\mathfrak{D}$ to be a graded R''-algebra $R''(\mathbf{C}\mathfrak{D})$, where $R(\mathbf{C}\mathfrak{D}) \cong R''(\mathbf{C}\mathfrak{D}) \otimes_{R''} R^{\mathrm{egs}}$. We define $\overline{\mathfrak{X}''} := \operatorname{Proj} R''(\mathbf{C}\mathfrak{D})$. □

We may now construct families over Spec of every ring in the diagram (4.21) via pullback of $\overline{\mathfrak{X}''}$. In particular, let $\overline{\mathfrak{X}^{\#}}$ be the pullback of $\overline{\mathfrak{X}''}$ via $\operatorname{Spec} R^{\#} \to \operatorname{Spec} R''$. Let η^{egs}, η'' and $\eta^{\#}$ denote the respective generic points of $\operatorname{Spec} R^{\mathrm{egs}}$, $\operatorname{Spec} R''$ and $\operatorname{Spec} R^{\#}$. The residue fields $\kappa(\eta^{\mathrm{egs}})$ and $\kappa(\eta^{\#})$ are both extensions of $\kappa(\eta'')$, so the generic fibre $\overline{\mathfrak{X}}_{\eta^{\mathrm{egs}}}$ is smooth if and only if $\overline{\mathfrak{X}''}_{\eta''}$ is smooth, if and only if $\overline{\mathfrak{X}^{\#}}_{\eta^{\#}}$ is smooth. Thus Theorem 5.1.1 follows from Lemma 5.1.3 below. □

Lemma 5.1.3. *The families* $\overline{\mathfrak{X}^{\#}} \longrightarrow \operatorname{Spec} R^{\#}$ *are smooth over the generic point.*

Proof. Let $\mathfrak{D}_{\mathrm{can}}$ denote the canonical wall structure for the log Calabi–Yau pair associated to $\mathfrak{D}$. The discussion in Section 2.3 shows that there is a truncation of $\mathfrak{D}_{\mathrm{can}}$ such that we have an equivalence of wall structures over R^{gs}

$$\overline{\mathfrak{D}_{\mathrm{can}}} \equiv \begin{cases} \mathbf{C}\mathfrak{D} & \text{in the first case } (\tilde{Y}, \tilde{D}) \\ \mathfrak{D} & \text{in the second two cases } (Y_f, D_f) \text{ and } (Y_g, D_g). \end{cases}$$

Moreover, the isomorphism of the R^{gs}-algebras of theta functions induced by the equivalence of wall structures actually extends to an isomorphism of R''-algebras by Claim 5.1.2. Therefore, the family $\overline{\mathfrak{X}^{\#}}$ is equal to the algebraisation of the canonical *formal* family

$$\hat{\mathfrak{X}} := \operatorname{colim}_{\sqrt{I} = \mathfrak{m}^{\#}} \operatorname{Proj} R(\overline{\mathfrak{D}_{\mathrm{can}}}/I) \xrightarrow{\hat{\pi}} \operatorname{Spf} R^{\#}.$$

The family $\overline{\mathfrak{X}^{\#}} \to \operatorname{Spec} R^{\#}$ is smooth over the generic point if and only if the formal family $\hat{\mathfrak{X}} \to \operatorname{Spf} R^{\#}$ is smooth over the generic point in the sense of [21, Definition 4.2]. We may therefore work in local charts and to finite order to show that, for a sufficiently large $\mathfrak{m}^{\#}$-primary ideal I, the singular locus of

$$\overline{\mathfrak{X}^{\#}}/I \xrightarrow{\hat{\pi}_I} W_I := \operatorname{Spec} R^{\#}/I$$

does not surject scheme-theoretically onto the base $\operatorname{Spec} R^{\#}/I$.

For every joint $\mathfrak{j}$ in the wall structure $\mathfrak{D}$, we may define an affine scheme

$U_{\mathfrak{j}} := \operatorname{Spec} R_{\mathfrak{j}}$ such that $\overline{\mathfrak{X}^{\#}}/I$ is covered by all $U_{\mathfrak{j}}$ such that $\mathfrak{j}$ appears as a joint of $\mathfrak{D}$ modulo I. The scheme $U_{\mathfrak{j}}$ extends the local construction of $\mathfrak{X}^{\circ}(\mathfrak{D}/I)$ about $\mathfrak{j}$ over codimension two: it is the affine completion of the glued scheme

$$\mathfrak{X}^{\circ}(\mathfrak{D}/I \cap B_{\mathfrak{j}}) := \bigcup_{\substack{\tau \in \mathscr{P}_{\mathfrak{D}} \\ \mathfrak{j} \in \tau}} \operatorname{Spec} R_{\tau} \subset \mathfrak{X}^{\circ}(\mathfrak{D}).$$

where $B_{\mathfrak{j}}$ is as in Definition 1.4.6. Furthermore, a subcover of $\overline{\mathfrak{X}^{\#}}/I$ is given by the set of $U_{\mathfrak{j}}$ such that $\mathfrak{j}$ is a either a boundary joint or a codimension two joint of $\mathfrak{D}/I$. Indeed, it follows from consistency in codimensions zero and one that when $\mathfrak{j}$ is an interior joint, we have $\mathfrak{X}^{\circ}(\mathfrak{D}/I \cap B_{\mathfrak{j}}) \cong \operatorname{Spec} R_{\mathfrak{u}}$ for any chamber $\mathfrak{u}$ containing $\mathfrak{j}$ if $\operatorname{codim} \mathfrak{j} = 0$, and $\mathfrak{X}^{\circ}(\mathfrak{D}/I \cap B_{\mathfrak{j}}) \cong \operatorname{Spec} R_{\mathfrak{b}}$ where $\mathfrak{b}$ is either of the two slabs containing $\mathfrak{j}$ if $\operatorname{codim} \mathfrak{j} = 1$. Thus we have

$$U_{\mathfrak{j}} = \begin{cases} \operatorname{Spec} R_{\mathfrak{u}} & \text{if } \operatorname{codim} \mathfrak{j} = 0 \\ \operatorname{Spec} R_{\mathfrak{b}} & \text{if } \operatorname{codim} \mathfrak{j} = 1 \end{cases}$$

and so $U_{\mathfrak{j}} \subset \mathfrak{X}^{\circ}(\mathfrak{D}/I \cap B_{\mathfrak{j}'})$ for some interior joint $\mathfrak{j}'$ of codimension two.

There are therefore three types of joints in $\mathfrak{D}$ which make up an affine cover of $\overline{\mathfrak{X}^{\#}}/I$: (1) boundary joints $\mathfrak{j} \in \partial\mathfrak{D}$; (2) the origin; (3) an additional interior joint. Case 3 only occurs when $\mathfrak{D} = \tilde{\alpha}(\overline{\mathfrak{D}_{\text{pert}}})$, and is the point $\mathfrak{j} = (-1, -1)$.

Case 1: Boundary joints. Suppose the boundary joint $\mathfrak{j}$ is contained in chambers $\mathfrak{u}_+$ and $\mathfrak{u}_-$, separated by a wall $\mathfrak{d}$ such that $\mathfrak{j} = \mathfrak{d} \cap \partial B$. The scheme $U_{\mathfrak{j}}$ is defined to be the affine completion of the glued scheme

$$\operatorname{Spec} R^{\partial}_{\mathfrak{u}_-} \hookleftarrow \operatorname{Spec} R_{\mathfrak{u}_-} \hookrightarrow \operatorname{Spec} R_{\mathfrak{d}} \hookleftarrow \operatorname{Spec} R_{\mathfrak{u}_+} \hookrightarrow \operatorname{Spec} R^{\partial}_{\mathfrak{u}_+}.$$

The ring of regular functions on this scheme is given by $R_{\mathfrak{j}} = \Bbbk[(\mathcal{P}^+_{\varphi})_x]$, where φ is the $Q^{\#}$-valued piecewise-linear function on $(B, \mathscr{P})$ and $(\mathcal{P}^+_{\varphi})_x$ is the stalk of the associated sheaf of monoids at the point $x = \mathfrak{j}$. It may be written as

$$R_{\mathfrak{j}} \cong (R^{\#}/I)[X, Y, Z]/(XY - t^{\kappa_{\mathfrak{d}}} Z^r f_{\mathfrak{d}}) \tag{5.5}$$

for some $r \in \mathbb{N}$, where $f_{\mathfrak{d}}$ is the wall function on $\mathfrak{d}$ and

$$\kappa_{\mathfrak{d}} = \begin{cases} \kappa_\rho(\varphi) & \text{if } \mathfrak{d} \text{ is a slab contained in } \rho \in \mathscr{P}^{[n-1]} \\ 0 & \text{if } \mathfrak{d} \text{ is a wall of codimension zero.} \end{cases}$$

The generators of $R_{\mathfrak{j}}$ represent the vectors in $(\mathcal{P}_\varphi^+)_x$ given by

$$X = z^{\xi_+} t^{\varphi^*(\xi_+)}, \quad Y = z^{\xi_-} t^{\varphi^*(\xi_-)}, \quad Z = z^{\xi_\rho} t^{\varphi^*(\xi_\rho)},$$

where $\xi_+ \in \Lambda_{\partial\mathfrak{u}_+}$, $\xi_- \in \Lambda_{\partial\mathfrak{u}_-}$ and $\xi_\rho \in \Lambda_\rho$ are all generators pointing away from $\mathfrak{j}$. Then the number $r \in \mathbb{N}$ is given by the equation

$$\xi_+ + \xi_- = r \cdot \xi_\rho \quad \in \Lambda_x.$$

Note that although $f_{\mathfrak{d}} \in (R^\#/I)[Z, Z^{-1}]$, we know by convexity of the joint $\mathfrak{j}$ that $Z^r f_{\mathfrak{d}} \in (R^\#/I)[Z]$.

If $\mathfrak{j}$ is a boundary joint in any of the three wall structures (5.2), then $r \in \{0, 1\}$. Furthermore, we know that when $r = 1$ we have

$$Z f_{\mathfrak{d}} = (Z + t^{-E_i}) \cdot g_{\mathfrak{d}}$$

for some $i \in \{a, b, c, d\}$, where $g_{\mathfrak{d}} \in (R^\#/I)[Z]$ and $g_{\mathfrak{d}} \equiv 1 \mod \mathfrak{m}'$. By [21, Definition-Lemma 4.1], the singular locus $\mathrm{Sing}(\hat{\pi}_{\mathfrak{j}}) \subset U_{\mathfrak{j}}$ is defined by the 2^{nd} fitting ideal of $\Omega^1_{U_{\mathfrak{j}}/W_I}$. This is the ideal $J = (X, Y, F(Z)) \subset R_{\mathfrak{j}}$, where

$$F(Z) := \begin{cases} t^{\kappa_{\mathfrak{d}}} \frac{\partial f_{\mathfrak{d}}}{\partial Z} & \text{if } r = 0 \\ t^{\kappa_{\mathfrak{d}}} \left(g_{\mathfrak{d}} + (Z + t^{-E_i}) \frac{\partial g_{\mathfrak{d}}}{\partial Z} \right) & \text{if } r = 1. \end{cases} \tag{5.6}$$

When $\kappa_{\mathfrak{d}} = 0$, we claim that $\mathrm{Sing}(\hat{\pi}_{\mathfrak{j}})$ is empty. Indeed, if $\kappa_{\mathfrak{d}} = r = 0$, then $f_{\mathfrak{d}} = XY$ is contained in the fitting ideal J. Since $f_{\mathfrak{d}} \equiv 1 \mod \mathfrak{m}^\#$ and $\sqrt{I} = \mathfrak{m}^\#$, $f_{\mathfrak{d}}$ is a unit in $R_{\mathfrak{j}}$ and so $J = R_{\mathfrak{j}}$. If $\kappa_{\mathfrak{d}} = 0$ and $r = 1$, then $(Z + t^{-E_i}) g_{\mathfrak{d}} = XY \in J$. Since $g_{\mathfrak{d}} \equiv 1 \mod \mathfrak{m}^\#$, it is also a unit in $R_{\mathfrak{j}}$, and so $(Z + t^{-E_i}) \in J$. But now $g_{\mathfrak{d}} = F - (Z + t^{-E_i}) \frac{\partial g_{\mathfrak{d}}}{\partial Z} \in J$, so $J = R_{\mathfrak{j}}$.

When $\kappa_{\mathfrak{d}} \neq 0$, we claim that $\mathrm{Sing}(\hat{\pi}_{\mathfrak{j}})$ does not surject scheme-theoretically onto

W_I. If $\kappa_\mathfrak{d} \neq 0$ and $r = 0$, the ideal J contains $t^{\kappa_\mathfrak{d}} f_\mathfrak{d} = XY$. Since $f_\mathfrak{d}$ is a unit in $R_\mathfrak{j}$, we have $t^{\kappa_\mathfrak{d}} f_\mathfrak{d} \cdot (f_\mathfrak{d})^{-1} \in J$, and so $t^{\kappa_\mathfrak{d}}$ is contained in the kernel of $R^\#/I \to R_\mathfrak{j}/J$. Since $\mathrm{Sing}(\hat{\pi}_\mathfrak{j}) = \mathrm{Spec}\, R_\mathfrak{j}/J$, we see that the singular locus fails to surject onto W_I. If $\kappa_\mathfrak{d} \neq 0$ and $r = 1$, we have $F = t^{\kappa_\mathfrak{d}} G \in J$, where G is the expression in brackets in (5.6). Since $g_\mathfrak{d} \equiv 1 \mod \mathfrak{m}'$, we have $G \equiv 1 \mod \mathfrak{m}^\#$, and so G is a unit in $R_\mathfrak{j}$. Therefore $t^{\kappa_\mathfrak{d}}$ is again an element of the kernel of $R^\#/I \to R_\mathfrak{j}/J$, and the claim follows.

Case 2: The origin. If the origin is the only interior joint in $\mathfrak{D}$, then $R_\mathfrak{j}$ is isomorphic to the localisation $R(\mathbf{C}\mathfrak{D})_{\vartheta_{(0,0,1)}}$. We can therefore apply the discussion around Claim 2.4.3 to show that $R_\mathfrak{j} \cong R(\mathfrak{D}_{\mathrm{can}})$. The main result of [21, Section 4] is that the generic fibre of the formal family defined by $R(\mathfrak{D}_{\mathrm{can}})$ is smooth.

The remaining case, where we consider the origin in $\tilde{\alpha}(\overline{\mathfrak{D}_{\mathrm{pert}}})$, is not covered by [21, Theorem 4.6] because the monomial carried by the incoming ray arising from $\mathfrak{d}_{cd}$ is not contained in $\mathfrak{m}'$. (That is, because not all of the incoming rays have order zero with respect to $\mathfrak{m}'$, the wall structure is not equivalent to a canonical wall structure.) Nevertheless, the argument of [21, Section 4] goes through verbatim in this setting, and shows that the generic fibre of the family associated to $R_\mathfrak{j}$ is smooth.

Case 3: The additional interior joint. As discussed, this case only arises when $\mathfrak{D} = \tilde{\alpha}(\overline{\mathfrak{D}_{\mathrm{pert}}})$ and $\mathfrak{j} = (-1,-1)$. The open set $U_\mathfrak{j}$ is the spectrum of $R(\mathfrak{D}_\mathfrak{j})$, and $\mathfrak{D}_\mathfrak{j}$ is the image under $\tilde{\alpha}$ of a neighbourhood N of $\mathfrak{j}$ in $\mathfrak{D}_{\mathrm{pert}}$ that contains no other joints (see Definition 1.4.6). As a scattering diagram, $N \equiv \mathrm{Scatter}(E)$, where E is the scattering diagram containing only those walls of N that are incoming to $\mathfrak{j}$. See Figure 5.1. The diagram E has exactly five incoming walls, and we consider a perturbed scattering diagram E' obtained by shifting three of the incoming walls as shown in Figure 5.2. Set $\mathfrak{E}' := \tilde{\alpha}(\mathrm{Scatter}(E'))$. The asymptotic scattering diagram associated to $\mathrm{Scatter}(E')$ is equivalent to N, and therefore we have an equivalence of wall structures $(\mathfrak{E}')_{\mathrm{as}} \equiv \mathfrak{D}_\mathfrak{j}$. Applying Lemma 4.3.12, we see that $U_\mathfrak{j}$ is the spectrum of the algebra of theta functions on $\mathfrak{E}'$.

Note that $\mathfrak{E}'$ contains only one joint of codimension two: the joint at $(-1-1)$, which we denote $\mathfrak{j}'$. It therefore suffices to show that $U_{\mathfrak{j}'}$ has smooth generic fibre. But the wall structure $\mathfrak{E}'_{\mathfrak{j}'}$ is finite and easy to describe – in fact, we claim that $\mathfrak{E}'_{\mathfrak{j}'}$ is equivalent to the algorithmic scattering diagram $\alpha(\mathfrak{D}_{(Y_\Sigma,H)})$ associated to a log Calabi–Yau pair (Y, D), where Y is a smooth del Pezzo surface of degree seven, as we now describe.

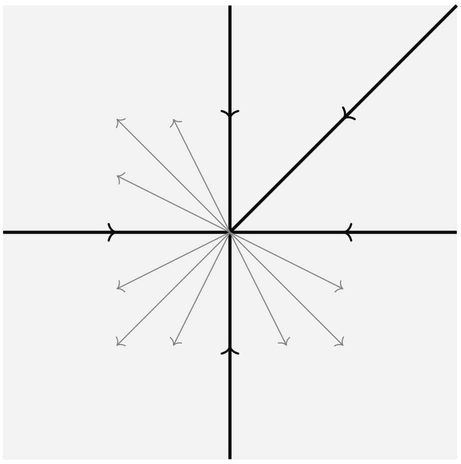

Figure 5.1: The scattering diagram N, with the support of E shown in black. The infinite collection of rays in every quadrant but the positive one is represented by grey arrows.

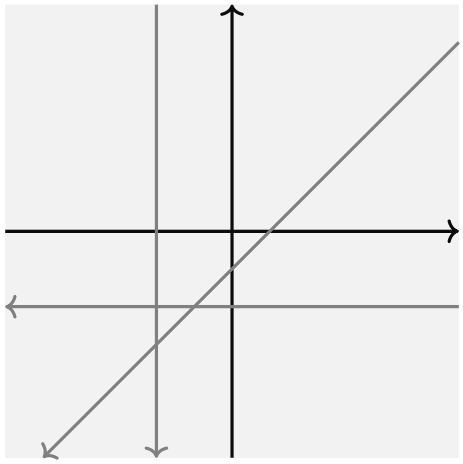

Figure 5.2: The scattering diagram E', where the perturbed lines are shown in grey.

We see from Figure 5.2 that $\mathfrak{E}'_{\mathfrak{j}'}$ is simply the image of the finite scattering diagram Scatter$\{\mathfrak{d}_c, \mathfrak{d}_d\}$ under $\tilde{\alpha}$. This is pictured in Figure 5.3. Thus $\mathfrak{E}'_{\mathfrak{j}'}$ consists of five walls, three of which coincide with codimension one cells in the polyhedral decomposition $\mathscr{P}_{\mathfrak{j}'}$. Each of these cells carries the kink $\tilde{E} \in \tilde{Q}$, and the two incoming walls to $\mathfrak{j}'$ carry the wall functions

$$\tilde{\alpha}(f_{\mathfrak{d}_c}) = 1 + t^{-E_c} x^{-1} \quad \text{and} \quad \tilde{\alpha}(f_{\mathfrak{d}_d}) = 1 + t^{-E_d} y^{-1}.$$

Thus we may consider Y to be a hypersurface of $\tilde{Y}$ with the divisorial log structure D induced by the divisorial log structure $\tilde{D}$. More concretely, $Y \subset \tilde{Y}$ is the strict transform of the toric divisor $Y_{\mathbb{R}_{\geq 0}(-1,-1,1)} \subset Y_{\tilde{\Sigma}}$ under the blowup $\mathrm{Bl}_{\tilde{H}} : \tilde{Y} \to Y_{\tilde{\Sigma}}$, and $D \subset \tilde{D}$ is the strict transform of the toric boundary of $Y_{\mathbb{R}_{\geq 0}(-1,-1,1)}$. We see that Y is a smooth del Pezzo surface of degree seven, and D is a union of three lines on it. The toric model of (Y, D) is induced by the toric model of $(\tilde{Y}, \tilde{H})$ – the toric variety is $Y_{\mathbb{R}_{\geq 0}(-1,-1,1)} \cong \mathbb{P}^2$ and the locus of the blowup is $\tilde{H} \cap Y_{\mathbb{R}_{\geq 0}(-1,-1,1)}$, which consists of two points: each on the interior of one of the toric boundary components of $\mathbb{P}^2$.

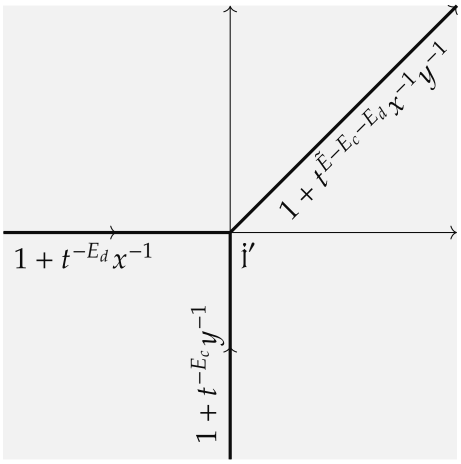

Figure 5.3: The wall structure $\mathfrak{C}'$, with the three slabs shown in bold.

The result of [21, Theorem 4.6] therefore also applies to the wall structure $\mathfrak{C}'_{\mathfrak{j}'}$; we conclude that $U_{\mathfrak{j}'}$ has smooth generic fibre. □

Remark 5.1.4. The algebra $R(\mathfrak{C}'_{\mathfrak{j}'})$ is calculated by Gross–Hacking–Keel–Kontsevich in [22, Example 8.31]. Since $\mathfrak{C}'_{\mathfrak{j}'}$ is finite, we may consider $U_{\mathfrak{j}'}$ to be a family over $\mathbb{A}^1$ with generic fibre isomorphic to the complement of five lines in a smooth del Pezzo surface of degree five.

Notation 5.1.5. The monoid maps ι_f and ι_g defined in Construction 4.3.1 induce ring homomorphisms

$$\iota_f : R^{\mathrm{egs}}_f \hookrightarrow R^{\mathrm{egs}}_\sim \quad \text{and} \quad \iota_g : R^{\mathrm{egs}}_g \hookrightarrow R^{\mathrm{egs}}_\sim \tag{5.7}$$

where R^{egs}_f is the ring R^{egs} associated to Q^{egs}_f, and similarly for R^{egs}_g and $R^{\mathrm{egs}}_\sim$. We denote the three families, previously denoted by $\overline{\mathfrak{X}}$, by $\overline{\mathfrak{X}_\sim}$, $\overline{\mathfrak{X}_f}$ and $\overline{\mathfrak{X}_g}$.

Lemma 5.1.6. *Consider the pullback families* $\iota_f^*\overline{\mathfrak{X}_f}$ *and* $\iota_g^*\overline{\mathfrak{X}_g}$ *over* $\operatorname{Spec} R^{\mathrm{egs}}_\sim$*, and the morphisms*

$$\begin{array}{ccc} & \overline{\mathfrak{X}_\sim} & \\ \swarrow & & \searrow \\ \iota_f^*\overline{\mathfrak{X}_f} & & \iota_g^*\overline{\mathfrak{X}_g}. \end{array} \tag{5.8}$$

defined in Proposition 4.2.15. These morphisms are birational over $\operatorname{Spec} R^{\mathrm{egs}}_\sim$.

Proof. The three families in (5.8) are all projective schemes over $\operatorname{Spec} R^{\mathrm{egs}}_\sim$. By the Stein factorization theorem, therefore, each of the two morphisms is a composition

$\pi \circ p$ of a proper morphism p with connected fibres and a finite morphism π. We claim that the finite morphism π is of degree one. Since being of fixed degree is an open condition, it suffices to show the existence of a point in $\iota_f^* \overline{\mathfrak{X}_f}$ and $\iota_g^* \overline{\mathfrak{X}_g}$ with fibre in $\overline{\mathfrak{X}_\sim}$ equal to a single point.

The central fibre of $\operatorname{Spec} R(\mathfrak{D}_{f\leftrightarrow g})$ is a union of eight irreducible components, seven of which are smooth toric varieties. Each of these toric varieties is isomorphic to $\operatorname{Spec} K[C(\mathbb{Z})]$, where $K = (R^{\mathrm{egs}}_\sim/\mathfrak{m}^{\mathrm{egs}})$, C is one of the ten domains of linearity of the piecewise linear function $\tilde{\psi}$ on $B_{f\leftrightarrow g}$, and $C(\mathbb{Z})$ denotes the integral points of C. (The seven domains of linearity that we consider here arise from the seven chambers in Figure 4.5 which are not in the lower left quadrant.) Let us consider $C = C_t + C_s$, where $C_t = \operatorname{Cone}\langle(0,0,1,0),(0,-1,1,0),(1,-1,1,0)\rangle$ and $C_s = \operatorname{Cone}\langle(0,0,0,1),(0,-1,0,1),(1,-1,0,1)\rangle$. The restrictions of the maps (5.8) to the component of the central fibre defined by C are induced by the inclusions $C_t, C_s \hookrightarrow C$:

$$\begin{array}{ccc} & \operatorname{Spec} K[C(\mathbb{Z})] & \\ \swarrow & & \searrow \\ \operatorname{Spec} K[C_t(\mathbb{Z})] & & \operatorname{Spec} K[C_s(\mathbb{Z})]. \end{array} \tag{5.9}$$

The GIT quotients of these three affine toric varieties by $\mathbb{G}_m^2$ (as specified in the proof of Proposition 4.2.15), are all isomorphic to $\mathbb{P}^2$. More specifically, the GIT quotients are realised as $\operatorname{Spec} K[C_t(\mathbb{Z})]/\mathbb{G}_m^2 \cong \mathbb{P}^2_K$, $\operatorname{Spec} K[C_s(\mathbb{Z})]/\mathbb{G}_m^2 \cong \mathbb{P}^2_K$ and $\operatorname{Spec} K[C(\mathbb{Z})]/\mathbb{G}_m^2 \cong \Delta \subset \mathbb{P}^2_K \times \mathbb{P}^2_K$, where the morphisms of (5.9) descend to the projections

$$\begin{array}{ccc} & \Delta \subset \mathbb{P}^2_K \times \mathbb{P}^2_K & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{P}^2_K & & \mathbb{P}^2_K. \end{array} \tag{5.10}$$

Hence both morphisms in (5.8) are generically of degree one. □

5.2 The main result

Definition 5.2.1. Let η denote the generic point of $\operatorname{Spec} \Bbbk[[t]]$, and denote the generic fibre of a family $\mathfrak{Z} \to \operatorname{Spec} \Bbbk[[t]]$ by $\mathfrak{Z}_\eta$. We will say that a one-parameter family $\gamma : \operatorname{Spec} \Bbbk[[t]] \to \operatorname{Spec} \tilde{R}^{\mathrm{egs}}$ is *general* if the generic fibre $(\gamma^* \overline{\mathfrak{X}})_\eta$ is smooth for each of the three families $\overline{\mathfrak{X}} \to \operatorname{Spec} \tilde{R}^{\mathrm{egs}}$ given by $\overline{\mathfrak{X}_\sim}$, $\iota_f^* \overline{\mathfrak{X}_f}$ and $\iota_g^* \overline{\mathfrak{X}_g}$.

Note that almost all one-parameter families γ are general. The main result of this thesis is:

Theorem 5.2.2. *For a general one-parameter family* $\gamma : \operatorname{Spec} \Bbbk[\![t]\!] \to \operatorname{Spec} \tilde{R}^{\mathrm{egs}}$*, the morphisms of (5.8) induce a diagram*

$$\begin{array}{ccc} & \left(\gamma^* \overline{\mathfrak{X}_\sim}\right)_\eta & \\ \swarrow & & \searrow \\ \left(\gamma^* \iota_f^* \overline{\mathfrak{X}_f}\right)_\eta & \longleftarrow & \left(\gamma^* \iota_g^* \overline{\mathfrak{X}_g}\right)_\eta \end{array} \tag{5.11}$$

where the bottom left corner is isomorphic to $\mathbb{P}^2_{\Bbbk(\!(t)\!)}$*, the other two entries in the diagram are isomorphic to* $\mathrm{Bl}_{pt} \mathbb{P}^2_{\Bbbk(\!(t)\!)}$*, and both leftward-pointing maps are the blowup in a point on the boundary.*

Theorem 5.2.2 follows immediately from the two lemmas below.

Lemma 5.2.3. *For a general one-parameter family* $\gamma : \operatorname{Spec} \Bbbk[\![t]\!] \to \operatorname{Spec} \tilde{R}^{\mathrm{egs}}$*, the generic fibre* $\left(\gamma^* \iota_f^* \overline{\mathfrak{X}_f}\right)_\eta$ *is isomorphic to* $\mathbb{P}^2_{\Bbbk(\!(t)\!)}$*.*

Proof. The generic fibre $\left(\gamma^* \iota_f^* \overline{\mathfrak{X}_f}\right)_\eta$ is a smooth del Pezzo surface. Since the Brauer group of $\Bbbk(\!(t)\!)$ is trivial for $\Bbbk$ of characteristic zero, it suffices to prove that this del Pezzo surface has degree nine; cf. [23, Example 6.0.2]. Consider the diagram of rings

$$\begin{array}{ccc} R_f^{\mathrm{egs}} & \overset{\iota_f}{\hookrightarrow} & \tilde{R}^{\mathrm{egs}} \\ & & \uparrow \\ & & \tilde{R}'' \hookrightarrow \tilde{R}^{\#}. \end{array} \tag{5.12}$$

The family $\iota_f^* \overline{\mathfrak{X}_f} \to \operatorname{Spec} \tilde{R}^{\mathrm{egs}}$ arises by pullback from a family $\mathfrak{W} \to \operatorname{Spec} \tilde{R}''$, because all the wall functions of $\iota_f\left(\alpha_f(\overline{\mathfrak{D}_{(\Sigma_f, H_f)}})\right)$ lie in $\tilde{R}''$. Consider the pullback of $\mathfrak{W}$ to $\operatorname{Spec} \tilde{R}^{\#}$ via the lower horizontal map in (5.12). It suffices to show that fibres of this family have degree nine – since the degree is constant on fibres, we compute the degree of the central fibre. The central fibre is the union of toric varieties with moment polytopes as pictured in Figure 5.4. This is also the central fibre of a Mumford degeneration with general fibre $\mathbb{P}^2$, and hence has degree nine. □

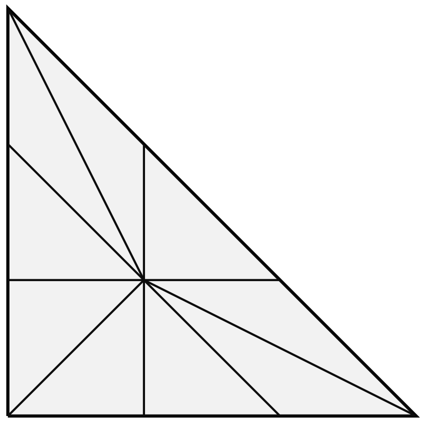

Figure 5.4: The moment polytopes of the toric varieties whose union forms the central fibre of the pullback of $\mathfrak{W}$ to Spec $\tilde{R}^{\#}$.

Lemma 5.2.4. *For a general one-parameter family* $\gamma : \operatorname{Spec} \Bbbk[\![t]\!] \to \operatorname{Spec} \tilde{R}^{\mathrm{egs}}$*, both the generic fibres* $\left(\gamma^* \overline{\mathfrak{X}_{\sim}}\right)_\eta$ *and* $\left(\gamma^* \iota_g^* \overline{\mathfrak{X}_g}\right)_\eta$ *are isomorphic to* $\mathrm{Bl}_{pt} \mathbb{P}^2_{\Bbbk(\!(t)\!)}$.

Proof. As above, both generic fibres are smooth del Pezzo surfaces. It suffices to prove that these del Pezzo surfaces have degree eight, since any such del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2$ and furthermore:

- $\left(\gamma^* \overline{\mathfrak{X}_{\sim}}\right)_\eta$ admits a birational morphism to $\mathbb{P}^2_{\Bbbk(\!(t)\!)}$ and hence cannot be isomorphic to $\mathbb{P}^1_{\Bbbk(\!(t)\!)} \times \mathbb{P}^1_{\Bbbk(\!(t)\!)}$;
- $\left(\gamma^* \iota_g^* \overline{\mathfrak{X}_g}\right)_\eta$ receives a birational morphism from $\left(\gamma^* \overline{\mathfrak{X}_{\sim}}\right)_\eta \cong \mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2_{\Bbbk(\!(t)\!)}$, and hence cannot be isomorphic to $\mathbb{P}^1_{\Bbbk(\!(t)\!)} \times \mathbb{P}^1_{\Bbbk(\!(t)\!)}$.

Here we used Lemma 5.1.6.

To see that these del Pezzo surfaces have degree eight, we argue exactly as above, computing the degree of the central fibres of families over Spec $\tilde{R}^{\#}$. These central fibres are once again unions of toric varieties, which also occur as central fibres of Mumford degenerations – this time of $\mathrm{Bl}_{\mathrm{pt}} \mathbb{P}^2$. They therefore have degree eight. ◻

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