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Non-commutative nature of ℓ-adic vanishing cycles

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Abstract. Let $p: X \to S$ be a flat (proper) and regular scheme of finite type over a strictly henselian discrete valuation ring. We prove that the singularity category of the special fiber with its natural two-periodic structure allows to recover the ℓ-adic vanishing cohomology of p. Along the way, we compute homotopy-invariant non-connective algebraic K-theory with compact support of certain embeddings $X_t \hookrightarrow X_T$ in terms of the motivic realization of the dg-category of relatively perfect complexes.

1. Introduction

1.1. Posing the problem.

1.1.1. It is well known, and well documented in the existing literature, that differential graded (dg) categories of singularities are intimately related to vanishing cohomology. For instance, see [5, 12, 13, 27, 31].

1.1.2. In particular, let W be a complex smooth quasi-projective variety and suppose that $f: W \to \mathbb{A}^1_{\mathbb{C}}$ is a regular map. In [13], it is proven that the vanishing cohomology of f *together with its monodromy action* can be recovered as the periodic cyclic homology of the singularity category of $f^{-1}(0)$, with the extra datum given by a *Getzler–Gauss–Manin connection*. The latter was introduced in [16] and written down explicitly in [32].

1.1.3. In this paper, we deal with the ℓ-adic analogue of the above phenomenon, where the extra datum of the Getzler–Gauss–Manin connection is replaced by a natural (left) module structure on the dg-category of singularities of the special fiber.

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1.1.4. Let $S = \operatorname{Spec}(\mathscr{O}_{\mathrm{K}})$ be the spectrum of an excellent strictly henselian discrete valuation ring, with closed point $i_S : s \to S$ and inertia group I_{K}. We assume that the residue field is perfect and fix a prime number ℓ different from the residue characteristic of $\mathscr{O}_{\mathrm{K}}$. Let $p: X \to S$ be a proper, flat and regular S-scheme. The main result of [5] shows that it is possible to recover the homotopy I_{K}-fixed points of the ℓ-adic vanishing cohomology of p by means of derived and non-commutative algebraic geometry as follows.

1.1.5. In [5], Blanc–Robalo–Toën–Vezzosi construct the *ℓ-adic realization of dg-categories* functor

$$\mathrm{r}_S^{\ell} : \mathrm{dgCat}_S \to \mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)}\big(\mathrm{Shv}_{\mathbb{Q}_\ell}(S)\big),$$

where the right-hand side is the ∞-category of modules over $\mathbb{Q}_{\ell,S}(\beta) = \bigoplus_{j \in \mathbb{Z}} \mathbb{Q}_{\ell,S}(j)[2j]$ in the ∞-category of ℓ-adic sheaves on S. More details will be provided in Section 2.3.

1.1.6. Consider the quotient dg-category

$$\mathsf{D}_{\mathsf{sg}}(X_s) := \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)}{\mathsf{D}_{\mathsf{perf}}(X_s)}.$$

This is called the dg-category of singularities of the special fiber and it is naturally a module over the convolution monoidal dg-category $\mathsf{D}_{\mathsf{sg}}(G)$, where $G = s \times_S s$ is the derived self-intersection of the special point.

1.1.7. The main theorem of [5] states that there is an equivalence

$$\mathrm{r}_S^{\ell}\big(\mathsf{D}_{\mathsf{sg}}(X_s)\big) \simeq i_{S*}\mathrm{H}^*_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,X}(\beta))\big)^{\mathrm{I}_{\mathrm{K}}}[-1],$$

where $\Phi_p(\mathbb{Q}_{\ell,X}(\beta))$ is the ℓ-adic sheaf of vanishing cycles of p with $\mathbb{Q}_{\ell,X}(\beta) = p^*\mathbb{Q}_{\ell,S}(\beta)$ coefficients. Moreover, this equivalence respects the natural actions of the algebra

$$\mathrm{r}_S^{\ell}\big(\mathsf{D}_{\mathsf{sg}}(G)\big) \simeq i_{S*}\mathbb{Q}_{\ell,s}(\beta)^{\mathrm{I}_{\mathrm{K}}}$$

on both sides.

It is natural to ask the following.

Question 1.1.8. Is it possible to recover the vanishing cohomology

$$\mathrm{H}^*_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,X}(\beta))\big),$$

with its natural continuous I_{K}-action, as the ℓ-adic realization of a dg-category?

1.2. Our main results. The goal of this paper is to provide an affirmative answer to the question above.

1.2.1. Let $T = \operatorname{Spec}(\mathscr{O}_{\mathrm{L}}) \to S$ be a (necessarily totally ramified) extension of excellent strictly henselian discrete valuation rings. Let I_{L} denote the absolute Galois group of the generic point Spec(L) of T. Let $i_{X_T} : X_t \hookrightarrow X_T$ be the pullback of the closed immersion $X_s \hookrightarrow X$ along $T \to S$. This morphism, being closed and quasi-smooth, induces a dg-functor

$$i_{X_T*} : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)$$

which preserves perfect complexes. In particular, it induces a dg-functor

$$i_{X_T*}\colon \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t)}{\mathsf{D}_{\mathsf{perf}}(X_t)} = \mathsf{D}_{\mathsf{sg}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T) = \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)}{\mathsf{D}_{\mathsf{perf}}(X_T)}$$

at the level of the singularity dg-categories. Let us consider the dg-category of relative singularities of $X_t \hookrightarrow X_T$ ([6, 14]),

$$\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T) := \mathrm{Ker}\big(i_{X_T*}\colon \mathsf{D}_{\mathsf{sg}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T)\big).$$

Our main theorem reads as follows.

Theorem A. *Let* $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ *denote the (finite) quotient of* I_{K} *by* I_{L}. *There is an equivalence*

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \simeq i_{S*}\mathrm{H}^{*}_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,X}(\beta))\big)^{\mathrm{I}_{\mathrm{L}}}[-1]$$

of $i_{S*}\mathbb{Q}^{\mathrm{I}_{\mathrm{L}}}_{\ell,s}(\beta)$*-modules, compatible with the natural* $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$*-actions.*

1.2.2. Given this result, it is then easy to answer to Question 1.1.8 as follows.

Let $\mathscr{E}$ be the filtered category of finite extensions of discrete valuation rings $T \to S$ as above. For two extensions $U \to T \to S$ as above, the pullback along $X_u \to X_t$ induces a dg-functor

$$\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T) \to \mathsf{D}_{\mathsf{sg}}(X_u \xrightarrow{i_{X_U}} X_U).$$

This construction induces a diagram of dg-categories indexed by $\mathscr{E}$. The actions of the finite quotients $\mathrm{I}_{\mathrm{K}}/\mathrm{I}_{\mathrm{L}}$ are compatible with this diagram and induce a continuous action of I_{K} on the colimit

$$\mathfrak{S} := \varinjlim_{T \in \mathscr{E}} \mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T).$$

Theorem B. *There is an equivalence*

$$\mathrm{r}^{\ell}_{S}(\mathfrak{S}) \simeq i_{S*}\mathrm{H}^{*}_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,X}(\beta))\big)[-1]$$

of $i_{S*}\mathbb{Q}_{\ell,s}(\beta)$*-modules, compatible with the natural (continuous)* I_{K}*-actions.*

Remark 1.2.3. In Theorems A and B, it is not really necessary to assume that $p\colon X \to S$ is proper. In Appendix A, we explain how to remove this hypothesis.

1.3. Strategy of the proof of Theorem A.

1.3.1. Observe that there is an equivalence of dg-categories

$$\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T) \simeq \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)}{\mathsf{D}_{\mathsf{perf}}(X_t)},$$

where

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T) \subset \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t)$$

denotes the full subcategory spanned by objects $E \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t)$ such that $i_{X_T*}(E) \in \mathsf{D}_{\mathsf{perf}}(X_T)$.

This is the dg-category of relatively perfect complexes we alluded to in the abstract. In order to prove Theorem A, one needs to compute the motivic realization of

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T).$$

1.3.2. We now notice that we have a localization sequence

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)$$

of dg-categories. Moreover, this is a localization sequence of left $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G)$-modules. Here, $a_1 \colon G_t \to t$ is the pullback of $i_S \colon s \to S$ along $t \to S$ and the dg-functor $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)$ is induced by the pushforward along a_1 (notice that a_1 is proper and quasi-smooth).

1.3.3. Since the dg-category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}}$ admits a right $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G)$-module structure, we can then apply the functor $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G)} -$ (i.e. the relative tensor product) and obtain the localization sequence

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G)} \big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)\big).$$

1.3.4. After computing these tensor products, we recognize that the rightmost dg-functor identifies with $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T)$, the composition of $i_{X_T *} \colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)$ with the quotient dg-functor $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T) \to \mathsf{D}_{\mathsf{sg}}(X_T)$. As a consequence, we deduce that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G)} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)$$

and that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T) \tag{1.1}$$

is a localization sequence.

Remark 1.3.5. This fact is nontrivial: even if

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)$$

is by definition the kernel of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T)$, the equivalence

$$\frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)} \xrightarrow{\simeq} \mathsf{D}_{\mathsf{sg}}(X_T)$$

is not obvious.

1.3.6. Now consider the *motivic realization of dg-categories*, that is, the functor

$$\mathcal{M}^{\vee}_S \colon \mathrm{dgCat}_S \to \mathcal{SH}_S$$

introduced in [5]; see Section 2.3 for the details. A fundamental property of $\mathcal{M}^{\vee}_S$ is that it sends localizations sequences to exact triangles. Using (1.1) as a key ingredient, we obtain that

$$\mathcal{M}^{\vee}_S\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \simeq \mathcal{M}^{\vee}_S\big(\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t}\big), \tag{1.2}$$

where $\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t}$ denotes the dg-category of perfect complexes on X_T with set-theoretic support contained in X_t.

Remark 1.3.7. The equivalence (1.2) could be regarded as a form of *dévissage for homotopy-invariant non-connective algebraic K-theory* and seems to be a new result interesting on its own; see Theorem 4.2.3.

1.3.8. Once the above computation of

$$\mathcal{M}_S^\vee(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T))$$

is settled, we can proceed similarly to [5] and conclude the proof of Theorem A.

Remark 1.3.9. This work is the second in a series of three papers whose goal is to prove the Deligne–Milnor conjecture following the vision of Toën–Vezzosi. The first and third paper of the series are [3] and [4], respectively.

2. Preliminaries

2.1. Notation. We fix here some notation that we will adopt in the main body of the paper.

2.1.1. Let $\mathscr{O}_\mathrm{K}$ be a complete[1)] strict discrete valuation ring and $\mathrm{K} \supseteq \mathscr{O}_\mathrm{K}$ its fraction field. We assume that the residue field is perfect.

We fix once and for all a uniformizing element $\pi_\mathrm{K} \in \mathscr{O}_\mathrm{K}$ and denote by $\mathrm{k} = \mathscr{O}_\mathrm{K}/(\pi_\mathrm{K})$ the (algebraically closed) residue field. We also fix a separable closure $\overline{\mathrm{K}}$ of K and denote by $\mathrm{I}_\mathrm{K} = \mathscr{G}al(\overline{\mathrm{K}}/\mathrm{K})$ the absolute Galois group of K, which coincides with the inertia group in this case. Moreover, let S (resp. s, η, $\bar{\eta}$) be the spectrum of $\mathscr{O}_\mathrm{K}$ (resp. k, K, $\overline{\mathrm{K}}$),

$$s \xrightarrow{i_S} S \xleftarrow{j_S} \eta \longleftarrow \bar{\eta}.$$

2.1.2. Let $\mathrm{K} \subseteq \mathrm{L}$ be a finite Galois extension (viewed inside $\overline{\mathrm{K}}$), which is necessarily totally ramified, and assume that the ring of integers $\mathscr{O}_\mathrm{L}$ of L is still a (strictly henselian) trait. In this case, for a fixed uniformizing element $\pi_\mathrm{L} \in \mathscr{O}_\mathrm{L}$, there is a unit $u \in \mathscr{O}_\mathrm{L}^\times$ such that

$$\pi_\mathrm{K} = u \cdot \pi_\mathrm{L}^\mathrm{e},$$

where $\mathrm{e} = [\mathrm{L} : \mathrm{K}]$ is the degree of the extension (which agrees with the ramification degree in this case).

2.1.3. Denote by $\mathrm{I}_\mathrm{L} = \mathscr{G}al(\overline{\mathrm{K}}/\mathrm{L})$ the absolute Galois group of L: this is an open normal subgroup of I_K. Let $\mathrm{G}_{\mathrm{L}/\mathrm{K}} \simeq \mathscr{G}al(\mathrm{L}/\mathrm{K})$ be the (finite) quotient group $\mathrm{I}_\mathrm{K}/\mathrm{I}_\mathrm{L}$. Set $T := \mathrm{Spec}(\mathscr{O}_\mathrm{L})$ and denote by t (resp. η_L, $\bar{\eta}_\mathrm{L}$) the pullback of s (resp. η, $\bar{\eta}$) along $T \to S$. We thus have

[1)] In the introduction, we only assumed $\mathscr{O}_\mathrm{K}$ to be excellent and strictly henselian. This further assumption on $\mathscr{O}_\mathrm{K}$ is harmless in view of [11, Exposé XIII, Proposition 2.1.12].

Cartesian squares

$$\begin{array}{ccccccc} t & \xrightarrow{i_T} & T & \xleftarrow{j_T} & \eta_{\mathrm{L}} & \longleftarrow & \bar{\eta}_{\mathrm{L}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ s & \xrightarrow{i_S} & S & \xleftarrow{j_S} & \eta & \longleftarrow & \bar{\eta}. \end{array}$$

Notice that $t \simeq \mathrm{Spec}(\mathscr{O}_{\mathrm{L}}/(\pi_{\mathrm{K}}))$ is a nil-thickening of Spec(k), while

$$\eta_{\mathrm{L}} = \mathrm{Spec}(\mathrm{L}) \quad \text{and} \quad \bar{\eta}_{\mathrm{L}} \simeq \mathrm{G}_{\mathrm{L/K}} \times \bar{\eta}.$$

2.1.4. We denote by $G_t \xrightarrow{a_1} t$ the pullback of $G \hookrightarrow s$ (i.e. the first projection $s \times_S s \to s$) along $t \to s$. In fact, it is easy to see that we can write G_t as the (derived) pullback $t \times_T t$ and that a_1 agrees with the first projection,

$$\begin{array}{ccc} t \times_T t & \xrightarrow{a_1} & t \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle i_T} \\ t & \xrightarrow{i_T} & T. \end{array}$$

Under the equivalence $t \times_T t \simeq s \times_S s \times_S T$, the map a_1 corresponds to the projection

$$\widehat{\mathrm{pr}}_{13}\colon s \times_S s \times_S T \to s \times_S T$$

onto the first and third component.

2.1.5. Throughout this paper, we will consider a proper and flat S-scheme $p\colon X \to S$, which is moreover assumed to be regular (and generically smooth). We denote by $p_s\colon X_s \to s$ (resp. $p_{\mathrm{K}}\colon X_{\mathrm{K}} \to \eta$, $p_{\overline{\mathrm{K}}}\colon X_{\overline{\mathrm{K}}} \to \bar{\eta}$) the pullback of $p\colon X \to S$ along $s \hookrightarrow S$ (resp. $\eta \hookrightarrow S$, $\bar{\eta} \to S$), so that we obtain a diagram

$$X_s \xrightarrow{i_X} X \xleftarrow{j_X} X_{\mathrm{K}} \longleftarrow X_{\overline{\mathrm{K}}}.$$

2.1.6. Similarly, we denote by $p_T\colon X_T \to T$ (resp. $p_t\colon X_t \to t$, $p_{\mathrm{L}}\colon X_{\mathrm{L}} \to \eta_{\mathrm{L}}$) the pullback of $p\colon X \to S$ along $T \to S$ (resp. $t \to S$, $\eta_T \to S$), and get the open-closed decomposition

$$X_t \xrightarrow{i_{X_T}} X_T \xleftarrow{j_{X_T}} X_{\mathrm{L}}.$$

2.2. Higher categories.

2.2.1. We will freely use the theory of higher categories; see [18, 19]. All functors are implicitly derived. Morphisms between ∞-categories are simply called "functors", instead of the more precise "∞-functors".

2.2.2. We work in the framework of dg-categories *up to Morita equivalences*. We refer to [17, 34, 35] for exhaustive accounts.

2.2.3. Let $\mathrm{dgCat}_S^{\mathrm{s}}$ denote the (ordinary) category of small $\mathscr{O}_{\mathrm{K}}$-linear dg-categories (i.e. categories enriched in cochain complexes of $\mathscr{O}_{\mathrm{K}}$-modules). A dg-functor is then just a functor compatible with these enrichments.

2.2.4. For a dg-category T, its homotopy category is the ($\mathscr{O}_K$-linear) category hT with the same objects of T and such that $\mathrm{Hom}_{\mathsf{hT}}(x, y) = \mathrm{H}^0(\mathrm{Hom}_{\mathsf{T}}(x, y))$ for any objects x, y.

2.2.5. Among all dg-functors, we consider the collection $\mathrm{W}_{\mathrm{Mor}}$ of *Morita equivalences*, that is, those dg-functors $F\colon \mathsf{T} \to \mathsf{U}$ such that F induces a quasi-isomorphism

$$\mathrm{Hom}_{\mathsf{T}}(x, y) \to \mathrm{Hom}_{\mathsf{U}}\big(F(x), F(y)\big)$$

for all pairs of objects $x, y \in \mathsf{T}$ and the image of F generates the Karoubi completion $\widehat{\mathsf{U}}_c$ of U (recall that $\mathsf{U} \subseteq \widehat{\mathsf{U}}_c$) under cones, shifts and retracts. Then we consider the ∞-localization of $\mathrm{dgCat}^{\mathrm{s}}_S$ along $\mathrm{W}_{\mathrm{Mor}}$, $\mathrm{dgCat}_S := \mathrm{dgCat}^{\mathrm{s}}_S[\mathrm{W}_{\mathrm{Mor}}^{-1}]$.

Remark 2.2.6. This ∞-localization has a model. Indeed, in [33], G. Tabuada exhibits a model category structure on $\mathrm{dgCat}^{\mathrm{s}}_S$ where weak equivalences are *quasi-equivalences*, i.e. dg-functors inducing quasi-isomorphisms on the hom complexes and which induce equivalences on the homotopy categories. Every quasi-equivalence is a Morita equivalence and one can take the associated Bousfield localization, which is a model category whose associated ∞-category is equivalent to dgCat_S.

2.2.7. In [34], B. Toën showed that there is a well behaved theory of dg-localizations. In other words, for every $\mathsf{T} \in \mathrm{dgCat}_S$ and every (saturated) collection of morphisms $\mathsf{W} \subset \mathsf{T}$, there exists a dg-category $\mathsf{T}[\mathsf{W}^{-1}] \in \mathrm{dgCat}_S$ endowed with a dg-functor $\mathsf{T} \to \mathsf{T}[\mathsf{W}^{-1}]$ which has the following universal property: it induces a fully faithful embedding of functor ∞-categories

$$\mathrm{Fun}_{\mathrm{dgCat}_S}(\mathsf{T}[\mathsf{W}^{-1}], \mathsf{U}) \to \mathrm{Fun}_{\mathrm{dgCat}_S}(\mathsf{T}, \mathsf{U})$$

for every $\mathsf{U} \in \mathrm{dgCat}_S$, whose essential image consists of dg-functors $\mathsf{T} \to \mathsf{U}$ mapping every morphism in W to an equivalence.

2.2.8. There is also a theory of dg-quotients: for a sub-dg-category $\mathsf{U} \subset \mathsf{T}$, the dg-quotient T/U is the dg-localization of T along those morphisms $x \to y$ in T whose fiber belongs to U. More generally, for a dg-functor $F\colon \mathsf{U} \to \mathsf{T}$, the dg-quotient T/U is defined as the dg-quotient of T by the full sub-dg-category spanned by the essential image of F.

2.2.9. Of major relevance for the purposes of this paper is the notion of *localization sequence* in dgCat_S. We say that a diagram $\mathsf{T}_1 \to \mathsf{T}_2 \to \mathsf{T}_3$ in dgCat_S is a localization sequence if the composition is homotopic to 0, the induced dg-functor $\mathsf{T}_2/\mathsf{T}_1 \to \mathsf{T}_3$ is a Morita equivalence and T_1 is the kernel of $\mathsf{T}_2 \to \mathsf{T}_3$.

2.3. Motivic and ℓ-adic realizations of dg-categories. We recall here some of the main constructions of [5].

2.3.1. We denote by $\mathcal{SH}_S$ the stable homotopy category of S-schemes introduced by F. Morel and V. Voevodsky in [20] (or rather its ∞-categorical version; see [30]). This is a stable symmetric monoidal presentable ∞-category endowed with a symmetric monoidal functor

$$\Sigma^{\infty}_{+}\colon \mathrm{Sm}_S \to \mathcal{SH}_S$$

which enjoys the following universal property (see [30]). Suppose that $\mathcal{C}$ is a stable presentable symmetric monoidal ∞-category endowed with a symmetric monoidal functor $F\colon \mathrm{Sm}_S \to \mathcal{C}$ such that

- F satisfies Nisnevich descent (i.e. it sends Nisnevich squares in Sch_S to pullbacks in $\mathcal{C}$),
- the canonical map $\mathbb{A}^1_S \to S$ is mapped to an equivalence in $\mathcal{C}$,
- the fiber of $F(S \xrightarrow{\infty} \mathbb{P}^1_S)$ is an invertible object in $\mathcal{C}$;

then F must factor (essentially uniquely) through Σ^∞_+.

2.3.2. Among motivic spectra (that is, objects in $\mathcal{SH}_S$), there is $\mathrm{B}\mathbb{U}_S$, the spectrum which represents homotopy-invariant non-connective algebraic K-theory. This object enjoys the *algebraic Bott periodicity*, i.e. there is a canonical equivalence $\mathrm{B}\mathbb{U}_S \simeq \mathrm{B}\mathbb{U}_S(1)[2]$.

2.3.3. In [30], M. Robalo constructs a non-commutative analogue of $\mathcal{SH}_S$ (see also [9, 10] for an alternative construction). This is a stable symmetric monoidal presentable ∞-category $\mathcal{SH}^{\mathrm{nc}}_S$ equipped with a symmetric monoidal functor $\iota\colon \mathrm{dgCat}^{\mathrm{ft,op}}_S \to \mathcal{SH}^{\mathrm{nc}}_S$ from the opposite ∞-category of dg-categories of finite type (see [36]) which enjoys the analogue universal property of $\mathcal{SH}_S$: for every symmetric monoidal functor $F\colon \mathrm{dgCat}^{\mathrm{ft,op}}_S \to \mathcal{C}$, where $\mathcal{C}$ is a stable presentable symmetric monoidal ∞-category, such that

- F sends Nisnevich squares of dg-categories (see [30]) to pullbacks in $\mathcal{C}$,
- the morphism $\mathsf{D}_{\mathsf{perf}}(\mathbb{A}^1_S) \to \mathsf{D}_{\mathsf{perf}}(S)$ in $\mathrm{dgCat}^{\mathrm{ft,op}}_S$ induced by pullback along the projection map is mapped to an equivalence in $\mathcal{C}$,
- the fiber of $F(\mathsf{D}_{\mathsf{perf}}(S) \xrightarrow{\infty^*} \mathsf{D}_{\mathsf{perf}}(\mathbb{P}^1_S))$ is an invertible object in $\mathcal{C}$,

then F factors (essentially uniquely) through ι.

2.3.4. The composition

$$\mathrm{Sm}_S \xrightarrow{\mathsf{D}_{\mathsf{perf}}} \mathrm{dgCat}^{\mathrm{ft,op}}_S \xrightarrow{\iota} \mathcal{SH}^{\mathrm{nc}}_S$$

is symmetric monoidal and enjoys all the properties listed above. By the universal property of $\mathcal{SH}_S$, we thus obtain a functor $\mathcal{R}_{\mathrm{pe}}\colon \mathcal{SH}_S \to \mathcal{SH}^{\mathrm{nc}}_S$, called the *perfect realization*. This is a (symmetric monoidal) colimit preserving functor between presentable stable ∞-categories; thus it admits a (lax-monoidal) right adjoint $\mathcal{M}_S\colon \mathcal{SH}^{\mathrm{nc}}_S \to \mathcal{SH}_S$. As proved in [30], this functor maps the unit object $\mathbf{1}^{\mathrm{nc}}_S$ of $\mathcal{SH}^{\mathrm{nc}}_S$ to $\mathrm{B}\mathbb{U}_S$.

2.3.5. In [5], the following "dual" version of $\mathcal{M}_S$ is considered:

$$\mathcal{M}^\vee_S\colon \mathrm{dgCat}^{\mathrm{ft}}_S \xrightarrow{\iota^{\mathrm{op}}} \mathcal{SH}^{\mathrm{nc,op}}_S \xrightarrow{\mathrm{Hom}_{\mathcal{SH}^{\mathrm{nc}}_S}(-,\mathbf{1}^{\mathrm{nc}}_S)} \mathcal{SH}^{\mathrm{nc}}_S \xrightarrow{\mathcal{M}_S} \mathcal{SH}_S.$$

Since $\mathcal{SH}_S$ is presentable and $\mathrm{Ind}(\mathrm{dgCat}^{\mathrm{ft}}_S) \simeq \mathrm{dgCat}_S$ (see [36]), we can extend this (lax-monoidal) functor to dgCat_S,

$$\mathcal{M}^\vee_S\colon \mathrm{dgCat}_S \to \mathcal{SH}_S.$$

This is called the *motivic realization of dg-categories*. As it is lax-monoidal, we actually get a functor $\mathcal{M}^\vee_S\colon \mathrm{dgCat}_S \to \mathrm{Mod}_{\mathrm{B}\mathbb{U}_S}(\mathcal{SH}_S)$. For a dg-category T, the motivic spectrum underlying $\mathcal{M}_S^\vee(\mathsf{T})$ is a functor $\mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Sp}$ (here Sp denotes the stable presentable symmetric monoidal ∞-category of spectra) defined on objects by the assignment

$$Y \mapsto \mathrm{HK}\big(\mathsf{T} \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(Y)\big),$$

where HK denotes homotopy-invariant non-connective algebraic K-theory.

2.3.6. The motivic realization of dg-categories enjoys the following properties:

- it preserves filtered colimits;
- for every qcqs S-scheme $q\colon Y \to S$ of finite type, $\mathcal{M}_S^\vee(\mathsf{D}_{\mathsf{perf}}(Y)) \simeq q_*\mathrm{B}\mathbb{U}_Y$;
- it sends localization sequences in dgCat_S to fiber-cofiber sequences in $\mathcal{SH}_S$.

2.3.7. Let ℓ be a prime number invertible in $\mathcal{O}_{\mathrm{K}}$. The authors of [5] considered also the ℓ-adic realization

$$\mathcal{R}_S^\ell\colon \mathcal{SH}_S \xrightarrow{-\otimes \mathrm{H}\mathbb{Q}} \mathrm{Mod}_{\mathrm{H}\mathbb{Q}}(\mathcal{SH}_S) \longrightarrow \mathrm{Shv}_{\mathbb{Q}_\ell}(S),$$

where $\mathrm{H}\mathbb{Q}$ is the spectrum of rational singular cohomology. The second functor is constructed in [5] (based on the rigidity theorems due to Ayoub and Cisinski–Déglise; see [1,7]). It is a symmetric monoidal functor with values in the ∞-category of ind-constructible ℓ-adic sheaves. It follows from results of J. Riou (see [29]) that $\mathcal{R}_S^\ell(\mathrm{B}\mathbb{U}_S) \simeq \mathbb{Q}_{\ell,S}(\beta) = \bigoplus_{j\in\mathbb{Z}} \mathbb{Q}_\ell(j)[2j]$.

2.3.8. The composition

$$\mathrm{r}_S^\ell\colon \mathrm{dgCat}_S \xrightarrow{\mathcal{M}_S^\vee} \mathrm{Mod}_{\mathrm{B}\mathbb{U}_S}(\mathcal{SH}_S) \xrightarrow{\mathcal{R}_S^\ell} \mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)} = \mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)}\big(\mathrm{Shv}_{\mathbb{Q}_\ell}(S)\big)$$

is a lax-monoidal functor which enjoys the same properties of $\mathcal{M}_S^\vee$ (*mutatis mutandis*) and it is called the *ℓ-adic realization of dg-categories*.

2.4. Some dg-categories of interest. We will be interested in some very specific dg-categories.

2.4.1. Let Y denote a (possibly derived) scheme of finite type over S. One associates to it its dg-category of quasi-coherent complexes $\mathsf{D}_{\mathsf{qcoh}}(Y)$. We will need to consider the following two sub-dg-categories:

- the full subcategory $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y)$, spanned by those complexes with coherent and bounded cohomology sheaves;
- the full subcategory $\mathsf{D}_{\mathsf{perf}}(Y)$ of perfect complexes.

Under the mild hypothesis that the structure sheaf $\mathcal{O}_Y$ is bounded, we have a fully faithful embedding $\mathsf{D}_{\mathsf{perf}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y)$. In this case, the *dg-category of (absolute) singularities of Y* is defined as the dg-quotient

$$\mathsf{D}_{\mathsf{sg}}(Y) := \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y)}{\mathsf{D}_{\mathsf{perf}}(Y)}.$$

2.4.2. We will also need to consider the following. Let $j\colon Z \hookrightarrow Y$ be a quasi-smooth closed embedding of (derived) schemes of finite type over S. Then the pushforward

$$j_*\colon \mathsf{D}_{\mathsf{qcoh}}(Z) \to \mathsf{D}_{\mathsf{qcoh}}(Y)$$

induces dg-functors $j_*\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y)$ and $j_*\colon \mathsf{D}_{\mathsf{perf}}(Z) \to \mathsf{D}_{\mathsf{perf}}(Y)$. Therefore, it induces a dg-functor $j_*\colon \mathsf{D}_{\mathsf{sg}}(Z) \to \mathsf{D}_{\mathsf{sg}}(Y)$. The *dg-category of relative singularities of* $Z \hookrightarrow Y$ is defined as the kernel of this dg-functor,

$$\mathsf{D}_{\mathsf{sg}}(Z \xrightarrow{j} Y) := \mathrm{Ker}\big(j_*\colon \mathsf{D}_{\mathsf{sg}}(Z) \to \mathsf{D}_{\mathsf{sg}}(Y)\big).$$

2.4.3. This dg-category also admits an alternative description. Let $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z \xrightarrow{j} Y)$ denote the kernel of the dg-functor

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z) \xrightarrow{j_*} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y) \twoheadrightarrow \mathsf{D}_{\mathsf{sg}}(Y).$$

This is the full subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z)$ spanned by those complexes $E \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z)$ whose image along j_* is a perfect complex of Y. Since j_* preserves perfect complexes, all perfect complexes over Z lie in this subcategory,

$$\mathsf{D}_{\mathsf{perf}}(Z) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z \xrightarrow{j} Y).$$

Thus, there is an equivalence

$$\frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z \xrightarrow{j} Y)}{\mathsf{D}_{\mathsf{perf}}(Z)} \xrightarrow{\simeq} \mathsf{D}_{\mathsf{sg}}(Z \xrightarrow{j} Y).$$

2.5. The monoidal dg-categories B^+ and B. Following [37], we now introduce two important monoidal dg-categories.

2.5.1. Consider the derived fiber product

$$G := s \times_S s,$$

i.e. the spectrum of the simplicial Koszul algebra $\mathrm{K}(\mathscr{O}_{\mathrm{K}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$. This is a derived groupoid scheme over s (i.e. $\mathrm{K}(\mathscr{O}_{\mathrm{K}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ is a Hopf algebroid). The composition $G \times_s G \to G$ corresponds to the projection onto the first and third factor under the equivalence

$$G \times_s G \simeq s \times_S s \times_S s,$$

while the unit corresponds to the canonical morphism $u\colon s \to G$.

2.5.2. This derived groupoid structure induces a monoidal convolution $\odot$ product on $\mathsf{B}^+ := \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G)$. Roughly, this is defined as the dg-functor

$$\begin{aligned} - \odot -\colon \mathsf{B}^+ \otimes \mathsf{B}^+ &\to \mathsf{B}^+, \\ (M, N) &\to \mathrm{pr}_{13*}(\mathrm{pr}_{12}^* M \otimes \mathrm{pr}_{23}^* N), \end{aligned}$$

where $\mathrm{pr}_{ij}\colon G \times_s G \simeq s \times_S s \times_S s \to G$ denotes the projection onto the i-th and j-th factors (which is a proper quasi-smooth map). The unit of this convolution product is $u_*\mathcal{O}_s$; in other words, it is k with the obvious $\mathcal{O}_G$-module structure.

Remark 2.5.3. Beware that this convolution product is associative and unital (up to coherent homotopy), but not commutative in general. In other words, B^+ is just an $\mathbb{E}_1$-algebra in dgCat_S.

2.5.4. The above convolution product is compatible with perfect complexes, i.e.

$$M \odot N \in \mathsf{D}_{\mathsf{perf}}(G)$$

as soon as M or N lies in $\mathsf{D}_{\mathsf{perf}}(G)$. Therefore, $\odot$ induces a similarly defined convolution product on the dg-category of singularities $\mathsf{B} := \mathsf{D}_{\mathsf{sg}}(G)$.

2.5.5. We will denote the ∞-category of left (resp. right) B^+-modules by $\mathrm{dgCat}_{\mathsf{B}^+}$ (resp. $\mathrm{dgCat}^{\mathsf{B}^+}$). An analogous notation will be employed for left (resp. right) B-modules.

3. A useful localization sequence

The goal of this section is to construct a localization sequence of dg-categories

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t),$$

and then show that it is B^+-linear for the natural B^+-module structures on all terms.

3.1. Explicit models.

3.1.1. Recall from [26] that we have the following explicit models for $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$. The simplicial algebra $\mathrm{K}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ corresponds, under the Dold–Kan equivalence, to the dg-algebra (also denoted $\mathrm{K}(\mathscr{O}_L, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$, by abuse of notation)

$$\mathscr{O}_{\mathrm{L}} \cdot h_1 h_2 \to \mathscr{O}_{\mathrm{L}} \cdot h_1 \oplus \mathscr{O}_{\mathrm{L}} \cdot h_2 \to \mathscr{O}_{\mathrm{L}}$$

placed in (cohomological) degrees $[-2, 0]$ and with differential characterized by the requirement that $h_1, h_2 \mapsto \pi_{\mathrm{K}}$. Also, the variables h_1, h_2 anticommute and square to zero.

3.1.2. Similarly, the simplicial algebra $\mathrm{K}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})$ corresponds, under the Dold–Kan equivalence, to the dg-algebra (also denoted $\mathrm{K}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})$) $\mathscr{O}_{\mathrm{L}} \cdot h \to \mathscr{O}_{\mathrm{L}}$ placed in (cohomological) degrees $[-1, 0]$ and with differential characterized by $h \mapsto \pi_{\mathrm{K}}$. The variable h squares to zero.

3.1.3. For $i = 1, 2$, the morphism

$$a_i : t \times_T t \simeq \mathrm{Spec}\big(\mathrm{K}(\mathscr{O}_L, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))\big) \to \mathrm{Spec}\big(\mathrm{K}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})\big) \simeq t$$

corresponds to the morphism of simplicial algebras $\mathrm{K}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}}) \to \mathrm{K}(\mathscr{O}_L, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ uniquely determined by $h \mapsto h_i$.

3.1.4. Let $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ denote the strict $\mathscr{O}_{\mathrm{K}}$-dg-category of dg-modules over $\mathrm{K}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ with strictly perfect underlying $\mathscr{O}_{\mathrm{L}}$-dg-modules. More explicitly, it is defined as follows.

- The objects of $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ are tuplets $(E, d, \{h_1, h_2\})$, where (E, d) is a strictly perfect cochain complex of $\mathscr{O}_{\mathrm{L}}$-modules (i.e. degreewise projective of finite type and strictly bounded) and each h_i is a $\mathscr{O}_{\mathrm{L}}$-linear morphism $h_i : E \to E[-1]$ of degree -1.

These data are subject to the following requirements:

(1) $h_i \circ h_i = 0$ for $i = 1, 2$;

(2) $[h_1, h_2] = 0$;

(3) $[d, h_i] = \pi_K \cdot \mathrm{id}_E$.

- For two such objects

$$\mathbb{E} = (E, d_E, \{h_{i,E}\}_{i=1,2}), \quad \mathbb{F} = (F, d_F, \{h_{i,F}\}_{i=1,2})$$

and for each $n \in \mathbb{Z}$, the $\mathscr{O}_K$-module of degree n morphisms $\mathrm{Hom}^n(\mathbb{E}, \mathbb{F})$ is the submodule of

$$\bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_L}(E^j, F^{j+n})$$

(which is isomorphic to $\prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_L}(E^j, F^{j+n})$ as the complexes are strictly bounded) spanned by those elements $\{\phi^j : E^j \to F^{j+n}\}_{j \in \mathbb{Z}}$ verifying the equations

$$\phi^j \circ h_{i,E}^{j+1} = h_{i,F}^{j+n+1} \circ \phi^{j+1}, \quad i = 1, 2.$$

As usual, these modules form a cochain complex by considering the differential

$$\mathrm{Hom}^n(\mathbb{E}, \mathbb{F}) \to \mathrm{Hom}^{n+1}(\mathbb{E}, \mathbb{F}),$$
$$\{\phi^j : E^j \to F^{j+n}\}_{j \in \mathbb{Z}} \mapsto \{d_F^{j+n} \circ \phi^j + (-1)^{n+1} \phi^{j+1} \circ d_E^j : E^j \to F^{j+n+1}\}_{j \in \mathbb{Z}}.$$

Remark 3.1.5. Since strictly perfect cochain complexes of $\mathscr{O}_L$-modules are degreewise projective of finite rank and strictly bounded, the $\mathscr{O}_L$-module $\bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_L}(E^j, F^{j+n})$ is projective of finite rank for each $n \in \mathbb{Z}$. As $\mathscr{O}_L$ is a principal ideal domain, this means that $\bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_L}(E^j, F^{j+n})$ is free of finite rank for each $n \in \mathbb{Z}$. Therefore,

$$\mathrm{Hom}^n(\mathbb{E}, \mathbb{F}) \subseteq \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_L}(E^j, F^{j+n})$$

is free of finite rank for each $n \in \mathbb{Z}$ as well. Since $\mathscr{O}_L$ is a (faithfully) flat $\mathscr{O}_K$-algebra, it follows that $\mathrm{Coh}^s(\mathscr{O}_L, (\pi_K, \pi_K))$ is a locally flat $\mathscr{O}_K$-dg-category.

3.1.6. We have an analogous model for $\mathsf{D}^b_{\mathsf{coh}}(t)$. Let $\mathrm{Coh}^s(\mathscr{O}_L, \pi_K)$ denote the strict $\mathscr{O}_K$-dg-category of $\mathrm{K}(\mathscr{O}_L, \pi_K)$ dg-modules with strictly perfect underlying $\mathscr{O}_L$-dg-module. This dg-category can be described explicitly as well.

- The objects of $\mathrm{Coh}^s(\mathscr{O}_L, \pi_K)$ are tuplets (E, d, h), where (E, d) is a strictly perfect complex of $\mathscr{O}_L$-modules (i.e. degreewise projective of finite rank and strictly bounded) and $h: E \to E[-1]$ is a $\mathscr{O}_L$-linear morphism of degree -1 such that

  (1) $h^2 = 0$;

  (2) $[d, h] = \pi_K \cdot \mathrm{id}_E$.

- For two such objects

$$\mathbb{E} = (E, d_E, h_E), \quad \mathbb{F} = (F, d_F, h_F)$$

and for each $n \in \mathbb{Z}$, the $\mathscr{O}_K$-module $\mathrm{Hom}^n(\mathbb{E}, \mathbb{F})$ is the submodule of

$$\bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_L}(E^j, F^{j+n})$$

(which is isomorphic to $\prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_{\mathrm{L}}}(E^j, F^{j+n})$ as the complexes are strictly bounded) spanned by those elements $\{\phi^j : E^j \to F^{j+n}\}_{j\in\mathbb{Z}}$ such that

$$\phi^j \circ h_E^{j+1} = h_F^{j+n+1} \circ \phi^{j+1}.$$

The $\mathscr{O}_{\mathrm{K}}$-module $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(\mathbb{E}, \mathbb{F})$ is equipped with the same differential as above.

Remark 3.1.7. Just as in the previous remark, $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})$ is a locally flat $\mathscr{O}_{\mathrm{K}}$-dg-category.

3.1.8. According to [5, 26], the strict dg-categories $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$, $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})$ are strict models for $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$.

Lemma 3.1.9. *Let* W_{qi} *denote the class of quasi-isomorphisms in both the dg-categories* $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ *and* $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})$. *Then*

$$\mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big)[\mathrm{W}_{\mathrm{qi}}^{-1}]_{\mathrm{dg}} \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t),$$
$$\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})[\mathrm{W}_{\mathrm{qi}}^{-1}]_{\mathrm{dg}} \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t).$$

On the left-hand side, we consider the localization of dg-categories introduced by B. Toën [34].

3.1.10. Using these strict models, it is easy to give strict models of the dg-functors

$$a_i^* : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t), \quad i = 1, 2,$$
$$a_{i*} : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t), \quad i = 1, 2.$$

Indeed, the pushforward along a_i $(i = 1, 2)$ corresponds to the dg-functor

$$\mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) \to \mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})$$

defined on objects by $(E, d, \{h_s\}_{s=1,2}) \to (E, d, h_i)$ and on morphisms by the inclusion

$$\begin{aligned}&\mathrm{Hom}\big((E, d_E, \{h_{E,s}\}_{s=1,2}), (F, d_F, \{h_{F,s}\}_{s=1,2})\big)\\ &\quad\subseteq \mathrm{Hom}\big((E, d_E, \{h_{E,i}\}), (F, d_F, \{h_{F,i}\})\big)\\ &\quad\subseteq \bigoplus_{j,n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{O}_{\mathrm{L}}}(E^j, F^{j+n}).\end{aligned}$$

This is obviously compatible with the differentials, with the identities and with the composition of morphisms. Moreover, it obviously preserves quasi-isomorphisms and its dg-localization along W_{qi} is equivalent to

$$a_{i*} : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t), \quad i = 1, 2.$$

3.1.11. The pullback along a_i $(i = 1, 2)$ can be "strictified" as well. Consider the dg-functor

$$\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}}) \to \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big)$$

defined by sending an object (E, d, h) to $E \oplus E[1]$ with differential

$$\begin{bmatrix} d & \pi_{\mathrm{K}} \cdot \mathrm{id}_E \\ 0 & -d \end{bmatrix} : E \oplus E[1] \to E[1] \oplus E[2]$$

and where $h_i \in \mathrm{K}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ acts via

$$\begin{bmatrix} h & 0 \\ 0 & -h \end{bmatrix} : E \oplus E[1] \to E[-1] \oplus E,$$

while $h_j \in \mathrm{K}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ ($j \in \{1,2\} \smallsetminus \{i\}$) acts via

$$\begin{bmatrix} 0 & 0 \\ \mathrm{id}_E & 0 \end{bmatrix} : E \oplus E[1] \to E[-1] \oplus E.$$

It is defined on morphisms as

$$\phi : E \to F \mapsto \begin{bmatrix} \phi & 0 \\ 0 & \phi[1] \end{bmatrix} : E \oplus E[1] \to F \oplus F[1].$$

It is straightforward to verify that this is compatible with the differentials, with the identities and with the composition. This dg-functor also preserves quasi-isomorphisms and its localization along W_{qi} is equivalent to

$$a_i^* : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t), \quad i = 1, 2.$$

Remark 3.1.12. The two pairs of dg-functors

$$a_i^* : \mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}}) \rightleftarrows \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) : a_{i*}$$

are adjunctions.

Remark 3.1.13. The dg-functor $a_2^* : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ factors through the full embedding $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \subset \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$. This is an immediate consequence of the base-change equivalence $a_{1*} \circ a_2^* \simeq i_T^* \circ i_{T*}$ and of the regularity of T.

3.2. The main computation. We use the above explicit models to prove Corollary 3.3.4, which is the main ingredient for Theorem 4.2.3.

Remark 3.2.1. The chain

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)$$

is a localization sequence if and only if so is

$$\mathsf{D}_{\mathsf{sg}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}_{\mathsf{sg}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t).$$

Lemma 3.2.2. *The dg-category*

$$\frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)} \simeq \frac{\mathsf{D}_{\mathsf{sg}}(G_t)}{\mathsf{D}_{\mathsf{sg}}(G_t \xrightarrow{a_1} t)}$$

is 2*-periodic, i.e. there is a natural equivalence of dg-functors*

$$\mathrm{id} \simeq [2] : \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)} \to \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)}.$$

Here $[2] = [1] \circ [1]$ *denotes the double shift functor.*

Proof. We will show that, for every $E \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$, there is a functorial exact triangle

$$a_2^* a_{2*} E \longrightarrow E \xrightarrow{u_E} E[2]$$

in $\mathsf{D}_{\mathsf{sg}}(G_t)$, where the first morphism is the counit of the adjunction (a_2^*, a_{2*}).[2)]

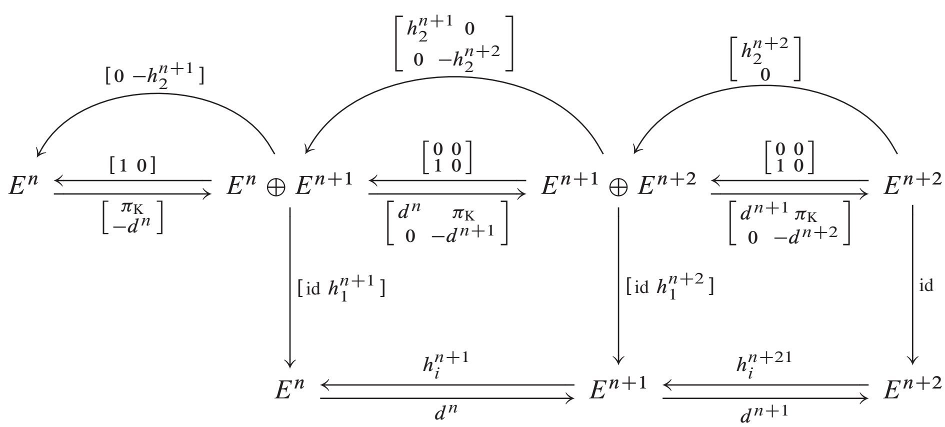
In the diagrams below, we will write horizontally from left to right the differentials of a complex and from right to left the homotopies which are part of the datum for an object in $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$. Morphisms in this category will be written vertically. Notice that it suffices to consider an object in $\mathsf{D}_{\mathsf{sg}}(G_t)$ represented by some

$$(E, d, \{h_1, h_2\}) \in \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big)$$

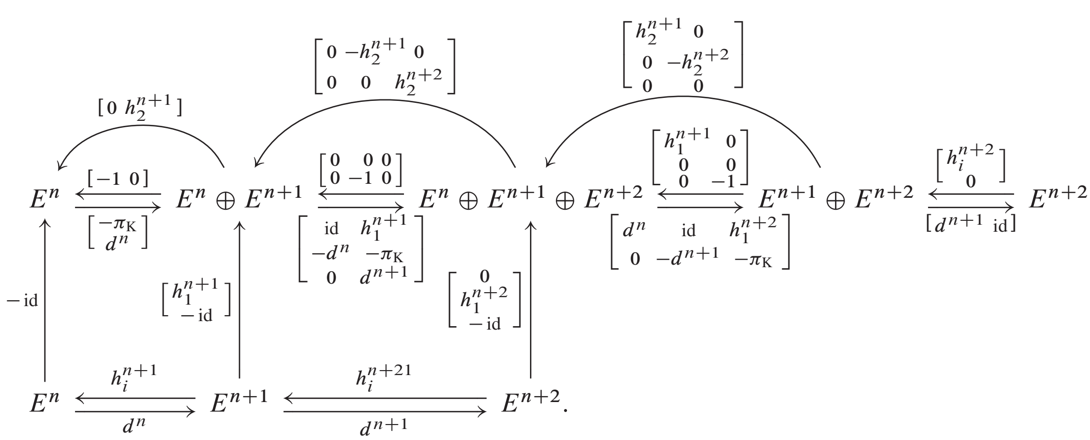
with E concentrated in three degrees at most, say $[n, n+2]$ (see [26, Theorem 2.7]),

$$E^n \underset{d^n}{\overset{h_i^{n+1}}{\rightleftarrows}} E^{n+1} \underset{d^{n+1}}{\overset{h_i^{n+21}}{\rightleftarrows}} E^{n+2}.$$

In this case, using the strict models above, one computes that $a_2^* a_{2*} E \to E$ is



Then we have the following morphism from $E[2]$ to the cone of the morphism displayed above:



[2)] By abuse of notation, we still denote by $E \in \mathsf{D}_{\mathsf{sg}}(G_t)$ the image of $E \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ along

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(G_t).$$

It is easy to check that the latter induces isomorphisms on cohomology groups. Now recall that the object $a_2^* a_{2*} E$ belongs to $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$ and it is thus zero in the dg-quotient

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)/\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t).$$

In other words, the morphism $u_E\colon E \to E[2]$ becomes an equivalence in

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)/\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t).$$

This shows that there is a canonical equivalence id $\simeq [2]$ on the dg-quotient of

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$$

computed in $\mathrm{dgCat}^{\mathrm{s}}_S[\mathrm{W}_{\mathrm{qe}}^{-1}]$, the ∞-localization of $\mathrm{dgCat}^{\mathrm{s}}_S$ with respect to quasi-equivalences. The objects of this quotient dg-category are in correspondence with those of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$. Since

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)/\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$$

is Karoubi generated by the image of the canonical functor

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)/\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$$

and the morphism u_E is functorial in E, we obtain a natural equivalence

$$u\colon \mathrm{id} \xrightarrow{\simeq} [2]\colon \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)} \to \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)}$$

as claimed. □

3.2.3. We will also need the following characterization of the objects in the quotient of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ by $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$.

Lemma 3.2.4. *The dg-category* $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)/\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$ *is Karoubi generated by the images along*

$$\mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) \to \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)}$$

of objects $(E, d, \{h_i\}_{i=1,2})$*, where* (E, d) *is a cochain complex concentrated in at most two degrees.*

Proof. Thanks to the equivalence

$$\frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)} \simeq \frac{\mathsf{D}_{\mathsf{sg}}(G_t)}{\mathsf{D}_{\mathsf{sg}}(G_t \xrightarrow{a_1} t)},$$

it follows from [26, Theorem 2.7] that this dg-quotient is Karoubi generated by the images of objects $(E, d, \{h_i\}_{i=1,2})$, where (E, d) is a cochain complex concentrated in at most three degrees,

$$E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1}.$$

Now, E^{n-1}, E^n and E^{n+1} are finitely generated projective $\mathscr{O}_{\mathrm{L}}$-modules. The characterization of finitely generated modules over a principal ideal domain guarantees that each is free of finite rank. Now, $\mathrm{Ker}(d^n) \subset E^n$ and $\mathrm{Im}(d^n) \subset E^{n+1}$ are submodules of free $\mathscr{O}_{\mathrm{L}}$-modules of finite rank. Thus, they are both finitely generated and torsion free; hence they are also free $\mathscr{O}_{\mathrm{L}}$-modules of finite rank. It follows that the cochain complexes (with obvious $\mathrm{K}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$-module structures)

$$E^{n-1} \xrightarrow{d^{n-1}} \mathrm{Ker}(d^n), \quad \mathrm{Im}(d^n) \hookrightarrow E^{n+1}$$

belong to $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$.

Moreover, $(E, d, \{h_i\}_{i=1,2})$ is clearly an extension of $\mathrm{Im}(d^n) \hookrightarrow E^{n+1}$ by

$$E^{n-1} \xrightarrow{d^{n-1}} \mathrm{Ker}(d^n).$$

We conclude that $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)/\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$ is Karoubi generated by the images of those objects $(E, d, \{h_i\}_{i=1,2}) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ such that (E, d) is a cochain complex concentrated in two degrees at most. □

Remark 3.2.5. Suppose that an object

$$(E, d, \{h_i\}_{i=1,2}) = (E^n \xrightarrow{d} E^{n+1}) \in \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big)$$

is concentrated in two degrees. Then $h_1 = h_2$. In fact, the equations imposed on d and h_i, combined with the fact that $\pi_{\mathrm{K}} \in \mathscr{O}_{\mathrm{L}}$ is not a zero-divisor, imply that d is injective and that

$$d \circ h_1 = d \circ h_2.$$

Lemma 3.2.6. *In Lemma* 3.2.4*, it suffices to consider those complexes generated in degrees* [0, 1].

Proof. By the two-periodicity of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)/\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$ (see Lemma 3.2.2) and by the proof of Lemma 3.2.4, we see that it suffices to show that, for every object A represented by an object $\mathbb{E} = (E, d, \{h_i\}_{i=1,2})$ of $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ concentrated in two degrees, $A[1]$ can be represented by some object of $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ *concentrated in the same two degrees*. We consider the dg-functor

$$\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}}) \to \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big),$$
$$(E, d, h) \mapsto (E, d, \{h, h\}).$$

This is a strict model for the pushforward along the diagonal map $\delta\colon t \to G_t \simeq t \times_T t$. As $a_1 \circ \delta = \mathrm{id}_t$, this induces a dg-functor

$$\mathsf{D}_{\mathsf{sg}}(t) \to \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)}.$$

By Remark 3.2.5, every object A as above is in the image of this functor. Therefore, it suffices to compute $A[1]$ before applying the functor, i.e. in $\mathsf{D}_{\mathsf{sg}}(t)$. Then [26, Corollary 3.7] is exactly what we want. □

Proposition 3.2.7. *The dg-functor* $a_{1*}\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$ *induces an equivalence*

$$\frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)} \xrightarrow{\simeq} \mathsf{D}_{\mathsf{sg}}(t)$$

in dgCat_S

Proof. The dg-functor

$$\mathfrak{Q} := \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)} \to \mathsf{D}_{\mathsf{sg}}(t)$$

is induced by the dg-functor of strict models

$$a_{1*}\colon \mathrm{Coh}^{\mathrm{s}}\big(\mathcal{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) \to \mathrm{Coh}^{\mathrm{s}}(\mathcal{O}_{\mathrm{L}}, \pi_{\mathrm{K}}),$$
$$(E, d, \{h_i\}_{i=1,2}) \mapsto (E, d, h_1).$$

Moreover, as the dg-categories involved are triangulated (see [35]), it suffices to show that it induces an equivalence at the level of homotopy categories, $\mathscr{F}\colon \mathrm{h}(\mathfrak{Q}) \to \mathrm{h}(\mathsf{D}_{\mathsf{sg}}(t))$. Let $\widehat{\mathfrak{Q}}$ and $\widehat{\mathsf{D}_{\mathsf{sg}}(t)}$ denote the dg-quotients of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1}) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ and $\mathsf{D}_{\mathsf{perf}}(t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$ computed in $\mathrm{dgCat}^{\mathrm{s}}_S[\mathrm{W}_{\mathrm{qe}}^{-1}]$. In particular, $\mathfrak{Q}$ and $\mathsf{D}_{\mathsf{sg}}(t)$ are the triangulated hulls of $\widehat{\mathfrak{Q}}$ and $\widehat{\mathsf{D}_{\mathsf{sg}}(t)}$ respectively. Recall that the homotopy category of $\widehat{\mathsf{D}_{\mathsf{sg}}(t)}$ has the following explicit description:

$$\mathrm{h}\big(\widehat{\mathsf{D}_{\mathsf{sg}}(t)}\big) \simeq \mathrm{h}\big(\mathrm{MF}(\mathcal{O}_{\mathrm{L}}, \pi_{\mathrm{K}})\big),$$

where $\mathrm{h}(\mathrm{MF}(\mathcal{O}_{\mathrm{L}}, \pi_{\mathrm{K}}))$ denotes the triangulated category of *matrix factorizations* (see [24]). This is the category whose objects are tuplets

$$\mathbb{E} = (E^0, E^1, d\colon E^0 \to E^1, h\colon E^1 \to E^0),$$

where E^0, E^1 are projective $\mathcal{O}_{\mathrm{L}}$-modules of finite rank and d, h are $\mathcal{O}_{\mathrm{L}}$-linear maps such that $d \circ h = \pi_{\mathrm{K}} \cdot \mathrm{id}_{E_1}$ and $h \circ d = \pi_{\mathrm{K}} \cdot \mathrm{id}_{E_0}$. For two such objects $\mathbb{E}, \mathbb{E}'$, the $\mathcal{O}_{\mathrm{K}}$-module of morphisms $\mathbb{E} \to \mathbb{E}'$ is the set of pairs of $\mathcal{O}_{\mathrm{L}}$-linear maps $\phi = (\phi^0\colon E^0 \to E'^0, \phi^1\colon E^1 \to E'^1)$ commuting with d, d', h, h' in the obvious sense, endowed with the obvious $\mathcal{O}_{\mathrm{K}}$-module structure. The shift functor is

$$[1]\colon \mathrm{h}\big(\mathrm{MF}(\mathcal{O}_{\mathrm{L}}, \pi_{\mathrm{K}})\big) \to \mathrm{h}\big(\mathrm{MF}(\mathcal{O}_{\mathrm{L}}, \pi_{\mathrm{K}})\big),$$
$$(E^0, E^1, d\colon E^0 \to E^1, h\colon E^1 \to E^0) \mapsto (E^1, E^0, -h\colon E^1 \to E^0, -d\colon E^0 \to E^1)$$

and the cone of a morphism $\phi\colon \mathbb{E} \to \mathbb{E}'$ as above is

$$\mathrm{coFib}(\phi) = \left(E^1 \oplus E'^0, E^0 \oplus E'^1, \begin{bmatrix} \phi^1 & d' \\ -h & 0 \end{bmatrix}, \begin{bmatrix} 0 & -d \\ h' & \phi^0 \end{bmatrix}\right).$$

The distinguished triangles are those isomorphic to triangles of the form

$$\mathbb{E} \xrightarrow{\phi} \mathbb{E}' \to \mathrm{coFib}(\phi).$$

The equivalence $\mathrm{h}(\widehat{\mathsf{D}_{\mathsf{sg}}(t)}) \simeq \mathrm{h}(\mathrm{MF}(\mathcal{O}_{\mathrm{L}}, \pi_{\mathrm{K}}))$ is induced by the functor

$$(E, d, h) \mapsto \Big(\bigoplus_{i \in \mathbb{Z}} E^{2i}, \bigoplus_{i \in \mathbb{Z}} E^{2i+1}, d + h, d + h\Big).$$

See [26, Corollary 3.11]. Therefore, we get a triangulated functor

$$\mathscr{G}\colon \mathrm{h}\big(\mathrm{MF}(\mathcal{O}_{\mathrm{L}}, \pi_{\mathrm{K}})\big) \to \mathrm{h}(\widehat{\mathfrak{Q}}),$$
$$\mathbb{E} = (E^0, E^1, d, h) \mapsto \mathscr{G}(\mathbb{E}) = (E^0 \xrightarrow{d} E^1, \{h, h\}),$$
$$(\phi^0, \phi^1) \mapsto (\phi^0, \phi^1),$$

where the object $\mathscr{G}(\mathbb{E})$ is concentrated in degrees $[0,1]$. The composition $\widehat{\mathscr{F}} \circ \mathscr{G}$ is then the identity functor. Here $\widehat{\mathscr{F}}$ denotes the dg-functor

$$\begin{aligned} \mathrm{h}(\widehat{\mathfrak{Q}}) &\to \mathrm{h}\big(\mathrm{MF}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})\big), \\ (E, d, \{h_i\}_{i=1,2}) &\mapsto \Big(\bigoplus_{i\in\mathbb{Z}} E^{2i}, \bigoplus_{i\in\mathbb{Z}} E^{2i+1}, d+h_1, d+h_1\Big). \end{aligned}$$

The composition $\mathscr{G} \circ \widehat{\mathscr{F}}$ is also equivalent to the identity, as it is so on those objects concentrated in degrees $[0,1]$, to which every object in $\mathrm{h}(\widehat{\mathfrak{Q}})$ is isomorphic. Thus, we have proved that $\mathrm{h}(\widehat{\mathfrak{Q}}) \simeq \mathrm{h}(\mathrm{MF}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}}))$ and the claim of the proposition follows immediately. □

Corollary 3.2.8. *The following are localization sequences in* dgCat_S*:*

$$\begin{aligned} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) &\hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t), \\ \mathsf{D}_{\mathsf{sg}}(G_t \xrightarrow{a_1} t) &\hookrightarrow \mathsf{D}_{\mathsf{sg}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t). \end{aligned}$$

Proof. By Remark 3.2.1, it suffices to show that the first one is a localization sequence. This follows from Proposition 3.2.7 and from the observation that $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$ is by definition the kernel of the dg-functor $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)$ induced by a_{1*}. □

3.3. The structure of left B^+-modules. In this section, we show that the above localization sequences are compatible with the natural B^+-module structures.

3.3.1. Recall from [37] that $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$ are both equipped with natural left actions of B^+. Since these actions preserve the full subcategories of perfect complexes, the quotient dg-categories $\mathsf{D}_{\mathsf{sg}}(G_t)$ and $\mathsf{D}_{\mathsf{sg}}(t)$ are left B^+-modules, too.

3.3.2. Let $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{K}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$ be the strict model for $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G)$, defined (*mutatis mutandis*) just as $\mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}}))$. There is a pseudo-action

$$\begin{aligned} -\odot-\colon \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{K}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) \otimes \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) &\to \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big), \\ (M, E) &\mapsto M \odot E := M \otimes_{\mathrm{K}(\mathscr{O}_{\mathrm{K}}, \pi_{\mathrm{K}})} E, \end{aligned}$$

where M and E are seen as a $\mathrm{K}(\mathscr{O}_{\mathrm{K}}, \pi_{\mathrm{K}})$-module by forgetting the actions of h_2 and h_1 respectively and by restricting scalars on E. Similarly, there is a pseudo-action

$$\begin{aligned} -\odot-\colon \mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{K}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) \otimes \mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}}) &\to \mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}}), \\ (M, E) &\mapsto M \odot E := M \otimes_{\mathrm{K}(\mathscr{O}_{\mathrm{K}}, \pi_{\mathrm{K}})} E, \end{aligned}$$

where M is seen as a $\mathrm{K}(\mathscr{O}_{\mathrm{K}}, \pi_{\mathrm{K}})$-module by forgetting the action of h_2 and by restring scalars on E.

These are strict models for the left B^+-module structures on $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$; see [37, Section 4.1].

Lemma 3.3.3. *The dg-functor*

$$a_{1*}\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$$

is a morphism of left B^+*-modules.*

Proof. This follows immediately from the strict models. In fact, as already mentioned above, $a_{1*}\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$ can be "strictified" by the dg-functor

$$\mathrm{Coh}^{\mathrm{s}}\big(\mathscr{O}_{\mathrm{L}}, (\pi_{\mathrm{K}}, \pi_{\mathrm{K}})\big) \to \mathrm{Coh}^{\mathrm{s}}(\mathscr{O}_{\mathrm{L}}, \pi_{\mathrm{K}})$$

which forgets the action of h_2. This is obviously compatible with the pseudo-actions described above. □

Corollary 3.3.4. *The sequence*

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)$$

is a localization sequence of left B^+*-modules.*

Proof. As $a_{1*}\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$ and $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t) \to \mathsf{D}_{\mathsf{sg}}(t)$ are morphisms of left B^+-modules, their composition is B^+-linear too. It remains to show that the inclusion

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$$

is B^+-linear. Observe that as $a_{1*}\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$ is B^+-linear and the action of B^+ on $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$ preserves $\mathsf{D}_{\mathsf{perf}}(t) \subset \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t)$, the dg-category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t)$ inherits a left B^+-module structure from $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$. It is then clear that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \subset \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$$

is B^+-linear. □

4. A dévissage-like result

In this section, we compute the motivic realization of the dg-category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)$.

4.1. Another useful localization sequence.

4.1.1. Recall from [37, Section 2.1] and [19, Section 4.4] that there is a functor

$$\mathrm{dgCat}^{\mathsf{B}^+} \times \mathrm{dgCat}_{\mathsf{B}^+} \to \mathrm{dgCat}_S$$

which sends a right B^+-module $\mathscr{R}$ and a left B^+-module $\mathscr{L}$ to

$$\mathscr{R} \otimes_{\mathsf{B}^+} \mathscr{L} := (\mathscr{R} \otimes_S \mathscr{L}) \otimes_{\mathsf{B}^{+,\mathrm{e}}} \mathsf{B}^{+,\mathrm{L}},$$

where $\mathsf{B}^{+,\mathrm{e}}$ denotes the "enveloping" algebra $\mathsf{B}^{+,\mathrm{rev}} \otimes_S \mathsf{B}^+$, and $\mathsf{B}^{+,\mathrm{L}}$ the dg-category B^+ endowed with its natural left $\mathsf{B}^{+,\mathrm{e}}$-module structure.

4.1.2. Also recall (see [37, Remark 2.1.4, Proposition 4.1.7]) that $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)$ is cotensored over B^+, so that $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}}$ is a right B^+-module. We can therefore consider the functor

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} -\colon \mathrm{dgCat}_{\mathsf{B}^+} \to \mathrm{dgCat}_S\,.$$

Lemma 4.1.3. *The functor* $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} -\colon \mathrm{dgCat}_{\mathsf{B}^+} \to \mathrm{dgCat}_S$ *sends localization sequences of left* B^+*-modules to localization sequences in* dgCat_S.

Proof. By [19, Corollary 4.4.2.15] (applied to $\mathcal{C} = \mathrm{dgCat}_S$, $A = C = \mathsf{D}_{\mathsf{perf}}(S)$ and $B = \mathsf{B}^+$), we know that this functor preserves colimits. In particular, if $\mathcal{L}_1 \hookrightarrow \mathcal{L}_2 \to \mathcal{L}_3$ is a localization sequence in $\mathrm{dgCat}_{\mathsf{B}^+}$, then

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_3 \simeq \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_2}{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_1}.$$

It remains to prove that the induced arrow $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_1 \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_2$ is fully faithful. Objects in the tensor product $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_1$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_2$) are pairs (E, L), where $E \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)$ and $L \in \mathcal{L}_1$ (resp. $L \in \mathcal{L}_2$). For two such objects (E, L), (E', L'), the hom complex of morphisms from (E, L) to (E', L') in $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_1$ (resp. $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathcal{L}_2$) is computed as

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}}}(E, E') \otimes_{k[u]} \mathrm{Hom}_{\mathcal{L}_1}(L, L')$$
$$(\text{resp. } \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}}}(E, E') \otimes_{k[u]} \mathrm{Hom}_{\mathcal{L}_2}(L, L')).$$

Here $k[u]$ is the algebra of endomorphisms of the unit object of B^+.

For two objects $L, L' \in \mathcal{L}_1$, the morphism $\mathrm{Hom}_{\mathcal{L}_1}(L, L') \to \mathrm{Hom}_{\mathcal{L}_2}(L, L')$ is a quasi-isomorphism. Therefore, its image along $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}}}(E, E') \otimes_{k[u]} -$ is a quasi-isomorphism as well and the claim follows. □

Corollary 4.1.4. *The sequence*

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}_{\mathsf{sg}}(t)$$

is a localization sequence in dgCat_S.

Proof. This follows immediately from Lemma 4.1.3 and Corollary 3.3.4. □

4.1.5. Our next goal is to identify the localization sequence above with a more explicit one. Recall from [37, Section 4.2] that, for two regular and flat S-schemes Y and Z, there is an equivalence

$$\overline{\overline{\mathfrak{F}}}_{Y,Z} := j_*\big(\mathbb{D}_{Y_s}(-) \boxtimes_s (-)\big)\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Z_s) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y \times_S Z)_{Y_s \times_s Z_s},$$

where $- \boxtimes_s -$ denotes the external tensor product over s,

$$\mathbb{D}_{Y_s}(-) := \underline{\mathrm{Hom}}_{Y_s}(-, \mathcal{O}_{Y_s})$$

the Grothendieck duality functor and $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y \times_S Z)_{Y_s \times_s Z_s}$ the subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y \times_S Z)$ spanned by those complexes supported on the closed subscheme $j\colon Y_s \times_s Z_s \hookrightarrow Y \times_S Z$.

Remark 4.1.6. Actually, we observe that the proof of [37, Lemma 4.2.3] only requires that Y and Z are Gorenstein S-schemes of finite type. The flatness assumption is never really used (it is actually there only to guarantee that the derived special fibers agree with the usual ones), while regularity is only needed to guarantee that this functor is compatible with perfect complexes and thus induces a functor at the level of dg-categories of singularities.

Proposition 4.1.7. *The localization sequence*

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)\big)$$

of Corollary 4.1.4 *identifies with*

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \xrightarrow{i_{X_T *}} \mathsf{D}_{\mathsf{sg}}(X_T).$$

Proof. We will proceed in steps.

Step 1. By Remark 4.1.6, we have an equivalence

$$\overline{\mathfrak{F}}_{X,t} \colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S t)_{X_s \times_s G_t}.$$

However, $X_s \times_s G_t \hookrightarrow X \times_S t \simeq X_t$ is a closed embedding with empty open complement (as $G_t \xrightarrow{a_1} t$ is so). In particular, we have $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S t)_{X_s \times_s G_t} \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t)$, and thus we obtain an equivalence

$$\overline{\mathfrak{F}}_{X,t} \colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t).$$

Step 2. Starting from the localization sequence $\mathsf{D}_{\mathsf{perf}}(t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t) \to \mathsf{D}_{\mathsf{sg}}(t)$ of left B^+-modules, Lemma 4.1.3 implies that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}_{\mathsf{perf}}(t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}_{\mathsf{sg}}(t)$$

is a localization sequence in dgCat_S. As explained in the proofs of [37, Lemma 4.2.3, Theorem 4.2.1], the functor $\overline{\mathfrak{F}}_{X,T}$ yields a commutative diagram

$$\begin{array}{ccc} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}_{\mathsf{perf}}(t) & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(t) \\ \big\downarrow \overline{\mathfrak{F}}_{X,t} & & \big\downarrow \mathfrak{F}_{X,T} \\ \mathsf{D}_{\mathsf{perf}}(X_T)_{X_t} & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t}, \end{array}$$

where the vertical arrows are equivalences. Therefore,

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}_{\mathsf{sg}}(t) \simeq \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t}}{\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t}} \simeq \mathsf{D}_{\mathsf{sg}}(X_T).$$

Here we used that X_T has smooth generic fiber, so that

$$\frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t}}{\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t}} \simeq \frac{\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)}{\mathsf{D}_{\mathsf{perf}}(X_T)} = \mathsf{D}_{\mathsf{sg}}(X_T).$$

Thus, $\overline{\mathfrak{F}}_{X,T}$ induces an equivalence

$$\mathfrak{F}_{X,T} \colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}_{\mathsf{sg}}(t) \xrightarrow{\simeq} \mathsf{D}_{\mathsf{sg}}(X_T).$$

Step 3. Consider now the square

$$\begin{array}{ccc} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) & \xrightarrow{\mathrm{id} \otimes_{\mathsf{B}^+} a_{1*}} & \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}_{\mathsf{sg}}(t) \\ \big\downarrow \overline{\mathfrak{F}}_{X,t} & & \big\downarrow \mathfrak{F}_{X,T} \\ \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) & \xrightarrow{i_{X_T *}} & \mathsf{D}_{\mathsf{sg}}(X_T), \end{array}$$

and recall from the two previous steps that both vertical arrows are equivalences. In this step, we prove that this square is commutative. So we need to show that dg-functor $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T)$ corresponding to $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} (\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t))$ identifies with the pushforward along $i_{X_T} \colon X_t \hookrightarrow X_T$ composed with the projection

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T) \twoheadrightarrow \mathsf{D}_{\mathsf{sg}}(X_T).$$

For this, we will use the diagram

$$\begin{array}{ccccc}
X \times_S s & & X \times_S s & & \\
\uparrow{\scriptstyle \mathrm{pr}_{12}} & & \uparrow{\scriptstyle \widetilde{\mathrm{pr}}_{12}} & & \\
X \times_S s \times_S s \times_S T & \xrightarrow{\mathrm{pr}_{134}} & X \times_S s \times_S T & \xrightarrow{i_{X_T} = \widetilde{\mathrm{pr}}_{13}} & X \times_S T \\
\downarrow{\scriptstyle \mathrm{pr}_{234}} & & \downarrow{\scriptstyle \widetilde{\mathrm{pr}}_{23}} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\
s \times_S s \times_S T & \xrightarrow{a_1 = \widehat{\mathrm{pr}}_{13}} & s \times_S T & \xrightarrow{i_T} & T.
\end{array}$$

Recall that $\overline{\mathfrak{F}}_{X,T}$ is defined as

$$\begin{aligned}(\widetilde{p}_{13})_*\big(\widetilde{\mathrm{pr}}_{12}^*\mathbb{D}_{X_s}(-) \otimes \widetilde{\mathrm{pr}}_{23}^*(-)\big) \colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(s \times_S T) \\ \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S T)_{X \times_S s \times_S T}\end{aligned}$$

and that $\overline{\mathfrak{F}}_{X, s \times_S T}$ is defined as

$$\begin{aligned}(\mathrm{pr}_{134})_*\big(\mathrm{pr}_{12}^*\, \mathbb{D}_{X_s}(-) \otimes \mathrm{pr}_{234}^*(-)\big) \colon \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(s \times_S s \times_S T) \\ \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S s \times_S T).\end{aligned}$$

Let $E \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)$ and $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t)$. To identify $\overline{\mathfrak{F}}_{X,T}(E, a_{1*}M)$ with $i_{X_T *}\overline{\mathfrak{F}}_{X,t}(E, M)$, we compute

$$\begin{aligned}
i_{X_T *}\overline{\mathfrak{F}}_{X, s\times_S T}(E, M) &= (\widetilde{\mathrm{pr}}_{13})_*(\mathrm{pr}_{134})_*\big(\mathrm{pr}_{12}^*\, \mathbb{D}_{X_s}(E) \otimes \mathrm{pr}_{234}^*\, M\big) \\
&\simeq (\widetilde{\mathrm{pr}}_{13})_*(\mathrm{pr}_{124})_*\big(\mathrm{pr}_{124}^*\, \widetilde{\mathrm{pr}}_{12}^*\mathbb{D}_{X_s}(E) \otimes \mathrm{pr}_{234}^*\, M\big) \\
&\simeq (\widetilde{\mathrm{pr}}_{13})_*\big(\widetilde{\mathrm{pr}}_{12}^*\mathbb{D}_{X_s}(E) \otimes (\mathrm{pr}_{124})_*\, \mathrm{pr}_{234}^*\, M\big) \\
&\simeq (\widetilde{\mathrm{pr}}_{13})_*\big(\widetilde{\mathrm{pr}}_{12}^*\mathbb{D}_{X_s}(E) \otimes \widetilde{\mathrm{pr}}_{23}^*(\widehat{\mathrm{pr}}_{13})_* M\big) \\
&= \mathfrak{F}_{X,T}(E, a_{1*}M) \quad \text{(when considered as an object in } \mathsf{D}_{\mathsf{sg}}(X_T)\text{)}.
\end{aligned}$$

The first equivalence follows from the obvious identities

$$\begin{aligned}
\widetilde{\mathrm{pr}}_{13} \circ \mathrm{pr}_{124} &\simeq \widetilde{\mathrm{pr}}_{13} \circ \mathrm{pr}_{134}, \\
\mathrm{pr}_{12} &\simeq \widetilde{\mathrm{pr}}_{12} \circ \mathrm{pr}_{124},
\end{aligned}$$

the second equivalence is the projection formula and the last equivalence follows from the base-change

$$\widetilde{\mathrm{pr}}_{23}^*(\widehat{\mathrm{pr}}_{13})_* \simeq (\mathrm{pr}_{124})_*\, \mathrm{pr}_{234}^*\,.$$

This shows that the diagram of dg-functors

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(s \times_S s \times_S T) & \xrightarrow{\mathrm{id} \otimes a_{1*}} & \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(s \times_S T) \\
\downarrow{\scriptstyle \overline{\mathfrak{F}}_{X,t}} & & \downarrow{\scriptstyle \overline{\mathfrak{F}}_{X,T}} \\
\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S s \times_S T) & \xrightarrow{i_{X_T *}} & \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X \times_S T)_{X \times_S s \times_S T}
\end{array}$$

is commutative. As an immediate consequence, we obtain

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)\big) \simeq \big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \xrightarrow{i_{X_T *}} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T) \twoheadrightarrow \mathsf{D}_{\mathsf{sg}}(X_T)\big).$$

Step 4. We can now conclude the proof. The commutativity of the above square implies that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+} \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1} t) \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)$$

as the left-hand side has to be equivalent to

$$\mathrm{Ker}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \xrightarrow{i_{X_T *}} \mathsf{D}_{\mathsf{sg}}(X_T)\big) = \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T. \qquad \square$$

4.2. Motivic realization of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i} X_T)$.

4.2.1. The motivic spectrum underlying $\mathcal{M}^{\vee}_S(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X))$ is easy to describe: it is the functor $\mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Sp}$ that sends a smooth S-scheme Y to

$$\mathrm{HK}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X) \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(Y)\big).$$

4.2.2. We think of the following statement as a kind of *dévissage for homotopy-invariant non-connective algebraic* K*-theory*. We therefore state it as a theorem, as it seems to be a new result interesting on its own.

Theorem 4.2.3. *With the same notation as in the previous sections, there are equivalences*

$$\mathcal{M}^{\vee}_S\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \simeq \mathcal{M}^{\vee}_S\big(\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t}\big) \simeq q_{T*} i_{X_T *} i^!_{X_T} \mathrm{B}\mathbb{U}_{X_T}$$

in $\mathcal{SH}_S$, where $q_T\colon X_T \to S$ is the composition $X_T \xrightarrow{p_T} T \longrightarrow S$.

Proof. Proposition 4.1.7 immediately yields a commutative diagram

$$\begin{array}{ccccc}
\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T) & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) & \longrightarrow & \mathsf{D}_{\mathsf{sg}}(X_T) \\
\big\downarrow{\scriptstyle i_{X_T *}} & & \big\downarrow{\scriptstyle i_{X_T *}} & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t} & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t} & \longrightarrow & \mathsf{D}_{\mathsf{sg}}(X_T),
\end{array}$$

where the rows are localization sequences in dgCat_S. As $\mathcal{M}^{\vee}_S$ sends localization sequences to fiber-cofiber sequences, we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathcal{M}^{\vee}_S\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)\big) & \longrightarrow & \mathcal{M}^{\vee}_S\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t)\big) & \longrightarrow & \mathcal{M}^{\vee}_S\big(\mathsf{D}_{\mathsf{sg}}(X_T)\big) \\
\big\downarrow{\scriptstyle \mathcal{M}^{\vee}_S(i_{X_T *})} & & \big\downarrow{\scriptstyle \mathcal{M}^{\vee}_S(i_{X_T *})} & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\mathcal{M}^{\vee}_S\big(\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t}\big) & \longrightarrow & \mathcal{M}^{\vee}_S\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t}\big) & \longrightarrow & \mathcal{M}^{\vee}_S\big(\mathsf{D}_{\mathsf{sg}}(X_T)\big),
\end{array} \tag{4.1}$$

where the rows are fiber-cofiber sequences. We know that

$$\mathcal{M}^{\vee}_S(i_{X_T *})\colon \mathcal{M}^{\vee}_S\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t)\big) \to \mathcal{M}^{\vee}_S\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t}\big)$$

is an equivalence. For Y a smooth S-scheme, the map of spectra

$$\mathcal{M}_S^\vee(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t))(Y) \xrightarrow{\mathcal{M}_S^\vee(i_{X_T*})(Y)} \mathcal{M}_S^\vee(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t})(Y)$$

identifies with a map

$$\mathrm{HK}(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(Y)) \to \mathrm{HK}(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t} \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(Y)).$$

Since Y is a smooth S-scheme, $\mathsf{D}_{\mathsf{perf}}(Y) \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y)$ and it follows from [27, Proposition B.4.1] that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(Y) \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \times_S Y) \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \times_s Y_s).$$

Similarly, using the fact that $- \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(Y)$ preserves localization sequences (see [30, Proposition 3.19 2)]), [27, Proposition B.4.1] implies that $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T)_{X_t} \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(Y)$ identifies with the kernel of the localization dg-functor

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T \times_S Y) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_{\mathrm{L}} \times_S Y) \simeq \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_{\mathrm{L}} \times_\eta Y_\eta),$$

that is, with $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T \times_S Y)_{X_t \times_s Y_s}$. It follows that the map $\mathcal{M}_S^\vee(i_{X_T*})(Y)$ identifies with

$$\underbrace{\mathrm{HK}(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \times_s Y_s))}_{=\mathrm{G}(X_t \times_s Y_s)} \xrightarrow{\mathrm{HK}((X_t \times_s Y_s \to X_T \times_S Y)_*)} \underbrace{\mathrm{HK}(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T \times_S Y)_{X_t \times_s Y_s})}_{=\mathrm{G}(X_T \times_S Y)_{X_t \times_s Y_S}},$$

which is an equivalence by the theorem of the heart (see [2, 21–23]) and by dévissage in G-theory (see [28, §5, Theorem 4]).

Since the middle and rightmost vertical arrows in diagram (4.1) are equivalences,

$$\mathcal{M}_S^\vee(i_{X_T*}) \colon \mathcal{M}_S^\vee(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)) \to \mathcal{M}_S^\vee(\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t})$$

is an equivalence as well. To show that these motivic spectra identify with $q_{T*} i_{X_T*} i^!_{X_T} \mathrm{B}\mathbb{U}_{X_T}$, we consider the localization sequence $\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t} \hookrightarrow \mathsf{D}_{\mathsf{perf}}(X_T) \to \mathsf{D}_{\mathsf{perf}}(X_{\mathrm{L}})$. Combining this with the equivalence

$$\mathcal{M}_S^\vee(\mathsf{D}_{\mathsf{perf}}(X_T) \to \mathsf{D}_{\mathsf{perf}}(X_{\mathrm{L}})) \simeq q_{T*}(\mathrm{B}\mathbb{U}_{X_T} \to j_{X_T*}\mathrm{B}\mathbb{U}_{X_{\mathrm{L}}})$$

(see [5]), we deduce that

$$\mathcal{M}_S^\vee(\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t} \hookrightarrow \mathsf{D}_{\mathsf{perf}}(X_T) \to \mathsf{D}_{\mathsf{perf}}(X_{\mathrm{L}}))$$

is a fiber-cofiber sequence that identifies with the localization sequence

$$q_{T*} i_{X_T*} i^!_{X_T} \mathrm{B}\mathbb{U}_{X_T} \to q_{T*}\mathrm{B}\mathbb{U}_{X_T} \to q_{T*} j_{X_T*}\mathrm{B}\mathbb{U}_{X_{\mathrm{L}}}$$

associated to the open-closed decomposition $i_{X_T} \colon X_t \to X_T \leftarrow X_{\mathrm{L}} : j_{X_T}$. □

Remark 4.2.4. The above theorem agrees with the prediction, stated in [5, 25], that

$$\mathcal{M}_S^\vee(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y_s \xrightarrow{i_Y} Y)) \simeq \mathcal{M}_S^\vee(\mathsf{D}_{\mathsf{perf}}(Y)_{Y_s})$$

for every qcqs flat S-scheme Y of finite type.

5. The main theorems

As already mentioned in Section 1 (and as already pointed out in [5, Remark 4.46]), once Theorem 4.2.3 is established, the proof of [5, Theorem 4.39] works essentially unchanged. In this section, we spell out the minor changes needed for the proof of Theorem A.

5.1. ℓ-adic vanishing cycles and $\mathrm{I_L}$-homotopy fixed points. We recall here the definition of the vanishing cohomology introduced in [11, 15].

5.1.1. Let $\bar{j}_X : X_{\overline{\mathrm{K}}} \to X$ be the pullback of $\bar{\eta} \to S$ along $p: X \to S$. We denote by $\mathrm{Shv}_{\mathbb{Q}_\ell}(X_s)^{\mathrm{I_K}}$ the ∞-category of ℓ-adic sheaves on X_s endowed with a continuous action of $\mathrm{I_K}$. The functor of *nearby cycles* is defined by

$$\Psi_p : \mathrm{Shv}_{\mathbb{Q}_\ell}(X_{\mathrm{K}}) \to \mathrm{Shv}_{\mathbb{Q}_\ell}(X_s)^{\mathrm{I_K}},$$
$$E \mapsto i_X^* \bar{j}_{X*}(E_{|X_{\overline{\mathrm{K}}}}),$$

with the $\mathrm{I_K}$-action induced by transport of structure from the natural $\mathrm{I_K}$-action on $E_{|X_{\overline{\mathrm{K}}}}$.

Remark 5.1.2. We do not spell out the details of this construction here. These are provided for example in [8] for finite coefficients. Then one can take a limit and invert ℓ to get $\mathbb{Q}_\ell$-coefficients.

5.1.3. For an ℓ-adic sheaf E on X, there is a functorial morphism

$$\mathrm{sp}_E : i_X^*(E) \to \Psi_p(E_{|X_{\mathrm{K}}})$$

called the *specialization morphism*, induced by the counit of the adjunction $(\bar{j}_X^*, \bar{j}_{X*})$. This morphism is $\mathrm{I_K}$-equivariant if we endow $i_X^*(E)$ with the trivial $\mathrm{I_K}$-action.

5.1.4. The *vanishing cycles* functor

$$\Phi_p : \mathrm{Shv}_{\mathbb{Q}_\ell}(X) \to \mathrm{Shv}_{\mathbb{Q}_\ell}(X_s)^{\mathrm{I_K}}$$

is defined as

$$\Phi_p(E) := \mathrm{coFib}(\mathrm{sp}_E),$$

where the cofiber is computed in $\mathrm{Shv}_{\mathbb{Q}_\ell}(X_s)^{\mathrm{I_K}}$.

5.1.5. Let us recall an explicit description of the homotopy $\mathrm{I_L}$-fixed points of $\Phi_p(\mathbb{Q}_{\ell,X})$. Let $v_X : X_T \to X$ be the pullback of $T \to S$ along $p: X \to S$.

Lemma 5.1.6. *There is a canonical equivalence*

$$\Phi_p(\mathbb{Q}_{\ell,X})^{\mathrm{I_L}} \simeq \mathrm{coFib}\big(\mathbb{Q}_{\ell,X_s} \oplus \mathbb{Q}_{\ell,X_s}(-1)[-1] \to i_X^* v_{X*} j_{X_T *} \mathbb{Q}_{\ell,X_{\mathrm{L}}}\big),$$

compatible with the natural actions of $\mathrm{G_{L/K}}$ *on both sides.*

Proof. As taking $\mathrm{I_L}$-fixed points is an exact functor, we have an equivalence

$$\Phi_p(\mathbb{Q}_{\ell,X})^{\mathrm{I_L}} \simeq \mathrm{coFib}\big(\mathbb{Q}_{\ell,X_s}^{\mathrm{I_L}} \xrightarrow{\mathrm{sp}^{\mathrm{I_L}}_{\mathbb{Q}_{\ell,X}}} \Psi_p(\mathbb{Q}_{\ell,X_{\mathrm{K}}})^{\mathrm{I_L}}\big).$$

We will start by proving that there are equivalences

$$\mathbb{Q}_{\ell,X_s} \oplus \mathbb{Q}_{\ell,X_s}(-1)[-1] \simeq \mathbb{Q}_{\ell,X_s}^{\mathrm{I_L}},$$
$$\Psi_p(\mathbb{Q}_{\ell,X_{\mathrm{K}}})^{\mathrm{I_L}} \simeq i_X^* v_{X*} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}},$$

compatible with the $\mathrm{G_{L/K}}$-actions. In the first equivalence, both members are equipped with the trivial $\mathrm{G_{L/K}}$-action; in the second one, the first member carries the canonical $\mathrm{G_{L/K}}$-action and the second one the action induced by transport of structure.

The first equivalence follows from the computations in [5] (applied to $p_T : X_T \to T$). This is tautologically compatible with the $\mathrm{G_{L/K}}$-actions because these actions are trivial on both sides.

The second equivalence is a form of Galois descent as in [5, Proposition 4.31]. Indeed, the morphism $u_{\mathrm{L}} : X_{\bar{\mathrm{K}}} \to X_{\mathrm{L}}$ induces an equivalence

$$u_{\mathrm{L}}^* : \mathrm{Shv}_{\mathbb{Q}_\ell}(X_{\mathrm{L}})^{\mathrm{G_{L/K}}} \leftrightarrows \mathrm{Shv}_{\mathbb{Q}_\ell}(X_{\bar{\mathrm{K}}})^{\mathrm{I_K}} : u_{\mathrm{L}*}(-)^{\mathrm{I_L}}$$

between the ∞-category of $\mathbb{Q}_\ell$-adic sheaves on X_{L} endowed with a $\mathrm{G_{L/K}}$-action and the ∞-category of $\mathbb{Q}_\ell$-adic sheaves on $X_{\bar{\mathrm{K}}}$ endowed with a continuous $\mathrm{I_K}$-action. In particular, $\mathbb{Q}_{\ell,X_{\mathrm{L}}} \simeq u_{\mathrm{L}*}(\mathbb{Q}_{\ell,X_{\bar{\mathrm{K}}}})^{\mathrm{I_L}}$. It follows that

$$i_X^* v_{X*} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}} \simeq i_X^* v_{X*} j_{X_T*} u_{\mathrm{L}*}(\mathbb{Q}_{\ell,X_{\bar{\mathrm{K}}}})^{\mathrm{I_L}} \simeq i_X^*(v_{X*} j_{X_T*} u_{\mathrm{L}*} \mathbb{Q}_{\ell,X_{\bar{\mathrm{K}}}})^{\mathrm{I_L}},$$

where the latter equivalence holds since the functor $(-)^{\mathrm{I_L}}$ commutes with pushforwards. Using the continuity of the $\mathrm{I_L}$-action as in [5, Proposition 4.31], we deduce that

$$(i_X^* v_{X*} j_{X_T*} u_{\mathrm{L}*} \mathbb{Q}_{\ell,X_{\bar{\mathrm{K}}}})^{\mathrm{I_L}} = \Psi_p(\mathbb{Q}_{\ell,X_{\mathrm{L}}})^{\mathrm{I_L}}.$$

It remains to construct an homotopy between $\mathrm{sp}_{\mathbb{Q}_{\ell,X}}^{\mathrm{I_L}}$ and

$$\mathbb{Q}_{\ell,X_s} \oplus \mathbb{Q}_{\ell,X_s}(-1)[-1] \to i_X^* v_{X*} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}.$$

For this, it suffices to observe that both morphisms pre-composed with

$$\mathbb{Q}_{\ell,X_s} \to \mathbb{Q}_{\ell,X_s} \oplus \mathbb{Q}_{\ell,X_s}(-1)[-1]$$

are homotopic to the morphism $\mathbb{Q}_{\ell,X_s} \to i_X^* v_{X*} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}$ induced by the unit of the adjunction

$$\big((v_X \circ j_{X_T})^*, (v_X \circ j_{X_T})_*\big).$$

Now, the fact that both $\mathrm{sp}_{\mathbb{Q}_{\ell,X}}^{\mathrm{I_L}}$ and $\mathbb{Q}_{\ell,X_s} \oplus \mathbb{Q}_{\ell,X_s}(-1)[-1] \to i_X^* v_{X*} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}$ are $\mathbb{Q}_{\ell,X_s}^{\mathrm{I_L}}$-linear concludes the proof. Notice that the latter morphism is $\mathrm{G_{L/K}}$-equivariant, as it factors through $i_X^* j_{X*} \mathbb{Q}_{\ell X_{\mathrm{K}}} \simeq \Psi_p(\mathbb{Q}_{\ell,X_{\mathrm{K}}})^{\mathrm{I_K}} \simeq (\Psi_p(\mathbb{Q}_{\ell,X_{\mathrm{K}}})^{\mathrm{I_L}})^{\mathrm{G_{L/K}}}$. □

Remark 5.1.7. A similar result holds (with the same proof) if we replace $\mathbb{Q}_{\ell,X}$ with $\mathbb{Q}_{\ell,X}(\beta)$.

5.2. The action of $\mathrm{G_{L/K}}$.

5.2.1. Recall that $\mathrm{G_{L/K}}$ denotes the (finite) group $\mathrm{I_K}/\mathrm{I_L}$. Explicitly, $\mathscr{O}_{\mathrm{L}}$ is isomorphic to the quotient of the polynomial ring $\mathscr{O}_{\mathrm{K}}[x]$ by an Eisenstein polynomial $E(x) \in \mathscr{O}_{\mathrm{K}}[x]$ of degree e. The group $\mathrm{G_{L/K}} \simeq \mathscr{G}al(\mathrm{L/K})$ permutes the roots of $E(x)$ and thus acts on $\mathscr{O}_{\mathrm{L}}$.

We thus obtain actions of $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ on the S-schemes T, X_T, η_{L}, X_{L}, t and X_t. These actions are compatible in the natural way.

5.2.2. We obtain actions (induced by pullbacks) of $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ on

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_T), \quad \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t), \quad \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T), \quad \text{etc.}$$

In turn, these immediately yield actions of $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ on the motivic and ℓ-adic realizations of such dg-categories.

Lemma 5.2.3. *There is a* $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$*-equivariant equivalence*

$$\mathcal{M}^{\vee}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_{\mathrm{L}})\big) \simeq q_{T*} j_{X_T *} \mathrm{B}\mathbb{U}_{X_{\mathrm{L}}}$$

in $\mathrm{Mod}_{\mathrm{B}\mathbb{U}_S}(\mathcal{SH}_S)$*. In particular, there is a* $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$*-equivariant equivalence*

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_{\mathrm{L}})\big) \simeq q_{T*} j_{X_T *} \mathbb{Q}_{\ell, X_{\mathrm{L}}}(\beta)$$

in $\mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)}(\mathrm{Shv}_{\mathbb{Q}_\ell}(S))$.

Proof. The first equivalence is one of the main features of the motivic realization of dg-categories. By functoriality, it is obviously compatible with the actions of $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$: these are both induced by the $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$-action on X_{L}. The second equivalence follows immediately from the first one. □

Lemma 5.2.4. *There is an equivalence*

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big) \simeq p_* i_{X*} \mathbb{Q}_{\ell, X_s}(\beta)$$

in $\mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)}(\mathrm{Shv}_{\mathbb{Q}_\ell}(S))$*. The group* $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ *acts trivially on both sides.*

Proof. Notice that $r\colon X_s \to X_t$ is a closed embedding (induced by $s = (t)_{\mathrm{red}} \to t$) with empty open complement. The localization sequence in ℓ-adic cohomology implies that

$$r_* \mathbb{Q}_{\ell, X_s}(\beta) \simeq \mathbb{Q}_{\ell, X_t}(\beta).$$

Moreover, we have that

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big) \simeq q_{T*} i_{X_T *} \mathbb{Q}_{\ell, X_t}(\beta), \quad \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_s)\big) \simeq p_* i_{X*} \mathbb{Q}_{\ell, X_s}(\beta).$$

Then the desired equivalence follows from $q_T \circ i_{X_T} \circ r = p \circ i_X$.

It remains to show that $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ acts trivially on $\mathrm{r}^{\ell}_{S}(\mathsf{D}_{\mathsf{perf}}(X_t))$. This is clear since the action is induced by pullbacks along the isomorphisms $h\colon X_t \to X_t$, which verify the equations $r = h \circ r$. □

5.3. The ℓ-adic realization of $\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)$. We now approach the proof of our main theorem.

Proposition 5.3.1. *The ℓ-adic realization of* $\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)$ *lives naturally in the following fiber-cofiber sequence:*

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \to p_* i_{X_*}\big(\mathbb{Q}_{\ell, X_s}(\beta) \oplus \mathbb{Q}_{\ell, X_s}(\beta)[1]\big) \to q_{T*} i_{X_T *} i^*_{X_T} j_{X_T *} \mathbb{Q}_{\ell, X_{\mathrm{L}}}(\beta).$$

Here, $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ *acts trivially on the middle term and naturally on the right one.*

Proof. By applying r^{ℓ}_{S} to the localization sequence

$$\mathsf{D}_{\mathsf{perf}}(X_t) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T) \to \mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T),$$

together with Theorem 4.2.3 and Lemma 5.2.4, we get the fiber-cofiber sequence

$$p_* i_{X*}\mathbb{Q}_{\ell,X_s}(\beta) \to q_{T*} i_{X_T*} i^{!}_{X_T}\mathbb{Q}_{\ell,X_T}(\beta) \to \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big)$$

in $\mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)}(\mathrm{Shv}_{\mathbb{Q}_\ell}(S))$. In particular, we observe that

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \simeq i_{S*} i^{*}_{S}\, \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big)$$

is supported on s. Consider now the diagram

$$\begin{array}{ccccc}
\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big) & \longrightarrow & \mathrm{r}^{\ell}_{S}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_t}} X_T)\big) & \longrightarrow & \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_t}} X_T)\big) \\
 & & \downarrow & \searrow^{\xi} & \\
p_* v_{X*} j_{X_T!}\mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta) & \longrightarrow & p_* v_{X*}\mathbb{Q}_{\ell,X_T}(\beta) & \longrightarrow & p_* v_{X*} i_{X_T*}\mathbb{Q}_{\ell,X_t}(\beta) \\
 & \searrow^{\zeta} & \downarrow & & \\
 & & p_* v_{X*} j_{X_T*}\mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta) & &
\end{array} \tag{5.1}$$

and observe that the two rows and the column in the middle are fiber-cofiber sequences. This has already been remarked for the first row. The second row is just localization in ℓ-adic sheaves. As for the column in the middle, one observes that the map

$$p_* v_{X*}\big(\mathbb{Q}_{\ell,X_T}(\beta) \to j_{X_T*}\mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)\big)$$

identifies with

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_T) \xrightarrow{j^{*}_{X_T}} \mathsf{D}_{\mathsf{perf}}(X_{\mathrm{L}})\big).$$

The latter has fiber equal to

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_T)_{X_t}\big) \simeq \mathrm{r}^{\ell}_{S}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)\big).$$

Consider now the composition

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big) \to \mathrm{r}^{\ell}_{S}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \xrightarrow{\xi} p_* v_{X*} i_{X_T*}\mathbb{Q}_{\ell,X_t}(\beta),$$

which we claim to be homotopic to zero. Indeed, the pushforwards along $X_t \to X_s$, $X_T \to X$ induce a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big) & \longrightarrow & \mathrm{r}^{\ell}_{S}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_t}} X_T)\big) & \xrightarrow{\mathrm{r}^{\ell}_{S}(i_{X_T*})} & \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_T)\big) & \xrightarrow{\mathrm{r}^{\ell}_{S}(i^{*}_{X_T})} & \underbrace{\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big)}_{\simeq p_* v_{X*} i_{X_T*}\mathbb{Q}_{\ell,X_t}(\beta)} \\
\downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \simeq \\
\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_s)\big) & \longrightarrow & \mathrm{r}^{\ell}_{S}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)\big) & \xrightarrow{\mathrm{r}^{\ell}_{S}(i_{X*})} & \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X)\big) & \xrightarrow{\mathrm{r}^{\ell}_{S}(i^{*}_{X})} & \underbrace{\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_s)\big)}_{\simeq p_* i_{X*}\mathbb{Q}_{\ell,X_s}(\beta)},
\end{array} \tag{5.2}$$

where the vertical morphisms at the extremes are equivalences. By [5, Lemma 3.26], the bottom composition is homotopic to zero and the claim follows.

Notice also that $\mathrm{coFib}(\xi) \simeq \mathrm{coFib}(\zeta)$. This is a general fact about diagrams like the one above in a stable ∞-category.

We now apply the octahedron construction to the composition

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big) \to \mathrm{r}^{\ell}_{S}\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \xrightarrow{\xi} p_* v_{X*} i_{X_T*} \mathbb{Q}_{\ell,X_t}(\beta)$$

and obtain the fiber-cofiber sequence

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \to p_* v_{X*} i_{X_T*} \mathbb{Q}_{\ell,X_t}(\beta) \oplus \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big)[1] \to \mathrm{coFib}(\zeta).$$

Observe now that all objects are supported on s and that

$$p_* v_{X*} i_{X_T*} \mathbb{Q}_{\ell,X_t}(\beta) \oplus \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big)[1] \simeq p_* i_{X*}\big(\mathbb{Q}_{\ell,X_s}(\beta) \oplus \mathbb{Q}_{\ell,X_s}(\beta)[1]\big).$$

Moreover, by proper base-change, we have

$$i_{S*} i_S^*\big(\mathrm{coFib}(\zeta)\big) \simeq q_{T*} i_{X_T*} i_{X_T}^* j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta).$$

We deduce that there is a fiber-cofiber sequence

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \to p_* i_{X*}\big(\mathbb{Q}_{\ell,X_s}(\beta) \oplus \mathbb{Q}_{\ell,X_s}(\beta)[1]\big) \to q_{T*} i_{X_T*} i_{X_T}^* j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta).$$

To conclude, observe that the term in the middle, which is equivalent to

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big) \oplus \mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{perf}}(X_t)\big)[1],$$

carries the trivial action of $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$ by Lemma 5.2.4. Therefore, it is equivalent to

$$p_* i_{X*} \mathbb{Q}_{\ell,X_s}(\beta) \otimes_{\mathbb{Q}_{\ell,S}} \mathbb{Q}_{\ell,S}^{\mathrm{I_L}}$$

by Lemma 5.1.6. □

Notation 5.3.2. We will denote the morphism appearing in Proposition 5.3.1 by

$$\mathrm{can}_{X_T} \colon p_* i_{X*}\big(\mathbb{Q}_{\ell,X_s}(\beta) \oplus \mathbb{Q}_{\ell,X_s}(\beta)[1]\big) \to q_{T*} i_{X_T*} i_{X_T}^* j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta).$$

Remark 5.3.3. The morphism $p_* i_{X*} \mathbb{Q}_{\ell,X_s}(\beta) \to q_{T*} i_{X_T*} i_{X_T}^* j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)$ obtained by restriction from the second map in the fiber-cofiber sequence of Proposition 5.3.1 corresponds to the one induced by the unit $\mathbb{Q}_{\ell,X_T}(\beta) \to j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)$ under the equivalence

$$p_* i_{X*} \mathbb{Q}_{\ell,X_s}(\beta) \simeq q_{T*} i_{X_T*} i_{X_T}^* \mathbb{Q}_{\ell,X_T}(\beta).$$

In particular, as

$$p_{s*}\big(\mathbb{Q}_{\ell,X_s}(\beta) \oplus \mathbb{Q}_{\ell,X_s}(\beta)[1]\big) \simeq p_{s*} \mathbb{Q}_{\ell,X_s}(\beta) \otimes_{\mathbb{Q}_{\ell,s}} \mathbb{Q}_{\ell,s}^{\mathrm{I_L}}$$

(see [5, (4.3.43)]), we see that can_{X_T} is obtained from the unit morphism

$$\mathbb{Q}_{\ell,X_T}(\beta) \to j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)$$

by recognizing that $q_{T*} i_{X_T*} i_{X_T}^* j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)$ has a natural $i_{S*}\mathbb{Q}_{\ell,s}^{\mathrm{I_L}}$-module structure.

In particular, we can write the fiber-cofiber sequence of Proposition 5.3.1 as

$$\mathrm{r}^{\ell}_{S}\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \longrightarrow i_{S*} p_{s*} \mathbb{Q}_{\ell,X_s}(\beta) \otimes_{\mathbb{Q}_{\ell,s}} \mathbb{Q}_{\ell,s}^{\mathrm{I_L}} \xrightarrow{\mathrm{can}_{X_T}} i_{S*} p_{s*} \Psi_p\big(\mathbb{Q}_{\ell,X_{\mathrm{K}}}(\beta)\big)^{\mathrm{I_L}}.$$

Remark 5.3.4. Diagram (5.2) also shows that the map

$$\mathrm{can}_{X_T}: p_* i_{X_*}\big(\mathbb{Q}_{\ell,X_s}(\beta) \oplus \mathbb{Q}_{\ell,X_s}(\beta)[1]\big) \to q_{T*} i_{X_T*} i^*_{X_T} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)$$

is $\mathrm{G}_{\mathrm{L/K}}$-equivariant. Indeed, combined with the version of (5.1) for X_T replaced by X, it implies that this map factors through

$$p_* i^*_X j_{X*} \mathbb{Q}_{\ell,X_{\mathrm{K}}}(\beta) \simeq \big(q_{T*} i_{X_T*} i^*_{X_T} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)\big)^{\mathrm{G}_{\mathrm{L/K}}}.$$

5.3.5. Proof of Theorem A. We are finally ready to prove our first main theorem. We will do so by showing that there is a homotopy of morphisms of algebras

$$\mathrm{sp}^{\mathrm{I_L}} \simeq \mathrm{can}_{X_T}: \underbrace{p_* i_{X_*}\big(\mathbb{Q}_{\ell,X_s}(\beta) \oplus \mathbb{Q}_{\ell,X_s}(\beta)[1]\big)}_{\simeq p_* i_{X*}\mathbb{Q}_{\ell,X_s}(\beta)\otimes_{\mathbb{Q}_{\ell,S}} \mathbb{Q}^{\mathrm{I_L}}_{\ell,S}} \to q_{T*} i_{X_T*} i^*_{X_T} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta).$$

Notice that $\mathrm{sp}^{\mathrm{I_L}}$ is homotopic to the morphism

$$p_* i_{X*} \mathbb{Q}_{\ell,X_s}(\beta) \otimes_{\mathbb{Q}_{\ell,s}} \mathbb{Q}^{\mathrm{I_L}}_{\ell,s} \to q_{T*} i_{X_T*} i^*_{X_T} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)$$

which corresponds, under the adjunction $(- \otimes_{\mathbb{Q}_{\ell,s}} \mathbb{Q}^{\mathrm{I_L}}_{\ell,s}, \mathrm{Forget})$, to the morphism

$$p_* i_{X*} \mathbb{Q}_{\ell,X_s}(\beta) \to q_{T*} i_{X_T*} i^*_{X_T} j_{X_T*} \mathbb{Q}_{\ell,X_{\mathrm{L}}}(\beta)$$

induced by the adjunction $(j^*_{X_T}, j_{X_T*})$. This is Lemma 5.1.6. The same is true for can_{X_T}, as one can see from diagram (5.1).

5.4. Non-commutative nature of ℓ-adic vanishing cycles. In this subsection, we prove Theorem B.

5.4.1. Recall from [5] that the category of LG models over S is the ordinary category of pairs (Y, f), where Y is a flat S-scheme and $f: Y \to \mathbb{A}^1_S$ is a function. A morphism $(Y, f) \to (Z, g)$ between LG models is a morphism of S-schemes $Y \to Z$ compatible with f and g in the obvious sense.

The assignment $(Y, f) \mapsto \mathsf{D}_{\mathrm{sg}}(Y_0 \xrightarrow{i_Y} Y)$, where $i_Y: Y_0 \to Y$ is the closed embedding of the fiber over zero of f in Y, can be promoted to a functor $\mathsf{D}_{\mathrm{sg}}: \mathrm{LG}^{\mathrm{op}}_S \to \mathrm{dgCat}_S$, where the transition maps are induced by pullbacks. See [5, §2.3.15].

Notation 5.4.2. In this section, we will adopt the notation

$$\mathsf{D}_{\mathrm{sg}}(Y, f) := \mathsf{D}_{\mathrm{sg}}(Y_0 \xrightarrow{i_Y} Y).$$

5.4.3. Let $\mathscr{E}$ denote the filtered category of finite extensions $\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}$ of complete strict discrete valuations rings, like the one considered in Section 2.1.2. For an S-scheme Y, we will denote by (Y, π_{K}) the LG model over S given by Y with the function $Y \longrightarrow S \xrightarrow{\pi_{\mathrm{K}}} \mathbb{A}^1_S$.

5.4.4. Consider a proper flat regular S-scheme X (generically smooth). For an extension $\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}$, let $X_{\mathscr{O}_{\mathrm{L}}}$ denote the pullback $X \times_S \mathrm{Spec}(\mathscr{O}_{\mathrm{L}})$. Then we get the following diagram of LG models over S:

$$\mathscr{E}^{\mathrm{op}} \to \mathrm{LG}_S,$$
$$(\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}) \mapsto (X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}}).$$

Notice that, for every chain of extensions $\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}} \subseteq \mathscr{O}_{\mathrm{M}}$, the morphism of LG models

$$(X_{\mathscr{O}_{\mathrm{M}}}, \pi_{\mathrm{K}}) \to (X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})$$

is H_{M}-equivariant, where the H_{M}-action on $X_{\mathscr{O}_{\mathrm{L}}}$ is induced by the quotient

$$\mathscr{G}al(\mathrm{M}/\mathrm{K}) = \mathrm{H}_{\mathrm{M}} \to \mathrm{G}_{\mathrm{L}/\mathrm{K}} = \mathscr{G}al(\mathrm{L}/\mathrm{K}).$$

5.4.5. Composing this diagram with the functor D_{sg}, we get a diagram

$$\mathfrak{d}\colon \mathscr{E} \to \mathrm{dgCat}_S$$

defined on objects by $(\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}) \mapsto \mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})$.

Remark 5.4.6. The dg-category $\mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})$ is precisely the dg-category denoted by

$$\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)$$

in the previous sections (for $T = \mathrm{Spec}(\mathscr{O}_{\mathrm{L}})$).

5.4.7. It follows immediately from functoriality that each $\mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})$ carries a canonical $\mathrm{G}_{\mathrm{L}/\mathrm{K}}$-action and that the dg functors

$$\mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}}) \to \mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{M}}}, \pi_{\mathrm{K}})$$

are compatible with these actions for every chain of extensions $\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}} \subseteq \mathscr{O}_{\mathrm{M}}$.

5.4.8. Recall that dgCat_S is a cocomplete ∞-category. We consider the colimit

$$\mathfrak{S} := \varinjlim_{(\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}) \in \mathscr{E}} \mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})$$

of the diagram $\mathfrak{d}$. It follows immediately that this dg-category carries a *continuous* action of

$$\mathrm{I}_{\mathrm{K}} \simeq \varprojlim_{(\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}) \in \mathscr{E}} \mathrm{G}_{\mathrm{L}/\mathrm{K}}.$$

Roughly, this means that, for every object $A \in \mathfrak{S}$, there exists some $(\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}) \in \mathscr{E}$ such that $\mathrm{I}_{\mathrm{L}} \subseteq \mathrm{I}_{\mathrm{K}}$ acts trivially on the full subcategory $(A) \subseteq \mathfrak{S}$ generated by A.

5.4.9. Proof of Theorem B. Notice that, for every chain $\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}} \subseteq \mathscr{O}_{\mathrm{M}}$, there is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{r}^{\ell}_S\big(\mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}},\pi_{\mathrm{K}}})\big) & \longrightarrow & i_{S*}p_{s*}\mathbb{Q}_{\ell,X_s}(\beta)^{\mathrm{I}_{\mathrm{L}}} & \xrightarrow{\mathrm{sp}^{\mathrm{I}_{\mathrm{L}}}} & i_{S*}\Psi_p\big(\mathbb{Q}_{\ell,X_{\mathrm{K}}}(\beta)\big)^{\mathrm{I}_{\mathrm{L}}} \\
\Big\downarrow{\scriptstyle ((X_{\mathscr{O}_{\mathrm{M}}}\to X_{\mathscr{O}_{\mathrm{L}}})\times_S s)^*} & & \Big\downarrow & & \Big\downarrow \\
\mathrm{r}^{\ell}_S\big(\mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{M}},\pi_{\mathrm{K}}})\big) & \longrightarrow & i_{S*}p_{s*}\mathbb{Q}_{\ell,X_s}(\beta)^{\mathrm{I}_{\mathrm{M}}} & \xrightarrow{\mathrm{sp}^{\mathrm{I}_{\mathrm{M}}}} & i_{S*}\Psi_p\big(\mathbb{Q}_{\ell,X_{\mathrm{K}}}(\beta)\big)^{\mathrm{I}_{\mathrm{M}}},
\end{array}$$

where the rows are fiber-cofiber sequences of Theorem A and the middle and rightmost vertical morphisms are the canonical maps from I_{L}-homotopy fixed points to I_{M}-homotopy fixed points.

Therefore, the filtered diagram

$$\mathrm{r}^{\ell}_S \circ \mathfrak{d} \colon \mathscr{E} \to \mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)}(\mathrm{Shv}_{\mathbb{Q}_\ell}(S))$$

is equivalent to the filtered diagram

$$\begin{gathered}\mathscr{E} \to \mathrm{Mod}_{\mathbb{Q}_{\ell,S}(\beta)}\big(\mathrm{Shv}_{\mathbb{Q}_\ell}(S)\big),\\ (\mathscr{O}_{\mathrm{K}} \subseteq \mathscr{O}_{\mathrm{L}}) \mapsto i_{S*}\mathrm{H}^*_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,S}(\beta))\big)^{\mathrm{I_L}}[-1].\end{gathered}$$

5.4.10. Recall that r^{ℓ}_S commutes with filtered colimits. Since the equivalences

$$\mathrm{r}^{\ell}_S\big(\mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})\big) \simeq i_{S*}\mathrm{H}^*_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,S}(\beta))\big)^{\mathrm{I_L}}[-1]$$

are compatible with the $\mathrm{G}_{\mathrm{L/K}}$-actions, we get that

$$\begin{aligned}
\mathrm{r}^{\ell}_S(\mathfrak{S}) &= \mathrm{r}^{\ell}_S\Big(\varinjlim_{(\mathscr{O}_{\mathrm{K}}\subseteq\mathscr{O}_{\mathrm{L}})\in\mathscr{E}} \mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})\Big) && \text{(definition of } \mathfrak{S})\\
&\simeq \varinjlim_{(\mathscr{O}_{\mathrm{K}}\subseteq\mathscr{O}_{\mathrm{L}})\in\mathscr{E}} \mathrm{r}^{\ell}_S\big(\mathsf{D}_{\mathsf{sg}}(X_{\mathscr{O}_{\mathrm{L}}}, \pi_{\mathrm{K}})\big) && (\mathrm{r}^{\ell}_S \text{ commutes with filtered colimits})\\
&\simeq \varinjlim_{(\mathscr{O}_{\mathrm{K}}\subseteq\mathscr{O}_{\mathrm{L}})\in\mathscr{E}} i_{S*}\mathrm{H}^*_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,S}(\beta))\big)^{\mathrm{I_L}}[-1] && \text{(Theorem A)}\\
&\simeq i_{S*}\mathrm{H}^*_{\text{ét}}\big(X_s, \Phi_p(\mathbb{Q}_{\ell,X}(\beta))\big)[-1] && \text{(continuity of the action of } \mathrm{I_K}).
\end{aligned}$$

This concludes the proof of Theorem B.

A. Remarks on the properness hypothesis

In this final section, we briefly comment on the properness hypothesis for the morphism $p\colon X \to S$. This assumption is superfluous, provided that one is willing to work at the level of ℓ-adic sheaves on X. This observation is the analogue of [37, footnote 8, page 503] in the case where Galois actions are taken into account.

A.1. An attentive reader might have noticed that the properness hypothesis is used only once throughout the paper: in the proof of Proposition 5.3.1 in order to invoke proper base-change. This is needed because we work with the ℓ-adic realization functor r^{ℓ}_S. However, as explained in [37, Remark 2.2.2], r^{ℓ}_S admits a relative version

$$\mathrm{r}^{\ell}_X\colon \mathrm{dgCat}_X \to \mathrm{Mod}_{\mathbb{Q}_{\ell,X}(\beta)}\big(\mathrm{Shv}_{\mathbb{Q}_\ell}(X)\big)$$

with the same properties of r^{ℓ}_S. The computations and the proofs in this paper all work *mutatis mutandis* by applying r^{ℓ}_X in place of r^{ℓ}_S. Only the fact that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T}} X_T) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T)$$

is a localization sequence of X-linear dg-categories deserves a bit of explanation.

A.2. The functor

$$p_*\colon \mathrm{dgCat}_X \to \mathrm{dgCat}_S$$

admits a (symmetric monoidal) left adjoint

$$p^* = - \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(X)\colon \mathrm{dgCat}_S \to \mathrm{dgCat}_X$$

which preserves localization sequences.

A.3. By Corollary 3.3.4, we obtain that

$$\big(\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1*} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)\big) \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(X)$$

is a localization sequence of (left) B^+_X-modules, where $\mathsf{B}^+_X := \mathsf{B}^+ \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(X)$.

A.4. Clearly, $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)$ (regarded as an X-linear dg-category) admits a left B^+_X-module structure. As a consequence, $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}}$ admits a right B^+_X-module structure.

A.5. One sees that

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_s)^{\mathrm{op}} \otimes_{\mathsf{B}^+_X} \big((\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t \xrightarrow{a_1*} t) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(G_t) \to \mathsf{D}_{\mathsf{sg}}(t)) \otimes_{\mathsf{D}_{\mathsf{perf}}(S)} \mathsf{D}_{\mathsf{perf}}(X)\big)$$

identifies with the diagram of X-linear dg-categories

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t \xrightarrow{i_{X_T*}} X_T) \hookrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X_t) \to \mathsf{D}_{\mathsf{sg}}(X_T). \tag{A.1}$$

In particular, this is a localization sequence in dgCat_X (the proofs of Lemma 4.1.3 and Proposition 4.1.7 can be adapted easily to the X-linear situation).

Remark A.5.1. If we apply the forgetful functor $p_*\colon \mathrm{dgCat}_X \to \mathrm{dgCat}_S$ to (A.1), we find the localization sequence of S-linear dg-categories obtained in Proposition 4.1.7.

A.6. Given this key ingredient, the proofs of the computations of motivic and ℓ-adic realizations in the main body of the paper apply *before taking* p_*. Hence, we can avoid any reference to proper base-change, and in particular, we obtain the following.

Theorem. *Let $p\colon X \to S$ be a flat and generically smooth morphism of finite type. Assume that X is regular. There is an equivalence of $i_{X*}\mathbb{Q}^{\mathrm{I_L}}_{\ell,X}(\beta)$-modules*

$$\mathrm{r}^{\ell}_X\big(\mathsf{D}_{\mathsf{sg}}(X_t \xrightarrow{i_{X_T}} X_T)\big) \simeq i_{X*}\Phi_p\big(\mathbb{Q}_{\ell,X}(\beta)\big)^{\mathrm{I_L}}[-1],$$

compatible with the natural $\mathrm{G}_{\mathrm{L/K}}$*-actions.*

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