

# Virial theorem in clusters of galaxies with MOND

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## ABSTRACT

A specific modification of Newtonian dynamics known as MOND has been shown to reproduce the dynamics of most astrophysical systems at different scales without invoking non-baryonic dark matter (DM). There is, however, a long-standing unsolved problem when MOND is applied to rich clusters of galaxies in the form of a deficit (by a factor around two) of predicted dynamical mass derived from the virial theorem with respect to observations. In this article, we approach the virial theorem using the velocity dispersion of cluster members along the line of sight rather than using the cluster temperature from X-ray data and hydrostatic equilibrium. Analytical calculations of the virial theorem in clusters for Newtonian gravity + DM and MOND are developed, applying pressure (surface) corrections for non-closed systems. Recent calibrations of DM profiles, baryonic ratio, and baryonic ( $\beta$  model or others) profiles are used, while allowing free parameters to range within the observational constraints. It is shown that solutions exist for MOND in clusters that give similar results to Newton + DM – particularly in the case of an isothermal  $\beta$  model for  $\beta = 0.55$ – $0.70$  and core radii  $r_c$  between 0.1 and 0.3 times  $r_{500}$  (in agreement with the known data). The disagreements found in previous studies seem to be due to the lack of pressure corrections (based on inappropriate hydrostatic equilibrium assumptions) and/or inappropriate parameters for the baryonic matter profiles.

**Key words:** gravitation – galaxies: clusters – dark matter.

## 1 INTRODUCTION

In present-day astrophysics, many lines of investigation support the existence of large amounts of non-baryonic dark matter (DM hereafter) in galaxies and in the Universe at large, the most obvious example being the asymptotically flat rotation curve of galaxies, which indicates the existence of massive DM haloes. Considerable fine tuning is required, however, to justify their observed properties, the most striking example possibly being the baryonic Tully–Fisher relation (see e.g. McGaugh (2012) and reference therein). Because of this, over the years more than one proposal has been made to find alternative explanations not involving DM. In particular, it has been shown that a specific modification of Newtonian dynamics, known as MOND (Milgrom 1983a, b, c), is able to describe many kinds of behaviour of galaxies and other cosmic structures generally ascribed to the presence of DM. The basic idea of MOND is that an acceleration ( $a_0$ ) exists, below which Newtonian dynamics is no longer valid.

The MOND hypothesis has profound and far-reaching implications. Since the seminal papers by Milgrom (1983a, b, c), MOND has been applied to several astrophysical objects including (in increasing order of size) wide binary stars (Hernández, Jiménez & Allen 2012; Hernández, Cookson & Cortes 2022), globular clusters (Scarpa, Marconi & Gilmozzi 2003; Scarpa & Falomo 2010; Scarpa et al. 2011; Hernández & Lara-D I 2020), dwarf galaxies (Milgrom 1995; McGaugh & Milgrom 2013; Sanders 2021), gas dominated galaxies (McGaugh 2012; Sanders 2019), spiral galaxies (Sanders 1996; Milgrom & Sanders 2007; Gentile, Famaey & de Blok 2011) including our Milky Way (Chrobáková et al. 2020), elliptical galaxies

(Milgrom & Sanders 2003; Tian & Ko 2016; Duraro et al. 2018), satellites around galaxies (Angus et al. 2008; Klypin & Prada 2009), pairs of galaxies (Milgrom 1983c; Scarpa, Falomo & Treves 2022), groups of galaxies (Milgrom 2019; McGaugh et al. 2021), gravitational lenses (Sanders 2014), and cluster of galaxies (Sanders 1999, 2003). In all cases except one, MOND may describe the observations without the need for DM. The problematic case being rich clusters of galaxies, which are a long-standing problem, thus far unsolved by MOND, and on which we try to shed some light here.

We know the virial theorem works in clusters of galaxies for standard Newtonian gravity within the usual assumption of the existence of non-baryonic DM as predicted by  $\Lambda$ CDM models (e.g. Evrard et al. 2008; Zhang et al. 2011; Munari et al. 2013), but it has not worked for MOND so far. Using a hydrostatic isothermal model with temperatures derived from X-ray data, the MOND mass prediction falls short by a factor  $\sim 2$  (Sanders 1999; Pointecouteau & Silk 2005). A more recent analysis by Ettori et al. (2019) finds that MOND scenarios underestimate hydrostatic masses of cluster by 40 per cent at  $r_{1000}$  ( $r_x$  being the radius of the sphere for which the average density inside it is  $x$  times the critical density  $\rho_c$ ), but with a decreasing tension as the radius increases, and reaches  $\sim 15$  per cent at  $r_{200}$ . However, this hydrostatic model has certain drawbacks which, according to some authors, may lead to important systematic errors of up to a factor 2 for the mass ((Bartelmann & Steinmetz 1996; Balland & Blanchard 1997; section 4.2 Sadat 1997).

Other applications of the virial theorem within the framework of MOND are discussed in several works (Milgrom 1994, 2010, 2014; Fabris & Velten 2009). One different method is the application of the virial theorem using the velocity dispersion of cluster members along the line of sight. A study of this kind has been carried out by Fabris & Velten (2009) for the Coma cluster revealing within

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Newtonian gravity a mass-to-light ratio  $M/L \sim 200$  in solar units (in agreement with estimates based on different methods (Carlberg, Yee & Ellingson 1997)), whereas for MOND it is three times lower (still problematic for MOND). Instead of using optical surface brightness to trace the baryonic mass (and non-baryonic mass for Newtonian gravity), a derivation of baryonic mass calibrated with X-ray data would be more accurate.

Here, by allowing free parameters to range within the observational constraints, we revisit the application of the virial theorem in clusters of galaxies using the velocity dispersion of cluster members along the line of sight. Recent calibrations of DM profiles, baryonic ratio, and baryonic (the  $\beta$  model or others) profiles are used. We also apply pressure corrections for non-closed systems (usually overlooked in the literature). Both Newtonian gravity and MOND are considered, although we pay special interest to the latter case.

## 2 APPLICATION OF THE VIRIAL THEOREM

### 2.1 Virial theorem

We assume spherical symmetry in a rich cluster with mass density distribution  $\rho(r)$  and mass interior to each radius  $r$

$$M(r) \equiv 4\pi \int_0^r dx x^2 \rho(x). \quad (1)$$

The potential energy with MOND or Newtonian gravity within a radius  $r_{\max}$  is (Fabris & Velten 2009)

$$V(r_{\max}) = -4\pi G \int_0^{r_{\max}} dr r \sqrt{1 + \left(\frac{r}{r_{\text{cM}}(r)}\right)^2} \rho(r) M(r), \quad (2)$$

with  $r_{\text{cM}}(r) \rightarrow \infty$  for Newtonian gravity. The distance  $r_{\text{cM}}(r)$  is related to the usual parameter  $a_0$  by means of

$$r_{\text{cM}}(r) = \sqrt{\frac{GM(r)}{a_0}} \quad (3)$$

and  $a_0 = 1.2 \times 10^{-10} \text{ m s}^{-2}$  for MOND or  $a_0 = 0$  for Newton.

The kinetic energy is

$$K(r_{\max}) = 3\pi \int_0^{r_{\max}} dR R \sigma_v^2(R) \Sigma(R), \quad (4)$$

$$\Sigma(R) = 2 \int_0^{\sqrt{r_{\max}^2 - R^2}} dz \rho(\sqrt{R^2 + z^2}), \quad (5)$$

where  $\sigma_v(R)$  is the line-of-sight velocity dispersion (in the rest-frame of the cluster) as a function of the projected distance  $R$ , and  $\Sigma(R)$  is the surface mass density. This can also be expressed as:

$$K(r_{\max}) = \frac{3}{2} M(r_{\max}) \sigma_{v,r < r_{\max}}^2. \quad (6)$$

In a virialized cluster in the limit of  $r_{\max} \rightarrow \infty$ , the following condition should be followed:

$$2K(r_{\max}) + V(r_{\max}) = 0. \quad (7)$$

Hence,

$$\sigma_{v,r < r_{\max}} = \sqrt{\frac{-V(r_{\max})}{3M(r_{\max})}}. \quad (8)$$

### 2.2 Corrections to the virial theorem that include the pressure term

None of the above considerations can be exact unless we make  $r_{\max} = \infty$ , which would make the cluster a closed system. For a finite virial

radius  $r_{\max}$ , there is a pressure term due to exchange of galaxies and other types of matter between the sphere within radius  $r_{\max}$  and the space beyond it. The relevance of the surface (pressure) term in standard gravity is known (e.g. The & White 1986; Carlberg et al. 1996; Carlberg et al. 1997; Girardi et al. 1998).

When the pressure term is included, the expression of the dispersion of velocities would be (see Appendix A):

$$\sigma_{v,r < r_{\max}}^2 = \sigma_{v,r < r_{\max}, P=0}^2 + \frac{4\pi}{(3 - 2\beta_a)} r_{\max}^3 \rho(r_{\max}) \sigma_{v,r=r_{\max}}^2 \frac{1}{M(r_{\max})}, \quad (9)$$

where  $\beta_a$  is the velocity anisotropy parameter:  $\beta_a = 1 - \frac{\sigma_\theta^2}{\sigma_r^2}$ . The second term is associated with the pressure. Assuming  $\beta_a = 1/4$  (Klypin & Prada 2009) and neglecting the variation of  $\sigma_v$  with the radius, we get for the virial radius  $r_{\max} = r_{200}$  (which by definition follows  $200\rho_c = \frac{M(r_{200})}{\frac{4}{3}\pi r_{200}^3}$ , with  $\rho_c$  the critical density)

$$\sigma_{v,r < r_{200}} \approx \sigma_{v,r < r_{200}, P=0} \times F_P, \quad (10)$$

$$F_P = \left[ 1 - \frac{6}{5} \frac{\rho(r_{200})}{200\rho_c} \right]^{-1/2}.$$

Note that both  $\rho$  and  $\rho_c$  should refer to the same matter, including either non-baryonic DM (in Newton + DM) or only baryonic (in MOND). This corrective factor by pressure,  $F_P$ , depends on the profile.

### 2.3 Newtonian gravity

For standard Newtonian gravity, a good description of the hydrostatic mass profile of clusters as derived from X-ray data is obtained with Navarro–Frenk–White (NFW) profiles (Navarro, Frenk & White 1997) with scale  $r_{200}$  (Ettori et al. 2019).

Using  $\rho(r)$ ,  $M(r)$  of NFW profiles (formulae of appendix Section B, with concentration index  $C$ ) and  $a_0 = 0$ , we get potential energy

$$V(r_{\max}) = -\frac{4\pi G \rho_0}{C \left[ \ln(1+C) - \frac{C}{1+C} \right]} r_{200}^2 M_{200} \times \int_0^{r_{\max}} dx \frac{[\ln(1+Cx) - \frac{Cx}{1+Cx}]}{(1+Cx)^2}. \quad (11)$$

Taking the values of  $r_{200}$  and  $\rho_0$  from equation (B1):

$$V(r_{\max}) = -(8.707 \times 10^{55} \text{ J}) \frac{C^2}{\left[ \ln(1+C) - \frac{C}{1+C} \right]^2} \left( \frac{M_{200}}{10^{14} M_\odot} \right)^{5/3} \times \int_0^{r_{\max}} dx \frac{[\ln(1+Cx) - \frac{Cx}{1+Cx}]}{(1+Cx)^2}. \quad (12)$$

The kinetic energy is

$$K(r_{\max}) = (2.985 \times 10^{50} \text{ J}) [\sigma_{v,r_{\max}} (\text{km s}^{-1})]^2 \left( \frac{M_{200}}{10^{14} M_\odot} \right) \times \frac{\left[ \ln\left(1 + \frac{Cr_{\max}}{r_{200}}\right) - \frac{Cr_{\max}}{r_{200} + Cr_{\max}} \right]}{\left[ \ln(1+C) - \frac{C}{1+C} \right]}. \quad (13)$$

For the virial radius  $r_{\max} = r_{200}$ , and applying the virial theorem equation (7), with the pressure correction referred at equation (10) ( $F_P$  is independent of the mass), we get

$$\begin{aligned} \sigma_{v,r200,\text{Newton+DM}} &= (382 \text{ km s}^{-1}) \left( \frac{M_{200}}{10^{14} M_{\odot}} \right)^{1/3} \\ &\times \frac{C}{\left[ \ln(1+C) - \frac{C}{1+C} \right]} \\ &\times \left[ \int_0^1 dx \frac{[\ln(1+Cx) - \frac{Cx}{1+Cx}]}{(1+Cx)^2} \right]^{1/2} \times F_P(C), \\ F_P &= \left[ 1 - 0.400 \frac{C^2}{(1+C)^2 \left[ \ln(1+C) - \frac{C}{1+C} \right]} \right]^{-1/2}. \end{aligned} \quad (14)$$

This dependence with the mass to the power of 1/3 is also well known from simulations (Evrard et al. 2008; Munari et al. 2013).

Throughout this paper, we shall calculate the dispersion of velocities as a function of  $M_{500} \equiv M(r_{500})$ . This amount might also be related to other parameters or measurements; for instance, the Sunyaev–Zel’dovich effect amplitude ( $Y_{\text{SZ}}$ ) (Arnaud et al. 2010; López-Corredoira, Gutiérrez & Génova-Santos 2017; Aguado-Barahona et al. 2022). In terms of  $M_{500}$  [with  $M_{200}/M_{500}$  derived from equation (B1)], and including all the dependence of  $C$  in a single factor,

$$\sigma_{v,r200,\text{Newton+DM}} = A_{\text{Newton,NFW}}(C) \left( \frac{M_{500}}{10^{14} M_{\odot}} \right)^{1/3}, \quad (15)$$

where  $A_{\text{Newton,NFW}}(C)$  is plotted in Fig. 1. The dependence on  $C$  is quite small for  $C$  between 2 and 8. Several values are given in the literature: from  $C = 2.9 \pm 0.2$  (Lin, Mohr & Stanford 2004; Macciò & van den Bosch 2008), derived from analyses of observational X-ray data, or  $C = 4.6_{-1.1}^{+1.8}$  from purely theoretical dynamical models in Prada et al. (2012) (the error bars represent here the r.m.s., not the error of the average; assuming that the 10, 90 per cent percentiles of fig. 13 in Prada et al. (2012) are 1.28 times the r.m.s., as it corresponds to a Gaussian distribution). The concentration index  $C$  has a modest dependence on mass (Lin et al. 2004; Macciò, Dutton & van den Bosch 2008; Prada et al. 2012; Ettori et al. 2019). The above range  $2.7 < C < 6.4$  gives a variation of only  $\sim 2$  per cent of  $A_{\text{Newton,NFW}}(C)$ , which is negligible compared to other sources of errors. We take in the following as default  $C = 3$ , for which  $A_{\text{Newton,NFW}}(C) = 522 \text{ km s}^{-1}$  ( $F_P = 1.244$ ).

## 2.4 MOND with only baryonic matter

We set  $a_0 = 1.2 \times 10^{-10} \text{ m s}^{-2}$ . For MOND, there is only baryonic mass. To calculate the amount of baryonic mass, we use the relationship obtained by González et al. (2013):

$$\begin{aligned} M_{\text{bar},500}(M_{500}) &= (0.117 \pm 0.004) \times 10^{14} \\ &\times \left( \frac{M_{500}}{10^{14} M_{\odot}} \right)^{1.16 \pm 0.04} M_{\odot}. \end{aligned} \quad (16)$$

For a baryonic density distribution of the type  $\rho_{\text{bar}}(r) = \rho_0 J\left(\frac{r}{r_c}\right)$  with  $J$ , a generic function and cluster core radius  $r_c$  proportional to  $r_{500}$  [we define the parameter independent of the mass  $x_{500} \equiv \frac{r_c}{r_{500}}$ ; (Pcaud et al. 2016)], the potential and kinetic energies are

$$\begin{aligned} V(r_{\text{max}}) &= -(6.366 \times 10^{55} \text{ J}) \frac{1}{x_{500} I^2(r_{500})} \left( \frac{M_{\text{bar},500}}{10^{14} M_{\odot}} \right)^{5/3} \\ &\times \int_0^{r_{\text{max}}} dx x J(x) I(x r_c) R(x), \end{aligned} \quad (17)$$

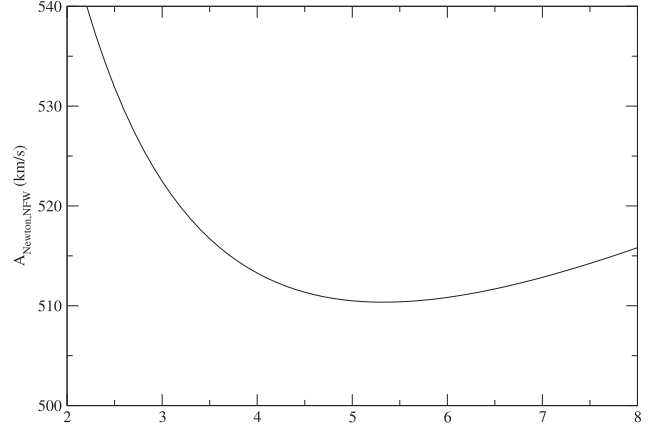


Figure 1. Value of  $A_{\text{Newton,NFW}}(C)$  in equation (15) as a function of  $C$ .

$$\begin{aligned} R(x) &= \sqrt{1 + 15.57 I(r_{500}) x_{500}^2 \left( \frac{M_{\text{bar},500}}{10^{14} M_{\odot}} \right)^{-1/3} \frac{x^2}{I(x r_c)}}, \\ I(r) &= \int_0^{r/r_c} dx x^2 J(x), \\ K(r_{\text{max}}) &= (2.980 \times 10^{50} \text{ J}) [\sigma_{v,r_{\text{max}}} (\text{km s}^{-1})]^2 \left( \frac{M_{\text{bar},500}}{10^{14} M_{\odot}} \right) \frac{I(r_{\text{max}})}{I(r_{500})}. \end{aligned} \quad (18)$$

Note that the radii  $r_{500}$  or  $r_{200}$  are here quite similar to the one obtained from Newton + DM + NFW. For instance, with equations (16), (C1), we get

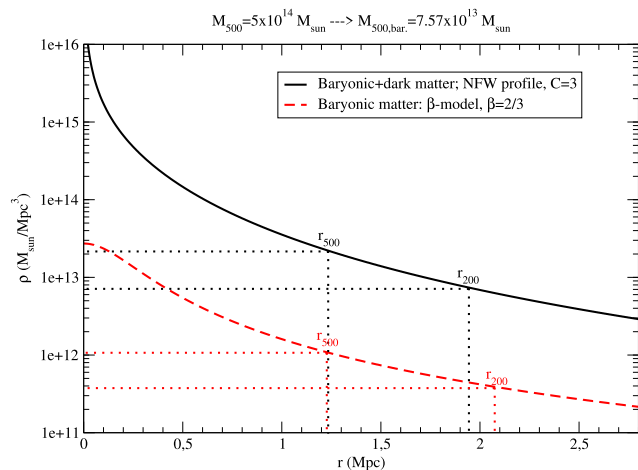
$$\begin{aligned} r_{500} &= 1.345 \text{ Mpc} \times \left( \frac{M_{\text{bar},500}}{10^{14} M_{\odot}} \right)^{1/3} \\ &= 0.658 \text{ Mpc} \times \left( \frac{M_{500}}{10^{14} M_{\odot}} \right)^{0.386}, \end{aligned} \quad (19)$$

which is similar although slightly lower than the  $r_{500}$  from equation (B1) of Newton + DM. In Fig. 2, we offer a plot with a numerical example for  $M_{500} = 5 \times 10^{14} M_{\odot}$ . This approximate coincidence is expected because the ratio of baryonic/total matter in the cluster is similar to the ratio of baryonic/total matter in the Universe ( $\Omega_b/\Omega_m$  and 1, respectively, for Newton + DM and MOND). The fact that this ratio is the same one in clusters and in the Universe implies that the central cluster density in MOND is similar to the cluster central density in standard gravity times  $\Omega_b/\Omega_m$ , thus leading to similar  $r_{500}$ 's in the two models. The fact that the similarity is tighter for  $r_{500}$  than for  $r_{200}$  is a coincidence.

For  $r_{\text{max}} = r_{200}$ , and only baryonic density (we take critical baryonic density  $\rho_{\text{cb}} = 1.33 \times 10^{-27} \text{ kg m}^{-3}$ ; see Appendix C), the correction of the virial theorem due to the pressure is a factor

$$F_P = \left[ 1 - \frac{3}{(3-2\beta_a)} \frac{5J\left(\frac{r_{200}}{x_{500}r_{500}}\right)}{6x_{500}^3 I(r_{500})} \right]^{-1/2}, \quad (20)$$

and we assume, as previously,  $\beta_a = 1/4$ ; except in the extreme case of very low-concentrated (almost flat) profiles with very high densities at  $r_{200}$  [where  $R_2 \equiv \frac{\rho_{\text{bar}}(r_{200})}{200\rho_{\text{cb}}} = \frac{5J\left(\frac{r_{200}}{x_{500}r_{500}}\right)}{6x_{500}^3 I(r_{500})}$  is larger than 0.75], for which we assume a  $\beta_a$  between 0 and 0.25, in a linear dependence with  $R_2$  ( $\beta_a = 1 - R_2$ ), in order to avoid a negative root square.



**Figure 2.** Numerical example of profiles with NFW model with  $C = 3$  for baryonic + DM and  $\beta$  model with  $\beta = 2/3$  for baryonic matter for a mass  $M_{500} = 5 \times 10^{14} M_{\odot}$ , equivalent to  $M_{\text{bar},500} = 0.773 \times 10^{14} M_{\odot}$ . In dotted lines, the respective positions of  $r_{500}$ ,  $r_{200}$ ,  $\rho(r_{500})$ ,  $\rho(r_{200})$ ; where  $r_x$  is the radius of the sphere for which the average density inside it is  $x$  times a critical density  $\rho_c = 8.5 \times 10^{-27} \text{ kg m}^{-3}$  for the case with DM or  $\rho_{\text{cb}} = 1.33 \times 10^{-27} \text{ kg m}^{-3}$  with only baryonic density. Note the similarity of  $r_{500}$  for both distributions.

#### 2.4.1 $\beta$ isothermal model

We now assume a  $\beta$  model for the baryonic matter profile, which is usually adopted to fit the intracluster gas distribution (Arnaud 2009):

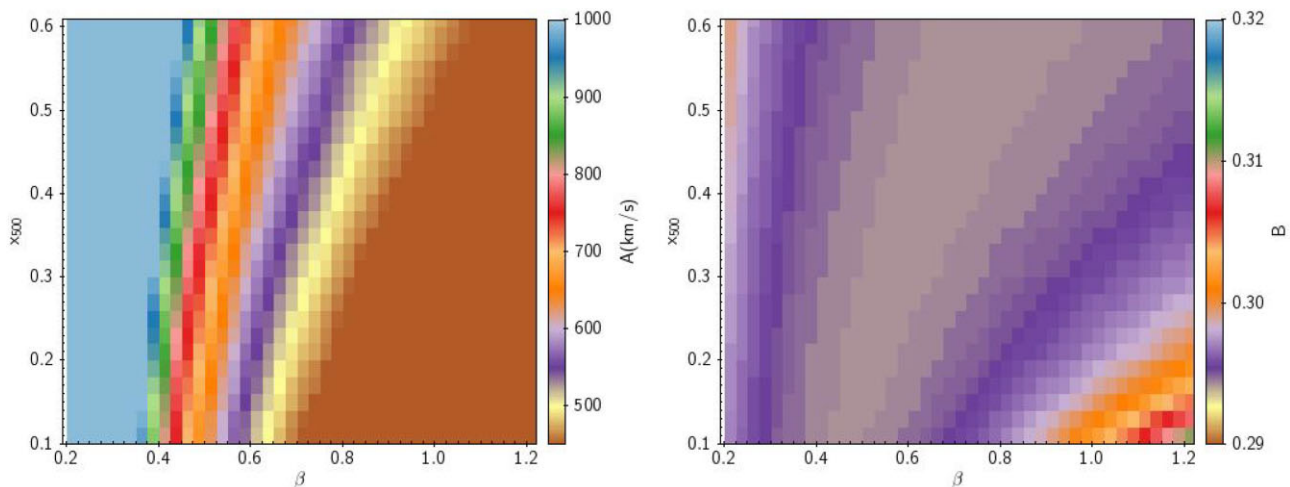
$$J(x) = \frac{1}{(1+x^2)^{1.5\beta}}, \quad (21)$$

where  $x = r/r_c$ .

Following the formulae of Appendix C, for  $r_{\text{max}} = r_{200}$ , applying the virial theorem equation (7), with the pressure correction referred at equation (10) [we obtain  $F_p$  given by equation (20), independent of the mass], we get a dependence that is fitted in the range  $M_{500} = (1-10) \times 10^{14} M_{\odot}$  with high accuracy by

$$\sigma_{v,r_{200},\text{MOND}} \approx A(\beta, x_{500}) \left( \frac{M_{500}}{10^{14} M_{\odot}} \right)^{B(\beta, x_{500})}, \quad (22)$$

where  $A(\beta, x_{500})$  and  $B(\beta, x_{500})$  are plotted at Fig. 3. The exponent  $B$



**Figure 3.** Values of  $A$ ,  $B$  in equation (22) as a function of  $\beta$  and  $x_{500}$  such that the cluster core radius  $r_c = x_{500} r_{500}$  in a  $\beta$ -model of the baryonic matter + MOND.

is almost constant, between 0.29 and 0.30 for most of the cases. The amplitude  $A$  is however quite dependent on the parameters of the  $\beta$  model.

For the usual values of  $\beta = 2/3$ ,  $x_{500} = 0.15$  (Pacaud et al. 2016),  $A = 493 \text{ km s}^{-1}$ ,  $B = 0.295$ . The pressure factor correction is  $F_p = 1.357$ . For comparison, the prediction of the velocity dispersion with only baryonic matter following  $\beta$ -model profile and Newtonian gravitation [equivalent to substituting 15.57 for 0 within the root square inside the integral of equation (17)], for the same parameters  $\beta = 2/3$ ,  $x_{500} = 0.15$ , and the same range of masses is  $A = 206 \text{ km s}^{-1}$ ,  $B = 0.387$ . Clearly, the effect of MOND is quite significant. It is almost enough to compensate for the absence of non-baryonic DM: it is 5–15 per cent (depending on the mass) lower than  $\sigma_v$  for Newton + DM for  $C = 3.0$ .

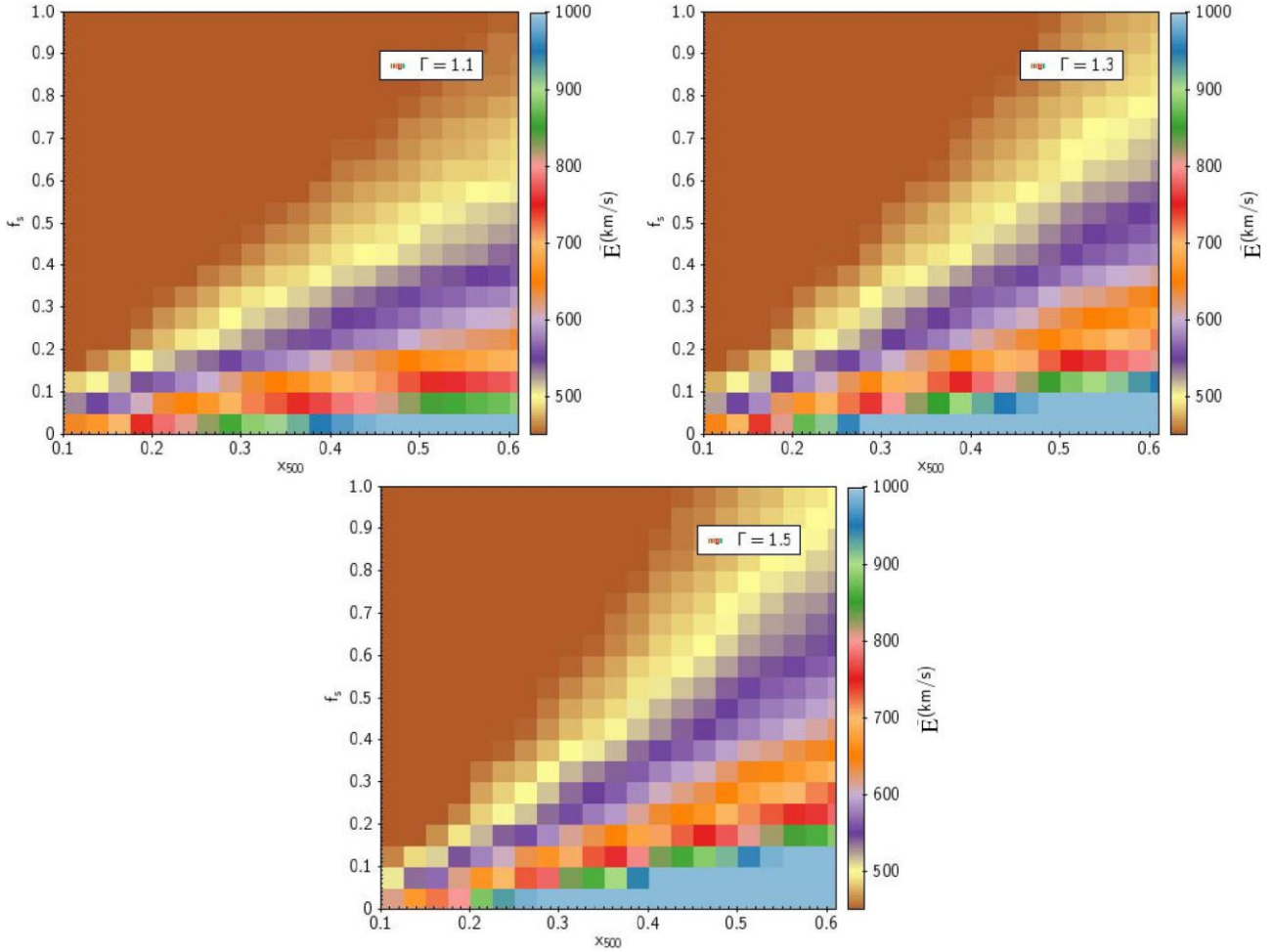
For other values of the parameters  $\beta$ ,  $x_{500}$  can also approximately reproduce the dynamics of Newton + DM. Values of  $A$  similar to the amplitude of Newton + DM ( $= 522 \text{ km s}^{-1}$ ) are obtained in the yellow–violet colour area of the left plot of Fig. 3:  $\beta$  between 0.55 and 0.80. For reasonable values of  $0.1 < x_{500} < 0.3$  [which lead through equation (19) to  $0.12 < r_c < 0.36 \text{ Mpc}$  for an average mass of  $M_{500} = 5 \times 10^{14} M_{\odot}$ ; of the order of  $r_c = 0.25 \text{ Mpc}$  given by Jones & Forman (1984)], the values of  $\beta$  are constrained between 0.55 and 0.70 in order to match the Newtonian amplitude. In Fig. 5, we plot the dispersion of velocities for the parameters  $\beta = 0.65$ ,  $x_{500} = 0.3$ , which are very close to the Newton + DM results, giving  $A = 553 \text{ km s}^{-1}$ ,  $B = 0.294$ ,  $F_p = 1.448$  (note that the pressure factor here is 16 per cent higher than with Newton). The values of  $\beta$  in the literature (e.g. Bahcall & Lubin 1994; Henning et al. 2009) are of the same order, between 0.50 and 0.65 for rich clusters.

#### 2.4.2 The Patej & Loeb (2015) model

The isothermal  $\beta$ -model is known to be insufficient for characterizing the range of cluster gas distributions (Vikhlinin et al. 2006; Patej & Loeb 2015). Other profiles could be used that give a better fit to the gas distribution. Here, we use the one given by Patej & Loeb (2015):

$$\rho_{\text{bar.}}(r) = \Gamma f_g \left( \frac{r}{s} \right)^{3\Gamma-3} \rho_{\text{DM}} \left[ s \left( \frac{r}{s} \right)^{\Gamma} \right], \quad (23)$$

where  $f_g$  is the fraction of gas with respect the total (in Newton + DM), i.e.  $f_g = \frac{M_{\text{bar},500}}{M_{500}}$ ;  $\rho_{\text{DM}}(r)$  is the profile of the total mass including DM, in our case given by the NFW profile (see Section B)



**Figure 4.** Values of  $E$  in equation (25) as a function of  $\Gamma$ ,  $x_{500}$ , and  $f_s$  in the Patej & Loeb (2015) model of the baryonic matter + MOND.

with scale  $r_s$  ( $= \frac{r_{200}}{C}$ , with concentration index  $C$ ), and  $\Gamma$ ,  $s$  are two extra free parameters. For  $\Gamma = 1$ , we would have that the baryonic mass traces the DM [ $\rho_{\text{bar.}}(r) = f_g \rho_{\text{DM}}(r)$ ]. Like the  $\beta$  model, the above expression is also motivated on a theoretical basis within standard Newtonian gravity. Here, with MOND, we use it because it simply fits the observational profile of gas in clusters of galaxies as a function that describes baryonic matter, and the theoretical derivation would have no sense.

This gives a

$$J(x) = \frac{x^{2\Gamma-3}}{(1 + f_s x^\Gamma)^2},$$

$$f_s = \frac{s}{r_s}, \quad r_c = s, \quad (24)$$

and applying the virial theorem from equations (7) with  $V$  and  $K$  from equations (17) and (18), we get a dependence that is fitted in the range  $M_{500} = (1-10) \times 10^{14} M_\odot$  with high accuracy by

$$\sigma_{v,r_{200},\text{MOND}} \approx E(\Gamma, x_{500}, f_s) \left( \frac{M_{500}}{10^{14} M_\odot} \right)^{D(\Gamma, x_{500}, f_s)}. \quad (25)$$

For the range  $1 < \Gamma \leq 2$ ,  $0 < x_{500} \leq 1$ ,  $0 < f_s \leq 0.4$ , the exponent  $D(\Gamma, x_{500}, f_s)$  falls always in the range between 0.29 and 0.34 in approximate agreement with Newton + DM. The amplitude  $E(\Gamma, x_{500}, f_s)$  is plotted at Fig. 4 for  $\Gamma = 1.1, 1.3, 1.5$ , which are within the constraints obtained by Patej & Loeb (2015) of  $1 < \Gamma \leq 1.5$ . There

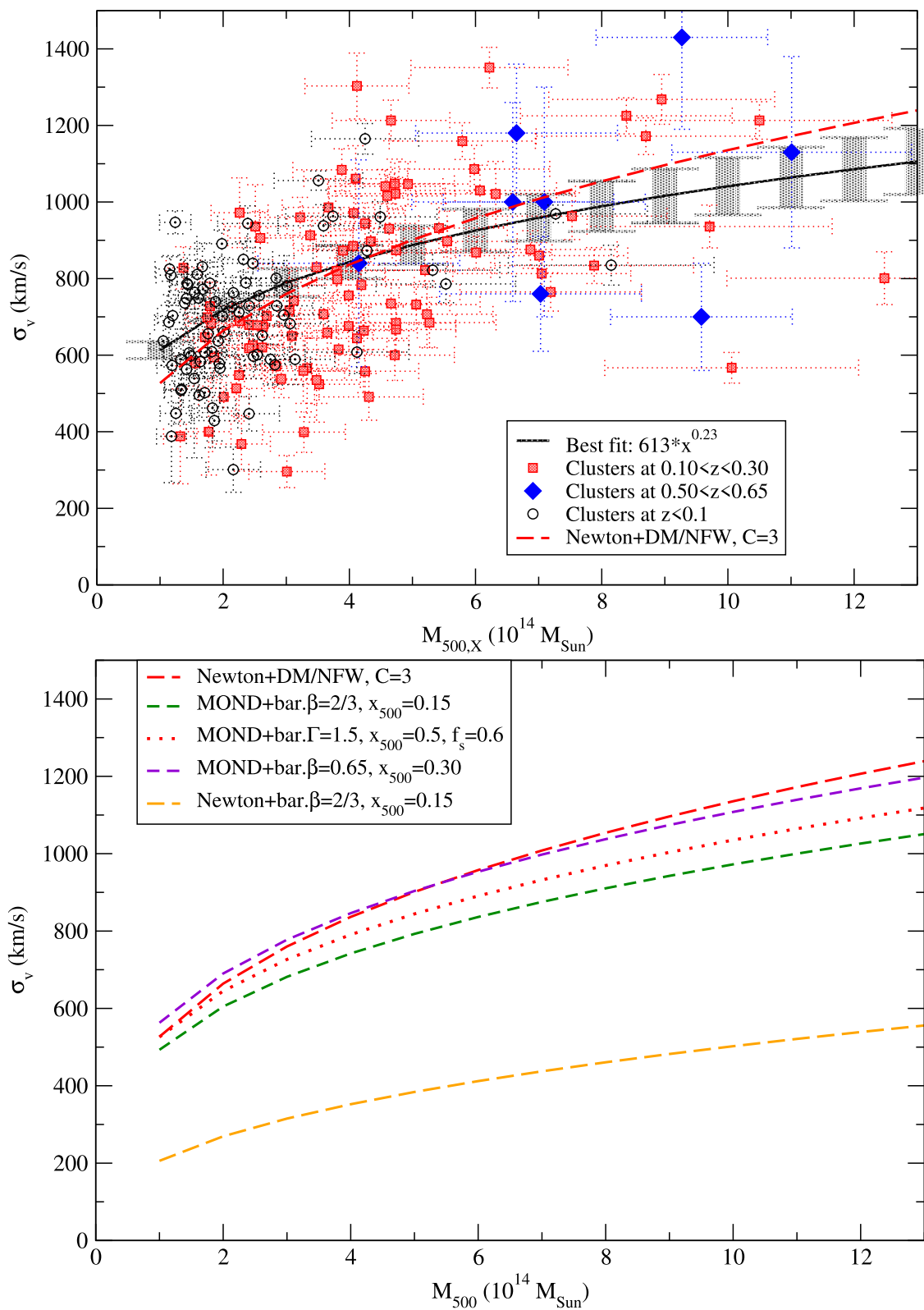
is a wide range of possible values compatible with Newton + DM. For instance, if we assume an average value of  $\Gamma = 1.5$ ,  $x_{500} = 0.5$  (hence,  $s = 0.5 r_{500}$ ),  $E = 646 \text{ km s}^{-1}$  implies  $f_s \approx 0.6$  (hence,  $r_s \sim 0.8 r_{500}$ , equivalent to a concentration index  $C \sim 2$ ).

## 2.5 Comparison with observations

Estimates of masses and velocity dispersions carried out by other teams for some clusters are shown in Fig. 5.

At low ( $z < 0.10$ ) and intermediate ( $0.10 < z < 0.30$ ) redshifts, we use velocity dispersion data within  $r_{200}$  from clusters of Sohn et al. (2020, table 2), including the mass  $M_{500}$ , within this table estimated from X-ray observations by Piffaretti et al. (2011). We use only the clusters with  $M_{500} \geq 10^{14} M_\odot$ . These comprise 74 clusters with  $z < 0.10$  and 96 clusters with  $0.10 < z < 0.30$ . The  $M_{500}$  errors are not provided; here, we assume they have a 20 per cent of error, which is typical of other estimates of X-ray masses (Vikhlinin et al. 2006; Walker et al. 2012; Martino et al. 2014; Haines et al. 2018; Whelan et al. 2022).

X-ray data for  $M_{500}$ ,  $\sigma_v$ , error ( $\sigma_v$  are obtained for eight high-redshift clusters ( $0.50 < z < 0.65$ ) from the NIKA2 cluster survey (Mayet et al. 2020): see Table 1. Rest-frame velocity dispersions were calculated using public Sloan Digital Sky Survey (SDSS) data of galaxies' velocities and applying a biweight technique (Beers, Flynn & Gebhardt 1990). X-ray masses were derived from REFLEX



**Figure 5.** Upper panel: Rest-frame dispersion of velocities along the line of sight as a function of the  $M_{500}$  mass in 178 clusters of galaxies.  $M_{500}$  represents the total mass within the radius  $r_{500}$  for Newtonian gravity + DM, whereas in MOND, it is related to the total (baryonic) mass within  $r_{500}$  through  $M_{\text{bar},500} = 0.117 \times 10^{14} \times \left(\frac{M_{500}}{10^{14} M_{\odot}}\right)^{1.16} M_{\odot}$ . The lines represent the power-law fit (the shaded area covers the range within a  $1\text{-}\sigma$  error of the fit), or the predictions with the standard Newtonian gravity + non-baryonic DM following a NFW profile. Bottom panel: predictions with the standard Newtonian gravity + non-baryonic DM following a NFW profile or MOND for different baryonic matter profiles, or Newtonian gravity with only baryonic matter.

**Table 1.** Clusters at high redshift,  $z > 0.5$ , used for our analysis, ordered by increasing redshift. The third column indicates the number of galaxies used for the measurement of the rest-frame velocity dispersion (column 5). Column 4 gives the estimated masses using either X-rays.

Name	Redshift	Nr. of galaxies	$M_{500}$ ( $10^{14} M_{\odot}$ )	$\sigma_v$
PSZ2 G211.21+38.66	0.503	25	$7.0 \pm 1.6$	$760 \pm 150$
PSZ2 G212.44+63.19	0.532	15	$4.2 \pm 1.6$	$840 \pm 270$
PSZ2 G201.50-27.31	0.534	47	$9.3 \pm 1.4$	$1430 \pm 240$
PSZ2 G094.56+51.03	0.541	55	$6.6 \pm 1.6$	$1180 \pm 180$
PSZ2 G228.16+75.20	0.542	26	$11.0 \pm 1.9$	$1130 \pm 250$
PSZ2 G111.61-45.71	0.547	30	$9.6 \pm 1.4$	$700 \pm 140$
PSZ2 G183.90+42.99	0.559	21	$6.6 \pm 1.6$	$1000 \pm 260$
PSZ2 G099.86+58.45	0.618	13	$7.1 \pm 1.6$	$1000 \pm 300$

(Böhringer et al. 2004) and REXCESS (Böhringer et al. 2007) cluster surveys applying the method by Arnaud et al. (2010), whose error bars are estimated with the relative error bar of the X-ray luminosities (when available, or the average value of similar clusters of this sample otherwise).

Note that the values of  $M_{500}$  from X-ray data correspond always to the estimations using the standard model Newton + DM. As it was remarked throughout the paper, in a MOND model, it would be approximately related to the total (baryonic) mass within  $r_{500}$  through  $M_{\text{bar},500} = 0.117 \times 10^{14} \times \left(\frac{M_{500}}{10^{14} M_{\odot}}\right)^{1.16} M_{\odot}$ .

In cases with small numbers of galaxies, there may be some important biases in the galaxy cluster velocity dispersion (Ferragamo et al. 2020). Here, we do not introduce any correction to take them into account, since the number of galaxies per cluster is high enough and the corrections of the statistics for small numbers are negligible. For the comparison with the theoretical predictions, we also assume that the r.m.s. of  $\sigma_v$  is much smaller than its average value within  $r_{200}$ , as is usually the case (Ferragamo et al. 2020). The observed velocities may be slightly different from the average because the average line-of-sight velocities were measured within a radius smaller than  $r_{200}$ , and our approximation of almost constant dispersion of velocities with radius might introduce some higher values of dispersion in the observations than in the theory. We assume that these differences are lower than the error bars.

Other effects could produce a few small systematics (Krížek, Krížek & Somer 2014): relativistic effects of high velocities, gravitational redshift, and gravitational lensing in a curved space, which would decrease the Hubble–Lemaître parameter, intergalactic baryonic matter, gravitational aberration, etc.

In Fig. 5, we see that the points for 178 clusters are close to the predictions of virial theorem within a virial radius of  $r_{200}$  for Newtonian gravity, or for MOND with some parameters. The best power-law weighted fit (taking into account both the errors of masses and velocities) is:

$$\sigma_{v,\text{best fit data}} = (613 \pm 22) \left(\frac{M_{500}}{10^{14} M_{\odot}}\right)^{0.230 \pm 0.027} \text{ km s}^{-1} \quad (26)$$

The data present a correlation between  $X = \ln\left(\frac{M_{500}}{10^{14} M_{\odot}}\right)$  and  $Y = \ln\left(\frac{\sigma_v}{\text{km s}^{-1}}\right)$ , including the same weighting factors:

$$C \pm \Delta C = \left(\frac{\langle XY \rangle}{\langle X \rangle \langle Y \rangle} - 1\right) \pm \left(\frac{\sigma_X \sigma_Y}{\sqrt{N} \langle X \rangle \langle Y \rangle}\right) \\ = (9.3 \pm 1.3) \times 10^{-3}, \quad (27)$$

a correlation at  $7.2\sigma$  level. This sigma-level does not strictly correspond to a Gaussian distribution, but practically indistinguishable from a Gaussian one. Perhaps the complement to one of the

confidence level might be somewhat larger than the seven sigmas Gaussian one ( $\sim 10^{-12}$ ), but it may certainly be said that in the present case, the null hypotheses may be rejected with a confidence level larger than 99.99 per cent. The estimator of the correlation coefficient is a sum of some 178 terms. Each of these terms is the product of two Gaussian variables, assuming that the errors of both mass and velocity dispersion conform to Gaussian statistics. Therefore, each of these terms are random variables following a Rayleigh distribution. The Rayleigh distribution is somewhat more extended than a Gaussian, but the central limit theorem assures that the sum of many variable following that distribution, or any distribution with well defined mean and variance tends to a Gaussian. In fact, the sum of only four of them is already quite close to a Gaussian, although not in the farthest positions of the wings.

For the respective redshift ranges of  $z$  (low:  $< 0.10$ ; intermedite:  $0.10 \leq z < 0.30$ ; high:  $\geq 0.30$ ), we get

$$\sigma_{v,\text{best fit data}, z < 0.10} = (636 \pm 28) \left(\frac{M_{500}}{10^{14} M_{\odot}}\right)^{0.203 \pm 0.045} \text{ km s}^{-1}, \quad (28)$$

$$\sigma_{v,\text{best fit data}, 0.10 \leq z < 0.30} = (550 \pm 43) \left(\frac{M_{500}}{10^{14} M_{\odot}}\right)^{0.294 \pm 0.050} \text{ km s}^{-1}, \quad (29)$$

$$\sigma_{v,\text{best fit data}, z \geq 0.50} = (690 \pm 560) \left(\frac{M_{500}}{10^{14} M_{\odot}}\right)^{0.19 \pm 0.39} \text{ km s}^{-1}. \quad (30)$$

There is no significant difference in the trend between low- and high-redshift clusters.

### 3 DISCUSSION AND CONCLUSIONS

The relationship between velocity dispersion and masses in clusters was known to work properly within Newton + DM (e.g. Evrard et al. 2008; Zhang et al. 2011; Munari et al. 2013), and also in the case of some modifications of gravity different from MOND without including DM (e.g. Brownstein & Moffat 2006). However, they did not work in MOND (Sanders 1999; Pointecouteau & Silk 2005; Ettori et al. 2019). We explored here the reason for this inconsistency and make major improvements in the application of the virial theorem. In particular, our virial theorem analytical relationship of velocity dispersion in galaxies with given mass profiles includes a pressure (surface) term, which, although its relevance is recognized in some literature (e.g. The & White 1986; Carlberg et al. 1996, 1997; Girardi et al. 1998) is not usually considered in analytical calculations, although it is implicitly taken into account when carrying out

numerical simulations. We also applied an updated calibration of non-baryonic mass in the NFW profile, baryonic ratio, and baryonic profiles, either with an isothermal  $\beta$  model or a Patej & Loeb (2015) model.

Our results show that we can reconcile MOND with the virial theorem in clusters. This agreement is obtained when: 1) the pressure term is taken into account in the virial theorem, which gives a 10–15 per cent higher velocity dispersion for MOND than for Newton + DM; 2) we explore a range of possible parameters in the baryonic matter profile rather than adopting a fixed one. In particular for MOND, we predict velocity dispersions equivalent to Newton + DM by adopting a  $\beta$  model with  $\beta = 0.55$ – $0.70$ , and core radii  $r_c < 0.30r_{500}$ , which is in agreement with the known data. Lower concentration favours a higher MOND effect, so  $x_{500} = \frac{r_c}{r_{500}} = 0.3$  increases the dispersion of velocities by a factor 10–15 per cent with respect to  $x_{500} = 0.15$  for the same  $\beta \approx 2/3$ ; decreasing  $\beta$  with  $x_{500} = 0.15$  also decreases the concentration and produces similar results. This last effect is easy to understand in MOND since lower concentrations enhance the MOND effect because the galaxies spend a longer time during their orbits in the MOND regime of low ( $<a_0$ ) accelerations. Also, the greater pressure term for MOND is due to a lower concentration of baryons than DM. Calculations without pressure and with default parameters ( $x_{500}$  much lower than 0.3 and  $\beta = 2/3$ ) would give a  $\sigma_v$  15–25 per cent lower than Newton + DM. Given that the dynamical mass is proportional to  $\sigma_v^3$ , this means dynamical masses 40–60 per cent lower, and this would explain the discrepancies found in previous studies.

MOND in the regime of very low accelerations creates a field [‘phantom mass’; (Milgrom 1986, 2009; Wu & Kroupa 2015; López-Corredoira & Betancort-Rijo 2021)] which has an effect dynamically similar to the presence of non-baryonic DM in Newtonian gravity. Here, we observe that MOND fits the predictions of the virial theorem in rich clusters of galaxies, which should not be surprising, given that the MOND phantom mass effect is equivalent to the non-baryonic DM. If some inconsistency arises, a revision of our knowledge of the distribution of baryons would be needed because, with appropriate profiles and calibrations of  $\frac{M_{\text{bar}}}{M_{\text{total}}}$ , there is always a mathematical solution able to mimic non-baryonic DM.

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## DATA AVAILABILITY

The data forming the basis of this article are available in the article or given references.

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## APPENDIX A: PRESSURE (SURFACE) TERM IN THE VIRIAL THEOREM

When the virial theorem is applied to a portion of a stable gravitating system, it takes a different form when it is applied to the whole system. In the latter case, the pressure term, which is present in general cancels (asymptotically), and the familiar result of  $2K + V = 0$  [equation (7)] holds, where  $K$  is the kinetic energy and  $V$  is the potential energy. However, the theorem is usually applied to the inner parts of a more extended system. This is the case, for instance, when the entities forming the objects (i.e. galaxies in the case of cluster of galaxies) are increasingly more difficult to discriminate from the interlopers. The relevance of this pressure (surface) term in standard gravity is known (e.g. The & White 1986; Carlberg et al. 1996, 1997; Girardi et al. 1998).

For the general form, we must use the general scalar virial theorem, which takes the form

$$\sum_i \vec{F}_i \vec{r}_i = \sum_i m_i |\vec{v}_i|^2, \quad (\text{A1})$$

where  $\vec{F}_i$  is the force acting on  $i$ -th particle and  $\vec{r}_i$  its position vector. The sum  $i$  extends to all particles. The right hand side of the equation is  $2K$ , while the left hand side is equal to  $-V$  plus the ‘pressure term’. This last term appears because the force  $\vec{F}_i$  acting on the  $i$ -th particle is due, not only to the gravitational fields but also to pressure for particles at the boundary of the system.

The contribution of this term is simply  $3P_r V_r$ , where  $P_r$  is the radial pressure at the boundary of the virial radius  $r_{\max}$ , assumed to be constant over it; and  $V_r$  is the volume within it. Assuming a stationary system with null average velocity, for which

$$\langle v_i^2 \rangle = 3\sigma_v^2, \quad (\text{A2})$$

where  $\sigma_v$  is the rms of the velocities along the line of sight. Hence, after dividing by the total mass, the virial theorem reads

$$\sigma_{v,r < r_{\max}}^2 = \sigma_{v,r < r_{\max}, P=0}^2 + \frac{P_r(r_{\max}) \frac{4}{3} \pi r_{\max}^3}{M(r_{\max})},$$

$$\sigma_{v,r < r_{\max}, P=0}^2 = -\frac{V(r_{\max})}{3M(r_{\max})}. \quad (\text{A3})$$

For isotropic pressure,

$$P_r(r_{\max}) = \sigma_{v,r=r_{\max}}^2 \rho(r_{\max}). \quad (\text{A4})$$

For a more general case, when the pressure is not isotropic,

$$P_r(r_{\max}) = \frac{3}{3 - 2\beta_a} \sigma_{v,r=r_{\max}}^2 \rho(r_{\max}), \quad (\text{A5})$$

where  $\beta_a$  is the anisotropy parameter:

$$\beta_a = 1 - \frac{\sigma_\theta^2}{\sigma_r^2},$$

$$3\sigma_v^2 = \sigma_r^2 + 2\sigma_\theta^2, \quad (\text{A6})$$

and where we have used the fact that for spherical systems  $\sigma_\theta^2 = \sigma_\phi^2$  ( $r, \theta, \phi$  denote radial, declinational, and azimuthal directions).

## APPENDIX B: THE NFW PROFILE

Assuming a critical density of  $\rho_c = 8.5 \times 10^{-27} \text{ kg m}^{-3}$  (for  $H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ), a virial radius equal to  $r_{200}$  and concentration index  $C$ , the Navarro–Frenk–White (NFW, Navarro et al. (1997)) profile follows these relationships for the mass density  $\rho(r)$  and the mass within the sphere of radius  $r$   $M(r)$ :

$$M(r) = M_{200} \times \frac{\left[ \ln \left( 1 + \frac{C r}{r_{200}} \right) - \frac{C r}{r_{200} + C r} \right]}{\left[ \ln(1 + C) - \frac{C}{1 + C} \right]^{1/3}},$$

$$r_{200} = 0.9834 \text{ Mpc} \times \left( \frac{M_{200}}{10^{14} M_\odot} \right)^{1/3},$$

$$r_{500} = 0.7246 \text{ Mpc} \times \left( \frac{M_{500}}{10^{14} M_\odot} \right)^{1/3},$$

$$\rho(r) = \frac{\rho_0}{\frac{C r}{r_{200}} \left( 1 + \frac{C r}{r_{200}} \right)^2},$$

$$\rho_0 = 8.368 \times 10^{12} \frac{C^3}{\left[ \ln(1 + C) - \frac{C}{1 + C} \right]} M_\odot \text{ Mpc}^{-3}, \quad (\text{B1})$$

where  $M_x \equiv M(r_x)$  and  $r_x$  is the radius of the sphere for which the average density inside it is  $x$  times the critical density  $\rho_c$ . That is,  $x \rho_c = \frac{M(r_x)}{\frac{4}{3} \pi r_x^3}$ .

## APPENDIX C: PROFILE FOR BARYONIC MATTER

We assume a critical density of baryonic matter  $\rho_{\text{cb}} = \rho_c \frac{\Omega_b}{\Omega_m}$  with  $\rho_c = 8.5 \times 10^{-27} \text{ kg m}^{-3}$  (for  $H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ );  $\Omega_m = 0.315$ ,  $\Omega_b = 0.0493$  (Planck Collaboration 2020). For a mass density profile  $\rho_{\text{bar}}(r) = \rho_0 J \left( \frac{r}{r_c} \right)$ , the mass within the sphere of radius  $r$  is:

$$M_{\text{bar}}(r) = \frac{M_{\text{bar},500}}{I(r_{500})} I(r),$$

$$I(r) = \int_0^{r/r_c} dx x^2 J(x),$$

$$r_{200} = 1.357 r_{500} \left( \frac{I(r_{200})}{I(r_{500})} \right)^{1/3},$$

$$r_{500} = 1.345 \text{ Mpc} \times \left( \frac{M_{\text{bar},500}}{10^{14} M_\odot} \right)^{1/3}$$

$$\rho_0 = \frac{M_{\text{bar},500}}{4\pi r_c^3 I(r_{500})}. \quad (\text{C1})$$

where  $M_{\text{bar},500} \equiv M_{\text{bar}}(r_{500})$  and  $r_x$  is the radius of the sphere for which the average density inside it is  $x$  times the critical baryonic density  $\rho_{\text{cb}}$ . The value of  $r_{200}$  is solved iteratively.

An isothermal  $\beta$  model is usual in the description of gas in clusters of galaxies (Cavaliere & Fusco-Femiano 1997), where

$J(x) = \frac{1}{(1+x^2)^{1.5\beta}}$ . For  $\beta = 2/3$ , which is usual the assumed value (Arnaud 2009),  $I(r)$  has an analytical solution:  $I(r) [\beta = 2/3] = \frac{r}{r_c} - \tan^{-1} \left( \frac{r}{r_c} \right)$ . A value of a cluster core radius scale equal to  $r_c \sim 0.25$  Mpc is expected (Jones & Forman 1984); although a dependence on the mass is also expected.

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