

THE FIRST EIGENVALUE OF THE LAPLACIAN ON ORIENTABLE SURFACES

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ABSTRACT. The famous Yang-Yau inequality provides an upper bound for the first eigenvalue of the Laplacian on an orientable Riemannian surface solely in terms of its genus γ and the area. Its proof relies on the existence of holomorphic maps to $\mathbb{C}\mathbb{P}^1$ of low degree. Very recently, Ros was able to use certain holomorphic maps to $\mathbb{C}\mathbb{P}^2$ in order to give a quantitative improvement of the Yang-Yau inequality for $\gamma = 3$. In the present paper, we generalize Ros' argument to make use of holomorphic maps to $\mathbb{C}\mathbb{P}^n$ for any $n > 0$. As an application, we obtain a quantitative improvement of the Yang-Yau inequality for all genera except for $\gamma = 4, 6, 8, 10, 14$.

1. INTRODUCTION

1.1. **Yang-Yau inequality.** Let (Σ, g) be a closed orientable Riemannian surface. The Laplace-Beltrami operator, or Laplacian, is defined as $\Delta_g = \delta_g d$, where δ_g is the formal adjoint of the differential d . For closed manifolds the spectrum of Δ_g consists only of eigenvalues and forms the following sequence

$$0 = \lambda_0(\Sigma, g) < \lambda_1(\Sigma, g) \leq \lambda_2(\Sigma, g) \leq \dots \nearrow \infty,$$

where eigenvalues are written with multiplicities.

Consider the normalized eigenvalues

$$\bar{\lambda}_k(\Sigma, g) = \lambda_k(\Sigma, g) \text{Area}(\Sigma, g).$$

The problem of geometric optimization of eigenvalues consists in determining the exact values of the following quantities

$$\Lambda_k(\Sigma) = \sup_g \bar{\lambda}_k(\Sigma, g).$$

We refer to [KNPP, K3] and references therein for a detailed survey of recent developments on the problem.

In the present paper we focus on the case $k = 1$. The first general upper bound on $\Lambda_1(\Sigma)$ was obtained by Yang and Yau in [YY] who proved that if Σ has genus γ , then

$$\Lambda_1(\Sigma) \leq 8\pi(\gamma + 1)$$

However, it was soon remarked in [ESI] that the same proof yields the following improved bound

$$(1.1) \quad \Lambda_1(\Sigma) \leq 8\pi \left\lfloor \frac{\gamma + 3}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ is the floor function or the integer part of x . In the following we refer to (1.1) as the Yang-Yau inequality. Since the original paper [YY], alternative proofs of (1.1) have appeared in [BLY, LY].

According to the results of Hersch [H] and Nayatani-Shoda [NS], the Yang-Yau inequality is sharp for $\gamma = 0$ and $\gamma = 2$ respectively. Apart from that, the exact value of $\Lambda_1(\mathbb{T}^2)$ was computed by Nadirashvili in [N1],

$$\Lambda_1(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}.$$

At the same time, it is known that for $\gamma \neq 0, 2$ the Yang-Yau inequality is strict, see [K2]. The existence of metrics achieving $\Lambda_1(\Sigma)$ has been recently established in [MS].

Remark 1.1. The problem of determining $\Lambda_1(\Sigma)$ makes sense for non-orientable Σ , see [K1] for a generalization of (1.1) to this setting. We refer to [LY] and [EGJ, JNP, CKM] for the exact values of Λ_1 on the projective plane and Klein bottle respectively. The existence result of [MS] continues to hold for non-orientable surfaces.

Finally, in a recent paper [R1] Ros obtained a quantitative improvement of (1.1) for $\gamma = 3$. Namely, he proved that if Σ is an orientable surface of genus 3, then

$$\Lambda_1(\Sigma) \leq 16(4 - \sqrt{7})\pi \approx 21.668\pi < 24\pi,$$

where 24π is the bound given by the Yang-Yau inequality. The results of the present paper are heavily inspired by the work of Ros and are essentially a generalization of [R1] to higher genera.

1.2. Main results. Our main result can be stated as follows.

Theorem 1.2. *Let Σ_γ be a compact orientable surface of genus γ . Then one has*

$$(1.2) \quad \Lambda_1(\Sigma_\gamma) \leq \frac{2\pi}{13 - \sqrt{15}} \left(\gamma + (33 - 4\sqrt{15}) \left\lceil \frac{5\gamma}{6} \right\rceil + 4(41 - 5\sqrt{15}) \right),$$

where $\lceil \cdot \rceil$ is the ceiling function. In particular,

$$\Lambda_1(\Sigma_\gamma) < 8\pi(0.43\gamma + 2.86).$$

We show in Lemma 3.6 that the bound (1.2) is an improvement over the Yang-Yau inequality (1.1) as soon as $\gamma \geq 25$. Furthermore, it is easy to compute the asymptotic behaviour of the r.h.s in (1.2). We formulate the corresponding result as follows.

Corollary 1.3. *Let Σ_γ be a compact orientable surface of genus γ . Then one has*

$$(1.3) \quad \limsup_{\gamma \rightarrow \infty} \frac{\Lambda_1(\Sigma_\gamma)}{8\pi\gamma} \leq \frac{5}{6} - \frac{89}{6(52 - 4\sqrt{15})} \approx 0.42703.$$

At the same time, the Yang-Yau inequality only yields $\frac{1}{2}$ in the r.h.s of (1.3).

Finally, we remark that the bound (1.2) is a particular member of a family of inequalities proved in Proposition 3.1. We choose to state Theorem 1.2 in its present form due to the fact that (1.2) is the best bound in the family for $\gamma \geq 102$ and, thus, yields the best constant in (1.3). However, for small γ , other members of the family yield a better bound. In particular, this approach gives a quantitative improvement over the Yang-Yau inequality for all γ except for $\gamma = 4, 6, 8, 10, 14$. We refer to Section 3 and Table 1 for more details.

1.3. Sketch of the proof. The proof is based on a particular construction of a balanced map from the surface Σ to the Euclidean sphere. Such maps are commonly used in the geometric optimization of eigenvalues, we refer e.g. to [BLY, H, KS, LY, N2] for applications to various problems. The coordinates of balanced maps are good test-functions for λ_1 provided the energy of the map is controlled. For example, in [BLY] the authors consider a particular minimal isometric embedding $A: \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{S}^{(n+1)^2-2}$. Precomposing A with a full holomorphic map $f: \Sigma \rightarrow \mathbb{C}\mathbb{P}^n$ gives the map $A_f = A \circ f$ to a sphere, whose energy is controlled by the degree of f . Finally, they use linear transformations on $\mathbb{C}\mathbb{P}^n$ to arrange A_f to be balanced. Among other things, this construction can be used to prove (1.1). In [R1] Ros builds up on this construction by considering a perturbation $\phi_a = A_f + aH_f$, where H_f is the mean curvature vector of A_f and $a \in \mathbb{R}$ is a parameter. He then takes $f: \Sigma \rightarrow \mathbb{C}\mathbb{P}^2$ to be the quartic realization of any non-hyperelliptic genus 3 surface Σ and argues that for certain range of a the map ϕ_a can still be arranged to be balanced. Finally, it turns out that the parameter a can be chosen so that the energy $E(\phi_a) < E(\phi_0)$, thus, yielding an improvement over (1.1).

We generalize the argument of Ros by considering f to be a full map induced by an arbitrary linear system of dimension n . The Brill-Noether theory provides an existence of such linear systems with degree bounded in terms of the genus of Σ . This allows to extend Ros' approach to surfaces of arbitrary genus. An additional upside is that one is free to choose n . In fact, $n = 5$ turns out to be optimal for large γ and corresponds to the bound (1.2). A novel feature compared to [R1] is that our maps f could have branch points. We show that the contribution of the branch points to the energy $E(\phi_a)$ has a negative sign and, thus, could be discarded in the proof of upper bound.

In conclusion, let us provide some geometric intuition for the definition of ϕ_a . Since the goal is to minimize the energy of a balanced map, it is natural to deform the map in the direction of the negative gradient of the energy. Since A_f is a conformal map, this gradient coincides with the mean curvature vector. Thus, the correspondence $A_f \mapsto \phi_a$ can be seen as one step of the discrete harmonic map heat flow or, equivalently, mean curvature flow. The surprising part of the calculation in [R1] is that the energy of ϕ_a only depends on a , genus γ and the degree of f . It would be interesting to see if this method can be refined further by considering two steps of the discrete flow or a continuous flow.

Organization of the paper. In section 2 we prove a general upper bound for the first eigenvalue in terms of a holomorphic map to $\mathbb{C}\mathbb{P}^n$, see Theorem 2.7. The content of this section is to the large extent a direct generalization of the results in [R1]. Section 3 is devoted to investigating the bound obtained in Theorem 2.7. Namely, we show that taking $n = 5$ yields the best bound for large γ and show that in this case the bound reduces to (1.2). Table 1 contains the optimal values of n for small γ . The section is elementary, but computationally heavy.

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2. EIGENVALUE BOUNDS FROM HOLOMORPHIC MAPS TO $\mathbb{C}\mathbb{P}^n$

In the following (Σ, g) denotes an orientable Riemannian surface of genus γ . The metric g (together with a choice of an orientation) induces a complex structure Σ and, thus, we view Σ as a Riemann surface throughout this section. The complex structure only depends on the conformal class $[g] = \{e^{2\omega}g, \omega \in C^\infty(M)\}$. Furthermore, our first result, Theorem 2.7 gives a bound on

$$\Lambda_1(\Sigma, [g]) := \sup_{h \in [g]} \bar{\lambda}_1(\Sigma, g).$$

The study of $\Lambda_1(\Sigma, [g])$ is of independent interest due to its connection with the theory of harmonic maps, see e.g. [KNPP].

2.1. Branched holomorphic curves in complex projective spaces.

Let $f: \Sigma \rightarrow \mathbb{C}\mathbb{P}^n$ be a holomorphic map. One can define the order of the differential at a point p denoted by $\text{ord}_p df$ as the order of the vector function f'_z , where $df = f'_z dz$ in some local coordinates on Σ and $\mathbb{C}\mathbb{P}^n$. The total ramification of f is defined as

$$(2.1) \quad \beta = \sum_p \text{ord}_p df = \sum_p (\text{ord}_p f - 1).$$

If it happens to be nonzero, the metric $h = f^*g_{FS}$ induced by the Fubini-Study metric has conical singularities, i.e. in a local coordinate z centered at

p the metric is given by $h = |z|^{2k} \rho^2 dz \otimes d\bar{z}$, where $\rho(0) \neq 0$, $k = \text{ord}_p f'_z$. Here we assume that g_{FS} is normalized to have holomorphic sectional curvature 1. Denote its Gauss curvature by K and the volume form by dv_h . After some calculations, we have

$$K dv_h = 4\partial_z \partial_{\bar{z}} \log(|z|^k \rho) dx \wedge dy = 4\partial_z \partial_{\bar{z}} \log \rho dx \wedge dy,$$

which means the Gauss curvature is integrable. Hence, the following formula holds, see e.g. [F],

$$(2.2) \quad \frac{1}{2\pi} \int_{\Sigma} K dv_h = 2 - 2g + \beta.$$

The degree of the map $f: \Sigma \rightarrow \mathbb{C}P^n$ is defined as the positive integer d that corresponds to the integral homology class $f_*[\Sigma] \in H_2(\mathbb{C}P^n) \simeq \mathbb{Z}$. Let ω_{FS} be the Kähler 2-form associated with Fubiny-Study metric. Then $\frac{1}{4\pi} \omega_{FS}$ represents the canonical basis in integral cohomology $H^2(\mathbb{C}P^n)$. Since $f^*(\omega_{FS}) = dv_h$, we conclude that

$$(2.3) \quad \text{Area}(\Sigma, h) = 4\pi d.$$

We are interested in finding full holomorphic maps of relatively small degree. Recall that $f: \Sigma \rightarrow \mathbb{C}P^n$ is called *full* if its image is not contained in a hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$. For this purpose, we use Brill–Noether theory, see e.g. section Special Linear Systems IV in [GH], which assures the existence of a linear system of degree d' and dimension n as long as $\gamma \geq (n+1)(\gamma+n-d')$. Removing the base points if necessary, we obtain a holomorphic map $f: \Sigma \rightarrow \mathbb{C}P^n$ of degree $d \leq d'$. This implies the following lemma.

Lemma 2.1. *For any positive n there exists a full holomorphic map $f: \Sigma \rightarrow \mathbb{C}P^n$ of degree d such that*

$$d \leq \left\lceil \frac{n\gamma}{n+1} \right\rceil + n.$$

2.2. An embedding of $\mathbb{C}P^n$ into the Euclidean space. Denote by $HM_1(m)$ the space of Hermitian operators of trace 1 acting on \mathbb{C}^m , and consider the Euclidean metric

$$\langle A, B \rangle = 2 \text{tr} AB \quad \forall A, B \in HM_1(n+1).$$

Then $\mathbb{C}P^n$ can be considered as the space of all orthogonal projectors of rank 1 in $HM_1(n+1)$. Using the matrix notation, this embedding $A: \mathbb{C}P^n \rightarrow HM_1(n+1)$ is given by the formula

$$(2.4) \quad z \mapsto \frac{1}{z^* z} z z^*,$$

where z is a column of homogeneous coordinates and $z^* = \bar{z}^t$. This embedding is also $U(n+1)$ -equivariant, i.e. it intertwines the actions

$$(2.5) \quad (P, z) \mapsto Pz, \quad (P, A) \mapsto PAP^*$$

on $\mathbb{C}P^n$ and $HM_1(n+1)$ respectively.

In what follows, we will slightly abuse the notation and denote by the same letter A both the embedding $A: \mathbb{C}\mathbb{P}^n \rightarrow HM_1(n+1)$ and the points of $\mathbb{C}\mathbb{P}^n$ regarded as a subset of $HM_1(n+1)$. The map A has been studied in detail in [R2]. The following properties are of particular importance to us,

- (1) the map A is an isometry. Recall that $\mathbb{C}\mathbb{P}^n$ is endowed with the Fubini-Study metric of holomorphic section curvature 1;
- (2) the image of $\mathbb{C}\mathbb{P}^n$ is a minimal submanifold of the sphere $S^{(n+1)^2-2} \subset HM_1(n+1)$ centered at $\frac{1}{n+1}I$ with radius $\sqrt{\frac{2n}{n+1}}$, where I is the identity matrix.

Now let $f: \Sigma \rightarrow \mathbb{C}\mathbb{P}^n$ be a holomorphic curve. The composition of the previous embedding with f produces the map $A_f: \Sigma \rightarrow HM_1(n+1)$. The map A_f is an immersion on the complement $\overset{\circ}{\Sigma}$ to the set of the branched points of f . Thus, on this open set $\overset{\circ}{\Sigma}$, one can apply all the local formulae from differential geometry by considering $\overset{\circ}{\Sigma}$ as an immersed Riemannian manifold with the induced metric h , also denoted by $\langle \cdot, \cdot \rangle$.

Let σ be the second fundamental form of the immersion $f: \overset{\circ}{\Sigma} \rightarrow \mathbb{C}\mathbb{P}^n$. Since $\mathbb{C}\mathbb{P}^n$ with the Fubini-Study metric is a Kähler manifold, its Levi-Civita connection commutes with the complex structure, i.e. $J\nabla = \nabla J$. As a consequence, we have

$$(2.6) \quad \sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y).$$

An isometric immersion of a Riemannian manifold M into another one \overline{M} , yields the relation between their sectional curvatures given as follows,

$$(2.7) \quad K(\xi, \eta) = \overline{K}(\xi, \eta) + \frac{\langle \sigma(\xi, \xi), \sigma(\eta, \eta) \rangle - |\sigma(\xi, \eta)|^2}{|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2},$$

where $\xi, \eta \in T_p M$ are linearly independent. Take $M = \overset{\circ}{\Sigma}$, $\overline{M} = \mathbb{C}\mathbb{P}^n$ and $\eta = J\xi$ so that $\{\xi, \eta\}$ form an orthonormal basis of $T_p M$. Then one obtains that $K(\xi, \eta) = K$ is just the Gaussian curvature of $\overset{\circ}{\Sigma}$ and $\overline{K}(\xi, \eta) = 1$ is the holomorphic sectional curvature of $\mathbb{C}\mathbb{P}^n$ with the Fubini-Study metric. Applying (2.6), one sees that $\sigma(\xi, \xi) = -\sigma(\eta, \eta)$ and $\sigma(\xi, \eta) = J\sigma(\xi, \xi)$. This immediately implies

$$(2.8) \quad K = 1 - \frac{1}{2}|\sigma|^2.$$

Since $HM_1(n+1)$ is an affine space, the coordinate-wise Hessian of A_f coincides with its second fundamental form. In particular, we have

$$(2.9) \quad -\Delta A_f = \text{tr Hess } A_f = 2H =: B,$$

where H stands for mean curvature of the map $A_f: \overset{\circ}{\Sigma} \rightarrow HM_1(n+1)$.

Lemma 2.2. *The following relations hold:*

$$\begin{cases} |I|^2 = 2(n+1), & \langle A_f, I \rangle = 2, & |A_f|^2 = 2, \\ \langle B, I \rangle = 0, & \langle B, A_f \rangle = -2, & |B|^2 = 4, \\ \langle \Delta_h B, A_f \rangle = -4, & \langle \Delta_h B, B \rangle = 8 + 2|\sigma|^2. \end{cases}$$

Proof. The first row follows from the definition of the isometric embedding $\mathbb{C}\mathbb{P}^n \subset HM_1(n+1)$. The next equation is obtained by noticing that $\text{tr } B = 0$ for any $B \in T_A HM_1(n+1)$. The rest follows from Lemma 3.2 in [R3]. \square

From formulae (2.2), (2.3), and (2.8) one has,

$$(2.10) \quad \begin{cases} \int_{\Sigma} |\sigma|^2 dv_h = 8\pi \left(d + \gamma - 1 - \frac{1}{2}\beta \right), \\ \int_{\Sigma} 1 dv_h = 4\pi d. \end{cases}$$

Fix a point $a \in \mathbb{R}$ and consider the map $\phi_a: \overset{\circ}{\Sigma} \rightarrow HM_1(n+1)$,

$$\phi_a(A) = A + aB(A).$$

Lemma 2.3. *The image of ϕ_a lies in the sphere with center at $\frac{1}{n+1}I$, namely,*

$$\left| \phi_a - \frac{1}{n+1}I \right|^2 = (2a-1)^2 + \frac{n-1}{n+1}.$$

The energy of ϕ_a is

$$\int_{\Sigma} |d\phi_a|_h^2 dv_h = 8\pi d [(2a-1)^2 + 2a^2\delta],$$

where $\delta = 1 + \frac{g-1-\frac{1}{2}\beta}{d} \geq 0$.

In particular, the energy remains invariant under projective transformations of Σ in $\mathbb{C}\mathbb{P}^n$. Furthermore, ϕ_a regarded as a vector-valued function belongs to the Sobolev space $W^{1,2}(\Sigma, g)$ for all smooth metrics g compatible with the complex structure on Σ .

Proof. Both conclusion are obtained using Lemma 2.2. For the first one we have

$$(2.11) \quad \begin{aligned} \left| \phi_a - \frac{1}{n+1}I \right|^2 &= \left| aB + \left(A_f - \frac{1}{n+1}I \right) \right|^2 = \\ &= a^2|B|^2 + 2a\langle B, A_f \rangle + \left| A_f - \frac{1}{n+1}I \right|^2 = 4a^2 - 4a + 1 + \frac{n-1}{n+1}. \end{aligned}$$

For the second, we observe that

$$(2.12) \quad \begin{aligned} \int_{\Sigma} |d\phi_a|_h^2 dv_h &= \int_{\Sigma} \langle \Delta_h \phi_a, \phi_a \rangle dv_h = \int_{\Sigma} \langle -B + a\Delta_h B, A_f + aB \rangle dv_h = \\ &= \int_{\Sigma} 2(1-2a)^2 + 2a^2|\sigma|^2 dv_h. \end{aligned}$$

An application of formulae (2.10) completes the proof of the identities.

To show that $\phi_a \in W^{1,2}(\Sigma, g)$ it is sufficient to observe that the Dirichlet integral is conformally invariant and that ϕ_a is a bounded function. \square

2.3. The center of mass. Let us denote by \mathcal{H} the convex hull of $\mathbb{C}P^n \subset HM_1(n+1)$. Recalling the fact that $\mathbb{C}P^n$ is embedded to $HM_1(n+1)$ as a set of all one-dimensional orthogonal projectors and the fact that Hermitian operators are diagonalizable, one has:

- $\mathcal{H} = \{A \in HM_1(n+1) \mid A \geq 0\}$, where $A \geq 0$ means that A is positive semi-definite;
- $\text{int } \mathcal{H} = \{A \in HM_1(n+1) \mid A > 0\}$;
- $\partial\mathcal{H} = \{A \in \mathcal{H} \mid \text{rk } A \leq n\}$.

Using these facts, we can prove the following lemma.

Lemma 2.4. *The distance between $\partial\mathcal{H}$ and the point $\frac{1}{n+1}I$ equals $\sqrt{\frac{2}{n(n+1)}}$.*

Proof. Let $A \in \partial\mathcal{H}$ be a point that realizes the distance. As $U(n+1)$ acts by isometries, we may suppose that A is diagonal, i.e.

$$(2.13) \quad A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \quad a_i \geq 0, \quad \sum_i a_i = 1.$$

Remark that $\frac{1}{n+1}I$ is the center of the n -dimensional simplex, A lies on a one of its faces, and so A must be the center of the face, i.e. $a_i = \frac{1}{n}$. Thus,

$$(2.14) \quad \left| A - \frac{1}{n+1}I \right|^2 = 2 \left(\frac{1}{(n+1)^2} + n \frac{1}{(n+1)^2 n^2} \right) = \frac{2}{n(n+1)}.$$

\square

Any point $P \in \text{int } \mathcal{H} \subset HM_1(n+1)$ defines a projective transformation of $\mathbb{C}P^n$. If the map $A_f: \Sigma \rightarrow HM_1(n+1)$ corresponds to the holomorphic map $f: \Sigma \rightarrow \mathbb{C}P^n$, then we denote by A_{f_P} the map corresponding to $f_P = P \circ f$. Let B_P be its mean curvature vector.

Given a metric g on Σ we define the map $\Phi_a: \text{int } \mathcal{H} \rightarrow HM_1(n+1)$ by the formula

$$(2.15) \quad \Phi_a(P) = \frac{1}{\text{Area}(\Sigma, g)} \int_{\Sigma} (A_{f_P} + aB_P) dv_g.$$

To each transformation P the map Φ_a assigns the center of mass of $A_{f_P} + aB_P$ with respect to the metric g . Our goal is to show that there exists P_0 such that $\Phi_a(P_0) = \frac{1}{n+1}I$.

We consider a as a parameter. Lemma 2.3 implies that $A_{f_P} + aB_P$ is bounded provided a is bounded. It follows from the dominated convergence theorem that the map Φ depends continuously on a and P . The case $a = 0$

is studied in [BLY] and it turns out that the map Φ_0 possesses the following properties,

- it extends to the continuous map $\Phi_0: \mathcal{H} \rightarrow HM_1(n+1)$;
- $\Phi_0(\partial\mathcal{H}) \subset \partial\mathcal{H}$, and the restriction of this extension $\Phi_0|_{\partial\mathcal{H}}$ has non-zero degree.

Remark 2.5. This is the only point of the proof that requires f to be full. The fullness assumption is used in [BLY] to prove the properties above.

Lemma 2.6. *If $|a| < \frac{1}{\sqrt{2n(n+1)}}$, then there exists $P_0 \in \text{int } \mathcal{H}$ such that $\Phi_a(P_0) = \frac{1}{n+1}I$.*

Proof. We "shrink" \mathcal{H} a little bit and call it \mathcal{H}_ε ,

$$\mathcal{H}_\varepsilon = \left\{ (1 - \varepsilon)P + \varepsilon \frac{1}{n+1}I \mid P \in \mathcal{H} \right\} \subset \text{int } \mathcal{H}.$$

We claim that $\Phi_a(\partial\mathcal{H}_\varepsilon)$ does not contain $\frac{1}{n+1}I$ when ε is small enough. Indeed, if $\Phi_a(P_\varepsilon) = \frac{1}{n+1}I$, then one has

$$0 = \Phi_a(P_\varepsilon) - \frac{1}{n+1}I = \left(\Phi_0(P_\varepsilon) - \frac{1}{n+1}I \right) + \frac{a}{\text{Area}(\Sigma, g)} \int_\Sigma B_{P_\varepsilon} d\mu_g,$$

so by Lemma 2.2

$$\left| \Phi_0(P_\varepsilon) - \frac{1}{n+1}I \right| \leq 2|a| < \text{dist}\left(\partial\mathcal{H}, \frac{1}{n+1}I\right),$$

but $\Phi_0(P_\varepsilon) \rightarrow \partial\mathcal{H}$ as $\varepsilon \rightarrow 0$, which will be a contradiction.

It remains to notice that $\Phi_a: \partial\mathcal{H}_\varepsilon \rightarrow HM_1(n+1) \setminus \left\{ \frac{1}{n+1}I \right\}$ homotopic to $\Phi_0: \partial\mathcal{H} \rightarrow \partial\mathcal{H}$, therefore, has the same degree, which is non-zero by [BLY]. Hence, $\Phi_a(\mathcal{H}_\varepsilon)$ must contain $\frac{1}{n+1}I$, and this concludes the proof. \square

2.4. Upper bound on λ_1 . Finally, we are ready to prove the upper bound on $\Lambda_1(\Sigma, [g])$. Recall that

$$(2.16) \quad \lambda_1(\Sigma, g) = \inf \left\{ \frac{\int_\Sigma |d\varphi|^2 dv_g}{\int_\Sigma \varphi^2 dv_g}, \varphi \in W^{1,2}(\Sigma, g) \setminus \{0\}, \int_\Sigma \varphi dv_g = 0 \right\}.$$

The following theorem is a direct generalization of the bound of Ros in [R1].

Theorem 2.7. *Let (Σ, g) be a compact oriented Riemannian surface of genus γ endowed with the compatible complex structure. Suppose that there exists a full holomorphic map $f: \Sigma \rightarrow \mathbb{C}P^n$ with total ramification $\beta = \sum_p (\text{ord}_p f - 1)$. Then for any $|a| \leq \frac{1}{\sqrt{2n(n+1)}}$ one has*

$$(2.17) \quad \Lambda_1(\Sigma, [g]) \leq 8\pi \deg(f) \left(1 + \frac{2a^2\delta - \frac{n-1}{n+1}}{(2a-1)^2 + \frac{n-1}{n+1}} \right),$$

where $\delta = 1 + \frac{\gamma-1-\frac{1}{2}\beta}{\deg(f)}$.

Proof. Fix $|a| < \frac{1}{\sqrt{2n(n+1)}}$ and consider a metric $g \in [g]$. We use Lemma 2.6 and replace f with f_P . It has the same degree and total ramification, but in addition, one has

$$\int_{\Sigma} \left(\phi_a - \frac{1}{n+1} I \right) dv_g = 0.$$

This allows us to use the coordinates of $\left(\phi_a - \frac{1}{n+1} I \right)$ as test-functions in (2.16), which are in $W^{1,2}(\Sigma, g)$ by Lemma 2.3. Let $G(a)$ be the r.h.s in (2.17). Thus, by identities in Lemma 2.3 one obtains

$$\lambda_1(\Sigma, g) \leq \frac{\int_{\Sigma} |d\phi_a|_g^2 dv_g}{\int_{\Sigma} \left| \phi_a - \frac{1}{n+1} I \right|^2 dv_g} = \frac{\int_{\Sigma} |d\phi_a|_h^2 dv_h}{\int_{\Sigma} \left| \phi_a - \frac{1}{n+1} I \right|^2 dv_g} = \frac{G(a)}{\text{Area}(\Sigma, g)},$$

where we used that $\left| \phi_a - \frac{1}{n+1} I \right|$ is constant and the energy of ϕ_a does not change within the class of conformal metrics. The case of $|a| = \frac{1}{\sqrt{2n(n+1)}}$ follows from the continuity of $F(a)$. \square

3. THE PROOF OF THE MAIN THEOREM

To effectively apply inequality (2.17) one needs to find full holomorphic maps $f: \Sigma \rightarrow \mathbb{C}\mathbb{P}^n$ of low degree. Such maps are given by Lemma 2.1. Note that the r.h.s of (2.17) is increasing in $\deg(f)$ and decreasing in β , therefore one has the following proposition.

Proposition 3.1. *Let Σ_{γ} be an orientable surface of genus γ . Then for any $n \in \mathbb{Z}$, $n > 0$ and any $|a| \leq \frac{1}{\sqrt{2n(n+1)}}$ one has*

$$\Lambda_1(\Sigma_{\gamma}) \leq 8\pi d \left(1 + \frac{2a^2\delta - \frac{n-1}{n+1}}{(2a-1)^2 + \frac{n-1}{n+1}} \right) =: F(a, n, \gamma),$$

where $d = d(n, \gamma) = \left\lceil \frac{n\gamma}{n+1} \right\rceil + n$, $\delta = \delta(n, \gamma) = 1 + \frac{\gamma-1}{d}$.

The remainder of this section is devoted to choosing the values (a, n) that lead to the best bound for each fixed γ . One first observes that for $n = 1$ the minimum of $F(a, 1, \gamma)$ is achieved for $a = 0$ and the resulting bound is the Yang-Yau bound (1.1). In the following we assume $n \geq 2$.

Lemma 3.2. *For fixed (n, γ) , the function $F(a, n, \gamma)$ has exactly two critical points on \mathbb{R} , a local minimum at*

$$a_{\min} = a_{\min}(n, \gamma) = \frac{\xi - \sqrt{\xi^2 - 2(n^2 - 1)\delta}}{2\delta(n+1)}$$

and a local maximum at

$$a_{\max} = a_{\max}(n, \gamma) = \frac{\xi + \sqrt{\xi^2 - 2(n^2 - 1)\delta}}{2\delta(n + 1)},$$

where $\xi = n\delta + (n - 1)$. Furthermore, $a_{\max} > \frac{1}{\sqrt{2n(n+1)}}$. Thus, the minimum of $F(a, n, \gamma)$ on $\left[-\frac{1}{\sqrt{2n(n+1)}}, \frac{1}{\sqrt{2n(n+1)}}\right]$ is achieved either at $a = a_{\min}$ or at $a = \frac{1}{\sqrt{2n(n+1)}}$.

Proof. The derivative $\partial_a F(a, n, \gamma)$ has the form $\frac{p_{n,\gamma}(a)}{q_{n,\gamma}(a)^2}$, where $p_{n,\gamma}(a)$ is a quadratic polynomial in a and $q_{n,\gamma}(a) = (2a - 1)^2 + \frac{n-1}{n+1}$ never vanishes. Therefore, the first assertion follows from the formula for the roots of a quadratic polynomial.

Recall that we are assuming $n \geq 2$, therefore, one has

$$a_{\max} \geq \frac{\xi}{2\delta(n+1)} > \frac{n}{2(n+1)} \geq \frac{1}{3} > \frac{1}{2\sqrt{3}} > \frac{1}{\sqrt{2n(n+1)}}.$$

□

Lemma 3.3. *For any $\gamma \geq 0$ and $n \geq 5$ one has $a_{\min}(n, \gamma) > \frac{1}{\sqrt{2n(n+1)}}$. Furthermore, $a_{\min} < \frac{1}{\sqrt{2n(n+1)}}$ provided that (a) $n = 2$, $\gamma \geq 0$; (b) $n = 3$, $\gamma \geq 4$; (c) $n = 4$, $\gamma \geq 40$.*

Proof. The condition $a_{\min} < \frac{1}{\sqrt{2n(n+1)}}$ can be rewritten as follows

$$\xi - \sqrt{2\frac{n+1}{n}}\delta < \sqrt{\xi^2 - 2(n^2 - 1)\delta}.$$

Provided $n \geq 2$, one can take squares, and direct calculation yield

$$(3.1) \quad \frac{\sqrt{n}(n-1) \left(\sqrt{n(n+1)} - \sqrt{2} \right)}{n\sqrt{2n} - \sqrt{n+1}} < \delta.$$

Note that for any $\gamma \geq 0$ the inequality $\delta(n, \gamma) \leq 2 + \frac{1}{n}$ holds. Thus, for $n \geq 5$ one has

$$(3.2) \quad \begin{aligned} & \frac{\sqrt{n}(n-1) \left(\sqrt{n(n+1)} - \sqrt{2} \right)}{n\sqrt{2n} - \sqrt{n+1}} \geq \frac{(n-1) \left(\sqrt{n(n+1)} - \sqrt{2} \right)}{\sqrt{2n} - 1} \geq \\ & \geq \frac{\sqrt{n(n+1)} - \sqrt{2}}{\sqrt{2}} \geq 2 + \frac{1}{5} \geq 2 + \frac{1}{n} \geq \delta(n, \gamma) \end{aligned}$$

for any $\gamma \geq 0$. As a result, by (3.1) one obtains $a_{\min} > \frac{1}{\sqrt{2n(n+1)}}$ for $n \geq 5$.

Similarly, for all $\gamma \geq 0$ one has $\delta(n, \gamma) \geq 1 + \frac{(n+1)(\gamma-1)}{n\gamma+(n+1)(n+2)}$. For a fixed n the r.h.s is an increasing function of γ , therefore, as long as

$$(3.3) \quad \frac{\sqrt{n}(n-1) \left(\sqrt{n(n+1)} - \sqrt{2} \right)}{n\sqrt{2n} - \sqrt{n+1}} < 1 + \frac{(n+1)(\gamma_0-1)}{n\gamma_0 + (n+1)(n+2)}$$

the condition $a_{\min}(n, \gamma) < \frac{1}{\sqrt{2n(n+1)}}$ is satisfied for $\gamma \geq \gamma_0$. Finally, a direct calculation shows that (3.3) holds for $(n, \gamma_0) = (2, 0), (3, 4), (4, 40)$. \square

Remark 3.4. By a direct computation using (3.1) one can see that, in fact, $a_{\min} < \frac{1}{\sqrt{2n(n+1)}}$ for $n = 3, \gamma \geq 3$ and $n = 4, \gamma \geq 30$. This observation is not necessary for the following arguments, since the low genus case is treated separately in Table 1.

Having determined the optimal value of a , we define

$$(3.4) \quad F_n(\gamma) = \begin{cases} F\left(\frac{1}{\sqrt{2n(n+1)}}, n, \gamma\right), & \text{if } n \geq 5; \\ F(a_{\min}(n, \gamma), n, \gamma), & \text{if } n = 2; n = 3, \gamma \geq 4; n = 4, \gamma \geq 40; \\ F(0, 1, \gamma) = 8\pi \left(\left\lceil \frac{\gamma}{2} \right\rceil + 1 \right), & \text{if } n = 1, \end{cases}$$

where $F_1(\gamma)$ is the r.h.s in the Yang-Yau inequality (1.1).

A direct computation yields that

$$(3.5) \quad F_5(\gamma) = \frac{2\pi}{13 - \sqrt{15}} \left(\gamma + (33 - 4\sqrt{15}) \left\lceil \frac{5\gamma}{6} \right\rceil + 4(41 - 5\sqrt{15}) \right).$$

Estimating $\lceil x \rceil \leq x + 1$ in (3.5) gives the following useful estimate,

$$(3.6) \quad \frac{F_5(\gamma)}{8\pi} \leq \left(\frac{89}{6(4\sqrt{15} - 52)} + \frac{5}{6} \right) \gamma + \left(\frac{115}{4\sqrt{15} - 52} + 6 \right) =: a\gamma + b,$$

where $a \leq 0.4271$, $b \leq 2.8501$.

The following lemma states that taking $n \geq 6$ is never optimal.

Lemma 3.5. *For any $\gamma \geq 0$ and any $n > 5$ one has*

$$F_n(\gamma) > F_5(\gamma).$$

Proof. Since

$$d(n, \gamma) = \left\lceil \frac{n\gamma}{n+1} \right\rceil + n \geq \frac{n\gamma}{n+1} + n$$

one has that for $n > 5$

$$\frac{F_n(\gamma)}{8\pi} \geq \frac{n\gamma}{n+1} + n + \frac{1}{2} \frac{\gamma \left(2n - n^2 + \frac{1}{n+1} \right) - (n-1)^2(n+1)}{n^2 - \sqrt{2n(n+1)} + 1} =: a_n\gamma + b_n.$$

To complete the proof we estimate a_n, b_n separately to show that $a_n > a$ and $b_n > b$ for $n \geq 6$, where a, b are as in (3.6).

Value of n	Genera γ , for which n is optimal
1	4, 6, 8, 10, 14
2	3, 7, 9, 18, 19
3	5, 12, 13, 16, 17, 20, 24, 28, 29, 32
4	11, 15, 21, 22, 23, 25-27, 30, 31, 33 - 41, 45 - 47, 50 - 53, 55 - 61, 65, 70, 71, 75 - 77, 80 - 82, 95, 100, 101
5	42 - 44, 48, 49, 54, 62-69, 72 - 74, 78, 79, 83 - 94, 96 - 99, $\gamma \geq 102$

TABLE 1. Optimal values of n for low genera. The value $n = 1$ corresponds to the Yang-Yau inequality, i.e. for $\gamma = 4, 6, 8, 10, 14$ our results do not improve on (1.1).

One has

$$(3.7) \quad a_n = \frac{n}{n+1} + \frac{2n - n^2 + \frac{1}{n+1}}{2(n^2 - \sqrt{2n(n+1)} + 1)}.$$

Thus,

$$(3.8) \quad a_n \geq \frac{n}{n+1} - \frac{n^2 - 2n}{2(n^2 + 1 - 2(n+1))} = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n^2 - 2n - 1} \geq \frac{1}{2} - \frac{1}{n+1}.$$

The r.h.s of (3.8) is increasing in n . As a result, it is straightforward to compute that (3.8) implies $a_n > a$ for $n \geq 13$. For $5 < n < 13$ one proves $a_n > a$ by a direct computation using (3.7).

Similarly, one has

$$(3.9) \quad b_n = n - \frac{(n-1)^2(n+1)}{2(n^2 - \sqrt{2n(n+1)} + 1)}.$$

Thus,

$$(3.10) \quad b_n \geq n - \frac{(n-1)^2(n+1)}{2(n^2 - 2n - 1)} = \frac{n-3}{2} - \frac{6n+2}{2(n^2 - 2n - 1)} \geq \frac{n-5}{2},$$

where in the last inequality we used $n \geq 6$. As a result, (3.10) implies $b_n > b$ for $n \geq 11$. For $5 < n < 11$ one proves $b_n > b$ by a direct computation using (3.9). □

Finally, we investigate the range $1 \leq n \leq 5$.

Lemma 3.6. *One has the following*

- (1) *If $\gamma \geq 25$, then $F_1(\gamma) > F_5(\gamma)$;*
- (2) *If $\gamma \geq 41$, then $F_2(\gamma) > F_5(\gamma)$;*

- (3) If $\gamma \geq 109$, then $F_3(\gamma) > F_5(\gamma)$;
(4) If $\gamma \geq 389$, then $F_4(\gamma) > F_5(\gamma)$.

Proof. Let us first deal with $\gamma = 1$. One obtains $F_1(\gamma) \geq 4\pi(\gamma + 2)$ and a direct comparison with (3.6) gives $F_1(\gamma) > F_5(\gamma)$ for $\gamma \geq 26$. The case $\gamma = 25$ can be checked directly.

If x_0 is a critical point of a function $\frac{f(x)}{g(x)}$, then $\frac{f(x_0)}{g(x_0)} = \frac{f'(x_0)}{g'(x_0)}$. Therefore, for $n = 2, 3, 4$ one has

$$(3.11) \quad F_n(\gamma) = 8\pi d \left(1 - \frac{\delta a_{\min}}{1 - 2a_{\min}} \right).$$

Recall that

$$\frac{n\gamma}{n+1} + (n+1) \leq d(n, \gamma) = \left\lceil \frac{n\gamma}{n+1} \right\rceil + n + 1 \leq \frac{n\gamma}{n+1} + (n+2),$$

$$\delta(n, \gamma) = 1 + \frac{\gamma - 1}{d(n, \gamma)}; \quad \xi(n, \gamma) = n\delta(n, \gamma) + n - 1.$$

Case 1: $n = 2$. One has

$$\delta_-(\gamma) = \frac{5\gamma + 9}{2\gamma + 12} \leq \delta \leq \frac{5\gamma + 6}{2\gamma + 9} \leq \frac{5}{2} = \delta_+; \quad \xi \leq 2\delta_+ + 1 = 6 = \xi_+.$$

Then one has

$$(3.12) \quad a_{\min} \leq \frac{\xi_+ - \sqrt{\xi_+^2 - 6\delta_+}}{6\delta_-(\gamma)} = \frac{(6 - \sqrt{21})(\gamma + 6)}{3(5\gamma + 9)} = a_+(\gamma).$$

We observe that $a_+(\gamma)$ is a decreasing function of γ . Therefore, by (3.11) one has

$$\frac{F_2(\gamma)}{8\pi} \geq \left(\frac{2\gamma}{3} + 3 \right) \left(1 - \frac{\delta_+ a_+(\gamma)}{1 - 2a_+(\gamma)} \right) \geq \left(\frac{2\gamma}{3} + 3 \right) \left(1 - \frac{\delta_+ a_+(\gamma_0)}{1 - 2a_+(\gamma_0)} \right)$$

for all $\gamma \geq \gamma_0$. Taking $\gamma_0 = 41$, a direct computation combined with the expression (3.6) yields the claim.

Case 2: $n = 3$. One has

$$\delta_-(\gamma) = \frac{7\gamma + 16}{3\gamma + 20} \leq \delta \leq \frac{7\gamma + 12}{3\gamma + 16} \leq \frac{7}{3} = \delta_+; \quad \xi \leq 3\delta_+ + 2 = 9 = \xi_+.$$

Then one has

$$a_{\min} \leq \frac{\xi_+ - \sqrt{\xi_+^2 - 16\delta_+}}{8\delta_-(\gamma)} = \left(9 - \sqrt{81 - \frac{112}{3}} \right) \frac{3\gamma + 20}{8(7\gamma + 16)} = a_+(\gamma).$$

We observe that $a_+(\gamma)$ is a decreasing function of γ . Therefore, by (3.11) one has

$$\frac{F_3(\gamma)}{8\pi} \geq \left(\frac{3\gamma}{4} + 4 \right) \left(1 - \frac{\delta_+ a_+(\gamma_0)}{1 - 2a_+(\gamma_0)} \right)$$

for all $\gamma \geq \gamma_0$. Taking $\gamma_0 = 109$, a direct computation combined with the expression (3.6) yields the claim.

Case 3: $n = 4$. One has

$$\delta_-(\gamma) = \frac{9\gamma + 25}{4\gamma + 30} \leq \delta \leq \frac{9\gamma + 20}{4\gamma + 25} \leq \frac{9}{4} = \delta_+; \quad \xi \leq 4\delta_+ + 3 = 12.$$

Then one has

$$a_{\min} \leq \frac{\xi_+ - \sqrt{\xi_+^2 - 30\delta_+}}{10\delta_-(\gamma)} = \left(12 - \sqrt{144 - \frac{135}{2}}\right) \frac{4\gamma + 30}{10(9\gamma + 25)} = a_+(\gamma).$$

We observe that $a_+(\gamma)$ is a decreasing function of γ . Therefore, by (3.11) one has

$$\frac{F_4(\gamma)}{8\pi} \geq \left(\frac{4\gamma}{5} + 5\right) \left(1 - \frac{\delta_+ a_+(\gamma_0)}{1 - 2a_+(\gamma_0)}\right)$$

for all $\gamma \geq \gamma_0$. Taking $\gamma_0 = 389$, a direct computation combined with the expression (3.6) yields the claim. \square

The results of this section implies that choosing $n = 5$ is optimal for large γ . The optimal values of n for small γ are collected in Table 1.

REFERENCES

- [BLY] J.P. Bourguignon, P. Li and S.T. Yau, Upper bound for the first eigenvalue of algebraic submanifolds, *Comment. Math. Helv.* 69 (1994), 199–207.
- [CKM] D. Cianci, M. Karpukhin, V. Medvedev, On branched minimal immersions of surfaces by first eigenfunctions. *Annals of Global Analysis and Geometry* 56:4 (2019), 667–690. Preprint [arXiv:1711.05916](#).
- [EGJ] A. El Soufi, H. Giacomini, M. Jazar, A unique extremal metric for the least eigenvalue of the Laplacian on the Klein bottle. *Duke Mathematical Journal* 135:1 (2006), 181–202.
- [ESI] A. El Soufi, S. Ilias, Le volume conforme et ses applications d’après Li et Yau, *Sém. Théorie Spectrale et Géométrie*, Institut Fourier, 1983–1984, No.VII, (1984).
- [F] H. Fang, The Gauss-Bonnet formula for a conformal metric with finitely many cone or cusp singularities on a compact Riemann surface, [arXiv:1912.01187v1 \[math.DG\]](#)
- [GH] P. A. Griffiths, J. Harris, *Principles of Algebraic Geometry*. Wiley Classics Library. Wiley Interscience, 1994.
- [H] J. Hersch, Quatre propriétés isopérimétriques de membranes sphériques homogènes. *C. R. Acad. Sci. Paris Sér A-B* 270 (1970), A1645–A1648.
- [JNP] D. Jakobson, N. Nadirashvili, and I. Polterovich, Extremal metric for the first eigenvalue on a Klein bottle. *Canadian J. of Mathematics* 58:2 (2006), 381–400.
- [K1] M. Karpukhin, Upper bounds for the first eigenvalue of the Laplacian on non-orientable surfaces. *Int. Math. Research Notices*, 20 (2016), 6200–6209.
- [K2] M. Karpukhin, On the Yang-Yau inequality for the first Laplace eigenvalue. *Geometric and Functional Analysis* 29:6 (2019), 1864–1885. Preprint [arXiv:1902.03473](#).
- [K3] M. Karpukhin, Index of minimal surfaces and isoperimetric eigenvalue inequalities. *Inventiones Mathematicae* 223 (2021), 335 – 377.
- [KNPP] M. Karpukhin, N. Nadirashvili, A. V. Penskoï, I. Polterovich, Conformally maximal metrics for Laplace eigenvalues on surfaces. *Surveys in Differential Geometry*, 24:1 (2019), 205–256. Preprint [arXiv:2003.02871](#).
- [KS] M. Karpukhin, D. L. Stern, Min-max harmonic maps and a new characterization of conformal eigenvalues. Preprint [arXiv:2004.04086](#).

- [LY] P. Li, S.-T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Inventiones mathematicae*, **69**:2 (1982), 269–291.
- [MS] H. Matthiesen, R. Petrides, Handle attachment and the normalized first eigenvalue. Preprint [arXiv:1909.03105](https://arxiv.org/abs/1909.03105).
- [N1] N. Nadirashvili, Berger’s isoperimetric problem and minimal immersions of surfaces. *Geom. Funct. Anal.*, **6**:5 (1996), 877–897.
- [N2] N. Nadirashvili, Isoperimetric inequality for the second eigenvalue of a sphere. *J. Differential Geom.* **61**:2 (2002), 335–340.
- [NS] S. Nayatani, T. Shoda, Metrics on a closed surface of genus two which maximize the first eigenvalue of the Laplacian. *Comptes Rendus Mathematique*, **357**:1 (2019), 84–98.
- [R1] A. Ros, On the first eigenvalue of the Laplacian on compact surfaces of genus three. *J. of the Math. Society of Japan* **1**:1 (2021), 1–16. Preprint [arXiv:2010.14857](https://arxiv.org/abs/2010.14857)
- [R2] A. Ros, Spectral geometry of CR-minimal submanifolds in the complex projective space, *Kodai Mathematical Journal* **6**:1 (1983), 88–99.
- [R3] A. Ros, On spectral geometry of Kaehler submanifolds, *J. Math. Soc. Japan* **36**:3 (1984), 433–448.
- [YY] P. C. Yang, S.-T. Yau, Eigenvalues of the laplacian of compact Riemann surfaces and minimal submanifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **7**:1 (1980), 55–63.

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