

ON THE WITHIN-PERFECT NUMBERS

by CHUNG-HANG KWAN[†]  and STEVEN J. MILLER[‡]

Department of Mathematics, University College London, 25 Gordon Street, London WC1H 0AY, United Kingdom

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267, United States of America

[†]Corresponding author. E-mail: ucahckw@ucl.ac.uk

[‡]sjm1@williams.edu; Steven.Miller.MC.96@aya.yale.edu

ABSTRACT

Motivated by the works of Erdős, Wirsing, Pomerance, Wolke and Harman on the sum-of-divisor function $\sigma(n)$, we study the distribution of a special class of natural numbers closely related to (multiply) perfect numbers which we term ‘ $(\ell; k)$ -within-perfect numbers’, where $\ell > 1$ is a real number and $k : [1, \infty) \rightarrow (0, \infty)$ is an increasing and unbounded function.

1. INTRODUCTION

A natural number n is said to be *perfect* if $\sigma(n) = 2n$, ℓ -*perfect* if $\sigma(n) = \ell n$ (with $\ell > 1$ being rational) and *multiply perfect* if $n \mid \sigma(n)$, where $\sigma(n)$ represents the sum of all positive divisors of n . An outstanding conjecture, originating from the ancient Greeks (300 BC), asserts that there are infinitely many even perfect numbers but no odd perfect numbers (see the studies by Euclid [12, Proposition IX.36], Dickson [7, Chapter I], Guy [13, Chapter B1]). This conjecture is well-supported by probabilistic heuristics due to Pomerance [27, pp. 249, 258–259]. For $\ell \in \{2, 3, \dots, 11\}$, there are known examples of ℓ -perfect numbers (see [13, Chapter B1]); however, for other values of ℓ , the (non)existence of ℓ -perfect numbers remains entirely open.

Starting in the mid-20th century, considerable interest emerged in understanding the statistical distribution of perfect numbers. These numbers are particularly *rare*, as demonstrated by the works of Erdős [10], Volkmann [37], Kanold [20, 21], Hornfeck [17] and Hornfeck-Wirsing [18], culminating in the sharpest known *upper bound* for the number of perfect numbers up to x in the study by Wirsing [38]. The bound obtained in [38] is of the order $\exp(O(\frac{\log x}{\log \log x}))$ as $x \rightarrow \infty$, which possesses the pleasant feature of *uniform* applicability to ℓ -perfect numbers for any rational ℓ . When restricting to the class of odd perfect numbers, there is the celebrated *Dickson’s finiteness theorem* [8] which asserts the following: given a natural number k , there are only finitely many odd perfect numbers with exactly k distinct prime divisors. This was later refined by Pomerance [26] and Heath-Brown [15].

Subsequently, it evolved into an active research area to investigate special classes of natural numbers closely linked to perfect numbers, see [13, Chapter B2–B3]. For instance, Sierpiński [34] introduced the notion of ‘*pseudo-perfect numbers*’. In a companion article [5], we studied a subclass of these

numbers known as ‘near-perfect numbers’, proposed by Pollack-Shevelev [32]. In [5], we strengthened the results and analysis of [32] by employing recursive partitions and sieve-theoretic techniques.

In this article, we investigate another class of ‘approximate’ perfect numbers with a somewhat different flavour, which we call the $(\ell; k)$ -within-perfect numbers, where $\ell > 1$ is a real number and $k = k(y)$ is a certain threshold function. More precisely, a natural number n is said to be $(\ell; k)$ -within-perfect if the Diophantine inequality

$$|\sigma(n) - \ell n| < k(n) \quad (1.1)$$

holds. There are two distinct origins of these numbers. On one hand, Erdős ([13, p. 46]) and Makowski [22] were interested in the case when $\ell = 2$ and k is a constant. On the other hand, the inequality (1.1) arises naturally in the field of *Diophantine approximations* for arithmetic functions. Wolke [39] studied (1.1) for any real $\ell > 1$ and function $k(y)$ of the form y^c . His result was improved by Harman [14] and very recently by Järvinieniemi [19]. In [19], it was shown that for any real $\ell > 1$ and any $c \in (0.45, 1)$, there exist *infinitely many* $(\ell; y^c)$ -within-perfect numbers. The range of c can be extended to $(0.39, 1)$ under the *Riemann Hypothesis* as indicated in [14]. The results of [14, 19] rely on deep inputs from the distributions of primes in short intervals as well as that of the differences in consecutive primes, see [4, 16, 19, 35]. Interested readers are also referred to the studies by Alkan-Ford-Zaharescu [1, 2] for settings more general than (1.1) and the related questions in Diophantine approximations.

The main results of this article concern the class of threshold functions $k = k(y)$ which are complementary to those considered in [14, 19, 39]. Moreover, we are also interested in estimating the size of the set

$$W(\ell; k; x) := \{n \leq x : |\sigma(n) - \ell n| < k(n)\}.$$

Consequently, our work employs a different set of techniques compared to the earlier mentioned works.

THEOREM 1.1 *Let $c \in (0, 1/3)$ be given. Suppose $k : [1, \infty) \rightarrow (0, \infty)$ is an increasing and unbound function satisfying $k(y) \leq y^c$ for $y \geq 1$. Let Σ be the set $\{\sigma(m)/m : m \geq 1\}$. Then*

(a) *If $\ell = a/b \in (\mathbb{Q} \cap (1, \infty)) \setminus \Sigma$ with $(a, b) = 1$, then*

$$\#W(\ell; k; x) = O\left(ab^3 x^{2/3+c+o(1)}\right) \quad (1.3)$$

for $1 < \ell \leq x^c$ and $x \geq 1$, where the implicit constants are absolute.

(b) *If $\ell = a/b \in \Sigma$ with $(a, b) = 1$ and there exists $\delta > 0$ such that $k(y) \geq y^\delta$ for $y \geq 1$, then*

$$\lim_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x/\log x} = \sum_{\sigma(m)=\ell m} \frac{1}{m}. \quad (1.4)$$

REMARK 1.2 We also have an analogous result for $c = 0$ and its proof is actually simpler, see Proposition 2.7 and Remark 3.1.

REMARK 1.3 Firstly, the infinite series of (1.4) converges by Wirsing’s Theorem ([38]), which will be applied in various ways throughout the course of proving Theorem 1.1.

Secondly, a key strength of our theorem is that all dependencies in (1.3) are made entirely explicit. Our bound remains non-trivial even if the numerator or denominator of ℓ grows with x at a controlled rate although this necessitates appropriately shrinking the admissible range for c .

Thirdly, while it is possible that the dependence on ℓ (that is, the factor ab^3) in (1.3) can be improved, it does not appear that this dependence can be removed. This is in contrast to Wirsing’s Theorem.

1.1. Notations

We use the following notations throughout this article:

- $f(x) \asymp g(x)$ if there exist constants $c_1, c_2 > 0$ such that $c_1g(x) < f(x) < c_2g(x)$ for sufficiently large x ,
- $f(x) \sim g(x)$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$,
- $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists a constant $C > 0$ such that $f(x) < Cg(x)$ for sufficiently large x ,
- $f(x) = o(g(x))$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$,
- subscripts indicate the dependence of implied constants on other parameters,
- p always denotes a prime number,
- $\omega(n)$ denotes the number of distinct prime factors of n .

2. PRELIMINARY DISCUSSIONS AND PREPARATIONS

2.1. Distribution function and phase transition

We begin by briefly explaining why the sublinear regime for $k = k(y)$ is of the greatest interest in the study of within-perfectness. For this, we recall the definition of a distribution function.

DEFINITION 2.1 Let $-\infty \leq a < b \leq \infty$.

- (1) A function $F : (a, b) \rightarrow \mathbb{R}$ is a distribution function if F is increasing, right continuous, $F(a+) = 0$ and $F(b-) = 1$.
- (2) An arithmetic function $f : \mathbb{N} \rightarrow \mathbb{R}$ has a distribution function if there exists a distribution function F such that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{n \leq x : f(n) \leq u\} = F(u)$$

at all points of continuity of F .

A celebrated theorem of Davenport [6], [27, Theorem 8.5] asserts that $\sigma(n)/n$ possesses a continuous and strictly increasing distribution function on $[1, \infty)$. Let $D(\cdot)$ be the distribution function of $\sigma(n)/n$. We have $D(1) = 0$ and $D(\infty) = 1$, and for convenience, we extend the definition of $D(\cdot)$ to \mathbb{R} by setting $D(u) = 0$ for $u < 1$. More generally, a necessary and sufficient criterion for the existence of distribution functions for additive functions was established by Erdős-Wintner [11], see [36, Chapter I.5, III.4] for further details.

The following result is an elementary consequence of Davenport's theorem. It describes the phase transition of the asymptotic density of the set of all $(\ell; k)$ -within-perfect numbers, which we denote by $W(\ell; k)$.

PROPOSITION 2.2 Let $D(\cdot)$ be the distribution function of $\sigma(n)/n$. Then

- If $k(n) = o(n)$, then $W(\ell; k)$ has asymptotic density 0.
- If $k(n) \sim cn$ for some $c > 0$, then $W(\ell; k)$ has asymptotic density equal to $D(\ell + c) - D(\ell - c)$.
- If $k(n) \asymp n$, then $W(\ell; k)$ has positive lower density and upper density strictly less than 1.
- If $n = o(k(n))$, then $W(\ell; k)$ has asymptotic density 1.

Sketch The proof is elementary, and we will only indicate the details for part (d). For any $j \in \mathbb{N}$, there exists $n_j \in \mathbb{N}$ such that $n/k(n) < 1/j$ for any $n \geq n_j$. We have

$$\frac{1}{x} \#\{n \leq x : |\sigma(n) - \ell n| < \} \leq \frac{n_j}{x} + \frac{1}{x} \#\{n \leq x : |\sigma(n) - \ell n| < k(n)\} \quad (2.1)$$

whenever $x \geq n_j$. If $j > \ell$, the left-hand side of (2.1) converges to $D(\ell + j)$ as $x \rightarrow \infty$, and hence

$$\liminf_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x} \geq D(\ell + j).$$

The desired result follows at once by taking $j \rightarrow \infty$, using the fact that $D(\infty) = 1$. \square

2.2. On congruences involving $\sigma(n)$

Henceforth, our focus shifts to the sublinear thresholds, where we crucially make use of the techniques developed by Pomerance and his co-authors over the years. In 1975, Pomerance [23] initiated the study of the equation

$$\sigma(n) = \ell n + k, \tag{2.2}$$

with ℓ, k being integers and $\ell > 1$. Central to [23] is the following important concept, which proves to be very useful in many Erdős-style problems (see [30]):

DEFINITION 2.3 (Regular/Sporadic) The solutions to the congruence $\sigma(n) \equiv k \pmod{n}$ of the form

$$n = pm, \quad \text{where } p \nmid m, \quad m \mid \sigma(m), \quad \sigma(m) = k,$$

are called *regular*. All other solutions are called *sporadic*.

The main observation is that sporadic solutions occur much less frequently than regular solutions. In a series of works [3, 23–25, 29, 32], this was quantified with various degrees of precision and uniformity. We summarize the progress made in this direction.

Let $\text{Spor}_k(x)$ be the set of sporadic solutions of $\sigma(n) \equiv k \pmod{n}$ up to x . In [23], [32], [3] and [29], it was shown, respectively, that the following bounds hold as $x \rightarrow \infty$:

- $\#\text{Spor}_k(x) = O_{\beta,k}(x \exp(-\beta \sqrt{\log x \log \log x}))$ for fixed k and fixed $\beta < 1/\sqrt{2}$,
- $\#\text{Spor}_k(x) = O(x^{2/3+o(1)})$ uniformly in $|k| < x^{2/3}$,
- $\#\text{Spor}_k(x) = O(x^{1/2+o(1)})$ uniformly in $|k| < x^{1/4}$ and
- $\#\text{Spor}_k(x) \cap \{\sigma(n) \text{ is odd}\} = O(|k|x^{1/4+o(1)})$ uniformly in $0 < |k| \leq x^{1/4}$.

When $k = 0$, we have a much stronger bound for (2.2) as in the study by Wirsing ([38]).

Unfortunately, the aforementioned results are not sufficient for our study of within-perfect numbers. We must consider a more general congruence

$$b\sigma(n) \equiv k \pmod{n}, \tag{2.4}$$

where b and k are integers with $b \geq 1$. Accordingly, the definitions of regular and sporadic solutions should be extended as follows:

DEFINITION 2.4 Suppose $b \mid k$. Then n is said to be a *regular solution* to (2.4) if

$$n = pm, \quad \text{where } p \nmid m, \quad m \mid b\sigma(m), \quad \sigma(m) = \frac{k}{b}.$$

All other solutions are called *sporadic*. In the case when $b \nmid k$, all solutions to (2.4) are considered *sporadic*.

We record the following generalization of [32, Lemma 8], which will be used in the proof of Theorem 1.1 and may also be of independent interest. For the convenience of the readers, we include a sketch of proof of this proposition here.

PROPOSITION 2.5 *The number of sporadic solutions $n \leq x$ to the congruence $b\sigma(n) \equiv k \pmod{n}$ is $O(b^2x^{2/3+o(1)})$ for any $x \geq 1$ and integer k satisfying $|k| < bx^{2/3}$. The implicit constants are absolute.*

Our intended applications take into account the uniformity of the range and the strength of the upper bound in counting sporadic solutions. In light of Wirsing’s theorem ([38]) and Remark 1.3, it is also desirable to maintain all implicit constants absolute, but this can be somewhat subtle, see [3, Remark 1] and [29]. After a careful examination of existing strategies, the authors believe that the approach of [32] is the most suitable for attaining our desired generalization (for example, ℓ can be rational), incorporating all the favourable features mentioned earlier. Their method softly utilizes the unique factorization of a natural number into its square-free and square-full parts, the anatomy of unitary divisors (Note: m is a unitary divisor of n if n has a decomposition of the form $n = mm'$, where $(m, m') = 1$), and the following result from [28, Theorem 1.3]:

$$\sum_{x^{1/3} < m \leq x^{2/3}} \frac{(m, \sigma(m))}{m^2} \leq 3x^{-1/3+o(1)}, \tag{2.6}$$

which turns out to be a nice application of Wirsing’s Theorem!

Proof of Proposition 2.5 We can certainly assume that $x \geq b$; otherwise the count is trivially bounded by b , which is acceptable in view of the bound claimed in Proposition 2.5. Let $|k| < bx^{2/3}$. Suppose $n \leq x$ is a sporadic solution to the congruence $b\sigma(n) \equiv k \pmod{n}$. One can simultaneously assume that $n > x^{2/3}$ and the square-full part of n is bound by $x^{2/3}$. Indeed, the contribution from the complement can be easily seen to be $O(x^{2/3})$. We then consider the following two cases.

- (1) Suppose $p := P^+(n) > x^{1/3}$. By the assumption made, we have $n = pm$ with $p \nmid m$ and $m < x^{2/3}$. The congruence can be written as $b\sigma(n) = qn + k$ for some integer $q \geq 0$. It follows that

$$b(p+1)\sigma(m) = b\sigma(n) = qpm + k$$

and

$$p(b\sigma(m) - qm) = k - b\sigma(m).$$

If $k - b\sigma(m) = 0$, then n is a regular solution, which is a contradiction! So, $k - b\sigma(m) \neq 0$. For each m , the number of such p is $O(\log |k - b\sigma(m)|) = O(\log(bx^{2/3} \log \log x)) = O(\log bx)$ because of $p \mid (k - b\sigma(m))$. Therefore, the number of such n is $O(x^{2/3} \log bx) = O(x^{2/3} \log x)$, which is acceptable.

- (2) Suppose $p := P^+(n) \leq x^{1/3}$. Such n must have a unitary divisor m in the interval $(x^{1/3}, x^{2/3}]$, see [32]. We have $\sigma(n) \equiv 0 \pmod{\sigma(m)}$ and $b\sigma(n) \equiv k \pmod{m}$. By the Chinese Remainder Theorem, we have $b\sigma(n) \equiv a_n \pmod{[m, \sigma(m)]}$ for some unique $0 \leq a_n < [m, \sigma(m)]$. Given $m \in (x^{1/3}, x^{2/3}]$, the number of possible values for $b\sigma(n)$ is $\leq 1 + (2bx \log x) / [m, \sigma(m)]$. Summing over $m \in (x^{1/3}, x^{2/3}]$, the total number of possible values of $b\sigma(n)$ is

$$\begin{aligned} &\leq \sum_{x^{1/3} < m \leq x^{2/3}} \left(\frac{2bx \log x}{[m, \sigma(m)]} + 1 \right) \\ &\leq x^{2/3} + (2bx \log x)(3x^{-1/3+o(1)}) \leq 7bx^{2/3+o(1)} \end{aligned}$$

from (2.6). Moreover, the size of $q = (b\sigma(n) - k)/n$ is clearly

$$\ll b \log \log x + \frac{|k|}{n} < b \log \log x + b \ll b \log \log x$$

by the assumptions $n > x^{2/3}$ and $|k| < bx^{2/3}$. Since $b\sigma(n) = qn + k$, the number of possible values of n is at most

$$(7bx^{2/3+o(1)})(b \log \log x) \leq 7b^2x^{2/3+o(1)}.$$

The desired result follows by putting the conclusions of the two cases together. \square

2.3. A simple application

Let k, a, b be integers such that $k \neq 0$, $a > b \geq 1$, and $(a, b) = 1$. Denote by $S(a, b; k)$ the set of all solutions to the Diophantine equation $b\sigma(n) = an + k$ that generalizes (2.2). Let $S(a, b; k; x) := S(a, b; k) \cap [1, x]$. The main result of [23] is stated as follows.

THEOREM 2.6 *As $x \rightarrow \infty$, we have*

$$\#S(a, 1; k; x) \ll_{k,a} \frac{x}{\log x}. \quad (2.7)$$

Motivated by Pomerance's theorem, Davis-Klyve-Kraght [9] recently performed an extensive numerical investigation on the true size of $S(a, 1; k; x)$. As a first application of Proposition 2.5, we sharpen Pomerance's theorem and confirm some of the observations and speculations made in [9]. This will also serve as the base case for Theorem 1.1, see Section 3.2.

PROPOSITION 2.7

(a) *If $k \geq 1$, $ab \mid k$ and $\sigma(k/a) = k/b$, then*

$$\#S(a, b; k; x) \sim \frac{a}{k} \frac{x}{\log(ax/k)} \quad (2.8)$$

as $x \rightarrow \infty$.

(b) *Otherwise, we have*

$$\#S(a, b; k; x) = O(b^2x^{2/3+o(1)}) \quad (2.9)$$

for any $x \geq 1$, where the implicit constants are absolute. In particular, the bound (2.9) is uniform in a .

Proof. Suppose $n \in S(a, b; k)$. Then $b\sigma(n) \equiv k \pmod{n}$. If $b \nmid k$, then all solutions are sporadic by Definition 2.4 and (2.9) follows at once from Proposition 2.5.

Suppose $b \mid k$ and n is a regular solution. Then $k(1+p) = b(1+p)\sigma(m) = apm + k$. Hence, $b\sigma(m) = k = am$ and in particular, we have

$$a \mid k \quad \text{and} \quad \sigma(k/a) = k/b (= \sigma(m)). \quad (2.10)$$

In other words, if (2.10) is violated, then all solutions are sporadic and once again (2.9) holds. This proves part (b) of Proposition 2.7.

Suppose $ab \mid k$ and $\sigma(k/a) = k/b$. Then the set $\{n \in \mathbb{N} : n = p(k/a), p \nmid (k/a)\}$ consists of all regular solutions and is contained in $S(a, b; k)$. Using Proposition 2.5, the Prime Number Theorem and the bound $\omega(n) = O(\log n)$, it follows that

$$\begin{aligned} \#S(a, b; k; x) &= \pi(ax/k) + O(\log |k|) + O(b^2x^{2/3+o(1)}) \\ &\sim \frac{a}{k} \frac{x}{\log(ax/k)} \end{aligned}$$

as $x \rightarrow \infty$. Hence, part (a) follows and this completes the proof of Proposition 2.7. \square

3. PROOF OF THEOREM 1.1

3.1. Upper bound

We begin by proving the harder part of Theorem 1.1, that is, the upper bounds for $W(\ell; k; x)$. Fix $c \in (0, 1/3)$. Let $k_c(y) := y^c$ and $k(y) \leq k_c(y)$ for $y \geq 1$. Given a function f , we write $\widetilde{W}(\ell; f; x) := \{n \leq x : |\sigma(n) - \ell n| < f(x)\}$. The following inequality is apparent:

$$\#W(\ell; k; x) \leq \#\widetilde{W}(\ell; k; x) \leq \#\widetilde{W}(\ell; k_c; x). \tag{3.1}$$

Let $\ell = a/b > 1$ be in the lowest term. For $n \in \widetilde{W}(\ell; k_c; x)$, we have $b\sigma(n) - an = k$ for some integer k such that $|k| < bx^c$. In particular, we have

$$b\sigma(n) \equiv k \pmod{n} \quad \text{for} \quad |k| < bx^c. \tag{3.2}$$

By Proposition 2.5, the number of $n \in \widetilde{W}(\ell; k_c; x)$ which is a sporadic solution (see Definition 2.4) is $O(bx^c) \cdot O(b^2x^{2/3+o(1)}) = O(b^3x^{2/3+c+o(1)})$, which is acceptable in view of Theorem 1.1. On the other hand, the number of regular solutions $n = pm \in \widetilde{W}(\ell; k_c; x)$ with $p \leq bx^c$ is $O(bx^{c+o(1)})$ by Wirsing's Theorem ([38] or see Section 1). The contribution is clearly negligibly small.

Suppose $p > bx^c$ and $b\sigma(m) = rm$ for some integer $r \geq a + 1$. Then

$$\begin{aligned} |b\sigma(n) - an| &= |b(1+p)\sigma(m) - apm| = |(1+p)rm - apm| = m|r + p(r-a)| \\ &\geq m(r+p) > bx^c. \end{aligned}$$

This contradicts with (3.2)!

Suppose $p > bx^c$ and $b\sigma(m) = rm$ for some integer $r \leq a - 1$. Then

- If $r + p(r - a) \geq 0$ (which implies $p < a$), then a contradiction with $p > bx^c$ arises whenever $x^c > \ell$.
- If $r + p(r - a) < 0$, then $|b\sigma(n) - an| < bx^c \iff m[(a - r)p - r] < bx^c$. By Merten's theorem, the number of such n is at most

$$\begin{aligned} \sum_{2 \leq r \leq a-1} \sum_{bx^c < p \leq x} \frac{bx^c}{(a-r)p-r} &\leq abx^c \sum_{bx^c < p \leq x} \frac{1}{p-(a-1)} \\ &\leq 2abx^c \sum_{p \leq x} \frac{1}{p} \ll abx^c \log \log x \end{aligned}$$

whenever $x^c \geq 2\ell$. The contribution is once again negligible.

Hence, we are left to consider the case when $r = a$, that is,

$$n = pm \in \widetilde{W}(\ell; k_c; x) \quad \text{such that} \quad p > bx^c \quad \text{and} \quad \sigma(m) = \ell m. \tag{3.3}$$

If $\ell \notin \Sigma$, there is clearly no such n and thus $\#W(\ell; k; x) = O(ab^3x^{2/3+c+o(1)})$ by taking into account the paragraphs right above. This proves part (a) of Theorem 1.1.

Suppose $\ell \in \Sigma$. Firstly, observe from partial summation and Wirsing's theorem that

$$\sum_{\sigma(m)=\ell m} \frac{\log m}{m} = \int_1^\infty \frac{\log t}{t} dP_\ell(t) = \lim_{t \rightarrow \infty} \frac{\log t}{t^{1-o(1)}} + \int_1^\infty \frac{\log t}{t^{2-o(1)}} dt \ll 1, \tag{3.4}$$

where $P_\ell(t) := \#\{m \leq t : \sigma(m) = \ell m\}$. As a result, both the series

$$\sum_{\sigma(m)=\ell m} \frac{\log m}{m} \quad \text{and} \quad \sum_{\sigma(m)=\ell m} \frac{1}{m} \tag{3.5}$$

are readily seen to be convergent. Notice that bound (3.4) is uniform in ℓ .

Secondly, we have $m < x^{1-c}$ since $x \geq n = pm > x^c m$. Then

$$0 < \frac{\log m}{\log x} \leq 1 - c < 1$$

and

$$\left(1 - \frac{\log m}{\log x}\right)^{-1} = 1 + O_c\left(\frac{\log m}{\log x}\right). \quad (3.6)$$

Let $\beta > 1$ be given. The Prime Number Theorem implies the existence of a constant $X_0 = X_0(\beta) > 0$ such that $\pi(x) < \beta x / \log x$ whenever $x \geq X_0$. For $x \geq X_0^{1/c}$, the number of n satisfying (3.3) is at most

$$\begin{aligned} \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^{1-c}}} \pi(x/m) &< \beta \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^{1-c}}} \frac{x/m}{\log(x/m)} \\ &< \frac{\beta x}{\log x} \sum_{\sigma(m)=\ell m} \frac{1}{m} + O_c\left(\frac{\beta x}{(\log x)^2} \sum_{\sigma(m)=\ell m} \frac{\log m}{m}\right) \\ &< \frac{\beta x}{\log x} \sum_{\sigma(m)=\ell m} \frac{1}{m} + O_c\left(\frac{\beta x}{(\log x)^2}\right), \end{aligned}$$

with the help of (3.6) and the convergence of the series in (3.5). Therefore, we have

$$\limsup_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x/\log x} \leq \beta \sum_{\sigma(m)=\ell m} \frac{1}{m} \quad (3.7)$$

for any $\beta > 1$. Let $\beta \rightarrow 1+$ in (3.7), it follows that

$$\limsup_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x/\log x} \leq \sum_{\sigma(m)=\ell m} \frac{1}{m}.$$

This proves the upper bound of Theorem 1.1(b).

3.2. Lower bound

The proof of the lower bound for Theorem 1.1(b) is relatively straightforward. Suppose $\ell \in \Sigma$. Based on the experience of Pomerance et al. (see Section 2.2), a lower bound for $\#W(\ell; k; x)$ can be obtained by estimating the size of the set

$$\mathcal{L}_\ell(\delta) := \{n \leq x : n = pm, p \nmid m, \sigma(m) = \ell m, \ell m < (pm)^\delta\},$$

provided that $k(y) \geq y^\delta$ for any $y \geq 1$, where $\delta \in (0, 1)$. We have

$$\#\mathcal{L}_\ell(\delta) = \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \sum_{\substack{(\ell m)^{1/\delta}/m < p \leq x/m \\ p \nmid m}} 1.$$

Using the bound $\omega(m) = O(\log m)$, Wirsing's theorem and partial summation, it follows that

$$\#\mathcal{L}_\ell(\delta) = \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \sum_{(\ell m)^{1/\delta}/m < p \leq x/m} 1 + O_\delta(x^{o(1)})$$

and

$$\sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \sum_{p \leq (\ell m)^{1/\delta}/m} 1 \ll \ell^{1/\delta} \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta}} m^{1/\delta-1} \ll \ell^{1/\delta} x^{1-\delta+o(1)}.$$

Hence,

$$\#\mathcal{L}_\ell(\delta) = \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \pi(x/m) + O_\delta\left(\ell^{1/\delta} x^{1-\delta+o(1)}\right).$$

Let $\alpha < 1$. By the Prime Number Theorem, there exists $x_0 = x_0(\alpha) > 0$ such that $\pi(x) > \alpha x / \log x$ whenever $x \geq x_0$. Thus, if $x > (x_0)^{1/(1-\delta)}$, then

$$\begin{aligned} \#\mathcal{L}_\ell(\delta) &> \frac{\alpha x}{\log x} \sum_{\substack{\sigma(m)=\ell m \\ m \leq x^\delta/\ell}} \frac{1}{m} + O_\delta\left(\ell^{1/\delta} x^{1-\delta+o(1)}\right) \\ &= \frac{\alpha x}{\log x} \left(\sum_{\sigma(m)=\ell m} \frac{1}{m} + O\left((\ell/x^\delta)^{1+o(1)}\right) \right) + O_\delta\left(\ell^{1/\delta} x^{1-\delta+o(1)}\right). \end{aligned}$$

From this, we may deduce that

$$\liminf_{x \rightarrow \infty} \frac{\#W(\ell; k; x)}{x/\log x} > \alpha \sum_{\sigma(m)=\ell m} \frac{1}{m}.$$

Since this holds for any $\alpha < 1$, the lower bound for Theorem 1.1(b) follows.

REMARK 3.1 Suppose $\ell \in \Sigma$. Proposition 2.7 implies that when $k \equiv k_0 \geq 1$ is a constant function, we have

$$\frac{\#W(\ell; k_0; x)}{x/\log x} = \sum_{\substack{|k| < bk_0 \\ b|m \\ \sigma(m)=\ell m}} \frac{\#S(a, b; k; x)}{x/\log x} \sim \sum_{\substack{0 < m < k_0/\ell \\ b|m \\ \sigma(m)=\ell m}} \frac{1}{m} \tag{3.11}$$

as $x \rightarrow \infty$. The rightmost quantity of (3.11) converges to $\sum_{\substack{\sigma(m)=\ell m \\ b|m}} 1/m$ as $k_0 \rightarrow \infty$. In

particular, when $\ell \in \mathbb{Z}$ and k is an increasing unbound function, one readily observes that

$$\frac{\#W(\ell; k; x)}{x/\log x} \sim \lim_{k_0 \rightarrow \infty} \frac{\#W(\ell; k_0; x)}{x/\log x} \tag{3.12}$$

as $x \rightarrow \infty$. However, the asymptotic (3.12) is not necessarily true when $\ell \notin \mathbb{Z}$ because of the restriction $b \mid m$ present in (3.11)!

4. CONCLUDING DISCUSSIONS, NUMERICS AND FURTHER DIRECTIONS

Building upon the method of [3], Pollack–Pomerance–Thompson [31] recently proved a variant of the main result of [3], albeit with a weaker error term and uniformity. Specifically, they proved that if $\ell \in \mathbb{Z}$ is kept fixed, the number of sporadic solutions to the equation $\sigma(n) = \ell n + k$ up to x is

$O(x^{3/5+o_\ell(1)})$ as $x \rightarrow \infty$ and for any integer k . To the best of the authors' knowledge, there seem to be a number of subtleties in generalizing the method of [31] to the equation $b\sigma(n) = an + k$. Additionally, as noted in [3], it appears that obtaining an estimate that is fully uniform in all of a, b, k would require considerable effort.

If ℓ is restricted to be an integer and is kept fixed, the same argument from Theorem 1.1 along with the main theorem of [31] should yield the slightly improved admissible range $c \in (0, 2/5)$. However, the barrier of our method seems to be $c \in (0, 1/2)$, see [31, Conjecture 4.3]. When ℓ is not an integer, it is unclear what the barrier should be and likely somewhat smaller than $(0, 1/2)$.

Theorem 1.1 leads to several interesting consequences which are stated as follows. As usual, $k : [1, \infty) \rightarrow (0, \infty)$ is an increasing function such that

$$y^\delta \leq k(y) \leq y^c \quad \text{for} \quad y \geq 1. \tag{4.1}$$

Firstly, it is natural to consider the following quantity:

$$\mathcal{D}_c(\ell) := \lim_{x \rightarrow \infty} \frac{\#W(\ell; y^c; x)}{x/\log x} =: \lim_{x \rightarrow \infty} \mathcal{D}_c(\ell; x)$$

for $\ell \in [1, \infty)$ and $c \in (0, 1)$. In view of Proposition 2.2, this new 'distribution function' is arguably well-suited to study the within-perfect numbers with respect to the sublinear threshold. However, this distribution function behaves quite differently from the ones described in Definition 2.1.

PROPOSITION 4.1 *The function $\ell \mapsto \mathcal{D}_c(\ell)$ is discontinuous on a dense subset of $[1, \infty)$, for any $c \in (0, 1/3)$.*

Proof. It follows from a theorem of Anderson (see [27, p. 270]) that $(\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$ is dense in $[1, \infty)$. Observe that \mathcal{D}_c takes the value 0 on $(\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$, but it takes positive values on Σ by Theorem 1.1. So, \mathcal{D}_c is discontinuous on Σ . It is a well-known theorem that Σ is again dense in $[1, \infty)$ (see [27, p. 275]). This completes the proof. □

Secondly, a real number $\ell > 1$ is said to be Σ -approximable if there exist a function $f(x) \rightarrow \infty$ and a sequence of positive integers $(m_i)_{i \geq 1}$ such that $|\ell - \sigma(m_i)/m_i| < 1/f(m_i)$ for any $i \geq 1$. It is clear that

PROPOSITION 4.2 *If $\ell > 1$ is Σ -approximable, then $\#W(\ell; k; x) \gg x/\log x$ on an unbound set of x .*

In fact, for any function $f(x) \rightarrow \infty$, there are irrational numbers $\ell > 1$ that are Σ -approximable by f . This follows from the standard nested interval argument and the theorems of Anderson used in the proof of Proposition 4.1.

We conclude this article with some numerics and open problems for further investigation.

Table 1. $\mathcal{D}_c(2; x)$ for various values of x and c

$k(y)$	$x = 1,000,000$	$x = 10,000,000$	$x = 20,000,000$
$y^{0.9}$	3.661860	3.305180	3.196040
$y^{0.8}$	1.141480	0.945623	0.908751
$y^{0.7}$	0.494278	0.435395	0.426470
$y^{0.6}$	0.311567	0.274586	0.267904
$y^{0.5}$	0.276559	0.259482	0.255962
$y^{0.4}$	0.264968	0.252956	0.250063
$y^{0.3}$	0.225980	0.247837	0.247299
$y^{0.2}$	0.151238	0.195911	0.197430

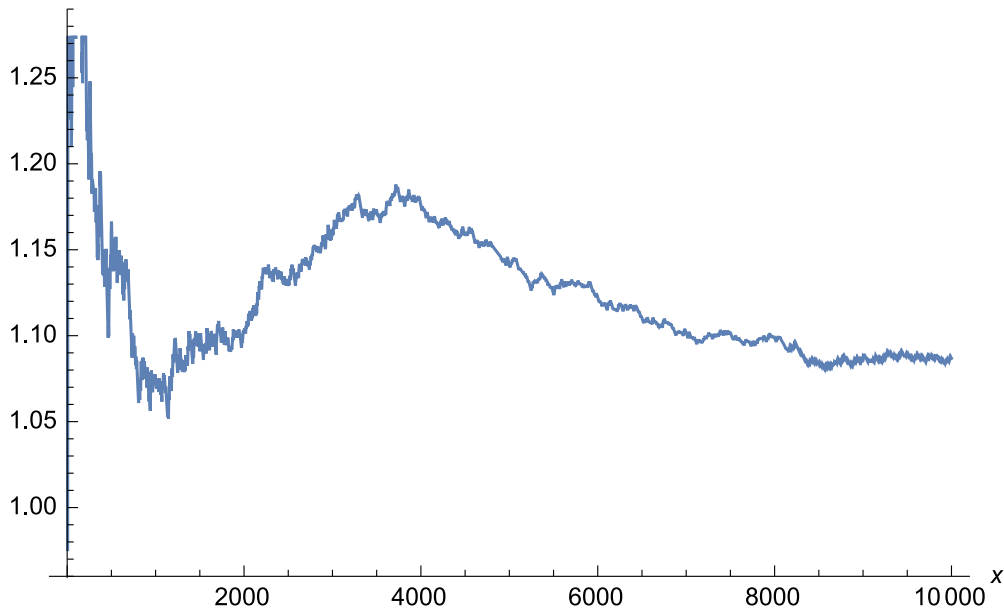


Figure 1. This plot shows the quantity $\#W(2; k; x)/(x/\log x)$ with $k(y) = y/\log y$ for $x = 2$ to $10,000$.

We calculate the quotient of $D_c(2; x)$ for various values of $c \in (0, 1)$ and at $x = 1,000,000$, $x = 10,000,000$ and $x = 20,000,000$. Note: $\sum_{\sigma(m)=2m} \frac{1}{m} \approx 0.2045$.

It is natural to ask the following.

PROBLEM 4.3 When $c \in (1/2, 1)$, does the correct order of magnitude for $\#W(\ell; k; x)$, with $\ell \in \Sigma$ and k satisfying (4.1), continue to be $x/\log x$ as $x \rightarrow \infty$?

In between the sublinear and linear regime, for example, $k(y) = y/\log y$, Proposition 2.2 gives no conclusion. Consider the plot of $x \mapsto \#W(2; k; x)/(x/\log x)$ for such k from $x = 2$ to $x = 10,000$:

PROBLEM 4.4 What is the order of magnitude of $\#W(\ell; k; x)$ if the function k satisfies $y^c = o(k(y))$ for any $c \in (0, 1)$?

PROBLEM 4.5 What is the order of magnitude for $\#W(\ell; k; x)$ for irrational ℓ ?

PROBLEM 4.6 Determine the set of points of continuity for the distribution function $\ell \mapsto \mathcal{D}_c(\ell)$.

ACKNOWLEDGMENTS

This material is based upon the work supported by the Engineering and Physical Sciences Research Council (EPSRC) grant: EP/W009838/1, the Williams SMALL REU Program under the National Science Foundation (NSF) grants DMS2341670, DMS1265673, DMS1561945 and DMS1347804, as well as the Swedish Research Council under grant no. 2021-06 594 while the first author was in residence at Institut Mittag-Leffler in Djursholm, Sweden, during the year 2024. Both the authors would like to express their gratitude to the participants of the SMALL REU (2015–16) for their constant encouragement and also for the generous hospitality in Williamstown and Djursholm. It is a great pleasure to thank the reviewer for a careful reading of the article and his/her valuable comments.

REFERENCES

1. Alkan, Emre; Ford, Kevin; Zaharescu, Alexandru. Diophantine approximation with arithmetic functions, I, *Trans. Am. Math. Soc.* **361** (2008), 2263–2275, [10.1090/S0002-9947-08-04822-8](https://doi.org/10.1090/S0002-9947-08-04822-8).
2. Alkan, Emre; Ford, Kevin; Zaharescu, Alexandru. Diophantine approximation with arithmetic functions, II, *Bull. Lond. Math. Soc.* **41** (2009), 676–682, [10.1112/blms/bdp051](https://doi.org/10.1112/blms/bdp051).
3. Anavi, Aria; Pollack, Paul; Pomerance, Carl. On congruences of the form $\sigma(n) \equiv a \pmod{n}$, *Int. J. Number Theory* **9** (2013), 115–124, [10.1142/S1793042112501266](https://doi.org/10.1142/S1793042112501266).
4. Baker, R. C.; Harman, G.; Pintz, J. The difference between consecutive primes, II, *Proc. Lond. Math. Soc.* **83** (2001) 532–562, [10.1112/plms/83.3.532](https://doi.org/10.1112/plms/83.3.532).
5. Cohen, Peter; Cordwell, Katherine; Epstein, Alyssa; Kwan, C.-H.; Lott, Adam; Miller, S J. On near-perfect numbers, *Acta Arith.* **194** (2020), 341–366, [10.4064/aa180821-11-10](https://doi.org/10.4064/aa180821-11-10).
6. Davenport, H. Über numeri abundantes, *S.-Ber. Preuß. Akad. Wiss., math.-nat. Kl.* (1933), 830–837.
7. Dickson, Leonard Eugene. *History of the Theory of Numbers. Vol. I: Divisibility and Primality*, Chelsea Publishing Co., New York, NY, 1966, xii+486.
8. Dickson, L E. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors, *Am. J. Mathematics* **35** (1913), 413–422, [10.2307/2370405](https://doi.org/10.2307/2370405).
9. Davis, Nichole; Klyve, Dominic; Kraght, Nicole. On the difference between an integer and the sum of its proper divisors, *Involve J. Math.* **6** (2013), 493–504, [10.2140/involve.2013.6.493](https://doi.org/10.2140/involve.2013.6.493).
10. Erdős, P. On perfect and multiply perfect numbers, *Ann. Mat. Pura Appl.* **42** (1956), 253–258, [10.1007/BF02411879](https://doi.org/10.1007/BF02411879).
11. Erdos, Paul; Wintner, Aurel. Additive arithmetical functions and statistical independence, *Am. J. Math.* **61** (1939), 713–721, [10.2307/2371326](https://doi.org/10.2307/2371326).
12. Euclid of Alexandria. Number theory: book 9 proposition 36, *The Elements*, Clay Mathematics Institute (available at: https://www.claymath.org/euclid_index/number-theory/?chapter=35, Date accessed 20 August 2024).
13. Guy, Richard K. Unsolved problems in number theory, *Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, I*, 2nd edn, Springer-Verlag, New York, NY, 1994, xvi+285.
14. Harman, Glyn. Diophantine approximation with multiplicative functions, *Mon. hefte Math.* **160** (2010), 51–57, [10.1007/s00605-008-0071-1](https://doi.org/10.1007/s00605-008-0071-1).
15. Heath-Brown, D. R. Odd perfect numbers, *Math. Proc. Camb. Phil. Soc.* **115** (1994), 191–196, [10.1017/S0305004100072030](https://doi.org/10.1017/S0305004100072030).
16. Heath-Brown, D. R. The differences between consecutive primes, V, *Int. Math. Res. Not.* **2021** (2021), 17514–17562, [10.1093/imrn/rnz295](https://doi.org/10.1093/imrn/rnz295).
17. Hornfeck, Bernhard. Zur Dichte der Menge der vollkommenen Zahlen, *Arch. Math.* **6** (1955), 442–443, [10.1007/BF01901120](https://doi.org/10.1007/BF01901120).
18. Hornfeck, B.; Wirsing E. Über die Häufigkeit vollkommener Zahlen, *Math. Ann.* **133** (1957), 431–438, [10.1007/BF01343756](https://doi.org/10.1007/BF01343756).
19. Järviniemi, Olli. On large differences between consecutive primes, 2022, arXiv:2212.10965 [math.NT].
20. Kanold, H.-J. Über die Dichten der Mengen der vollkommenen und der befreundeten Zahlen, *Math. Z.* **61** (1954), 180–185, [10.1007/BF01181341](https://doi.org/10.1007/BF01181341).
21. Kanold, H.-J. Über die Verteilung der vollkommenen Zahlen und allgemeinerer Zahlenmengen, *Math. Ann.* **132** (1957), 442–450, [10.1007/BF01350158](https://doi.org/10.1007/BF01350158).
22. Makowski, A. Some equations involving the sum of divisors, *Elem. Math.* **34** (1979), 82.
23. Pomerance, Carl. On the congruences $\sigma(n) \equiv a \pmod{n}$ and $n \equiv a \pmod{\phi(n)}$, *Acta Arith.* **26** (1975), 265–272, [10.4064/aa-26-3-265-272](https://doi.org/10.4064/aa-26-3-265-272).
24. Pomerance, Carl. On composite n for which $\phi(n)|n-1$, *Acta Arith.* **28** (1976), 387–389, [10.4064/aa-28-4-387-389](https://doi.org/10.4064/aa-28-4-387-389).
25. Pomerance, Carl. On composite n for which $\phi(n)|n-1$, II, *Pac. J. Math.* **69** (1977), 177–186, [10.2140/pjm.1977.69.177](https://doi.org/10.2140/pjm.1977.69.177).
26. Pomerance, Carl. Multiply perfect numbers, Mersenne primes, and effective computability, *Math. Ann.* **226** (1977), 195–206, [10.1007/BF01362422](https://doi.org/10.1007/BF01362422).
27. Pollack, Paul. *Not Always Buried Deep. A Second Course in Elementary Number Theory*, American Mathematical Society, Providence, RI, 2009, xvi+303.
28. Pollack, Paul. On the greatest common divisor of a number and its sum of divisors, *Michigan Math. J.* **60** (2011), 199–214, [10.1307/mmj/1301586311](https://doi.org/10.1307/mmj/1301586311).
29. Pollack, Paul; Pomerance, Carl. On the distribution of some integers related to perfect and amicable numbers, *Coll. Math.* **130** (2013), 169–182, [10.4064/cm130-2-3](https://doi.org/10.4064/cm130-2-3).
30. Pollack, Paul; Pomerance, Carl. Some problems of Erdős on the sum-of-divisors function, *Trans. Am. Math. Soc.* **B 3** (2016), 1–26, [10.1090/btran/10](https://doi.org/10.1090/btran/10).

31. Pollack, Paul; Pomerance, Carl; Thompson, Lola. Divisor-sum fibers, *Mathematika* **64** (2018), 330–342, [10.1112/S0025579317000535](https://doi.org/10.1112/S0025579317000535).
32. Pollack, Paul; Shevelev Vladimir, On perfect and near-perfect numbers, *J. Number Theory* **132** (2012), 3037–3046, [10.1016/j.jnt.2012.06.008](https://doi.org/10.1016/j.jnt.2012.06.008).
33. Schinzel, A. On functions $\phi(n)$ and $\sigma(n)$, *Bull. Acad. Pol. Sci. Cl. III* **3** (1955), 415–419.
34. Sierpiński, W. Sur les nombres pseudoparfaits, *Mat. Vesnik* **2** (1965), 212–213.
35. Stadlmann, Julia. On the mean square gap between primes, 2022 (arXiv:[2212.10867](https://arxiv.org/abs/2212.10867) [math.NT]).
36. G. Tenenbaum, *Introduction to Analytic and Probabilistic Number theory*, Cambridge University Press, Cambridge, 1995, English edition.
37. B. Volkmann, Ein Satz über die Menge der vollkommenen Zahlen, *J. Reine Angew. Math.* **1955** (1955), 152–155, [10.1515/crll.1955.195.152](https://doi.org/10.1515/crll.1955.195.152).
38. E. Wirsing, Bemerkung zu der Arbeit Über vollkommene Zahlen in *Math. Ann.* Bd. 133, S. 431-438 (1957), *Math. Ann.* **137** (1959), 316–318, [10.1007/BF01360967](https://doi.org/10.1007/BF01360967).
39. D. Wolke, Eine Bemerkung über die Werte der Funktion $\sigma(n)$, *Mon. hefte Math.* **83** (1977), 163–166, [10.1007/BF01534638](https://doi.org/10.1007/BF01534638).