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ORIGINAL ARTICLE

A Continuous Proof of Zassenhaus's Solubility Theorem

F. E. A. Johnson[1]

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Abstract
A class $\mathcal{S}$ of soluble groups is *D-bounded* when there exists a uniform upper bound for the lengths $d(\Gamma)$ of the derived series for $\Gamma \in \mathcal{S}$. A theorem of Zassenhaus (Abh. Math. Semin. Hansisch. Univ. **12**, 289–312, 1938) states that for each n the class of soluble subgroups of $GL(n, \mathbb{C})$ is D-bounded. Although Zassenhaus's theorem is fundamental to the study infinite discrete linear groups the proof given here is located within the theory of continuous groups and the only discrete groups which appear are finite.

Keywords Linear group · Soluble group · Compact Lie group

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1 Introduction

If Γ is a soluble group with derived series $\{D_r(\Gamma)\}_{1\le r}$ its *derived length* $d(\Gamma)$ is

$$d(\Gamma) = \min\{n \mid D_n(\Gamma) = \{1\}\}.$$

A class $\mathcal{S}$ of soluble groups is said to be *D-bounded* when there exists an integer n such that $d(\Gamma) \le n$ for all $\Gamma \in \mathcal{S}$. The theorem of Zassenhaus [13] is of fundamental importance in studying infinite discrete subgroups of Lie groups. It states that for any given positive integer n the class of soluble subgroups of $GL(n, \mathbb{C})$ is D-bounded. Despite its fundamental importance, its coverage in the literature has been rather neglected. In this paper, we shall re-prove Zassenhaus's theorem in the form:

Theorem 1.1 *If Γ is a soluble subgroup of $GL(n, \mathbb{C})$ then $d(\Gamma) \le (2n^2 - 3)\log_2(n) + 6$.*

Zassenhaus's proof is intricate. Its essential feature is the iterated use of the theorem of A.H. Clifford [3]. Our proof follows the same strategy as that of Zassenhaus but with different tactics. The inclusion $\Gamma \subset GL(n, \mathbb{C})$ defines an n-dimensional representation of Γ and Clifford's theorem applies only in the special case where this representation and its restrictions to subgroups are completely reducible.

✉ F. E. A. Johnson
feaj@math.ucl.ac.uk

1 Department of Mathematics, University College London, Gower Street, WC1E 6BT London, U.K.

In Zassenhaus' account, more work is required to reduce the general case to the above special case. However, as we show, this aspect of Zassenhaus's proof can be circumvented by a straightforward application of the work of Borel on linear algebraic groups, [1], which was not available when Zassenhaus wrote his paper. Using Borel's work and a theorem of Mostow [9] we can, in a single step, reduce the problem to the case where Γ is a compact Lie subgroup of $GL(n, \mathbb{C})$. Then by the Peter–Weyl theorem ([11, 12]) the finite dimensional representations of Γ and its closed subgroups are all completely reducible and Clifford's theorem applies immediately.

As is implicit in the title of [13], Zassenhaus's theorem was motivated by the study subgroups of Lie groups which are *infinite and discrete*. It is perhaps paradoxical therefore that the proof given here is located within the theory of continuous groups and that the only discrete groups which appear are finite.

To describe our approach in detail we introduce the following notation:

$$\begin{aligned} \mathcal{S}(n): &\quad \text{the class of soluble subgroups of } GL(n,\mathbb{C}),\\ \mathcal{C}(n): &\quad \text{the class of compact soluble subgroups of } GL(n,\mathbb{C}),\\ \Sigma_n: &\quad \text{the group of permutations of } \{1,\ldots,n\},\\ \Pi(n): &\quad \text{the set of soluble subgroups of } \Sigma_n. \end{aligned}$$

As $\Pi(n)$ is finite it is D-bounded and we denote by $\pi(n)$ its D bound:

$$\pi(n) = \max\{d(H) \mid H \in \Pi(n)\}.$$

The class $\mathcal{C}(1)$ consists of the 1-dimensional torus $U(1)\{z \in \mathbb{C} : |\lambda| = 1\}$ together with its finite subgroups. As these are all abelian then $d(\Gamma) = 1$ for all nontrivial $\Gamma \in \mathcal{C}(1)$. The essence of the proof is then to show inductively that:

Theorem 1.2 *$\mathcal{C}(n)$ is D-bounded by* $c(n) \le \max\{c(n-1) + \pi(n) + 1, \pi(n^2) + 3\}$.

That being so, it follows from a theorem of Mostow [9] that:

Theorem 1.3 *$\mathcal{S}(n)$ is D-bounded by* $s(n) \le c(n) + \log_2(n) + 2$.

It remains to estimate the size of $c(n)$, for which it is first necessary to do the same for $\pi(n)$ and $\pi(n^2)$. A straightforward, if crude, estimate shows that:

$$\begin{cases} \pi(n) \le \log_2(n!) \le (n-2)\log_2(n) + 1, \\ \pi(n^2) \le \log_2(n^2!) \le 2(n^2-2)\log_2(n) + 1. \end{cases}$$

With these estimates Theorem 1.1 follows directly from Theorems 1.2 and 1.3.

Whilst these estimates are by no means best possible they are nevertheless better than cubic in n. By contrast, in Zassenhaus's paper the bounds are not stated explicitly although one of the preliminary bounds is already beyond astronomical, for example $(n^{n^2+1})!$ ([13, p. 294]) which, for $n = 3$, already vastly exceeds the number of atoms in the Milky Way. The best estimates are rather complicated but are essentially of first order in n. They depend upon a far more detailed analysis of soluble subgroups of Σ_n; for a detailed discussion on this point we refer the reader to the paper of M.F. Newman [10].

2 Soluble Groups

If G is a group we denote by $(D_r(G))_{1\le r}$ its *derived series*; that is

$$D_0(G) = G, \quad D_{r+1}(G) = [D_r(G), D_r(G)].$$

G is *soluble* when $D_m(G) = \{1\}$ for some m; the *derived length* $d(G)$ is then

$$d(G) = \min\{m \mid D_m(G) = \{1\}\}.$$

We note the following:

(2.1) Let H be a subgroup of G. If G is soluble then so is H and $d(H) \leq d(G)$.
(2.2) Let $\varphi : G \to Q$ be a surjective group homomorphism. If G is soluble then so is Q and $d(Q) \leq d(G)$.

If $1 \to K \to G \to Q \to 1$ is an exact sequence of groups then G is soluble if and only if K and Q are both soluble, in which case:

$$d(G) \leq d(K) + d(Q). \tag{2.3}$$

In the special case where the extension is a direct product we have:

$$d(K \times Q) = \max\{d(K), d(Q)\}. \tag{2.4}$$

We denote by $(L_r(G))_{1 \leq r}$ the *lower central series* of G, that is

$$L_1(G) = G, \quad L_{r+1}(G) = [G, L_r(G)].$$

G is *nilpotent* when $L_\mu(G) = \{1\}$ for some μ; the *nilpotent length* $l(G)$ is then

$$l(G) = \min\{m \mid L_{m+1}(G) = \{1\}\}.$$

The basic relation between the derived series and the lower central series is that $D_n(G) \subset L_{2^n}(G)$ (cf. [5, p. 15]). It follows easily that

(2.5) If G is nilpotent then G is soluble and $d(G) \leq \log_2(l(G) + 1) + 1$.

3 Soluble Groups of Restricted Type

If G is a group we denote its centre by $\mathcal{Z}(G)$. Let n be an integer ≥ 1. By an $\mathcal{R}(n)$ *structure* we mean a triple (H, Z, A), where

(I) H is a finite soluble group and $Z = \mathcal{Z}(H)$ is isomorphic to C_m for some m;
(II) if N is an abelian normal subgroup of H then $N \subset Z$;
(III) H/Z has a maximal abelian normal subgroup A such that $|A| = n$.

We say that the soluble group H is *restricted of type* n when it admits such an $\mathcal{R}(n)$ structure and we denote by $\mathcal{R}(n)$ the class of such groups. Given an $\mathcal{R}(n)$ structure (H, Z, A) we denote by $\pi : H \to H/Z$ the canonical homomorphism. We define

(IV) $\Gamma = \pi^{-1}(A)$;
(V) $C = \{c \in H \mid c\gamma = \gamma c \text{ for all } \gamma \in \Gamma\}$

and denote by

(VI) $\mathcal{Z}^1(A, Z)$ the group of 1-cocycles of A with values in Z.

We now suppose given an $\mathcal{R}(n)$ structure (H, Z, A).

Proposition 3.1 $C = Z$.

Proof We first show that C is normal in H; thus let $h \in H$, $c \in C$ and $\gamma \in \Gamma$. As Γ is normal in G then $(h^{-1}\gamma h) \in \Gamma$ and so $c(h^{-1}\gamma h) = (h^{-1}\gamma h)c\}$ and hence

$$(hch^{-1})\gamma = h\{c(h^{-1}\gamma g\}h^{-1} = h\{(h^{-1}\gamma h)c\}h^{-1} = \gamma(hch^{-1}).$$

As this is true for all $\gamma \in \Gamma$ then $hch^{-1} \in C$ and C is normal as claimed.

As H is soluble then so is C so define $k = d(C)$ and put $C_r = D_r(C)$ for $1 \le r \le k$. We claim that C is abelian. Thus suppose not so that $k \ge 2$ and C_{k-2} is nonabelian. As C_r is a characteristic subgroup of C each C_r is normal in H. As each C_r commutes with Γ then $C_r \cap \Gamma$ is an abelian normal subgroup. Observe that $\Gamma \subset \Gamma \cdot C_{k-2}$. If $\Gamma = \Gamma \cdot C_{k-2}$ then $C_{k-2} \subset \Gamma$ and hence $C_{k-2} \subset C_{k-2} \cap \Gamma$. This is a contradiction as C_{k-2} is nonabelian and $C_{k-2} \cap \Gamma$ is abelian.

Thus $\Gamma \ne \Gamma \cdot C_{k-2}$ and we may choose $x \in \Gamma \cdot C_{k-2}$ such that $x \notin \Gamma$. We claim that $\pi(x) \notin p(\Gamma)$. Otherwise, if $\pi(x) = \pi(\gamma)$ for some $\gamma \in \Gamma$ then $x\gamma^{-1} \in \mathrm{Ker}(\pi) = Z \subset \Gamma$ yielding the contradiction that $x \in \Gamma$. Thus $\pi(\Gamma)$ is a proper subgroup of $\pi(\Gamma \cdot C_{k-2})$. As A is abelian it follows from the exact sequence $1 \to Z \to \Gamma \to A \to 1$ that $[\Gamma, \Gamma] \subset Z$. Also $[C_{k-2}, C_{k-2}] = C_{k-1}$ is an abelian normal subgroup of H so that by (II), we see that $[C_{k-2}, C_{k-2}] \subset Z$. Let $x_1, x_2 \in C_{k-2}$ and $\gamma_1, \gamma_2 \in \Gamma$. As C_{k-2} commutes with Γ then

$$[\gamma_1 x_1, \gamma_2 x_2] = [\gamma_1, \gamma_2][x_1, x_2] \in Z.$$

Hence $[\Gamma \cdot C_{k-2}, \Gamma \cdot C_{k-2}] = \{1\}$ and so also $[\pi(\Gamma \cdot C_{k-2}), \pi(\Gamma \cdot C_{k-2})] = \{1\}$. Thus $\pi(\Gamma \cdot C_{k-2})$ is an abelian normal subgroup of H/Z which properly contains $A = \pi(\Gamma)$ thereby contradicting (III). Hence, C is abelian as claimed. It now follows from (II) that $C \subset Z$. Evidently $Z \subset C$ so that $C = Z$. □

Proposition 3.2 *H/Z is an extension $1 \to K \to H/Z \to Q \to 1$, where K is a subgroup of $\mathcal{Z}^1(A, Z)$ and Q is a subgroup of* $\mathrm{Aut}(Z) \times \mathrm{Aut}(A)$.

Proof Clearly $Z \subset \mathcal{Z}(\Gamma)$. However, as $\mathcal{Z}(\Gamma)$ centralizes Γ then $\mathcal{Z}(\Gamma) \subset C$. Thus $Z = \mathcal{Z}(\Gamma)$ by Proposition 3.1 and so Z is characteristic subgroup of Γ. Thus any automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ gives rise to an automorphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & \Gamma & \longrightarrow & A & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_1(\alpha)} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \rho_2(\alpha)} & & \\ 1 & \longrightarrow & Z & \longrightarrow & \Gamma & \longrightarrow & A & \to & 1. \end{array}$$

Putting $\rho = (\rho_1, \rho_2)$ we have exact sequence (cf. [7, pp. 204–205]).

$$1 \to \mathcal{Z}^1(A, Z) \to \mathrm{Aut}(\Gamma) \xrightarrow{\rho} \mathrm{Aut}(Z) \times \mathrm{Aut}(A).$$

Let $c : H \to \mathrm{Aut}(\Gamma)$ be the conjugation homomorphism $c(h)(\gamma) = h\gamma h^{-1}$. As $\mathrm{Ker}(c) = C = Z$, c induces an injective homomorphism $c_* : H/K \to \mathrm{Aut}(\Gamma)$. Hence, we have an exact sequence $1 \to K \to H/Z \to Q \to 1$, where $K = \mathcal{Z}^1(A, Z) \cap \mathrm{Im}(c_*)$ and $Q = \mathrm{Im}(\rho \circ c_*)$. □

As in the Introduction we denote by $\Pi(n)$ the set of soluble subgroups of Σ_n. The degenerate case $n = 1$ has derived length zero. To avoid this we put

$$\pi(n) = \max\{1, d(H) \mid H \in \Pi(n)\}.$$

The first few values are $\pi(1) = 1$, $\pi(2) = 1$, $\pi(3) = 2$, $\pi(4) = 3$, after which matters become progressively more complicated. We give a crude though effective upper bound for $\pi(n)$ in Section 7.

Theorem 3.3 *If* $H \in \mathcal{R}(n)$ *then* $d(H) \le \pi(n) + 2$.

Proof From the exact sequence $1 \to K \to H/Z \to Q \to 1$ it follows that $d(H/Z) \le d(K) + d(Q)$ and hence $d(H) \le d(Z) + d(K) + d(Q)$. As Z is abelian then $d(Z) \le 1$. As K is a subgroup of the abelian group $\mathcal{Z}^1(A, Z)$ then $d(K) \le 1$. As $Z \cong C_m$ then $\mathrm{Aut}(Z)$ is the cyclic group of order $\phi(m)$, where ϕ is Euler's totient function, and so $d(\mathrm{Aut}(Z)) \le 1$. Thus $d(H) \le d(Q) + 2$. As Q is a subgroup of $\mathrm{Aut}(Z) \times \mathrm{Aut}(A)$ then $d(Q) \le \max\{d(\mathrm{Aut}(Z)), d(\mathrm{Aut}(A))\}$. As $\mathrm{Aut}(Z)$ is abelian then $d(\mathrm{Aut}(Z))1$. If $n = 1$ then $\mathrm{Aut}(A)$ is trivial and $d(\mathrm{Aut}(A)) \le 1$. If $n \ne 2$ then $\mathrm{Aut}(A)$ is a subgroup of Σ_n and $d(\mathrm{Aut}(A)) \le \pi(n)$. Either way, $d(Q) \le \pi(n)$ and $d(H) \le \pi(n) + 2$. □

The cases which arise in practice are groups of type $\mathcal{R}(n^2)$ in which case one gets:

(3.4) If $H \in \mathcal{R}(n^2)$ then $d(H) \le \pi(n^2) + 2$.

4 Lie Groups and Algebraic Groups

We recall some standard facts about Lie groups; proofs of these can be found in many places, for example [2, 6]. Thus, let G be a Lie group, then

(4.1) G admits a unique real analytic structure with respect to which any continuous homomorphism $f : G \to G'$ of Lie groups is real analytic;
(4.2) If K is a closed subgroup of G then K is a real analytic submanifold of G and hence is a Lie group in its own right;
(4.3) The centre $\mathcal{Z}(G)$ of G is a closed subgroup;
(4.4) If G has only finitely many components then its identity component G_0 is a closed normal subgroup and G is an extension $1 \to G_0 \to G \to \Phi \to 1$, where Φ is a finite group; in general this extension is nonsplit.

One sees easily that a compact Lie group has only finitely many connected components. We appeal to the following which in the connected case is due to E. Cartan but in this level of generality is due to Mostow [9].

(4.5) If G has only finitely many connected components then G contains a maximal compact subgroup K such that G/K is diffeomorphic to a Euclidean space. In particular, G/K is connected.

When G is compact then (cf. [11, 12]) any continuous representation $\rho : G \to GL(n, \mathbb{C})$ decomposes as a direct sum $(G, \rho) \cong \bigoplus_{i=1}^{e} (G, \rho_i)$, where each (G, ρ_i) is an irreducible representation. In particular,

(4.6) If G is a compact abelian Lie group then any continuous representation $\rho : G \to GL(n, \mathbb{C})$ decomposes as a direct sum $(G, \rho) \cong \bigoplus_{i=1}^{e} (G, \rho_i)$, where each $\rho_i : G \to \mathbb{C}^*$ is a 1-dimensional representation.

We denote by $U(1) = \{z \in \mathbb{C} : |z| = 1\}$ is the 1-dimensional torus, then

(4.7) If G is compact and soluble then, $G_0 \cong \underbrace{U(1) \times \cdots \times U(1)}_{m}$ for some m.

We next recall some standard facts about linear algebraic groups for which the standard reference is [1]. Thus, if N is a positive integer a subset $X \subset \mathbb{C}^N$ is *algebraic* when it is defined by the vanishing of a finite set of polynomial equations in the variables $(x_i)_{1 \le i \le N}$.

Let $M_n(\mathbb{C})$ be the ring of $n \times n$ matrices over $\mathbb{C}$; we denote a typical element of $M_n(\mathbb{C})$ by $X = (X_{ij})_{1 \le i,j \le n}$ and a typical element of $M_n(\mathbb{C}) \times \mathbb{C}$ by $((X_{ij}), y)$. We identify $M_n(\mathbb{C}) \times \mathbb{C}$ with $\mathbb{C}^{n^2+1}$ by re-indexing coordinates as follows:

$$X_{ij} \longleftrightarrow x_{n(i-1)+j}, \quad y \longleftrightarrow x_{n^2+1}.$$

Let $\nu : M_n(\mathbb{C}) \times \mathbb{C} \xrightarrow{\simeq} \mathbb{C}^{n^2+1}$ be the linear isomorphism so obtained. A subset $A \subset M_n(\mathbb{C}) \times \mathbb{C}$ is then said to be algebraic when $\nu(A)$ is an algebraic subset of $\mathbb{C}^{n^2+1}$. This allows us to describe $GL(n, \mathbb{C})$ as an algebraic set as follows:

$$GL(n, \mathbb{C}) = \{(X, y) \in M_n(\mathbb{C}) \times \mathbb{C} \mid \det(X)y = 1\}.$$

A subgroup $\Gamma \subset G$ is *linear algebraic* when Γ is an algebraic subset of $M_n(\mathbb{C}) \times \mathbb{C}$. We note:

(4.8) A linear algebraic subgroup of $GL(n, \mathbb{C})$ is a Lie group with finitely many connected components.

We likewise transfer the Zariski topology from $\mathbb{C}^{n^2+1}$ to $M_n(\mathbb{C}) \times \mathbb{C}$ by requiring $A \subset M_n(\mathbb{C}) \times \mathbb{C}$ to be Zariski closed when $\nu(A) \subset \mathbb{C}^{n^2+1}$ is Zariski closed. We denote by $\widehat{\Gamma}$ the Zariski closure of $\Gamma \subset GL(n, \mathbb{C})$.

(4.9) If $\Gamma \subset GL(n, \mathbb{C})$ is a soluble subgroup then $\widehat{\Gamma} \subset GL(n, \mathbb{C})$ is a soluble linear algebraic subgroup.

We denote by $\mathfrak{T}(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ the subgroup of upper triangular matrices:

$$\mathfrak{T}(n, \mathbb{C}) = \{X \in GL(n, \mathbb{C}) | X_{ij} = 0 \text{ if } i > j\},$$

and by $\mathfrak{N}(n, \mathbb{C})$ the subgroup of $\mathfrak{T}(n, \mathbb{C})$ consisting of unipotent matrices:

$$\mathfrak{N}(n, \mathbb{C}) = \{X \in \mathfrak{T}(n, \mathbb{C}) \,| X_{ii} = 1 \text{ for all } i\}.$$

$\mathfrak{N}(n, \mathbb{C})$ is nilpotent with nilpotent length $n - 1$ (cf. [5, p. 16]); from (2.5) we see

(4.10) $d(\mathfrak{N}(n, \mathbb{C})) \le \log_2(n) + 1$.

Finally, let $\mathfrak{D}(n, \mathbb{C})$ denote the subgroup of $\mathfrak{T}(n, \mathbb{C})$ consisting of diagonal matrices:

$$\mathfrak{D}(n, \mathbb{C}) = \{X \in \mathfrak{T}(n, \mathbb{C}) | X_{ij} = 0 \text{ whenever } i \ne j\}.$$

(4.11) $\mathfrak{T}(n, \mathbb{C})$ is the semidirect product $\mathfrak{T}(n, \mathbb{C}) = \mathfrak{N}(n, \mathbb{C}) \rtimes \mathfrak{D}(n, \mathbb{C})$.

It follows from (2.3) that $d(\mathfrak{T}(n, \mathbb{C})) \le d(\mathfrak{N}(n, \mathbb{C})) + d(\mathfrak{D}(n, \mathbb{C}))$. As $\mathfrak{D}(n, \mathbb{C})$ is abelian then by (4.10):

(3.12) $d(\mathfrak{T}(n, \mathbb{C})) \le \log_2(n) + 2$.

The following is essentially due to Lie but formally due to Kolchin [8]:

(4.13) If Γ is a connected soluble Lie subgroup of $GL(n, \mathbb{C})$ then Γ is isomorphic to a subgroup of $\mathfrak{T}(n, \mathbb{C})$.

Hence, we see that

(4.14) If Γ is a connected soluble Lie subgroup of $GL(n, \mathbb{C})$ then $d(\Gamma) \le \log_2(n) + 2$.

5 Clifford's Theorem for Compact Lie Groups

Let Σ_k denote the group of permutations of $\{1, \ldots, k\}$; we say that a homomorphism $\theta : G \to \Sigma_k$ is *transitive* when $\theta(G)$ acts transitively on $\{1, \ldots, k\}$. In the context of continuous representations of compact Lie groups Clifford's theorem takes the form:

Proposition 5.1 *Let $i : H \hookrightarrow G$ be the inclusion of a closed normal subgroup of the compact Lie group G and let $\mathcal{V} = (\mathbb{C}^n, \rho)$, where $\rho : G \to GL(n, \mathbb{C})$ is a continuous simple finite dimensional representation of G; then are simple continuous $\mathbb{C}[H]$ modules $(W_r)_{1\le r\le k}$, where $k \le n$ and positive integers m, e such that*

i) $k \le n$;
ii) $W_r \ncong W_s$ *when* $r \ne s$;
iii) $\dim_{\mathbb{C}}(W_r) = m$ *for all* r;
iv) *there are isotypic* $\mathbb{C}[H]$ *submodules* $\mathcal{U}_1, \ldots, \mathcal{U}_k$ *of* $i^*(\mathcal{V})$ *such that* $\mathcal{U}_r \cong W_r^{(e)}$ *and* $i^*(\mathcal{V})$ *is the internal direct sum* $i^*(\mathcal{V}) = \mathcal{U}_1 \dot{+} \cdots \dot{+} \mathcal{U}_k$;
v) *there is a transitive homomorphism* $\pi : G \to \Sigma_k$ *such that* $g \cdot \mathcal{U}_r = \mathcal{U}_{\pi(g)(r)}$;
vi) $n = e \cdot m \cdot k$.

Proof As the category of finite dimensional continuous $\mathbb{C}[H]$ modules is semisimple then $i^*(\mathcal{V})$ has an isotypic decomposition

$$i^*(\mathcal{V}) \cong W_1^{(e_1)} \oplus \cdots \oplus W_k^{(e_k)},$$

where $(W_r)_{1\le r\le k}$ are simple continuous $\mathbb{C}[H]$ modules such that $W_r \ncong W_s$ when $r \ne s$.

As each $W_r \ne 0$ then $k \le n$. For each r, let U_r be a simple $\mathbb{C}[H]$ submodule of $i^*(\mathcal{V})$ such that $U_r \cong_{\mathbb{C}[H]} W_r$. It follows that

a) if U is a simple $\mathbb{C}[H]$ submodule of $i^*(\mathcal{V})$ then $U \cong U_r$ for some integer r such that $1 \le r \le k$;
b) $U_r \cong U_s \iff r = s$;
c) $i^*(\mathcal{V})$ is the internal direct sum $i^*(\mathcal{V}) \cong \mathcal{U}_1 \dot{+} \cdots \dot{+} \mathcal{U}_k$, where $\mathcal{U}_r$ is a $\mathbb{C}[H]$ submodule such that $U_r \subset \mathcal{U}_r$ and $\mathcal{U}_r \cong W_r^{(e_r)} \cong U_r^{(e_r)}$.

If $g \in G$ then $g \cdot U_1$ is a simple $\mathbb{C}[H]$ submodule of $i^*(\mathcal{V})$. Thus, there is a mapping $\theta : G \to \{1, \ldots, k\}$ such that $g \cdot U_1 \cong U_{\theta(g)}$. Observe that $\sum_{g\in G} g \cdot U_1$ is a nonzero $\mathbb{C}[G]$-submodule of $\mathcal{V}$. As $\mathcal{V}$ is simple then

$$\sum_{g\in G} g \cdot U_1 = \mathcal{V}$$

so that, as $\mathbb{C}[H]$-modules $\sum_{g\in G} g \cdot U_1 = \mathcal{U}_1 \dot{+} \cdots \dot{+} \mathcal{U}_k$. As each $g \cdot U_1$ is simple over $\mathbb{C}[H]$ then for each r, there exists $g \in G$ such that $g \cdot U_1 \subset \mathcal{U}_r$. Hence, $g \cdot U_1 \cong U_r$ and so $\theta(g) = r$ and $\theta : G \to \{1, \ldots, k\}$ is surjective. Put $m = \dim_{\mathbb{C}}(U_1)$ and for each r choose $g \in G$ such that $\theta(g) = r$. As g induces a $\mathbb{C}$ linear mapping $U_1 \to U_{\theta(g)} = U_r$ with inverse $g^{-1} : U_{\theta(g)} \to U_1$ then $\dim_{\mathbb{C}}(U_r) = m$ and hence

$$\dim_{\mathbb{C}}(W_r) = \dim_{\mathbb{C}}(U_r) = \dim_{\mathbb{C}}(U_1) = m.$$

If U is a simple submodule of $\mathcal{U}_1$ then $g \cdot U \cong g \cdot U_1 \cong U_r \subset \mathcal{U}_r$. Hence, $g \cdot \mathcal{U}_1 \subset \mathcal{U}_r$ and g induces a $\mathbb{C}$-linear mapping $\mathcal{U}_1 \to \mathcal{U}_r$. Likewise g^{-1} induces a $\mathbb{C}$-linear mapping $\mathcal{U}_r \to \mathcal{U}_1$. As $g \cdot g^{-1} = g^{-1} \circ g = \mathrm{Id}$ then $\dim_{\mathbb{C}}(\mathcal{U}_r) = \dim_{\mathbb{C}}(\mathcal{U}_1)$. However, $\dim_{\mathbb{C}}(\mathcal{U}_r) = e_r \cdot m$

and $\dim_{\mathbb{C}}(\mathcal{U}_1) = e_1 \cdot m$. Thus $e_r = e_1$ for all i. Let e denote the common value of e_r, then $\dim_{\mathbb{C}}(\mathcal{U}_r) = e \cdot m$ and hence

$$n = \dim_{\mathbb{C}}(\mathcal{V}) = \sum_{r=1}^{k} \dim_{\mathbb{C}}(\mathcal{U}_r) = e \cdot m \cdot k.$$

Finally, if $g \in G$ then $g \cdot U_r$ is $\mathbb{C}[H]$-simple. Hence $g \cdot U_r \cong U_s$ for some $s \in \{1, \ldots, k\}$. We obtain a homomorphism $\pi : G \to \Sigma_k$ on writing $g \cdot U_r \cong U_{\pi(g)(r)}$. It follows that

$$g \cdot \mathcal{U}_r \cong \mathcal{U}_{\pi(g)(r)},$$

thereby giving an action of G on the isotypic components of $i^*(\mathcal{V})$. To see this action is transitive, given $r, s \in \{1, \ldots, k\}$ choose $\gamma, \delta \in G$ such that $\theta(\gamma) = r$ and $\theta(\delta) = s$; then $\pi(\delta\gamma^{-1})(r) = s$. □

Let G be a compact Lie group and let $\rho : G \to GL(n, \mathbb{C})$ be a continuous representation. We say that ρ is *primitive* when $\mathcal{R}\mathrm{es}_N^G(\rho)$ is isotypic for every normal subgroup N of finite index in G; otherwise, we say that ρ is *imprimitive*.

Proposition 5.2 *Let $\rho : G \to GL(n, \mathbb{C})$ be a continuous faithful irreducible representation of the compact Lie group G. If ρ is imprimitive there is a normal subgroup Γ of index $\leq n!$ in G and a faithful representation $\sigma : \Gamma \to GL(\nu, \mathbb{C})$, where $\nu < n$.*

Proof Put $\mathcal{V} = (\mathbb{C}^n, \rho)$ and let $i : N \hookrightarrow H$ be the inclusion of a closed normal subgroup such that $i^*(\mathcal{V})$ is *not* isotypic. By iv) of Proposition 5.1 above $i^*(\mathcal{V})$ decomposes as the internal direct sum of k summands, $1 < k \leq n$,

$$i^*(\mathcal{V}) = \mathcal{W}_1 \dot{+} \cdots \dot{+} \mathcal{W}_k,$$

where for each r there exists a simple $\mathbb{C}[N]$-module W_r such that $\mathcal{W}_r \cong W_r^{(e)}$, the exponent e being the same for each isotypic summand. Moreover, if $m_r = \dim_{\mathbb{C}}(W_r)$ then $m_1 = m_2 = \cdots = m_k$. Put $\nu = e \cdot m_1$ then $\dim_{\mathbb{C}}(\mathcal{W}_i) = \nu$ and

$$n = \nu \cdot k.$$

As $k > 1$ then $\nu < n$. Moreover, the right action of G permutes the isotypic summands $\mathcal{W}_r$ transitively; in particular, there exists a homomorphism $\sigma : G \to \Sigma_k$ such that $\mathcal{W}_r \cdot g = \mathcal{W}_{\sigma(g)(r)}$. Put $\Gamma = \mathrm{Ker}(\sigma)$, then G is an extension

$$1 \to \Gamma \to G \to G/\Gamma \to 1,$$

where G/Γ is isomorphic to a subgroup of $\Sigma_k \subset \Sigma_n$. Put

$$\Gamma_1 = \{g \in G \mid \mathcal{W}_1 \cdot g = \mathcal{W}_1\}.$$

Then the action of Γ_1 on $\mathcal{W}_1$ defines a representation $\tau : \Gamma_1 \to GL(\nu, \mathbb{C})$. Put $\mathcal{U} = (\mathbb{C}^\nu, \tau)$. Then $\mathcal{V} = j_*(\mathcal{U})$ where $j : \Gamma_1 \hookrightarrow G$ is the inclusion. Thus $j^*(\mathcal{V}) = j^* j_*(\mathcal{U}) = \underbrace{\mathcal{U} \oplus \cdots \oplus \mathcal{U}}_{k}$ so that, as $\mathcal{V}$ is faithful, so is $\mathcal{U}$. As $\Gamma \subset \Gamma_1$ then $\sigma = \mathcal{R}\mathrm{es}_\Gamma^{\Gamma_1}(\tau) : \Gamma \to GL(\nu, \mathbb{C})$ is a faithful representation and G/Γ, being isomorphic to a subgroup of Σ_n, has $|G/\Gamma| \leq n!$. □

6 Compact Soluble Lie Groups

Let $\mathcal{C}(n)$ denote the class of compact soluble Lie groups G which admit a faithful continuous representation $\rho : G \to GL(n, \mathbb{C})$. We shall prove

Theorem 6.1 *The class $\mathcal{C}(n)$ is D-bounded; in particular, there exists a nondecreasing sequence $c(k)_{1 \leq k}$ of positive integers such that for all $G \in \mathcal{C}(n)$,*

$$d(G) \leq \max\{c(n-1) + \pi(n) + 1, \pi(n^2) + 3\}.$$

We proceed by induction on n; thus let $\mathfrak{P}(n)$ be the following statement:

$\mathfrak{P}(n)$: There exists a nondecreasing sequence $c(k)_{1 \leq k \leq n}$ of positive integers such that

$$c(k) = \min\{d(G) \mid G \in \mathcal{C}(k)\}.$$

(6.2) $\mathfrak{P}(n)$ is true for all $n \geq 1$.

Observe that if G admits a faithful representation $\rho : G \to GL(1, \mathbb{C})$ then G is abelian so that $d(G) \leq 1$; in particular:

(6.3) $\mathfrak{P}(1)$ is true.

Thus assume that $n \geq 2$ and that $\mathfrak{P}(n-1)$ is true:

Proposition 6.4 *Let $\rho : H \to GL(n, \mathbb{C})$ be a continuous faithful representation of the compact soluble Lie group H. If ρ is not simple then $d(H) \leq c(n-1)$.*

Proof (H, ρ) decomposes into a direct sum of simple representations

$$(H, \rho) \cong \bigoplus_{i=1}^{k} (H, \rho_i),$$

where $\rho_i : H \to GL(m_i, \mathbb{C})$ and $n = \sum_{i=1}^{k} m_i$. As (H, ρ) is not simple then $k > 1$ and each $m_i \leq n-1$. Putting $H_i = \mathrm{Im}(\rho_i)$ then H_i is a subgroup of $GL(m_i, \mathbb{C})$ so, by hypothesis $\mathfrak{P}(n-1)$, $d(H_i) \leq c(m_i) \leq c(n-1)$. However H imbeds as a subgroup of $H_1 \times \cdots \times H_k$ so that $d(H) \leq \max\{d(G_i) \mid 1 \leq i \leq k\} \leq c(n-1)$. □

Proposition 6.5 *Let $\rho : H \to GL(n, \mathbb{C})$ be a continuous faithful simple representation of the compact soluble Lie group H. If ρ is not primitive then*

$$d(H) \leq c(n-1) + \pi(n).$$

Proof As ρ is not primitive then by (4.1) there exists a normal subgroup of H such that $H/\Gamma \in \Pi(n)$ and a faithful representation $\sigma : \Gamma \to GL(m, \mathbb{C})$, where $m < n$. Consequently $d(\Gamma) \leq c(m) \leq c(n-1)$. As $H/\Gamma \in \Pi(n)$ then $d(H/\Gamma) \leq \pi(n)$. From the exact sequence $1 \to \Gamma \to H \to H/\Gamma \to 1$, we see that

$$d(H) \leq d(\Gamma) + d(H/\Gamma) \leq c(n-1) + \pi(n).$$

□

We denote by $\mathcal{Z}(H)$ the centre of H and by C_m the cyclic group of order m. Suppose $\rho : H \to SL(n, \mathbb{C})$ is a simple faithful unimodular representation. By Schur's lemma, $\mathrm{End}_H(\mathbb{C}^n, \rho) = \mathcal{Z}(M_n(\mathbb{C}))$. If $z \in \mathcal{Z}(H)$ then $\rho(z) \in \mathrm{End}_H(\mathbb{C}^n, \rho)$, so we can write

$$\rho(z) = \begin{pmatrix} \lambda(z) & & & & \\ & \lambda(z) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda(z) \end{pmatrix} \in \mathcal{Z}(M(n, \mathbb{C})).$$

However, as ρ is unimodular then $\det(\rho(z)) = 1$ and $\rho(Z) \subset \mathcal{Z}(SL(n, \mathbb{C})) \cong C_n$. As ρ is faithful then $Z \cong \rho(Z) \cong C_m$ where $m|n$; to summarize:

(6.6) Let $\rho : H \to SL(n, \mathbb{C})$ be a continuous faithful unimodular representation of the compact soluble Lie group H; if $(\mathbb{C}^n, \rho)$ is simple then $\mathcal{Z}(H) \cong C_m$, where $m|n$.

We define $\mathfrak{U}(n)$ to be the class of pairs (H, ρ), where H is a soluble compact Lie group and $\rho : H \to SL(n, \mathbb{C})$ is a faithful *unimodular* representation. We partition $\mathfrak{U}(n)$ into three classes according to the following properties:

Case I: (H, ρ) is *not simple*;
Case II: (H, ρ) is simple but *not primitive*;
Case III: (H, ρ) is *simple and primitive*.

By Proposition 6.4 it follows that:

(6.7) If (H, ρ) is in Case I then $d(H) \le c(n-1)$.

Likewise, it follows from Proposition 6.5 that

Proposition 6.8 *If (H, ρ) is in Case II then $d(H) \le c(n-1) + \pi(n)$.*

Let H_0 be the identity component of H. We note that

Proposition 6.9 *If (H, ρ) is in Case III then H is finite, $\mathcal{Z}(H) \cong C_m$, where m is a positive integral divisor of n and $H \in \mathcal{R}(k^2)$, where $k \le n$.*

Proof We have already observed in (6.6) that $Z = \mathcal{Z}(H) \cong C_m$, where $m|n$. We note that the identity component H_0 of H is an abelian normal subgroup of H. Thus let N be an abelian normal subgroup of H which contains H_0 and put $\mathcal{W} = j^*(\mathcal{V})$, where $j : N \hookrightarrow H$ is the inclusion. As $\mathcal{V}$ is primitive then $\mathcal{W}$ is isotypic so write $\mathcal{W} = W^{(\mu)}$, where W is simple. As N is abelian them $\dim_{\mathbb{C}}(W) = 1$ so that $\mu = n$. Let $\sigma : N \to \mathbb{C}^*$ be the representation associated with W and let $\tau : N \to SL_n(\mathbb{C})$ be the representation associated with $\mathcal{W}$. Then τ has the form

$$\tau(x) = \underbrace{\begin{pmatrix} \sigma(x) & & & & \\ & \sigma(x) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sigma(x) \end{pmatrix}}_{n} \in \mathcal{Z}(SL(n, \mathbb{C})),$$

from which it follows that $N \subset Z \cong C_m$ and so $H_0 \subset C_m$. Thus $\dim(H_0) = 0$, $H_0 = \{1\}$ and H is finite. Moreover, the argument also shows that Z is the unique maximal abelian normal subgroup of H.

Let A be a maximal abelian subgroup of G/Z. Let $\pi : H \to H/Z$ be the canonical homomorphism, put $\Gamma = \pi^{-1}(A)$ and put $\mathcal{U} = \mathcal{R}\mathrm{es}^G_\Gamma(\mathcal{V})$. By hypothesis on $\mathcal{V}$, $\mathcal{U}$ is isotypic; that is, there exist a simple Γ module U and a positive integer e such that $\mathcal{U} \cong U^{(e)}$ and $n = e \cdot k$ where $k = \dim_{\mathbb{C}}(U)$. It follows from Burnside's theorem ([4, (3.32) p. 51]) that $|A| = k^2$. Thus (H, Z, A) is an $\mathcal{R}(k^2)$ structure on H. □

It follows from (3.4) that for (H, ρ) in Case III, $d(H) \leq \pi(k^2) + 2$, where $k \leq n$. As the right-hand side of this inequality does not decrease with k we see that:

Corollary 6.10 *If (H, ρ) is in Case III then $d(H) \leq \pi(n^2) + 2$.*

If $G \in \mathcal{C}(n)$ and $\rho : G \to GL(n, \mathbb{C})$ is a faithful representation put

$$H = \mathrm{Ker}(\det \circ \rho : G \to \mathbb{C}^*), \quad \rho_0 = \rho_{|H}.$$

As G/H is abelian then $d(G) \leq d(H) + 1$. Moreover, $(H, \rho_0) \in \mathfrak{U}(n)$ so that by (6.7), Proposition 6.5 and Corollary 6.10, $d(H) \leq \max\{c(n-1) + \pi(n), \pi(n^2) + 2\}$ and hence

$$d(G) \leq \max\{c(n-1) + \pi(n) + 1, \pi(n^2) + 3\} \quad \text{for all } G \in \mathcal{C}(n). \tag{6.11}$$

In particular $\mathcal{C}(n)$ is D-bounded. We define $c(n) = \max\{d(G) \mid G \in \mathcal{C}(n)\}$. As $\mathcal{C}(n-1) \subset \mathcal{C}(n)$ it follows that $c(n-1) \leq c(n)$. Thus we have shown that $\mathfrak{P}(n-1) \Rightarrow \mathfrak{P}(n)$, completing the proof of (6.2).

We can represent the above argument by the flowchart in Fig. 1.

7 Estimating $c(n)$

Proposition 7.1 $\log_2(n!) \leq (n-2)\log_2(n) + 1$ *for all* $n \geq 1$.

Proof The inequality is trivially true for $n = 1, 2$. For $n \geq 3$, we have

$$\log_2(n!) = \sum_{r=1}^{n} \log_2(r) = 0 + 1 + \sum_{r=3}^{n} \log_2(r) \leq 1 + (n-2)\log_2(n).$$

□

Proposition 7.2 *If Φ is a finite soluble group then $d(\Phi) \leq \log_2(|\Phi|)$.*

Proof If $d(\Phi) = m$ then $|\Phi| = \prod_{r=0}^{m-1} |D_r(\Phi)/D_{r+1}(\Phi)|$. As $2 \leq |D_r(\Phi)/D_{r+1}(\Phi)|$ when $0 \leq r \leq m-1$ then $2^m \leq |\Phi|$ and $m \leq \log_2(|\Phi|)$. □

It now follows directly from Propositions 7.1 and 7.2 that

(7.3) If Φ is a soluble subgroup of the symmetric group Σ_n then

$$d(\Phi) \leq (n-2)\log_2(n) + 1.$$

As $\log_2(n^2) = 2\log_2(n)$ it follows that

(7.4) If Φ is a soluble subgroup of the symmetric group Σ_{n^2} then

$$d(\Phi) \leq 2(n^2 - 2)\log_2(n) + 1.$$

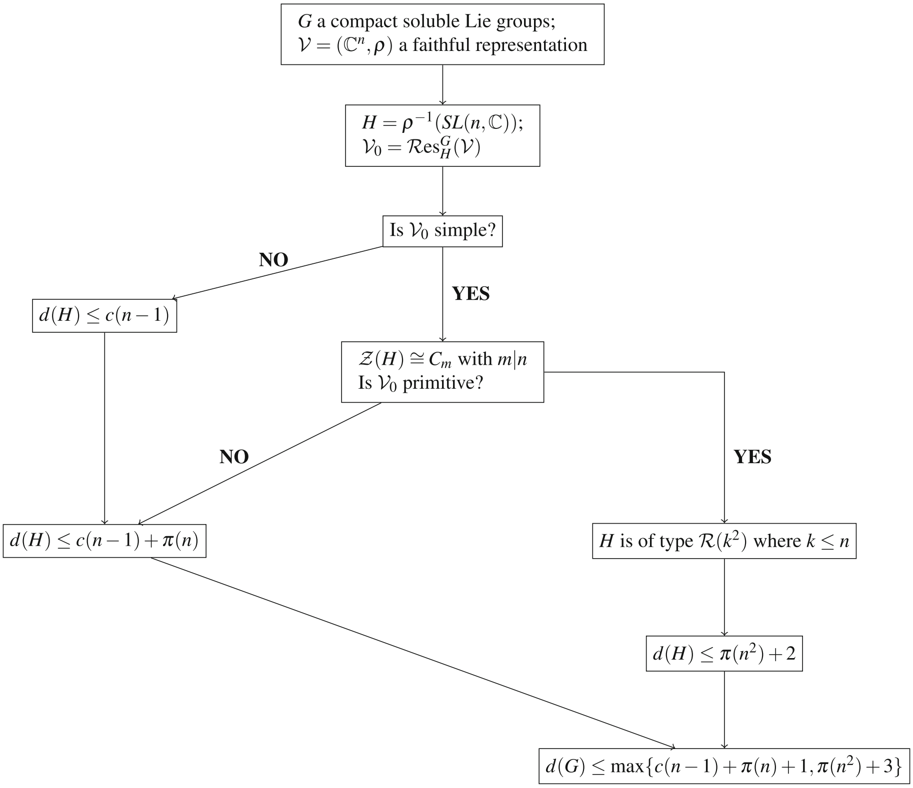

Fig. 1 Summary of argument

As in Section 3 we denote by $\pi(n)$ the D-bound of the set $\Pi(n)$ of soluble subgroups of Σ_n. It follows from (7.3) that

$$\pi(n) \le (n-2)\log_2(n) + 1. \tag{7.5}$$

Likewise from (7.4)

$$\pi(n^2) \le 2(n^2-2)\log_2(n) + 1. \tag{7.6}$$

Define $g(n) = 2(n^2-2)\log_2(n) + 4$, then $g(1) = 0 + 4 = 4$ and $g(2) = 4 + 4 = 8$.

Proposition 7.7 *For $n \ge 2$, $g(n-1) + (n-2)\log_2(n) + 2 < g(n)$.*

Proof Let $\mathcal{P}(n)$ be the inequality '$g(n-1) + (n-2)\log_2(n) + 2 < g(n)$'. Then $\mathcal{P}(2)$ is true as $g(1) + (1-2)\log_2(1) + 2 = 6 < 8 = g(2)$. Suppose $\mathcal{P}(n-1)$ is true for $n \ge 3$. Noting that $0 < n^2 - 2n - 1$ we see that

$$\begin{aligned} g(n-1) &= 2\{(n-1)^2 - 2\}\log_2(n-1) + 4 \\ &= 2(n^2 - 2n - 1)\log_2(n-1) + 4 \\ &\le 2(n^2 - 2n - 1)\log_2(n) + 4, \end{aligned}$$

so that as $3n\log_2(n) - 2 > 0$

$$\begin{aligned} g(n-1) + (n-2)\log_2(n) + 2 &\leq 2(n^2 - 2n - 1)\log_2(n) + 4 + (n-2)\log_2(n) + 2 \\ &= 2(n^2 - 2)\log_2(n) + 4 - \{3n\log_2(n) - 2\} \\ &= g(n) - \{3n\log_2(n) - 2\} \\ &< g(n) \end{aligned}$$

and $g(n-1) + (n-2)\log_2(n) + 2 < g(n)$ as claimed. □

We note in preparation that

$$\pi(n^2) + 3 \leq g(n). \tag{7.8}$$

We claim that

Theorem 7.9 $c(n) \leq 2(n^2 - 2)\log_2(n) + 4$.

Proof That is, we must show $c(n) \leq g(n)$ for all $n \geq 1$. Observe that

$$c(1) = 1 < 4 = g(1)$$

so that the statement is true for $n = 1$. Also, from (6.11) and (7.8),

$$c(2) \leq \max\{c(1) + 2, \pi(2^2) + 3\} = 6 \leq \max\{3, g(2)\} = 8,$$

so the statement is also true for $n = 2$. Assume it is true for $n - 1$, then by (6.11), (7.5), (7.8) and induction we see that

$$\begin{aligned} c(n) &\leq \max\{c(n-1) + \pi(n) + 1, \pi(n^2) + 3\} \\ &\leq \max\{c(n-1) + (n-2)\log_2(n) + 2, g(n)\} \\ &\leq \max\{g(n-1) + (n-2)\log_2(n) + 2, g(n)\}. \end{aligned}$$

However by Proposition 7.7, $g(n-1) + (n-2)\log_2(n) + 2 < g(n)\}$ so that, as claimed,

$$c(n) \leq g(n) = 2(n^2 - 2)\log_2(n) + 4.$$

□

8 Proof of Zassenhaus's Theorem

(8.1) Let $\mathbb{G} \subset GL(n, \mathbb{C})$ be a soluble linear algebraic subgroup, then

$$d(\mathbb{G}) \leq (2n^2 - 3)\log_2(n) + 6.$$

Proof Let $\mathbb{G}_0$ denote the identity component of $\mathbb{G}$, then $\mathbb{G}_0$ is a normal subgroup of $\mathbb{G}$ so that taking K to be a maximal compact subgroup of $\mathbb{G}$ we may form the semidirect product $\mathbb{G}_0 \rtimes K$ with multiplication

$$(\gamma_1, k_1) \cdot (\gamma_2, k_2) = (\gamma_1 \cdot (k_1 \cdot \gamma_2 \cdot k_1^{-1}), k_1 \cdot k_2).$$

Moreover, we have a group homomorphism $\mu : \mathbb{G}_0 \rtimes K \to \mathbb{G}$ given by $\mu(\gamma, k) = \gamma \cdot k$. We claim that μ is surjective. To see this, observe that we have an exact sequence $\pi_0(K) \overset{i_*}{\to} \pi_0(\mathbb{G}) \overset{\pi_*}{\to} \pi_0(\mathbb{G}/K)$. As $\mathbb{G}/K$ is connected then $\pi_0(\mathbb{G}/K) = \{1\}$ so giving a surjection $i_* : \pi_0(K) \twoheadrightarrow \pi_0(\mathbb{G})$. Now let $g \in \mathbb{G}$ and denote by $[g]$ the connected component of $\mathbb{G}$ to which g belongs. As $i_* : \pi_0(K) \to \pi_0(\mathbb{G})$ is surjective we may choose $k \in K$

such that $i_*([k]) = [g]$; that is, k belongs to the same connected component of $\mathbb{G}$ as g. Let $p : [0, 1] \to \mathbb{G}$ be a path such that $p(0) = k$ and $p(1) = g$ and let $q : [0, 1] \to \mathbb{G}$ be the path

$$q(t) = p(t) \cdot k^{-1}.$$

Then q is a path from $1_{\mathbb{G}}$ to $g \cdot k^{-1}$. Hence $g \cdot k^{-1} \in \mathbb{G}_0$. Writing $\gamma = g \cdot k^{-1}$ we see that $g \in \mathbb{G}$ can be written in the form $g = \gamma \cdot k$, where $\gamma \in \mathbb{G}_0$ and $k \in K$. That is, μ is surjective as claimed. As G_0 and K are subgroups of the soluble group $\mathbb{G}$ then both are soluble. Hence $d(\mathbb{G}_0 \rtimes K) \leq d(\mathbb{G}_0) + d(K)$. By (4.14), $d(\mathbb{G}_0) \leq \log_2(n) + 2$ whilst by Theorem 7.9, $d(K) \leq 2(n^2 - 2)\log_2(n) + 4$. Hence $d(\mathbb{G}_0 \rtimes K) \leq (2n^2 - 3)\log_2(n) + 6$. However, as $\mathbb{G}$ is the surjective image of $\mathbb{G}_0 \rtimes K$ under μ then $d(\mathbb{G}) \leq (2n^2 - 3)\log_2(n) + 6$. □

In consequence we now have

Theorem 8.2 *Let R be a subring of the field of complex numbers. If Γ is a soluble subgroup of $GL(n, R)$ then $d(\Gamma) \leq (2n^2 - 3)\log_2(n) + 6$.*

Proof As $R \subset \mathbb{C}$ then $\Gamma \subset GL(n, \mathbb{C})$. Let $\widehat{\Gamma}$ denote the Zariski closure of Γ; by (4.9) $\widehat{\Gamma}$ is a linear algebraic subgroup of $GL(n, \mathbb{C})$. Moreover, by (4.10), as Γ is soluble then so is $\widehat{\Gamma}$ and $d(\Gamma) \leq d(\widehat{\Gamma})$. The conclusion follows from (8.1). □

Proposition 8.3 *Let G be a connected Lie group of dimension n, then every soluble subgroup Γ of G has derived length $d(\Gamma) \leq (2n^2 - 3)\log_2(n) + 7$.*

Proof G occurs in an extension $1 \to \mathcal{Z} \overset{\pi}{\hookrightarrow} G \overset{\pi}{\twoheadrightarrow} \mathrm{Ad}(G) \to 1$, where $\mathrm{Ad}(G)$ is the adjoint group of G and $\mathcal{Z}$ is central in G. If Γ is a soluble subgroup of G then $d(\Gamma) \leq d(\pi(\Gamma)) + 1$. However, $\mathrm{Ad}(G)$ imbeds in $GL(n, \mathbb{C})$ so that, by Proposition 7.2, $d(\pi(\Gamma)) \leq (2n^2 - 3)\log_2(n) + 6$. Thus $d(\Gamma) \leq (2n^2 - 3)\log_2(n) + 7$. □

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