Spectral Analysis of Pseudo-Differential Operators with Discontinuous Symbols

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Declaration

I, Alexey Derkach, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

The spectral theory of Pseudo-Differential Operators (ΨDOs) with smooth symbols is quite mature. Many aspects are well studied including asymptotic formulae of eigenvalues (see [**4**], [**9**], [**14**] and the references therein). There are two types of results: those for unbounded ΨDOs (e.g. [**14**]) and those for compact ΨDOs [**4**]. We focus on the latter ones.

There are significantly more results for associated operators, for the case when the symbol of a ΨDO is smooth, or when ΨDOs are defined on modulation spaces. However, considerably less is known about the cases with discontinuous symbols, even for the simplest type of discontinuity.

This work is devoted to spectral properties of compact ΨDOs $\mathrm{Op}_1^{\mathrm{W}}(\sigma)$ with a certain type of symbols, namely Weyl symbols with jump discontinuity. These symbols are considered indicator functions $\sigma = \chi_\Lambda$ of given bounded regions Λ in phase space.

The general goal is to understand how the rate of eigenvalues decay depends on the geometry of the boundary $\partial\Lambda$.

Impact Statement

The main impact of the dissertation is to Time-Frequency analysis where ΨDOs of Weyl's type are intensively studied. The operators we consider in this work (those with symbols/signals equal to indicator functions χ_Λ, i.e. concentrated in a localised domain Λ in the time-frequency plane/phase space) are of interest to Signal Processing theory. For instance, the asymptotic decay of eigen/singular values of such ΨDOs (called *time–frequency localisation operators* in corresponding applied literature) determines the characteristics of time-frequency filters (see [**7**], [**11**]).

In quantum mechanics, finding the exact upper and lower bounds of quasi-probability integrals (1.2.1) leads to estimates of the maximal and minimal eigenvalues of the corresponding ΨDO and is also of interest to many researchers (see [**21**]). In 1988 Flandrin [**7**] conjectured that under some conditions the quasi-probability integral does not exceed 1.

However, this conjecture was recently rejected by Delourme, Duyckaerts and Lerner [**2**], and an explicit counterexample was provided. This required to study a bounded non-compact ΨDO (along with its eigenvalues) with signal concentrated on the first quarter of the time-frequency space. In the thesis we focus on the asymptotic behaviour of eigenvalues of an operator similar to that in [**2**] but naturally adjusted to be compact. The result of the study is an asymptotic formula (1.2.2) for the case when the signal is concentrated on an angular domain (see Chapter 5), as well as asymptotic estimates when we extend the domain to any polygonal region.

To achieve this, we develop some auxiliary techniques (see Chapter 3), which can be presented as an independent result in time-frequency theory and the second main result of the dissertation.

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Contents

Chapter 1

Introduction

1.1 General notations

PART A.

χ_Ω	indicator function of $\Omega \subseteq \mathbb{R}^d$	
$\partial\Omega$	boundary of $\Omega \subseteq \mathbb{R}^d$	
$C_0^\infty(\Omega)$	space of smooth (infinitely differentiable) functions on Ω with a compact support	
$W_2^2(\Omega)$	Sobolev space (of order 2) of functions in $\mathrm{L}^2(\Omega)$	Section 4.1, p.60
$\langle f, g\rangle$	inner product on $\mathrm{L}^2(\Omega)$ space	
$\langle \mathbf{x}\rangle := \sqrt{1+\lvert\mathbf{x}\rvert^2}$	Japanese brackets on $\mathbb{R}^d$	
$\zeta_\delta(\cdot) = \zeta(\cdot)$	smooth cut-off function	Definition 3.2.6, p.42
D_A	domain of an operator A	Section 4.1, p.60
$\overline{A}$	closure of an operator A	
A^*	adjoint operator	
$\nu_\pm(A)$	deficiency indices of a symmetric operator A	Definition 4.1.2, p.61

$\#(\lambda, A)$	spectral counting function of a lower semibounded operator A	Section 4.1, p.61
$\#(\lambda; V; (a, b))$	counting function of one-dimensional Schrödinger operator on (a, b) with potential V	Section 4.1, p.61
$n_{\pm}(\lambda, \sigma) = n_{\pm}(\lambda)$	spectral counting functions of a compact ΨDO with symbol σ	Definition 2.1.1, p.21
$n(\lambda, A)$	number of singular values of a self-adjoint compact operator A exceeding λ	Section 4.1, p.61
$\lambda_k^+(T) \quad (\lambda_k^-(T))$	positive (negative) eigenvalues of a self-adjoint compact operator T	Section 2.1, p.21
$s_k(T)$	singular values of a compact operator T	Section 2.2, p.24
$\mathrm{l}^{\gamma,\infty}, \quad \|\cdot\|_{\gamma,\infty}$	weak-l^γ quasi-normed space and weak-l^γ quasi-norm	Definition 3.1.1, p.31
$\mathrm{l}^{\gamma,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$, $\|\cdot\|_{q,\gamma,\infty}$	weak lattice quasi-normed space and weak lattice quasi-norm	Definition 3.1.2, p.32
$\mathrm{l}^{\gamma}(\mathrm{L}^q)(\mathbb{R}^d)$ $\|\cdot\|_{q,\gamma}$	lattice quasi-normed space and lattice quasi-norm	Definition 3.1.3, p.32
$\mathbb{S}_\gamma$	Schatten operator ideal class	Section 2.2, p.24
$\mathbb{S}_{\gamma,\infty}$	weak-Schatten operator class	Section 2.2, p.24
$\mathfrak{S}_\gamma$	Weidl quasi-normed space	Section 2.2.3, p.26, Definition 2.2.9, p.27
$\mathcal{F}, \quad \mathcal{F}^{-1}$	Fourier and inverse Fourier transform	Part B, p.10

$\mathrm{Op}^{\mathrm{a}}_{\alpha}(p)$	pseudo-differential operator with amplitude p	Part B, p.10
$\mathrm{Op}^{\mathrm{W}}_{\alpha}(\sigma)$	pseudo-differential operator with symbol σ of Weyl's type	Part B, p.10

Part B. Pseudo-differential operators and differential notations

We use the following notation for the pseudo-differential operator on $\mathrm{L}^2(\mathbb{R}^d)$ with amplitude $p = p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})$

$$\mathrm{Op}^{\mathrm{a}}_{\alpha}(p)u(\mathbf{x}) := \Big(\frac{\alpha}{2\pi}\Big)^d \iint_{\mathbb{R}^{2d}} e^{i\alpha(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})u(\mathbf{y})d\mathbf{y}d\boldsymbol{\xi} \tag{1.1.1}$$

If in addition $p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) = \sigma\Big(\frac{\mathbf{x}+\mathbf{y}}{2}, \boldsymbol{\xi}\Big)$, we use the notation

$$\mathrm{Op}^{\mathrm{W}}_{\alpha}(\sigma) = \mathrm{Op}^{\mathrm{W}}_{\alpha}(\sigma(\mathbf{t}, \boldsymbol{\xi})) := \mathrm{Op}^{\mathrm{a}}_{\alpha}\Big(\sigma\Big(\frac{\mathbf{x}+\mathbf{y}}{2}, \boldsymbol{\xi}\Big)\Big),$$

and p is called *symbol of Weyl's type.*

Notations $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and the inverse Fourier transform, respectively, defined by

$$\mathcal{F}(f)(\cdot) = \Big(\frac{1}{2\pi}\Big)^{\frac{d}{2}} \int_{\mathbb{R}^d} f(\mathbf{t})e^{-i\mathbf{t}\cdot} d\mathbf{t}, \quad \mathcal{F}^{-1}(f)(\cdot) = \Big(\frac{1}{2\pi}\Big)^{\frac{d}{2}} \int_{\mathbb{R}^d} f(\mathbf{t})e^{i\mathbf{t}\cdot} d\mathbf{t}.$$

To specify the arguments in the Fourier transform of functions of the type $f = f(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we define

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}}[f(\mathbf{x}, \mathbf{y})] := g(\boldsymbol{\xi}, \mathbf{y}),$$

where

$$g(\boldsymbol{\xi}, \mathbf{y}) = \Big(\frac{1}{2\pi}\Big)^{\frac{d}{2}} \int_{\mathbb{R}^d} f(\mathbf{t}, \mathbf{y})e^{-i\mathbf{t}\boldsymbol{\xi}} d\mathbf{t}.$$

Notations $\mathcal{F}_{\mathbf{y}\to\boldsymbol{\xi}}$, $\mathcal{F}^{-1}_{\boldsymbol{\xi}\to\mathbf{x}}$ are defined accordingly.

Note, that if the amplitude $p = p(\mathbf{x}, \boldsymbol{\xi})$ (in this case p is called *symbol*),

$$\mathrm{Op}^{\mathrm{a}}_{1}(p)u(\mathbf{x}) = \mathcal{F}^{-1}_{\boldsymbol{\xi}\to\mathbf{x}}\big[p(\mathbf{x}, \boldsymbol{\xi})\mathcal{F}_{\mathbf{y}\to\boldsymbol{\xi}}[u(\mathbf{y})]\,\big].$$

Consider a multivariable function $f : \mathbb{R}^d \to \mathbb{R}$.

The gradient of $f(\mathbf{x}) = f(x_1, x_2, ..., x_d)$ is denoted by

$$\nabla f = \big(\partial_{x_1} f, \partial_{x_2} f, ..., \partial_{x_d} f\big),$$

where we use a short notation for the partial derivative,

$$\partial_{x_j} f := \frac{\partial f}{\partial x_j}, \quad j = 1, 2, ..., d.$$

To describe the control of the partial derivatives the following notation is used

$$\big|\nabla^m f\big| := \sqrt{\sum_{\substack{0 \le j_1, j_2, ..., j_d \le m \\ j_1 + j_2 + ... + j_d = m}} \Big|\partial_{x_1}^{j_1} \partial_{x_2}^{j_2} ... \partial_{x_d}^{j_d} f\Big|^2}.$$

For functions with more than one multidimensional variable, e.g. if $f = f(\mathbf{x}, \boldsymbol{\xi}) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, consider the following notations

$$\nabla_{\mathbf{x}} f = \big(\partial_{x_1} f(\mathbf{x}, \boldsymbol{\xi}), \partial_{x_2} f(\mathbf{x}, \boldsymbol{\xi}), ..., \partial_{x_d} f(\mathbf{x}, \boldsymbol{\xi})\big),$$

$$\nabla_{\boldsymbol{\xi}} f = \big(\partial_{\xi_1} f(\mathbf{x}, \boldsymbol{\xi}), \partial_{\xi_2} f(\mathbf{x}, \boldsymbol{\xi}), ..., \partial_{\xi_d} f(\mathbf{x}, \boldsymbol{\xi})\big),$$

$$\big|\nabla_{\mathbf{x}}^k \nabla_{\boldsymbol{\xi}}^m f\big| = \sqrt{\sum_{\substack{0 \le j_1, j_2, ..., j_d \le m, \ 0 \le t_1, t_2, ..., t_d \le k \\ j_1 + j_2 + ... + j_d = m, \ t_1 + t_2 + ... + t_d = k}} \Big|\partial_{x_1}^{t_1} \partial_{x_2}^{t_2} ... \partial_{x_d}^{t_d} \partial_{\xi_1}^{j_1} \partial_{\xi_2}^{j_2} ... \partial_{\xi_d}^{j_d} f(\mathbf{x}, \boldsymbol{\xi})\Big|^2}.$$

Part C. Asymptotic notations

Big-O and little-o notation.

Let $\big(g(x; \mathbf{A})\big)_{\mathbf{A} \in \mathcal{A}}$ be a parametric family (indexed by $\mathcal{A} \subseteq \mathbb{R}^m$) of real valued functions $g(x; \mathbf{A})$ on $\Omega \subseteq \mathbb{R}$.
We use the notation

$$g(x; \mathbf{A}) = \underline{O}(f(x; \mathbf{A})),$$

if $g(x; \mathbf{A}) = b(x; \mathbf{A}) f(x; \mathbf{A})$, where for any fixed parameters $(a_1, a_2, ..., a_m) = \mathbf{A}$ the function $b(x) = b(x; \mathbf{A})$ is a bounded function on Ω.
Notation $\underline{O}(f_k; \mathbf{A})$ is defined similarly.

Remark. Mostly, in the thesis we use notation $\underline{O}(f(x))$ or $\underline{O}(f_k)$, i.e. it is clear what stands for the variable x (index k) and what stands for the parameter a_j.
Few exceptions, like $s_k(T) = \underline{O}(k^{-p})$, always have the form of a log-power function with a parameter in the power, e.g. $f_k = k^{a_1} \log^{a_2}(k+1)$.

The notation $\overline{o}(f(x))$, $x \to x_0$, $x_0 \in \mathbb{R} \cup \{\infty\}$, stands for $\alpha(x)f(x)$, where $\alpha(x) \to 0$, $x \to x_0$.
If f is a parameterised function, $\overline{o}(f(x;\mathbf{A})) = \alpha(x;\mathbf{A})f(x;\mathbf{A})$, where $\alpha(x;\mathbf{A}) \to 0$, $x \to x_0$ for any fixed $\mathbf{A}$.
Notations $\overline{o}(f_n)$, $\overline{o}(f_n;\mathbf{A})$, $n \to \infty$ are defined similarly.

Domination of functions.

Let $\big(f(\mathbf{x};\mathbf{A})\big)_{\mathbf{A}\in\mathcal{A}}$ and $\big(g(\mathbf{x};\mathbf{A})\big)_{\mathbf{A}\in\mathcal{A}}$be two parametric families (indexed by $\mathcal{A} \subseteq \mathbb{R}^m$) of real valued functions on $\Omega \subseteq \mathbb{R}^d$.
Let $\mathbf{A_o} = (a_{i_1}, a_{i_2}, ..., a_{i_t})$, $1 \leq i_1 < i_2 < ... < i_t \leq m$, be a list of some parameters of $\mathbf{A} = (a_1, a_2, ..., a_m)$.
Notation $f(\mathbf{x};\mathbf{A}) \lesssim_{\mathbf{A_o}} g(\mathbf{x};\mathbf{A})$ is used if

$$|f(\mathbf{x};\mathbf{A})| \leq C_{\mathbf{A_o}}|g(\mathbf{x};\mathbf{A})|, \quad \mathbf{x} \in \Omega,\ \mathbf{A} \in \mathcal{A},$$

where $C_{\mathbf{A_o}}$ depends on $\mathbf{A_o} = (a_{i_1}, a_{i_2}, ...a_{i_t})$ only (does not depend on $\mathbf{x}$ and other parameters a_j, $j \notin \{i_1, i_2, ..., i_t\}$).

$f(\mathbf{x};\mathbf{A}) \lesssim g(\mathbf{x};\mathbf{A})$ means that there exists $C > 0$ such that

$$|f(\mathbf{x};\mathbf{A})| \leq C|g(\mathbf{x};\mathbf{A})|, \quad \mathbf{x} \in \Omega,\ \mathbf{A} \in \mathcal{A}.$$

For sequences $(f_\mathbf{n})_{\mathbf{n}\in\mathbb{Z}^d}$, $(g_\mathbf{n})_{\mathbf{n}\in\mathbb{Z}^d}$ the notations $f_\mathbf{n} \lesssim_{\mathbf{A_o}} g_\mathbf{n}$ and $f_\mathbf{n} \lesssim g_\mathbf{n}$ are defined similarly.

Remark. Mostly, in the thesis $\mathcal{A} \subseteq \mathbb{R}^m$ for some m, i.e. a_j are real numbers. However, sometimes parameter a_j might stand for a set (e.g. $s_k(\mathrm{Op}_1^{\mathrm{W}}(\chi_\Omega)) \lesssim_\Omega k^{-\frac{3}{4}}$ in the proof of Corollary 3.2.4) or for a function (e.g. $s_k(T_m) \lesssim_{m,a} k^{-m}$ in Corollary 3.2.7).

Equivalence of functions.

Two real valued functions f and g are equivalent,
$f(x) \sim g(x)$, $x \to x_0 \in \mathbb{R} \cup \infty$, if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$.

1.2 Motivation

The study of time–frequency localisation operators $\mathrm{Op}_1^{\mathrm{W}}(\chi_\Lambda)$ (ΨDO with Weyl symbol $\sigma(t,\xi) = \chi_\Lambda(t,\xi),\ \Lambda \subseteq \mathbb{R}_t \times \mathbb{R}_\xi$) helps to investigate signal energy concentration in Signal Processing. The operator establishes the following connection with the quasi-probability distribution function, the Wigner function $W(\cdot,\cdot)$ (see [**8**]), by

$$\langle \mathrm{Op}_{2\pi}^{\mathrm{W}}(\chi_\Lambda)u, v\rangle = \iint \chi_\Lambda(t,\xi) W(u,v)(t,\xi) dt d\xi = \iint_\Lambda W(u,v)(t,\xi) dt d\xi \tag{1.2.1}$$

The properties of the quasi-probability integral 1.2.1 are important for the signal energy distribution [**7**]. Unlike classical mechanics where the value of the integral of the standard probability density function is always between 0 and 1 (inclusive), in quantum mechanics the integration of $W(\cdot,\cdot)$ may give a result which is negative or greater than 1.

Flandrin conjectured [**7**] that for any convex domain Λ the integral (1.2.1) does not exceed 1 for any normalised $v = u$.

However, Delourme, Duyckaerts and Lerner in [**2**] rejected the hypothesis by finding an eigenfunction of $\mathrm{Op}_{2\pi}^{\mathrm{W}}(\chi_\Lambda)$ where $\Lambda = \{t \geq 0, \xi \geq 0\} \subseteq \mathbb{R}_t \times \mathbb{R}_\xi$ is the first quarter in the phase space, corresponding to the maximal eigenvalue of the operator.
While in [**2**] Flandrin's conjecture was invalidated via numerical arguments, later Lerner in [**12**] provided a theoretical proof.

It turns out the bounds of the integral (1.2.1) coincide with the minimal and maximal eigenvalues of $\mathrm{Op}_{2\pi}^{\mathrm{W}}(\chi_\Lambda)$.

For some domains Λ (namely, concave cones on the phase space), Lerner described the extreme values of the spectrum (see [**12, Ch. 7**]) and made hypotheses about more general cases of convex regions.

The symbol of the main operator we consider in this work is the function studied by Lerner, $\chi_{\{t,\xi\geq 0\}}$, multiplied by a function $a(t,\xi) \in C_0^\infty(\mathbb{R}^2)$, i.e. a smooth compactly supported function. Unlike Lerner's work **[12]** where the norm of the bounded non-compact operator $\mathrm{Op}_{2\pi}^{\mathrm{W}}(\chi_{\{t,\xi\geq 0\}})$ is studied, we focus on the asymptotic behaviour of eigenvalues of the compact operator $\mathrm{Op}_1^{\mathrm{W}}(a(t,\xi)\chi_{\{t,\xi\geq 0\}})$.

The main goal of this work is to obtain asymptotic formulae and estimates for the eigenvalues of operators when the signal is concentrated on a polygonal domain and to develop tools for obtaining such formulae. In particular, if Λ represents a bounded angular region in the phase space, we prove that the k^{th} positive (negative) eigenvalue

$$\lambda_k^{\pm} = \frac{1}{4\pi^2} k^{-1} \log(k+1)\big(1 + \overline{o}(1)\big) \tag{1.2.2}$$

Splitting an arbitrary polygon Λ into triangles, we can obtain a weaker result, $\lambda_k^{\pm} = \underline{O}(k^{-1}\log(k+1))$. However, (1.2.2) may be no longer be true for any $n-$sided polygonal, since the method (to obtain an asymptotic formula) introduced in Chapter 5 can be applied to signals the "main part" of which is the indicator function of an angular region only.

Generally, considering a compact operator $\mathrm{Op}_1^{\mathrm{W}}(a\chi_\Lambda)$ on $\mathrm{L}^2(\mathbb{R})$, we notice that the rate of decay of eigenvalues depends on the curvature of the boundary $\partial\Lambda$.

If the boundary can be described by a straight line (i.e. we consider symbol $a \cdot \chi_\Lambda$ where Λ is a set $\{(t,\xi)|\xi \leq c_1 t + c_2\}$, the rate of decay is $\underline{O}(k^{-1})$. For a polygonal boundary the rate does not exceed $\underline{O}\big(\frac{\log(k+1)}{k}\big)$. The "angles in the boundary" slow down the decay comparing with the case mentioned above, and it is not established yet if this estimate is sharp.

Finally, when the curvature of the boundary becomes non-zero at least at some points, the decay slows down even more and can reach $c_k k^{-\frac{3}{4}}$ where $c_k \in [t_1, t_2] \subseteq (0,\infty)$ for annular regions **[13, Prop. 10]**. Ramanathan and Topiwala **[13, Th. 9]** proved the estimate $\underline{O}(k^{-\frac{3}{4}})$ for any C^1-boundary.

The following paragraph describes existing methods and tools to estimate singular values of compact ΨDOs (which implies the same estimates in a self-adjoint case) and compares them with the toolkit proposed in this work.

1.3 Review of existing methods

Considering compact integral operators, there is a variety of methods to estimate their singular values. One of the simplest approaches is to use Hilbert-Schmidt (H-S) norms for appropriate operators, i.e. when the kernel K is an L^2-integrable function with $\|K\|^2_{\mathrm{L}^2} = \sum s_n^2$, where s_n are singular values of the operator arranged in a non-increasing order (i.e. $s_1 \geq s_2 \geq ... \geq s_n > 0$). This gives the estimate for the n^{th} singular value s_n of the kind $s_n = \overline{o}(n^{-1/2}\|K\|_{\mathrm{L}^2})$. Unfortunately, this method might give a rather rough estimate. However, sometimes it is convenient to split the operator into several complementary parts, the "main body" and the "remainder". While the techniques used for the "main" part differ, an appropriate estimate for the remainder part (usually it is an operator with a kernel which is either discontinuous or has an unbounded support) may be obtained with the help of the H-S norm, which might give the best possible estimate particularly for the remainder part.

Another approach is to approximate the operator using a sequence of finite rank operators and Satz III of Weyl's paper **[20]**. This provides an estimate of the form

$$n^{1/2}s_n \leq \|K - K_n\|_{L_2},$$

where $\{K_n\}$ are kernels of the operators in the approximation sequence (i.e. Rank $K_n \leq n$).

An advanced version of this approach was used in **[13]** where the authors successfully applied the technique to the case when the symbol is the indicator function supported on a domain with a smooth boundary.
It turns out (see **[13, Th. 9]**) that if the boundary of the region Λ has a piecewise C^1-boundary, then the k^{th} eigenvalue can be estimated $\lambda_k^{\pm} = \underline{O}(k^{-\frac{3}{4}})$. The estimate is sharp when the boundary consists of radial circles. However, it can be strengthened for some other regions.

Unfortunately, the finite rank approximation does not provide precise estimates for other symbols (when the symbol is an indicator of a domain other than an annular region). The approximation can be enhanced if the uniform grid for the main part (which Ramanathan and Topiwala used) is replaced with a more advanced one, for instance, the gradient grid. Although this may provide a better result, it still does not give the precise estimate we expect.

Rewriting the ΨDO as an integral operator in an equivalent (in some sense) form can reduce the question to the case of smooth symbols where relevant techniques are available (see **[18]**, **[4]**). If the kernel K is supported on a compact set Ω, the degree l of its smoothness defines the order of the estimate [**4, Ch.11, §8, Th.4, p.273**],

$$s_n = C_{\Omega,l}\Big(\int_\Omega \|K(\cdot,y)\|^2_{W^l_2(\Omega)}dy\Big)^{\frac{1}{2}} \cdot \overline{o}\left(\frac{1}{n^{\frac{1}{2}+l}}\right).$$

For kernels with unbounded support one can use other estimates, which require certain regularity conditions of the symbol. In Section 3.2 we extend this estimate to a wider class of kernels (see Theorem 3.2.3). This completely covers the result of Ramanathan and Topiwala in [**13**] as we will see in Corollary 3.2.4.

Belonging to trace-ideal operator classes (Schatten classes $\mathbb{S}_p, \mathbb{S}_{p,\infty}$) (see Section 2 for notations) immediately implies polynomial estimates of the type $s_n \lesssim n^{-\frac{1}{p}}$.

Some spectral estimates for ΨDOs with smooth symbols are obtained in [**18, Th. 2.5 - 2.7**].

Sometimes the symbol $\sigma = \sigma(t,\xi)$ is not necessarily smooth but belongs to $\mathrm{L}^p(\mathbb{R}^d \times \mathbb{R}^d)$ class.

In this case the following estimate holds (see [**10, Th 2.2**])

$$\|\mathrm{Op}^{\mathrm{a}}_1(\sigma)\|_{\mathbb{S}_{q,\infty}} \le C_p\|\sigma\|_p,$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p\in[1,2]$.

However, the best estimate provided by this theorem is $\overline{o}(n^{-\frac{1}{2}})$ when $p=2$ and the operator is Hilbert-Schmidt.

We focus on the case when the variables are separated,
i.e. $\sigma(x,\xi) = f(x)g(\xi)$, where f and g belong to some specific functional spaces. There are several known results.

For instance, if $p>2$, $f\in \mathrm{L}^p(\mathbb{R}^d)$ and $g\in \mathrm{L}^p_w(\mathbb{R}^d)$, the weak-$\mathrm{L}^p-$space, the Cwikel (see [**5**]) theorem holds,

$$\|\mathrm{Op}^{\mathrm{a}}_1(fg)\|_{\mathbb{S}_{p,\infty}} \le C_p\cdot\|f\|_p\cdot\|g\|_{p,w}.$$

Birman and Solomyak introduced a special lattice-norm class $\mathrm{l}^p(\mathrm{L}^q)(\mathbb{R}^d)$ to cover the case $0<p<2$ (see [**3, Th. 11.1**]) and obtained the estimate

$$\|\mathrm{Op}^{\mathrm{a}}_1(fg)\|_{\mathbb{S}_p} \le C_p\cdot\|f\|_{2,p}\cdot\|g\|_{2,p}, \tag{1.3.1}$$

where $f,g\in \mathrm{l}^p(\mathrm{L}^2)(\mathbb{R}^d)$ (see Chapter 3 for notations).

Simon in [**15, Th. 4.6**] extended the result above on so called weak-lattice quasi-normed spaces (however restricting the interval of values

p). If one of the functions (for example, g) belongs to a weak quasi-normed space, $\mathrm{l}^{p,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$, then the following weak operator norm estimate holds

$$\|\mathrm{Op}_1^{\mathrm{a}}(fg)\|_{\mathbb{S}_{p,\infty}} \leq C_p \cdot \|f\|_{2,p} \cdot \|g\|_{2,p,\infty}, \tag{1.3.2}$$

where $f \in \mathrm{l}^p(\mathrm{L}^2)(\mathbb{R}^d)$, $g \in \mathrm{l}^{p,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$, $p \in (1,2)$.

In this work we obtain a new result (Theorem 3.3.1), which covers Simon's result (see Remark 3.3.2), and extend the estimate above to a wider interval, $p \in (0,2)$, and for a wider class of functions, namely, when both f and g belong to possibly different weak-lattice quasi-normed spaces $\mathrm{l}^{\gamma_1,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$ and $\mathrm{l}^{\gamma_2,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$, respectively. Unlike Simon's estimate (1.3.2), the operator is no longer in $\mathbb{S}_{p,\infty}$ (i.e. the estimate is weaker) in the case when $\gamma_1 = \gamma_2$ (i.e. when f and g both belong to the same weak-lattice space). To describe the asymptotic behaviour of the operator singular values, we consider Weidl operator classes $\Sigma_{f,p}$ introduced in Chapter 2. In the next paragraph we compare the obtained result with the Simon's estimate (1.3.2) in more details.

There is a spectral asymptotic formula for some types of ΨDOs obtained by Dauge and Robert in [**6**]. However, his method and computations are quite cumbersome. What we achieve below obeys his results, however requires much simpler derivations. A different method is introduced, which is independent from the approach of Dauge and Robert.

1.4 Main results

The main result refers to the Weyl discontinuous symbol of a special type (indicator of an angular region) and can briefly be represented as

$$\lambda_k^{\pm}\Big(\mathrm{Op}_1^{\mathrm{W}}(\chi_{\{(\xi,t)\,|\,\xi\leq ct,\, t\geq 0\}}a(t,\xi))\Big) = \frac{1}{4\pi^2}\frac{\log(k+1)}{k}\big(1+\bar{o}(1)\big) \tag{1.4.1}$$

where $c \in \mathbb{R}$ and $a(\cdot,\cdot) \in C_0^\infty(\mathbb{R}^2)$.

The ΨDO $\mathrm{Op}_1^{\mathrm{W}}(\chi_{\{(\xi,t)\,|\,\xi\leq ct,\, t\geq 0\}}a(t,\xi))$ is the *main operator* studied in this work.

The estimate (1.4.1) is not a Schatten-wise estimate. While Schatten operator classes $\mathbb{S}_{p,\infty}$ provide polynomial estimates $\underline{O}(k^{-1/p})$, the main operator eigenvalues asymtotics can be described more precisely using Weidl operator classes $\mathfrak{S}_p$ consisting of the operators with singular values decay rate $\underline{O}(k^{-\frac{1}{p}}\log^{-\frac{1}{p}}(k+1))$ (see Definition 2.2.3 in Section 2).

The spectral analysis of this operator requires some preliminary work and some techniques for operators of the form $\mathrm{Op}_1^{\mathrm{a}}(f(x)g(\xi))$. As mentioned before, this was investigated in **[4]** and **[15]**. However these results are not enough and we need to generalise them on a wider class of functions f, g, the lattice (quasi-)normed spaces $\mathrm{l}^{p,\infty}(\mathrm{L}^q)(\mathbb{R})$ introduced in Chapter 3.

In Theorem 3.3.1 we consider symbols which admit separation of variables.

If $\gamma_1, \gamma_2 \in (0,2)$, $\gamma = \max\{\gamma_1, \gamma_2\}$, $p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) = f(\mathbf{x})g(\boldsymbol{\xi})$
where $f \in \mathrm{l}^{\gamma_1,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$, $g \in \mathrm{l}^{\gamma_2,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$, then

$$s_k\big(\mathrm{Op}_1^{\mathrm{a}}(p)\big) \le C_\gamma k^{-\frac{1}{\gamma}} \log^{-\frac{1}{\gamma}}(k+1) \cdot \|f\|_{2,\gamma,\infty} \cdot \|g\|_{2,\gamma,\infty}, \text{ if } \gamma_1 = \gamma_2,$$

and

$$s_k\big(\mathrm{Op}_1^{\mathrm{a}}(p)\big) \le C_{\gamma_1,\gamma_2} k^{-\frac{1}{\gamma}} \cdot \|f\|_{2,\gamma_1,\infty} \cdot \|g\|_{2,\gamma_2,\infty}, \text{ if } \gamma_1 \neq \gamma_2.$$

Taking f and g equal to $\frac{\zeta(t)}{t}$ or $\langle t\rangle^{-1} = \frac{1}{\sqrt{t^2+1}}$ we obtain

$$s_k\big(\mathrm{Op}_1^{\mathrm{a}}(fg)\big) = \underline{O}(k^{-1}\log(k+1))$$

We call each of these four operators $\mathrm{Op}_1^{\mathrm{a}}(fg)$ the *model operator.*
It turns out the spectral analysis of the operator $\mathrm{Op}_1^{\mathrm{W}}(\chi_P)$ where P refers to a polygon in the phase space (along with the main operator), can be reduced (see Chapter 5, Section 5.4) to the model operator.

Theorem 3.3.1 is an independent result and can be considered as the secong main result of the work.

According to Remark 3.3.2 this theorem covers Simon's result (1.3.2) and generalises it for $p = \gamma \in (0,1]$ and also for the case when functions f and g belong to different weak-lattice spaces.
The idea used in **[15, Th. 4.6]** repeats Cwikel's approach (see **[5, p.3]**) of splitting the support of the symbol into two parts (split into dyadic cubes each), which are described by inequalities
$f_n(x)g_k(\xi) \ge R$ (corresponding to the "big" values of the symbol) and $f_n(x)g_k(\xi) < R$ (corresponding to the "small" values of the symbol), where f_n, g_k are the corresponding norms of f, g on the n^{th} and k^{th} dyadic cube, respectively.
For the "big" part, Simon uses the trace norm ($\|\cdot\|_{\mathbb{S}_1}$) estimate, which might not exist in case $p \in (0,1]$). For the "small" part the Hilbert-Schmidt norm ($\|\cdot\|_{\mathbb{S}_2}$) estimate is used, which diverges when both functions lie in a weak-lattice space.
Unlike Simon's idea, we split the support in a different way. Instead of

the "hyperbolic partition" we split separately function f and function g into f_{R-}, f_{R+} and g_{R-}, g_{R+}, respectively, where subscript $R-$ refers to the "big" values and $R+$ to the small ones.
It turns out that for such "rectangular" partition the Hilbert-Schmidt estimate gives a more precise result. Moreover, for the "big" part we use $\mathbb{S}_q-$norm estimate, $q < \gamma$, following (1.3.1), which, in case $\gamma \leq 1$, gives a more precise result than the trace-norm estimate in **[15, Th. 4.6]**.
The combination of the smart partition and appropriately chosen auxiliary facts is the key ingredient leading to stronger results than the existing ones.

1.5 How the paper is structured

In Chapter 2 we introduce notations of Weidl operator classes and state the general result of the perturbation theory, Theorem 2.2.14, which allows to state asymptotic formulae for the operators we study reducing them to the model operators with a known asymptotic formula.

Chapter 3 consists of some generalisations of known results with focus on Theorem 3.3.1 and Theorem 3.3.14.
Theorem 3.3.1 describes the case when the symbol $p(\mathbf{x}, \boldsymbol{\xi}) = f(\mathbf{x})g(\boldsymbol{\xi})$, where $f \in \mathrm{l}^{\gamma_1,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$, $g \in \mathrm{l}^{\gamma_2,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$ for some $\gamma_1, \gamma_2 \in (0, 2)$. It turns out that the "weakest decaying function" dictates the estimate for the singular value, i.e. $s_k \lesssim_{\gamma_1,\gamma_2} k^{-\frac{1}{\gamma}}$, where $\gamma = \max\{\gamma_1, \gamma_2\}$ and $\gamma_1 \neq \gamma_2$. Otherwise, when f and g have the same "rate of decay"(i.e. $\gamma_1 = \gamma_2 = \gamma$) $s_k \lesssim_\gamma k^{-\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}(1 + k)$.
In Theorem 3.3.14 we prove an auxiliary result of the reduction process. Under certain restriction, the ΨDO with the Weyl symbol
$p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) = a(\frac{\mathbf{x}+\mathbf{y}}{2})b(\boldsymbol{\xi})$ belongs to the same trace-ideal class as the one with the symbol $p(\mathbf{x}, \boldsymbol{\xi}) = a(\mathbf{x})b(\boldsymbol{\xi})$.
The model operators we appeal to, $\mathrm{Op}_1^{\mathrm{W}}(\langle t\rangle^{-1}\langle\xi\rangle^{-1})$, $\mathrm{Op}_1^{\mathrm{W}}(\zeta(t)t^{-1}\langle\xi\rangle^{-1})$ etc, are introduced in Remark 3.3.16.

Chapter 4 describes the theory of self-adjoint operators of a specific type in terms of operator decoupling. These results help to establish asymptotic formulae for the model operator (see Chapter 5, Section 5.3).

The main chapter, Chapter 5, consists of the reduction process (from the main operator (5.0.1) to the model operator $\mathrm{Op}_1^{\mathrm{W}}(\zeta(t)t^{-1}\langle\xi\rangle^{-1})$) in terms of obtaining an asymptotic estimate (Lemma 5.1.4 and Theorem 5.1.9) Using two different approaches (Dauge-Robert formula and

Birman-Schwinger principle), we obtain the asymptotic formula (1.2.2) in Sections 5.2 and 5.3, respectively.

Chapter 2

Weidl classes and asymptotical formulae

2.1 Notations and auxiliary results

We denote the positive and negative eigenvalues of a compact self-adjoint operator T by

$$\lambda_1^+ \geq \lambda_2^+ \geq \ldots \lambda_k^+ \geq ... > 0 > ... - \lambda_k^- \geq -\lambda_2^- \geq \ldots \geq -\lambda_1^-$$

counted with their multiplicities.

Consider a self-adjoint compact pseudo-differential operator $T = \mathrm{Op}_1^{\mathrm{W}}(\sigma)$ with real valued Weyl symbol σ.

We introduce the spectral counting and volume functions as follows.

Definition 2.1.1. Spectral counting functions n_+, n_- of a compact pseudo-differential operator $\mathrm{Op}_1^{\mathrm{W}}(\sigma)$ are defined by

$$n_\pm(\lambda) = n_\pm(\lambda;\sigma) := \Big|\{k| \ \pm\lambda_k^\pm \geq \lambda\}\Big|.$$

Definition 2.1.2. Spectral volume function $V_+ : [0,\infty) \to [0,\infty)$ of a compact pseudo-differential operator is defined by

$$V_\pm(\lambda;) = V_\pm(\lambda;\sigma) := \frac{1}{(2\pi)^d} \int_{\pm\sigma(t,\xi)>\lambda} dtd\xi.$$

We focus on V_+, n_+ functions, but the same results hold for the V_-, n_- analogues.

Due to [6, Th. (1.3)] under some assumptions about the symbol σ

$$n_+(\lambda;\sigma) = V_+(\lambda;\sigma)(1+\overline{o}(1)), \quad \lambda \to 0+. \tag{2.1.1}$$

Definition 2.1.3. Symbol σ satisfying (2.1.1) is called *"Weyl asymptotics symbol"* (see [9, §9, p.67]).

The following lemma is an auxiliary result describing the connection between the asymptotic expansion of the counting function $n_\pm(\lambda)$ of the operator $\mathrm{Op}_1^{\mathrm{W}}(\sigma)$ and the sequence of its eigenvalues, $\lambda_k^\pm$.
Note that $n_\pm(\lambda_k^\pm) = k$. Hence, for any Weyl asymptotics symbol

$$V_+(\lambda_k^+) = k(1 + \overline{o}(1)), \quad k \to \infty \tag{2.1.2}$$

A natural question is whether we can invert the formula below and under what restrictions one can consider $\lambda_k^+ \sim V_+^{-1}(k)$.

Lemma 2.1.4. *Let $f : (0, \infty) \to (0, \infty)$ be a differential function satisfying*

$$\frac{f'(x)}{f(x)} > \frac{C_1}{x} \tag{2.1.3}$$

for some constant $C_1 > 0$ and for all $x > 0$.
Moreover, let $(\beta_k)_{k\geq 1}$ be a sequence of positive numbers such that $\beta_k \to \infty$ and

$$f(\beta_k) = k(1 + \overline{o}(1)), \quad k \to \infty.$$

Then

$$\beta_k = f^{-1}(k)(1 + \overline{o}(1)), \quad k \to \infty,$$

where f^{-1} is the inverse function of f.

Proof. Since $f(x) > 0$ for all $x > 0$, we deduce $f'(x) > 0$ from (2.1.3). Thus, f is a strictly increasing function which has a differentiable inverse f^{-1} with $\big(f^{-1}\big)'(k) = \frac{1}{f'(x)} > 0$ for any k and x such that $k = f(x) > 0$. Moreover, the inverse function f^{-1} also obeys (2.1.3) but with another constant. Indeed,

$$\frac{1}{k\big(f^{-1}\big)'(k)} = \frac{f'(x)}{f(x)} > \frac{C_1}{x} = \frac{C_1}{f^{-1}(k)}.$$

Thus,

$$\big(\log f^{-1}(\cdot)\big)'(k) = \frac{\big(f^{-1}\big)'(k)}{f^{-1}(k)} < \frac{1}{C_1 k} \tag{2.1.4}$$

With the help of the Mean value theorem for f^{-1}

$$\beta_k = f^{-1}(f(\beta_k)) = f^{-1}(k + \overline{o}(k)) = f^{-1}(k) + f^{-1}(k + \overline{o}(k)) - f^{-1}(k)$$

$$= f^{-1}(k) + \big(f^{-1}\big)'(k_0) \cdot \overline{o}(k) = f^{-1}(k)\Big(1 + \frac{k}{f^{-1}(k)} \cdot \big(f^{-1}\big)'(k_0) \cdot \overline{o}(1)\Big),$$

where $k_0 \in (k, k+\overline{o}(k))$ (without loss of generality we assume that $\overline{o}(k) > 0$)
It remains to prove that $\frac{k}{f^{-1}(k)} \cdot \left(f^{-1}\right)'(k_0)$ is a bounded function. Indeed, using the Mean value theorem for $\log(f^{-1})$ and (2.1.4)

$$0 < \frac{k}{f^{-1}(k)} \cdot \left(f^{-1}\right)'(k_0) < \frac{k}{f^{-1}(k)} \cdot \frac{f^{-1}(k_0)}{C_1 k_0}$$

$$= \frac{1}{C_1} \cdot \frac{k}{k+\overline{o}(k)} \cdot e^{\log(f^{-1}(k_0)) - \log(f^{-1}(k))}$$

$$\lesssim C_1^{-1} e^{(\log f^{-1}(\cdot))'(t)(k_0-k)} < C_1^{-1} e^{\frac{k_0-k}{C_1 t}} < C_1^{-1} e^{\frac{k_0-k}{C_1 k}} = C_1^{-1} e^{\overline{o}(1)} = \underline{O}(1),$$

where $t \in (k, k_0)$.

□

Remark 2.1.5. The statement of the lemma is also true for $f : (d_1, \infty) \to (d_2, \infty)$, $d_1, d_2 > 0$. Condition (2.1.3) for $f(x) = V_+\left(\frac{1}{x}\right)$ is a consequence of condition (T) in [6, p.93] for $V(\lambda)$, $\lambda = \frac{1}{x}$.

Theorem 2.1.6. *If function $f(x) = V_+\left(\frac{1}{x}; \sigma\right)$ satisfies (2.1.3) and σ is a Weyl asymptotics symbol, then*

$$\lambda_k^+ = V_+^{-1}(k)(1+\overline{o}(1)), \quad k \to \infty.$$

Proof. Since $f^{-1}(k) = \frac{1}{V_+^{-1}(k)}$, taking $\beta_k = \frac{1}{\lambda_k^+} \to \infty, k \to \infty$, using (2.1.2), lemma 2.1.4 implies the asymptotics $\beta_k = \frac{1}{V_+^{-1}(k)}(1+\overline{o}(1))$, which leads to the result. □

Example. Log-power functions (see [9, §9, p.65]). It appears that $V_+(\lambda) = -C \cdot \lambda^{-a} \log \lambda$, $\lambda \in (0,1)$ with $a, C > 0$ satisfies (2.1.3).
Indeed, $V_+\left(\frac{1}{x}\right) = C \cdot x^a \log x$. Hence, for $x > 1$

$$\frac{V_+'(x)}{V_+(x)} = \frac{Ca \cdot x^{a-1} \log x + C x^{a-1}}{C \cdot x^a \log x} = \frac{a}{x} + \frac{1}{x \log x} \gtrsim \frac{a}{x}.$$

However, there is no explicit formula for V_+^{-1}. Instead we can obtain an asymptotic expression of $\log \lambda$ in terms of V_+.

$$\log V_+(\lambda) = \log C - a\log(\lambda) + \log\log(\lambda)^{-1} = -a \log \lambda (1+\overline{o}(1)),\ \lambda \to 0+,$$

or equivalently

$$\log \lambda = -\frac{\log V_+(\lambda)}{a} \cdot (1+\overline{o}(1)).$$

Therefore,

$$\lambda^{-a} = -\frac{1}{C} \cdot \frac{V_+(\lambda)}{\log \lambda} = \frac{a}{C} \cdot \frac{V_+(\lambda)}{\log V_+(\lambda)} \cdot (1+\overline{o}(1)).$$

Expressing $\lambda = V_+^{-1}(t)$ as the inverse function of $t = V_+(\lambda)$,

$$V_+^{-1}(t) = \Big(\frac{C}{a}\Big)^{\frac{1}{a}} \cdot \Big(\frac{\log t}{t}\Big)^{\frac{1}{a}} \cdot (1 + \overline{o}(1)).$$

Thus, finally, due to Theorem 2.1.6

$$\lambda_k^+ \sim \Big(\frac{C}{a}\Big)^{\frac{1}{a}} \cdot \Big(\frac{\log k}{k}\Big)^{\frac{1}{a}}, \quad k \to \infty.$$

Remark 2.1.7. In the case when V_+ does not satisfy (2.1.3), sometimes the main part of the asymptotic expansion of V_+ might obey (2.1.3). In this case the statement of the theorem is still true.
Indeed, if $V_+\big(\frac{1}{x}\big) = \tilde{V}_+\big(\frac{1}{x}\big)(1+\overline{o}(1))$, $x \to \infty$ and $\tilde{V}_+\big(\frac{1}{x}\big)$ obey (2.1.3), then, since $k = n_+(\lambda_k^+) \sim V_+(\lambda_k^+) \sim \tilde{V}_+(\lambda_k^+)$, condition (2.1.2) also holds. We apply Theorem 2.1.6 to the function $\tilde{V}_+\big(\frac{1}{x}\big)$ to obtain

$$\lambda_k^+ \sim \tilde{V}_+^{-1}(k), \quad k \to \infty.$$

In the example above the function $\tilde{V}_+(\lambda)$, the inverse of function
$\tilde{V}_+^{-1}(k) = \Big(\frac{C}{a}\Big)^{\frac{1}{a}} \cdot \Big(\frac{\log k}{k}\Big)^{\frac{1}{a}}$ is asymptotically equivalent to
$V_+(\lambda) = -C \cdot \lambda^{-a} \log \lambda$ as $\lambda \to 0+$.

2.2 Weidl operator classes

We denote by $\{s_k(T)\}_{k\geq 1}$ the set of *singular values* of a compact operator T arranged in a non-increasing order counted with multiplicities, i.e. $s_k(T) = \lambda_k^+\big(\sqrt{T^*T}\big)$, $s_1(T) \geq s_2(T) \geq ...$
First recall the Schatten operator class $\mathbb{S}_p$, $p > 0$, the collection of all compact operators T with $\{s_k(T)\}_{k\geq 1} \in l^p$. The Schatten quasi-norm (norm for $p \geq 1$) $\|\cdot\|_{\mathbb{S}_p}$ is defined as follows

$$\|T\|_{\mathbb{S}_p} := \|\{s_k(T)\}\|_p = \Big(\sum_{k=1}^{\infty} s_k^p(T)\Big)^{\frac{1}{p}},$$

$$\|T\|_{\mathbb{S}_\infty} := \|\{s_k(T)\}\|_\infty = s_1(T) = \|T\|.$$

Belonging to a Schatten class $\mathbb{S}_p$ provides a trace-class estimate $\sum_{k=1}^{\infty} s_k^p(T) \leq C$, and, as a consequence of this, $s_k(T) = \overline{o}\big(k^{-\frac{1}{p}}\big)$.
Indeed, since $\{s_k(T)\}_{k\geq 1}$ is a decreasing sequence, and the series $\sum_{k=1}^{\infty} s_k^p(T)$ converges, the part of its remainder $\sum_{k=n+1}^{2n} s_k^p(T) \to 0, \ \ n \to \infty$. Therefore,

$$n s_{2n}^p(T) \leq \sum_{k=n+1}^{2n} s_k^p(T) = \overline{o}(1).$$

Hence, $s_{2n}^p(T) = \bar{o}(n^{-1})$, which implies $s_k(T) = \bar{o}\big(k^{-\frac{1}{p}}\big)$ for both odd and even indices k.
An important generalisation of the Schatten operator class is the quasi-normed space $\mathbb{S}_{p,q}$ where the norm is defined by

$$\|T\|_{\mathbb{S}_{p,q}} := \|\{s_k(T)\}\|_{p,q} := \begin{cases} \Big(\sum_{k=1}^{\infty} k^{\frac{q}{p}-1} s_k^q(T)\Big)^{\frac{1}{q}}, & q \in (0,\infty) \\ \sup_k k^{\frac{1}{p}} s_k(T), & q = \infty. \end{cases}$$

We focus on the space $\mathbb{S}_{p,\infty}$ (the weak-Schatten class). For any compact operator T in this space we obtain a pointwise polynomial estimate of the form $s_k(T) = \underline{O}\big(k^{-\frac{1}{p}}\big)$.
Since $\|\cdot\|_{\mathbb{S}_{p,\infty}}$ is not a norm, the standard triangle inequality does not hold. However, we can state a weaker result, the quasi-triangle inequality stated in the Proposition below (see [3, Lemma 1.1]).

Proposition 2.2.1. *For any $p \in (0,1)$ and a finite or countable set of $T_n \in \mathbb{S}_{p,\infty}$*

$$\Big\|\sum_n T_n\Big\|_{\mathbb{S}_{p,\infty}}^p \leq C \sum_n \|T_n\|_{\mathbb{S}_{p,\infty}}^p$$

where $C = C(p)$ does not depend on the number of T_n.
Moreover, if $T_m^* T_k = O$ for any $m \neq k$, then the inequality holds for $p \in (0,2)$.

The full set of relationships between Schatten and weak-Schatten operator classes, the triangle and quasi-triangle inequalities for their norms and quasi-norms respectively can be found in [3, §1].

Next, we generalise the weak-Schatten class $\mathbb{S}_{p,\infty}$ by introducing a non-polynomial scale of estimates.
Let's introduce a certain class of functions f, the class $\mathcal{B}$, satisfying

- $f : [0,\infty) \to [0,\infty)$
- $f(0) = 0,\ f(1) = 1, \lim_{x\to\infty} f(x) = \infty$
- f is increasing and concave

All functions in this class obey

Lemma 2.2.2. *If $f \in \mathcal{B}$, then $f(n+1) \sim f(n)$, i.e. $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$, and f is a subadditive function, i.e.*

$$f(n+m) \leq f(n) + f(m) \quad \textit{for any } n, m \geq 0.$$

Proof. Since, f is an increasing concave function, the function $F(x) = \frac{f(x+m)-f(x)}{m}$, $m > 0$, is decreasing. Thus, for any $n, m > 0$ $F(n) < F(0)$, which is equivalent to $f(n+m) \leq f(n) + f(m)$.
Due to the monotonicity and subadditivity $f(n) \leq f(n+1) \leq f(n)+f(1)$. Thus, $1 \leq \frac{f(n+1)}{f(n)} \leq 1 + \frac{f(1)}{f(n)} \to 1$, $n \to \infty$. Therefore, $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$. □

For $f \in \mathcal{B}$ and for a positive parameter $\gamma > 0$ we recall the following operator class (see [19, p.119])

Definition 2.2.3.

$$\Sigma_{f,\gamma} := \{T-\text{compact operator}|\sup_k s_k(T)f(k)^{\frac{1}{\gamma}} < \infty\}$$

The notations of two important examples of $\Sigma_{f,\gamma}$ considered in the work are below. If $f(x) = x$, then $\Sigma_{f,\gamma} = \mathbb{S}_{\gamma,\infty}$. If $f(x) = \frac{\log 2}{1-\log 2} \cdot \left(\frac{x}{\log(1+x)} - 1\right)$, then $\Sigma_{f,\gamma} = \mathfrak{S}_\gamma$.

Definition 2.2.4. For a compact operator T the spectral counting functions n is defined by

$$n(\lambda; T) := \left|\{k|s_k(T) \geq \lambda\}\right|$$

Lemma 2.2.5. *For any compact operator T and any $f \in \mathcal{B}$ the upper limit $\limsup_{s\to 0+} f(n(s;T))s^\gamma$ exists if and only if there exists $\limsup_{k\to\infty} f(k)s_k^\gamma$. Moreover, the values of the limits are the same.*

Remark 2.2.6. The statement is also true if we replace s_k with $\lambda_k^\pm$.

Proof. If $\limsup_{s\to 0+} f(n(s;T))s^\gamma$ exists, then for any sequence $x_k \to 0$ (including $x_k = s_k$) $\limsup_{k\to\infty} f(n(x_k;T))x_k^\gamma = \limsup_{s\to 0+} f(n(s;T))s^\gamma$. Since $n(s_k;T) = k$, $\limsup_{k\to\infty} f(k)s_k^\gamma = \limsup_{s\to 0+} f(n(s;T))s^\gamma$.

Let $\limsup_{k\to\infty} f(k)s_k^\gamma = D$.
Take any $s \in (0, s_1)$. There exists such n that $s \in [s_{n+1}, s_n)$. Since f is an increasing function,

$$f(n+1)s_n^\gamma = f(n(s_{n+1};T))s_n^\gamma \geq s^\gamma f(n(s;T)) \geq s_{n+1}^\gamma f(n(s_n;T)) = f(n)s_{n+1}^\gamma \tag{2.2.1}$$

Since $f(n+1) \sim f(n)$,

$$\lim_{n\to\infty} f(n+1)s_n^\gamma = \lim_{n\to\infty} f(n)s_n^\gamma = \lim_{n\to\infty} f(n)s_{n+1}^\gamma = D.$$

Now, applying the squeeze theorem in (2.2.1), we obtain

$$\lim_{n\to\infty} f(n(s;T))s^\gamma = D.$$

□

Let's introduce the following seven functionals on this operator class.

Definition 2.2.7. [19, (1.3), (1.4)]

$$|T|_{f,\gamma} := \sup_k f(k) s_k^\gamma(T),$$

$$D_{f,\gamma}(T) = \limsup_{s\to 0+} f(n(s;T)(s))s^\gamma = \limsup_{k\to\infty} f(k)s_k^\gamma,$$

$$d_{f,\gamma}(T) = \liminf_{s\to 0+} f(n(s;T)(s))s^\gamma = \liminf_{k\to\infty} f(k)s_k^\gamma,$$

$$\Delta^\pm_{f,\gamma}(T) = \limsup_{s\to 0+} f(n_\pm(s;T)(s))s^\gamma = \limsup_{k\to\infty} f(k)(\lambda_k^\pm)^\gamma,$$

$$\delta^\pm_{f,\gamma}(T) = \liminf_{s\to 0+} f(n_\pm(s;T)(s))s^\gamma = \liminf_{k\to\infty} f(k)(\lambda_k^\pm)^\gamma.$$

Remark 2.2.8. Due to Lemma 2.2.5 the functionals are well defined. The functional $|\cdot|_{f,\gamma}$ describes the estimates of the singular values

$$s_k(T) \le \frac{|T|_{f,\gamma}^{\frac{1}{\gamma}}}{f(k)^{\frac{1}{\gamma}}} \text{ for any } k.$$

The functionals $D_{f,\gamma}, d_{f,\gamma}$ ($\Delta^\pm_{f,\gamma}, \delta^\pm_{f,\gamma}$) help to describe the asymptotic formula of singular values (eigenvalues) as $k \to \infty$,

$$s_k(T) = \frac{A_k}{f(k)^{\frac{1}{\gamma}}} \cdot (1 + \overline{o}(1)), \text{ where } d_{f,\gamma} \le A_k \le D_{f,\gamma} \le |T|_{f,\gamma},$$

$$\lambda_k^\pm(T) = \frac{B_k}{f(k)^{\frac{1}{\gamma}}} \cdot (1 + \overline{o}(1)), \text{ where } \delta^\pm_{f,\gamma} \le B_k \le \Delta^\pm_{f,\gamma}.$$

While Schatten operator class provides the polynomial scale $\{k^{-1/\gamma}\}_\gamma$ and corresponding estimates $\underline{O}(k^{-1/\gamma})$ for the k^{th} singular value, Weidl operator classes give more precise scales $\{f(k)^{-1/\gamma}\}_\gamma$ for the estimates using slowly increasing functions $f \in \mathcal{B}$.

Definition 2.2.9. To describe the rate of eigenvalues decay it is convenient to use the following functionals, which are quasi-norms,

$$\|T\|_{\Sigma_{f,\gamma}} := |T|_{f,\gamma}^{\frac{1}{\gamma}} = \sup_k f^{\frac{1}{\gamma}}(k) s_k(T), \quad f \in \mathcal{B}.$$

In particular,

$$\|T\|_{\mathfrak{S}_\gamma} = \left(\frac{\log 2}{1-\log 2}\right)^{\frac{1}{\gamma}} \sup_k \left(\frac{k}{\log(1+k)} - 1\right)^{\frac{1}{\gamma}} s_k(T).$$

Remark. If $f(x) = x$, $\|\cdot\|_{\Sigma_{f,\gamma}} = \|\cdot\|_{\mathbb{S}_{\gamma,\infty}}$

Definition 2.2.10. $\overset{\circ}{\Sigma}_{f,\gamma} := \{T \in \Sigma_{f,\gamma} \mid D_{f,\gamma} = 0\}$.

The following theorem is a triangle inequality analogue for the functionals introduced above.

Theorem 2.2.11. *[19, (1.7)] If $T_1, T_2 \in \Sigma_{f,p}$ where $f \in \mathcal{B}$, then*

$$D_{f,p}(T_1 + T_2)^{\frac{1}{p+1}} \leq D_{f,p}(T_1)^{\frac{1}{p+1}} + D_{f,p}(T_2)^{\frac{1}{p+1}}$$

Proof. We use the following inequality (see [4, Ch. 11, §1, (17), p.245])

$$n(s; T_1 + T_2) \leq n(\theta s; T_1) + n((1-\theta)s; T_2), \quad \theta \in (0, 1).$$

Thus, using the properties of the functions in class $\mathcal{B}$ (monotonicity and Lemma 2.2.2),

$$f(n(s; T_1 + T_2)) \leq f(n(\theta s; T_1) + n((1-\theta)s; T_2))$$

$$\leq f(n(\theta s; T_1) + f(n((1-\theta)s; T_2), \quad \theta \in (0, 1).$$

Hence, multiplying by s^p,

$$s^p \cdot f(n(s; T_1 + T_2)) \leq \frac{(s\theta)^p f(n(\theta s; T_1))}{\theta^p} + \frac{(s(1-\theta))^p f(n((1-\theta)s; T_2)}{(1-\theta)^p}.$$

Taking $\limsup_{s \to 0+}$ we obtain

$$D_{f,p}(T_1 + T_2) \leq \theta^{-p} D_{f,p}(T_1) + (1-\theta)^{-p} D_{f,p}(T_2).$$

Then, taking $\inf_{\theta \in (0,1)}$, we obtain the result.

□

Remark 2.2.12. The statement of the Theorem 2.2.11 is also true for the functionals $\Delta^{\pm}_{f,p}, \delta^{\pm}_{f,p}$ and $d_{f,p}$.

Applying the triangle inequality above for T_1 and $T_2 - T_1$, we get

$$\left| D_{f,p}(T_1)^{\frac{1}{p+1}} - D_{f,p}(T_2)^{\frac{1}{p+1}} \right| \leq D_{f,p}(T_1 - T_2)^{\frac{1}{p+1}}.$$

Since $D_{f,p}(T_1 - T_2) \leq |T_1 - T_2|_{f,p}$, the following corollary (for the functional $D_{f,\gamma}(\cdot)$ and for the other five) holds.

Corollary 2.2.13. *If $T_1, T_2 \in \Sigma_{f,p}$ where $f \in \mathcal{B}$, then*

$$\left|\Delta^{\pm}_{f,p}(T_1)^{\frac{1}{p+1}} - \Delta^{\pm}_{f,p}(T_2)^{\frac{1}{p+1}}\right|, \quad \left|D_{f,p}(T_1)^{\frac{1}{p+1}} - D_{f,p}(T_2)^{\frac{1}{p+1}}\right|,$$

$$\left|\delta^{\pm}_{f,p}(T_1)^{\frac{1}{p+1}} - \delta^{\pm}_{f,p}(T_2)^{\frac{1}{p+1}}\right|, \quad \left|d_{f,p}(T_1)^{\frac{1}{p+1}} - d_{f,p}(T_2)^{\frac{1}{p+1}}\right| \leq |T_1 - T_2|_{f,p}^{\frac{1}{p+1}}.$$

The following theorem provides a useful result of perturbation theory. It turns out that the asymptotic formulae are the same for two operators (T and T_m) which differ from each other by an operator T_r with a higher rate of its eigenvalues decay.

Theorem 2.2.14. *If* $T = T_m + T_r$*, where* $T_m \in \Sigma_{f,p}$*,* $T_r \in \Sigma_{g,p}$ *with* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$*,* $f, g \in \mathcal{B}$*, then*

$$T \in \Sigma_{f,p}, \quad D_{f,p}(T) = D_{f,p}(T_m),$$

$$T_r = T - T_m \in \overset{\circ}{\Sigma}_{f,p} .$$

Proof. Using the result above

$$\left|D_{f,p}(T)^{\frac{1}{p+1}} - D_{f,p}(T_m)^{\frac{1}{p+1}}\right|^{p+1} \leq D_{f,p}(T_r) = \limsup_k s_k^{\frac{1}{p}}(T_r) g(k) \cdot \frac{f(k)}{g(k)}$$

$$\leq D_{g,p}(T_r) \cdot \lim_{k\to\infty} \frac{f(k)}{g(k)} = 0.$$

□

Remark 2.2.15. The statement of the Theorem is also true for the functionals $\Delta^{\pm}_{f,p}$, $\delta^{\pm}_{f,p}$ and $d_{f,p}$.

If the asymptotics for singular values of two operators, T_1 and T_2, is the same, e.g. $(f(k))^{-1}(1 + \overline{o}(1))$, the sum $T_1 + T_2$ might have a different asymptotic formula $(g(k))^{-1}(1 + \overline{o}(1))$, $f = \overline{o}(g)$ (indeed, check $T_2 = -T_1$).

However, an important fact about singular values estimation for the sum of two compact operators T_1 and T_2, namely, the Ky Fan's inequality (see [4, Ch.11, §1, p.3]), may help

$$s_{2n-1}(T_1 + T_2) \leq s_n(T_1) + s_n(T_2) \tag{2.2.2}$$

Remark 2.2.16. Note that for any sum of a fixed finite number N_o of operators T_k, $k = 1, 2, ..., N_o$, with the log-polynomial rate of singular values decay, i.e. $s_n(T_k) \leq Cn^{-a} \log^b n$, where $a > 0$ and b are two real constants, we state the same type of estimate, i.e.

$$s_n\Big(\sum_{k=1}^{N_o} T_k\Big) \leq C_{a,b,N_o} \cdot n^{-a} \log^b n, \tag{2.2.3}$$

where C_{a,b,N_o} depends on a, b, N_o only.

Indeed, if $s_n(T_1), s_n(T_2) \leq C\frac{\log^b n}{n^a}$, then $s_{2n}(T_1 + T_2) \leq s_{2n-1}(T_1 + T_2) \leq$

$2C\frac{\log^b n}{n^a}$. Therefore,

$$s_n(T_1 + T_2) \le C2^{a+1}\frac{\log^b \frac{n}{2}}{n^a} \le C_{a,b}\frac{\log^b n}{n^a}.$$

Note, that a result similar to (2.2.3) (corresponding to a different rate of decay) might not be true when singular values decay exponentially, i.e. of $s_n \lesssim_a e^{-an}$.

If singular values decay polynomially, we can sometimes state an estimate for an infinite sum of operators using Proposition 2.2.1.

Chapter 3

General estimates for singular values of ΨDOs with symbols of specific form

3.1 Basic concepts and definitions

In this section we introduce the notions for some sequence and function spaces, which help to specify the ΨDO's symbols we use in this work. We also define the representation

$$f = f_{R-} + f_{R+}$$

for the functions in these spaces, which helps to split the symbol (and the ΨDO) into the *main* and *remainder* parts.

Definition 3.1.1. For a number $\gamma > 0$ the space $\mathrm{l}^{\gamma,\infty}$ (sometimes called the *weak*-l^{γ}) is defined as follows

$$\mathrm{l}^{\gamma,\infty} := \Big\{ (a_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^d} \mid \|a\|_{\gamma,\infty} := \sup_{E>0} E \cdot \Big(\#\big\{ \mathbf{n} : \ |a_{\mathbf{n}}| > E \big\} \Big)^{\frac{1}{\gamma}} < \infty \Big\},$$

where $\#\{...\}$ stands for the number of elements in a set.

A canonical example of a normalised vector of this space is the sequence $a_n = n^{-\frac{1}{\gamma}}$, $n \in \mathbb{Z}$.
Consider here and thereafter the following partition of $\mathbb{R}^d$ into unit $d-$dimensional cubes

$$\mathbb{R}^d = \bigsqcup_{\mathbf{n}\in Z^d} Q_{\mathbf{n}} \ ,$$

where $Q_{\mathbf{n}} = [0,1)^d + \mathbf{n} := \{x + \mathbf{n} | \ x \in [0,1)^d\}$.

Definition 3.1.2. For numbers $q, \gamma > 0$ the *weak lattice quasi-norm* is defined as follows

$$\|f\|_{q,\gamma,\infty} := \Big\| \Big(\|f\chi_{Q_{\mathbf{n}}}\|_q \Big)_{\mathbf{n}\in\mathbb{Z}^d} \Big\|_{\gamma,\infty},$$

where $\|\cdot\|_q$ is defined by $\|f\|_q := \Big(\int_{\mathbb{R}^d} |f(x)|^q dx \Big)^{\frac{1}{q}}$ and refers to the standard L^q-norm whenever $q \geq 1$.
The quasi-normed space of functions with finite quasi-norm $\|\cdot\|_{q,\gamma,\infty}$ is denoted by $\mathrm{l}^{\gamma,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$.

Definition 3.1.3. For numbers $q, \gamma > 0$ the *lattice quasi-norm* is defined as follows

$$\|f\|_{q,\gamma} := \Big\| \Big(\|f\chi_{Q_{\mathbf{n}}}\|_q \Big)_{\mathbf{n}\in\mathbb{Z}^d} \Big\|_{\gamma},$$

where $\|\cdot\|_\gamma$ is defined by $\|(a_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^d}\|_\gamma := \Big(\sum_{\mathbf{n}\in\mathbb{Z}^d} |a_{\mathbf{n}}|^\gamma \Big)^{\frac{1}{\gamma}}$ and refers to the standard $\mathrm{l}^\gamma-$norm whenever $\gamma \geq 1$. The corresponding quasi-normed space is denoted by $\mathrm{l}^\gamma(\mathrm{L}^q)(\mathbb{R}^d)$.

Properties of lattice quasi-norms

Proposition 3.1.4. *For a fixed $f \in l^\gamma(L^q)(\mathbb{R}^d)$, $\gamma, q > 0$ the quasi-norm function $\mathcal{N}_{f,\gamma}(\cdot) = \|f\|_{\cdot,\gamma}$ is well defined and increasing on $(0, q)$.*
The quasi-norm function $\mathfrak{N}_{f,q}(\cdot) = \|f\|_{q,\cdot}$ is well defined and decreasing on (γ, ∞).
For any $\delta > \gamma > 0$ the following inclusion holds $l^{\gamma,\infty}(L^q)(\mathbb{R}^d) \subseteq l^\delta(L^q)(\mathbb{R}^d)$.

Proof. Indeed, the Hölder inequality with weights $\frac{\alpha}{\beta}$ and $1 - \frac{\alpha}{\beta}$, where $0 < \alpha < \beta < q$, implies that

$$\|f\|_{\alpha,\gamma}^\gamma = \sum_{\mathbf{n}\in\mathbb{Z}^d} \Big(\int_{Q_{\mathbf{n}}} |f(x)|^\alpha dx \Big)^{\frac{\gamma}{\alpha}}$$

$$\leq \sum_{\mathbf{n}\in\mathbb{Z}^d} \Big(\int_{Q_{\mathbf{n}}} |f(x)|^\beta dx \Big)^{\frac{\gamma}{\alpha}\cdot\frac{\alpha}{\beta}} \cdot \Big(\int_{Q_{\mathbf{n}}} 1 \; dx \Big)^{\frac{\gamma}{\alpha}\cdot(1-\frac{\alpha}{\beta})} = \|f\|_{\beta,\gamma}^\gamma.$$

Both quasi-norms $\|f\|_{\alpha,\gamma}, \|f\|_{\beta,\gamma}$ are finite, since in the same spirit $\|f\|_{\beta,\gamma} \leq \|f\|_{q,\gamma} < \infty$.
Note that $\mathfrak{N}_{f,q}(\cdot) = \|f\|_{q,\cdot}$ is a decreasing function due to the corresponding property of l^p-(quasi-)norms.

Let $f \in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$, i.e. $\|f\|_{q,\gamma,\infty} < \infty$.
The number of $a_{\mathbf{n}} = \|f\chi_{Q_{\mathbf{n}}}\|_q \in \mathrm{l}^{\gamma,\infty}$ which exceed $2^{\frac{k}{\gamma}}$ can be estimated by $\Big|\{\mathbf{n} | a_{\mathbf{n}}^\gamma > 2^k\}\Big| \leq 2^{-k} \cdot \|(a_{\mathbf{n}})\|_{\gamma,\infty}^\gamma = 2^{-k}\|f\|_{q,\gamma,\infty}^\gamma$.

Since $\left|\{\mathbf{n}|a_{\mathbf{n}}^{\gamma} > \|f\|_{q,\gamma,\infty}^{\gamma} + \epsilon\}\right| < 1$, $a_{\mathbf{n}} \leq \|f\|_{q,\gamma,\infty}$ for all $\mathbf{n}$.
Consider the following partition of the positive half line,

$$(0,\infty) = \bigsqcup_{-\infty<k\leq -1} (2^k, 2^{k+1}] \sqcup (1,\infty)$$

to obtain the estimate for $\|f\|_{q,\delta}^{\delta}$.

$$\begin{aligned}
\|f\|_{q,\delta}^{\delta} &= \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^{\gamma}>1} \|f\chi_{Q_{\mathbf{n}}}\|_q^{\delta} + \sum_{k\leq 0} \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^{\gamma}\in(2^k,2^{k+1}]} \|f\chi_{Q_{\mathbf{n}}}\|_q^{\delta} \\
&\leq \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^{\gamma}>1} \|f\|_{q,\gamma,\infty}^{\delta} + \sum_{k\leq 0} \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^{\gamma}\in(2^k,2^{k+1}]} 2^{(k+1)\cdot\frac{\delta}{\gamma}} \\
&\leq \|f\|_{q,\gamma,\infty}^{\gamma+\delta} + \sum_{k\leq 0} 2^{(k+1)\cdot\frac{\delta}{\gamma}} \cdot 2^{-k} \|f\|_{q,\gamma,\infty}^{\gamma} \\
&\leq \|f\|_{q,\gamma,\infty}^{\gamma+\delta} + 2^{\frac{\delta}{\gamma}} \|f\|_{q,\gamma,\infty}^{\gamma} \cdot \sum_{k\leq 0} 2^{k(\frac{\delta}{\gamma}-1)} \\
&\lesssim_{\delta,\gamma} \|f\|_{q,\gamma,\infty}^{\delta+\gamma} + \|f\|_{q,\gamma,\infty}^{\gamma} < \infty
\end{aligned} \tag{3.1.1}$$

□

Definition 3.1.5. For a function $f \in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$, where $q,\gamma > 0$, and for any positive real number R the following representation

$$f = f_{R-} + f_{R+} \tag{3.1.2}$$

is defined, where

$$\begin{aligned}
f_{R-} &= \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^{\gamma}>R^{-1}} f\chi_{Q_{\mathbf{n}}}, \\
f_{R+} &= \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^{\gamma}\leq R^{-1}} f\chi_{Q_{\mathbf{n}}}.
\end{aligned}$$

Lemma 3.1.6. *Let $f \in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^q)(\mathbb{R}^d)$, where γ and q are some positive numbers. Then for any β, δ s.t. $\gamma > \beta > 0$ and for any $R > 1$ the quasi-norms $\|f_{R-}\|_{q,\beta}$ and $\|f_{R-}\|_{q,\gamma}$ are finite and the following inequalities hold*

$$\begin{aligned}
\|f_{R-}\|_{q,\beta} &\leq C_{\beta,\gamma} \|f\|_{q,\gamma,\infty}^{\frac{\gamma}{\beta}} \cdot R^{-\frac{1}{\gamma}+\frac{1}{\beta}}, \\
\|f_{R-}\|_{q,\gamma} &\leq C_{\gamma} \|f\|_{q,\gamma,\infty} \Big(\log^{\frac{1}{\gamma}}(R+1) + \log^{\frac{1}{\gamma}} \big(\|f\|_{q,\gamma,\infty} + 1\big)\Big),
\end{aligned}$$

Moreover, if in addition $q \geq 2 > \gamma > 0$, then $\|f_{R+}\|_2$ is finite and

$$\|f_{R+}\|_2 \leq M_{\gamma} \|f\|_{q,\gamma,\infty}^{\frac{\gamma}{2}} \cdot R^{-\frac{1}{\gamma}+\frac{1}{2}}.$$

The constants $C_{\beta,\gamma}$, C_γ and M_γ are defined in the proof and depend on parameters β and γ only.

Proof. Let's proceed with the first estimate. After the substitution $r = R^{-\frac{1}{\gamma}}$ the inequality can be reduced to a homogeneous one. Indeed, rewrite the statement as follows

$$C^{\beta}_{\beta,\gamma} \geq \frac{\sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma > R^{-1}} \|f\chi_{Q_{\mathbf{n}}}\|_q^\beta}{\|f\|^\gamma_{q,\gamma,\infty} \cdot R^{-\frac{\beta}{\gamma}+1}} = \frac{\sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma > r^\gamma} \|f\chi_{Q_{\mathbf{n}}}\|_q^\beta}{\|f\|^\gamma_{q,\gamma,\infty} \cdot r^{-\gamma} \cdot r^\beta}$$

$$= \frac{\sum_{\|r^{-1}f\chi_{Q_{\mathbf{n}}}\|_q^\gamma > 1} \|r^{-1}f\chi_{Q_{\mathbf{n}}}\|_q^\beta}{\|r^{-1}f\|^\gamma_{q,\gamma,\infty}}.$$

Due to homogeneity, since the expression above can be expressed as a function of $\frac{f}{r} = \frac{f}{R^{-\frac{1}{\gamma}}}$, we can consider $\|f\|_{q,\gamma,\infty} = 1$ by applying new variables, $\tilde{f} = f \cdot \|f\|^{-1}_{q,\gamma,\infty}$, $\tilde{R} := R \cdot \|f\|^\gamma_{q,\gamma,\infty}$.
Now, using $(R^{-1}, \infty) = \bigsqcup_{k=0}^\infty (2^k R^{-1}, 2^{k+1} \cdot R^{-1}]$ we split the f_{R-}-representation as follows

$$f_{R-} = \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma > R^{-1}} f\chi_{Q_{\mathbf{n}}} = \sum_{k\geq 0} \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \in (2^k R^{-1}, 2^{k+1}\cdot R^{-1}]} f\chi_{Q_{\mathbf{n}}}.$$

The number of $a_{\mathbf{n}} = \|f\chi_{Q_{\mathbf{n}}}\|_q \in \mathrm{l}^{\gamma,\infty}$ which exceed $2^{\frac{k}{\gamma}} R^{-\frac{1}{\gamma}}$ can be estimated by

$$\left|\{\mathbf{n} | a_{\mathbf{n}}^\gamma > 2^k R^{-1}\}\right| \leq 2^{-k} R \cdot \|(a_{\mathbf{n}})\|^\gamma_{\gamma,\infty} \tag{3.1.3}$$

Thus, for a positive $\beta < \gamma$

$$\|f_{R-}\|^\beta_{q,\beta} = \sum_{k\geq 0} \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \in (2^k R^{-1}, 2^{k+1}\cdot R^{-1}]} \|f\chi_{Q_{\mathbf{n}}}\|_q^\beta$$

$$\leq \sum_{k\geq 0} \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \in (2^k R^{-1}, 2^{k+1}\cdot R^{-1}]} 2^{(k+1)\cdot\frac{\beta}{\gamma}} \cdot R^{-\frac{\beta}{\gamma}}$$

$$\leq \sum_{k\geq 0} 2^{(k+1)\cdot\frac{\beta}{\gamma}} \cdot R^{-\frac{\beta}{\gamma}} \cdot 2^{-k} R \|f\|^\gamma_{q,\gamma,\infty} \leq 2^{\frac{\beta}{\gamma}} \cdot R^{-\frac{\beta}{\gamma}+1} \cdot \sum_{k\geq 0} 2^{k(\frac{\beta}{\gamma}-1)} = T_{\beta,\gamma} \cdot R^{-\frac{\beta}{\gamma}+1},$$

where $T_{\beta,\gamma} = \frac{2^{\frac{\beta}{\gamma}+1}}{2-2^{\frac{\beta}{\gamma}}}$.

In the same spirit we deal with the 3^{rd} estimate for $q \geq 2$. Using Proposition 3.1.4, the monotonicity of the quasi-norm function, we obtain

$$\|f_{R+}\|_2^2 = \|f_{R+}\|^2_{2,2} \leq \|f_{R+}\|^2_{q,2} = \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \leq R^{-1}} \|f\chi_{Q_{\mathbf{n}}}\|_q^2$$

$$= \sum_{k\geq 0} \sum_{R^{-1}\cdot 2^{-(k+1)}<\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \leq R^{-1}\cdot 2^{-k}} \|f\chi_{Q_{\mathbf{n}}}\|_q^2$$

$$\leq \sum_{k\geq 0} \sum_{R^{-1}\cdot 2^{-(k+1)}<\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \leq R^{-1}\cdot 2^{-k}} R^{-\frac{2}{\gamma}} \cdot 2^{-\frac{2k}{\gamma}}$$

$$\leq \sum_{k\geq 0} R^{-\frac{2}{\gamma}} \cdot 2^{-\frac{2k}{\gamma}} \cdot 2^{k+1} R = 2R^{2(-\frac{1}{\gamma}+\frac{1}{2})} \cdot \sum_{k\geq 0} 2^{k(1-\frac{2}{\gamma})} = T_\gamma \cdot R^{2(-\frac{1}{\gamma}+\frac{1}{2})},$$

where $T_\gamma = \frac{2^{\frac{2}{\gamma}+1}}{2^{\frac{2}{\gamma}}-2}$.

The 2^{nd} estimate cannot be reduced to a homogeneous inequality, however, we repeat the same estimates as for $\|f_{R-}\|_{\beta,\gamma}$.
Consider $R\|f\|_{q,\gamma,\infty}^\gamma > 1$ (otherwise $f_{R-} = 0$).

$$\|f_{R-}\|_{q,\gamma}^\gamma = \sum_{k\geq 0} \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \in (2^k R^{-1}, 2^{k+1}\cdot R^{-1}]} \|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma$$

$$\leq \sum_{k\geq 0} \sum_{\|f\chi_{Q_{\mathbf{n}}}\|_q^\gamma \in (2^k R^{-1}, 2^{k+1}\cdot R^{-1}]} 2^{k+1} \cdot R^{-1}$$

Note that the number of $a_{\mathbf{n}} = \|f\chi_{Q_{\mathbf{n}}}\|_q \in l^{\gamma,\infty}$ which exceed $2^{\frac{k}{\gamma}} R^{-\frac{1}{\gamma}}$ with $R\|f\|_{q,\gamma,\infty}^\gamma < 2^k$ is zero due to (3.1.3).
Thus, we can consider $k \leq \log_2(R\|f\|_{q,\gamma,\infty}^\gamma) =: k_0$ in the upper bound above. For these values of k we use the same estimations as for $\|f_{R-}\|_{\beta,\gamma}$ to obtain

$$\|f_{R-}\|_{q,\gamma}^\gamma \leq \sum_{k_0\geq k\geq 0} 2^{-k} R\|f\|_{q,\gamma,\infty}^\gamma \cdot 2^{k+1} \cdot R^{-1} = 2\|f\|_{q,\gamma,\infty}^\gamma \cdot \sum_{k_0\geq k\geq 0} 1$$

$$= 2\|f\|_{q,\gamma,\infty}^\gamma k_0 = 2(\log 2)^{-1}\|f\|_{q,\gamma,\infty}^\gamma (\log R + \gamma \log \|f\|_{q,\gamma,\infty})$$
$$\leq 2(\log 2)^{-1}\|f\|_{q,\gamma,\infty}^\gamma \big(\log R + \gamma \log(\|f\|_{q,\gamma,\infty}+1)\big).$$

Thus,

$$\|f_{R-}\|_{q,\gamma} \leq 2^{\frac{1}{\gamma}}(\log 2)^{-\frac{1}{\gamma}}\|f\|_{q,\gamma,\infty}\Big(\log R + \gamma \log\big(\|f\|_{q,\gamma,\infty}+1\big)\Big)^{\frac{1}{\gamma}}.$$

Note that since $\gamma^{\frac{1}{\gamma}} < 2$,

$$\big(\log R + \gamma \log\big(\|f\|_{q,\gamma,\infty}+1\big)\big)^{\frac{1}{\gamma}} \leq \big(2\max\{\log R, \gamma \log(\|f\|_{q,\gamma,\infty}+1)\}\big)^{\frac{1}{\gamma}}$$

$$= 2^{\frac{1}{\gamma}} \max\{(\log R)^{\frac{1}{\gamma}}, \gamma^{\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}\big(\|f\|_{q,\gamma,\infty}+1\big)\}$$
$$\leq 2^{\frac{1}{\gamma}}(\log R)^{\frac{1}{\gamma}} + 2^{1+\frac{1}{\gamma}} \log^{\frac{1}{\gamma}}\big(\|f\|_{q,\gamma,\infty}+1\big).$$

Therefore,

$$\|f_{R-}\|_{q,\gamma} \leq C_\gamma \Big(\|f\|_{q,\gamma,\infty} (\log^{\frac{1}{\gamma}}(R+1) + \|f\|_{q,\gamma,\infty} \log^{\frac{1}{\gamma}} \left(\|f\|_{q,\gamma,\infty} + 1\right) \Big).$$

The exact formulae for the constans are $C_\gamma = 2\Big(\frac{4}{\log 2}\Big)^{\frac{1}{\gamma}}$, $C_{\beta,\gamma} = T_{\beta,\gamma}^{\frac{1}{\beta}} = \frac{2^{\frac{1}{\gamma}+\frac{1}{\beta}}}{\left(2-2^{\frac{\beta}{\gamma}}\right)^{\frac{1}{\beta}}}$ and $M_\gamma = \sqrt{T_\gamma} = \frac{2^{\frac{1}{\gamma}+\frac{1}{2}}}{\sqrt{2^{\frac{2}{\gamma}}-2}}$.

□

3.2 Domain boundary. Known and new results.

Recall that we use the following notation for the ΨDO on $L^2(\mathbb{R})$ with amplitude $p = p(x, y, \xi)$

$$\mathrm{Op}^{\mathrm{a}}_\alpha(p)u(x) = \frac{\alpha}{2\pi} \iint_{\mathbb{R}^2} e^{i\alpha(x-y)\xi} p(x, y, \xi)u(y)dyd\xi,$$

and we denote its k^{th} singular value (counted with multiplicity) by

$$s_k = s_k\big(\mathrm{Op}^{\mathrm{a}}_\alpha(p)\big),$$

i.e. $s_1 \geq s_2 \geq ... \geq 0$.
If in addition $p(x, y, \xi) = \sigma\Big(\frac{x+y}{2}, \xi\Big)$, we use the notation

$$\mathrm{Op}^{\mathrm{W}}_\alpha(\sigma) = \mathrm{Op}^{\mathrm{W}}_\alpha(\sigma(t, \xi)) = \mathrm{Op}^{\mathrm{a}}_\alpha\Big(\sigma\Big(\frac{x+y}{2}, \xi\Big)\Big)$$

Let's focus on the case $\sigma = \chi_\Lambda$ for some domains $\Lambda \subseteq \mathbb{R}^2$ in the phase space.

As it was mentioned in the Introduction, the rate of singular values decay depends on the curvature of the boundary $\partial\Lambda$. If the boundary can be described by a straight line (see the details in Theorem 3.2.8), the rate of decay is $\underline{O}(k^{-1})$. For an angular boundary (when $\Lambda = \{(t, \xi)|c_1 t \leq \xi \leq c_2 t\}$) the rate is $\frac{1}{4\pi^2} \cdot \frac{\log(k+1)}{k}$, and for general polygonal boundary the rate does not exceed $\underline{O}\big(\frac{\log(k+1)}{k}\big)$. It is not established yet if this estimate is sharp.
The decay is $c_k k^{-\frac{3}{4}}$ where $c_k \in [t_1, t_2] \subseteq (0, \infty)$ for annular regions [13, Prop. 10]. Ramanathan and Topiwala [13, Th. 9] proved the estimate $\underline{O}(k^{-\frac{3}{4}})$ for any C^1-boundary.
In this chapter we give another proof of the estimate $\underline{O}(k^{-\frac{3}{4}})$ using a

more advanced technique. We rewrite the considered ΨDO as an integral operator and study its kernel using known results (see Theorem 3.2.1). We generalise then this theorem extending domain Ω to the whole $\mathbb{R}$ (see Theorem 3.2.3), which also helps to prove the estimate $\underline{O}(k^{-1})$ for a straight line boundary case.
However, to prove the estimate $\underline{O}\big(\frac{\log(k+1)}{k}\big)$ when the boundary consists of angles, we need to apply a completely different approach, which is presented in Chapter 5. In some sense this is an intermediate rate of decay in terms of comparison with the straight line boundary case and C^1-boundary case and requires Weidl operator class scales instead of Schatten ones (see Remark 2.2.8).

First, recall a corollary of the Theorem [4, Ch. 11, §8, Th.4, p.273] for one-dimensional case.

Theorem 3.2.1. *Let s_k be the k^{th} singular value of the integral operator $Au(x) = \int_\Omega K(x,y)u(y)dy$ where Ω is a finite interval and for almost all y the function $K(\cdot,y) \in W_2^l(\Omega)$ with $\int_\Omega \|K(\cdot,y)\|^2_{W_2^l(\Omega)}dy < \infty$.*
Then $A \in \mathbb{S}_{p,\infty}$ where $\frac{1}{p} = \frac{1}{2} + l$, and the following estimate holds

$$s_k(A) \leq \frac{C_{|\Omega|,l}}{k^{\frac{1}{2}+l}}\Big(\int_\Omega \|K(\cdot,y)\|^2_{W_2^l(\Omega)}dy\Big)^{\frac{1}{2}}.$$

Corollary 3.2.2. *Consider an integral operator A defined on $\mathrm{L}^2(\mathbb{R})$ by*

$$Au(x) = \int_{\mathbb{R}} K(x,y)u(y)dy,$$

where kernel $K(\cdot,\cdot) \in C_0^\infty(\mathbb{R}^2)$.
Then singular values of A decay superpolynomially, i.e. for any integer $n > 0$ there exists such constant C_n that

$$s_k(A) \leq \frac{C_n}{k^n}.$$

Let's extend the theorem above to a wider class of kernels.

Theorem 3.2.3. *Consider an integral operator Q defined on $\mathrm{L}^2(\mathbb{R})$ by*

$$Qu(x) = \int K(x,y)u(y)dy$$

where kernel K satisfies the following conditions:

$$\operatorname{supp} K \subseteq \{(x,y)|\, |x+y| \leq R\} \text{ for some } R > 0,$$

$$|K(x,y)| \leq C\langle x-y\rangle^{-m_2}, \text{ for some } C > 0,\ m_2 > \frac{1}{2},$$

$$K(\cdot,y)\in C^l(\mathbb{R})\text{ for almost all }y,$$
$$\text{moreover, }\sum_{1\le n\le l}\left|\frac{\partial^n K}{\partial x^n}(x,y)\right|\le C\langle x-y\rangle^{-m_1},\text{ where }m_1\le m_2.$$

If $m_1 < l+\frac{1}{2}$, *then* $Q\in\mathbb{S}_{\frac{1}{m},\infty}$ *(i.e.* $s_k(Q)\le C_1k^{-m}$*), where*

$$m=\frac{1}{2}+l\cdot\frac{m_2-\frac{1}{2}}{m_2-m_1+l}.$$

If $m_1 > l+\frac{1}{2}$, *then* $Q\in\mathbb{S}_{\frac{1}{m},\infty}$, *where* $m=\frac{1}{2}+l$.
Finally, if $m_1 = l+\frac{1}{2}=m$, *then* $Q\in\mathfrak{S}_{\frac{1}{m}}$, *i.e.*

$$s_k(Q)\le C_2\left(\frac{\log(k+1)}{k}\right)^m.$$

The constants C_1, C_2 *in the estimates depend on* R, C, l, m_1 *and* m_2 *only.*

Remark. *In this Section we are interested in two cases. The first case is* $m_2=l=1$, $m_1=0$, *which corresponds to* C^1*-boundary of the domain* Λ *(see Corollary 3.2.4). The estimate gives* $s_k(Q)=\underline{O}(k^{-\frac{3}{4}})$.
The second case is $m_1=m_2=l=1$ *and describes the operators when the boundary of* Λ *is a straight line (see Corollary 3.2.8).*

The statement is also true for $l=0$ *if we conventionally take* $m_1=0$. *In this case we have* $s_k(Q)=\underline{O}(\frac{1}{\sqrt{k}})$, *the estimate for singular values which is true for any Hilbert-Schmidt operator.*
For all positive values of l *we obtain an asymptotic decay close or equal to* $k^{-\frac{1}{2}-l}$ *as in Theorem 3.2.1.*

Proof. Consider the flip operator J defined by $Ju(x)=u(-x)$. Since J is a unitary operator, $s_k(Q)=s_k(QJ)$. Rewrite $T=QJ$ as follows

$$(Tu)(x)=(QJu)(x)=\int K(x,y)u(-y)dy=\int K(x,-y)u(y)dy.$$

Define $K_n(x,y):=K(x,-y)\chi_{[n,n+1]}(y)$. Note that since

$$\text{supp }K\subseteq\{(x,y)|\ |x+y|\le R\},$$

supp $K_n\subseteq\{(x,y)|\ x,y\in[n-R,n+1+R]\}$. Therefore, we can define the following integral operator on $\mathrm{L}^2(n-R,n+1+R)$ by

$$(T_nu)(x):=\int K_n(x,y)u(y)dy=\int K(x,-y)\chi_{[n,n+1]}(y)u(y)dy$$

Split the operator T as follows

$$Tu(\cdot) = T\chi_{|n|>N}u(\cdot) + \sum_{|n|\le N} T\chi_{[n,n+1]}u(\cdot) =: T\chi_{|n|>N}u(\cdot) + \sum_{|n|\le N} T_n u(\cdot).$$

Since for almost all y kernel $K_n(\cdot, y) \in C_0^l(\mathbb{R}^2)$ with $\frac{\partial^j K_n}{\partial x^j}\big(\cdot, y\big)$ $= \frac{\partial^j K}{\partial x^j}\big(\cdot, -y\big)\chi_{[n,n+1]}(y)$ for $j = 1, 2, ..., l$,

$$|K_n(x,y)| \le C\langle x+y\rangle^{-m_2}, \quad \Big|\frac{\partial^j K_n}{\partial x^j}\big(x,y\big)\Big| \le C\langle x+y\rangle^{-m_1}.$$

Thus,

$$\|K_n(\cdot,y)\|^2_{W_2^l(n-R,n+1+R)} \lesssim_{l,C} \int_{n-R}^{n+1+R} \frac{1}{\langle x+y\rangle^{2m_2}}dx + \int_{n-R}^{n+1+R} \frac{1}{\langle x+y\rangle^{2m_1}}dx$$

$$\lesssim_R \frac{1}{\langle n+y-R\rangle^{2m_1}},$$

Thus, the integral

$$\int_{[n-R,n+1+R]} \|K_n(\cdot,y)\|^2_{W_2^l(n-R,n+1)}dy \lesssim_R \int_{[n-R,n+1+R]} \frac{1}{\langle n+y-R\rangle^{2m_1}}dy$$

$$\lesssim_R \langle 2n-R\rangle^{-2m_1}$$

for any $y \in [n, n+1]$.
Thus, due to Theorem 3.2.1

$$\|T_n\|^2_{\mathbb{S}_{p,\infty}} \lesssim_{l,C,R} n^{-2m_1},$$

where $p = \frac{2}{2l+1}$.
Now consider three cases mentioned in the statement.
Case 1. Let $m_1 < l + \frac{1}{2}$. Thus, $m_1 p = \frac{2m_1}{2l+1} < 1$.
Now, using quasi-triangle inequality (see Proposition 2.2.1)

$$\Big\| \sum_{|n|\le N} T_n \Big\|^p_{\mathbb{S}_{p,\infty}} \lesssim_p \sum_{|n|\le N} \|T_n\|^p_{\mathbb{S}_{p,\infty}} \lesssim_{l,R} \sum_{0<|n|\le N} n^{-m_1 p} \lesssim_{p,m_1} N^{1-m_1 p} \tag{3.2.1}$$

Hence,

$$k^{\frac{1}{p}} s_k\Big(\sum_{|n|\le N} T_n\Big) \le \Big\|\sum_{|n|\le N} T_n\Big\|_{\mathbb{S}_{p,\infty}} \lesssim_{l,R,m_1} N^{\frac{1}{p}-m_1},$$

or equivalently

$$s_k\Big(\sum_{|n|\le N} T_n\Big) \lesssim_{l,R,m_1} \Big(\frac{N}{k}\Big)^{\frac{1}{p}} \cdot N^{-m_1}.$$

The Hilbert-Schmidt estimate for $N > R$ gives

$$ks_k^2(T\chi_{|y|>N}) \leq \Big\| \sum_{|n|>N} T_n \Big\|_{\mathbb{S}_2}^2 = \|K(x,-y)\chi_{|y|>N}\|_{\mathrm{L}^2(\mathbb{R}^2)}$$

$$\lesssim_C \int_N^\infty \int_{y-R}^{y+R} \frac{1}{\langle x+y\rangle^{2m_2}} dx\, dy \lesssim_R \int_N^\infty \frac{1}{\langle 2y-R\rangle^{2m_2}} dy \lesssim_R N^{-(2m_2-1)}.$$

Thus,

$$s_k(T\chi_{|x|>N}) \lesssim_{C,R} k^{-\frac{1}{2}} \cdot N^{-m_2+\frac{1}{2}}.$$

Finally, using Ky Fan inequality (2.2.2) for $N > R$

$$s_{2k-1}(Q) = s_{2k-1}(T) \leq s_k(T\chi_{|x|>N}) + s_k\Big(\sum_{|n|\leq N} T_n\Big) \lesssim_{C,R,m_1,l} k^{-\frac{1}{2}} N^{-m_2+\frac{1}{2}} + k^{-\frac{1}{p}} N^{\frac{1}{p}} N^{-m_1}.$$

To optimize this estimate we choose N such that

$$k^{-\frac{1}{p}} N^{\frac{1}{p}-m_1} = k^{-\frac{1}{2}} N^{-m_2+\frac{1}{2}},$$

i.e. $N = k^{\frac{1/p-1/2}{1/p-m_1+m_2-1/2}} = k^{\frac{l}{l+m_2-m_1}}$.
Finally, due to Remark 2.2.16

$$s_k(Q) \lesssim_{C,R,m_1,m_2,l} k^{-\frac{1}{p}} k^{\frac{(1/p-1/2)(1/p-1/m_1)}{1/p-m_1+m_2-1/2}} = k^{\frac{m_1-m_2(2l+1)}{2(m_2-m_1)+2l}} = k^{-\frac{1}{2}-\frac{l}{2}\cdot\frac{2m_2-1}{m_2-m_1+l}}.$$

Case 2. Let $m_1 p > 1$. The quasi-triangle inequality (3.2.1) rewrites as follows

$$\Big\| \sum_{|n|\leq N} T_n \Big\|_{\mathbb{S}_{p,\infty}}^p \lesssim_p \sum_{|n|\leq N} \|T_n\|_{\mathbb{S}_{p,\infty}}^p \lesssim_{l,R} \sum_{0<|n|\leq N} n^{-m_1 p} < \sum_{0<|n|<\infty} n^{-m_1 p} = C_{p,m_1}.$$

Therefore, repeating the same argument,

$$s_k(Q) = s_k(T) \lesssim_{C,R,m_1,m_2,l} s_k\Big(\sum_{|n|\leq N} T_n\Big) + s_k(T\chi_{|x|>N}) \lesssim_{C,l,R} k^{-\frac{1}{p}} + k^{-\frac{1}{2}} N^{-m_2+\frac{1}{2}}.$$

The optimal choice of N satisfies $k^{-\frac{1}{p}} = k^{-\frac{1}{2}} N^{-m_2+\frac{1}{2}}$ which leads to

$$s_k(Q) \lesssim_{C,l,m_1,m_2,R} k^{-\frac{1}{p}} = k^{-(\frac{1}{2}+l)}.$$

Case 3. Let $m_1 p = 1$. Then (3.2.1) gives

$$\Big\| \sum_{|n|\leq N} T_n \Big\|_{\mathbb{S}_{p,\infty}}^p \lesssim_{l,R} \log N.$$

Thus, taking N s.t. $k^{-\frac{1}{p}} = k^{-\frac{1}{2}} N^{-m_2+\frac{1}{2}}$,

$$s_k(Q) = s_k(T) \lesssim_{C,R,m_1,m_2} s_k\Big(\sum_{|n|\le N} T_n\Big) + s_k(T\chi_{|x|>N}) \lesssim_{C,l,R} k^{-\frac{1}{p}}(\log N)^{\frac{1}{p}} + k^{-\frac{1}{2}} N^{-m_2+\frac{1}{2}}$$

$$= k^{-\frac{1}{p}}((\log N)^{\frac{1}{p}} + 1) \lesssim_{m_1,m_2,p} \Big(k^{-1}\log(k+1)\Big)^{\frac{1}{p}} = k^{-m_1}\log^{m_1}(k+1).$$

□

An important consequence of Theorem 3.2.3 is Ramanathan and Topiwala's result describing a pseudo-differentiable operator whose symbol is nothing but an indicator function of a region with a smooth boundary.

Corollary 3.2.4. *[13, Th. 9] Consider operator*

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma)u(x) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{i\xi(x-y)} \cdot \sigma\Big(\frac{x+y}{2}, \xi\Big)u(y)dyd\xi,$$

where $\sigma(t,\xi) = \chi_\Omega(t,\xi)$ s.t. Ω is a region in $\mathbb{R}^2$ with piece-wise C^1-boundary. Then

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma) \in \mathbb{S}_{\frac{4}{3},\infty}$$

Proof. First, consider the case when the region Ω can be expressed as follows

$$\Omega = \{(t,\xi) \mid \alpha(t) \le \xi \le \beta(t),\ t \in [T_1,T_2]\} \qquad (3.2.2)$$

with $\alpha,\beta \in C^1(T_1,T_2)$, $\operatorname{supp}\alpha, \operatorname{supp}\beta \subseteq [T_1,T_2]$.
Using Fubini's theorem the operator with symbol $\chi_\Omega(t,\xi)$ can be represented as an integral operator

$$\mathrm{Op}_1^{\mathrm{W}}(\chi_\Omega)u(x) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{i\xi(x-y)} \cdot \chi_\Lambda\Big(\frac{x+y}{2}, \xi\Big)u(y)dyd\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\alpha\left(\frac{x+y}{2}\right)}^{\beta\left(\frac{x+y}{2}\right)} e^{i\xi(x-y)}d\xi\ u(y)dy = \int_{\mathbb{R}} K(x,y)u(y)dy$$

where kernel

$$K(x,y) = ie^{\frac{i}{2}(x-y)\left(\alpha\left(\frac{x+y}{2}\right)+\beta\left(\frac{x+y}{2}\right)\right)} \cdot \frac{\sin\left(\frac{1}{2}(x-y)\cdot\left(\beta\left(\frac{x+y}{2}\right)-\alpha\left(\frac{x+y}{2}\right)\right)\right)}{(x-y)}.$$

Indeed, $|K(x,y)| \le \frac{C}{\langle x-y\rangle}$ and $|K'_x(x,y)| \le C$. Therefore, applying Theorem 3.2.3 for $m_1 = 0, m_2 = l = 1$ we obtain the following upper bound for the k^{th} singular value

$$s_k \lesssim_\Omega k^{\frac{m_1 - m_2(2l+1)}{2(m_2-m_1)+2l}} = k^{-\frac{3}{4}}.$$

A general domain Ω with a piece-wise C^1-boundary can be split by a

finite number of Ω_j such that each of them after a rotation (if necessary) satisfies the property (3.2.2).
Since the estimate $s_k(\mathrm{Op}_1^{\mathrm{W}}(\chi_{\Omega_j})) \lesssim_{\Omega_j} k^{-\frac{3}{4}}$ is polynomial, using Ky Fan's inequality and Remark 2.2.16 we conclude $s_k(\mathrm{Op}_1^{\mathrm{W}}(\chi_\Omega)) \lesssim_\Omega k^{-\frac{3}{4}}$.

□

Remark 3.2.5. It turns out that the degree of smoothness of the boundary Ω does not make any impact on the estimate in the suggested method. Indeed, taking region Ω with C^l-boundary and applying Theorem 3.2.3 for $m_1 = -l+1, m_2 = 1$ we obtain again

$$s_k \lesssim_{\Omega,l} k^{-\frac{3l}{4l}} = k^{-\frac{3}{4}}.$$

Definition 3.2.6. By *cut-off function* $\zeta_\delta(t)$ with parameter δ we understand any even $C^\infty(\mathbb{R}^d)-$function such that $\zeta_\delta(t) = \chi_{|t|\geq\delta}$, whenever $|t| \notin (\delta, \delta+1)$ for some positive δ.
Without loss of generality consider $\|\zeta_\delta\|_\infty = 1$. If the value of δ does not matter, we use the notation $\zeta(t) := \zeta_1(t)$.

Corollary 3.2.7. *Consider a compact operator T_m $(m \geq 1)$ on $\mathrm{L}^2(\mathbb{R})$ defined by*

$$T_m u(x) = \int \frac{\zeta(x-y)}{(x-y)^m} a(x+y)u(y)dy,$$

where $a \in C_0^\infty(\mathbb{R})$.
Then $T_m \in \mathbb{S}_{\frac{1}{m},\infty}$ (i.e. $s_k(T_m) \lesssim_{m,a} k^{-m}$)

Proof. Indeed, applying Theorem 3.2.3 for $m_2 = m_1 = m$ and any $l \geq m$ we obtain the result. □

The following theorem describes the case when the "discontinuous part" of the domain boundary is a straight line.

Theorem 3.2.8. *Assume $\eta = \eta(t,\xi) \in C_0^\infty(\mathbb{R}^2)$, $\chi_\Lambda = \chi_\Lambda(t,\xi)$, where $\Lambda = \{(t,\xi) |\ \xi \leq at+b\}$. Then*

$$\mathrm{Op}_1^W(\chi_\Lambda\eta) \in \mathbb{S}_{1,\infty}.$$

Proof. Since shifts and rotations of region Λ on the phase space reduce ΨDO to a unitary equivalent operator (see [8, Ch.2, Prop. 2.13], [11, Th. 6, p.3327]), without loss of generality we can assume that $a = 1, b = 0$ and $\operatorname{supp} \eta \subseteq \mathbb{R}_+ \times \mathbb{R}_+$. The kernel K of the operator can be expressed as follows

$$2\pi K(x,y) = \int_{\mathbb{R}} e^{i\xi(x-y)} \chi_\Lambda\Big(\frac{x+y}{2}, \xi\Big)\eta\Big(\frac{x+y}{2}\Big)d\xi$$

$$= \eta\Big(\frac{x+y}{2}\Big) \cdot \int_0^{\frac{x+y}{2}} e^{i\xi(x-y)} d\xi = -i\eta\Big(\frac{x+y}{2}\Big)\Big(e^{i\cdot\frac{x^2-y^2}{2}} - 1\Big) \cdot \frac{1}{x-y}.$$

Let $\zeta(t) := \zeta_1(t)$ be a cut-off function on $\mathbb{R}$. Decompose the kernel K as follows

$$K(x,y) = S(x,y) + T(x,y),$$

where

$$S(x,y) = -\frac{i}{2\pi}\eta\Big(\frac{x+y}{2}\Big)\Big(e^{\frac{i}{2}(x^2-y^2)} - 1\Big) \cdot \frac{1}{(x^2-y^2)} \cdot (x+y)(1-\zeta(x-y))$$

is a C^∞-function (consider $\frac{e^\alpha - 1}{\alpha} = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{3!} + ... \in C^\infty$) supported on a bounded region $\Omega = \{(x,y)|\ |x+y| < const\} \cap \{(x,y)|\ |x-y| < 1\}$ and

$$T(x,y) = -\frac{i}{2\pi}\Big(e^{i\cdot\frac{x^2-y^2}{2}} - 1\Big) \cdot \frac{1}{x-y}\zeta(x-y)\eta\Big(\frac{x+y}{2}\Big).$$

Applying Corollary 3.2.2 and Corollary 3.2.7 we obtain

$$S \in \cap_{\gamma>0}\mathbb{S}_{\gamma,\infty}, \quad T \in \mathbb{S}_{1,\infty},$$

which completes the proof. □

Discontinuous kernel

There are some results for the case when the kernel is not a smooth function.

For instance, [4, Ch. 11, §8, Th. 6] implies $A_1, A_2 \in \bigcap_{p>0}\mathbb{S}_{p,\infty}$, where

$$A_1u(x) = \int_{\mathbb{R}} \chi_{\{x<x_0\}}K(x,y)u(y)dy, \quad A_2u(x) = \int_{\mathbb{R}} \chi_{\{y<y_0\}}K(x,y)u(y)dy$$

are integral operators defined on L^2 with kernel $K(\cdot,\cdot) \in C_0^\infty(\mathbb{R}^2)$. However, generally one can state a significantly weaker result about operator

$$A_\Omega u(x) = \int_{\mathbb{R}} K(x,y)\chi_\Omega(x,y)u(y)dy,$$

where Ω is an arbitrarily bounded region in $\mathbb{R}^2$ with C^1-boundary and function $K(x,y) \in C^\infty(\mathbb{R}^2)$.

Splitting the operator into the "main" and "remainder" parts (in the same spirit as in Theorem 3.2.3), one can show

$$A_\Omega \in \bigcap_{p>1}\mathbb{S}_{p,\infty},$$

i.e. unlike A_1 or A_2 (which have the superpolynomial rate of singular values decay, i.e. $s_k(A_j) \lesssim_p k^{-p}$ for any $p > 0$), the singular values of A_Ω satisfy

$$s_k(A_\Omega) \lesssim_\epsilon k^{-1+\epsilon} \quad \text{for any } \epsilon > 0.$$

For some cases when $\Omega = \{x + y \geq 0\}$ (the main operator (5.0.1) is reduced to this case in Chapter 5), the estimate might be $\underline{O}(k^{-1}\log(k+1))$. For the operator (5.0.1) this estimate is sharp. This is the case when the polynomial scale provided by Schatten operator classes gives a rough estimate $\underline{O}(k^{-1+\epsilon})$, while the Weidl class Σ_f, $f(x) = C\big(\frac{x}{\log(1+x)} - 1\big)$ gives exactly the precise result (see Remark 2.2.8).

3.3 Auxiliary results. Overview and proofs.

Consider the pseudo-differential operator on the $\mathrm{L}^2(\mathbb{R}^d)$ space

$$\mathrm{Op}_\alpha^{\mathrm{a}}(p)u(\mathbf{x}) = \left(\frac{\alpha}{2\pi}\right)^d \iint_{\mathbb{R}^{2d}} e^{i\alpha(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} p(\mathbf{x},\mathbf{y},\boldsymbol{\xi})u(\mathbf{y})d\boldsymbol{\xi}d\mathbf{y}.$$

The results of the section are represented by the estimates of the operator norms with the amplitude p of the form $p(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = a(\mathbf{x},\mathbf{y})b(\boldsymbol{\xi})$ for some suitable functions a and b (i.e. when the frequency variable ξ is separated from others).
Two main theorems are Theorem 3.3.1, where functions $a = a(\mathbf{x})$ and $b = b(\boldsymbol{\xi})$ belong to a lattice quasi-norm spaces, and Theorem 3.3.14 which helps to reduce the Weyl symbol $\sigma_W = a\big(\frac{\mathbf{x}+\mathbf{y}}{2}\big)b(\boldsymbol{\xi})$ to the one of the form $\sigma = a(\mathbf{x})b(\boldsymbol{\xi})$.

Theorem 3.3.1. *Let* $\gamma_1,\gamma_2 \in (0,2)$*,* $\gamma = \max\{\gamma_1,\gamma_2\}$ *and* $p(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = f(\mathbf{x})g(\boldsymbol{\xi})$*, where* $f \in \mathrm{l}^{\gamma_1,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$*,* $g \in \mathrm{l}^{\gamma_2,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$*.*
If $\gamma_1 = \gamma_2$*, then* $\mathrm{Op}_1^{\mathrm{a}}(p) \in \mathfrak{S}_\gamma$ *and*

$$\left\|\mathrm{Op}_1^{\mathrm{a}}(p)\right\|_{\mathfrak{S}_\gamma} \le C_\gamma \cdot \|f\|_{2,\gamma,\infty} \cdot \|g\|_{2,\gamma,\infty}.$$

If $\gamma_1 \neq \gamma_2$*, then* $\mathrm{Op}_1^{\mathrm{a}}(p) \in \mathbb{S}_{\gamma,\infty}$ *and*

$$\left\|\mathrm{Op}_1^{\mathrm{a}}(p)\right\|_{\mathbb{S}_{\gamma,\infty}} \le C_{\gamma_1,\gamma_2} \cdot \|f\|_{2,\gamma_1,\infty} \cdot \|g\|_{2,\gamma_2,\infty}.$$

Proof. Without loss of generality we might consider

$$\|f\|_{2,\gamma,\infty} = \|g\|_{2,\gamma,\infty} = 1.$$

We denote $f_{\mathbf{n}}(\mathbf{x}) = f\chi_{Q_{\mathbf{n}}}(\mathbf{x})$ and $g_{\mathbf{n}}(\boldsymbol{\xi}) = g\chi_{Q_{\mathbf{n}}}(\boldsymbol{\xi})$.

Due to Proposition 3.1.4 and (3.1.1) $f \in \mathrm{l}^{\gamma_1,\infty}(\mathrm{L}^2)(\mathbb{R}^d) \subseteq \mathrm{l}^2(\mathrm{L}^2)(\mathbb{R}^d) = \mathrm{L}^2(\mathbb{R}^d)$. Moreover, $\|f\|_2 = \|f\|_{2,2} = C_{\gamma_1}$. A similar result holds for $g \in \mathrm{L}^2(\mathbb{R}^d)$, i.e. $\|g\|_2 = C_{\gamma_2}$.

Part A.
Consider the first case, $\gamma_1 = \gamma_2 = \gamma$.

<u>*Idea of the proof*</u>

The idea of the proof is to split (with the help of the the decomposition (3.1.2) for functions in $\mathrm{l}^{\gamma,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$) the operator as follows

$$\mathrm{Op}_1^{\mathrm{a}}(fg) = \sum_{\|f_{\mathbf{n}}\|_2^\gamma > R^{-1}} \Big(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R-}) + \mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R+})\Big) + \mathrm{Op}_1^{\mathrm{a}}(f_{R+} g).$$

Then we estimate the singular values of operators $\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R-})$ using (1.3.1). The singular values of the operators $\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R+})$, $\mathrm{Op}_1^{\mathrm{a}}(f_{R+} g)$ are estimated with the help of Hilbert-Schmidt norm. Finally, Remark 2.2.16 helps to combine the mentioned estimates.

Let's proceed with the details.

Step 1. $s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R-}))$ *estimate*

Take any $q \in (0, \gamma)$. Then, using Lemma 3.1.6 for $f, g \in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$ we obtain the following estimates

$$\|f_{R-}\|_{2,q}, \|g_{R-}\|_{2,q} \lesssim_\gamma R^{-\frac{1}{\gamma}+\frac{1}{q}}, \tag{3.3.1}$$

$$\|f_{R-}\|_{2,\gamma}, \|g_{R-}\|_{2,\gamma} \lesssim_\gamma (\log(R+1))^{\frac{1}{\gamma}}, \tag{3.3.2}$$

$$\|f_{R+}\|_2, \|g_{R+}\|_2 \lesssim_\gamma R^{-\frac{1}{\gamma}+\frac{1}{2}}, \tag{3.3.3}$$

where $R > 1$ and

$$f_{R-} = \sum_{\|f_{\mathbf{n}}\|_2^\gamma > R^{-1}} f_{\mathbf{n}}, \quad f_{R+} = \sum_{\|f_{\mathbf{n}}\|_2^\gamma \le R^{-1}} f_{\mathbf{n}},$$

$$g_{R-} = \sum_{\|g_{\mathbf{n}}\|_2^\gamma > R^{-1}} g_{\mathbf{n}}, \quad g_{R+} = \sum_{\|g_{\mathbf{n}}\|_2^\gamma \le R^{-1}} g_{\mathbf{n}}.$$

Using (1.3.1), the Birman-Solomyak estimate, we obtain

$$\|\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R\_})\|_{\mathbb{S}_q} \lesssim_q \big\|f_{\mathbf{n}}\big\|_{2,q} \cdot \|g_{R\_}\|_{2,q} \lesssim_\gamma R^{\frac{1}{q}-\frac{1}{\gamma}} \big\|f_{\mathbf{n}}\big\|_{2,q}.$$

Hence, for any k and R

$$k^{\frac{1}{q}} s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R\_})) \lesssim_{\gamma,q} R^{\frac{1}{q}-\frac{1}{\gamma}} \big\|f_{\mathbf{n}}\big\|_{2,q} = R^{\frac{1}{q}-\frac{1}{\gamma}} \big\|f_{\mathbf{n}}\big\|_2,$$

and thus,

$$k^{\frac{1}{\gamma}} s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R\_})) \lesssim_{\gamma,q} \left(\frac{R}{k}\right)^{\frac{1}{q}-\frac{1}{\gamma}} \big\|f_{\mathbf{n}}\big\|_2 \tag{3.3.4}$$

Step 2. $s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R+}))$ *estimate*

Using the Hilbert-Schmidt estimate and (3.3.3), we obtain

$$k s_k^2(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R_+})) \le \sum_t s_t^2(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R_+})) = \|f_{\mathbf{n}}(\mathbf{x}) g_{R_+}(\boldsymbol{\xi})\|^2_{\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d)}$$

$$= \|f_{\mathbf{n}}\|_2^2 \cdot \|g_{R_+}\|_2^2 \lesssim_\gamma \|f_{\mathbf{n}}\|_2^2 \cdot R^{1-\frac{2}{\gamma}}.$$

Therefore,

$$k^{\frac{1}{\gamma}} s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R_+})) \lesssim_\gamma \|f_{\mathbf{n}}\|_2 \cdot \left(\frac{R}{k}\right)^{\frac{1}{2}-\frac{1}{\gamma}} \tag{3.3.5}$$

Step 3. $s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R-} g))$ *estimate*

Combining (3.3.4) and (3.3.5) together and using Fan's inequality (2.2.2),

$$k^{\frac{1}{\gamma}} s_{2k-1}(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g)) \le k^{\frac{1}{\gamma}} s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R_-})) + k^{\frac{1}{\gamma}} s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g_{R_+}))$$

$$\lesssim_\gamma \|f_{\mathbf{n}}\|_2 \cdot \left(\left(\frac{R}{k}\right)^{-\frac{1}{\gamma}+\frac{1}{q}} + \left(\frac{R}{k}\right)^{\frac{1}{2}-\frac{1}{\gamma}} \right) \tag{3.3.6}$$

Taking $R = k$ we obtain

$$\|\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g)\|_{\gamma,\infty} = \sup_k k^{\frac{1}{\gamma}} s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g)) \lesssim_\gamma \|f_{\mathbf{n}}\|_2.$$

Since $\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g)$ and $\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{m}} g)$ satisfy

$$\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g)^* \mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{m}} g) = \mathrm{Op}_1^{\mathrm{a}}(f g)^* \chi_{Q_{\mathbf{n}}} \chi_{Q_{\mathbf{m}}} \mathrm{Op}_1^{\mathrm{a}}(f g) = 0$$

for any $\mathbf{n} \neq \mathbf{m}$, using the quasi-triangle inequality for $\gamma < 2$ (see Proposition 2.2.1) and (3.3.2)

$$\left\|\mathrm{Op}_1^{\mathrm{a}}(f_{R-} g)\right\|^\gamma_{\mathbb{S}_{\gamma,\infty}} = \left\|\mathrm{Op}_1^{\mathrm{a}}(f_{k-} g)\right\|^\gamma_{\mathbb{S}_{\gamma,\infty}} = \Big\| \sum_{\|f_{\mathbf{n}}\|_2^\gamma > k^{-1}} \mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g) \Big\|^\gamma_{\mathbb{S}_{\gamma,\infty}}$$

$$\lesssim_\gamma \sum_{\|f_{\mathbf{n}}\|_2^\gamma > k^{-1}} \left\|\mathrm{Op}_1^{\mathrm{a}}(f_{\mathbf{n}} g)\right\|^\gamma_{\mathbb{S}_{\gamma,\infty}}$$

$$\lesssim_\gamma \sum_{\|f_{\mathbf{n}}\|_2^\gamma > k^{-1}} \|f_{\mathbf{n}}\|_2^\gamma = \|f_{k-}\|^\gamma_{2,\gamma} \lesssim_\gamma \log(k+1).$$

Thus,

$$s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R-}g)) = s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{k-}g)) \lesssim_\gamma \left(\frac{\log(k+1)}{k}\right)^{\frac{1}{\gamma}} \tag{3.3.7}$$

<u>*Step 4.* $s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g))$ *estimate and final result*</u>

Applying the Hilbert-Schmidt estimate again, with the help of (3.3.3) we obtain

$$ks_k^2(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g)) = ks_k^2(\mathrm{Op}_1^{\mathrm{a}}(f_{k+}g)) \leq \sum_t s_t^2(\mathrm{Op}_1^{\mathrm{a}}(f_{k+}g))$$

$$= \|f_{k+}g\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d)} = \|f_{k+}\|_2^2 \cdot \|g\|_2^2 \lesssim_\gamma k^{1-\frac{2}{\gamma}}.$$

Hence,

$$s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g)) = s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{k+}g)) \lesssim_\gamma \frac{1}{k^\gamma} \tag{3.3.8}$$

Finally, (3.3.7) and (3.3.8) and Ky Fan's inequality (2.2.2) imply

$$s_{2k-1}(\mathrm{Op}_1^{\mathrm{a}}(fg)) \leq s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R-}g)) + s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g)) \lesssim_\gamma \left(\frac{\log(k+1)}{k}\right)^\gamma + \frac{1}{k^\gamma}$$

$$\lesssim_\gamma \left(\frac{\log(k+1)}{k}\right)^\gamma.$$

PART B.

Now, let's consider the case $\gamma_1 > \gamma_2$ (the reasoning when $\gamma_1 < \gamma_2$ is absolutely similar)

We split the operator as follows

$$\mathrm{Op}_1^{\mathrm{a}}(fg) = \mathrm{Op}_1^{\mathrm{a}}(f_{R-}g_{R-}) + \mathrm{Op}_1^{\mathrm{a}}(f_{R+}g_{R-}) + \mathrm{Op}_1^{\mathrm{a}}(fg_{R+}).$$

Apply (1.3.1), the Birman-Solomyak estimate for $\mathrm{Op}_1^{\mathrm{a}}(f_{R-}g_{R-})$ and $\|\cdot\|_{\mathbb{S}_q}$-norm, where $q \in (\gamma_2, \gamma_1) \subseteq (0, 2)$:

$$\|\mathrm{Op}_1^{\mathrm{a}}(f_{R-}g_{R-})\|_{\mathbb{S}_q} \lesssim_q \left\|f_{R-}\right\|_{2,q} \cdot \|g_{R-}\|_{2,q}.$$

Proposition 3.1.4 implies $\mathrm{l}^{\gamma_2,\infty}(\mathrm{L}^2)(\mathbb{R}^d) \subseteq \mathrm{l}^q(\mathrm{L}^2)(\mathbb{R}^d)$. Thus, $\|g_{R-}\|_{2,q} \leq \|g\|_{2,q} = C_q$. Using (3.3.1) we obtain

$$k^{\frac{1}{q}} s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R-}g_{R-})) \lesssim_{\gamma_1,q} R^{-\frac{1}{\gamma_1}+\frac{1}{q}}.$$

In the same spirit as in the previous case we estimate $s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g_{R-}))$

and $\mathrm{Op}_1^{\mathrm{a}}(fg_{R+})$ using the Hilbert-Schmidt estimate

$$ks_k^2(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g_{R-})) \leq \sum_t s_t^2(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g_{R-})) = \|f_{R+}g_{R-}\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d)}$$

$$= \|f_{R+}\|_2^2 \cdot \|g_{R-}\|_2^2 \lesssim_{\gamma_1} R^{1-\frac{2}{\gamma_1}} \cdot \|g\|_2^2 \lesssim_{\gamma_1,\gamma_2} R^{1-\frac{2}{\gamma_1}}$$

and

$$ks_k^2(\mathrm{Op}_1^{\mathrm{a}}(fg_{R+})) \leq \sum_t s_t^2(\mathrm{Op}_1^{\mathrm{a}}(fg_{R+}))$$

$$= \|fg_{R+}\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d)} = \|f\|_2^2 \cdot \|g_{R+}\|_2^2 \lesssim_{\gamma_2} R^{1-\frac{2}{\gamma_2}} \cdot \|f\|_2^2 \lesssim_{\gamma_1,\gamma_2} R^{1-\frac{2}{\gamma_2}}.$$

Combining all of the above and using Fan's inequality (2.2.2), we finally obtain

$$s_{3k-2}(\mathrm{Op}_1^{\mathrm{a}}(fg)) \leq s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R-}g_{R-})) + s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R+}g_{R-})) + s_k(\mathrm{Op}_1^{\mathrm{a}}(fg_{R+}))$$

$$\lesssim_{\gamma_1,\gamma_2} \left(\frac{R}{k}\right)^{\frac{1}{q}} \cdot \left(\frac{1}{R}\right)^{\frac{1}{\gamma_1}} + \left(\frac{R}{k}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{R}\right)^{\frac{1}{\gamma_1}} + \left(\frac{R}{k}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{R}\right)^{\frac{1}{\gamma_2}}.$$

Taking $R = k$ completes the proof.

□

Remark 3.3.2. If in addition $f \in \mathrm{l}^\gamma(\mathrm{L}^2)(\mathbb{R}^d)$, i.e. f belongs to a strong lattice-normed space, then the estimate (3.3.7) can be strengthened. Indeed, in this case $\|f_{R-}\|_{2,\gamma} \leq \|f\|_{2,\gamma} < \infty$ and (3.3.7) can be rewritten as follows

$$s_k(\mathrm{Op}_1^{\mathrm{a}}(f_{R-}g)) \lesssim_\gamma \left(\frac{\|f_{R-}\|_{2,\gamma}}{k}\right)^{\frac{1}{\gamma}} \lesssim_\gamma k^{-\frac{1}{\gamma}}.$$

Repeating the remaining part of the proof we obtain the Simon's estimate (1.3.2), $\mathrm{Op}_1^{\mathrm{a}}(fg) \in \mathbb{S}_\gamma$, but for a wider interval of parameter γ values, $(0, 2)$.

For the next result, Theorem 3.3.14 (about the connection between the initial operator and the one with the corresponding Weyl symbol), we need some auxiliary tools.

The following two propositions correspond to some theorems in [18] which describe the estimates when a particular control of the amplitude $p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})$ is considered.

Proposition 3.3.3 (Theorem 2.6 in [18] for $\mathbf{w}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{x}+\mathbf{y}}{2}$ and $h_1 = h_2 \equiv 1$). *Suppose that $q \in (0,1]$, $F_n \in \mathrm{l}^q(\mathrm{L}^1)(\mathbb{R}^{2d})$ where*

$$F_n(\mathbf{w},\boldsymbol{\xi};p) = \sum_{k,t=0}^{n} |\nabla_{\mathrm{w}}^k \nabla_{\boldsymbol{\xi}}^t p(\mathbf{w},\boldsymbol{\xi})|, \quad n = [d \cdot q^{-1}] + 1.$$

Then

$$\|\mathrm{Op}_1^{\mathrm{W}}(p)\|_{\mathbb{S}_q} \leq C_q \|F_n(p)\|_{1,q}$$

where C_q depends on q only.

Example. The amplitude $p(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \frac{1}{\langle\boldsymbol{\xi}\rangle^2}\frac{1}{\langle\mathbf{x}+\mathbf{y}\rangle^3}$ along with all its derivatives belongs to $\mathrm{l}^{q,1}(\mathbb{R}^2)$ for any $q > \frac{1}{2}$. Thus, $\mathrm{Op}_1\Big(\frac{1}{\langle\boldsymbol{\xi}\rangle^2}\frac{1}{\langle\mathbf{x}+\mathbf{y}\rangle^3}\Big) \in \bigcap_{q>\frac{1}{2}} \mathbb{S}_q$.

Proposition 3.3.4 (Theorem 2.5 in [18]for $m = 0$, $T = I$ and $h_1 = h_2 \equiv 1$). *Suppose that $q \in (0,1]$, $P_n \in \mathrm{l}^q(\mathrm{L}^1)(\mathbb{R}^{3d})$, where*

$$P_n(\mathbf{x},\mathbf{y},\boldsymbol{\xi};p) = \sum_{k,t=0}^{n} |\nabla_{\mathbf{x}}^k \nabla_{\mathbf{y}}^t p(\mathbf{x},\mathbf{y},\boldsymbol{\xi})|, \quad n = [d \cdot q^{-1}] + 1.$$

Then

$$\|\mathrm{Op}_1(p)\|_{\mathbb{S}_q} \leq C_q \|P_n(p)\|_{1,q}.$$

The following lemma describes the estimate for a general case of the symbol of the form $a(\mathbf{x},\mathbf{y})b(\boldsymbol{\xi})$ under some control of a and b.

Lemma 3.3.5. *Consider $p(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = a(\mathbf{x},\mathbf{y})b(\boldsymbol{\xi})$, where*

$$a \in C^\infty(\mathbb{R}^{2d}), \quad b \in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^2)(\mathbb{R}^d), \ 0 < \gamma < 2,$$

$$|\nabla_{\mathbf{x}}^m \nabla_{\mathbf{y}}^t a| \in \mathrm{l}^\beta(\mathrm{L}^1)(\mathbb{R}^{2d}) \ \ for\ some \ \ \beta \in (0,\gamma) \cap (0,1]$$

$$and \ \ t, m = 0, 1, ..., n = [d\beta^{-1}] + 1.$$

Moreover, suppose that for some positive number M_a at least one of the following two conditions holds

$$(i) \qquad \sup_{\mathbf{s}} \|a(\mathbf{s}+\cdot,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \sup_{\mathbf{s}} \Big(\int |a(\mathbf{y}+\mathbf{s},\mathbf{y})|^2 d\mathbf{y}\Big)^{\frac{1}{2}} \leq M_a,$$

$$(ii) \qquad \sup_{\mathbf{s}} \|a(\cdot,\mathbf{s}+\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d)} = \sup_{\mathbf{s}} \Big(\int |a(\mathbf{x},\mathbf{x}+\mathbf{s})|^2 d\mathbf{x}\Big)^{\frac{1}{2}} \leq M_a.$$

Then

$$\mathrm{Op}_1^{\mathrm{a}}(p) \in \mathbb{S}_{\gamma,\infty}, \quad s_k(\mathrm{Op}_1^{\mathrm{a}}(p)) \leq C_{\beta,\gamma,d} \cdot \Big(D_a + M_a\Big) \cdot \|b\|_{2,\gamma,\infty} \cdot k^{-\frac{1}{\gamma}},$$

where $D_a = \max_{0\le m,t\le n} \left\| |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a| \right\|_{1,\beta}$ *and* $C_{\beta,\gamma,d}$ *depends on* β, γ, d *only.*

Proof. Without loss of generality consider

$$M_a := \sup_{\mathbf{s}} \|a(\mathbf{s} + \cdot, \cdot)\|_{\mathrm{L}^2(\mathbb{R}^d)} < \infty.$$

Let's split our operator as follows

$$\mathrm{Op}^{\mathrm{a}}_1(p) = \mathrm{Op}^{\mathrm{a}}_1(a \cdot b_{R-}) + \mathrm{Op}^{\mathrm{a}}_1(a \cdot b_{R+})$$

Proposition 3.3.4 for operator $\mathrm{Op}^{\mathrm{a}}_1(a \cdot b_{R-})$ implies

$$\|\mathrm{Op}_1(a \cdot b_{R-})\|^\beta_\beta \le C_\beta \cdot \Big\| \sum_{0\le m,t\le n} |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a(\cdot,\cdot) b_{R-}(\cdot)| \Big\|^\beta_{1,\beta}$$

$$= C_\beta \cdot \Big\| \sum_{0\le m,t\le n} |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a(\cdot,\cdot)| \Big\|^\beta_{1,\beta} \cdot \|b_{R-}\|^\beta_{1,\beta}.$$

Using monotonicity of lattice quasi-norm function and Lemma 3.1.6 we obtain

$$\|b_{R-}\|^\beta_{1,\beta} \le \|b_{R-}\|^\beta_{2,\beta} \le C^\beta_{\beta,\gamma} \cdot \|b\|^\gamma_{2,\gamma,\infty} \cdot R^{1-\frac{\beta}{\gamma}}.$$

The quasi-triangle inequality implies

$$\Big\| \sum_{0\le m,t\le n} |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a(\cdot,\cdot)| \Big\|^\beta_{1,\beta} \le C_{\beta,n} \sum_{0\le m,t\le n} \left\| |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a(\cdot,\cdot)| \right\|^\beta_{1,\beta}$$

$$\le C_{\beta,n} \cdot \max_{0\le m,t\le n} \left\| |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a| \right\|^\beta_{1,\beta}.$$

Hence,

$$k s^\beta_k(\mathrm{Op}_1(a \cdot b_{R-})) \le C_{\gamma,\beta,d} \cdot \|b\|^\gamma_{2,\gamma,\infty} \cdot \max_{0\le m,t\le n} \left\| |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a| \right\|^\beta_{1,\beta} \cdot R^{1-\frac{\beta}{\gamma}},$$

or equivalently

$$s_k(\mathrm{Op}_1(a \cdot b_{R-})) \le C_{\beta,\gamma,d} \cdot \|b\|^{\frac{\gamma}{\beta}}_{2,\gamma,\infty} \cdot \max_{0\le m,t\le n} \left\| |\nabla^m_{\mathbf{x}} \nabla^t_{\mathbf{y}} a| \right\|_{1,\beta} \cdot \left(\frac{R}{k}\right)^{\frac{1}{\beta}} \cdot R^{-\frac{1}{\gamma}} \tag{3.3.9}$$

Without loss of generality let's consider condition (i).

Let's rewrite $\mathrm{Op}^{\mathrm{a}}_1(a \cdot b_{R+})$ as an integral operator.

$$\mathrm{Op}_1(a b_{R+}) u(\mathbf{x}) = \left(\frac{1}{2\pi}\right)^d \iint_{\mathbb{R}^{2d}} e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} a(\mathbf{x},\mathbf{y}) b_{R+}(\boldsymbol{\xi}) u(\mathbf{y}) d\boldsymbol{\xi} d\mathbf{y}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} a(\mathbf{x},\mathbf{y}) \cdot (\mathcal{F}^{-1} b_{R+})(\mathbf{x}-\mathbf{y}) u(\mathbf{y}) d\mathbf{y},$$

where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform.

Now, the Hilbert-Schmidt estimates provides

$$ks_k^2(\mathrm{Op}_1(a{\cdot}b_{R+})) \le \|\mathrm{Op}_1^{\mathrm{a}}(a{\cdot}b_{R+})\|_2^2 \le \iint |a(\mathbf{x},\mathbf{y})|^2{\cdot}|(\mathcal{F}^{-1}b_{R+})(\mathbf{x}{-}\mathbf{y})|^2 d\mathbf{x}d\mathbf{y}$$

$$= \iint |a(\mathbf{y}+\mathbf{s},\mathbf{y})|^2 \cdot |(\mathcal{F}^{-1}b_{R+})(\mathbf{s})|^2 d\mathbf{y}d\mathbf{s}$$

$$\le \int |(\mathcal{F}^{-1}b_{R+})(\mathbf{s})|^2 \cdot \Big(\sup_{\mathbf{s}} \int |a(\mathbf{y}+\mathbf{s},\mathbf{y})|^2 d\mathbf{y}\Big) d\mathbf{s}$$

$$\le M_a^2 \cdot \|\mathcal{F}^{-1}b_{R+}\|_2^2 = M_a^2 \cdot \|b_{R+}\|_2^2 \le C_\gamma^2 \cdot M_a^2 \cdot \|b\|_{2,\gamma,\infty}^{\gamma} \cdot R^{1-\frac{2}{\gamma}}.$$

Therefore,

$$s_k(\mathrm{Op}_1^{\mathrm{a}}(a \cdot b_{R+}) \le C_\gamma \cdot M_a \cdot \left(\frac{R}{k}\right)^{\frac{1}{2}} \cdot \|b\|_{2,\gamma,\infty}^{\frac{\gamma}{2}} \cdot R^{-\frac{1}{\gamma}}. \tag{3.3.10}$$

Without loss of generality let's consider $\|b\|_{2,\gamma,\infty} \neq 0$.
Taking $R = k \cdot \|b\|_{2,\gamma,\infty}^{-\gamma}$ in (3.3.9), (3.3.10) and using Fan's inequality (2.2.2) we obtain

$$s_{2k-1}(\mathrm{Op}_1^{\mathrm{a}}(a \cdot b)) \le s_k(\mathrm{Op}_1^{\mathrm{a}}(a \cdot b_{R-})) + s_k(\mathrm{Op}_1^{\mathrm{a}}(a \cdot b_{R+}))$$

$$\le C_{\beta,\gamma,d} \cdot \|b\|_{2,\gamma,\infty}^{\frac{\gamma}{\beta}} \cdot \max_{0\le m,t\le n} \Big\||\nabla_x^m \nabla_y^t a|\Big\|_{1,\beta} \cdot \left(\frac{R}{k}\right)^{\frac{1}{\beta}} \cdot R^{-\frac{1}{\gamma}}$$

$$+ C_\gamma \cdot M_a \cdot \|b\|_{2,\gamma,\infty}^{\frac{\gamma}{2}} \cdot \left(\frac{R}{k}\right)^{\frac{1}{2}} \cdot R^{-\frac{1}{\gamma}}$$

$$= \|b\|_{2,\gamma,\infty}\Big(C_{\beta,\gamma,d} \cdot \max_{0\le m,t\le n} \Big\||\nabla_x^m \nabla_y^t a|\Big\|_{1,\beta} + C_\gamma \cdot M_a\Big) \cdot k^{-\frac{1}{\gamma}}.$$

□

Remark 3.3.6. In the case $\beta = \gamma \in (0,1]$ we can state

$$s_k(\mathrm{Op}_1^{\mathrm{a}}(p)) \le C_{\gamma,d} \cdot \Big(D_a + M_a\Big) \cdot \|b\|_{2,\gamma,\infty} \cdot \left(\frac{\log(k+1)}{k}\right)^{\frac{1}{\gamma}}.$$

The proof almost repeats the previous one, except the replacement of (3.3.9) with

$$s_k(\mathrm{Op}_1^{\mathrm{a}}(a \cdot b_{R-})) \le C_{\gamma,d} \cdot \|b\|_{2,\gamma,\infty} \cdot \max_{0\le m,t\le n} \Big\||\nabla_x^m \nabla_y^t a|\Big\|_{1,\gamma} \cdot \left(\frac{\log R}{k}\right)^{\frac{1}{\gamma}}.$$

Remark 3.3.7. If in addition $|\nabla_x^m \nabla_y^t a| \in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^1)(\mathbb{R}^d)$, then

$$s_k(\mathrm{Op}_1^{\mathrm{a}}(p)) \le C_{\gamma,d} \cdot \Big(D_a + M_a\Big) \cdot \|b\|_{2,\gamma,\infty} \cdot \left(\frac{\log^2(k+1)}{k}\right)^{\frac{1}{\gamma}}.$$

The following two lemmata (along with corollaries) give examples of functions in the lattice quasi-norm spaces we consider in the proof of Theorem 3.3.14.

Lemma 3.3.8. *If three vectors $(1,-1)^T$, $(c_1, c_2)^T$ and $(c_3, c_4)^T$ are pairwise linearly independent in $\mathbb{Q}^2$, then the series*

$$\sum_{\substack{-\infty<n,m<\infty\\ n\neq m\\ c_1 n + c_2 m \neq 0\\ c_3 n + c_4 m \neq 0}} \frac{1}{|n-m|^k |c_1 n + c_2 m|^s |c_3 n + c_4 m|^t}$$

converges for any $k > 1$ and $s, t \ge 0$, where $s + t > 1$.

Remark. *Vectors $(c_1, c_2)^T$ and $(c_3, c_4)^T$ can be collinear. Vector $(1,-1)^T$ can be replaced with any other $\mathbb{Q}^2$-vector preserving linear independence with each of the other two. We choose $(1,-1)^T$ with our further estimates in mind.*

Proof. We can consider that $c_i \in \mathbb{Z}$ (otherwise multiply by a common denominator of all 4 coefficients) After the substitution $w = n - m,\ v = c_1 n + c_2 m$, we reduce the series to

$$\sum_{\substack{-\infty<w,v<\infty\\ v\neq 0, w\neq 0\\ v+cw\neq 0}} \frac{1}{|w|^k |v|^s |v+cw|^t},$$

where c is a new constant. Without loss of generality consider $c > 0$.

$$\sum_{\substack{-\infty<w,v<\infty\\ v\neq 0, w\neq 0\\ v+cw\neq 0}} \frac{1}{w^k v^s |v+cw|^t} = \sum_{\substack{v>0,w>0\\ v+cw\neq 0}} \frac{2}{w^k v^s (v+cw)^t} + \sum_{\substack{v>0,w<0\\ v+cw\neq 0}} \frac{2}{|w|^k v^s |v+cw|^t}$$

$$\le \sum_{v,w>0} \frac{2}{w^k v^{s+t}} + \sum_{\substack{w<0,\ cw<u=v+cw\\ u\neq 0}} \frac{2}{|w|^k (u-cw)^s |u|^t}$$

$$\le \sum_{v>0} \frac{2}{w^k} \sum_{v>0} \frac{1}{v^{s+t}} + \sum_{w<0,\ cw<u<0} \frac{2}{|w|^k (u-cw)^s |u|^t} + \sum_{w<0<u} \frac{2}{|w|^k (u-cw)^s |u|^t}$$

$$\le \sum_{v>0} \frac{4}{w^k} \sum_{v>0} \frac{1}{v^{s+t}} + \sum_{w>0,\ cw>u>0} \frac{2}{w^k (cw-u)^s u^t}$$

$$= const + \sum_{w>0} \frac{2}{w^k} \Big(\sum_{cw>u>0} \frac{1}{(cw-u)^s u^t} \Big)$$

To finish the proof we establish the following estimate

$$\sum_{cw>u>0} \frac{1}{(cw-u)^s u^t} \lesssim_{c,t,s} (\log w)^s$$

for any fixed $c > 0$, $w > 1$.
Indeed, if $t > 1$ or $s > 1$, the sum is dominated by convergent series, $\sum_{n>0} \frac{1}{n^{\max\{t,s\}}}$. Suppose $t, s \leq 1$. If $s = 1$, then $\sum_{u \in (0,cw)} \frac{1}{cw-u} \lesssim_{c,t,s} \log w$. Let $s < 1$. Since $t > 1 - s$, using the Hölder inequality we obtain

$$\sum_{cw>u>0} \frac{1}{(cw-u)^s u^t} \leq \Big(\sum_{cw>u>0} \frac{1}{cw-u} \Big)^s \cdot \Big(\sum_{cw>u>0} \frac{1}{u^{\frac{t}{1-s}}} \Big)^{1-s} \lesssim_{c,t,s} (\log w)^s.$$

□

Corollary 3.3.9. *For any integer $d > 0$ and $k, s > d$ define*

$$f(\mathbf{x}, \mathbf{y}) = \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^k \langle \mathbf{x} + \mathbf{y} \rangle^s}, \quad g(\mathbf{x}, \mathbf{y}) = \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^k \langle \mathbf{x} \rangle^s}, \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

For any $\beta \in \big(\max\{\frac{1}{s}, \frac{1}{k}\} \cdot d, 1 \big)$ $f, g \in \mathrm{l}^\beta(\mathrm{L}^1)(\mathbb{R}^{2d})$.

Lemma 3.3.10. *For any two linear independent vectors $(c_1, c_2)^T$ and $(c_3, c_4)^T$ in $\mathbb{R}^2$ and any $k > 1$ the series*

$$\sum_{(n,m) \in \mathbb{Z}^2} \frac{\chi_{\{|c_1 n + c_2 m| < 1\}}}{\langle c_3 n + c_4 m \rangle^k}$$

is convergent.

Proof. Using the substitution $u = c_1 x + c_2 y$, $v = c_3 x + c_4 y$ with non-zero Jacobian determinant, we can estimate

$$\sum_{(n,m) \in \mathbb{Z}^2} \frac{\chi_{\{|c_1 n + c_2 m| < 1\}}}{\langle c_3 n + c_4 m \rangle^k} \lesssim_{k,c_1,c_2,c_3,c_4} \int_{\mathbb{R}^2} \frac{\chi_{\{|c_1 x + c_2 y| < 1\}}}{\langle c_3 x + c_4 y \rangle^k} dx dy$$

$$\lesssim_{k,c_1,c_2,c_3,c_4} \int_{|u| \leq 1} du \int_{\mathbb{R}} \frac{dv}{\langle v \rangle^k} < \infty.$$

□

Corollary 3.3.11. *For any two linear independent vectors $(c_1, c_2)^T$ and $(c_3, c_4)^T$ in $\mathbb{R}^2$, for any $k > 1$ and $\beta \in \big(\frac{1}{k}, 1\big)$*

$$\frac{\chi_{\{|c_1 x + c_2 y| < 1\}}}{\langle c_3 x + c_4 y \rangle^k} \in \mathrm{l}^\beta(\mathrm{L}^1)(\mathbb{R}^2).$$

The auxiliary lemma below provides a convenient substitution for the symbol to avoid singularities after the integration by parts.

Lemma 3.3.12. *Let* $p = p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *be an amplitude satisfying*

$$p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) = p(\boldsymbol{\xi}) \in C^m(\mathbb{R}^d), \qquad \lim_{|\boldsymbol{\xi}| \to \infty} \left| \nabla^k_{\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \right| = 0$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ *and* $k = 0, 1, ..., m-1$.
Moreover, let

$$\left| \nabla^k_{\boldsymbol{\xi}} p(\mathbf{x}, \cdot, \cdot) \right| \in \mathrm{L}^1(\mathbb{R}^{2d})$$

for any $\mathbf{x} \in \mathbb{R}^d$ *and* $k = 0, 1, ..., m-1$.
Then

$$\mathrm{Op}^{\mathrm{a}}_1(p) = \mathrm{Op}^{\mathrm{a}}_1\left(\frac{(1 - i(\mathbf{x} - \mathbf{y})\nabla_{\boldsymbol{\xi}})^m p}{\langle \mathbf{x} - \mathbf{y} \rangle^{2m}} \right).$$

Proof. Integration by parts gives

$$(x_k - y_k) \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} \cdot \partial_{\xi_k} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) d\boldsymbol{\xi} = i(x_k - y_k)^2 \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) d\boldsymbol{\xi}$$

for any $k = 1, 2, ..., d$.

Thus,

$$\int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} \big(p - i(\mathbf{x} - \mathbf{y})\nabla_{\boldsymbol{\xi}} p\big) d\boldsymbol{\xi} = \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \, d\boldsymbol{\xi}$$

$$-i \sum_{k=1}^{d} (x_k - y_k) \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} \cdot \partial_{\xi_k} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) d\boldsymbol{\xi}$$

$$= \big(1 + |\mathbf{x} - \mathbf{y}|^2\big) \cdot \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \, d\boldsymbol{\xi} = \langle \mathbf{x} - \mathbf{y} \rangle^2 \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \, d\boldsymbol{\xi}.$$

Therefore,

$$\int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \, d\boldsymbol{\xi} = \int_{\mathbb{R}^d} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} \frac{\big(1 - i(\mathbf{x} - \mathbf{y})\nabla_{\boldsymbol{\xi}}\big) p(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})}{\langle \mathbf{x} - \mathbf{y} \rangle^2} d\boldsymbol{\xi}.$$

Repeating this reduction m times and applying Fubini's theorem, we obtain the statement.

□

Remark 3.3.13. Using the same approach one can obtain the following formula in one dimensional case

$$\int_{[a_0,a_1]} e^{is\xi} p(s)\, ds = i\frac{\xi}{\langle\xi\rangle^2} p(a_0) + i\frac{\xi}{\langle\xi\rangle^2}\frac{(1+i\xi\partial_s)p}{\langle\xi\rangle^2}(a_0)$$

$$+\int_{[a_0,a_1]} e^{is\xi}\frac{(1+i\xi\partial_s)^2 p}{\langle\xi\rangle^4}(s)ds,$$

where $p \in C^2(\mathbb{R})$ with $p(a_1) = p'(a_1) = 0,\ k = 0,1.$

Finally,

Theorem 3.3.14. *Consider functions $g = g(\mathbf{s}), b = b(\boldsymbol{\xi}) \in C^\infty(\mathbb{R}^d)$, with parameters $\alpha > d$, $\gamma > \frac{1}{\alpha}$ satisfying*

$$|\nabla^j g(\mathbf{s})| \lesssim \langle \mathbf{s}\rangle^{-\alpha}, \ \ j = 1,2,...,l = l(\alpha,\gamma),$$

$$|\nabla^k b| \in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^2)(\mathbb{R}^d), \ \ k = 0,1,2,...,[2\alpha]+1.$$

Then

$$\mathrm{Op}_1^{\mathrm{a}}\Big(\Big(g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big) - g(\mathbf{x})\Big)b(\boldsymbol{\xi})\Big) \in \mathbb{S}_{\gamma,\infty}.$$

Remark. This theorem reduces the Weyl symbols $\sigma_W = g\big(\frac{\mathbf{x}+\mathbf{y}}{2}\big)b\big(\boldsymbol{\xi}\big)$ to one of the form $\sigma = g(\mathbf{x})b(\boldsymbol{\xi})$ when singular values are estimated. Then we can apply Theorem 3.3.1. The parameter $l \geq \max\{\frac{d}{\gamma}, d\}+1$.

Proof. Let's define $a(\mathbf{x},\mathbf{y}) = g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big) - g(\mathbf{x})$.

Applying Lemma 3.3.12, after the multiple integration by parts we represent the kernel of the operator as follows

$$\mathrm{Op}_1^{\mathrm{a}}(ab) = \frac{1}{(2\pi)^d}\iint_{\mathbb{R}^{2d}} e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} a(\mathbf{x},\mathbf{y})b(\boldsymbol{\xi})u(y)dyd\boldsymbol{\xi}$$

$$= \frac{1}{(2\pi)^d}\iint_{\mathbb{R}^{2d}} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\cdot\frac{a(\mathbf{x},\mathbf{y})}{\langle\mathbf{x}-\mathbf{y}\rangle^{2m}}\cdot(1-i(\mathbf{x}-\mathbf{y})\nabla_{\boldsymbol{\xi}})^m b(\boldsymbol{\xi})u(y)dyd\boldsymbol{\xi}$$

$$= \sum_{|\mathbf{j}|=k}\mathrm{Op}_1^{\mathrm{a}}\big(A_{\mathbf{j}}\cdot\partial_{\boldsymbol{\xi}}^{\mathbf{j}}\ b\big),$$

where

$$A_{\mathbf{j}}(\mathbf{x},\mathbf{y}) = C_{\mathbf{j}}\cdot\frac{\prod_{1\le t\le d}(x_t-y_t)^{j_k}}{\langle\mathbf{x}-\mathbf{y}\rangle^{2m}}\cdot a(\mathbf{x},\mathbf{y}).$$

Let's prove that for a fixed $m \geq 2\alpha+1$ each $\mathrm{Op}_1^{\mathrm{a}}\big(A_{\mathbf{j}}\cdot\partial_{\boldsymbol{\xi}}^{\mathbf{j}}\ b\big) \in \mathbb{S}_{\gamma,\infty}$, checking that all conditions in Lemma 3.3.5 are satisfied.

Let's estimate $|\nabla_x^r\nabla_y^k A_{\mathbf{j}}|$ using the Leibniz rule and the fact that

$|x_t - y_t|^u < 1 + |x_t - y_t|^w$ for any $0 < u < w$.

$$|\nabla_x^r \nabla_y^k A_{\mathbf{j}}| \lesssim_{\mathbf{j}, C_{\mathbf{j}}, r, s} \frac{\prod_{1 \le t \le d}(1 + |x_t - y_t|)^{j_t}}{\langle \mathbf{x} - \mathbf{y} \rangle^{2m}} \cdot \sum_{\substack{0 \le t \le r \\ 0 \le u \le k}} \left| \nabla_x^u \nabla_y^t \Big(g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big) - g(\mathbf{x}) \Big) \right|.$$

Note that since $\frac{1+|x_t - y_t|}{2} \le \sqrt{\frac{1+|x_t-y_t|^2}{2}} \le \frac{\langle \mathbf{x}-\mathbf{y} \rangle}{\sqrt{2}}$,

$$\frac{\prod_{1 \le t \le d}(1 + |x_t - y_t|)^{j_t}}{\langle \mathbf{x} - \mathbf{y} \rangle^{2m}} \lesssim_d \frac{\langle \mathbf{x} - \mathbf{y} \rangle^{\sum_{t=0}^d j_t}}{\langle \mathbf{x} - \mathbf{y} \rangle^{2m}} = \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^{2m-k}} \le \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^{m}}.$$

Therefore,

$$|\nabla_x^r \nabla_y^k A_{\mathbf{j}}| \lesssim_{\mathbf{j}, C_{\mathbf{j}}, r, k, d} \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^{m}} \cdot \sum_{\substack{0 \le t \le r \\ 0 \le u \le k \\ (t,u) \ne (0,0)}} \left| \nabla_x^u \nabla_y^t \Big(g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big) - g(\mathbf{x}) \Big) \right|$$

$$+ \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^{m}} \cdot \left| g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big) - g(\mathbf{x}) \right|$$

$$\lesssim_{\alpha, r, k} \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^{m} \langle \mathbf{x} + \mathbf{y} \rangle^{\alpha}} + \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^{m} \langle \mathbf{x} \rangle^{\alpha}} + \frac{1}{\langle \mathbf{x} - \mathbf{y} \rangle^{m}} \cdot \left| g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big) - g(\mathbf{x}) \right|.$$

The first two terms in the upper bound above belong to $\mathrm{l}^\beta(\mathrm{L}^1)(\mathbb{R}^{2d})$ for $\beta \in (\frac{1}{\alpha}, \min\{\gamma, 1\})$. Indeed, since $m \ge 2\alpha + 1 > d$, we can directly apply Corollary 3.3.9. It remains to estimate the last term.

Representing the difference $g(\mathbf{x}) - g(0)$ as the line integral, we obtain

$$|g(\mathbf{x}) - g(0)| = \Big| \int_{\{t\mathbf{x} | 0 \le t \le 1\}} \nabla g \cdot dr \Big| = \Big| \int_0^1 \sum_{j=1}^d \partial_{x_j} g(\mathbf{x}t) \cdot x_j dt \Big|$$

$$\le \sum_{j=1}^d |x_j| \cdot \int_0^1 \frac{1}{\langle t\mathbf{x} \rangle^\alpha} dt = \frac{\sum_{j=1}^d |x_j|}{|\mathbf{x}|} \cdot \int_0^{|\mathbf{x}|} \frac{1}{\langle t \rangle^\alpha} dt \lesssim_{\alpha, d} \frac{\sum_{j=1}^d |x_j|}{\sqrt{\sum_{j=1}^d |x_j|^2}} \le \sqrt{d}.$$

Thus, $g \in \mathrm{L}^\infty(\mathbb{R}^d)$.

Mean Value Theorem implies the estimate

$$\left| g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big) - g(\mathbf{x}) \right| \le \frac{|\mathbf{y} - \mathbf{x}|}{2} \cdot \max_{0 \le t \le 1} \left| \nabla g\Big(\frac{\mathbf{x}+\mathbf{y}}{2} \cdot t + (1-t)\mathbf{x}\Big) \right|$$

$$\lesssim |\mathbf{y} - \mathbf{x}| \cdot \frac{1}{\min_{0 \le t \le 1} \left\langle \mathbf{x} + \frac{\mathbf{y}-\mathbf{x}}{2} \cdot t \right\rangle^\alpha}.$$

Note that, if $|\mathbf{x} - \mathbf{y}| \le |\mathbf{x}|$, then for any $t \in (0,1)$

$$\left| \mathbf{x} + \frac{y - \mathbf{x}}{2} \cdot t \right| \ge |\mathbf{x}| - t \left| \frac{y - \mathbf{x}}{2} \right| \ge |\mathbf{x}| - \left| \frac{\mathbf{x}}{2} \right| = \left| \frac{\mathbf{x}}{2} \right|$$

and
$$\frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^m}\Big|g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big)-g(\mathbf{x})\Big| \lesssim_\alpha \frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^{m-1}}\cdot\frac{1}{|\mathbf{x}|^\alpha}.$$

Otherwise, if $|\mathbf{x}-\mathbf{y}|>|\mathbf{x}|$, then $|\mathbf{x}-\mathbf{y}|^m > |\mathbf{x}-\mathbf{y}|^{\frac{m}{2}}|\mathbf{x}|^{\frac{m}{2}}$ and, since $g\in \mathrm{L}^\infty(\mathbb{R}^d)$,

$$\frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^m}\Big|g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big)-g(\mathbf{x})\Big| \lesssim_{\alpha,d} \frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^{\frac{m}{2}}}\frac{1}{\langle\mathbf{x}\rangle^{\frac{m}{2}}}.$$

Therefore, since $m\geq 2\alpha+1>2d+1$, for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$

$$\frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^m}\Big|g\Big(\frac{\mathbf{x}+\mathbf{y}}{2}\Big)-g(\mathbf{x})\Big| \lesssim_{\alpha,d} \frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^{m-1}}\frac{1}{|\mathbf{x}|^\alpha}+\frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^{\frac{m}{2}}}\frac{1}{\langle\mathbf{x}\rangle^{\frac{m}{2}}}$$

$$\lesssim_{\alpha,d} \frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^{2d}}\cdot\frac{1}{\langle\mathbf{x}\rangle^\alpha}+\frac{1}{\langle \mathbf{x}-\mathbf{y}\rangle^{\alpha}}\frac{1}{\langle\mathbf{x}\rangle^\alpha}\in \mathrm{l}^\beta(\mathrm{L}^1)(\mathbb{R}^{2d})$$

due to Corollary 3.3.9.

Therefore, $D_{A_\mathbf{j}}=\max_{0\leq r,k\leq n}\Big\|\big|\nabla^r_\mathbf{x}\nabla^k_y A_\mathbf{j}\big|\Big\|_{1,\beta}<\infty$, where $n=[d\beta^{-1}]+1$. Moreover, using the same estimates and triangle inequality

$$M_{A_\mathbf{j}}=\sup_\mathbf{s}\|A_\mathbf{j}(\mathbf{s}+\cdot,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d)}=\sup_\mathbf{s}\|A_\mathbf{j}(\mathbf{s}+\mathbf{y},\mathbf{y})\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$\leq \sup_\mathbf{s}\Big\|\frac{1}{\langle\mathbf{s}\rangle^{2d+1}\langle\mathbf{s}+2\mathbf{y}\rangle^\alpha}+\frac{1}{\langle\mathbf{s}\rangle^{2d+1}\langle\mathbf{s}+\mathbf{y}\rangle^\alpha}$$

$$+\frac{1}{\langle\mathbf{s}\rangle^{2d}}\cdot\frac{1}{\langle\mathbf{s}+\mathbf{y}\rangle^\alpha}+\frac{1}{\langle\mathbf{s}\rangle^{\alpha}}\frac{1}{\langle\mathbf{s}+\mathbf{y}\rangle^\alpha}\Big\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$\lesssim_{\alpha,d}\sup_\mathbf{s}\frac{1}{\langle\mathbf{s}\rangle^d}\left(\Big\|\frac{1}{\langle\mathbf{s}+2\mathbf{y}\rangle^\alpha}\Big\|_{\mathrm{L}^2(\mathbb{R}^d)}+\Big\|\frac{1}{\langle\mathbf{s}+\mathbf{y}\rangle^\alpha}\Big\|_{\mathrm{L}^2(\mathbb{R}^d)}\right)$$

$$\lesssim_{\alpha,d}\sup_\mathbf{s}\frac{1}{\langle\mathbf{s}\rangle^d}\Big\|\frac{1}{\langle\mathbf{y}\rangle^\alpha}\Big\|_{\mathrm{L}^2(\mathbb{R}^d)}<\infty.$$

Finally, $\partial^{j_1}_{\xi_1}...\partial^{j_d}_{\xi_d}b\in \mathrm{l}^{\gamma,\infty}(\mathrm{L}^2)(\mathbb{R}^d)$, which implies the claimed result.

□

Using the result above for $d=1$, $\alpha=2$, $\gamma=1$, $g(t)=\frac{\zeta(t)}{t}$ and $\beta(\xi)=\frac{\zeta(\xi)}{\xi}$, we obtain

Corollary 3.3.15.

$$\mathrm{Op}^\mathrm{a}_1\left(\left(\frac{\zeta(x)}{x}-\frac{2\zeta\big(\frac{x+y}{2}\big)}{x+y}\right)\frac{\zeta(\xi)}{\xi}\right)\in\mathbb{S}_{1,\infty}.$$

The obtained result is fully compliant with the asymptotic formula [6, p. 95, (2.1)'] and is a core tool for proving the main result in Chapter 5 (see Theorem 5.1.9 about an angular domain).

Model operator

Remark 3.3.16. Since $\mathbb{S}_{1,\infty} \subseteq \mathfrak{S}_1$ (recall that $T \in \mathfrak{S}_1$ means $s_k(T) \lesssim \frac{\log(k+1)}{k}$, see Definition 2.2.3) this Corollary allows to reformulate the result of Theorem 3.3.1 in the following way

$$\mathrm{Op}_1^{\mathrm{W}}\Big(\frac{\zeta(t)}{t}\frac{\zeta(\xi)}{\xi}\Big) \in \mathfrak{S}_1.$$

The same is true for operators $\mathrm{Op}_1^{\mathrm{W}}(\langle t\rangle^{-1}\langle\xi\rangle^{-1}), \mathrm{Op}_1^{\mathrm{W}}(\zeta(t)t^{-1}\langle\xi\rangle^{-1})$, $\mathrm{Op}_1^{\mathrm{W}}(\zeta(2t)t^{-1}\langle\xi\rangle^{-1}) \in \mathfrak{S}_1$, which might be considered as model operators, which we appeal to in Chapter 5.

Indeed,

$$\frac{\zeta(t)}{t}\frac{1}{\langle\xi\rangle} - \frac{\zeta(t)\zeta(\xi)}{t\xi} = \frac{\zeta(t)}{t}\frac{\xi - \langle\xi\rangle\zeta(\xi)}{\xi\langle\xi\rangle} = \frac{\zeta(t)}{t}\cdot F(\xi),$$

where $F \in \mathrm{l}^{\frac{1}{3},\infty}(\mathrm{L}^2)$.
Thus, by Theorem 3.3.1 $\mathrm{Op}_1^{\mathrm{a}}\Big(\frac{\zeta(x)}{x}\frac{1}{\langle\xi\rangle} - \frac{\zeta(x)\zeta(\xi)}{x\xi}\Big) \in \mathbb{S}_{1,\infty}$. Similarly, $\mathrm{Op}_1^{\mathrm{a}}\Big(\frac{1}{\langle x\rangle\langle\xi\rangle} - \frac{\zeta(x)}{x\langle\xi\rangle}\Big) \in \mathbb{S}_{1,\infty}$. Note that

$$\frac{1}{\langle t\rangle\langle\xi\rangle} - \frac{\zeta(t)\zeta(\xi)}{t\xi} = \frac{1}{\langle t\rangle\langle\xi\rangle} - \frac{\zeta(t)}{t\langle\xi\rangle} + \frac{\zeta(t)}{t\langle\xi\rangle} - \frac{\zeta(t)\zeta(\xi)}{t\xi}.$$

Thus, $\mathrm{Op}_1^{\mathrm{a}}(\langle x\rangle^{-1}\langle\xi\rangle^{-1}), \mathrm{Op}_1^{\mathrm{a}}(\zeta_\delta(x)x^{-1}\langle\xi\rangle^{-1}) \in \mathfrak{S}_1$.
Therefore, due to Corollary $\mathrm{Op}_1^{\mathrm{W}}(\langle t\rangle^{-1}\langle\xi\rangle^{-1}), \mathrm{Op}_1^{\mathrm{W}}(\zeta(t)t^{-1}\langle\xi\rangle^{-1}) \in \mathfrak{S}_1$.

The inclusion $\mathrm{Op}_1^{\mathrm{W}}(\zeta(2t)t^{-1}\langle\xi\rangle^{-1}) \in \mathfrak{S}_1$ follows from the fact that $\frac{\zeta(2t)}{t} - \frac{\zeta(t)}{t} \in C_0^\infty$. Therefore,

$$\mathrm{Op}_1^{\mathrm{a}}\Big(\frac{\zeta(2x)}{x\langle\xi\rangle} - \frac{\zeta(x)}{x\langle\xi\rangle}\Big) \in \mathbb{S}_{1,\infty} \tag{3.3.11}$$

Chapter 4

Self-adjoint differential operators

4.1 Introduction

The spectral analysis of unbounded operators requires quite a different toolkit compared to the one described in the previous chapter. Some estimates for the number of eigenvalues of this type of operators are obtained in [17] However, the operators discussed in the [17] are considered on a finite interval (a, b). Below we discuss some techniques of decoupling an unbounded operator defined on $\mathbb{R}$ to reduce it to the case of a finite interval (a, b) applying spectral aspects of the theory of self-adjoint extensions. The theory of self-adjoint extensions is well-described in [4, Ch. 10].
We start with the model theorems about the adjoint of the Laplacian $-\Delta$ defined on the set of functions with Dirichlet boundary conditions (Lemmata 4.2.1 and 4.2.4) and then extent these results. After that we introduce the machinery which helps to compute the spectral counting functions of the main ΨDO (introduced in Chapter 5) using spectral results of some specific differential operators (Proposition 4.4.1).

Notations used

We denote by $W_2^2(\Omega)$ the *Sobolev space* equipped with the norm $\|f\|_{W_2^2(\Omega)} = \left(\|f\|^2_{\mathrm{L}^2(\Omega)} + \|f'\|^2_{\mathrm{L}^2(\Omega)} + \|f''\|^2_{\mathrm{L}^2(\Omega)}\right)^{\frac{1}{2}}$, where f' and f'' are weak derivatives of order 1 and 2, respectively.

Domain, D_A, of an operator A is considered as a dense subspace in L^2 space.

Definition 4.1.1. We define operator $\mathrm{H}_{(a,b)} := H_0 - p$, where

$$H_0 = -\Delta = -\frac{d^2}{dx^2}, \quad p \in \mathrm{L}^\infty(\mathbb{R}),$$

the domain of the operator

$$D_{\mathrm{H}_{(a,b)}} = \{y|\ y \in W_2^2(a,b), y(a) = y(b) = 0\},$$

where $a, b \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$.

Remark. $\mathrm{H}_{(a,b)}$ is a differential operator corresponding to the differential expression $l(y) = -y'' - py$.

Spectral counting functions $\#(\cdot, A), n(\cdot, A), \#(\cdot; -p; (a,b))$

We use the following notations:

$\#(\lambda; -p; (a,b))$ for the counting function of the one-dimensional Schrödinger operator $\mathrm{H}_{(a,b)}$;

Spectral counting function of a lower semibounded operator
$\#(\lambda, A) := \Big|\{k|\lambda_k(A) < \lambda\}\Big|$;

Number of singular values of a self-adjoint compact operator
$n(\lambda, A) := \Big|\{k|s_k(A) > \lambda\}\Big|$.

Definition 4.1.2. Deficiency indices $\nu_-(A), \nu_+(A)$ of a symmetric operator A are defined as follows

$$\nu_\pm(A) := \dim \mathrm{Ker}\Big(\overline{A}^* \mp iI\Big),$$

where I denotes the identity operator.

As shown in Corollaries 4.3.4, 4.3.6 all the results in this chapter (about the adjoint operators and deficiency indices) are valid for any potential function $p \in \mathrm{L}^\infty(\mathbb{R})$, i.e. do not change under small perturbations. Thus, we consider $p = 0$.

4.2 Restrictions of differential operators with Dirichlet boundary conditions

In this section we study the properties of *decouplings*, differential operators with a wider set of boundary conditions. Being a common restriction, such operators connect two other differential operators (their extensions), one of which leads to the model operator, while the other has a well-estimated counting function (see Proposition 4.4.1 below)

Let $R \in (a,b)$, $|a|,|b| < \infty$. Consider differential operator $T_R = -\Delta$ on a finite interval (a,b) with domain

$$D_R = \{y \in W_2^2(a,R) \oplus W_2^2(R,b) |\ y(a) = y(R) = y(b) = 0,\ y' \in C(a,b)\ \}$$

Lemma 4.2.1. *The adjoint operator* $T_R^* = -\Delta$ *and*

$$D_R^* = \{y \in W_2^2(a,R) \oplus W_2^2(R,b) |\ y(a) = y(b) = 0,\ y(R-0) = y(R+0)\ \}$$

Proof. Since for any $f, g \in D_R$ the derivative f' is absolutely continuous on (a,b), $g' \in \mathrm{L}^2(a,b) \subseteq \mathrm{L}^1(a,b)$, we can apply integration by parts and obtain

$$\langle T_R f, g\rangle = -\int_a^b f''(t)\overline{g(t)}dt = -f'\overline{g}\Big|_a^b + f\overline{g'}\Big|_a^b + \langle f, T_R g\rangle = \langle f, T_R g\rangle.$$

Thus, T_R is symmetric. Let's find the domain of its adjoint.

The idea of the proof is based on the approach proposed in [1, Ch. 4, p. 163] with some adjustments.

Step 1.

First, we prove the inclusion
$D_R^* \subseteq \{y \in W_2^2(a,R) \oplus W_2^2(R,b) |\ y(a) = y(b) = 0,\ y(R-0) = y(R+0)\}$.
For any $g \in D_R^*$ define $T_R^* g = g^* \in \mathrm{L}^2$ and consider $f = f_1 \chi_{[a,R]}$, where $f_1 \in D_R$ with $f_1'(R) = 0$. By definition

$$-\int_a^R f''(t)\overline{g(t)}dt = \langle T_R f, g\rangle = \langle f, T_R^* g\rangle = \int_a^R f(t)\overline{g^*(t)}dt$$

Let $G_{c_1}(t) = \int_a^t g^*(z)dz + c_1$. Since G is absolutely continuous, integration by parts gives

$$\int_a^b f(t)\overline{g^*(t)}dt = \int_a^R f(t)\overline{G_{c_1}'(t)}dt = -\int_a^R f'(t)\overline{G_{c_1}(t)}dt + f(t)\overline{G_{c_1}(t)}\Big|_a^R$$

$$= -\int_a^R f'(t)\overline{G_{c_1}(t)}dt = -\int_a^R f'(t)\overline{\Big(\int_a^t G_{c_1}(s)ds + c_2\Big)'}dt$$

$$= \int_a^R f''(t)\overline{\Big(\int_a^t G_{c_1}(s)ds + c_2\Big)}\, dt$$

$$-f'(t)\overline{\Big(\int_a^t G_{c_1}(s)ds\Big)}\Big|_a^R = \int_a^R f''(t)\overline{\Big(\int_a^t G_{c_1}(s)ds + c_2\Big)}\, dt.$$

Thus,

$$\int_a^R f''(t) \cdot \overline{\Big(g(t) + \int_a^t G_{c_1}(s)ds + c_2\Big)}\, dt = 0. \qquad (4.2.1)$$

Take

$$f_1(t) = \int_a^t \int_a^w \Big(g(s) + \int_a^s G_{c_1}(u)du + c_2\Big) ds\, dw$$

We claim that $f \in D_R$.
Indeed, $f(a) = 0$ and $f''(t) = g(t) + \int_a^t G_{c_1}(s)ds + c_2 \in \mathrm{L}^2(a, R)$. Let's compute $f(R)$ and $f'(R)$.

$$f'(R) = \int_a^R \Big(g(s)+\int_a^s G_{c_1}(u)du+c_2\Big)ds = I_{t,g}+I_{t,g^*}+c_1\cdot\frac{(R-a)^2}{2}+c_2(R-a),$$

$$f(R) = \int_a^R \big(I_{w,g} + I_{w,g^*}\big)dw + c_1 \cdot \frac{(R-a)^3}{6} + c_2 \cdot \frac{(R-a)^2}{2},$$

where $I_{t,g} = \int_a^t g(s)ds$ and $I_{t,g^*} = \int_a^t \int_a^s \int_a^u g^*(z)dz\, du\, ds$.
Consider the system of independent linear equations $f(R) = f'(R) = 0$, which has an unique solution in terms of c_1, c_2.

Now (4.2.1) can be rewritten

$$\Big\| g(t) + \int_a^t G_{c_1}(s)ds + c_2 \Big\|_{\mathrm{L}^2(a,R)} = 0.$$

Thus, $g(t) = -\int_a^t G_{c_1}(s)ds + c_2$ a.e. on (a, R). This implies $g'(t) = -G_{c_1}(t) = -\int_a^t g^*(z)dz - c_1 \in AC(a, R) \subseteq \mathrm{L}^2(a, R)$.
Differentiating once again we obtain $g'' = -g^* \in \mathrm{L}^2(a, R)$. Thus, $\chi_{(a,R)}g \in W_2^2(a, R)$, $T_R^* g = -g''$.
Similarly, taking $f = f_2\chi_{[R,b]}$ with appropriate $f_2 \in D_R$ we obtain $\chi_{[R,b]}g \in W_2^2(R, b)$, $T_R^* g = -g''$.
Let's rewrite $\langle T_R f, g\rangle = \langle f, T_R^* g\rangle$ for $f \in D_R,\ g \in W_2^2(a, R) \oplus W_2^2(R, b)$ using integration by parts separately on each of the segments $[a, R]$ and $[R, b]$.

$$\langle T_R f, g\rangle = -\int_a^R f''(t)\overline{g(t)}dt - \int_R^b f''(t)\overline{g(t)}dt$$

$$= -f'\overline{g}\Big|_a^R + f\overline{g'}\Big|_a^R - f'\overline{g}\Big|_R^b + f\overline{g'}\Big|_R^b - \int_a^R f(t)\overline{g''(t)}dt - \int_R^b f(t)\overline{g''(t)}dt$$

$$= -f'(R)\overline{g(R-0)} - f'(b)\overline{g(b)} + f'(a)\overline{g(a)} + f'(R)\overline{g(R+0)} + \langle f, T_R^* g\rangle$$

$$= \langle f, T_R^* g \rangle.$$

Therefore,

$$f'(R)\overline{g(R-0)} + f'(b)\overline{g(b)} = f'(a)\overline{g(a)} + f'(R)\overline{g(R+0)}.$$

Taking f such that $f'(R) = 0$ we obtain $f'(b)\overline{g(b)} = f'(a)\overline{g(a)}$. If in addition $f'(b) = 1$, $f'(a) = 0$, then $g(b) = 0$. Similarly, when $f'(a) = 1$, $f'(b) = 0$, $g(a) = 0$.

Finally, $f'(R)\overline{g(R-0)} = f'(R)\overline{g(R+0)}$. Thus, if $f'(R) = 1$, $g(R-0) = g(R+0)$, which completes the proof for the inclusion

$$D_R^* \subseteq \{ y \in W_2^2(a,R) \oplus W_2^2(R,b) |\ y(a) = y(b) = 0,\ y(R-0) = y(R+0)\ \}.$$

Step 2.

Conversely, if $g \in \{ y \in W_2^2(a,R) \oplus W_2^2(R,b) |\ y(a) = y(b) = 0,\ y(R-0) = y(R+0)\ \}$, $f \in D_R$, the integration by parts twice gives

$$\langle T_R f, g \rangle = -\int_a^R f''(t)\overline{g(t)}dt - \int_R^b f''(t)\overline{g(t)}dt$$

$$= -f'\overline{g}\Big|_a^R + f\overline{g'}\Big|_a^R - f'\overline{g}\Big|_R^b + f\overline{g'}\Big|_R^b - \int_a^R f(t)\overline{g''(t)}dt - \int_R^b f(t)\overline{g''(t)}dt$$

$$= -f'(R-0)\overline{g(R-0)} + f'(R+0)\overline{g(R+0)} - \int_a^b f(t)\overline{g''(t)}dt = \langle f, T_R^* g \rangle.$$

Thus, $g \in D_R^*$, and we obtain the inverse inclusion.

□

Remark 4.2.2. There is another way to prove $T_R^* g = -g''$ by using the definitions of Sobolev space and generalized derivatives.

For any $f \in C_0^\infty(a,R) \subseteq D_R$ and $g \in D_R^*$

$$-\int_a^R f''(t)\overline{g(t)}dt = \langle T_R f, g \rangle = \langle f, T_R^* g \rangle = \int_a^R f(t)\overline{g^*(t)}dt,$$

where $g^* = T_R^* g$.

Therefore, $-g^*$, the L^2-function, is the 2^{nd} weak derivative of the function g on $(-\infty, R)$ by definition, i.e. $-g'' = T_R^* g$ on (a,R).

Lemma 4.2.3. *The deficiency indices* $\nu_\pm(T_R) = 1$.

Proof. First, let's prove that T_R is closed. Since D_R and D_R^* are dense in L^2, T_R^{**} exists and $\overline{T_R} = T_R^{**}$ (see [4, Ch.3, §3, p.70, Th.7]). The inclusion

$\overline{T_R} \supseteq T_R$ holds, as for any operator. Thus, it is sufficient to explain why $\overline{T_R} = T_R^{**} \subseteq T_R$.

Let's $f \in D_R^*, g \in D_R^{**}$(the domain of T_R^{**}). Since $T_R \subseteq T_R^*$,$T_R^{**} \subseteq T_R^*$, and thus, $f(a) = f(b) = g(a) = g(b) = 0$. Moreover, we can define $f(R) := f(R-0) = f(R+0)$ and similarly $g(R) := g(R-0) = g(R+0)$. Now, after the integration by parts on (a, R) and (R, b), as previously, the condition $\langle T_R^* f, g\rangle = \langle f, T_R^{**} g\rangle$ becomes

$$-\int_a^b f''(t)\overline{g(t)}dt = -f'\overline{g}\Big|_a^R + f\overline{g'}\Big|_a^R - f'\overline{g}\Big|_R^b + f\overline{g'}\Big|_R^b - \int_a^b f(t)\overline{g''(t)}dt$$

$$= -f(R)\cdot\overline{\big(g'(R+0) - g'(R-0)\big)} + g(R)\cdot\overline{\big(f'(R+0) - f'(R-0)\big)}$$

$$-\int_a^b f(t)\overline{g''(t)}dt = -\int_a^b f(t)\overline{g''(t)}dt.$$

Thus, for any $f \in D_R^*$

$$f(R)\cdot\overline{\big(g'(R+0) - g'(R-0)\big)} = g(R)\cdot\overline{\big(f'(R+0) - f'(R-0)\big)}.$$

Now, taking f such that $f(R) = f'(R+0) = 0$, $f'(R-0) = 1$ we obtain $g(R) = 0$. Taking f with $f(R) = 1$ we get

$$g'(R+0) - g'(R-0) = \overline{g(R)}\cdot(f'(R+0) - f'(R-0)) = 0.$$

Therefore, $g \in D_R$ which proves the inclusion $D_R^{**} \subseteq D_R$.

Let's find the deficiency indices. Since, T_R is a closed operator,

$$\nu_-(T_R) = \dim Ker(T_R^* + iI) = \dim\{y \in D_R^* | \; y'' = iy\}$$

In $W_2^2(a, R) \oplus W_2^2(R, b)$ the equation $y'' = iy$ has solution of the form

$$y = c_1\chi_{(a,R]}e^{\lambda x} + c_2\chi_{(a,R]}e^{-\lambda x} + c_3\chi_{(R,b)}e^{\lambda x} + c_4\chi_{(R,b)}e^{-\lambda x},$$

where $\pm\lambda$ are square roots of i ($\Re\lambda > 0$). Thus, the conditions $y(a) = y(b) = 0$, $y(R-0) = y(R+0)$ form a system of 3 linear independent equations

$$\begin{cases} c_1e^{\lambda a} + c_2e^{-\lambda a} = 0 \\ c_3e^{\lambda b} + c_4e^{-\lambda b} = 0 \\ (c_1 - c_3)e^{\lambda R} + (c_2 - c_4)e^{-\lambda R} = 0 \end{cases}$$

which has a one-dimensional space of solutions in terms of (c_1, c_2, c_3, c_4). Therefore, $\nu_-(T_R) = 1$. Similarly, $\nu_+(T_R) = 1$.

□

The lemmata above can be extended to the case $a = -\infty,\ b = \infty$.
Consider differential operator $T_R = -\Delta$ with

$$D_R = \{y \in W_2^2(-\infty, R) \oplus W_2^2(R, \infty) |\ y(R) = 0,\ y' \in C(R)\ \}$$

Lemma 4.2.4. *The operator T_R is symmetric, the adjoint operator T_R^* is defined by the same expression, i.e. $T_R^* = -\Delta$, and its domain is defined by*

$$D_R^* = \{y \in W_2^2(-\infty, R) \oplus W_2^2(R, \infty) |\ y(R-0) = y(R+0)\ \}.$$

Proof. The first part of the proof repeats Lemma 4.4.4.
Step 1.
Recall that for $f, g \in D_R$ the derivatives f' and g' are absolutely continuous. Moreover, $f, g, f', g', f'', g'' \in \mathrm{L}^2(\mathbb{R})$, thus (due to the same argument as in [4, Ch.4, §8, Lemma 1, p.120]),

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} f'(x) = \lim_{|x|\to\infty} g'(x) = \lim_{|x|\to\infty} g(x) = 0 \tag{4.2.2}$$

Thus, we can apply integration by parts

$$\langle T_R f, g\rangle = -\int_{-\infty}^{\infty} f''(t)\overline{g(t)}dt = -f'\overline{g}\Big|_{-\infty}^{\infty} + f\overline{g'}\Big|_{-\infty}^{\infty} + \langle f, T_R g\rangle = \langle f, T_R g\rangle.$$

Thus, T_R is symmetric. Let's find the domain of its adjoint.

Step 2.

Let's prove the inclusion $D_R^* \subseteq \{y \in W_2^2(-\infty, R) \oplus W_2^2(R, \infty) |\ y(R-0) = y(R+0)\}$.
For any $f \in C_0^\infty(-\infty, R) \subseteq D_R$ and $g \in D_R^*$

$$-\int_{-\infty}^{R} f''(t)\overline{g(t)}dt = \langle T_R f, g\rangle = \langle f, T_R^* g\rangle = \int_{-\infty}^{R} f(t)\overline{g^*(t)}dt,$$

where $g^* = T_R^* g$.
Therefore, $-g^*$, the L^2-function, is the 2^{nd} weak derivative of the function g on $(-\infty, R)$ by definition, i.e. $g'' = -T_R^* g$ on $(-\infty, R)$.
The existence of $g' \in \mathrm{L}^2(-\infty, R)$ immediately follows from the equivalence of the Sobolev space norms (see [16, Ch. 1, §112]). However, we prove this directly.

We claim the existence of g' in the weak sense and that this is an L^2-function.

Indeed, since $g, g'' \in L^2(\mathbb{R})$, their Fourier transforms, $\mathcal{F}(g), \mathcal{F}(g'')(\xi) = -\xi^2\mathcal{F}(\xi) \in L^2(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} |\mathcal{F}(g)(\xi)|^2 d\xi, \quad \int_{\mathbb{R}} |\xi|^4 |\mathcal{F}(g)(\xi)|^2 d\xi < \infty.$$

Additionally, since $2|\xi|^2 \leq 1 + |\xi|^4$,

$$\int_{\mathbb{R}} |\xi|^2 |\mathcal{F}(g)(\xi)|^2 d\xi < \infty.$$

Therefore $\xi\mathcal{F}(g) \in L^2(\mathbb{R})$. Denote $h = \mathcal{F}^{-1}\big(i\xi\mathcal{F}(g)\big) \in L^2(\mathbb{R})$.
For any $\phi \in C_0^\infty(\mathbb{R})$ by the Plancherel theorem

$$\langle g, \phi' \rangle = \langle \mathcal{F}(g), \mathcal{F}(\phi') \rangle = \langle \mathcal{F}(g), i\xi\mathcal{F}(\phi) \rangle = -\langle i\xi\mathcal{F}(g), \mathcal{F}(\phi) \rangle = -\langle h, \phi \rangle,$$

which by definition means that h is the 1^{st} weak derivative of function g.

In the same spirit we deal with functions g defined on (R, ∞).
The remaining part of the proof repeats the proof of Lemma 4.2.1.
Let's rewrite $\langle T_R f, g \rangle = \langle f, T_R^* g \rangle$ for $f \in D_R$, $g \in W_2^2(-\infty, R) \oplus W_2^2(R, \infty)$ using integration by parts separately on each of the intervals $(-\infty, R)$ and (R, ∞).

$$\langle T_R f, g \rangle = -\int_{-\infty}^{R} f''(t)\overline{g(t)}dt - \int_{R}^{\infty} f''(t)\overline{g(t)}dt$$

$$= -f'\overline{g}\Big|_{-\infty}^{R} + f\overline{g'}\Big|_{-\infty}^{R} - f'\overline{g}\Big|_{R}^{\infty} + f\overline{g'}\Big|_{R}^{\infty} - \int_{-\infty}^{R} f(t)\overline{g''(t)}dt - \int_{R}^{\infty} f(t)\overline{g''(t)}dt.$$

Since f and g satisfy (4.2.2) and $f(R-0) = f(R+0) = 0$,

$$f\overline{g'}\Big|_{-\infty}^{R} = f\overline{g'}\Big|_{R}^{\infty} = 0.$$

Due to $f'(R-0) = f'(R+0) = f'(R)$

$$f'\overline{g}\Big|_{-\infty}^{R} = f'(R)\overline{g(R-0)}, \quad f'\overline{g}\Big|_{R}^{\infty} = -f'(R)\overline{g(R+0)}$$

Therefore,

$$\langle T_R f, g \rangle = -f'(R)\overline{g(R-0)} + f'(R)\overline{g(R+0)} + \langle f, T_R^* g \rangle = \langle f, T_R^* g \rangle.$$

Therefore,

$$f'(R)\overline{g(R-0)} = f'(R)\overline{g(R+0)}.$$

Thus, if $f'(R) = 1$, then $g(R-0) = g(R+0)$, which completes the proof for the inclusion

$$D_R^* \subseteq \{y \in W_2^2(-\infty, R) \oplus W_2^2(R, \infty)|\ y(R-0) = y(R+0)\ \}.$$

Step 3.

Now let $g \in \{y \in W_2^2(-\infty, R) \oplus W_2^2(R, \infty)|\ y(R-0) = y(R+0)\ \}$ and $f \in D_R$.

The integration by parts twice on the intervals $(-\infty, R)$ and $(-R, \infty)$ gives

$$\langle T_R f, g\rangle = -\int_{-\infty}^{R} f''(t)\overline{g(t)}dt - \int_{R}^{\infty} f''(t)\overline{g(t)}dt$$

$$= -f'\overline{g}\Big|_{-\infty}^{R} + f\overline{g'}\Big|_{-\infty}^{R} - f'\overline{g}\Big|_{R}^{\infty} + f\overline{g'}\Big|_{R}^{\infty} - \int_{-\infty}^{R} f(t)\overline{g''(t)}dt - \int_{R}^{\infty} f(t)\overline{g''(t)}dt.$$

Since $g(R-0) = g(R+0)$, $f(R-0) = f(R+0) = 0$, $f'(R-0) = f'(R+0) = f'(R)$, and due to (4.2.2) the expression above equals

$$-\int_{-\infty}^{\infty} f(t)\overline{g''(t)}dt = \langle f, T_R^* g\rangle.$$

Thus, $g \in D_R^*$, and we obtain the other inclusion $\{y \in W_2^2(-\infty, R) \oplus W_2^2(R, \infty)|\ y(R-0) = y(R+0)\} \subseteq D_R^*$.

□

Lemma 4.2.5. *The deficiency indices $\nu_\pm(T_R) = 1$.*

Proof. The proof repeats the proof of Lemma 4.2.3.

In $W_2^2(-\infty, R) \oplus W_2^2(R, \infty)$ the equation $y'' = iy$ has two independent solutions, one per each half-line $(-\infty, R)$ and (R, ∞):

$$f_1(x) = c_1 e^{\lambda x}\chi_{(-\infty,R)}, \quad f_2(x) = c_2 e^{-\lambda x}\chi_{(R,\infty)},$$

where $\Re(\lambda) > 0$.

Since $f_1(R-0) = f_2(R+0)$,

$$c_1 e^{\lambda R} + c_2 e^{-\lambda R} = 0.$$

Therefore, it has a one-dimensional space of solutions in terms of (c_1, c_2). Hence, $\nu_-(T_R) = 1$. Similarly, $\nu_+(T_R) = 1$.

□

Definition 4.2.6. For $R > 0$ define operator $L_R := -\Delta$ on domain

$$D_R = \{y |\ y \in W_2^2(-\infty, -R) \oplus W_2^2(-R, R) \oplus W_2^2(R, \infty)\ , y(\pm R) = 0\}$$

and operator
$L_R^o := -\Delta$ on domain $\{y |\ y \in W_2^2(-R, R)\ , y(\pm R) = 0\}$.

A similar operator is considered later, in Proposition 4.4.1, the only difference being that now we consider the whole real number line $\mathbb{R}$. The following result explains why $L_R = -\Delta$ (unlike operators T_R discussed above) is a self-adjoint operator.

Lemma 4.2.7. *The operator $L_R = -\Delta$ defined on the domain D_R is self-adjoint.*

Proof. Note that for $f, g \in D_R$ the derivatives f' and g' are absolutely continuous. Moreover, $f, g, f', g', f'', g'' \in \mathrm{L}^2(\mathbb{R})$, thus,
(due to the same argument as in [4, Ch.4, §8, Lemma 1, p.120]), $\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} f'(x) = \lim_{|x|\to\infty} g'(x) = \lim_{|x|\to\infty} g(x) = 0$.
We claim that L_R is symmetric operator. Indeed, integration by parts twice for $f, g \in D_R$ gives

$$-\langle L_R f, g\rangle = \int_{-\infty}^{\infty} f''(t)\overline{g(t)}dt$$

$$= \int_{-\infty}^{-R} f''(t)\overline{g(t)}dt + \int_{-R}^{R} f''(t)\overline{g(t)}dt + \int_{R}^{\infty} f''(t)\overline{g(t)}dt$$

$$= f'\overline{g}\Big|_{-\infty}^{-R} - f\overline{g'}\Big|_{-\infty}^{-R} + f'\overline{g}\Big|_{-R}^{R} - f\overline{g'}\Big|_{-R}^{R} + f'\overline{g}\Big|_{R}^{\infty} - f\overline{g'}\Big|_{R}^{\infty}$$

$$+ \int_{-\infty}^{-R} f(t)\overline{g''(t)}dt + \int_{-R}^{R} f''(t)\overline{g(t)}dt + \int_{R}^{\infty} f''(t)\overline{g(t)}dt$$

$$= \int_{-\infty}^{\infty} f(t)\overline{g''(t)}dt = -\langle f, L_R g\rangle.$$

Let's prove that there exists 2^{nd} weak derivative f'' on each of the three intervals, $(-\infty, -R), (-R, R)$ and (R, ∞).
Take any $f \in C_0^\infty(-\infty, R) \subseteq D_R$ and $g \in D_R^*$ (the domain of the adjoint operator)

$$-\int_{-\infty}^{R} f''(t)\overline{g(t)}dt = \langle L_R f, g\rangle = \langle f, L_R^* g\rangle = \int_{-\infty}^{R} f(t)\overline{g^*(t)}dt,$$

where $g^* = L_R^* g$.
Therefore, $-g^*$, the L^2-function, is the 2^{nd} weak derivative of the function g on $(-\infty, R)$ by definition, i.e. $g'' = -L_R^* g$ on $(-\infty, R)$.

In the same spirit $g'' = -L_R^* g$ on $(-R, R)$ and on (R, ∞).

The existence of $g' \in \mathrm{L}^2(-\infty, R)$ immediately follows from the equivalence of the Sobolev space norms (see [Smi64, p.331]).

Therefore, $D_R^* \subseteq W_2^2(-\infty, -R) \oplus W_2^2(-R, R) \oplus W_2^2(R, \infty)$.

Since, L_R is symmetric, $D_R \subseteq D_R^*$. It remains to prove that $D_R^* \subseteq D_R$. Indeed, for any $f \in D_R, g \in D_R^*$ repeat integration by parts:

$$-\langle L_R f, g\rangle = \int_{-\infty}^{\infty} f''(t)\overline{g(t)}dt = \int_{-\infty}^{-R} f''(t)\overline{g(t)}dt + \int_{-R}^{R} f''(t)\overline{g(t)}dt$$

$$+ \int_{R}^{\infty} f''(t)\overline{g(t)}dt$$

$$= f'\overline{g}\Big|_{-\infty}^{-R} - f\overline{g'}\Big|_{-\infty}^{-R} + f'\overline{g}\Big|_{-R}^{R} - f\overline{g'}\Big|_{-R}^{R} + f'\overline{g}\Big|_{R}^{\infty} - f\overline{g'}\Big|_{R}^{\infty} + \int_{-\infty}^{\infty} f(t)\overline{g''(t)}dt$$

$$= f'(-R-0)\overline{g}(-R-0) + f'\overline{g}\Big|_{-R}^{R} - f'(R+0)\overline{g}(R+0) + \int_{-\infty}^{\infty} f(t)\overline{g''(t)}dt$$

$$= -\langle f, L_R^* g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g''(t)}dt.$$

Thus,
$f'(-R-0)\overline{g}(-R-0) + f'(R-0)\overline{g}(R-0) - f'(-R+0)\overline{g}(-R+0) - f'(R+0)\overline{g}(R+0) = 0.$
For a fixed f denote

$$f_1 = f'(-R-0), \quad f_2 = f'(-R+0), \quad f_3 = f'(R-0), \quad f_4 = f'(R+0) \tag{4.2.3}$$

Function f' need not be continuous at points R and $-R$. Therefore, for any four values f_i, $i = 1, 2, 3, 4$, one can find a proper function $f \in D_R$ satisfying (4.2.3).

Now, taking f such that $f_1 = 1, f_2 = f_3 = f_4 = 0$, we obtain $g(-R-0) = 0$. Similarly, one can prove $g(-R+0) = g(R-0) = g(R+0) = 0$, which implies that $D_R^* \subseteq D_R$. Therefore, L_R is a self-adjoint operator.

□

Definition 4.2.8. The *decoupling* of L_R is the operator $L_R^c = -\Delta$ on domain

$$D_R^c = \{y|\ y \in W_2^2(-\infty, R) \oplus W_2^2(-R, R) \oplus W_2^2(R, \infty),$$

$$y(\pm R) = 0,\ y'(\pm R - 0) = y'(\pm R + 0)\ \}$$

Remark 4.2.9. For the operator L_R^c we can obtain a similar result (as in previous Lemma 4.2.4), i.e. the domain of its adjoint

$$D_R^{c\,*} = \{y \in W_2^2(-\infty, -R) \oplus W_2^2(-R, R) \oplus W_2^2(R, \infty)|$$

$$y(-R+0) = y(-R-0),\ y(R+0) = y(R-0)\}$$

with $\nu_\pm(L_R^c) = 2$.
Indeed, if a function $y \in W_2^2(-\infty, -R) \oplus W_2^2(-R, R) \oplus W_2^2(R, \infty)$ satisfies $y'' = iy$, then

$$y = c_1\chi_{(-\infty,-R]}e^{\lambda x} + c_2\chi_{(-R,R)}e^{-\lambda x} + c_3\chi_{(-R,R)}e^{\lambda x} + c_4\chi_{[R,\infty)}e^{-\lambda x},$$

where $\pm\lambda$ are square roots of i ($\Re\lambda > 0$). In other words there are four independent solutions: one per each half-line $(-\infty, -R)$ and (R, ∞), two one the finite interval $(-R, R)$.
Thus, (c_1, c_2, c_3, c_4) must obey the following system of 2 linear independent linear equations

$$\begin{cases} y(-R-0) = c_1e^{-\lambda R} = c_2e^{\lambda R} + c_3e^{-\lambda R} = y(-R+0) \\ y(R-0) = c_4e^{-\lambda R} = c_2e^{-\lambda R} + c_3e^{\lambda R} = y(R-0) \end{cases},$$

which has exactly 2 independent solutions in terms of (c_1, c_2, c_3, c_4).

Remark 4.2.10. We single out two operators, L_R and $H_{(-\infty,\infty)} + p$ which are two self-adjoint extensions of the operator L_R^c. This helps to compute the spectral counting function of corresponding operators using the theory of self-adjoint extensions (Proposition 5.3.5)
Before the description of the reduction procedure in Chapter 5 we need to state some auxiliary results.

4.3 Small perturbations of DO

Let's recall the notion of operator domination.

Definition 4.3.1. (see [4, Ch.3, §4.1]) Let T and S be two operators defined on a Hilbert space H with the domains D_T and D_S, respectively, s. t. $D_T \subseteq D_S$.
S is said to be dominated by T if there exist such constants $a, b \geq 0$ that

$$\|Sf\| \leq a\|Tf\| + b\|f\|, \quad \forall f \in D_T$$

If in addition there exists $a < 1$ satisfying the estimate above, we say that S is strongly dominated by T.

Clearly, for any function $p \in \mathrm{L}^\infty(\mathbb{R})$ the multiplication operator p defined

on L^2 is strictly dominated by any other operator defined on L^2 with any constant $a \in [0,1)$ and $b = \sup |p|$.
The following three propositions ([4, Ch.3, §4, Th. 2, 3, p.74] and [4, Ch.4, §1, Th.9, p.100]) help to describe the basic properties of the DO perturbations.
It turns out that closeness, self-adjointness and deficiency indices are stable under strongly dominated perturbations.

Proposition 4.3.2. *Let T be a closed operator with the domain D_T and S be strongly dominated by T. Then $T + S$ defined on D_T is closed*

Proposition 4.3.3. *Let T be closed and densely defined. Suppose S and S^* are strongly dominated by T and T^* respectively. Then operators $(T + S)^*$ and T^* have the same domain and*

$$(T + S)^* = T^* + S^*.$$

Corollary 4.3.4. *If T is a self-adjoint operator, then for any $p \in L^\infty(\mathbb{R})$ the operator $T - p$ is self-adjoint as well. In particular, $L_R - p$ is a self-adjoint operator.*

Proposition 4.3.5. *Let S and T be symmetric operators. Suppose T is closed with the domain D_T and S is strongly dominated by T. Then the symmetric operator $S + T$ is closed on D_T and*

$$\nu_\pm(T) = \nu_\pm(S + T)$$

Corollary 4.3.6. *If $p = gV \in L^\infty(\mathbb{R})$, then the operator $L^c_R - p$ is closed and*

$$\nu_\pm(L^c_R - p) = \nu_\pm(L^c_R) = 2.$$

4.4 Schrödinger operators with Dirichlet boundary conditions

In this section we consider two Schrödinger operators with Dirichlet boundary conditions, which form an essential part of the reduction process.

The following result (see [17, Th. 6.4., p.31]) helps to estimate the counting function $\#(\lambda; -p; (-R, R))$ of a Schrödinger operator $-\Delta - p$ on a finite interval with Dirichlet boundary conditions $f(-R) = f(R) = 0$ (i.e. we cinsider operator $L^o_R - p$). We adapt the theorem for our needs and state the result in a suitable form.

Proposition 4.4.1. *Let $p \in C^1(\mathbb{R})$ be a non-negative function, and let $R > 0$. Then for any $\lambda \in \mathbb{R}$ one has*

$$\left|\#(\lambda; -p; (-R, R)) - \frac{1}{\pi}\int_{(-R,R)} \sqrt{p(t)}dt\right| \leq \int_{(-R,R)} \frac{|p'(t)|}{4\pi(p(t) + |\lambda|)}dt$$

$$+\frac{6\sqrt{|\lambda| + 1}}{\pi} \cdot R + 1.$$

We will see in Chapter 5 that this result corresponds to the Weyl asymptotics, (2.1.1), where the spectral counting function $\#(\cdot)$ is approximated by the spectral volume function $V(\frac{1}{x})$ which is exactly equal to $\frac{1}{\pi}\int_{(-x,x)} \sqrt{p(t)}dt$. The exact correspondence is provided by Birman-Schwinger principle, Theorem 5.3.4 in Section 5.3.

Corollary 4.4.2. *Suppose for all x the potential $p(x) = gV(x)$ satisfies the following conditions*

$$C_1\langle x\rangle^{-\alpha} \leq V(x) \leq C_2\langle x\rangle^{-\alpha},$$

$$|V'(x)| \leq C_0\langle x\rangle^{-(\alpha/2+1+\epsilon)}$$

$$\textit{where } C_0, C_1, C_2, \epsilon > 0,\ \alpha \geq 2,$$

Then for $R = g^{\frac{1}{\alpha}}$ the counting function
$\#(-\lambda; -p; (-R, R)) = \#(-\lambda, L_R^o - p)$ satisfies

$$C_{\alpha,1}g^{\frac{1}{2}}(1 + \overline{o}(1)) \leq \#(-\lambda, L_R^o - p) \leq C_{\alpha,2}g^{\frac{1}{2}}(1 + \overline{o}(1)),\ \textit{if } \alpha > 2,$$

$$\frac{\sqrt{C_1}}{\pi} \cdot g^{\frac{1}{2}} \log g\ (1 + \overline{o}(1)) \leq \#(-\lambda, L_R^o - p)$$

$$\leq \frac{\sqrt{C_2}}{\pi} \cdot g^{\frac{1}{2}} \log g\ (1 + \overline{o}(1)),\ \textit{if } \alpha = 2,$$

where $C_{\alpha,j}$ (depends on α only) are positive constants. All asymptotic estimates above are considered when $g \to \infty$.
If $p(x) = g\langle x\rangle^{-2}$, i.e. if $C_1 = C_2 = 1$, then

$$\#(-1; -p; (-R, R)) = \frac{1}{\pi}g^{\frac{1}{2}} \log g\ (1 + \overline{o}(1)).$$

Proof. First, we have to estimate the "main part" of the formula, the integral $\frac{1}{\pi}\int_{(-R,R)} \sqrt{p(t)}dt$, and then "the error term" consisting of two parts, the integral $\frac{1}{4\pi}\int_{(-R,R)} \frac{|p'(t)|}{p(t)+1}dt$, and the summand $\frac{6\sqrt{2}}{\pi} \cdot R = Cg^{\frac{1}{\alpha}}$. We will see that the error term is significantly less than the main part for $\alpha \geq 2$, however, it has the same order when $\alpha \in (0, 2)$.

Let's go through the details.

$$\sqrt{C_1}\cdot\sqrt{g}\int_{-R}^{R}\langle x\rangle^{-\frac{\alpha}{2}}dx \le \int_{(-R,R)}\sqrt{p(t)}dt \le \sqrt{C_2}\cdot\sqrt{g}\int_{-R}^{R}\langle x\rangle^{-\frac{\alpha}{2}}dx$$

where

$$\sqrt{g}\int_{0}^{R}\langle x\rangle^{-\frac{\alpha}{2}}dx = \begin{cases} g^{\frac{1}{2}}\log g(1+\overline{o}(1)),\ \alpha=2 \\ C_\alpha g^{\frac{1}{2}},\ \alpha\in(2,\infty)\end{cases}$$

To estimate the remainder we use the asymptotic properties of p' and p.

$$\int_{(-R,R)}\frac{|p'(t)|}{p(t)+1}dt \le \int_{(-R,R)}\frac{C_0 g\langle t\rangle^{-\alpha/2-1-\epsilon}}{C_1 g\langle t\rangle^{-\alpha}}dt \lesssim_{C_0,C_1,\alpha,\epsilon} R^{\alpha/2-\epsilon} = g^{1/2-\epsilon/\alpha}.$$

The following two estimates

$$g^{\frac{1}{2}}\log g \pm \left(\underline{O}(g^{\frac{1}{2}-\frac{\epsilon}{2}})+C\cdot R\right) = g^{\frac{1}{2}}\log g(1+\overline{o}(1)),$$

$$C_\alpha g^{\frac{1}{2}}\pm\left(\underline{O}(g^{\frac{1}{2}-\frac{\epsilon}{\alpha}})+C\cdot R\right) = C_\alpha g^{\frac{1}{2}}\log g(1+\overline{o}(1)) \text{ where } R=g^{\frac{1}{\alpha}}=\overline{o}(g^{\frac{1}{2}}),\ \alpha>2,$$

finish the proof.

□

Remark 4.4.3. The asymptotic formula

$$\#(-1;-p;(-R,R)) = \frac{1}{\pi}g^{\frac{1}{2}}\log g\ (1+\overline{o}(1)).$$

still holds for $V(x)=\frac{\zeta(x)^2}{x^2}\sim\frac{1}{\langle x\rangle^2},\ x\to\infty$. The same estimates in the proof of the Corollary can be applied.

It turns out that for a convenient potential function p the two operators mentioned above, L_R^o-p and L_R-p, have "almost" the same number of eigenvalues (for a proper potential p), since L_R-p can be decomposed in the orthogonal sum $L_R-p=\mathrm{H}_{(-\infty,-R)}\oplus\mathrm{H}_{(-R,R)}\oplus\mathrm{H}_{(R,\infty)}$, Namely, the following lemma holds.

Lemma 4.4.4. *Let* $R=g^{\frac{1}{\alpha}}$, $p=g\cdot\frac{1}{\langle x\rangle^\alpha}$ *or* $p=g\cdot\frac{\zeta^\alpha(x)}{x^\alpha}$, $\alpha>0$. *Then*

$$\#(-1,L_R-p)=\#(-1,L_R^o-p).$$

Proof. Note that $\#(-1;-p;(-\infty,-R))=\#(-1;-p;(-R,\infty))=0$. Indeed, consider $x>R=g^{\frac{1}{\alpha}}$. Then $-p>-I$. Since $-\Delta>0$, the operator $-\Delta-p+I>0$. Hence, all eigenvalues of this operator are positive.

Since every $f\in D_R$ can be represented as the orthogonal sum $f=f_1+f_2+f_3$, where $f_1=f\chi_{(-\infty,-R)}$, $f_2=f\chi_{(-R,R)}$ and $f_3=f\chi_{(R,\infty)}$, there is one-to-one correspondence between the eigenfunctions (and eigenvalues) of the operator L_R and the eigenfunctions (and eigenvalues) of $\mathrm{H}_{(-\infty,-R)} \oplus \mathrm{H}_{(-R,R)} \oplus \mathrm{H}_{(R,\infty)}$. Hence,

$$\#(-1, L_R - p) = \#(-1, L_R^o - p)$$

$$+ \#(-1; -p; (-\infty, -R)) + \#(-1; -p; (-R, \infty)) = \#(-1, L_R^o - p).$$

□

Chapter 5

Weyl discontinuous symbol. Asymptotic estimates and formulae.

In this Chapter we analyse the case when the signal is concentrated on an angular domain of the phase space.
Consider the following operator

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma)u(x) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{i\xi(x-y)}\cdot\sigma\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi \tag{5.0.1}$$

where symbol $\sigma(t,\xi) := \chi_{\{t\geq 0\}}(t)\cdot\chi_{\{\xi\leq ct\}}(t,\xi)\cdot a(t,\xi)$, $a\in C_0^\infty(\mathbb{R}\times\mathbb{R})$ with supp $a\subseteq[-R,R]^2$ and $c\in\mathbb{R}$.

We will see below (Remark 5.1.7) that the spectral properties of the operator do not depend on c, thus we can consider $c=0$, i.e. we can reduce the angular region to a region with a right angle. To estimate singular values of (5.0.1) we implement the following approach.

We reduce (5.0.1) to a different pseudo-differential operator with a smooth slowly decaying symbol $\tilde{\sigma}$ using the process described in Theorem 5.1.3. For this type of symbols we have a toolkit described in Section 3.3. Correctly splitting the symbol and the kernel of the operator (the process is described in detail in Lemma 5.1.4 and Theorem 5.1.9) we obtain the main part of the operator, which is unitarily equivalent to the operator $\frac{a(0,0)}{4\pi}\mathrm{Op}_1^{\mathrm{a}}\big(\frac{\zeta_\delta(t)}{t}\frac{1}{\langle\xi\rangle}\big)$. Then we apply Theorem 3.3.1 to obtain the asymptotic estimate $\underline{O}(\log(k+1)k^{-1})$. The remainder part belongs to the class $\mathbb{S}_{1,\infty}$.
ΨDO with symbol $\frac{\zeta_\delta(t)}{t}\frac{1}{\langle\xi\rangle}$ has the same asymptotic decay of singular values as ΨDO with symbol $\frac{1}{\langle t\rangle\langle\xi\rangle}$. Therefore, both symbols can be used as the main part of the symbol partition. However, the multiple $\frac{\zeta_\delta(t)}{t}$

containing the cut-off function ζ_δ (the parameter $\delta = 2R$ as we will see below) provides some essential properties of the operator kernel (see Example 5.1.2) making the spectrum symmetric.
At the end of the Chapter we derive the asymptotic formula for the eigenvalues of (5.0.1).

The Chapter is structured as follows. Section 5.1 contains some auxiliary facts (Lemma 5.1.1 about spectrum symmetry and Theorem 5.1.3 about symbol reassembling), which help to proceed with the process of splitting the model operator in such a way that the main part of this partition is a self-adjoint compact operator T_1 (see Theorem 5.1.9) with a symmetric spectrum, and, thus, has the same asymptotic estimates for its eigenvalues as for its singular values.
In the last two sections we derive the asymptotic formula (1.2.2) using two different methods. In Section 5.2 we use the Weyl asymptotic law described in Chapter 2. In Section 5.3 we use the results of Chapter 4 and the Birman-Schwinger principle to reduce the asymptotic formulae for DO (of a special type) to the one for the model ΨDO.

5.1 Eigenvalues and asymptotic estimates.

To estimate the eigenvalues of the operator (5.0.1) we split it in such a way that "the main part" has symmetric (with respect to the origin) spectrum, and thus, the estimates for positive eigenvalues $\lambda_1^+ \geq \lambda_2^+ \geq \dots \lambda_k^+ > 0$ (as well as for negative, $-\lambda_1^- \leq -\lambda_2^- \leq \dots - \lambda_k^- < 0$) are the same as for the singular values.
The auxiliary lemma below describes a general case of an operator having symmetric point spectrum.

Lemma 5.1.1. *Let W be a linear space which can be expressed as a direct sum $W = W_1 \oplus W_2$. Let T be a linear operator on this space such that $TW_1 \subseteq W_2$ and $TW_2 \subseteq W_1$.*
Then the point spectrum of T is a symmetric set, i.e. if λ is an eigenvalue of T, then $-\lambda$ is its eigenvalue as well. Moreover, the eigenvalues $\lambda, -\lambda$ of finite multiplicity have the same algebraic and geometric multiplicity.

Proof. Let V_λ be the eigenspace corresponding to the eigenvalue λ. Take any $v \in V_\lambda$. If v admits the following decomposition, $v = v_1 + v_2$ where $v_j \in W_j$, then

$$Tv_1 + Tv_2 = Tv = \lambda v = \lambda v_2 + \lambda v_1.$$

Since $Tv_1, v_2 \in W_2$ and $Tv_2, v_1 \in W_1$,

$$Tv_1 = \lambda v_2, \quad Tv_2 = \lambda v_1.$$

Therefore,

$$T(v_1 - v_2) = Tv_1 - Tv_2 = \lambda v_2 - \lambda v_1 = -\lambda(v_1 - v_2),$$

Hence, $u = v_1 - v_2$ is an eigenvector corresponding to the eigenvalue $-\lambda$. If $V_{-\lambda}$ is the corresponding eigenspace, then linear map $S : V_\lambda \to V_{-\lambda}$ defined by $S(v) = S(v_1 + v_2) = v_1 - v_2$ is a bijection, which implies that eigenspaces are of the same dimension.
Now let's prove that algebraic multiplicities coincide as well. It is sufficient to show that for any integer $n > 0$ the linear map S defined as previously (by $S(v) = S(v_1 + v_2) = v_1 - v_2$) is a bijective correspondence between subspaces $\operatorname{Ker}(T - \lambda I)^n$ and $\operatorname{Ker}(T + \lambda I)^n$. This implies the equality of algebraic multiplicities, since in this case

$$\dim \operatorname{Ker}(T - \lambda I)^n = \dim \operatorname{Ker}(T + \lambda I)^n.$$

Proceed by induction. If $n = 1$, then

$$v_1 + v_2 \in \operatorname{Ker}(T - \lambda I) = V_\lambda \iff v_1 - v_2 \in V_{-\lambda} = \operatorname{Ker}(T + \lambda I)$$

as previously shown.

Let the statement be true for some $n = k$.
Consider $v = v_1 + v_2 \in \mathrm{Ker}(T - \lambda I)^{k+1}$ where $v_j \in W_j$. Note that

$$0 = (T - \lambda I)^{k+1} v = (T - \lambda I)^k (T - \lambda I)(v_1 + v_2)$$

$$= (T - \lambda I)^k (T v_2 - \lambda v_1 + T v_1 - \lambda_1) = (T - \lambda I)^k (u_1 + u_2),$$

where $u_1 = T v_2 - \lambda v_1 \in W_1$ and $u_2 = T v_1 - \lambda v_2 \in W_2$.
Using the induction step, this implies that

$$u_1 - u_2 \in \mathrm{Ker}(T + \lambda I)^k,$$

which means

$$0 = (T + \lambda I)^k (u_2 - u_1) = (T + \lambda I)^k (T v_1 - \lambda v_2 - T v_2 + \lambda v_1)$$

$$= (T + \lambda I)^k (T + \lambda I)(v_1 - v_2) = (T + \lambda I)^{k+1} (v_1 - v_2).$$

Hence, $v = v_1 - v_2 \in \mathrm{Ker}(T + \lambda I)^{k+1}$.

□

Example 5.1.2. Consider the following $\mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ integral operator

$$(Tu)(x) = \int_{\mathbb{R}} K(x, y) u(y) dy,$$

where kernel $K(x, y)$ satisfies

$$\mathrm{supp} K \subseteq \{(x, y) |\ xy < 0\} \tag{5.1.1}$$

If $Tu = \lambda u$ for some function $u \neq 0$, then there exists a function $v \neq 0$ such that $Tv = -\lambda v$ (i.e. T has symmetric point spectrum with respect to the origin). Moreover, λ and $-\lambda$ have the same multiplicity.

Proof. Take $W_1 = \{f \in \mathrm{L}^2(\mathbb{R}) \mid \mathrm{supp}\ f \subseteq (-\infty, 0]\}$, $W_2 = \{f \in \mathrm{L}^2(\mathbb{R}) \mid \mathrm{supp}\ f \subseteq [0, \infty)\}$ and apply Lemma 5.1.1

□

An important technique presented in the theorem below describes the main idea of the reduction process. We "reassemble" the symbol σ such that the new symbol $\tilde{\sigma}$ is more convenient to work with in terms of spectral estimates.

Theorem 5.1.3. *If symbol $\sigma = \sigma(t, \xi) \in \mathrm{L}^2(\mathbb{R}^2)$, then*

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma) = \int_{\mathbb{R}} K\Big(\frac{x + y}{2}, x - y\Big) u(y) dy, \ \textit{where } K(\lambda, \mu) = \frac{1}{2\pi} \int e^{i\mu r} \sigma(\lambda, r) dr.$$

Moreover,

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma)P = \mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma}),$$

where $Pu(x) = u(-x)$ *is the flip operator and*

$$\tilde{\sigma}(t,\xi) = \int e^{-is\xi} K\Big(\frac{s}{2}, 2t\Big) ds = \frac{1}{\pi} \iint e^{i(2tr - s\xi)} \sigma(s,r) dr ds.$$

Let T *be an integral operator defined by* $Tu(x) = \int_{\mathbb{R}} K\Big(\frac{x+y}{2}, x-y\Big)u(y)dy$ *on* $\mathrm{L}^2(\mathbb{R})$ *where* K *has support in* $[a,b]\times\mathbb{R}$, $K(\cdot,\mu) \in C^2(\mathbb{R})$, $\partial_\lambda^m K(\lambda,\cdot) \in \mathrm{L}^1(\mathbb{R})$ *for any* $\lambda \in [a,b], \mu \in \mathbb{R}$ *and* $m = 0,1,2$. *Then*

$$s_k(T) = s_k\Big(\mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma})\Big).$$

REMARK. *Since* P *is a unitary operator,* $s_k\Big(\mathrm{Op}_1^{\mathrm{W}}(\sigma)\Big) = s_k\Big(\mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma})\Big)$.

Proof. Let's represent the pseudo-differential operator in the form of an integral operator. First, consider $\sigma \in C_0^\infty(\mathbb{R}^2)$. Due to Fubini's theorem

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma)u(x) = \frac{1}{2\pi} \cdot \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi} \cdot \sigma\Big(\frac{x+y}{2}, \xi\Big) u(y) dy d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi} \cdot \sigma\Big(\frac{x+y}{2}, \xi\Big) d\xi u(y) dy = \int_{\mathbb{R}} K\Big(\frac{x+y}{2}, x-y\Big) u(y) dy,$$

where kernel $K(\lambda,\mu) = \frac{1}{2\pi}\int e^{i\mu s}\sigma(\lambda,s)ds \in C^\infty(\mathbb{R}^2)$.

Let $Pu(y) = u(-y)$ be the flip operator. Since this is a unitary operator, $s_k\Big(\mathrm{Op}_1^{\mathrm{W}}(\sigma)\Big) = s_k\Big(\mathrm{Op}_1^{\mathrm{W}}(\sigma)P\Big)$. Hence, we can proceed with the latter one.

Using the Fourier inversion theorem,
$K(\frac{\lambda}{2},\mu) = \frac{1}{2\pi} \cdot \iint_{\mathbb{R}^2} e^{i(\lambda - t)\xi} K(\frac{t}{2},\mu) dt d\xi$, we obtain

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma)Pu(x) = \int_{\mathbb{R}} K\Big(\frac{x-y}{2}, x+y\Big) u(y) dy$$

$$= \frac{1}{2\pi} \cdot \int_{\mathbb{R}} \iint_{\mathbb{R}^2} \Big(e^{i(x-y-t)\xi} K\Big(\frac{t}{2}, x+y\Big) dt\, d\xi\Big) u(y) dy = \iint F(x,\xi,y) d\xi u(y) dy,$$

where $F(x,\xi,y) = (2\pi)^{-1}\int_{\mathbb{R}} e^{i(x-y-t)\xi} K\Big(\frac{t}{2}, x+y\Big)dt$. If we can interchange the order of integration, then

$$\iint F(x,\xi,y) d\xi u(y) dy = \iint F(x,\xi,y) u(y) dy d\xi$$

$$= \frac{1}{2\pi} \cdot \int_{\mathbb{R}^2} e^{i(x-y)\xi} \Big(\int_{\mathbb{R}} e^{-it\xi} K\Big(\frac{t}{2}, x+y\Big) dt\Big)\ u(y) dy d\xi$$

$$= \mathrm{Op}_1^{\mathrm{a}}\left(\int_{\mathbb{R}} e^{-it\xi} K\Big(\frac{t}{2}, x+y\Big) dt \right) u(x) = \mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma}) u(x),$$

which leads to the statement of the theorem.
It is sufficient to prove

$$\iint F(x,\xi,y)u(y)d\xi dy = \iint F(x,\xi,y)u(y)dyd\xi$$

for $u \in C_0^\infty(\mathbb{R})$ (on the Schwartz space) and then continuously extend the identity to $\mathrm{L}^2(\mathbb{R})$.

Since for any μ supp $K(\cdot,\mu) \subseteq$ supp $\sigma(\cdot,\mu) \subseteq [a,b]$ for some $a,b \in \mathbb{R}$, due to Corollary 3.3.13 $|F(x,y,\xi)| \lesssim_\sigma \langle y\rangle^{-2}\langle \xi\rangle^{-2}$ for any x. Hence, $F(x,\cdot,\cdot) \in \mathrm{L}^1(\mathbb{R}^2)$ for any x, and we can apply Fubini's theorem to change the order of integration. Therefore,

$$s_k\big(\mathrm{Op}_1^{\mathrm{W}}(\sigma)\big) = s_k\Big(\mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma})\Big).$$

Any $\sigma \in \mathrm{L}^2$ can be approximated by a sequence of smooth symbols $\sigma_n \in C_0^\infty$. Due to Plancherel theorem

$$\big\|\mathrm{Op}_1^{\mathrm{W}}(\sigma)\big\|_{\mathbb{S}_2} = \Big\|K\Big(\frac{x-y}{2}, x+y\Big)\Big\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \iint \Big|K\Big(\frac{x-y}{2}, x+y\Big)\Big|^2 dxdy$$

$$= \iint |K(\lambda,\mu)|^2 d\mu d\lambda = \frac{1}{2\pi}\iint |\mathcal{F}^{-1}(\sigma(\lambda,\cdot)(\mu))|^2 d\mu d\lambda$$

$$= \frac{1}{2\pi}\iint |\sigma(\lambda,\mu)|^2 d\mu d\lambda = \frac{1}{2\pi}\|\sigma\|^2_{\mathrm{L}^2(\mathbb{R}^2)}.$$

Thus, convergence $\sigma_n \to \sigma$ in L_2 implies the convergence $\mathrm{Op}_1^{\mathrm{W}}(\sigma_n) \to \mathrm{Op}_1^{\mathrm{W}}(\sigma)$ in the Hilbert-Schmidt norm, which implies the convergence in the operator norm. Note that $\sigma_n \to \sigma$ in L^2 also implies $\tilde{\sigma}_n \to \tilde{\sigma}$. Indeed, since

$$\tilde{\sigma}(t,\xi) = 2\mathcal{F}_{\lambda\to\xi}\Big[\mathcal{F}^{-1}_{\xi\to t}[\sigma(\lambda,\xi)]\Big(\frac{\lambda}{2}, 2t\Big)\Big](t,\xi),$$

due to the Plancherel theorem

$$\|\tilde{\sigma}_n - \tilde{\sigma}\|_{\mathrm{L}^2} = 2\|\sigma_n - \sigma\|_{\mathrm{L}^2} \to 0.$$

Therefore,

$$\mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma}_n) \to \mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma}),$$

which finishes the proof. □

Recall the main operator (5.0.1)

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma)u(x) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{i\xi(x-y)} \cdot \sigma\Big(\frac{x+y}{2}, \xi\Big)u(y)dyd\xi$$

with $\sigma(t,\xi) = \chi_{\{t\geq 0\}}(t) \cdot \chi_{\{\xi \leq ct\}}(t,\xi) \cdot a(t,\xi)$, where
$a \in C_0^\infty(\mathbb{R}\times\mathbb{R})$ with supp $a \subseteq [-R,R]^2$.
The initial symbol σ is discontinuous but compactly supported. Since the kernel of the operator $\mathrm{Op}_1^{\mathrm{W}}(\sigma)$ is a discontinuous function, we cannot apply Theorems 3.2.1, 3.2.3 directly. Instead, we can reduce symbol σ to a new equivalent one (in terms of spectral analysis), symbol $\tilde{\sigma}$, via Theorem 5.1.3. It turns out that the new symbol $\tilde{\sigma}$ is no longer compactly supported; however, it is a smooth function with a low rate of decay. The tools introduced in Section 3.3 help to estimate singular values for such type of symbols.

To estimate the eigenvalues of the model operator we split symbol
$\sigma\big(\frac{x+y}{2},\xi\big) = \sigma_1(x,y,\xi) + \sigma_2(x,y,\xi)$, where

$$\sigma_1 = \zeta_\delta(x-y)\sigma, \quad \sigma_2 = (1-\zeta_\delta(x-y))\sigma, \quad \delta = 2R \tag{5.1.2}$$

The first symbol σ_1 is supported outside the strip $\{|x-y| > \delta\}$, thus, we do not deal with the singularities when analysing the functions like $\frac{\sigma}{x-y}$ (for example, if we integrate by parts). Conversely, the other symbol, σ_2, allows to apply specific estimates since x and y are relatively close to each other.

We need this split to emphasise the main part of the symbol σ which corresponds to the operator $\mathrm{Op}_1^{\mathrm{a}}(\sigma_1)$. This operator, in turn, can be decomposed into a sum of two operators, $T_1 + T_2$, s.t. T_1 has a kernel supported on $\{xy < 0\}$ (see Example 5.1.2) and, thus, a symmetric spectrum (which implies the same rate of decay for eigenalues as for singular values, which is $\underline{O}(k^{-1}\log k)$ as we will see below). The operator T_2 along with $\mathrm{Op}_1^{\mathrm{a}}(\sigma_2)$ are the remainder terms, which have a faster rate of singular values decay, which is $\underline{O}(k^{-1})$.
As follows from Theorem 2.2.14, the spectral properties of the main operator $\mathrm{Op}_1^{\mathrm{W}}(\sigma)$ are the same as the spectral properties of its main part T_m in the representation

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma) = T_m + T_r,$$

where $T_m := T_1 \in \mathfrak{S}_1$ and $T_r := T_2 + \mathrm{Op}_1^{\mathrm{a}}(\sigma_2) \in \mathbb{S}_{1,\infty}$.

The lemmata below explain this in detail.

Lemma 5.1.4.

$$\mathrm{Op}_1^{\mathrm{a}}(\sigma_1) = T_1 + T_2,$$

where T_1 is a self-adjoint compact operator with a symmetric (with respect to the origin) spectrum satisfying (5.1.1) and defined by

$$T_1u(x) = -\frac{i}{2\pi}\int_{\mathbb{R}} \chi_{\{x+y\geq 0\}} \cdot e^{i\frac{c(x^2-y^2)}{2}} \cdot \frac{\zeta_{2R}(x-y)a\left(\frac{x+y}{2}, \frac{c(x+y)}{2}\right)}{x-y} u(y)dy,$$

$\mathrm{Op}_1^{\mathrm{a}}(\sigma_1), T_1 \in \mathfrak{S}_1,\ T_2 \in \mathbb{S}_{1,\infty}$.

Proof. Step 1.
Due to Theorem 5.1.3 the kernel $K\left(\frac{x+y}{2}, x-y\right)$ of the operator $\mathrm{Op}_1^{\mathrm{a}}(\sigma_1)$ is represented by

$$K(\lambda,\mu) = \frac{1}{2\pi}\chi_{\{\lambda\geq 0\}} \cdot \zeta_\delta(\mu)\int_{(-\infty,c\lambda]} e^{i\mu\xi}\cdot a(\lambda,\xi)d\xi$$

$$= -\frac{i}{2\pi}\chi_{\{\lambda\geq 0\}}e^{i\mu c\lambda}\sum_{j=0}^{N} i^j\partial_\xi^j a(\lambda,c\lambda)\frac{\zeta_\delta(\mu)}{\mu^{j+1}} + \chi_{\{\lambda\geq 0\}}F(\lambda,\mu)\frac{\zeta_\delta(\mu)}{\mu^N} \quad (5.1.3)$$

where $\delta = 2R$, $F(\cdot,\mu) \in C_0^\infty(\mathbb{R})$ and $\left|\partial_\lambda^l\partial_\mu^k F(\lambda,\mu)\right| \leq C_{l,k}\cdot\mu^{k-1}$ for any $l,k \geq 0$.

Indeed, the integration by parts N times (the number N will be defined later) when $\mu \neq 0$ gives

$$\int_{(-\infty,c\lambda]} e^{i\mu\xi}\cdot a(\lambda,\xi)d\xi = -e^{ic\lambda\mu}\sum_{j=0}^{N} i^{j+1}\partial_\xi^j a(\lambda,c\lambda)\frac{1}{\mu^{j+1}}$$

$$+\frac{i^N}{\mu^N}\int_{[-R,c\lambda]} e^{i\mu\xi}\partial_\xi^N a\big(\lambda,\xi\big)d\xi.$$

Taking $F(\lambda,\mu) := \frac{i^N}{2\pi}\int_{[-R,c\lambda]} e^{i\mu\xi}\cdot\partial_\xi^N a\big(\lambda,\xi\big)d\xi$ we obtain the result claimed above, since

$$\partial_\lambda^k F(\cdot,\mu) \lesssim_{a,c,k,N} \mu^{k-1}.$$

Therefore, we may split operator $\mathrm{Op}_1^{\mathrm{a}}(\sigma_1)$ into the sum $T_1 + T_2$, where T_1 is defined in the statement of the theorem, T_2 is an integral operator defined by

$$T_2u(x) = \sum_{j=1}^{N}\int_{\mathbb{R}} K_j\Big(\frac{x+y}{2}, x-y\Big)u(y)dy,$$

where

$$K_j(\lambda,\mu) = -\frac{i^{j+1}}{2\pi}\chi_{\{\lambda\geq 0\}}e^{ic\lambda\mu}\partial_\xi^j a(\lambda,c\lambda)\frac{\zeta_\delta(\mu)}{\mu^{j+1}},$$

$$j = 1,2,...,N-1,$$

$$K_N(\lambda,\mu) = \chi_{\{\lambda\geq 0\}} F(\lambda,\mu)\frac{\zeta_\delta(\mu)}{\mu^N}.$$

Step 2.

Note that for any compact operator T $s_k\big(w_1 T w_2\big) = s_k\big(T\big)$, if $|w_1(x)| = |w_2(y)| \equiv 1$ for almost all x and y, since operators T and $w_1 T w_2$ are unitarily equivalent (see [3, §1.6])

Thus, taking $w_1(x) = e^{i\frac{cx^2}{2}}$ and $w_2(y) = e^{-i\frac{cy^2}{2}}$, we may consider up to unitary equivalence

$$T_1 u(x) = -\frac{i}{2\pi}\int_{\mathbb{R}} \chi_{\{x+y\geq 0\}} a\Big(\frac{x+y}{2}, \frac{cx+cy}{2}\Big)\frac{\zeta_\delta(x-y)}{x-y}u(y)dy.$$

Using the remark to Theorem 5.1.3 we get $s_k(T_1) = s_k(\mathrm{Op}_1^{\mathrm{a}}(\tilde{\sigma_1}))$, where

$$\tilde{\sigma_1}(t,\xi) = \int e^{-is\xi} K_0\Big(\frac{s}{2}, 2t\Big)ds,$$

with

$$K_0(\lambda,\mu) = -\frac{i}{2\pi}\chi_{\{\lambda\geq 0\}}\cdot a(\lambda, c\lambda)\frac{\zeta_\delta(\mu)}{\mu}.$$

Applying Remark 3.3.13 for $[a_0, a_1] = [0, 2R]$

$$\int_{[0,2R]} e^{is\xi} K_0(s/2, 2t)\, ds = i\frac{\xi}{\langle\xi\rangle^2}K_0(0,2t) + i\frac{\xi}{\langle\xi\rangle^2}\frac{\big(1+i\xi\partial_\lambda\big)K_0}{\langle\xi\rangle^2}\big(0,2t\big)$$

$$+\int_{[0,2R]} e^{is\xi}\frac{\big(1+i\xi\partial_\lambda\big)^2 K_0}{\langle\xi\rangle^4}\big(s/2,2t\big)ds = R_0\cdot\frac{\zeta_\delta(t)}{t}\frac{1}{\langle\xi\rangle} + \frac{\zeta_\delta(t)}{t}\cdot\frac{R_1(\xi)}{\langle\xi\rangle^2},$$

where $R_0 = \frac{a(0,0)}{4\pi}$, $\partial^k R_1 \in \mathrm{L}^\infty(\mathbb{R})$ for any $k\geq 0$.

Since $\frac{\zeta_\delta(t)}{t} \in \mathrm{l}^{1,\infty}(\mathbb{R})$, using Theorems 3.3.1 and 3.3.14 we get

$$\mathrm{Op}_1^{\mathrm{a}}\Big(\frac{2R_0\zeta_\delta(x+y)}{x+y}\cdot\frac{1}{\langle\xi\rangle}\Big) \in \mathbf{S}_1, \quad \mathrm{Op}_1^{\mathrm{a}}\Big(\frac{\zeta_\delta(x+y)}{x+y}\cdot\frac{R_1(\xi)}{\langle\xi\rangle^2}\Big) \in \mathbb{S}_{1,\infty}.$$

Hence, the main part of the model operator is unitarily equivalent to a new ΨDO,

$$T_1 \simeq \mathrm{Op}_1^{\mathrm{a}}(\tilde{\sigma}_1) \in \mathbf{S}_1.$$

Step 3.

Similarly estimate the singular values of operators with kernels K_j, $j = 1, ..., N-1$ (which decay faster than K_0 as functions of μ). Each of them can be reduced to

$$\mathrm{Op}_1^{\mathrm{a}}\Big(\frac{\zeta_\delta(x+y)}{(x+y)^j}\cdot\frac{R_j(\xi)}{\langle\xi\rangle}\Big) \in \mathbb{S}_{1,\infty}, \quad \partial^k R_j \in \mathrm{L}^\infty(\mathbb{R}).$$

Slightly differently we deal with the kernel K_N, since $F(\lambda, \mu)$ might not allow the separation of variables λ and μ. Reduce to $\mathrm{Op}_1^{\mathrm{a}}(\tilde{\sigma}_N)$, where due to Remark 3.3.13

$$\tilde{\sigma}_N(t,\xi) = i\frac{\xi}{\langle\xi\rangle^2}\frac{F(0,2t)\zeta_\delta(t)}{t^N} + i\frac{\xi}{\langle\xi\rangle^2}\frac{\big(1+i\xi\partial_\lambda\big)F}{\langle\xi\rangle^2}\big(0,2t\big)\cdot\frac{\zeta_\delta(t)}{t^N}$$

$$+\frac{\zeta_\delta(t)}{t^N}\int_{[0,2R]} e^{is\xi}\frac{\big(1+i\xi\partial_\lambda\big)^2 F}{\langle\xi\rangle^4}\big(s/2,2t\big)ds.$$

Since $|\partial_\lambda^k F(s/2,2t)| \lesssim_{a,c,k,N} t^{k-1}$,

$$\tilde{\sigma}_N(t,\xi) = \frac{1}{\langle\xi\rangle}\frac{A(t)}{\langle t\rangle^N} + \frac{1}{\langle\xi\rangle^2}\frac{A_1(t)}{\langle t\rangle^{N-1}} + \frac{1}{\langle\xi\rangle^2}\frac{A_2(t,\xi)}{\langle t\rangle^N},$$

where $A, A_1, \partial_\xi^k A_2(t,\cdot) \in \mathrm{L}^\infty(\mathbb{R})$, $|\partial_t^k A_2(\cdot,\xi)| \lesssim_{a,c,k,N} t^{k+1}$. Therefore, using Proposition 3.3.3 with $q=1, n=2$

$$\mathrm{Op}_1^{\mathrm{W}}\Big(\frac{1}{\langle\xi\rangle^2}\frac{A_2(t,\xi)}{\langle t\rangle^N}\Big) \in \mathbb{S}_1$$

for $N \geq 5$. Indeed, in this case $|\partial_t^k(A_2(t,\xi)\langle t\rangle^{-N})| \lesssim_{a,c,k,N} \langle t\rangle^{3-N} \leq \langle t\rangle^{-2} \in \mathrm{L}^1(\mathbb{R})$, $k=0,1,2$.
Since $\mathrm{Op}_1^{\mathrm{W}}\Big(\frac{1}{\langle\xi\rangle}\frac{A_1(t)}{\langle t\rangle^N} + \frac{1}{\langle\xi\rangle^2}\frac{A_1(t)}{\langle t\rangle^{N-1}}\Big) \in \mathbb{S}_{1,\infty}$, $T_2 \in \mathbb{S}_{1,\infty}$.
Note that since supp $a \in [-R,R]^2$, the kernel of T_1 satisfies (5.1.1). Indeed, $|x+y| < 2R$ along with $|x-y| > 2R$ imply $xy < 0$. Thus, the spectrum of the operator is symmetric.

□

Remark 5.1.5.

Examples of the boundary $\partial\Omega$ when the method is applicable.

In Section 3.2 we discussed the case with smooth boundary (which gives estimate $\underline{O}(k^{-\frac{3}{4}})$). A natural question is whether we can consider a curved angular area, i.e. $\sigma(t,\xi) = \sigma_f(t,\xi) := \chi_{\{t\geq 0\}}(t)\cdot\chi_{\{\xi\leq f(t)\}}(t,\xi)\cdot a(t,\xi)$ where f is a smooth function.
It turns out that the method applied above is applicable only for linear functions f (and for the standard angular region). Indeed, as it was shown in the proof, the reduction of the main operator to a suitable unitarily equivalent one requires the identity of the form

$$e^{i(x+y)f(\frac{x-y}{2})} = e^{iG_1(x)}e^{iG_2(y)} = F_1(x)F_2(y),$$

which is equivalent to

$$(x-y)f\Big(\frac{x+y}{2}\Big) = G_1(x) + G_2(y) \tag{5.1.4}$$

for some functions G_1 and G_2

Lemma 5.1.6. *Representation (5.1.4) is equivalent to $f(t) = c_1 t + c_2$ for some constants c_1, c_2.*

Proof. Definitely, $f(t) = c_1 t + c_2$ admits the representation. To prove the other direction of the statement we denote $H_j(x) = \frac{G_j(2x)}{2}$. This implies

$$(x-y)f(x+y) = H_1(x) + H_2(y).$$

Taking $y = x$ we obtain $H_1(x) = -H_2(x) =: H(x)$.
Thus, for any real t, s the following holds

$$(s-t)f(s+t) = H(s) - H(t).$$

Take $t = 0$ and obtain $H(s) = sf(s) + H(0)$. Thus,

$$(s-t)f(s+t) = sf(s) - tf(t).$$

Thus,

$$(s-2)f(s) = (s-1)f(s-1) - f(1),$$

or equivalently

$$f(s) = \frac{s-1}{s-2}f(s-1) - \frac{f(1)}{s-2}.$$

Repeating this formula $s-1$ times (for $s, s-1, s-2, ..., 2$), obtain

$$f(s) = \frac{s-1}{s-2}f(s-1) - \frac{f(1)}{s-2} = \frac{s-1}{s-3}f(s-2) - \frac{f(1)(s-1)}{(s-2)(s-3)} - \frac{f(1)}{s-2}$$

$$= \frac{s-1}{s-4}f(s-3) - \frac{f(1)(s-1)}{(s-3)(s-4)} - \frac{f(1)(s-1)}{(s-2)(s-3)} - \frac{f(1)}{s-2} = ...$$

$$= \frac{s-1}{1}f(2) - f(1)(s-1)\cdot \sum_{1\le j\le s-2} \frac{1}{j(j+1)} = (f(2)-f(1))s + 2f(1) - f(2)$$

$$= c_1 s + c_2.$$

□

Remark 5.1.7. Since we consider $f(t) = ct$ and the singular values estimate does not depend on c, we can take $c = 0$ and consider the quadrant $\{t \geq 0, \xi \leq 0\}$ in the main operator description.

Lemma 5.1.8.

$$\mathrm{Op}_1^{\mathrm{a}}(\sigma_2) \in \mathbb{S}_{1,\infty},$$

where σ_2 is defined in (5.1.2).

Proof. To obtain the estimate $\mathrm{Op}_1^{\mathrm{a}}(\sigma_2) \in \mathbb{S}_{1,\infty}$, we proceed in the same way as in the Lemma 5.1.4, except we do not decompose the kernel $K(\lambda, \mu)$ of the operator $\mathrm{Op}_1^{\mathrm{a}}(\sigma_2)$ as in (5.1.3). Note that the kernel K is compactly supported on $\mathbb{R}^2$.
Thus, we reduce the symbol σ_2 to the symbol $\tilde{\sigma}_2$ (the operators $\mathrm{Op}_1^{\mathrm{a}}(\tilde{\sigma}_2) \simeq \mathrm{Op}_1^{\mathrm{a}}(\sigma_2)$ are unitarily equivalent), where due to Remark 3.3.13

$$\tilde{\sigma}_2(t,\xi) = i\frac{\xi}{\langle\xi\rangle^2}\frac{K(0,2t)\zeta_\delta(t)}{t^N} + i\frac{\xi}{\langle\xi\rangle^2}\frac{\big(1+i\xi\partial_\lambda\big)K}{\langle\xi\rangle^2}\big(0,2t\big)\cdot\frac{\zeta_\delta(t)}{t^N}$$

$$+\frac{\zeta_\delta(t)}{t^N}\int_{[0,2R]} e^{is\xi}\frac{\big(1+i\xi\partial_\lambda\big)^2 K}{\langle\xi\rangle^4}\big(s/2,2t\big)ds.$$

Therefore, since $\operatorname{supp} K(\cdot,\cdot) \in [0,2R]\times[-2R,2R]$,

$$\tilde{\sigma}_2(t,\xi) = \frac{A(t)}{\langle\xi\rangle} + \frac{A_1(t)}{\langle\xi\rangle^2} + \frac{A_2(t,\xi)}{\langle\xi\rangle^2},$$

where $A, A_1 \in C_0^\infty(\mathbb{R})$, $|\partial_t^m\partial_\xi^l A_2(\cdot,\cdot)| \in \mathrm{L}^\infty(\mathbb{R}^2)$. Therefore, using Proposition 3.3.3 with $q > \frac{1}{2}$

$$\mathrm{Op}_1^{\mathrm{W}}\Big(\frac{1}{\langle\xi\rangle^2}\frac{A_2(t,\xi)}{\langle t\rangle^N}\Big) \in \bigcap_{q>\frac{1}{2}} \mathbb{S}_q$$

Theorems 3.3.1 and 3.3.14 imply

$$\mathrm{Op}_1^{\mathrm{W}}\Big(\frac{A(t)}{\langle\xi\rangle}\Big) \in \mathbb{S}_{1,\infty}, \quad \mathrm{Op}_1^{\mathrm{W}}\Big(\frac{A_1(t)}{\langle\xi\rangle^2}\Big) \in \mathbb{S}_{\frac{1}{2},\infty}$$

□

Since all the auxiliary results have been derived, we can decompose the model operator as follows.

Theorem 5.1.9. *The operator (5.0.1) can be decomposed as follows*

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma) = T_m + T_r \in \mathfrak{S}_1,$$

where $T_r \in \mathbb{S}_{1,\infty}$, the main part of the representation, T_m, is a self-adjoint compact operator with a symmetric (with respect to the origin) spectrum defined by

$$T_m u(x) = -\frac{i}{2\pi}\int_{\mathbb{R}} \chi_{\{x+y\geq 0\}}\cdot e^{i\frac{c(x^2-y^2)}{2}}\cdot\frac{\zeta_{2R}(x-y)a\big(\frac{x+y}{2},\frac{c(x+y)}{2}\big)}{x-y}u(y)dy \in \mathfrak{S}_1$$

and is unitarily equivalent to $\mathrm{Op}_1^{\mathrm{W}}(\tilde\sigma)$, *where*

$$\tilde\sigma(t,\xi) = \frac{a(0,0)}{4\pi}\frac{\zeta_{2R}(2t)}{t}\cdot\frac{1}{\langle\xi\rangle} + \frac{\zeta_{2R}(2t)}{t}\cdot\underline{O}\Big(\frac{1}{\langle\xi\rangle^2}\Big).$$

Remark 5.1.10. For the spectral estimates due to (3.3.11) the cut-off function $\zeta_{2R}(2t)$ can be replaced with $\zeta_{2R}(t)$.

Remark 5.1.11. Let A, B, C be 3 non-collinear points on the $(t,\xi)-$plane and $\angle ABC$ be the interior of the corresponding angle on the phase space. Define $c := \tan\big(\sphericalangle ABC - \frac{\pi}{2}\big)$. Since singular values of an operator are invariant under translations and rotations (see [8, Ch.2, Lemma 2.14], [11, Th. 6, p.3327]),

$$s_k(\mathrm{Op}_1^{\mathrm{W}}(\tilde\zeta(t,\xi)\chi_{\angle ABC})) = s_k(\mathrm{Op}_1^{\mathrm{W}}(\zeta(t,\xi)\chi_{t\ge 0,\xi\le ct})) = \underline{O}\big(k^{-1}\log(k+1)\big),$$

where $\zeta(t,\xi), \tilde\zeta(t,\xi) \in C_0^\infty(\mathbb{R}^2)$ and $\zeta(t,\xi) = \tilde\zeta(t_1,\xi_1)$, where (t_1,ξ_1) are new variables (after the substitution which shifts the angular region $\angle ABC$ to the angular region $\{(t,\xi) \mid t\ge 0, \xi\le ct\}$).

Remark 5.1.12. Since T_1 is a self-adjoint operator satisfying (5.1.1), due to Example 5.1.2

$$\lambda_k^-\big(\mathrm{Op}_1^{\mathrm{W}}(\tilde\sigma_1)\big) = \lambda_k^+\big(\mathrm{Op}_1^{\mathrm{W}}(\tilde\sigma_1)\big) = s_{2k}\big(\mathrm{Op}_1^{\mathrm{W}}(\tilde\sigma_1)\big) = \underline{O}\Big(\frac{\log(k+1)}{k}\Big).$$

Remark 5.1.13. The estimate for the eigenvalues of the model operator can be obtained even in the case when the point spectrum is not symmetric. We appeal to the following fact.

Proposition 5.1.14. *[4, Ch.11, §5, Th.5, p.260] For any compact operator T and $p>0$*

$$\sum_{r=1}^{k}|\lambda_r(T)|^p \le \sum_{r=1}^{k} s_r(T)^p, \quad k=1,2,...,$$

where $\{\lambda_r(T)\}_{r\ge1}$ is the sequence of eigenvalues of T enumerated (counted with their multiplicity) so that $\{|\lambda_r(T)|\}$ is a non-increasing sequence.

Let's apply this Proposition for our operator. If $0<p<1$, then for $k\ge1$

$$k|\lambda_k^\pm(\mathrm{Op}_1^{\mathrm{W}}(\sigma)|^p \le \max\Big\{\sum_{r=1}^{k}|\lambda_r^+(\mathrm{Op}_1^{\mathrm{W}}(\sigma)|^p\,,\sum_{r=1}^{k}|\lambda_r^-(\mathrm{Op}_1^{\mathrm{W}}(\sigma)|^p\Big\}$$

$$\le \sum_{r=1}^{k}\Big(s_r(\mathrm{Op}_1^{\mathrm{W}}(\sigma))\Big)^p \lesssim_a \sum_{r=1}^{k}\frac{\log^p r}{r^p} \lesssim_p \int_1^k \frac{\log^p x}{x^p}dx = \int_0^{\log k} t^p e^{t-pt}dt$$

$$= \frac{t^p e^{t-pt}}{1-p}\Bigg|_0^{\log k} - p\int_0^{\log k} t^{p-1}e^{t-pt}dt \lesssim_p (\log(k+1))^p \cdot k^{1-p}.$$

Thus,

$$|\lambda_k^{\pm}(\mathrm{Op}_1^{\mathrm{W}}(\sigma)| \lesssim_a \frac{\log(k+1)}{k}.$$

Note that due to the Proposition for any compact operator T, the estimate $s_k(T) \lesssim k^{-p_1}(\log(k+1))^{p_2}$, $p_1 > 0$, implies $\lambda_k(T) \lesssim k^{-p_1}(\log(k+1))^{p_2}$.

5.2 Asymptotic formula via Dauge-Robert result

Recall that the main operator (5.0.1) can be decomposed (see Theorem 5.1.9) as follows

$$\mathrm{Op}_1^{\mathrm{W}}(\sigma) = T_m + T_r,$$

where $T_m \simeq T$ are unitarily equivalent,

$$T = \frac{a(0,0)}{4\pi}\mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma}) = \frac{a(0,0)}{4\pi}\mathrm{Op}_1^{\mathrm{W}}(\zeta(t)t^{-1}\langle\xi\rangle^{-1}) \in \mathfrak{S}_1$$

and

$$T_r \in \mathbb{S}_{1,\infty}.$$

Using tools from Chapter 2 we state

Theorem 5.2.1. *The following asymptotic formula holds*

$$\lambda_k^{\pm}\Big(\mathrm{Op}_1^{\mathrm{W}}(\sigma)\Big) = \frac{a(0,0)}{4\pi^2}\cdot\frac{\log k}{k} + \overline{o}\Big(\frac{\log k}{k}\Big)$$

as $k \to \infty$.

Proof. Let's compute the spectral volume function V_+ (see Definition 2.1.2) directly.

$$2\pi V_+\Big(\frac{a(0,0)}{4\pi}\lambda; \frac{a(0,0)}{4\pi}\zeta(t)t^{-1}\langle\xi\rangle^{-1}\Big) = \int_{2\geq t\geq 1;\ \zeta(t)t^{-1}\langle\xi\rangle^{-1}>\lambda} dtd\xi$$

$$+\int_{t\geq 2;\ t^{-1}\langle\xi\rangle^{-1}>\lambda} dtd\xi.$$

The first integral can be estimated as follows

$$\int_{2\geq t\geq 1;\ \zeta(t)t^{-1}\langle\xi\rangle^{-1}>\lambda} dtd\xi \leq \int_{2\geq t\geq 1;\ \langle\xi\rangle^{-1}>\lambda} dtd\xi \leq \sqrt{\frac{1}{\lambda^2}-1} < \frac{1}{\lambda}.$$

For the second integral, we have $\frac{1}{2\langle\xi\rangle} \geq \frac{1}{t\langle\xi\rangle} > \lambda$, thus, $|\xi| \in \Big[0, \sqrt{\frac{1}{4\lambda^2} - 1}\,\Big]$ and

$$\int_{t\geq 2;\, t^{-1}\langle\xi\rangle^{-1}>\lambda} dt d\xi = 2\int_0^{\sqrt{\frac{1}{4\lambda^2}-1}} \int_0^{\frac{1}{\langle\xi\rangle\lambda}} dt d\xi = \frac{2}{\lambda}\int_0^{\sqrt{\frac{1}{4\lambda^2}-1}} \frac{1}{\sqrt{\xi^2+1}} d\xi$$

$$= \frac{2}{\lambda}\log\Big(\sqrt{\frac{1}{4\lambda^2}-1} + \frac{1}{2\lambda}\Big) = \frac{2}{\lambda}\log\Big(\frac{1}{\lambda}\Big)\big(1+\bar{o}(1)\big), \quad \lambda \to 0+.$$

Therefore,

$$V_+\Big(\frac{a(0,0)}{4\pi}\lambda; \frac{a(0,0)}{4\pi}\zeta(t)t^{-1}\langle\xi\rangle^{-1}\Big) = \frac{1}{\pi}\cdot\frac{1}{\lambda}\log\frac{1}{\lambda} + \underline{O}\Big(\frac{1}{\lambda}\Big), \quad \lambda \to 0+$$

Finally, the volume spectral function for the main operator

$$V_+\Big(\frac{1}{x}; \tilde{\sigma}\Big) = \frac{a(0,0)}{4\pi^2} x\log x + \underline{O}(x), \quad x \to \infty \tag{5.2.1}$$

Using Remark 2.1.7 and the example of log-power function for Theorem 2.1.6,

$$\lambda_k^+(T_m) = \lambda_k^+\Big(\mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma})\Big) = \frac{a(0,0)}{4\pi^2}\frac{\log k}{k}(1+\bar{o}(1)), \quad k \to \infty.$$

Since $\mathrm{Op}_1^{\mathrm{W}}(\sigma) - T_m \in \mathbb{S}_{1,\infty}$, due to Theorem 2.2.14

$$\lambda_k^+\Big(\mathrm{Op}_1^{\mathrm{W}}(\sigma)\Big) = \lambda_k^+(T_m) = \frac{a(0,0)}{4\pi^2}\frac{\log k}{k}(1+\bar{o}(1)), \quad k \to \infty.$$

The same result holds for negative eigenvalues since the spectrum is symmetric. □

5.3 Asymptotic formula via reduction to model ΨDO from DO

In this paragraph we reduce the study of the spectral counting function of the model pseudo-differential operator to the analysis of differential operators (DO) using the Birman-Schwinger principle. The reduction process leads to a Schrödinger operator with negative potential defined on a real number line $\mathbb{R}$.

Let's start with some preliminary results. The following lemmata represent DO and DO-related operators in the ΨDO form.

Proposition 5.3.1. *[4, Ch.8, §5, p.198]*
Let $a_m \in \mathbb{C}$, $0 \leq m \leq M$ and D be a differential operator defined by

$$Du = \sum_{M \geq m \geq 0} a_m \cdot \frac{i^m d^m}{dx^m} u$$

Then

$$D = \mathcal{F}^{-1}\Big(\sum_{M \geq m \geq 0} a_m \xi^m \Big) \mathcal{F} = \mathrm{Op}_1^{\mathrm{a}}\Big(\sum_{M \geq m \geq 0} a_m \xi^m \Big),$$

where $\psi(\xi) = \sum_{M \geq m \geq 0} a_m \xi^m$ denotes the multiplication operator.

Lemma 5.3.2. *Operator $(H_0 + I)^{-\frac{1}{2}}$ defined on $W_2^2(\mathbb{R})$ admits the following representation*

$$(H_0 + I)^{-\frac{1}{2}} = \mathrm{Op}_1^{\mathrm{a}}\big(\langle \xi \rangle^{-1}\big)$$

Proof. Due to Proposition 5.3.1

$$(H_0 + I) = \mathcal{F}^{-1}(\xi^2 + 1)\mathcal{F} = \mathrm{Op}_1^{\mathrm{a}}\big(\xi^2 + 1\big).$$

Note that

$$\Big(\mathcal{F}^{-1} \sqrt{\xi^2 + 1} \mathcal{F} \Big)^2 = \mathcal{F}^{-1} \sqrt{\xi^2 + 1} \mathcal{F} \mathcal{F}^{-1} \sqrt{\xi^2 + 1} \mathcal{F}$$

$$= \mathcal{F}^{-1} \sqrt{\xi^2 + 1} \sqrt{\xi^2 + 1} \mathcal{F} = \mathcal{F}^{-1}(\xi^2 + 1)\mathcal{F} = H_0 + I.$$

Therefore,

$$(H_0 + I)^{\frac{1}{2}} = \mathcal{F}^{-1} \sqrt{\xi^2 + 1} \mathcal{F}.$$

Moreover,

$$\mathcal{F}^{-1} \sqrt{\xi^2 + 1} \mathcal{F} \mathcal{F}^{-1} \frac{1}{\sqrt{\xi^2 + 1}} \mathcal{F} = \mathcal{F}^{-1} \sqrt{\xi^2 + 1} \frac{1}{\sqrt{\xi^2 + 1}} \mathcal{F} = \mathcal{F}^{-1} \mathcal{F} = I.$$

Hence,

$$(H_0 + I)^{-\frac{1}{2}} = \mathcal{F}^{-1} \frac{1}{\sqrt{\xi^2 + 1}} \mathcal{F}.$$

Finally,

$$(H_0 + I)^{-\frac{1}{2}} u(x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} \frac{1}{\sqrt{\xi^2 + 1}} \hat{u}(\xi) d\xi = \mathrm{Op}_1^{\mathrm{a}}\Big(\frac{1}{\sqrt{\xi^2 + 1}} \Big).$$

□

Remark 5.3.3. The following representation holds

$$(H_0+I)^{-\frac{1}{2}}V(H_0+I)^{-\frac{1}{2}} = (H_0+I)^{-\frac{1}{2}}V^{\frac{1}{2}}V^{\frac{1}{2}}(H_0+I)^{-\frac{1}{2}} = A^*A,$$

where

$$A := \mathrm{Op}_1^{\mathrm{a}}\big(V(x)^{\frac{1}{2}}\langle\xi\rangle^{-1}\big).$$

The main part of the reduction process is based on the following principle. It turns out that the counting function of a specific pseudo-differential operator coincides with the counting function of a certain lower semi-bounded differential operator. Namely,

Theorem 5.3.4. *('Birman-Schwinger principle')*

$$\#(-1, H_0 - gV) = n\Big(\frac{1}{\sqrt{g}}, \mathrm{Op}_1^{\mathrm{a}}\Big(\frac{1}{\sqrt{g}}\frac{\sqrt{V(x)}}{\langle\xi\rangle}\Big)\Big).$$

Proof. Note that (see [4, Ch.9, §2, Th.6, p.213]) for any self-adjoint compact operator T and any number $\lambda > 0$

$$n(\lambda, T) = \max\{\dim M | M \text{ is a subspace of functions } \psi \text{ in } \mathrm{L}^2(\mathbb{R})$$
$$\text{with } \langle T\psi, \psi\rangle > \lambda \cdot \|\psi\|^2\}$$

and for any any self-adjoint and lower semibounded operator L and for any number $\mu < 0$

$$\#(\mu, L) = \max\{\dim M | M \text{ is a subspace of functions } \psi \text{ in } D_{L_R}$$
$$\text{with } \langle L\psi, \psi\rangle < \mu \cdot \|\psi\|^2\}.$$

Define

$$m_1 = \max\{\dim M_1 | M_1 \text{ is a subspace of functions } \psi \text{ in } W_2^2(\mathbb{R})$$
$$\text{with } \langle (H_0 - gV)\psi, \psi\rangle < -\|\psi\|^2\},$$

$$m_2 = \max\{\dim M_2 | M_2 \text{ is a subspace of functions } \psi \text{ in } (H_0+I)^{\frac{1}{2}}(W_2^2(\mathbb{R}))$$
$$\text{with } \langle -B\psi, \psi\rangle < -\frac{1}{g}\|\psi\|^2\},$$

where operator $B = (H_0+I)^{-\frac{1}{2}}V(H_0+I)^{-\frac{1}{2}}$ can be expressed as a product A^*A with $A = \sqrt{V}\cdot\mathrm{Op}_1^{\mathrm{a}}(\frac{1}{\sqrt{\xi^2+1}})$ due to Remark 5.3.3.

We claim that $m_1 = m_2$.

Indeed, if $\psi \in M_1$ (where M_1 is some subspace in the definition of m_1), then $-\|\psi\|^2 > \langle (H_0 - gV)\psi, \psi\rangle$, which is equivalent to

$$\langle gV\psi, \psi\rangle > \langle (H_0 + I)\psi, \psi\rangle = \langle (H_0 + I)^{\frac{1}{2}}\psi, (H_0 + I)^{\frac{1}{2}}\psi\rangle.$$

Let $w = (H_0 + I)^{\frac{1}{2}}\psi \in (H_0 + I)^{\frac{1}{2}}(W_2^2(\mathbb{R}))$.
Then, plugging $\psi = (H_0 + I)^{-\frac{1}{2}}w$ in the estimate above, we obtain

$$\langle w, w\rangle = \langle (H_0 + I)^{\frac{1}{2}}\psi, (H_0 + I)^{\frac{1}{2}}\psi\rangle < \langle gV(H_0 + I)^{-\frac{1}{2}}w, (H_0 + I)^{-\frac{1}{2}}w\rangle$$

$$= \langle g(H_0 + I)^{-\frac{1}{2}}V(H_0 + I)^{-\frac{1}{2}}w, w\rangle = \langle gBw, w\rangle,$$

Therefore,

$$\langle -Bw, w\rangle < -\frac{1}{g}\|w\|^2$$

and $w = (H_0 + I)^{\frac{1}{2}}\psi \in M_2$ (some subspace defined in the definition of m_2). Moreover, since $(H_0 + I)^{\frac{1}{2}}$ is invertible operator, the maximal number of linearly independent functions w satisfying $\langle -Bw, w\rangle < -\frac{1}{g}\|w\|^2$ is not less than the number of linearly independent functions $\psi \in M_1$. Thus, $m_2 \geq m_1$.

Conversely, if $w \in M_2$, then define $\psi = (H_0 + I)^{-\frac{1}{2}}w \in W_2^2(\mathbb{R})$. It follows that

$$-\|\psi\|^2 > \langle (H_0 - gV)\psi, \psi\rangle$$

and $\psi \in M_1$. Similarly, since $(H_0 + I)^{-\frac{1}{2}}$ is invertible, $m_1 \geq m_2$.

Hence,

$$\#(-1, H_0 + gV) = m_1 = m_2 = \#\Big(-\frac{1}{g}, -(H_0 + 1)^{-\frac{1}{2}}V(H_0 + 1)^{-\frac{1}{2}}\Big)$$

$$= \#\Big(-\frac{1}{g}, -A^*A\Big) = n\Big(\frac{1}{\sqrt{g}}, A\Big),$$

where A is defined in Remark 5.3.3.

□

Common restriction of 2 self-adjoint operators

The motivation to study different lower semibounded operators in Chapter 4 comes from the fact that they might have a common restriction. In this case one can infer some information about the spectral counting functions of these operators.

It turns out that the two self-adjoint extensions of a common closed symmetric operator have counting functions, which differ by a finite number. Namely, the following theorem holds

Proposition 5.3.5. *([4, Ch. 9, §3, Lemmata 2, 3, p.214-215]) Let A_0 be a closed symmetric operator with discrete spectrum and finite equal deficiency indices $\nu_{\pm}(A_0) = n$. Let $A = A^*$, $B = B^*$ be two its self-adjoint extensions. Then for $\lambda \in \rho(A) \cap \rho(B)$ (where $\rho(\cdot)$ stands for the resolvent set of an operator) the quantity $rank((A-\lambda I)^{-1}-(B-\lambda I)^{-1}) = r$ is finite and*

$$|\#(\lambda, A) - \#(\lambda, B)| \leq r < n.$$

In this paragraph consider

$$V(x) = \frac{\zeta^2(x)}{x^2} := \frac{\zeta_1^2(x)}{x^2}.$$

Since $V \in \mathrm{L}^\infty(\mathbb{R})$, the operator $p = gV$ is dominated by $-\Delta$, and we can apply the results from Chapter 4 to obtain the estimate for the counting function of the model operator $\mathrm{Op}_1^{\mathrm{a}}\Big(\sqrt{\frac{V(x)}{\xi^2+1}}\Big)$, which is in the same Weidl operator class as the main operator (5.0.1).

The following reasoning helps to establish the connection between the model operator and the operator whose spectral counting function has been estimated.
We consider a suitable Schrödinger operator $\mathrm{H}_{(-\infty,\infty)}$ with the same counting function (see Theorem 5.3.4), which we try to estimate with the same upper bound as for $\#(-1; -p; (a, b))$ (Corollary 4.4.2 and Remark 4.4.3) For this purpose, we take the decoupling L_R, whose counting function equals to $\#(-1; -p; (-R, R))$ and try to find a suitable common restriction of $\mathrm{H}_{(-\infty,\infty)}$ and $L_R - p$ (the operator $L_R^c - p$), since due to Proposition 5.3.5 this means that their counting functions differ by a constant number.
This reasoning leads us to the result, which is the theorem below

Theorem 5.3.6.

$$n\big(\mathrm{Op}\big(\zeta(t)t^{-1}\langle\xi\rangle^{-1}\big), x^{-1}\big) = \frac{2}{\pi} x \log x + \underline{O}(x), \quad x \to \infty.$$

Remark 5.3.7. This is in line with the result (5.2.1). Indeed,

$$n\big(\mathrm{Op}\big(\zeta(t)t^{-1}\langle\xi\rangle^{-1}\big), x^{-1}\big) \sim V_+\Big(\frac{1}{x}\Big) + V_-\Big(\frac{1}{x}\Big)$$

$$\sim \frac{2}{\pi} x \log x, \ x \to \infty.$$

Thus,

$$n\Big(\frac{a(0,0)}{4\pi}\mathrm{Op}\big(\zeta(t)t^{-1}\langle\xi\rangle^{-1}\big), x^{-1}\Big) \sim \frac{1}{2\pi^2}x\log x,\ x\to\infty.$$

Therefore,

$$s_k\Big(\frac{a(0,0)}{4\pi}\mathrm{Op}\big(\zeta(t)t^{-1}\langle\xi\rangle^{-1}\big)\Big) = \frac{a(0,0)}{2\pi^2}\frac{\log k}{k}(1+\overline{o}(1)), \quad k\to\infty.$$

Finally, $s_{2k} = \lambda_k^{\pm} \sim \frac{a(0,0)}{4\pi^2}\frac{\log k}{k}, \quad k\to\infty$ which repeats the result of Theorem 5.2.1.

Proof. Take $p = gV = g\cdot\frac{\zeta^2(x)}{x^2}$, $R=\sqrt{g}$ and $\lambda=-1$. The operators $\mathrm{H}_{(-\infty,\infty)}$ and L_R are self-adjoint extensions of the operator $L_R^c - gV$. Thus, due to Proposition 5.3.5 and Corollary 4.3.6

$$\Big|\#(-1;-p;(-\infty,\infty)) - \#(-1,L_R)\Big| \le \nu_+(L_R^c - gV) = 2.$$

Lemma 4.4.4 implies $\#(-1,L_R) = \#(-1,L_R^o)$.

Using Theorem 5.3.4, we obtain

$$n\big(\mathrm{Op}_1\big(\zeta(t)t^{-1}\langle\xi\rangle^{-1}\big), g^{-\frac{1}{2}}\big) = \#(-1, H_0 - gV)$$
$$= \#(-1;-p;(-\infty,\infty)) \in \Big(\#(-1,L_R^o)-2, \#(-1,L_R^o)+2\Big).$$

Remark 4.4.3 for $\alpha = 2$ implies

$$\#(-1, L^o_{\sqrt{g}}) = \frac{1}{\pi}\sqrt{g}\log g + \underline{O}(\sqrt{g}), \quad g\to\infty$$

which finishes the proof.

□

5.4 Additional remarks and conclusion

In this Section we describe two corollaries of the main result, Theorem 5.1.9.

Let's see how the asymptotic formulae change in the case of the angular boundary, if one of its sides becomes smooth (in terms of the properties of the corresponding smooth function a), i.e. the boundary of the domain is described by a straight line.

Recall the main result of the previous Chapter. If we impose some additional conditions on function a, namely, if symbol $\sigma(t,\xi) := \chi_{\{t\ge 0\}}(t)\cdot\chi_{\{\xi\le 0\}}(\xi)\cdot a(t,\xi)$, where $a\in C_0^\infty(\mathbb{R}_+\times\mathbb{R})$, i.e. supp $a\subseteq[\epsilon,R]\times[-R,R]$

for some positive numbers ϵ, R, then $\mathrm{Op}_1^{\mathrm{W}}(\sigma) \in \mathbb{S}_{1,\infty}$.

Indeed, in this case $a(0,0) = 0$ and the following representation (Theorem 5.1.9) of the main part of the symbol σ holds

$$\tilde{\sigma}(t,\xi) = \frac{a(0,0)}{4\pi}\frac{\zeta_{2R}(2t)}{t} \cdot \frac{1}{\langle\xi\rangle} + \frac{\zeta_{2R}(2t)}{t} \cdot \underline{O}\Big(\frac{1}{\langle\xi\rangle^2}\Big)$$
$$= \frac{\zeta_{2R}(2t)}{t} \cdot \underline{O}\Big(\frac{1}{\langle\xi\rangle^2}\Big).$$

Due to Theorem 3.3.1 $\mathrm{Op}_1^{\mathrm{W}}(\sigma)$ along with $\mathrm{Op}_1^{\mathrm{W}}(\tilde{\sigma}) \in \mathbb{S}_{1,\infty}$, which confirms the result of Theorem 3.2.8.

The following theorem states that the estimate $k^{-1}\log(k+1)$ holds for any polygonal region.

Theorem 5.4.1. *Let $\Omega = A_1A_2...A_n$ be the interior of an $n-$sided polygon on the (t,ξ)-phase space. Then*

$$\mathrm{Op}_1^{\mathrm{W}}(\chi_\Omega) \in \mathfrak{S}_1.$$

Proof. First we prove the statement for any triangle $\Lambda = A_1A_2A_3$, then split the polygon Ω into $n-2$ triangles, $\Omega = \bigcup_{1\le j\le n-2}\Lambda_j$, using Remark 2.2.16 and (2.2.3),

$$s_k(\mathrm{Op}_1^{\mathrm{W}}(\chi_\Omega)) \lesssim_n \sum_{1\le j\le n-2} s_k(\mathrm{Op}_1^{\mathrm{W}}(\chi_{\Lambda_j})) \lesssim_{\Lambda_1,\dots,\Lambda_{n-2}} k^{-1}\log(k+1).$$

To prove the theorem for the triangle Λ, we introduce a smooth function $a \in C_0^\infty(\mathbb{R}^2)$ s.t. $a \equiv 1$ on the interior of triangle Λ, $|a| \le 1$ outside the interior of Λ (see Fig. 5.1 below).

Split symbol χ_Λ as follows (Fig. 5.2)

$$\chi_\Lambda = a\chi_{\angle A_1A_2A_3} - a\chi_{\{t\le l_{A_1A_3}\}} + a\chi_{\angle B_1A_1B_2} + a\chi_{\angle B_2A_3B_4},$$

where $l_{A_1A_3}$ is the line passing through A_1 and A_3.

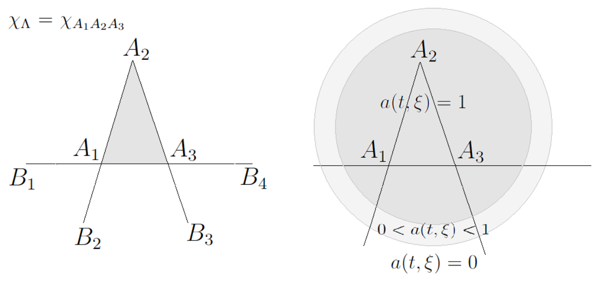

Figure 5.1: symbol χ_Λ and function $a = a(t,\xi)$

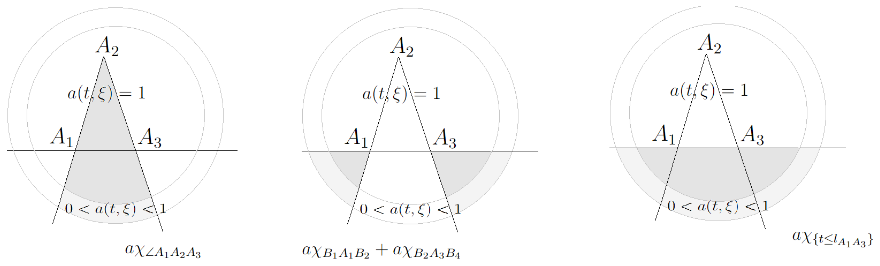

Figure 5.2: functions $a\chi_{\angle A_1A_2A_3}$, $a\chi_{\angle B_1A_1B_2} + a\chi_{\angle B_2A_3B_4}$ and $a\chi_{\{t\le l_{A_1A_3}\}}$

Theorem 5.1.9 implies $\mathrm{Op}_1^{\mathrm{W}}(a\chi_{\angle A_1A_2A_3})$, $\mathrm{Op}_1^{\mathrm{W}}(a\chi_{\angle B_1A_1B_2})$ and $\mathrm{Op}_1^{\mathrm{W}}(a\chi_{\angle B_2A_3B_4}) \in \mathfrak{S}_1$. Theorem 3.2.8 gives $\mathrm{Op}_1^{\mathrm{W}}(a\chi_{\{t\le l_{A_1A_3}\}}) \in \mathbb{S}_{1,\infty}$.

Finally, due to Theorem 2.2.14 $\mathrm{Op}_1^{\mathrm{W}}(\chi_\Lambda) \in \mathfrak{S}_1$.

□

In conclusion, we notice that for asymptotic estimates sometimes we treat ΨDO as an integral operator and study the properties of its kernel to apply corresponding tools (Section 3.2), sometimes we "reassemble" the symbol in such a way that a new symbol satisfies conditions of Theorems 3.3.1, 3.3.14 and continue to deal with a new ΨDO.

Splitting the kernel or the symbol into parts and applying Ky Fan's inequality, we can obtain asymptotic estimates for eigen/singular values, However, it is impossible to use the same approach to obtain an asymptotic formula if the parts have the same order of s-numbers decay, and there is no way to apply the results of perturbation theory, Theorem 2.2.14. That is why, if the boundary is a broken line and has at least 2 angles (e.g. the boundary is a triangle/polygon), the contribution of each angle to the main term of the asymptotic formula is $\frac{1}{4\pi^2}\frac{\log k}{k}$. Adding these parts might compensate the main terms, thus, we are not sure if the estimate $\underline{O}(k^{-1}\log(k+1))$ is sharp.

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