Proof-theoretic Semantics

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In *model-theoretic semantics* (M-tS), logical consequence is defined in terms of models; that is, abstract mathematical structures in which propositions are interpreted and their truth is judged. This includes, in particular, denotational semantics and Tarski's [51, 52] conception of logical consequence: a proposition ϕ follows model-theoretically from a context Γ iff every model of Γ is a model of ϕ,

$$\Gamma \models \phi \quad \text{iff} \quad \text{for all models } \mathcal{M}, \text{ if } \mathcal{M} \models \psi \text{ for all } \psi \in \Gamma, \text{ then } \mathcal{M} \models \phi$$

Proof-theoretic semantics (P-tS) [45, 10, 53] is an alternative approach to meaning and validity in which they are characterized in terms of *proofs* — understood as objects denoting collections of acceptable inferences from accepted premisses. It also concerns the semantics *of* proofs, understood as 'valid' arguments.

To be clear, P-tS is not about providing a proof system. As Schroeder-Heister [44] observes, since no formal system is fixed (only notions of inference) the relationship between semantics and provability remains the same as it has always been: soundness and completeness are desirable features of formal systems.

The semantic paradigm supporting P-tS is *inferentialism* — the view that meaning (or validity) arises from rules of inference (see Brandom [4]). This may be viewed as a particular instantiation of the *meaning-as-use* paradigm by Wittgenstein [55] in which 'use' in logic is understood as inferential rôle.

Heuristically, what differs is that (pre-logical) *proofs* in P-tS serve the rôle of *truth* in M-tS. This shift has substantial and subtle mathematical and conceptional consequences, as discussed below.

To illustrate the paradigmatic shift from M-tS to P-tS, consider the proposition 'Tammy is a vixen'. What does it mean? Intuitively, it means, somehow, '"Tammy is female" and "Tammy is a fox"'. On inferentialism, its meaning is given by the rules,

$$\frac{\text{Tammy is a fox} \quad \text{Tammy is female}}{\text{Tammy is a vixen}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is female}} \qquad \frac{\text{Tammy is a vixen}}{\text{Tammy is a fox}}$$

These merit comparison with the laws governing conjunction ($\wedge$), which justify the sense in which the above proposition is a conjunction,

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \qquad \frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi}$$

There are several branches of research within P-tS — see, for example, the discussion on proof-theoretic validity in the Dummett-Prawitz tradition by Schroeder-Heister [43] — see also Gheorghiu and Pym [15]. Here, we concentrate on two topics: *base-extension semantics* and *bilateralism*.

Before proceeding to the topic details, we outline some important questions for the development of P-tS:

- *To what extent should P-tS depend on paradigms of proof?* On the one hand, different logics are more naturally expressed in some format of proofs than others (e.g., substructural logics typically favour sequent presentations more than natural deduction) and their P-tS may be influenced by this bias. Moreover, P-tS gives up an opportunity to challenge the current foundations and received dogma of the very concept of 'proof' in logic. On the other hand, semantics ought to be syntax independent (in some sense). This may mean that a given notion of P-tS should be instantiable to different paradigms of proof, if none is taken as conceptually prior to the others (e.g., one may view a sequent calculus as providing 'constructions' and natural deduction as providing the genuine article).

- *What might we expect of the relationship to M-tS?* Since M-tS is a powerful way of looking at logics, one may strive to show that the usual properties of M-tS are not lost if one transitions to P-tS. In particular, one may desire that the behaviour of models be represented in P-tS in some way that remains to be made precise. On the contrary, P-tS may offer an entirely different meta-theory that gives access to entirely distinct understandings of logics while forbidding other, perhaps useful, features of their extant semantics.

- *What is the real value of P-tS?* Developing the last point in a particular direction, we should consider what mathematical and computational value P-tS holds beyond its philosophical significance. To this end, one may begin by investigating how meta-theoretic properties of logics (e.g., compactness, categoricity, decidability, and so on) may be proved from the point of view of P-tS.

There are, of course, many more questions that one could ask. For example, P-tS may lead us to consider entirely new logics that have an obscure M-tS (if they have one at all). We defer further discussion of these matters to another time.

Base-extension Semantics

This section is concerned with a formalism in P-tS called *base-extension semantics* (B-eS). It follows the tradition of Piecha et al. [26, 25, 28] and Sandqvist [41, 39, 40, 38].

The idea of B-eS begins with the notion of an *atomic system*. An atomic system is a collection of inferential relationships between *atoms*. They represent some beliefs that an agent may posses about the inferential relationship between thoughts. Piecha and Schroeder-Heister [27, 46] and Sandqvist [41] have given an analysis of them based on earlier work by Prawitz [29] and Schroeder-Heister [42].

Presently, we shall consider three types of atomic rules. Let $C, P_1, \ldots, P_n$ be atoms and $\mathbb{P}_1, \ldots, \mathbb{P}_n$ be finite, possibly empty, sets of atoms. The following are zero-, first-, and second-level atomic rules, respectively

$$\frac{}{C} \qquad \frac{P_1 \quad \ldots \quad P_n}{C} \qquad \frac{\begin{array}{c}[\mathbb{P}_1]\\ P_1\end{array} \quad \ldots \quad \begin{array}{c}[\mathbb{P}_n]\\ P_n\end{array}}{C}$$

The rules governing Tammy and her vixenhood above are atomic rules; specifically, they are *first-level* rules. Sandqvist [40] provides the following example of a second-level rule:

$$\frac{\text{A is a sibling of B} \qquad \begin{array}{c}[\text{A is a brother of B}]\\ P\end{array} \qquad \begin{array}{c}[\text{A is a sister of B}]\\ P\end{array}}{P}$$

Whether atomic rules correspond to 'knowledge' or 'definition' is a debated topic; we defer to Piecha and Schroeder-Heister [46, 27] and Sandqvist [41] for details.

Atomic rules are read essentially as natural deduction rules in the sense of Gentzen [50]. However, they are taken *per se* so that no substation is allowed. Thus, they are intuitively related to hereditary Harrop formulae in the sense of Miller [22] — see also Gheorghiu and Pym [14].

A collection of atomic rules is an *atomic system*. We may restrict attention to certain atomic systems, in which case we call them *bases* ($\mathcal{B}$). Their reading as natural deduction rules (without substitution) determines a notion of *derivability in a base* ($\vdash_{\mathcal{B}}$).

Relative to a notion of derivability in a base ($\vdash_{\mathcal{B}}$), a B-eS is determined by a judgement called *support* ($\Vdash_{\mathcal{B}}$) defined inductively according to the structure of formulae with the base case (i.e., the support of atoms) given by *provability in a base*. This induces a validity judgement by quantifying our bases,

$$\Gamma \Vdash \phi \qquad \text{iff} \qquad \Gamma \Vdash_{\mathcal{B}} \phi \text{ for any base } \mathcal{B}$$

We illustrate this idea below.

Define a base $\mathcal{B}$ to be an atomic system that only contains zero- and first-level rules,

$$\frac{}{C} \qquad \frac{P_1 \quad \ldots \quad P_n}{C}$$

Let $\overline{\mathbb{A}}$ denote the set of closed atoms and $\overline{\mathbb{T}}$ denote the set of closed terms.

Relative to this notion of base, define a support relation ($\Vdash$) as follows:

$$\Vdash_{\mathcal{B}} P \quad \text{iff} \quad \vdash_{\mathcal{B}} P \tag{At}$$

$$\Vdash_{\mathcal{B}} \bot \quad \text{iff} \quad \Vdash_{\mathcal{B}} P \text{ for any } P \in \overline{\mathbb{A}} \tag{$\bot$}$$

$$\Vdash_{\mathcal{B}} \phi \wedge \psi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \phi \text{ and } \Vdash_{\mathcal{B}} \psi \tag{$\wedge$}$$

$$\Vdash_{\mathcal{B}} \phi \to \psi \quad \text{iff} \quad \phi \Vdash_{\mathcal{B}} \psi \tag{$\to$}$$

$$\Vdash_{\mathcal{B}} \forall x \phi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \phi[x \mapsto t] \text{ for any } t \in \overline{\mathbb{T}} \tag{$\forall$}$$

$$\Gamma \Vdash_{\mathcal{B}} \phi \quad \text{iff} \quad \text{for any } \mathcal{C} \supseteq \mathcal{B}, \text{ if } \Vdash_{\mathcal{C}} \psi \text{ for } \psi \in \Gamma, \text{ then } \Vdash_{\mathcal{C}} \phi \tag{Inf}$$

Sandqvist [38, 39] (see also Makinson [21] and Gheorghiu [11]) have shown that this characterises classical logic; that is,

$$\Gamma \Vdash \phi \qquad \text{iff} \qquad \phi \text{ follows classically from } \Gamma$$

Interestingly, $\Gamma \Vdash \phi$ is equivalent to $\Gamma \Vdash_{\emptyset} \phi$, suggesting that logical validity corresponds to *analytic* knowledge.

To express intuitionistic logic, we require extending the language with disjunction ($\vee$) and the existential quantifier ($\exists$). To this end, we may propose the following clauses:

$$\Vdash_{\mathcal{B}} \phi \vee \psi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \phi \text{ or } \Vdash_{\mathcal{B}} \phi \tag{$\vee$}$$

$$\Vdash_{\mathcal{B}} \bot \quad \text{iff} \quad \Vdash_{\mathcal{B}} P \text{ for any } P \in \overline{\mathbb{A}} \tag{$\exists$}$$

Piecha et al. [26, 25, 28] have shown that, surprisingly, intuitionistic logic is *incomplete* for this semantics. Subsequently, Stafford [48] showed that, in the propositional case, it corresponds to an intermediate logic known as *(general) inquisitive logic*.

We now observe that in the B-eS above, absurdity ($\bot$) is defined by *ex falso quodlibet*. This is quite unlike its treatment in more traditional M-tS. A philosophical motivation for this clause has been given by Dummett [7]. Following this motivation, Sandqvist [40] suggests the following alternative clauses:

$$\Vdash_{\mathcal{B}} \phi \vee \psi \quad \text{iff} \quad \text{for any } \mathcal{C} \supseteq \mathcal{B} \text{ and } P \in \overline{\mathbb{A}}, \text{ if } \phi \Vdash_{\mathcal{C}} P \text{ and } \psi \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P \tag{$\vee$}$$

$$\Vdash_{\mathcal{B}} \exists x \phi \quad \text{iff} \quad \text{for any } \mathcal{C} \supseteq \mathcal{B} \text{ and } P \in \overline{\mathbb{A}}, \text{ if } \phi[x \mapsto t] \Vdash_{\mathcal{C}} P \text{ for any } t \in \overline{\mathbb{T}}, \text{ then } \Vdash_{\mathcal{C}} P \tag{$\exists$}$$

Here $\phi[x \mapsto t]$ is the result of replacing every free occurrence of x in ϕ by t.

To capture intuitionistic logic, some modification must be required at this point. To see this, consider Peirce's Law, $((P \to Q) \to P) \to P$. This formula is classically but not intuitionistically valid. Since it only contains implications and atoms, it is valid in the B-eS before the clauses for disjunction ($\vee$) and existential quantifier ($\exists$) were added, but that corresponds to classical logic. Hence, this intuitionsitic logic is not complete for this B-eS.

We require only a small but significant change for intuitionistic logic: we now permit second-level rules in bases,

$$\frac{\begin{matrix}[\mathbb{A}_1] \\ P_1\end{matrix} \quad \cdots \quad \begin{matrix}[\mathbb{A}_n] \\ P_n\end{matrix}}{c}$$

Sandqvist [41] (see also Gheorghiu [11]) have shown that the result indeed corresponds to intuitionistic logic,

$$\Gamma \Vdash \phi \qquad \text{iff} \qquad \phi \text{ follows intuitionistically from } \Gamma$$

Though B-eS appears to be closely related to M-tS (esp. possible world semantics in the sense of Beth [3] and Kripke [20]), the formal connection remains an enigma. Indeed, while Makinson [21] (resp. Eckhardt and Pym [9, 8]) have made formal connections between the M-tS and B-eS of classical logic (resp. normal modal logic), the analogous connections for intuitionistic logics are currently unknown. Part of the challenge is in the considerably different ways that disjunctive structures (i.e., $\bot$, $\vee$, $\exists$) are treated.

The above work on the B-eS of classical and intuitionistic logic has been extended by Eckhardt and Pym [9, 8] to modal logic, by Gheorghiu et al. [12, 13] and Buzoku [5] to substructural logic (namely, *intuitionistic Linear Logic* and *the logic of Bunched Implications*), and by Nascimento et al. [24] to ecumenical logic. Closely related approaches have also been developed by Goldfarb [16] and Nascimento and Stafford [49, 23].

Bilteralism

Logical bilateralism can be very generally described as an approach to meaning and consequence on the grounds of a symmetry between certain notions, like assertion and denial, proof and refutation or truth and falsity, in that both are taken as primitive and not, as in conventional 'unilateralist' approaches, merely reducing the latter to the former, more primary notion. In recent years, the field of logical bilateralism has seen significant development with various systems being developed that showcase a range of orientations within this framework. In Rumfitt's seminal paper [37], in which the term 'bilateralism' was introduced, he means to give a motivation for how the natural deduction rules of classical logic lay down the meaning of the connectives once we consider a calculus containing introduction and elimination rules determining not only the assertion conditions for the connectives but also the denial conditions.

This is realized by using signed formulas in the form of '+A' and '–A' where '+' and '–' are used as force indicators. Smiley [47] developed a similar approach and there are also other (see, for example, work by Humberstone [17] and Restall [34]) and earlier (see, for example, Price [30, 31]) works promoting general bilateralist ideas. While several works explore and refine this approach to bilateralism in that the main focus is on natural deduction style proof systems with assertion and denial conditions (see, for example, Incurvati and Schlöder [18, 19] and Del Valle-Inclan and Schlöder [6]), there have been developments in other directions, in which bilateralist considerations play an equally central rôle. Some propose a new way of reading a (classical) sequent calculus with multiple conclusions, namely by way of defining an inference, represented by a sequent, as valid if and only if it is incoherent to assert all the premises (i.e., the formulas on the left side of the sequent sign), while simultaneously denying all the conclusions (i.e., the formulas on the right side of the sequent sign) — see, for example, Restall [32, 33] and Ripley [35, 36]. Here, the bilateralist considerations do not arise in the design of a distinctive proof system, but in the interpretation of an already existing proof calculus by way of taking assertion and denial as dual notions.

The approach presented in the special session focuses not so much on the speech acts of assertion and denial but on a duality between different inferential relationships, which in turn give rise to motivating proof systems with dual derivability relations. Such proof systems displaying provability and refutability can be represented both in natural deduction and in sequent calculus style (see, for example, Wansing [54] and Ayhan [1, 2]). On such a view it can be asked, then, how these dual derivability relations can be implemented on a meta-level. In a sequent calculus setting, for example, this would mean not only to have signed sequents, displaying provability and refutability within sequents, but also displaying the dual relations between sequents.

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