# Diagrammatic Algebra of First Order Logic


Filippo Bonchi
University of Pisa
Italy

Alessandro Di Giorgio
University College London
United Kingdom

Nathan Haydon
Tallinn University of Technology
Estonia

Paweł Sobociński
Tallinn University of Technology
Estonia



## ABSTRACT

We introduce the calculus of neo-Peircean relations, a string diagrammatic extension of the calculus of binary relations that has the same expressivity as first order logic and comes with a complete axiomatisation. The axioms are obtained by combining two well known categorical structures: cartesian and linear bicategories.



## CCS CONCEPTS

• **Theory of computation** → **Logic**; **Categorical semantics**.

## KEYWORDS

calculus of relations, string diagrams, deep inference

**ACM Reference Format:**
Filippo Bonchi, Alessandro Di Giorgio, Nathan Haydon, and Paweł Sobociński. 2024. Diagrammatic Algebra of First Order Logic. In *39th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS '24), July 8–11, 2024, Tallinn, Estonia.* ACM, New York, NY, USA, 15 pages. https://doi.org/10.1145/3661814.3662078


## 1 INTRODUCTION

The modern understanding of first order logic (FOL) is the result of an evolution with contributions from many philosophers and mathematicians. Amongst these, particularly relevant for our exposition is the calculus of relations (CR) by Charles S. Peirce [59]. Peirce, inspired by De Morgan [53], proposed a relational analogue of Boole's algebra [13]: a rigorous mathematical language for combining relations with operations governed by algebraic laws.

With the rise of first order logic, Peirce's calculus was forgotten until Tarski, who in [77] recognised its algebraic flavour. In the introduction to [78], written shortly before his death, Tarski put much emphasis on two key features of CR: (a) its lack of quantifiers and (b) its sole deduction rule of substituting equals by equals. The calculus, however, comes with two great shortcomings: (c) it is strictly less expressive than FOL and (d) it is *not* axiomatisable.

Despite these limitations, CR had —and continues to have— a great impact in computer science, e.g., in the theory of databases [22] and in the semantics of programming languages [2, 39, 45, 46, 69]. Indeed, the lack of quantifiers avoids the usual burden of bindings, scopes and capture-avoid substitutions (see [27, 32, 35, 41, 63, 65] for some theories developed to address specifically the issue of bindings). This feature, together with purely equational proofs, makes CR particularly suitable for proof assistants [44, 66, 67].

Less influential in computer science, there are two others quantifier-free alternatives to FOL that are worth mentioning: first, *predicate functor logic* (PFL) [70] that was thought by Quine as the first order logic analogue of combinatory logic [24] for the $\lambda$-calculus; second, Peirce's *existential graphs* (EGs) [73] and, in particular, its variant named *system $\beta$*. In this system FOL formulas are *diagrams* and the deduction system consists of rules for their manipulation. Peirce's work on EGs remained unpublished during his lifetime.

Diagrams have been used as formal entities since the dawn of computer science, e.g. in the Böhm-Jacopini theorem [3]. More recently, the spatial nature of mobile computations led Milner to move from the traditional term-based syntax of process calculi to bigraphs [51]. Similarly, the impossibility of copying quantum information and, more generally, the paradigm-shift of treating data as a physical resource (see e.g. [33, 57]), has led to the use [1, 5, 7, 11, 23, 28, 29, 34, 54, 64, 76] of *string diagrams* [43, 75] as syntax. String diagrams, formally arrows of a freely generated symmetric (strict) monoidal category, combine the rigour of traditional terms with a visual and intuitive graphical representation. Like traditional terms, they can be equipped with a compositional semantics.

In this paper, we introduce the calculus of *neo-Peircean relations*, a string diagrammatic account of FOL that has several key features:

(1) Its diagrammatic syntax is closely related to Peirce's EGs, but it can also be given through a context free grammar equipped with an elementary type system;
(2) It is quantifier-free and, differently than FOL, its compositional semantics can be given by few simple rules: see (8);
(3) Terms and predicates are not treated as separate syntactic and semantic entities;
(4) Its sole deduction rule is substituting equals by equals, like CR, but differently, it features a complete axiomatisation;
(5) The axioms are those of well-known algebraic structures, also occurring in different fields such as linear algebra [12] or quantum foundations [23];
(6) It allows for compositional encodings of FOL, CR and PFL;
(7) String diagrams disambiguate interesting corner cases where traditional FOL encounters difficulties. One perk is that we allow empty models —forbidden in classical treatments— leading to (slightly) more general Gödel completeness;
(8) The corner case of empty models coincides with *propositional* models and in that case our axiomatisation simplifies to the deep inference Calculus of Structures [15, 36].



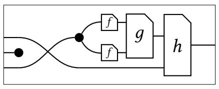
*LICS '24, July 8–11, 2024, Tallinn, Estonia*

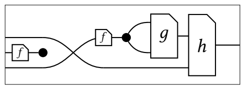


ACM ISBN 979-8-4007-0660-8/24/07
https://doi.org/10.1145/3661814.3662078

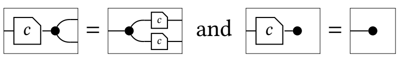
By returning to the algebraic roots of logic we preserve CR's benefits (a) and (b) while overcoming its limitations (c) and (d).

*Cartesian syntax.* To ease the reader into this work, we show how traditional terms appear as string diagrams. Consider a signature $\Sigma$ consisting of a unary symbol $f$ and two binary symbols $g$ and $h$. The term $h(\, g(f(x_3), f(x_3)), x_1\,)$ corresponds to the string diagram on the left below.




The explicit treatment of copying and discarding distinguishes diagrams from traditional syntax trees. The discharger informs us that the variable $x_2$ does not appear; the copier makes clear that the variable $x_3$ is shared by two sub-terms. The diagram on the right represents the same term if one admits the equations



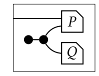
(Nat)

Fox [30] showed that (Nat) together with axioms asserting that copier and discard form a *comonoid* ((◀°-as), (◀°-un), (◀°-co) in Fig. 2) force the monoidal category of string diagrams to be *cartesian* ($\otimes$ is the categorical product): actually, it is the *free* cartesian category on $\Sigma$.

*Functorial semantics.* The work of Lawvere [47] illustrates the deep connection of syntax with semantics, explaining why cartesian syntax is so well-suited to functional structures, but also hinting at its limitations when denoting other structures, e.g. relations. Given an algebraic theory $\mathbb{T}$ in the universal algebraic sense, i.e., a signature $\Sigma$ with a set of equations $E$, one can freely generate a cartesian category $\mathbf{L}_{\mathbb{T}}$. *Models* –in the standard algebraic sense– are in one-to-one correspondence with cartesian functors $\mathcal{M}$ from $\mathbf{L}_{\mathbb{T}}$ to **Set**, the category of sets and functions. More generally, models of the theory in any cartesian category $\mathbf{C}$ are cartesian functors $\mathcal{M}\colon \mathbf{L}_{\mathbb{T}} \to \mathbf{C}$. By taking $\mathbf{C}$ to be $\mathbf{Rel}^\circ$, the category of sets and relations, one could wish to use the same approach for relational theories but any such attempt fails immediately since the cartesian product of sets is not the categorical product in $\mathbf{Rel}^\circ$.

*Cartesian bicategories.* An evolution of Lawvere's approach for relational structures is developed in [8, 10, 74]. Departing from cartesian syntax, it uses string diagrams generated by the *first* row of the grammar in Fig. 1, where $R$ is taken from a monoidal signature $\Sigma$ – a set of symbols equipped with both an arity and also a *coarity* – and can be thought of as akin to relation symbols of FOL. The diagrams are subject to the laws of cartesian bicategories [17] in Fig. 2: and form a comonoid, but the category of diagrams is not cartesian since the equations in (Nat) hold laxly ((◀°-nat), (!°-nat)). The diagrams and form a *monoid* ((▶°-as), (▶°-un), (▶°-co)) and are right *adjoint* to copier and discard. Monoids and comonoids together satisfy *special Frobenius* equations ((S°),(F°)). The category of diagrams $\mathbf{CB}_\Sigma$ is the free cartesian bicategory generated by $\Sigma$ and, like in Lawvere's functorial semantics, models are morphisms of cartesian bicategories $\mathcal{M}\colon \mathbf{CB}_\Sigma \to \mathbf{Rel}^\circ$. Importantly, the laws of cartesian bicategories provide a complete axiomatisation for $\mathbf{Rel}^\circ$, meaning that $c, d$ in $\mathbf{CB}_\Sigma$ are provably equal with the laws of cartesian bicategories iff $\mathcal{M}(c) = \mathcal{M}(d)$ for all models $\mathcal{M}$.

The (co)monoid structures allow one to express existential quantification: for instance, the FOL formula $\exists x_2.P(x_1, x_2) \wedge Q(x_2)$ is depicted as the diagram on the right. The expressive power of $\mathbf{CB}_\Sigma$ is, however, limited to the existential-conjunctive fragment of FOL.




*Cocartesian bicategories.* To express the universal-disjunctive fragment, we consider the category $\mathbf{CB}_\Sigma$ of string diagrams generated by the *second* row of the grammar in Fig. 1 and subject to the laws of *co*cartesian bicategories in Fig. 3: those of cartesian bicategories but with the reversed order $\geq$. The diagrams of $\mathbf{CB}_\Sigma$ are photographic negative of those in $\mathbf{CB}_\Sigma$. To explain this change of colour, note that sets and relations form *another* category: $\mathbf{Rel}^\bullet$. Composition ⨾ in $\mathbf{Rel}^\bullet$ is the De Morgan dual of the usual relational composition: $R ⨾ S \stackrel{\text{def}}{=} \{(x,z) \mid \exists y.(x,y) \in R \wedge (y,z) \in S\}$ while $R ⨾ S \stackrel{\text{def}}{=} \{(x,z) \mid \forall y.(x,y) \in R \vee (y,z) \in S\}$. While $\mathbf{Rel}^\circ$ is a cartesian bicategory, $\mathbf{Rel}^\bullet$ is *cocartesian*. Interestingly, this "black" composition ⨾ was used in Peirce's approach [58] to relational algebra.

Just as $\mathbf{CB}_\Sigma$ is complete with respect to $\mathbf{Rel}^\circ$, dually, $\mathbf{CB}_\Sigma$ is complete wrt $\mathbf{Rel}^\bullet$. The former accounts for the existential-conjunctive fragment of FOL; the latter for its universal-disjunctive fragment. This raises a natural question:

How do the white and black structures combine to form a complete account of first order logic?

*Linear bicategories.* Although $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$ have the same objects and arrows, there are two different compositions (⨾ and ⨾). The appropriate categorical structures to deal with these situations are *linear bicategories* introduced in [19] as a horizontal categorification of linearly distributive categories [21, 26]. The laws of linear bicategories are in Fig. 4: the key law is *linearly distributivity* of ⨾ over ⨾ (($\delta_l$), ($\delta_r$)), that was already known to hold for relations since the work of Peirce [58]. Another crucial property observed by Peirce is that for any relation $R \subseteq X \times Y$, the relation $R^\perp \subseteq Y \times X \stackrel{\text{def}}{=} \{(y,x) \mid (x,y) \notin R\}$ is its *linear adjoint*. This operation has an intuitive graphical representation: given $c$, take its mirror image $c$ and then its photographic negative $c$. For instance, the linear adjoint of $R$ is $R$.

*First order bicategories.* The final step is to characterise how cartesian, cocartesian and linear bicategories combine: *(i)* white and black (co)monoids are linear adjoints that *(ii)* satisfy a "linear" version of the Frobenius law. We dub the result *first order bicategories*. We shall see that this is a complete axiomatisation for first order logic, yet all of the algebraic machinery is compactly summarised at the right of Fig. 1.

*Functorial semantics for first order theories.* In the spirit of functorial semantics, we take the free first order bicategory $\mathbf{FOB}_{\mathbb{T}}$ generated by a theory $\mathbb{T}$ and observe that models of $\mathbb{T}$ in a first order bicategory $\mathbf{C}$ are morphisms $\mathcal{M}\colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$. Taking $\mathbf{C} = \mathbf{Rel}$, the first order bicategory of sets and relations, these are models in the sense of FOL with one notable exception: in FOL models with the empty domain are forbidden. As we shall wee, theories with empty models are exactly the propositional theories.

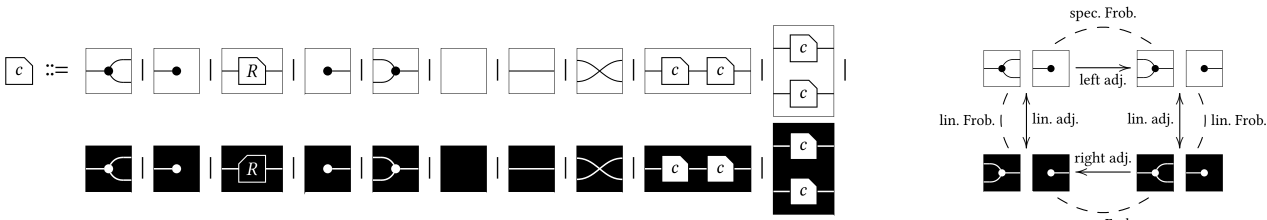
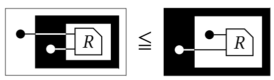


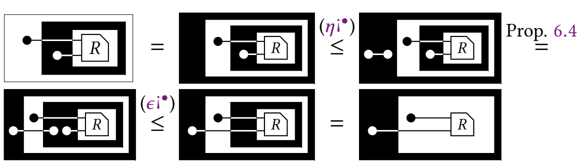

**Figure 1: Diagrammatic syntax of $\mathsf{NPR}_\Sigma$ (left) and a summary of its axioms (right)**

*Completeness.* We prove that the laws of first order bicategories provide a complete axiomatisation for first order logic. The proof is a diagrammatic adaptation of Henkin's proof [38] of Gödel's completeness theorem. However, in order to properly consider models with an empty domain, we make a slight additional step to go beyond Gödel completeness.

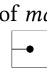
*A taste of diagrammatic logic.* Before we introduce the calculus of neo-Peircean relations, we start with a short worked example to give the reader a taste of using the calculus to prove a non-trivial result of first order logic. Doing so lets us illustrate the methodology of proof within the calculus, which is sometimes referred to as diagrammatic reasoning or string diagram surgery.

Let $R$ be a symbol with arity 2 and coarity 0. The two diagrams on the right correspond to FOL formulas $\exists x. \forall y. R(x,y)$ and $\forall y. \exists x. R(x,y)$: see § 9 for a dictionary of translating between FOL and diagrams. It is well-known that $\exists x. \forall y. R(x,y) \models \forall y. \exists x. R(x,y)$, i.e. in any model, if the first formula evaluates to true then so does the second. Within our calculus, this statement is expressed as the above inequality. This can be proved by mean of the axiomatisation we introduce in this work:






(1)

The central step relies on the particularly good behaviour of *maps*, intuitively those relations that are functional. In particular  is an example. The details are not important at this stage.

*Synopsis.* We begin by recalling CR in § 2. The calculus of neo-Peircean relations is introduced in § 3, together with the statement of our main result (Theorem 3.2). We recall (co)cartesian and linear bicategories in § 4 and § 5. The categorical structures most important for our work are first-order bicategories, introduced in § 6. In § 7 we consider first order theories, the diagrammatic version of the deduction theorem (Theorem 7.8) and some subtle differences with FOL that play an important role on the proof of completeness in § 8. Translations of CR and FOL into the calculus of neo-Peircean relations are given in § 8.1 and § 9. Fully detailed proofs and further additional material can be found in [6].

## 2 THE CALCULUS OF BINARY RELATIONS

The calculus of binary relations, in an original presentation given by Peirce in [58], features two forms of relational compositions $\fatsemi$ and $\bullet$, defined for all relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ as

$$\begin{aligned} R \fatsemi S &\stackrel{\text{def}}{=} \{(x,z) \mid \exists y \in Y \,.\, (x,y) \in R \wedge (y,z) \in S\} \subseteq X \times Z \text{ and} \\ R \,\bullet\, S &\stackrel{\text{def}}{=} \{(x,z) \mid \forall y \in Y \,.\, (x,y) \in R \vee (y,z) \in S\} \subseteq X \times Z \end{aligned} \quad (2)$$

with units the equality and the difference relations respectively, defined for all sets $X$ as

$$id^\circ_X \stackrel{\text{def}}{=} \{(x,y) \mid x = y\} \subseteq X \times X \text{ and } id^\bullet_X \stackrel{\text{def}}{=} \{(x,y) \mid x \neq y\} \subseteq X \times X. \quad (3)$$

Beyond the usual union $\cup$, intersection $\cap$, and their units $\bot$ and $\top$, the calculus also features two unary operations $(\cdot)^\dagger$ and $\overline{(\cdot)}$ denoting the opposite and the complement: $R^\dagger \stackrel{\text{def}}{=} \{(y,x) \mid (x,y) \in R\}$ and $\overline{R} \stackrel{\text{def}}{=} \{(x,y) \mid (x,y) \notin R\}$. In summary, its syntax is given by the following context free grammar

$$\begin{array}{lllllllllllll} E & ::= & R & \mid & id^\circ & \mid & E \fatsemi E & \mid & id^\bullet & \mid & E \bullet E & \mid \\ & & E^\dagger & \mid & \top & \mid & E \cap E & \mid & \bot & \mid & E \cup E & \mid & \overline{E} \end{array} \quad (\mathsf{CR}_\Sigma)$$

where $R$ is taken from a given set $\Sigma$ of generating symbols. The semantics (4) is defined wrt a *relational interpretation* $\mathcal{I}$, that is, a set $X$ together with a binary relation $\rho(R) \subseteq X \times X$ for each $R \in \Sigma$.

$$\begin{array}{lll} \langle R \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \rho(R) & \langle id^\circ \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^\circ_X & \langle E_1 \fatsemi E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \fatsemi \langle E_2 \rangle_{\mathcal{I}} \\ \langle E^\dagger \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E \rangle_{\mathcal{I}}^\dagger & \langle id^\bullet \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^\bullet_X & \langle E_1 \bullet E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \bullet \langle E_2 \rangle_{\mathcal{I}} \\ \langle \overline{E} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \overline{\langle E \rangle_{\mathcal{I}}} & \langle \bot \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \varnothing & \langle E_1 \cup E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}} \\ & \langle \top \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} X \times X & \langle E_1 \cap E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}} \end{array} \quad (4)$$

Two expressions $E_1, E_2$ are said to be *equivalent*, written $E_1 \equiv_{\mathsf{CR}} E_2$, if and only if $\langle E_1 \rangle_{\mathcal{I}} = \langle E_2 \rangle_{\mathcal{I}}$, for all interpretations $\mathcal{I}$. Inclusion, denoted by $\leq_{\mathsf{CR}}$, is defined analogously by replacing $=$ with $\subseteq$. For instance, the following inclusions hold, witnessing the fact that $\fatsemi$ *linearly distributes* over $\bullet$.

$$R \fatsemi (S \bullet T) \leq_{\mathsf{CR}} (R \fatsemi S) \bullet T \qquad (R \bullet S) \fatsemi T \leq_{\mathsf{CR}} R \bullet (S \fatsemi T) \quad (5)$$

Along with the boolean laws, in 'Note B' [58] Peirce states (5) and stresses the importance of this fact. However, since $R \bullet S \equiv_{\mathsf{CR}} \overline{\overline{R} \fatsemi \overline{S}}$ and $id^\bullet \equiv_{\mathsf{CR}} \overline{id^\circ}$, both $\bullet$ and $id^\bullet$ are often considered redundant, for instance by Tarski [77] and much of the later work.

Tarski asked whether $\equiv_{\mathsf{CR}}$ can be axiomatised, i.e., is there a basic set of laws from which one can prove all the valid equivalences? Unfortunately, there is no finite complete axiomatisations for the whole calculus [52] nor for several fragments, e.g., [4, 31, 40, 68, 72].

Our work returns to the same problem, but from a radically different perspective: we see the calculus of relations as a sub-calculus of a more general system for arbitrary (i.e. not merely binary) relations. The latter is strictly more expressive than $\mathsf{CR}_\Sigma$ – actually it is as expressive as first order logic (FOL)– but allows for an elementary complete axiomatisation based on the interaction of two influential algebraic structures: that of linear bicategories and cartesian bicategories.

## 3 NEO-PEIRCEAN RELATIONS

Here we introduce the calculus of *neo-Peircean relations* ($\mathsf{NPR}_\Sigma$).

The first step is to move from binary relations $R \subseteq X \times X$ to relations $R \subseteq X^n \times X^m$ where, for any $n \in \mathbb{N}$, $X^n$ denotes the set of row vectors $(x_1, \dots, x_n)$ with all $x_i \in X$. In particular, $X^0$ is the one element set $\mathbb{1} \stackrel{\text{def}}{=} \{\star\}$. Considering this kind of relations allows us to identify two novel fundamental constants: the *copier* $\blacktriangleleft^{\circ}_X \subseteq X \times X^2$ which is the diagonal function $\langle id^{\circ}_X, id^{\circ}_X \rangle \colon X \to X \times X$ (considered as a relation) and the *discharger* $!^{\circ}_X \subseteq X \times \mathbb{1}$ which is, similarly, the unique function from $X$ to $\mathbb{1}$. By combining them with opposite and complement we obtain, in total, 8 basic relations.

$$
\begin{array}{ll}
\blacktriangleleft^{\circ}_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x = y \wedge x = z\} & !^{\circ}_X \stackrel{\text{def}}{=} \{(x,\star) \mid x \in X\} \\
\blacktriangleright^{\circ}_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x = y \wedge x = z\} & \text{¡}^{\circ}_X \stackrel{\text{def}}{=} \{(\star,x) \mid x \in X\} \\
\hline
\blacktriangleleft^{\bullet}_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x \neq y \vee x \neq z\} & !^{\bullet}_X \stackrel{\text{def}}{=} \varnothing \\
\blacktriangleright^{\bullet}_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x \neq y \vee x \neq z\} & \text{¡}^{\bullet}_X \stackrel{\text{def}}{=} \varnothing
\end{array}
\tag{6}
$$

Together with $id^{\circ}_X$ and $id^{\bullet}_X$ and the compositions $\mathbin{;^{\circ}}$ and $\mathbin{;^{\bullet}}$ from (3), there are black and white *symmetries*: $\sigma^{\circ}_{X,Y} \stackrel{\text{def}}{=} \{(\ (x,y),(y,x)\ ) \mid x \in X, y \in Y\}$ and $\sigma^{\bullet}_{X,Y} \stackrel{\text{def}}{=} \overline{\sigma^{\circ}_{X,Y}}$. The calculus does *not* feature the boolean operators nor the opposite and the complement: these can be derived using the above structure and two *monoidal products* $\otimes$ and $\mathbin{\otimes^{\bullet}}$, defined for $R \subseteq X \times Y$ and $S \subseteq V \times W$ as

$$
\begin{array}{rcl}
R \otimes S & \stackrel{\text{def}}{=} & \{(\ (x,v),(y,w)\ ) \mid (x,y) \in R \wedge (v,w) \in S\} \\
R \mathbin{\otimes^{\bullet}} S & \stackrel{\text{def}}{=} & \{(\ (x,v),(y,w)\ ) \mid (x,y) \in R \vee (v,w) \in S\}\ .
\end{array}
\tag{7}
$$

*Syntax.* Terms are defined by the following context free grammar

$$
\begin{array}{rl}
c ::= & \blacktriangleleft^{\circ}_1 \mid !^{\circ}_1 \mid R^{\circ} \mid \text{¡}^{\circ}_1 \mid \blacktriangleright^{\circ}_1 \mid id^{\circ}_0 \mid id^{\circ}_1 \mid \sigma^{\circ}_{1,1} \mid c \mathbin{;^{\circ}} c \mid c \otimes c \mid \\
& \blacktriangleleft^{\bullet}_1 \mid !^{\bullet}_1 \mid R^{\bullet} \mid \text{¡}^{\bullet}_1 \mid \blacktriangleright^{\bullet}_1 \mid id^{\bullet}_0 \mid id^{\bullet}_1 \mid \sigma^{\bullet}_{1,1} \mid c \mathbin{;^{\bullet}} c \mid c \mathbin{\otimes^{\bullet}} c
\end{array}
\tag{$\mathsf{NPR}_\Sigma$}
$$

where $R$, like in $\mathsf{CR}_\Sigma$, belongs to a fixed set $\Sigma$ of *generators*. Differently than in $\mathsf{CR}_\Sigma$, each $R \in \Sigma$ comes with two natural numbers: arity $ar(R)$ and coarity $coar(R)$. The tuple $(\Sigma, ar, coar)$, usually simply $\Sigma$, is a *monoidal signature*. Intuitively, every $R \in \Sigma$ represents some relation $R \subseteq X^{ar(R)} \times X^{coar(R)}$.

In the first row of ($\mathsf{NPR}_\Sigma$) there are eight constants and two operations: white composition ($\mathbin{;^{\circ}}$) and white monoidal product ($\otimes$). These, together with identities ($id^{\circ}_0$ and $id^{\circ}_1$) and symmetry ($\sigma^{\circ}_{1,1}$) are typical of symmetric monoidal categories. Apart from $R^{\circ}$, the constants are the copier ($\blacktriangleleft^{\circ}_1$), discharger ($!^{\circ}_1$) and their opposite cocopier ($\blacktriangleright^{\circ}_1$) and codischarger ($\text{¡}^{\circ}_1$). The second row contains the "black" versions of the same constants and operations. Note that our syntax does not have variables, no quantifiers, nor the usual associated meta-operations like capture-avoiding substitution.

We shall refer to the terms generated by the first row as the *white fragment*, while to those of second row as the *black fragment*. Sometimes, we use the gray colour to be agnostic wrt white or black. The rules in top of Table 1 assigns to each term at most one type $n \to m$. We consider only those terms that can be typed. For all $n, m \in \mathbb{N}$, $id^{\circ}_n \colon n \to n$, $\sigma^{\circ}_{n,m} \colon n + m \to m + n$, $\blacktriangleleft^{\circ}_n \colon n \to n + n$, $\blacktriangleright^{\circ}_n \colon n + n \to n$, $!^{\circ}_n \colon n \to 0$ and $\text{¡}^{\circ}_n \colon 0 \to n$ are as in middle of Table 1.

*Semantics.* As for $\mathsf{CR}_\Sigma$, the semantics of $\mathsf{NPR}_\Sigma$ needs an interpretation $\mathcal{I} = (X, \rho)$: a set $X$, the *semantic domain*, and $\rho(R) \subseteq X^{ar(R)} \times X^{coar(R)}$ for each $R \in \Sigma$. The semantics of terms is:

$$
\begin{array}{llll}
\mathcal{I}^\sharp(\blacktriangleleft^{\circ}_1) \stackrel{\text{def}}{=} \blacktriangleleft^{\circ}_X & \mathcal{I}^\sharp(!^{\circ}_1) \stackrel{\text{def}}{=} !^{\circ}_X & \mathcal{I}^\sharp(\blacktriangleright^{\circ}_1) \stackrel{\text{def}}{=} \blacktriangleright^{\circ}_X & \mathcal{I}^\sharp(\text{¡}^{\circ}_1) \stackrel{\text{def}}{=} \text{¡}^{\circ}_X \\
\mathcal{I}^\sharp(id^{\circ}_0) \stackrel{\text{def}}{=} id^{\circ}_{\mathbb{1}} & \mathcal{I}^\sharp(id^{\circ}_1) \stackrel{\text{def}}{=} id^{\circ}_X & \mathcal{I}^\sharp(\sigma^{\circ}_{1,1}) \stackrel{\text{def}}{=} \sigma^{\circ}_{X,X} & \mathcal{I}^\sharp(R^{\circ}) \stackrel{\text{def}}{=} \rho(R) \\
\multicolumn{4}{l}{\mathcal{I}^\sharp(c \mathbin{;} d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \mathbin{;} \mathcal{I}^\sharp(d) \quad \mathcal{I}^\sharp(c \circledast d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \circledast \mathcal{I}^\sharp(d) \quad \mathcal{I}^\sharp(R^{\bullet}) \stackrel{\text{def}}{=} \overline{\rho(R)}^{\dagger}}
\end{array}
\tag{8}
$$

The constants and operations appearing on the right-hand-side of the above equations are amongst those defined in (2), (3), (6), (7). A simple inductive argument confirms that $\mathcal{I}^\sharp$ maps terms $c$ of type $n \to m$ to relations $R \subseteq X^n \times X^m$. In particular, $id^{\circ}_0 \colon 0 \to 0$ is sent to $id^{\circ}_{\mathbb{1}} \subseteq \mathbb{1} \times \mathbb{1}$, since $X^0 = \mathbb{1}$ independently of $X$. Note that there are only two relations on the singleton set $\mathbb{1} = \{\star\}$: the relation $\{(\star,\star)\} \subseteq \mathbb{1} \times \mathbb{1}$ and the empty relation $\varnothing \subseteq \mathbb{1} \times \mathbb{1}$. These are, by (3), $id^{\circ}_{\mathbb{1}}$ and $id^{\bullet}_{\mathbb{1}}$, embodying *truth* and *falsity*.

*Example 3.1.* Take $\Sigma$ with two symbols $R$ and $S$ with arity and coarity 1. From Table 1, the two terms below have type $1 \to 1$.

$$
!^{\circ}_1 \mathbin{;^{\circ}} \text{¡}^{\circ}_1 \qquad \blacktriangleleft^{\circ}_1 \mathbin{;^{\circ}} ((R^{\circ} \otimes S^{\circ}) \mathbin{;^{\circ}} \blacktriangleright^{\circ}_1)
\tag{9}
$$

For any interpretation $\mathcal{I} = (X, \rho)$, $\mathcal{I}^\sharp(!^{\circ}_1 \mathbin{;^{\circ}} \text{¡}^{\circ}_1)$ is the top $X \times X$:

$$
\begin{aligned}
\mathcal{I}^\sharp(!^{\circ}_1 \mathbin{;^{\circ}} \text{¡}^{\circ}_1) &= !^{\circ}_X \mathbin{;^{\circ}} \text{¡}^{\circ}_X = \{(x,\star) \mid x \in X\} \mathbin{;^{\circ}} \{(\star,x) \mid x \in X\} \\
&= \{(x,y) \mid x,y \in X\} = X \times X = \langle \top \rangle_{\mathcal{I}}.
\end{aligned}
$$

Similarly, $\mathcal{I}^\sharp(\blacktriangleleft^{\circ}_1 \mathbin{;^{\circ}} ((R^{\circ} \otimes S^{\circ}) \mathbin{;^{\circ}} \blacktriangleright^{\circ}_1) = \rho(R) \cap \rho(S) = \langle R \cap S \rangle_{\mathcal{I}}$.

Remark 1. *$\mathsf{NPR}_\Sigma$ is as expressive as FOL. We draw the reader's attention to the simplicity of the inductive definition of semantics compared to the traditional FOL approach where variables and quantifiers make the definition more involved. Moreover, in traditional accounts, the domain of an interpretation is required to be a non-empty set. In our calculus this is unnecessary and it is* not *a mere technicality: in § 7 we shall see that empty models capture the propositional calculus.*

Two terms $c, d \colon n \to m$ are *semantically equivalent*, written $c \equiv d$, if and only if $\mathcal{I}^\sharp(c) = \mathcal{I}^\sharp(d)$, for all interpretations $\mathcal{I}$. *Semantic inclusion* ($\leqq$) is defined analogously replacing $=$ with $\subseteq$.

By definition $\equiv$ and $\leqq$ only relate terms of the same type. Throughout the paper, we will encounter several relations amongst terms of the same type. To avoid any confusion with the relations denoted by the terms, we call them *well-typed relations* and use symbols $\mathbb{I}$ rather than the usual $R, S, T$. In the following, we write $c \mathbb{I} d$ for $(c, d) \in \mathbb{I}$ and $\mathsf{pc}(\mathbb{I})$ for the smallest precongruence (w.r.t. $\mathbin{;^{\circ}}$, $\mathbin{;^{\bullet}}$, $\otimes$ and $\mathbin{\otimes^{\bullet}}$) generated by $\mathbb{I}$, i.e., the relation inductively generated as

$$
\begin{array}{ccc}
\dfrac{c \mathbb{I} d}{c \,\mathsf{pc}(\mathbb{I})\, d}\ (id) & \dfrac{-}{c \,\mathsf{pc}(\mathbb{I})\, c}\ (r) & \dfrac{a \,\mathsf{pc}(\mathbb{I})\, b \quad b \,\mathsf{pc}(\mathbb{I})\, c}{a \,\mathsf{pc}(\mathbb{I})\, c}\ (t) \\[2ex]
\multicolumn{3}{c}{\dfrac{c_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \quad d_1 \,\mathsf{pc}(\mathbb{I})\, d_2}{c_1 \mathbin{;} d_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \mathbin{;} d_2}\ (;) \qquad \dfrac{c_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \quad d_1 \,\mathsf{pc}(\mathbb{I})\, d_2}{c_1 \circledast d_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \circledast d_2}\ (\circledast)}
\end{array}
\tag{10}
$$

*Diagrams.* The terms of our calculus enjoy a convenient graphical representation inspired by string diagrams [43, 75], formally arrows of a free symmetric (strict) monoidal category. A term $c \colon n \to m$ is depicted as a diagram with $n$ ports on the left and $m$ ports on the right; $\mathbin{;^{\circ}}$ is depicted as horizontal composition while $\otimes$ by vertically "stacking" diagrams. However, since there are two

**Table 1: Typing rules (top); inductive definitions of syntactic sugar (middle); structural congruence (bottom)**

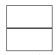
$$id^{\circ}_0 : 0 \to 0 \quad id^{\circ}_1 : 1 \to 1 \quad \sigma^{\circ}_{1,1} : 2 \to 2 \qquad \frac{ar(R) = n \quad coar(R) = m}{R^{\circ} : n \to m} \qquad \frac{ar(R) = n \quad coar(R) = m}{R^{\bullet} : m \to n} \qquad \frac{c : n_1 \to m_1 \quad d : n_2 \to m_2}{c \otimes d : n_1 + n_2 \to m_1 + m_2} \qquad \frac{c : n \to m \quad d : m \to o}{c \fatsemi d : n \to o}$$

$$\blacktriangleleft^{\circ}_1 : 1 \to 2 \quad !^{\circ}_1 : 1 \to 0 \quad \blacktriangleright^{\circ}_1 : 2 \to 1 \quad ¡^{\circ}_1 : 0 \to 1$$

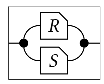
$$\blacktriangleleft^{\circ}_0 = id^{\circ}_0 \quad \blacktriangleleft^{\circ}_{n+1} = (\blacktriangleleft^{\circ}_1 \otimes \blacktriangleleft^{\circ}_n) \fatsemi (id^{\circ}_1 \otimes \sigma^{\circ}_{1,n} \otimes id^{\circ}_n) \qquad !^{\circ}_0 = id^{\circ}_0 \quad !^{\circ}_{n+1} = !^{\circ}_1 \otimes !^{\circ}_n \qquad id^{\circ}_0 = id^{\circ}_0 \qquad \sigma^{\circ}_{0,0} = id^{\circ}_0 \qquad \sigma^{\circ}_{1,0} = \sigma^{\circ}_{0,1} = id^{\circ}_1$$

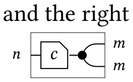
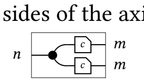
$$\blacktriangleright^{\circ}_0 = id^{\circ}_0 \quad \blacktriangleright^{\circ}_{n+1} = (id^{\circ}_1 \otimes \sigma^{\circ}_{1,n} \otimes id^{\circ}_n) \fatsemi (\blacktriangleright^{\circ}_1 \otimes \blacktriangleright^{\circ}_n) \qquad ¡^{\circ}_0 = id^{\circ}_0 \quad ¡^{\circ}_{n+1} = ¡^{\circ}_1 \otimes ¡^{\circ}_n \qquad id^{\circ}_{n+1} = id^{\circ}_1 \otimes id^{\circ}_n \qquad \sigma^{\circ}_{1,n+1} = (\sigma^{\circ}_{1,n} \otimes id^{\circ}_1) \fatsemi (id^{\circ}_n \otimes \sigma^{\circ}_{1,1}) \qquad \sigma^{\circ}_{m+1,n} = (id^{\circ}_1 \otimes \sigma^{\circ}_{m,n}) \fatsemi (\sigma^{\circ}_{1,n} \otimes id^{\circ}_m)$$

$$a \fatsemi (b \fatsemi c) = (a \fatsemi b) \fatsemi c \quad id^{\circ}_n \fatsemi c = c = c \fatsemi id^{\circ}_m \quad (a \otimes b) \otimes c = a \otimes (b \otimes c) \quad id^{\circ}_0 \otimes c = c = id^{\circ}_0 \otimes c \quad (a \otimes b) \fatsemi (c \otimes d) = (a \fatsemi c) \otimes (b \fatsemi d) \quad \sigma^{\circ}_{1,1} \fatsemi \sigma^{\circ}_{1,1} = id^{\circ}_2 \quad (c \otimes id^{\circ}_o) \fatsemi \sigma^{\circ}_{m,o} = \sigma^{\circ}_{n,o} \fatsemi (id^{\circ}_o \otimes c)$$

compositions $\fatsemi$ and $\bullet$ and two monoidal products $\otimes$ and $\circledast$, to distinguish them we use different colours. All constants in the white fragment have white background, mutatis mutandis for the black fragment: for instance $id^{\circ}_1$ and $id^{\bullet}_1$ are drawn  and . Indeed, the diagrammatic version of ($\mathsf{NPR}_\Sigma$) is given by the grammar on the left of Fig. 1.

Note that one diagram may correspond to more than one term: for instance the diagram on the right does not only represent the rightmost term in (9), namely $\blacktriangleleft^{\circ}_1 \fatsemi ((R^{\circ} \otimes S^{\circ}) \fatsemi \blacktriangleright^{\circ}_1)$, but also $(\blacktriangleleft^{\circ}_1 \fatsemi (R^{\circ} \otimes S^{\circ})) \fatsemi \blacktriangleright^{\circ}_1$. Indeed, it is clear that traditional term-based syntax carries more information than the diagrammatic one (e.g. associativity). From the point of view of the semantics, however, this bureaucracy is irrelevant and is conveniently discarded by the diagrammatic notation. To formally show this, we recall that diagrams capture only the axioms of symmetric monoidal categories [43, 75], illustrated in Table 1; we call *structural congruence*, written $\approx$, the well-typed congruence generated by such axioms and we observe that $\approx \subseteq \equiv$.




*Axioms.* Figs. 2, 3, 4, 5 illustrate the axioms of first-order bicategories, the categorical structure that we shall introduce in § 6. These laws also provide a complete axiomatisation for $\mathsf{NPR}_\Sigma$: from each picture of these figures, one can obtain a diagram by substituting the letters $X, Y$ with arbitrary $n, m \in \mathbb{N}$ and the letter $a, b, c, d$ with arbitrary diagrams of the appropriate type. For instance the left and the right hand sides of the axiom ($\blacktriangleleft^{\circ}$-nat) in Figure 2 become  and  for all $n, m \in \mathbb{N}$ and $c\colon n \to m$ generated by the grammar in Fig.1. [1] Let $\mathbb{FOB}$ be the well-typed relation containing the pairs of diagrams $(c, d)$ obtained via such substitutions for the axioms in Figs. 2, 3, 4, 5 and call its precongruence closure *syntactic inclusion*, written $\lesssim$. In symbols $\lesssim = \mathsf{pc}(\mathbb{FOB})$. We will also write $\cong \stackrel{\text{def}}{=} \lesssim \cap \gtrsim$. Our main result is:

Theorem 3.2. *For all diagrams $c, d\colon n \to m$, $c \lesssim d$ iff $c \leqq d$.*

Note that axiomatisation is far from minimal and is redundant in several respects. We chose the more verbose presentation in order to emphasise both the underlying categorical structures and the various dualities that we will highlight in the next sections.

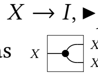
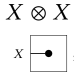
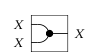
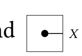
*Proofs as diagrams rewrites.* Proofs in $\mathsf{NPR}_\Sigma$ are rather different from those of traditional proof systems: since the only inference rules are those in (10), any proof of $c \lesssim d$ consists of a sequence of applications of axioms. As an example consider (1) from the Introduction (see [6, App. B.1] for a proof not using Prop. 6.4). Note that, when applying axioms, we are in fact performing diagram rewriting: an instance of the left hand side of an axiom is found within a larger diagram and replaced with the right hand side. Since such rewrites can happen anywhere, there is a close connection between proofs in $\mathsf{NPR}_\Sigma$ and *deep inference* [15, 36, 42]: see Ex. 7.7.

[1] An axiomatisation on terms, rather than on diagrams, is illustrated in [6, Fig. 9].

## 4 (CO)CARTESIAN BICATEGORIES

Although the term bicategory might seem ominous, the beasts considered in this paper are actually quite simple. We consider *poset enriched symmetric monoidal categories*: every homset carries a partial order $\leq$, and composition $\fatsemi$ and monoidal product $\otimes$ are monotone. That is, if $a \leq b$ and $c \leq d$ then $a \fatsemi c \leq b \fatsemi d$ and $a \otimes c \leq b \otimes d$. A *poset enriched symmetric monoidal functor* is a (strong, and usually strict) symmetric monoidal functor that preserves the order $\leq$. The notion of *adjoint arrows*, which will play a key role, amounts to the following: for $c\colon X \to Y$ and $d\colon Y \to X$, $c$ is *left adjoint* to $d$, or $d$ is *right adjoint* to $c$, written $d \vdash c$, if $id^{\circ}_X \leq c \fatsemi d$ and $d \fatsemi c \leq id^{\circ}_Y$. We extend such terminology to pairs of arrows: $(a, b)$ is left adjoint to $(c, d)$ iff $c \vdash a$ and $d \vdash b$.

For a symmetric monoidal bicategory $(\mathbf{C}, \otimes, I)$, we will write $\mathbf{C}^{\mathsf{op}}$ for the bicategory having the same objects as $\mathbf{C}$ but homsets $\mathbf{C}^{\mathsf{op}}[X, Y] \stackrel{\text{def}}{=} \mathbf{C}[Y, X]$: ordering, identities and monoidal product are defined as in $\mathbf{C}$, while composition $c \fatsemi d$ in $\mathbf{C}^{\mathsf{op}}$ is $d \fatsemi c$ in $\mathbf{C}$. Similarly, we will write $\mathbf{C}^{\mathsf{co}}$ to denote the bicategory having the same objects and arrows of $\mathbf{C}$ but equipped with the reversed ordering $\geq$. Composition, identities and monoidal product are defined as in $\mathbf{C}$. In this paper, we will often tacitly use the facts that, by definition, both $(\mathbf{C}^{\mathsf{op}})^{\mathsf{op}}$ and $(\mathbf{C}^{\mathsf{co}})^{\mathsf{co}}$ are $\mathbf{C}$ and that $(\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ is $(\mathbf{C}^{\mathsf{op}})^{\mathsf{co}}$.

All monoidal categories considered throughout this paper are tacitly assumed to be strict [48], i.e. $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $I \otimes X = X = X \otimes I$ for all objects $X, Y, Z$. This is harmless: strictification [48] allows to transform any monoidal category into a strict one, enabling the sound use of string diagrams. These will be exploited in this and the next two sections to describe the categorical structures of interest. In particular, in the following definition $\blacktriangleleft^{\circ}_X : X \to X \otimes X$, $!^{\circ}_X : X \to I$, $\blacktriangleright^{\circ}_X : X \otimes X \to X$ and $¡^{\circ}_X : I \to X$ are drawn, respectively, as  ,  ,  and  .

*Definition 4.1.* A *cartesian bicategory* $(\mathbf{C}, \otimes, I, \blacktriangleleft^{\circ}, !^{\circ}, \blacktriangleright^{\circ}, ¡^{\circ})$, shorthand $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$, is a poset enriched symmetric monoidal category $(\mathbf{C}, \otimes, I)$ and, for every object $X$ in $\mathbf{C}$, arrows $\blacktriangleleft^{\circ}_X : X \to X \otimes X$, $!^{\circ}_X : X \to I$, $\blacktriangleright^{\circ}_X : X \otimes X \to X$, $¡^{\circ}_X : I \to X$ s.t.
1. $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ is a comonoid and $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$ a monoid (i.e., ($\blacktriangleleft^{\circ}$-as), ($\blacktriangleleft^{\circ}$-un), ($\blacktriangleleft^{\circ}$-co) and ($\blacktriangleright^{\circ}$-as), ($\blacktriangleright^{\circ}$-un), ($\blacktriangleright^{\circ}$-co) in Fig. 2 hold);
2. arrows $c\colon X \to Y$ are lax comonoid morphisms (($\blacktriangleleft^{\circ}$-nat), ($!^{\circ}$-nat));
3. $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ are left adjoints to $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$ (($\eta \blacktriangleleft^{\circ}$), ($\epsilon \blacktriangleleft^{\circ}$), ($\eta !^{\circ}$), ($\epsilon !^{\circ}$));
4. $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ and $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$ form special Frobenius algebras (($\mathsf{F}^{\circ}$), ($\mathsf{S}^{\circ}$));

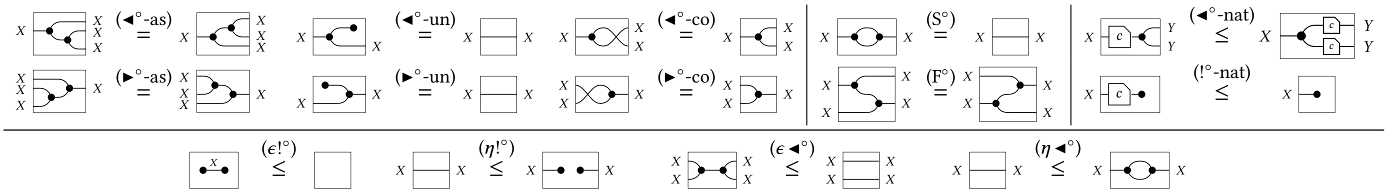


**Figure 2: Axioms of cartesian bicategories**

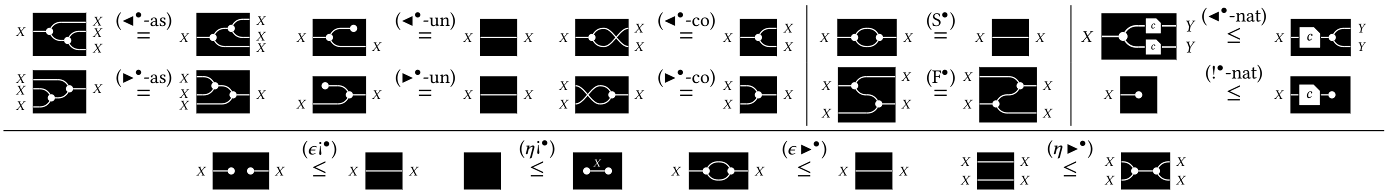


**Figure 3: Axioms of cocartesian bicategories**

5. $(\blacktriangleleft^\circ_X, !^\circ_X)$ and $(\blacktriangleright^\circ_X, ¡^\circ_X)$ satisfy the coherence conditions:[2]

$$\begin{array}{ll} \blacktriangleleft^\circ_I = id^\circ_I & \blacktriangleleft^\circ_{X\otimes Y} = (\blacktriangleleft^\circ_X \otimes \blacktriangleleft^\circ_Y) ; (id^\circ_X \otimes \sigma^\circ_{X,Y} \otimes id^\circ_Y) \\ \blacktriangleright^\circ_I = id^\circ_I & \blacktriangleright^\circ_{X\otimes Y} = (id^\circ_X \otimes \sigma^\circ_{X,Y} \otimes id^\circ_Y) ; (\blacktriangleright^\circ_X \otimes \blacktriangleright^\circ_Y) \\ !^\circ_I = id^\circ_I & !^\circ_{X\otimes Y} = !^\circ_X \otimes !^\circ_Y \qquad ¡^\circ_I = id^\circ_I \qquad ¡^\circ_{X\otimes Y} = ¡^\circ_X \otimes ¡^\circ_Y \end{array}$$

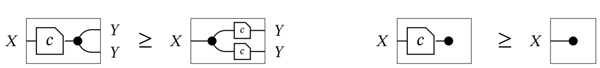
**C** is a *cocartesian bicategory* if $\mathbf{C}^{\mathsf{co}}$ is a cartesian bicategory. A *morphism of (co)cartesian bicategories* is a poset enriched strong symmetric monoidal functor preserving monoids and comonoids.

Remark 2. *The structures in the above definition were originally referred as "cartesian bicategory of relations" in [17].*

The archetypal example is $(\mathbf{Rel}^\circ, \blacktriangleleft^\circ, \blacktriangleright^\circ)$. $\mathbf{Rel}^\circ$ is the bicategory of sets and relations ordered by inclusion $\subseteq$ with white composition $;$ and identities $id^\circ$ defined as in (2) and (3). The monoidal product on objects is the cartesian product of sets with unit $I$ the singleton set $\mathbb{1}$. On arrows, $\otimes$ is defined as in (7). It is immediate to check that, for every set $X$, the arrows $\blacktriangleleft^\circ_X, !^\circ_X$ defined in (6) form a comonoid in $\mathbf{Rel}^\circ$, while $\blacktriangleright^\circ_X, ¡^\circ_X$ a monoid. Simple computations also proves all the (in)equalities in Fig. 2.

The fact that relations are lax comonoid homomorphisms is the most interesting to show: since $R ; \blacktriangleleft^\circ_Y = \{(x,(y,y)) \mid (x,y) \in R\}$ is included in $\{(x,(y,z)) \mid (x,y) \in R \wedge (x,z) \in R\} = \blacktriangleleft^\circ_X ;(R \otimes R)$ and $R ; !^\circ_Y = \{(x,\star) \mid \exists y \in X \,.\, (x,y) \in R\}$ in $\{(x,\star) \mid x \in X\} = !^\circ_X$ for any relation $R \subseteq X \times Y$, ($\blacktriangleleft^\circ$-nat) and ($!^\circ$-nat) hold. The reversed inclusions are interesting to consider: $R ; \blacktriangleleft^\circ_Y \supseteq \blacktriangleleft^\circ_X ;(R \otimes R)$ holds iff the relation $R$ is single valued, while $R ; !^\circ_Y \supseteq !^\circ_X$ iff $R$ a total. That is, the two inequalities in Definition 4.1.(2) are equalities iff the relation $R$ is a function. This justifies the following:

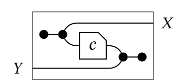
*Definition 4.2.* An arrow $c \colon X \to Y$ is a *map* if



It is easy to see that maps form a monoidal subcategory of **C** [17], hereafter denoted by $\mathsf{Map}(\mathbf{C})$. In fact, it is cartesian.

[2]Note that the coherence conditions are not in Fig. 2 since they hold in $\mathsf{NPR}_\Sigma$, given the inductive definitions in the middle of Tab. 1.

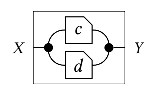
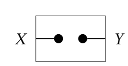
Given a cartesian bicategory $(\mathbf{C}, \blacktriangleleft^\circ, \blacktriangleright^\circ)$, one can take $\mathbf{C}^{\mathsf{op}}$, swap monoids and comonoids and thus, obtain a cartesian bicategory $(\mathbf{C}^{\mathsf{op}}, \blacktriangleright^\circ, \blacktriangleleft^\circ)$. Most importantly, there is an identity on objects isomorphism $(\cdot)^\dagger \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ defined for all arrows $c \colon X \to Y$ as

$c^\dagger \stackrel{\text{def}}{=}$  (11)

Proposition 4.3. $(\cdot)^\dagger \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ *is an isomorphism of cartesian bicategories, namely the laws in the first three rows of Table 2.(a) hold.*

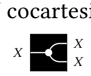
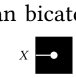
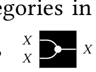
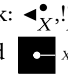
Hereafter, we write $c$ for $c^\dagger$ and we call it the *mirror image* of $c$. Note that in § 2, we used the same symbol $(\cdot)^\dagger$ to denote the converse relation. This is no accident: in the cartesian bicategory $(\mathbf{Rel}^\circ, \blacktriangleleft^\circ, \blacktriangleright^\circ)$, $R^\dagger$ as in (11) is exactly $\{(y,x) \mid (x,y) \in R\}$.

In a cartesian bicategory, one can also define, for all arrows $c, d \colon X \to Y$, $c \sqcap d$ and $\top$ as follows.

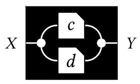
$c \sqcap d \stackrel{\text{def}}{=}$  $\top \stackrel{\text{def}}{=}$  (12)

We have already seen in Example 3.1 that these terms, when interpreted in $\mathbf{Rel}^\circ$, denote respectively intersection and top. It is easy to show that in any cartesian bicategory **C**, $\sqcap$ is associative, commutative, idempotent and has $\top$ as unit. Namely, $\mathbf{C}[X,Y]$ is a meet-semilattice with top. However, **C** is usually *not* enriched over meet-semilattices since $;$ distributes only laxly over $\sqcap$. Indeed, in $\mathbf{Rel}^\circ$, $R ; (S \cap T) \subseteq (R ; S) \cap (R ; T)$ holds but the reverse does not.

Let us now turn to *co*cartesian bicategories. Our main example is $(\mathbf{Rel}^\bullet, \blacktriangleleft^\bullet, \blacktriangleright^\bullet)$. $\mathbf{Rel}^\bullet$ is the bicategory of sets and relations ordered by $\subseteq$ with composition $\bullet$, identities $id^\bullet$ and $\otimes$ defined as in (2), (3) and (7). Comonoids $(\blacktriangleleft^\bullet_X, !^\bullet_X)$ and monoids $(\blacktriangleright^\bullet_X, ¡^\bullet_X)$ are those of (6). To see that $\mathbf{Rel}^\bullet$ is a cocartesian bicategory, observe that the complement $\overline{(\cdot)}$ is a poset-enriched symmetric monoidal isomorphism $\overline{(\cdot)} \colon (\mathbf{Rel}^\circ)^{\mathsf{co}} \to \mathbf{Rel}^\bullet$ preserving (co)monoids.

We draw arrows of cocartesian bicategories in black: $\blacktriangleleft^\bullet_X, !^\bullet_X$, $\blacktriangleright^\bullet_X$ and $¡^\bullet_X$ are drawn  ,  ,  and  . Following this convention, the axioms of cocartesian bicategories are in Fig. 3; they can also be obtained from Fig. 2 by inverting both the colours and the order.

It is not surprising that in a cocartesian bicategory $\mathbf{C}$, every homset $\mathbf{C}[X, Y]$ carries a join semi-lattice with bottom, where $c \sqcup d$ and $\bot$ are defined for all arrows $c, d \colon X \to Y$ as follows.

$c \sqcup d \stackrel{\text{def}}{=}$  $\quad \bot \stackrel{\text{def}}{=}$  (13)

## 5 LINEAR BICATEGORIES

We have seen that $\mathbf{Rel}^{\circ}$ forms a cartesian, and $\mathbf{Rel}^{\bullet}$ a cocartesian bicategory. Categorically, they are remarkably similar — as evidenced by the isomorphism $\overline{(\cdot)}$ — but from a logical viewpoint they represent two complementary parts of FOL: $\mathbf{Rel}^{\circ}$ the existential conjunctive fragment, and $\mathbf{Rel}^{\bullet}$ the universal disjunctive fragment. To discover the full story, we must merge them into one entity and study the algebraic interactions between them. However, the coexistence of two different compositions $\fatsemi$ and $\mathbin{\boldsymbol{;}}$ brings us out of the realm of ordinary categories. The solution is linear bicategories [19]. Here $\fatsemi$ linearly distributes over $\mathbin{\boldsymbol{;}}$, as in Peirce's calculus. To keep our development easier, we stick to the poset enriched case and rely on diagrams, using white and black to distinguish $\fatsemi$ and $\mathbin{\boldsymbol{;}}$.

*Definition 5.1.* A *linear bicategory* $(\mathbf{C}, \fatsemi, id^{\circ}, \mathbin{\boldsymbol{;}}, id^{\bullet})$ consists of two poset enriched categories $(\mathbf{C}, \fatsemi, id^{\circ})$ and $(\mathbf{C}, \mathbin{\boldsymbol{;}}, id^{\bullet})$ with the same objects, arrows and orderings but possibly different identities and compositions such that $\fatsemi$ linearly distributes over $\mathbin{\boldsymbol{;}}$ (i.e., $(\delta_l)$ and $(\delta_r)$ in Fig. 4 hold). A *symmetric monoidal linear bicategory* $(\mathbf{C}, \fatsemi, id^{\circ}, \mathbin{\boldsymbol{;}}, id^{\bullet}, \otimes, \sigma^{\circ}, \mathbin{\boldsymbol{\otimes}}, \sigma^{\bullet}, I)$, shortly $(\mathbf{C}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$, consists of a linear bicategory $(\mathbf{C}, \fatsemi, id^{\circ}, \mathbin{\boldsymbol{;}}, id^{\bullet})$ and two poset enriched symmetric monoidal categories $(\mathbf{C}, \otimes, I)$ and $(\mathbf{C}, \mathbin{\boldsymbol{\otimes}}, I)$ such that $\otimes$ and $\mathbin{\boldsymbol{\otimes}}$ agree on objects, i.e., $X \otimes Y = X \mathbin{\boldsymbol{\otimes}} Y$, share the same unit $I$ and
1. there are linear strengths for $(\otimes, \mathbin{\boldsymbol{\otimes}})$, (i.e., $(\nu_l^{\circ})$, $(\nu_r^{\circ})$, $(\nu_l^{\bullet})$, $(\nu_r^{\bullet})$);
2. $\mathbin{\boldsymbol{\otimes}}$ preserves $id^{\circ}$ colaxly and $\otimes$ preserves $id^{\bullet}$ laxly $((\otimes^{\bullet}), (\mathbin{\boldsymbol{\otimes}}^{\circ}))$.

A *morphism of symmetric monoidal linear bicategories* $\mathcal{F} \colon (\mathbf{C_1}, \otimes, \mathbin{\boldsymbol{\otimes}}, I) \to (\mathbf{C_2}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$ consists of two poset enriched symmetric monoidal functors $\mathcal{F}^{\circ} \colon (\mathbf{C_1}, \otimes, I) \to (\mathbf{C_2}, \otimes, I)$ and $\mathcal{F}^{\bullet} \colon (\mathbf{C_1}, \mathbin{\boldsymbol{\otimes}}, I) \to (\mathbf{C_2}, \mathbin{\boldsymbol{\otimes}}, I)$ that agree on objects and arrows: $\mathcal{F}^{\circ}(X) = \mathcal{F}^{\bullet}(X)$ and $\mathcal{F}^{\circ}(c) = \mathcal{F}^{\bullet}(c)$ for all objects $X$ and arrows $c$.

Remark 3. *In the literature* $\fatsemi$*,* $id^{\circ}$*,* $\mathbin{\boldsymbol{;}}$ *and* $id^{\bullet}$ *are written with the linear logic notation* $\otimes, \top, \oplus$ *and* $\bot$*. Modulo this, the traditional notion of linear bicategory (Definition 2.1 in [19]) coincides with the one in Definition 5.1 whenever the 2-structure is collapsed to a poset.*

*Monoidal products on linear bicategories are not much studied although the axioms in Definition 5.1.1 already appeared in [55]. They are the linear strengths of the pair* $(\otimes, \mathbin{\boldsymbol{\otimes}})$ *seen as a linear functor (Definition 2.4 in [19]), a notion of morphism that crucially differs from ours on the fact that the* $\mathcal{F}^{\circ}$ *and* $\mathcal{F}^{\bullet}$ *may not coincide on arrows. Instead the inequalities* $(\otimes^{\bullet})$ *and* $(\mathbin{\boldsymbol{\otimes}}^{\circ})$ *are, to the best of our knowledge, novel. Beyond being natural, they are crucial for Lemma 5.2 below.*

All linear bicategories in this paper are symmetric monoidal. We therefore omit the adjective *symmetric monoidal* and refer to them simply as linear bicategories. For a linear bicategory $(\mathbf{C}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$, we will often refer to $(\mathbf{C}, \otimes, I)$ as the *white structure*, shorthand $\mathbf{C}^{\circ}$, and to $(\mathbf{C}, \mathbin{\boldsymbol{\otimes}}, I)$ as the *black structure*, $\mathbf{C}^{\bullet}$. Note that a morphism $\mathcal{F}$ is a mapping of objects and arrows that preserves the ordering, the white and black structures; thus we write $\mathcal{F}$ for both $\mathcal{F}^{\circ}$ and $\mathcal{F}^{\bullet}$.

If $(\mathbf{C}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$ is linear bicategory then $(\mathbf{C}^{\mathsf{op}}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$ is a linear bicategory. Similarly $(\mathbf{C}^{\mathsf{co}}, \mathbin{\boldsymbol{\otimes}}, \otimes, I)$, the bicategory obtained from $\mathbf{C}$ by reversing the ordering and swapping the white and the black structure, is a linear bicategory.

Our main example is the linear bicategory $\mathbf{Rel}$ of sets and relations ordered by $\subseteq$. The white structure is the symmetric monoidal category $(\mathbf{Rel}^{\circ}, \otimes, \mathbb{1})$, introduced in the previous section and the black structure is $(\mathbf{Rel}^{\bullet}, \mathbin{\boldsymbol{\otimes}}, \mathbb{1})$. Observe that the two have the same objects, arrows and ordering. The white and black monoidal products $\otimes$ and $\mathbin{\boldsymbol{\otimes}}$ agree on objects and are the cartesian product of sets. As common unit object, they have the singleton set $\mathbb{1}$. We already observed in (5) that the white composition $\fatsemi$ distributes over $\mathbin{\boldsymbol{;}}$ and thus $(\delta_l)$ and $(\delta_r)$ hold. By using the definitions in (2), (3) and (7), the reader can easily check also the inequalities in Definition 5.1.1,2.

Lemma 5.2. *Let* $(\mathbf{C}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$ *be a linear bicategory. For all arrows* $a, b, c$ *the following hold:*

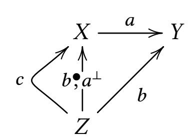
(1) $id_I^{\bullet} \leq id_I^{\circ}$ (2) $a \otimes b \leq a \mathbin{\boldsymbol{\otimes}} b$ (3) $(a \mathbin{\boldsymbol{\otimes}} b) \otimes c \leq a \mathbin{\boldsymbol{\otimes}} (b \otimes c)$

Remark 4. *As* $\otimes$ *linearly distributes over* $\mathbin{\boldsymbol{\otimes}}$*, it may seem that symmetric monoidal linear bicategories of Definition 5.1 are linearly distributive [21, 26]. Moreover (1), (2) of Lemma 5.2 may suggest that they are mix categories [20]. This is not the case: functoriality of* $\otimes$ *over* $\mathbin{\boldsymbol{;}}$ *and of* $\mathbin{\boldsymbol{\otimes}}$ *over* $\fatsemi$ *fails in general.*

*Closed linear bicategories.* In § 4, we recalled adjoints of arrows in bicategories; in linear bicategories one can define *linear* adjoints. For $a \colon X \to Y$ and $b \colon Y \to X$, $a$ is *left linear adjoint* to $b$, or $b$ is *right linear adjoint* to $a$, written $b \Vdash a$, if $id_X^{\circ} \leq a \mathbin{\boldsymbol{;}} b$ and $b \fatsemi a \leq id_Y^{\bullet}$.

Next we discuss some properties of right linear adjoints. Those of left adjoints are analogous but they do not feature in our exposition since in the categories of interest — in next section — left and right linear adjoint coincide. As expected, linear adjoints are unique.

Lemma 5.3. *If* $b \Vdash a$ *and* $c \Vdash a$*, then* $b = c$*.*




By virtue of the above result we can write $a^{\perp} \colon Y \to X$ for *the* right linear adjoint of $a \colon X \to Y$. With this notation one can write the *left residual* of $b \colon Z \to Y$ by $a \colon X \to Y$ as $b \mathbin{\boldsymbol{;}} a^{\perp} \colon Z \to X$. The left residual is the greatest arrow $Z \to X$ making the diagram on the right commute laxly in $\mathbf{C}^{\circ}$, namely if $c \fatsemi a \leq b$ then $c \leq b \mathbin{\boldsymbol{;}} a^{\perp}$. This can be equivalently expressed as:

Lemma 5.4 (Residuation). $a \leq b$ *iff* $id_X^{\circ} \leq b \mathbin{\boldsymbol{;}} a^{\perp}$*.*

*Definition 5.5.* A linear bicategory $(\mathbf{C}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$ is said to be *closed* if every $a \colon X \to Y$ has both a left and a right linear adjoint and the white symmetry is both left and right linear adjoint to the black symmetry, i.e. $(\tau\sigma^{\circ})$, $(\gamma\sigma^{\circ})$, $(\tau\sigma^{\bullet})$ and $(\gamma\sigma^{\bullet})$ in Fig. 4 hold.

$\mathbf{Rel}$ is a a closed linear bicategory: both left and right linear adjoints of a relation $R \subseteq X \times Y$ are given by $\overline{R}^{\dagger} = \{(y, x) \mid (x, y) \notin R\} \subseteq Y \times X$. With this, it is easy to see that $\sigma^{\bullet} \Vdash \sigma^{\circ} \Vdash \sigma^{\bullet}$ in $\mathbf{Rel}$.

Observe that if a linear bicategory $(\mathbf{C}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$ is closed, then also $(\mathbf{C}^{\mathsf{op}}, \otimes, \mathbin{\boldsymbol{\otimes}}, I)$ and $(\mathbf{C}^{\mathsf{co}}, \mathbin{\boldsymbol{\otimes}}, \otimes, I)$ are closed. The assignment $a \mapsto a^{\perp}$ gives rise to an identity on objects functor $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$.

Proposition 5.6. $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ *is a morphism of linear bicategories, i.e., the laws in the first two columns of Table 2.(b) hold.*

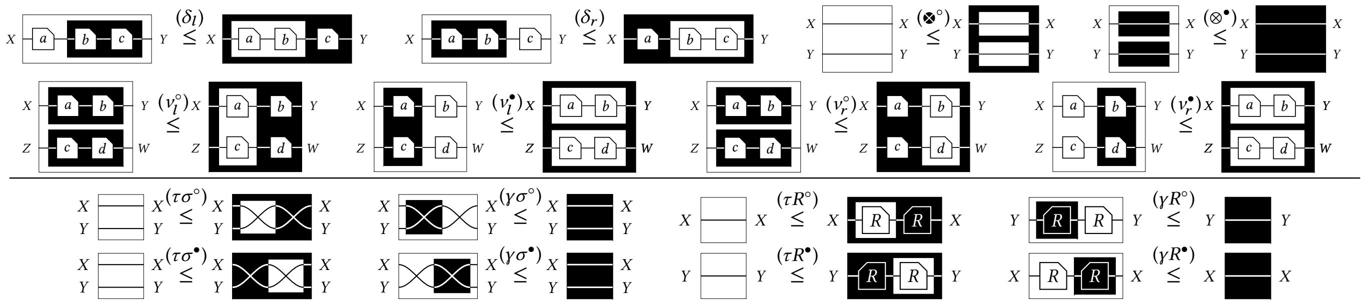



**Figure 4: Axioms of closed symmetric monoidal linear bicategories**

Hereafter, the diagram obtained from c, by taking its mirror image c and then its photographic negative c will denote $c^{\perp}$.

## 6 FIRST ORDER BICATEGORIES

Here we focus on the most important and novel part of the axiomatisation. Indeed, having introduced the two main ingredients, cartesian and linear bicategories, it is time to fire up the Bunsen burner. The remit of this section is to understand how the cartesian and the linear bicategory structures interact in the context of relations. We introduce *first order bicategories* that make these interactions precise. The resulting axioms echo those of cartesian bicategories but in the linear bicategory setting: recall that in a cartesian bicategory the monoid and comonoids are adjoint and satisfy the Frobenius law. Here, the white and black (co)monoids are again related, but by *linear* adjunctions; moreover, they also satisfy appropriate "linear" counterparts of the Frobenius equations.

*Definition 6.1.* A *first order bicategory* $(\mathbf{C}, \otimes, \otimes, I, ◂^{\circ}, !^{\circ}, ▸^{\circ}, ¡^{\circ}, ◂^{\bullet}, !^{\bullet}, ▸^{\bullet}, ¡^{\bullet})$, shorthand *fo-bicategory* $(\mathbf{C}, ◂^{\circ}, ▸^{\circ}, ◂^{\bullet}, ▸^{\bullet})$, consists of
1. a closed linear bicategory $(\mathbf{C}, \otimes, \otimes, I)$,
2. a cartesian bicategory $(\mathbf{C}, \otimes, I, ◂^{\circ}, !^{\circ}, ▸^{\circ}, ¡^{\circ})$ and
3. a cocartesian bicategory $(\mathbf{C}, \otimes, I, ◂^{\bullet}, !^{\bullet}, ▸^{\bullet}, ¡^{\bullet})$, such that
4. the white comonoid $(◂^{\circ}, !^{\circ})$ is left and right linear adjoint to black monoid $(▸^{\bullet}, ¡^{\bullet})$ and $(▸^{\circ}, ¡^{\circ})$ is left and right linear adjoint to $(◂^{\bullet}, !^{\bullet})$, i.e. the inequalities on the left of Figure 5 hold;
5. white and black (co)monoids satisfy the linear Frobenius laws, i.e. the equalities on the right of Fig. 5 hold.

A *morphism of fo-bicategories* is a morphism of linear bicategories *and* of (co)cartesian bicategories.

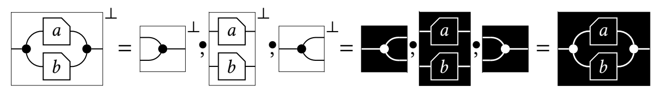
We have seen that **Rel** is a closed linear bicategory, $\mathbf{Rel}^{\circ}$ a cartesian bicategory and $\mathbf{Rel}^{\bullet}$ a cocartesian bicategory. Given (6), it is easy to confirm linear adjointness and linear Frobenius.

Now if $(\mathbf{C}, ◂^{\circ}, ▸^{\circ}, ◂^{\bullet}, ▸^{\bullet})$ is a fo-bicategory then $(\mathbf{C}^{\mathsf{op}}, ▸^{\circ}, ◂^{\circ}, ▸^{\bullet}, ◂^{\bullet})$ and $(\mathbf{C}^{\mathsf{co}}, ◂^{\bullet}, ▸^{\bullet}, ◂^{\circ}, ▸^{\circ})$ are fo-bicategories: the laws of Fig. 5 are closed under mirror-reflection and photographic negative. The fourth condition in Definition 6.1 entails that the linear bicategory morphism $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ (see Prop. 5.6) is a morphism of fo-bicategories and, similarly, the fifth condition that also $(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ (Prop. 4.3) is a morphism of fo-bicategories.

Proposition 6.2. *Let* $(\mathbf{C}, ◂^{\circ}, ▸^{\circ}, ◂^{\bullet}, ▸^{\bullet})$ *be a fo-bicategory. Then* $(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ *and* $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ *are isomorphisms of fo-bicategories, namely the laws in Table 2.(a) and (b) hold.*

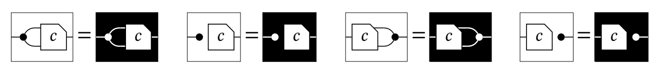
Corollary 6.3. *The laws in Table 2.(c) hold.*

The corollary follows from (12) and (13) and the laws in Tables 2.(a) and (b). For instance, $(a \sqcap b)^{\perp} = a^{\perp} \sqcup b^{\perp}$ is proved as:



The next result about maps (Definition 4.2) plays a crucial role.

Proposition 6.4. *For all maps* $f \colon X \to Y$ *and arrows* $c \colon Y \to Z$, $f \mathbin{⨾} c = (f^{\dagger})^{\perp} \mathbin{⨟} c$ *and thus*



Proof. Recall that, in any cartesian bicategory an arrow $f \colon X \to Y$ is a map iff it is a left adjoint (see e.g. [6, Prop. C.3]), namely

$$id_X^{\circ} \leq f \mathbin{⨾} f^{\dagger} \qquad f^{\dagger} \mathbin{⨾} f \leq id_Y^{\circ} \tag{14}$$

The following two derivations prove the two inclusions.

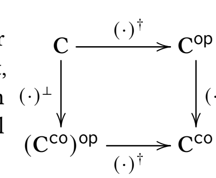
| | |
|---|---|
| $f \mathbin{⨾} c$ | $f \mathbin{⨾} c$ |
| $= id_X^{\circ} \mathbin{⨾} f \mathbin{⨾} c$ | $= f \mathbin{⨾} (id_X^{\bullet} \mathbin{⨟} c)$ |
| $\leq ((f^{\dagger})^{\perp} \mathbin{⨟} f^{\dagger}) \mathbin{⨾} f \mathbin{⨾} c$ $(f^{\dagger} \Vdash (f^{\dagger})^{\perp})$ | $\geq f \mathbin{⨾} ((f^{\dagger} \mathbin{⨾} (f^{\dagger})^{\perp}) \mathbin{⨟} c)$ $((f^{\dagger}) \Vdash (f^{\dagger})^{\perp})$ |
| $\leq (f^{\dagger})^{\perp} \mathbin{⨟} (f^{\dagger} \mathbin{⨾} f \mathbin{⨾} c)$ $(\delta_r)$ | $\geq f \mathbin{⨾} f^{\dagger} \mathbin{⨾} ((f^{\dagger})^{\perp} \mathbin{⨟} c)$ $(\delta_l)$ |
| $\leq (f^{\dagger})^{\perp} \mathbin{⨟} (id_Y^{\circ} \mathbin{⨾} c)$ (14) | $\geq id_X^{\circ} \mathbin{⨾} ((f^{\dagger})^{\perp} \mathbin{⨟} c)$ (14) |
| $= (f^{\dagger})^{\perp} \mathbin{⨟} c$ | $= (f^{\dagger})^{\perp} \mathbin{⨟} c$ |

Note that $f^{\dagger} \Vdash (f^{\dagger})^{\perp}$ holds since, by Prop. 6.2, in any fo-bicategory left and right linear adjoint coincide (namely $(a^{\perp})^{\perp} = a$).

To check the four equivalences, first observe that

$$c \mathbin{⨾} f^{\dagger} = (f \mathbin{⨾} c)^{\dagger} = ((f^{\dagger})^{\perp} \mathbin{⨟} c)^{\dagger} = c \mathbin{⨟} f^{\perp}$$

and conclude by taking as map $f$ either $◂^{\circ}$ or $!^{\circ}$. □




For fo-bicategory **C**, we have the four isomorphisms in the diagram on the right, which commutes by Corollary 6.3. We can thus define the complement as the diagonal of the square, namely $\overline{(\cdot)} \stackrel{\text{def}}{=} ((\cdot)^{\perp})^{\dagger}$.

In diagrams, given c, its negation is $(c^{\perp})^{\dagger}$ = c $^{\dagger}$ = c.

Clearly $\overline{(\cdot)} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{co}}$ is an isomorphism of fo-bicategories. Moreover, it induces a Boolean algebra on each homset of **C**.

**Table 2: Properties of first order bicategories.**

| (a) Properties of $(\cdot)^{\dagger} : (\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}) \to (\mathbf{C}^{\mathsf{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet})$ | | | |
|---|---|---|---|
| if $c \leq d$ then $c^{\dagger} \leq d^{\dagger}$ | | $(c^{\dagger})^{\dagger} = c$ | |
| $(c \mathbin{;^{\circ}} d)^{\dagger} = d^{\dagger} \mathbin{;^{\circ}} c^{\dagger}$ | $(id^{\circ}_X)^{\dagger} = id^{\circ}_X$ | $(\blacktriangleright^{\circ}_X)^{\dagger} = \blacktriangleleft^{\circ}_X$ | $(¡^{\circ}_X)^{\dagger} = !^{\circ}_X$ |
| $(c \otimes d)^{\dagger} = c^{\dagger} \otimes d^{\dagger}$ | $(\sigma^{\circ}_{X,Y})^{\dagger} = \sigma^{\circ}_{Y,X}$ | $(\blacktriangleleft^{\circ}_X)^{\dagger} = \blacktriangleright^{\circ}_X$ | $(!^{\circ}_X)^{\dagger} = ¡^{\circ}_X$ |
| $(c \mathbin{;^{\bullet}} d)^{\dagger} = d^{\dagger} \mathbin{;^{\bullet}} c^{\dagger}$ | $(id^{\bullet}_X)^{\dagger} = id^{\bullet}_X$ | $(\blacktriangleright^{\bullet}_X)^{\dagger} = \blacktriangleleft^{\bullet}_X$ | $(¡^{\bullet}_X)^{\dagger} = !^{\bullet}_X$ |
| $(c \otimes^{\bullet} d)^{\dagger} = c^{\dagger} \otimes^{\bullet} d^{\dagger}$ | $(\sigma^{\bullet}_{X,Y})^{\dagger} = \sigma^{\bullet}_{Y,X}$ | $(\blacktriangleleft^{\bullet}_X)^{\dagger} = \blacktriangleright^{\bullet}_X$ | $(!^{\bullet}_X)^{\dagger} = ¡^{\bullet}_X$ |

| (b) Properties of $(\cdot)^{\perp} : (\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}) \to ((\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ})$ | | | |
|---|---|---|---|
| if $c \leq d$ then $c^{\perp} \geq d^{\perp}$ | | $(c^{\perp})^{\perp} = c$ | |
| $(c \mathbin{;^{\circ}} d)^{\perp} = d^{\perp} \mathbin{;^{\bullet}} c^{\perp}$ | $(id^{\circ}_X)^{\perp} = id^{\bullet}_X$ | $(\blacktriangleright^{\circ}_X)^{\perp} = \blacktriangleleft^{\bullet}_X$ | $(¡^{\circ}_X)^{\perp} = !^{\bullet}_X$ |
| $(c \otimes d)^{\perp} = c^{\perp} \otimes^{\bullet} d^{\perp}$ | $(\sigma^{\circ}_{X,Y})^{\perp} = \sigma^{\bullet}_{Y,X}$ | $(\blacktriangleleft^{\circ}_X)^{\perp} = \blacktriangleright^{\bullet}_X$ | $(!^{\circ}_X)^{\perp} = ¡^{\bullet}_X$ |
| $(c \mathbin{;^{\bullet}} d)^{\perp} = d^{\perp} \mathbin{;^{\circ}} c^{\perp}$ | $(id^{\bullet}_X)^{\perp} = id^{\circ}_X$ | $(\blacktriangleright^{\bullet}_X)^{\perp} = \blacktriangleleft^{\circ}_X$ | $(¡^{\bullet}_X)^{\perp} = !^{\circ}_X$ |
| $(c \otimes^{\bullet} d)^{\perp} = c^{\perp} \otimes d^{\perp}$ | $(\sigma^{\bullet}_{X,Y})^{\perp} = \sigma^{\circ}_{Y,X}$ | $(\blacktriangleleft^{\bullet}_X)^{\perp} = \blacktriangleright^{\circ}_X$ | $(!^{\bullet}_X)^{\perp} = ¡^{\circ}_X$ |

| (c) Interaction of $\cdot^{\dagger}$ and $\cdot^{\perp}$ with $\sqcap$ and $\sqcup$ | |
|---|---|
| $(c \sqcap d)^{\dagger} = c^{\dagger} \sqcap d^{\dagger}$ | $\top^{\dagger} = \top$ |
| $(c \sqcup d)^{\dagger} = c^{\dagger} \sqcup d^{\dagger}$ | $\perp^{\dagger} = \perp$ |
| $(c \sqcap d)^{\perp} = c^{\perp} \sqcup d^{\perp}$ | $(\top)^{\perp} = \perp$ |
| $(c \sqcup d)^{\perp} = c^{\perp} \sqcap d^{\perp}$ | $(\perp)^{\perp} = \top$ |
| $(c^{\dagger})^{\perp} = (c^{\perp})^{\dagger}$ | |

| (d) Laws of Boolean algebras | |
|---|---|
| $c \sqcap (d \sqcup e) = (c \sqcap d) \sqcup (c \sqcap e)$ | |
| $c \sqcup (d \sqcap e) = (c \sqcup d) \sqcap (c \sqcup e)$ | |
| $\overline{(c \sqcap d)} = \overline{c} \sqcup \overline{d}$ | $\overline{\top} = \perp$ |
| $\overline{(c \sqcup d)} = \overline{c} \sqcap \overline{d}$ | $\overline{\perp} = \top$ |
| $c \sqcap \overline{c} = \perp$ | $c \sqcup \overline{c} = \top$ |

| (e) Enrichment over join-meet semilattices | | | | | | |
|---|---|---|---|---|---|---|
| | $c \mathbin{;^{\circ}} (d \sqcup e) = (c \mathbin{;^{\circ}} d) \sqcup (c \mathbin{;^{\circ}} e)$ | $(d \sqcup e) \mathbin{;^{\circ}} c = (d \mathbin{;^{\circ}} c) \sqcup (e \mathbin{;^{\circ}} c)$ | $c \mathbin{;^{\circ}} \perp = \perp = \perp \mathbin{;^{\circ}} c$ | $c \otimes (d \sqcup e) = (c \otimes d) \sqcup (c \otimes e)$ | $(d \sqcup e) \otimes c = (d \otimes c) \sqcup (e \otimes c)$ | $c \otimes \perp = \perp = \perp \otimes c$ |
| | $c \mathbin{;^{\bullet}} (d \sqcap e) = (c \mathbin{;^{\bullet}} d) \sqcap (c \mathbin{;^{\bullet}} e)$ | $(d \sqcap e) \mathbin{;^{\bullet}} c = (d \mathbin{;^{\bullet}} c) \sqcap (e \mathbin{;^{\bullet}} c)$ | $c \mathbin{;^{\bullet}} \top = \top = \top \mathbin{;^{\bullet}} c$ | $c \otimes^{\bullet} (d \sqcap e) = (c \otimes^{\bullet} d) \sqcap (c \otimes^{\bullet} e)$ | $(d \sqcap e) \otimes^{\bullet} c = (d \otimes^{\bullet} c) \sqcap (e \otimes^{\bullet} c)$ | $c \otimes^{\bullet} \top = \top = \top \otimes^{\bullet} c$ |

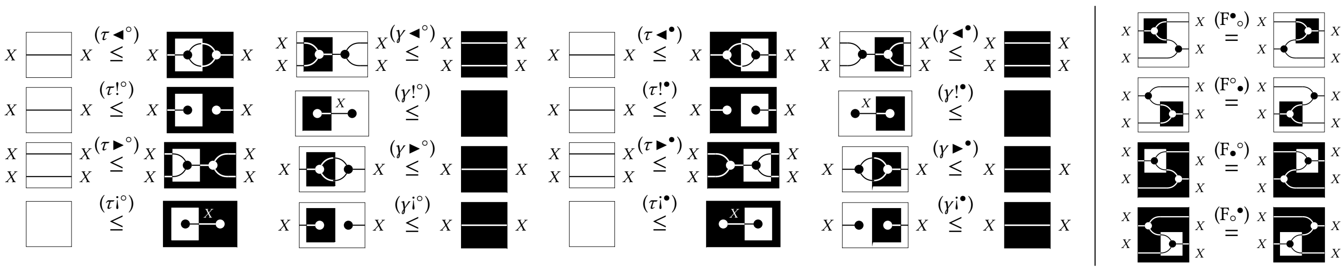


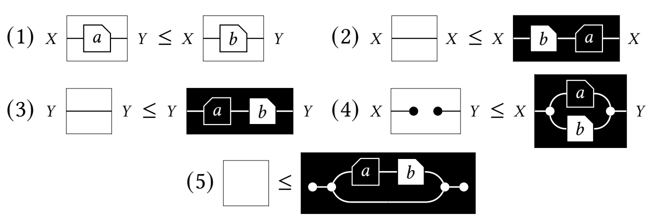
**Figure 5: Additional axioms for fo-bicategories**

Proposition 6.5. *Let $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ be a fo-bicategory. Then every homset of $\mathbf{C}$ is a Boolean algebra: the laws in Tab. 2.(d) hold. Further, $(\mathbf{C}, \otimes, I)$ is monoidally enriched over $\sqcup$-semilattices with $\perp$, while $(\mathbf{C}, \otimes^{\bullet}, I)$ over $\sqcap$-semilattices with $\top$: the laws in Tab. 2.(e) hold.*

The monoidal enrichment is interesting: as we mentioned in § 4, the white structure is not enriched over $\sqcap$, but it *is* enriched over $\sqcup$. In **Rel**, this is the fact that $R \mathbin{;} (S \cup T) = (R \mathbin{;} S) \cup (R \mathbin{;} T)$.

We conclude with a result that extends Lemma 5.4 with five different possibilities to express the concept of logical entailment.

Lemma 6.6. *In a fo-bicategory, the following are equivalent:*



## 6.1 The calculus of neo-Peircean relations as a freely generated first order bicategory

We now return to $\mathsf{NPR}_{\Sigma}$. Recall that $\lesssim$ is the precongruence obtained from the axioms in Figs. 2, 3, 4 and 5. Its soundness (half of Theorem 3.2) is immediate since **Rel** is a fo-bicategory.

Proposition 6.7. *For all terms $c, d \colon n \to m$, if $c \lesssim d$ then $c \leqq d$.*

Next, we show how $\mathsf{NPR}_{\Sigma}$ gives rise to a fo-bicategory $\mathbf{FOB}_{\Sigma}$. Objects are natural numbers and monoidal products $\otimes$ are defined as addition, with unit object 0. Arrows from $n$ to $m$ are terms $c \colon n \to m$ modulo syntactic equivalence $\cong$, namely $\mathbf{FOB}_{\Sigma}[n, m] \stackrel{\text{def}}{=} \{[c]_{\cong} \mid c \colon n \to m\}$. Observe that this is well defined since $\cong$ is well-typed. Since $\cong$ is a congruence, the operations $;$ and $\otimes$ on terms are well defined on equivalence classes: $[t_1]_{\cong} \mathbin{;} [t_2]_{\cong} \stackrel{\text{def}}{=} [t_1 \mathbin{;} t_2]_{\cong}$ and $[t_1]_{\cong} \otimes [t_2]_{\cong} \stackrel{\text{def}}{=} [t_1 \otimes t_2]_{\cong}$. By fixing as partial order the syntactic inclusion $\lesssim$, one can easily prove the following.

Proposition 6.8. **$\mathbf{FOB}_{\Sigma}$** *is a first order bicategory.*

A useful consequence is that, for any interpretation $\mathcal{I} = (X, \rho)$, the semantics $\mathcal{I}^{\sharp}$ gives rise to a morphism $\mathcal{I}^{\sharp} \colon \mathbf{FOB}_{\Sigma} \to \mathbf{Rel}$ of fo-bicategories: it is defined on objects as $n \mapsto X^n$ and on arrows by the inductive definition in (8). To see that it is a morphism, note that, by (8), all the structure of (co)cartesian bicategories and of linear bicategories is preserved (e.g. $\mathcal{I}^{\sharp}(\blacktriangleleft^{\circ}_1) = \blacktriangleleft^{\circ}_X$). Moreover, the ordering is preserved by Prop. 6.7. Note that, by construction,

$$\mathcal{I}^{\sharp}(1) = X \text{ and } \mathcal{I}^{\sharp}(R^{\circ}) = \rho(R) \text{ for all } R \in \Sigma. \tag{15}$$

Actually, $\mathcal{I}^{\sharp}$ is the unique such morphism of fo-bicategories. This is a consequence of a more general universal property: **Rel** can be replaced with an arbitrary fo-bicategory $\mathbf{C}$. To see this, we first need to generalise the notion of interpretation.

*Definition 6.9.* Let $\Sigma$ be a monoidal signature and $\mathbf{C}$ a first order bicategory. An *interpretation* $\mathcal{I} = (X, \rho)$ of $\Sigma$ in $\mathbf{C}$ consists of an object $X$ of $\mathbf{C}$ and an arrow $\rho(R) \colon X^n \to X^m$ for each $R \in \Sigma[n, m]$.

With this definition, we can state that $\mathbf{FOB}_{\Sigma}$ is the fo-bicategory freely generated by $\Sigma$.

Proposition 6.10. *Let $\Sigma$ be a monoidal signature, $\mathbf{C}$ a first order bicategory and $\mathcal{I} = (X, \rho)$ an interpretation of $\Sigma$ in $\mathbf{C}$. There exists a unique morphism of fo-bicategories $\mathcal{I}^{\sharp} \colon \mathbf{FOB}_{\Sigma} \to \mathbf{C}$ such that $\mathcal{I}^{\sharp}(1) = X$ and $\mathcal{I}^{\sharp}(R^{\circ}) = \rho(R)$ for all $R \in \Sigma$.*

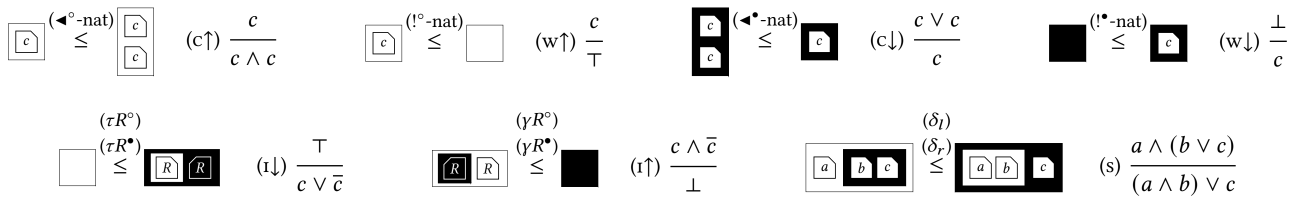


**Figure 6: The axioms in Figures 2, 3 and 4 reduce to those above for diagrams of type $I \to I$. In this case, the axioms correspond to rules of SKSg in [15]. By the laws of symmetric monoidal categories ⚲ and ⊗ coincide: they both correspond to ∧. Moreover they are associative, commutative and with unit $id_I^\circ$, corresponding to ⊤. Symmetrically ⚲ and ⊗ coincide and correspond to ∨.**

## 7 DIAGRAMMATIC FIRST ORDER THEORIES

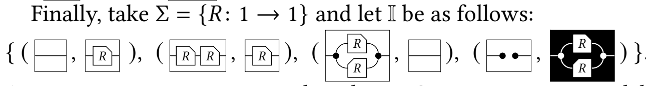
Here we take the first steps towards completeness and show that for first order theories, fo-bicategories play an analogous role to cartesian categories in Lawvere's functorial semantics [47].

A *first order theory* $\mathbb{T}$ is a pair $(\Sigma, \mathbb{I})$ where $\Sigma$ is a signature and $\mathbb{I}$ is a set of *axioms*: pairs $(c, d)$ for $c, d\colon n \to m$ in $\mathbf{FOB}_\Sigma$. A *model* of $\mathbb{T}$ is an interpretation $\mathcal{I}$ of $\Sigma$ where if $(c, d) \in \mathbb{I}$, then $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$.

*Example 7.1.* The simplest case is $\Sigma = \mathbb{I} = \varnothing$. An interpretation is a set: all sets, including the empty set $\varnothing$, are models.

Next take $\Sigma = \varnothing$ and $\mathbb{I} = \{( \ , \ )\}$. An interpretation $\mathcal{I}$ is a set $X$. By (8), $\mathcal{I}^\sharp( \ ) = \{(\star, x) \mid x \in X\} ⚲ \{(x, \star) \mid x \in X\}$, so $\mathcal{I}^\sharp( \ ) = \{(\star,\star)\}$ if $X \neq \varnothing$, but $\varnothing$ if $X = \varnothing$. Instead, $\mathcal{I}^\sharp( \ ) = \{(\star,\star)\}$ always, since $X^0$ is always $\mathbb{1}$. Succinctly, $\mathcal{I}^\sharp( \ ) \subseteq \mathcal{I}^\sharp( \ )$ iff $X \neq \varnothing$: models are *non-empty sets.*

Finally, take $\Sigma = \{R\colon 1 \to 1\}$ and let $\mathbb{I}$ be as follows:



An interpretation is a set $X$ and a relation $R \subseteq X \times X$. It is a model iff $R$ is an order, i.e., reflexive, transitive, antisymmetric and total.

Monoidal signatures $\Sigma$, differently from usual FOL alphabets, do not have function symbols. The reason is that, by adding the axioms below to $\mathbb{I}$, one forces a symbol $f\colon n \to 1 \in \Sigma$ to be a function.

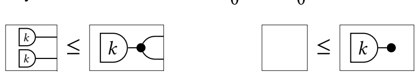
$(\mathbb{M}_f)$

Indeed, as we remarked in § 4, $f \subseteq X^n \times X$ satisfies $\mathbb{M}_f$ if and only if it is single valued and total, i.e. a function. We depict functions as $f$ and constants, being $0 \to 1$ functions, as $k$.

The axioms of a theory together with $\lesssim$ form a deduction system. Formally, the *deduction relation* induced by $\mathbb{T} = (\Sigma, \mathbb{I})$ is the closure (see (10)) of $\lesssim \cup \, \mathbb{I}$, i.e. $\lesssim_{\mathbb{T}} \stackrel{\text{def}}{=} \mathsf{pc}(\lesssim \cup \, \mathbb{I})$. We write $\cong_{\mathbb{T}}$ for $\lesssim_{\mathbb{T}} \cap \gtrsim_{\mathbb{T}}$.

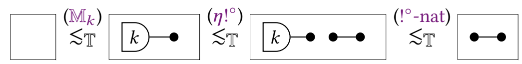
Proposition 7.2. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory. If $c \lesssim_{\mathbb{T}} d$, then $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$ for all models $\mathcal{I}$.*

*Example 7.3.* Consider the theory $\mathbb{T}$ with $\Sigma = \{k\colon 0 \to 1\}$ and axioms $\mathbb{M}_k$. By the definitions of $\blacktriangleleft_0^\circ$ and $!_0^\circ$ in Tab. 1, these are:



$(\mathbb{M}_k)$

An interpretation $\mathcal{I}$ of $\Sigma$ consists of a set $X$ and a relation $k \subseteq \mathbb{1} \times X$. An interpretation is a model iff $k$ is a function of type $\mathbb{1} \to X$. One can easily prove that in all models the domain is non-empty:



(16)

*Contradictory vs trivial theories.* The following two notions play a key role in the proof of our main result however, as we will explain in Remark 6, their distinction is invisible in FOL.

*Definition 7.4.* A theory $\mathbb{T}$ is *contradictory* if $\lesssim_{\mathbb{T}}$ . It is *trivial* if $\lesssim_{\mathbb{T}}$ .

Triviality implies all models have domain $\varnothing$: $\mathcal{I}^\sharp( \ ) = \{(\star, x) \mid x \in X\}$ is included in $\varnothing = \mathcal{I}^\sharp( \ )$ iff $X = \varnothing$. On the other hand, contradictory theories cannot have a model, not even when $X = \varnothing$: since $\mathcal{I}^\sharp( \ ) = \{(\star,\star)\}$ and $\mathcal{I}^\sharp( \ ) = \varnothing$ independently of $X$. As expected, every contradictory theory is trivial.

Lemma 7.5. *Let $\mathbb{T}$ be a theory. If $\mathbb{T}$ is contradictory then it is trivial.*

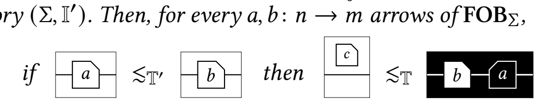
In trivial theories diagrams of type $0 \to 0$ can be quite interesting (see Example 7.7), while those with a different type collapse:

Lemma 7.6. *Let $\mathbb{T}$ be a trivial theory and $c\colon n \to m{+}1$, $d\colon m{+}1 \to n$ be arrows in $\mathbf{FOB}_\Sigma$. Then $\top \lesssim_{\mathbb{T}} c \lesssim_{\mathbb{T}} \bot$ and $\top \lesssim_{\mathbb{T}} d \lesssim_{\mathbb{T}} \bot$.*

*Example 7.7 (The trivial theory of propositional calculus).* Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be the theory where $\Sigma$ contains only symbols $P, Q, R \ldots$ of type $0 \to 0$ and $\mathbb{I} = \{( \ , \ )\}$. In any model of $\mathbb{T}$, the domain $X$ must be $\varnothing$, because of the only axiom in $\mathbb{I}$. A model is a mapping of each of the symbols in $\Sigma$ to either $\{(\star,\star)\}$ or $\varnothing$. In other words, $P, Q, R, \ldots$ act as propositional variables and any model is just an assignment of boolean values. By Lemma 7.6 all arrows collapse, with the exception of those of type $0 \to 0$, that are exactly propositional formulas (see [6, Prop. B.1]). In this case, our axiomatisation reduces to the one in Fig. 6 that, moreover, corresponds to the deep inference system SKSg in [15].

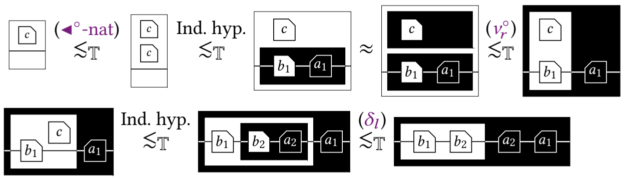
Diagrams $c\colon 0 \to 0$, which can be thought of as closed formulas of FOL, also play an important role in the following result: a diagrammatic analogue of the deduction theorem (the reader may check [6, App. F.1] for a detailed comparison with theories in FOL).

Theorem 7.8 (Deduction theorem). *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $c\colon 0 \to 0$ in $\mathbf{FOB}_\Sigma$. Let $\mathbb{I}' = \mathbb{I} \cup \{(id_0^\circ, c)\}$ and let $\mathbb{T}'$ denote the theory $(\Sigma, \mathbb{I}')$. Then, for every $a, b\colon n \to m$ arrows of $\mathbf{FOB}_\Sigma$,*



Proof. By induction on the rules of (10). We show only the case for ($\fatsemi$). The remaining ones can be found in [6, App. F].

Assume $a = a_1 \fatsemi a_2$ and $b = b_1 \fatsemi b_2$ for some $a_1, b_1 \colon n \to l, a_2, b_2 \colon l \to m$ such that $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$. By induction hypothesis $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} b_1 \,\bullet\, a_1^{\perp}$ and $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} b_2 \,\bullet\, a_2^{\perp}$. Thus:




□

Corollary 7.9. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory, $c \colon 0 \to 0$ in $\mathbf{FOB}_{\Sigma}$ and $\mathbb{T}' = (\Sigma, \mathbb{I} \cup \{(id_0^{\circ}, \overline{c})\})$. Then $id_0^{\circ} \lesssim_{\mathbb{T}} c$ iff $\mathbb{T}'$ is contradictory.*

## 7.1 Functorial semantics for first order theories

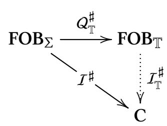
Recall that the notion of interpretation of a signature $\Sigma$ in $\mathbf{Rel}$ has been generalised in Definition 6.9 to an arbitrary fo-bicategory. As expected, the same is possible also with the notion of model.

*Definition 7.10.* Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $\mathbf{C}$ a first order bicategory. An interpretation $\mathcal{I}$ of $\Sigma$ in $\mathbf{C}$ is a model iff, for all $(c, d) \in \mathbb{I}$, $\mathcal{I}^{\sharp}(c) \leq \mathcal{I}^{\sharp}(d)$.

For any theory $\mathbb{T} = (\Sigma, \mathbb{I})$, one can build a fo-bicategory $\mathbf{FOB}_{\mathbb{T}}$: this is like $\mathbf{FOB}_{\Sigma}$, but homsets are now $\mathbf{FOB}_{\mathbb{T}}[n, m] = \{[d]_{\cong_{\mathbb{T}}} \mid d \in \mathbf{FOB}_{\Sigma}[n, m]\}$ ordered by $\lesssim_{\mathbb{T}}$. Since, by definition, $\lesssim \subseteq \lesssim_{\mathbb{T}}$, $\mathbf{FOB}_{\mathbb{T}}$ is a fo-bicategory. Thus, one can take an interpretation $\mathcal{Q}_{\mathbb{T}}$ of $\Sigma$ in $\mathbf{FOB}_{\mathbb{T}}$: the domain $X$ is 1 and $\rho(R) = [R^{\circ}]_{\cong_{\mathbb{T}}}$ for all $R \in \Sigma$. By Prop. 6.10, $\mathcal{Q}_{\mathbb{T}}$ induces a fo-bicategory morphism $\mathcal{Q}_{\mathbb{T}}^{\sharp} \colon \mathbf{FOB}_{\Sigma} \to \mathbf{FOB}_{\mathbb{T}}$.

Proposition 7.11. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory, $\mathbf{C}$ a fo-bicategory and $\mathcal{I}$ an interpretation of $\Sigma$ in $\mathbf{C}$. $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{C}$ iff $\mathcal{I}^{\sharp} \colon \mathbf{FOB}_{\Sigma} \to \mathbf{C}$ factors uniquely through $\mathcal{Q}_{\mathbb{T}}^{\sharp} \colon \mathbf{FOB}_{\Sigma} \to \mathbf{FOB}_{\mathbb{T}}$.*

In other words, there is a unique fo-bicategory morphism $\mathcal{I}_{\mathbb{T}}^{\sharp} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$ s.t. the diagram on the right commutes. The assignment $\mathcal{I} \mapsto \mathcal{I}_{\mathbb{T}}^{\sharp}$ yields a 1-to-1 correspondence between models and morphisms.




Corollary 7.12. *To give a model of $\mathbb{T}$ in $\mathbf{C}$ is to give a fo-bicategory morphism $\mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$.*

By virtue of the above, we can tacitly identify models and morphisms. Proposition 7.11 can also be used to obtain the following result, useful for showing completeness in the next section.

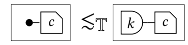
Lemma 7.13. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ and $\mathbb{T}' = (\Sigma', \mathbb{I}')$ be theories s.t. $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$. Then there exists an identity on objects fo-bicategory morphism $\mathcal{F} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{FOB}_{\mathbb{T}'}$ mapping each $d$ of $\mathbf{FOB}_{\mathbb{T}}$ to $[d]_{\cong_{\mathbb{T}'}}$.*

# 8 BEYOND GÖDEL'S COMPLETENESS

Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory. First, we prove Gödel completeness

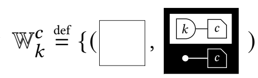
$$\text{if } \mathbb{T} \text{ is non-trivial, then } \mathbb{T} \text{ has a model} \qquad \text{(Gödel)}$$

by adapting Henkin's [38] proof to $\mathsf{NPR}_{\Sigma}$. We begin with two additional definitions. Note that when referring to arrows in the context of $\mathbb{T}$, we mean arrows of $\mathbf{FOB}_{\mathbb{T}}$ (or of $\mathbf{FOB}_{\Sigma}$, it is immaterial).

*Definition 8.1.* $\mathbb{T}$ is *syntactically complete* if for all $c \colon 0 \to 0$ either $id_0^{\circ} \lesssim_{\mathbb{T}} c$ or $id_0^{\circ} \lesssim_{\mathbb{T}} \overline{c}$. $\mathbb{T}$ has *Henkin witnesses* if for all $c \colon 1 \to 0$ there is a map $k \colon 0 \to 1$ s.t. .

These properties do not hold for the theories we have considered so far. In terms of $\mathsf{FOL}$, syntactic completeness means that closed formulas either hold in all models of the theory or in none. A Henkin witness is a term $k$ such that $c(k)$ holds: a theory has Henkin witnesses if for every true formula $\exists x.c(x)$, there exists such a $k$. We shall see in Theorem 8.3 that non-trivial theories can be expanded to have Henkin witnesses, be non-contradictory and syntactically complete. The key idea of Henkin's proof, Theorem 8.6, is that these three properties yield a model.

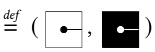
To add a witness for $c \colon 1 \to 0$, one adds a constant $k \colon 0 \to 1$ and the axiom $\mathbb{W}_k^c$, asserting that $k$ is a witness. This preserves non-triviality.

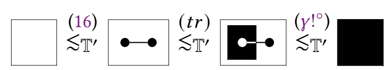


Lemma 8.2 (Witness Addition). *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and consider an arbitrary $c \colon 1 \to 0$. Let $\mathbb{T}' = (\Sigma \cup \{k \colon 0 \to 1\}, \mathbb{I} \cup \mathbb{M}_k \cup \mathbb{W}_k^c)$. If $\mathbb{T}$ is non-trivial then $\mathbb{T}'$ is non-trivial.*

Remark 5. *Observe that the distinction between trivial and contradictory theories is essential for the above development. Indeed, under the conditions of Lemma 8.2, it does* not *hold that*

*if $\mathbb{T}$ is non-contradictory, then $\mathbb{T}'$ is non-contradictory.*

*As counter-example, take as $\mathbb{T}$ the theory consisting only of the trivialising axiom $(tr) \stackrel{def}{=}$ . By definition $\mathbb{T}$ is trivial but non-contradictory. Instead, by Example 7.3, $\mathbb{T}'$ is contradictory:*

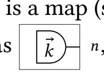



(17)

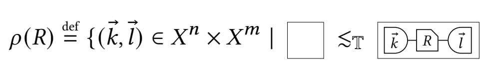
*This shows that adding Henkin witnesses to a non-contradictory theory may end up in a contradictory theory. Therefore, the usual Henkin proof for* $\mathsf{FOL}$ *works just for our* non-trivial *theories.*

By iteratively using Lemma 8.2, one can transform a non-trivial theory into a non-trivial theory with Henkin witnesses. To obtain a syntactically complete theory, we use the standard argument featuring Zorn's Lemma (see [6, Prop. G.4]). In summary:

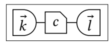
Theorem 8.3. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-trivial theory. There exists a theory $\mathbb{T}' = (\Sigma', \mathbb{I}')$ such that $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$; $\mathbb{T}'$ has Henkin witnesses; $\mathbb{T}'$ is syntactically complete; $\mathbb{T}'$ is non-contradictory.*

Before introducing Henkin's interpretation, observe that any map $c \colon 0 \to n$ can be decomposed as $k_1 \otimes \ldots \otimes k_n$ where each $k_i \colon 0 \to 1$ is a map (see [6, Prop. G.4]). We thus write such $c$ as $\vec{k}$, depicted as , to make explicit its status as a vector.

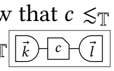
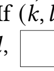
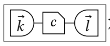
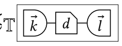
*Definition 8.4.* Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory. The *Henkin interpretation* $\mathcal{H}$ of $\Sigma$, consists of a set $X \stackrel{def}{=} \mathtt{Map}(\mathbf{FOB}_{\mathbb{T}})[0, 1]$ and a function $\rho$, defined for all $R \colon n \to m \in \Sigma$ as:



The domain is the set of constants of the theory. Then $R \colon n \to m$ is mapped to all pairs $(\vec{k}, \vec{l})$ of vectors that make $R$ true in $\mathbb{T}$. The following characterisation of $\mathcal{H}^{\sharp} \colon \mathbf{FOB}_{\Sigma} \to \mathbf{Rel}$ is crucial.

Proposition 8.5. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-contradictory, syntactically complete theory with Henkin witnesses. Then, for any $c\colon n \to m$,* $\mathcal{H}^\sharp(c) = \{(\vec{k}, \vec{l}) \in X^n \times X^m \mid$  $\lesssim_{\mathbb{T}}$  $\}$.

Theorem 8.6. *If $\mathbb{T}$ is non-contradictory, syntactically complete with Henkin witnesses, then $\mathcal{H}$ is a model.*

Proof. We show that $c \lesssim_{\mathbb{T}} d$ gives $\mathcal{H}^\sharp(c) \subseteq \mathcal{H}^\sharp(d)$. If $(\vec{k}, \vec{l}) \in \mathcal{H}^\sharp(c)$ then  $\lesssim_{\mathbb{T}}$  by Prop. 8.5. Since $c \lesssim_{\mathbb{T}} d$,  $\lesssim_{\mathbb{T}}$  $\lesssim_{\mathbb{T}}$  and by Prop. 8.5, $(\vec{k}, \vec{l}) \in \mathcal{H}^\sharp(d)$. □

Theorems 8.3 and 8.6 give us a proof for (Gödel).

Proof of (Gödel). Let $\mathbb{T}' = (\Sigma', \mathbb{I}')$ be obtained via Theorem 8.3. Since $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$, by Lemma 7.13, we have $\mathcal{F}\colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{FOB}_{\mathbb{T}'}$. Since $\mathbb{T}'$ has Henkin witnesses, is syntactically complete and non-contradictory, Theorem 8.6 gives $\mathcal{H}^\sharp_{\mathbb{T}'}\colon \mathbf{FOB}_{\mathbb{T}'} \to \mathbf{Rel}$. We thus have a morphism $\mathbf{FOB}_{\mathbb{T}} \to \mathbf{Rel}$. □

Now, we would like to conclude Theorem 3.2 by means of (Gödel), but this is not possible since, for the former one needs a model for all non-contradictory theories, while (Gödel) provides it only for non-trivial ones. Thankfully, the Henkin interpretation $\mathcal{H}$ gives us, once more, a model (see [6, Prop. G.13]) that allows us to prove

if $\mathbb{T}$ is trivial and non-contradictory, then $\mathbb{T}$ has a model. (Prop)

From (Prop) and (Gödel) we can prove general completeness

if $\mathbb{T}$ is non-contradictory, then $\mathbb{T}$ has a model (General)

and thus deduce our main result.

Proof of (General) and Theorem 3.2. To prove (General) take $\mathbb{T}$ to be a non-contradictory theory. If $\mathbb{T}$ is trivial, then it has a model by (Prop). Otherwise, it has a model by (Gödel). Now, by means of traditional FOL arguments exploiting Corollary 7.9, one can show that (General) entails Theorem 3.2 (see [6, Prop. G.14]). □

### 8.1 The Calculus of Binary Relations (revisited)

The map $\mathcal{E}(\cdot)$ defined in Table 3 is an encoding of the calculus of relations into $\mathsf{NPR}_\Sigma$. Since $\mathcal{E}(\cdot)$ preserves the semantics (see [6, Prop. G.15]), from Theorem 3.2 it follows that one can prove inclusions of expressions of $\mathsf{CR}_\Sigma$ by translating them into $\mathsf{NPR}_\Sigma$ via $\mathcal{E}(\cdot)$ and then using the axioms in Figs. 2, 3, 4 and 5.

Corollary 8.7. *For all $E_1, E_2$, $E_1 \leq_{\mathsf{CR}} E_2$ iff $\mathcal{E}(E_1) \lesssim \mathcal{E}(E_2)$.*

## 9 FIRST ORDER LOGIC WITH EQUALITY

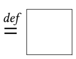
As we already mentioned in the introduction the white fragment of $\mathsf{NPR}_\Sigma$ is as expressive as the existential-conjunctive fragment of first order logic with equality (FOL). The semantic preserving encodings between the two fragments are illustrated in [10]. From the fact that the full $\mathsf{NPR}_\Sigma$ can express negation, we get immediately semantic preserving encodings between $\mathsf{NPR}_\Sigma$ and the full FOL. In this section we illustrate anyway a translation $\mathcal{E}(\cdot)\colon \mathsf{FOL} \to \mathsf{NPR}_\Sigma$ to emphasise the subtle differences between the two. To go in the other way, the reader is referred to [6, App. B.4].

To ease the presentation, we consider FOL formulas $\varphi$ to be typed in the context of a list of variables that are allowed (but not required) to appear in $\varphi$. Fixing $\mathbf{x}_n \stackrel{\text{def}}{=} \{x_1, \dots, x_n\}$ we write $n\colon\varphi$ if all free variables of $\varphi$ are contained in $\mathbf{x}_n$. It is standard to present FOL in two steps: first terms and then formulas. For every function symbol $f$ of arity $m$ in FOL, we have a symbol $f\colon m \to 1$ in the signature $\Sigma$ together with the equations $\mathbb{M}_f$ forcing $f$ to be interpreted as a function. The translation of a term $n\colon t$ to an $\mathsf{NPR}_\Sigma$ diagram $n \to 1$ is given inductively in the left part of Fig. 7.

Formulas $n\colon\varphi$ translate to $\mathsf{NPR}_\Sigma$ diagrams $n \to 0$. For every $n$-ary predicate symbol $R$ in FOL there is a symbol $R\colon n \to 0 \in \Sigma$. In order not to over-complicate the presentation with bureaucratic details, we assume that it is always the last variable that is quantified over. Additional variable manipulation can be introduced: see [6, App. B.3] for an encoding of Quine's predicate functor logic.

The full encoding in Fig. 7 should give the reader the spirit of the correspondence between $\mathsf{NPR}_\Sigma$ and traditional syntax. There is one aspect of the above translation that merits additional attention.

Remark 6. *By the definition of $!^\circ_n$ in Table 1, we have that:*

$\mathcal{E}(0\colon\top) \stackrel{def}{=}$  $\qquad \mathcal{E}(0\colon\bot) \stackrel{def}{=}$ 

*Thus $\top$ and $\bot$ translate to, respectively $id^\circ_0$, $id^\bullet_0$ in the absence of free variables or to $!^\circ_n$, $!^\bullet_n$, respectively, when $n > 0$. This can be seen as an ambiguity in the traditional FOL syntax, which obscures the distinction between inconsistent and trivial theories in traditional accounts, and as a side effect requires the assumption on non-empty models in formal statements of Gödel completeness. Instead, the syntax of $\mathsf{NPR}_\Sigma$ ensures that this pitfall is side-stepped.*

## 10 CONCLUDING REMARKS

The diagrammatic notation of $\mathsf{NPR}_\Sigma$ is closely related to system $\beta$ of Peirce's EGs [60–62, 73]. Consider the two diagrams on the left of Fig. 8 corresponding to the closed FOL formula $\exists x.\, p(x) \wedge \forall y.\, p(y) \to q(y)$. In existential graph notation the circle enclosure (dubbed 'cut' by Peirce) signifies negation. To move from EGs to diagrams of $\mathsf{NPR}_\Sigma$ it suffices to treat lines and predicate symbols in the obvious way and each cut as a color switch.

A string diagrammatic approach to existential graphs appeared in [37]. This exploits the white fragment of $\mathsf{NPR}_\Sigma$ with a primitive negation operator rendered as Peirce's cut, namely a circle around diagrams. However, this inhibits a fully compositional treatment since, for instance, negation is not functorial. As an example consider Peirce's (de)iteration rule in Fig. 8: in $\mathsf{NPR}_\Sigma$ on the center, and in [37] on the right. Note that the diagrams on the right require open cuts, a notational trick, allowing to express the rule for arbitrary contexts, i.e. any diagram eventually appearing inside the cut. In $\mathsf{NPR}_\Sigma$ this ad-hoc treatment of contexts is not needed as negation is not a primitive operation, but a derived one. A derivation of the law in the middle of Fig. 8 can be found in Fig. 9.

Other diagrammatic calculi of Peirce's EGs appear in [50] and [14]. The categorical treatment goes, respectively, through the notions of chiralities and doctrines. The formers consider a pair of categories $(\mathbf{Rel}_\bullet, \mathbf{Rel}_\circ)$ that are significantly different from our $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$: to establish a formal correspondence, it might be convenient to first focus on doctrines. To this aim, we plan to exploit the equivalence in [9] between cartesian bicategories and certain doctrines (elementary existential with comprehensive diagonals and unique choice [49]). Preliminary attempts suggests the same equivalence

**Table 3: The encoding $\mathcal{E}(\cdot)\colon \mathsf{CR}_\Sigma \to \mathsf{NPR}_\Sigma$**

| | | | | | |
|---|---|---|---|---|---|
| $\mathcal{E}(R) \stackrel{\text{def}}{=} R^\circ$ | $\mathcal{E}(id^\circ) \stackrel{\text{def}}{=} id_1^\circ$ | $\mathcal{E}(E_1 \mathbin{;^\circ} E_2) \stackrel{\text{def}}{=} \mathcal{E}(E_1) \mathbin{;^\circ} \mathcal{E}(E_2)$ | $\mathcal{E}(\top) \stackrel{\text{def}}{=} !_1^\circ \mathbin{;^\circ} ¡_1^\circ$ | $\mathcal{E}(E_1 \cap E_2) \stackrel{\text{def}}{=} \blacktriangleleft_1^\circ \mathbin{;^\circ} (\mathcal{E}(E_1) \otimes \mathcal{E}(E_2)) \mathbin{;^\circ} \blacktriangleright_1^\circ$ | $\mathcal{E}(\overline{E}) \stackrel{\text{def}}{=} \overline{\mathcal{E}(E)}$ |
| $\mathcal{E}(E^\dagger) \stackrel{\text{def}}{=} \mathcal{E}(E)^\dagger$ | $\mathcal{E}(id^\bullet) \stackrel{\text{def}}{=} id_1^\bullet$ | $\mathcal{E}(E_1 \mathbin{;^\bullet} E_2) \stackrel{\text{def}}{=} \mathcal{E}(E_1) \mathbin{;^\bullet} \mathcal{E}(E_2)$ | $\mathcal{E}(\bot) \stackrel{\text{def}}{=} !_1^\bullet \mathbin{;^\bullet} ¡_1^\bullet$ | $\mathcal{E}(E_1 \cup E_2) \stackrel{\text{def}}{=} \blacktriangleleft_1^\bullet \mathbin{;^\bullet} (\mathcal{E}(E_1) \otimes^\bullet \mathcal{E}(E_2)) \mathbin{;^\bullet} \blacktriangleright_1^\bullet$ | |

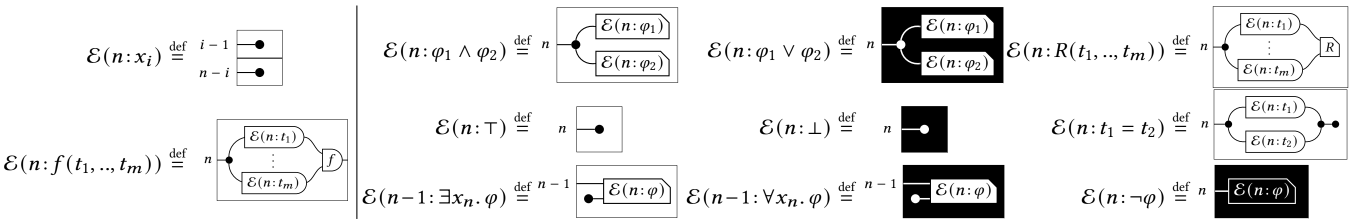


**Figure 7: FOL encoding in $\mathsf{NPR}_\Sigma$.**

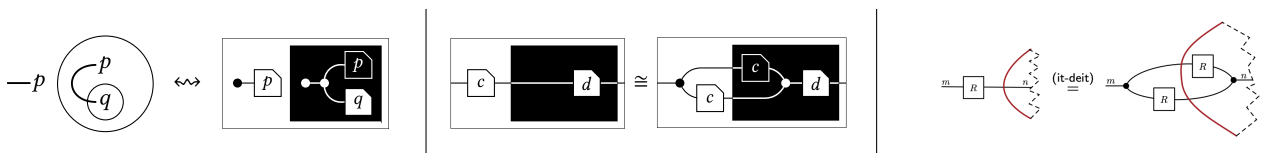


**Figure 8: An EG and its encoding in $\mathsf{NPR}_\Sigma$ (left); Peirce's (de)iteration rule in $\mathsf{NPR}_\Sigma$ (middle) and in [37] (right).**

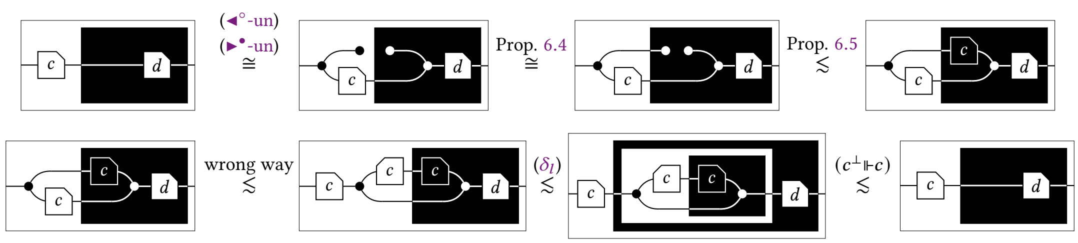



**Figure 9: Derivation of Peirce's (de)iteration rule in Figure 8. The step marked with "wrong way" holds in any cartesian bicategory, see e.g. [6, Lemma E.12].**

restrict to fo-bicategories and boolean hyperdoctrines but many details have to be carefully checked. A relational understanding of doctrines has been initiated in [25] with the notion of relational doctrines. The latter are as expressive as regular logic, i.e. the white fragment of $\mathsf{NPR}_\Sigma$, thus it might be interesting to understand the role of the black structure in this setting. The connection with allegories [31] is also worth to be explored: since cartesian bicategories are equivalent to unitary pretabular allegories, Prop. 6.5 suggests that fo-bicategories are closely related to Peirce allegories [56].

It is worth remarking that $\mathsf{NPR}_\Sigma$ only deals with *classical* FOL, as hinted by the fact that the homsets of a fo-bicategory are Boolean algebras (Prop. 6.5). Hopefully, the intuitionistic case might be handled by relaxing some of the conditions of Def. 6.1.

To conclude it is worth mentioning a further research direction. We plan to extend to FOL, the combinatorial characterisation of its regular fragment in terms of hypergraphs [18] and the associated rewriting approach [5]. In particular, we foresee the possibility of defining a deep inference system having as rules the inequalities of our axiomatisation and compare its proof theory with [16, 42, 71].

## ACKNOWLEDGMENTS

The authors would like to thank Matteo Acclavio and Francesco Gavazzo for insightful comments on the early version of the paper and the GAATI lab at the University of French Polynesia for providing an inspiring research environment. Bonchi is supported by the Ministero dell'Università e della Ricerca of Italy grant PRIN 2022 PNRR No. P2022HXNSC - RAP (Resource Awareness in Programming). Di Giorgio is supported by the EPSRC grant No. EP/V002376/1. Sobociński is supported by the Estonian Research Council grant PRG1210 and by the European Union under Grant Agreement No. 101087529.

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Received 16 January 2024; revised 19 March 2024; accepted 15 April 2024