Degree Sequences of Triangular Multigraphs

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Abstract

A simple graph is *triangular* if every edge is contained in a triangle. A sequence of integers is *graphical* if it is the degree sequence of a simple graph. Egan and Nikolayevsky recently conjectured that every graphical sequence whose terms are all at least 4 is the degree sequence of a triangular simple graph, and proved this in some special cases. In this paper we state and prove the analogous version of this conjecture for multigraphs.

Mathematics Subject Classifications: 05C07

1 Introduction

A graph is *simple* if it does not contain any loops or multiple edges. A sequence of integers $(d_1, \dots, d_n)$ is *graphical* if there exists a simple graph G on vertices $v_1, \dots, v_n$ such that $\deg(v_i) = d_i$ for all $i \in [n]$. The well-known Erdős-Gallai Theorem provides a complete characterisation of graphical sequences.

Theorem 1 (Erdős-Gallai Theorem [2])**.** *A sequence of positive integers $d_1 \geqslant \dots \geqslant d_n$ is graphical if and only if*

- $d_1 + \dots + d_n$ *is even and*
- $\sum_{i=1}^{k} d_i \leqslant k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}$ *for every* $k \in [n]$.

A *triangle* in a simple graph consists of three distinct vertices that are pairwise adjacent. A simple graph is *triangular* if every edge is contained in a triangle. Recently, Egan and Nikolayevsky [1] conjectured that any positive integer sequence whose terms are all at least 4, and satisfies the obvious necessary condition of being graphical, is the degree sequence of a triangular simple graph. By the Erdős-Gallai Theorem, this is equivalent to the following.

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Conjecture 2 (Egan and Nikolayevsky [1])**.** If $n \geqslant 3$ and $(d_1, \dots, d_n)$ is a sequence of integers satisfying

- $d_1 \geqslant \cdots \geqslant d_n \geqslant 4$,
- $d_1 + \cdots + d_n$ is even,
- $\sum_{i=1}^{k} d_i \leqslant k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}$ for every $k \in [n]$,

then it is the degree sequence of a triangular simple graph.

Egan and Nikolayevsky [1] proved this conjecture in the case when the degree sequence contains at most two distinct terms.

Theorem 3 (Egan and Nikolayevsky [1])**.** *Any graphical sequence of the form (a^p, b^q), where $a > b \geqslant 4$ and $p \geqslant 0, q > 0$ is the degree sequence of a triangular simple graph.*

In this paper, we state and prove the analogous version of Conjecture 2 for multigraphs. A *triangle* in a multigraph consists of three distinct vertices which are pairwise adjacent. A multigraph is *triangular* if every edge is contained in a triangle. The following lemma provides two necessary conditions for the degree sequences of triangular multigraphs.

Lemma 4. *If $n \geqslant 3$ and $d_1 \geqslant \cdots \geqslant d_n > 0$ is the degree sequence of a triangular multigraph on n vertices, then*

- $\sum_{i=1}^{n} d_i$ *is even,*
- $d_1 \leqslant \sum_{i=2}^{n} (d_i - 1)$.

Proof. $\sum_{i=1}^{n} d_i$ is even from the well-known handshake lemma. Now suppose for a contradiction that $d_1 > \sum_{i=2}^{n}(d_i - 1)$ and G is a triangular multigraph on vertices $v_1, \dots, v_n$ satisfying $\deg(v_i) = d_i$ for all $i \in [n]$. Note that any triangular multigraph is necessarily loopless. If for every $2 \leqslant i \leqslant n$, v_i is adjacent to a vertex that is not v_1, then since G is loopless, $\deg(v_1) \leqslant \sum_{i=2}^{n}(d_i - 1) < d_1$, contradiction. Hence, there must exist some $2 \leqslant i \leqslant n$ such that v_i is only adjacent to the vertex v_1. Since $d_i > 0$, the edge $v_1 v_i$ has positive multiplicity, but cannot be in a triangle, contradicting G is triangular. □

Our main result is that the analogue of Conjecture 2 holds for multigraphs. Any sequence of $n \geqslant 3$ integers, each at least 4, and satisfying the obvious necessary conditions in Lemma 4 is the degree sequence of a triangular multigraph.

Theorem 5. *If $n \geqslant 3$ and $(d_1, \dots, d_n)$ is a sequence of integers satisfying*

(i) $d_1 \geqslant \cdots \geqslant d_n \geqslant 4$,

(ii) $\sum_{i=1}^{n} d_i$ *is even,*

(iii) $d_1 \leqslant \sum_{i=2}^{n} (d_i - 1)$,

then it is the degree sequence of a triangular multigraph.

As evidenced by the following proposition, we cannot replace the number 4 in condition (i) by a smaller integer.

Proposition 6. *The degree sequence given by $d_i = 3$ for all $i \in [n]$ is the degree sequence of a triangular multigraph if and only if n is divisible by 4.*

Proof. Let G be a triangular multigraph on n vertices, all of which have degree 3. It suffices to show that every connected component of G is isomorphic to K_4.

Fix a connected component of G. Suppose there exists a vertex v_1 adjacent to three different vertices v_2, v_3, v_4. As edges v_1v_2, v_1v_3, v_1v_4 all need to be in triangles, we may, without loss of generality, assume edges v_2v_3, v_2v_4 are also in G. If edge v_3v_4 is also in G, then all of v_1, v_2, v_3, v_4 have degree 3, so the connected component containing them is isomorphic to K_4. Otherwise, vertex v_3 is adjacent to a new vertex v_5. But since v_1v_5, v_2v_5 are not in G, the edge v_3v_5 is not in a triangle, contradiction.

Suppose now there is no vertex in this connected component that is adjacent to three different vertices. Let v_1 be a vertex in this component. Either there is an edge v_1v_2 of multiplicity 3, which cannot be in a triangle, or we have an edge v_1v_2 with multiplicity 2 and an edge v_1v_3 with multiplicity 1. For edges v_1v_2, v_1v_3 to be in triangles, we must have edge v_2v_3 as well. One of edges v_1v_3, v_2v_3 must have multiplicity at least 2, as v_3 cannot be adjacent to three different vertices. But then one of v_1, v_2 will have degree at least 4, contradiction. □

2 Proof of Theorem 5

Suppose $n \geqslant 3$ and $(d_1, \ldots, d_n)$ is a sequence of integers satisfying (i)-(iii). The goal of Theorem 5 is to construct a triangular multigraph G on vertices $v_1, \ldots, v_n$, such that $\deg(v_i) = d_i$ for all $i \in [n]$. It turns out that $D = \sum_{i=1}^{n}(-1)^{i-1}d_i$ is a critical quantity that will guide our constructions. Note that D is non-negative by (i) and is even by (ii).

If $D \geqslant n-2$, we show in Lemma 7 that a fan-shaped construction (see Figure 1) works, with v_1 being the central vertex. If $D \leqslant 4$, we show in Lemma 8 that a construction based on modifying the square of the length n cycle (see Figure 2) works. Finally, we complete the proof of Theorem 5 by showing that in the intermediate case, $6 \leqslant D \leqslant n-3$, a combination of the above two constructions, with v_1 being the unique common vertex, works.

In order to combine these two constructions in the proof of Theorem 5, we will need to state and prove Lemma 7 and Lemma 8 in the slightly more general setting where we do not assume d_1 is the largest term of the sequence. Throughout the constructions in this section, the multiplicity of an edge v_iv_j in a multigraph G will be denoted by $m(v_i, v_j)$.

Lemma 7. *Let $n \geqslant 3$ and let $(d_1, \ldots, d_n)$ be a sequence of non-negative integers satisfying*

- $d_2 \geqslant \cdots \geqslant d_n \geqslant 4$,

- *$d_1 + \cdots + d_n$ is even,*
- *$d_1 \leqslant \sum_{i=2}^n (d_i - 1)$,*
- *$D = \sum_{i=1}^n (-1)^{i-1} d_i \geqslant n-2$,*

then there exists a triangular multigraph G with degree sequence $(d_1, \ldots, d_n)$.

Proof. We separate into two cases depending on the parity of n.

If n is odd, then $D \geqslant n-1$ as D is even. Let $\overline{d}_{n+2-2i} = d_{n+2-2i} - 2$ for each $1 \leqslant i \leqslant \frac{n-1}{2}$. Using $d_1 \leqslant \sum_{i=2}^n (d_i - 1)$, we have

$$\begin{aligned}
\frac{1}{2}\left(D - (n-1)\right) &= \frac{1}{2}\left(\sum_{i=1}^n (-1)^{i-1} d_i - (n-1)\right) \\
&\leqslant \frac{1}{2}\left(\sum_{i=2}^n d_i + \sum_{i=2}^n (-1)^{i-1} d_i - 2(n-1)\right) \\
&= \sum_{i=1}^{\frac{n-1}{2}} d_{n+2-2i} - (n-1) = \sum_{i=1}^{\frac{n-1}{2}} \overline{d}_{n+2-2i}.
\end{aligned}$$

Hence, there exists an index $1 \leqslant k \leqslant \frac{n-1}{2}$ such that $\sum_{i=1}^{k-1} \overline{d}_{n+2-2i} \leqslant \frac{1}{2}(D - (n-1)) \leqslant \sum_{i=1}^{k} \overline{d}_{n+2-2i}$. Let $\delta = \frac{1}{2}(D-(n-1)) - \sum_{i=1}^{k-1} \overline{d}_{n+2-2i}$, so $0 \leqslant \delta \leqslant \overline{d}_{n+2-2k} = d_{n+2-2k} - 2$. Consider the multigraph G on n vertices $v_1, \ldots, v_n$ whose edge multiplicities are given as follows (see also Figure 1a). For each $i \in [k-1]$, let $m(v_1, v_{n-2i+2}) = d_{n-2i+2} - 1$, $m(v_1, v_{n-2i+1}) = d_{n-2i+1} - 1$, and $m(v_{n-2i+1}, v_{n-2i+2}) = 1$. For each $k+1 \leqslant i \leqslant \frac{n-1}{2}$, let $m(v_1, v_{n-2i+2}) = 1$, $m(v_1, v_{n-2i+1}) = 1 + d_{n-2i+1} - d_{n-2i+2}$, and $m(v_{n-2i+1}, v_{n-2i+2}) = d_{n-2i+2} - 1$. Finally, let $m(v_1, v_{n-2k+2}) = 1 + \delta$, $m(v_1, v_{n-2k+1}) = 1 + \delta + d_{n-2k+1} - d_{n-2k+2}$, and $m(v_{n-2k+1}, v_{n-2k+2}) = d_{n-2k+2} - 1 - \delta$. Note that every edge mentioned so far has multiplicity at least 1. Let all other potential edges in G have multiplicity 0. It follows that $\deg(v_i) = d_i$ for all $2 \leqslant i \leqslant n$ and

$$\begin{aligned}
\deg(v_1) &= \sum_{i=2}^n m(v_1, v_i) \\
&= \sum_{i=1}^{k-1} (d_{n-2i+1} + d_{n-2i+2} - 2) + \sum_{i=k}^{\frac{n-1}{2}} (d_{n-2i+1} - d_{n-2i+2} + 2) + 2\delta \\
&= d_1 - D + 2\sum_{i=1}^{k-1} \overline{d}_{n-2i+2} + (n-1) + 2\delta = d_1,
\end{aligned}$$

where the last equality follows from the definition of δ. Hence, G is a multigraph with degree sequence $(d_1, \ldots, d_n)$. Moreover, as an edge in G has positive multiplicity if and only if it is of the form $v_1 v_i$ for $2 \leqslant i \leqslant n$, or of the form $v_{2i} v_{2i+1}$ for some $1 \leqslant i \leqslant \frac{n-1}{2}$, we see that G is triangular, completing the proof of the odd case.

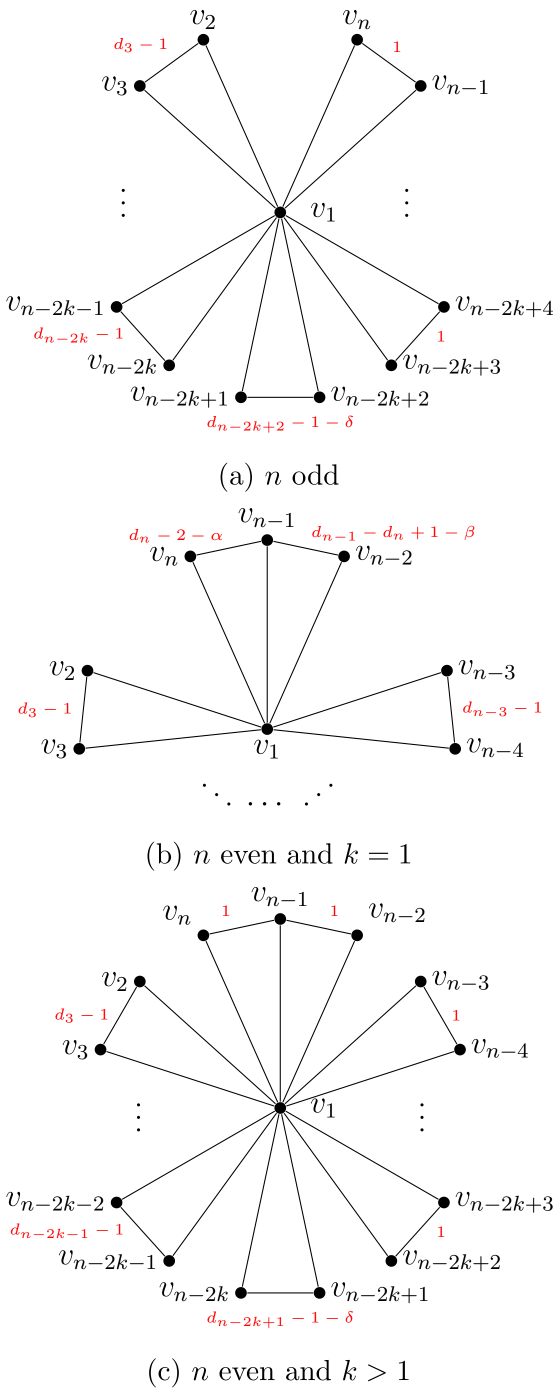


Figure 1: The fan-shaped constructions in Lemma 7. For simplicity only multiplicities of edges not containing v_1 are labelled. Multiplicities of edges containing v_1 are included in the proof and can be deduced using $\deg(v_i) = d_i$ for all $i \in [n]$.

If n is even, then $d_1-\sum_{i=2}^n d_i$ is also even, and thus $d_1 \leqslant \sum_{i=2}^n d_i-n$. Let $\overline{d}_{n-1} = d_{n-1}-3$, and for each $2 \leqslant i \leqslant \frac{n-2}{2}$, let $\overline{d}_{n+1-2i} = d_{n+1-2i} - 2$. It follows that

$$\begin{aligned}\frac{1}{2}\left(D-(n-2)\right) &= \frac{1}{2}\left(\sum_{i=1}^{n}(-1)^{i-1}d_i-(n-2)\right)\\ &\leqslant \frac{1}{2}\left(\sum_{i=2}^{n}d_i+\sum_{i=2}^{n}(-1)^{i-1}d_i-(2n-2)\right)\\ &= \sum_{i=1}^{\frac{n-2}{2}}d_{n+1-2i}-(n-1)=\sum_{i=1}^{\frac{n-2}{2}}\overline{d}_{n+1-2i}.\end{aligned}$$

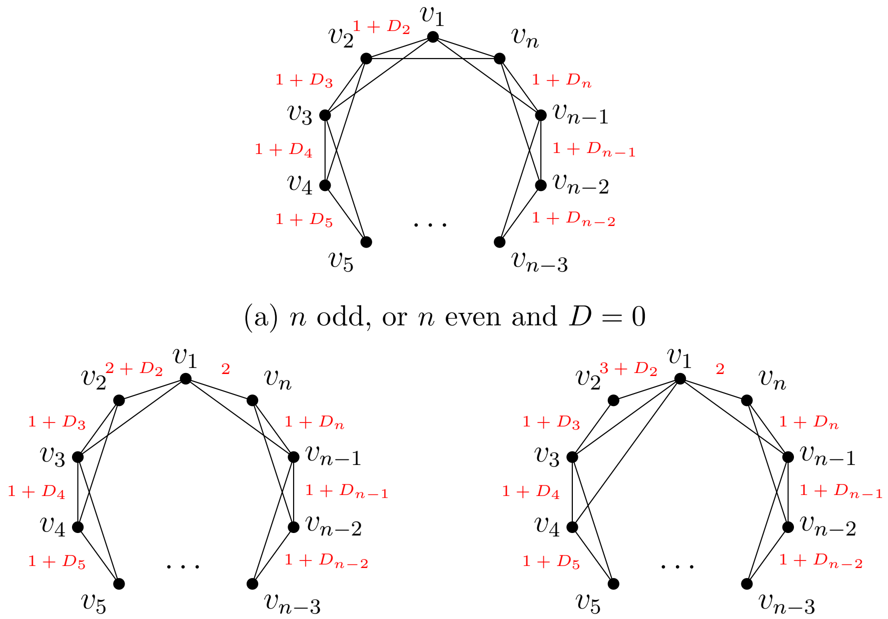
Hence, there exists an index $1 \leqslant k \leqslant \frac{n-2}{2}$ such that $\sum_{i=1}^{k-1}\overline{d}_{n+1-2i} \leqslant \frac{1}{2}(D-(n-2)) \leqslant \sum_{i=1}^{k}\overline{d}_{n+1-2i}$. Let $\delta = \frac{1}{2}(D-(n-2)) - \sum_{i=1}^{k-1}\overline{d}_{n+1-2i}$, so $0 \leqslant \delta \leqslant \overline{d}_{n+1-2k}$.

If $k = 1$, let α, β be any non-negative integers satisfying $\alpha \leqslant d_n-3$, $\beta \leqslant d_{n-1}-d_n$ and $\alpha+\beta = \delta$. Such α, β exists as $d_n-3+d_{n-1}-d_n = \overline{d}_{n-1} \geqslant \delta$. Consider the multigraph G on n vertices $v_1, \dots, v_n$ whose edge multiplicities are given as follows (see also Figure 1b). Let $m(v_1, v_n) = 2+\alpha$, $m(v_1, v_{n-1}) = 1+\alpha+\beta$, $m(v_1, v_{n-2}) = d_{n-2}-d_{n-1}+d_n-1+\beta$, $m(v_n, v_{n-1}) = d_n-2-\alpha$ and $m(v_{n-1}, v_{n-2}) = d_{n-1}-d_n+1-\beta$. For each $2 \leqslant i \leqslant \frac{n-2}{2}$, let $m(v_1, v_{n-2i+1}) = 1$, $m(v_1, v_{n-2i}) = 1+d_{n-2i}-d_{n-2i+1}$, and $m(v_{n-2i}, v_{n-2i+1}) = d_{n-2i+1}-1$. Note that every edge mentioned so far has multiplicity at least 1. Let all other potential edges in G have multiplicity 0. Then, $\deg(v_i) = d_i$ for all $2 \leqslant i \leqslant n$ and

$$\begin{aligned}\deg(v_1) &= 2+2\alpha+2\beta+d_{n-2}-d_{n-1}+d_n+\sum_{i=2}^{\frac{n-2}{2}}(2+d_{n-2i}-d_{n-2i+1})\\ &= 2+2\delta+d_1-D+(n-4) = d_1.\end{aligned}$$

If $k > 1$, consider the multigraph G on n vertices $v_1, \dots, v_n$ whose edge multiplicities are given as follows (see also Figure 1c). Let $m(v_1, v_n) = d_n-1$, $m(v_1, v_{n-1}) = d_{n-1}-2$, $m(v_1, v_{n-2}) = d_{n-2}-1$, and $m(v_n, v_{n-1}) = m(v_{n-1}, v_{n-2}) = 1$. For each $2 \leqslant i \leqslant k-1$, $m(v_1, v_{n-2i+1}) = d_{n-2i+1}-1$, $m(v_1, v_{n-2i}) = d_{n-2i}-1$, and $m(v_{n-2i}, v_{n-2i+1}) = 1$. For each $k+1 \leqslant i \leqslant \frac{n-2}{2}$, $m(v_1, v_{n-2i+1}) = 1$, $m(v_1, v_{n-2i}) = 1+d_{n-2i}-d_{n-2i+1}$, and $m(v_{n-2i}, v_{n-2i+1}) = d_{n-2i+1}-1$. Finally, let $m(v_1, v_{n-2k+1}) = 1+\delta$, $m(v_1, v_{n-2k}) = 1+\delta+d_{n-2k}-d_{n-2k+1}$, and $m(v_{n-2k}, v_{n-2k+1}) = d_{n-2k+1}-1-\delta$. Note that from assumptions, every edge mentioned so far has multiplicity at least 1. Let all other potential edges in G have multiplicity 0. Then $\deg(v_i) = d_i$ for all $2 \leqslant i \leqslant n$ and

$$\begin{aligned}\deg(v_1) &= d_n+d_{n-1}+d_{n-2}-4+\sum_{i=2}^{k-1}(d_{n-2i+1}+d_{n-2i}-2)+\sum_{i=k}^{\frac{n-2}{2}}(d_{n-2i}-d_{n-2i+1}+2)+2\delta\\ &= d_1-D+2\sum_{i=1}^{k-1}\overline{d}_{n-2i+1}+(n-2)+2\delta = d_1.\end{aligned}$$



(a) n odd, or n even and $D = 0$

(b) n even and $D = 2$

(c) n even and $D = 4$

Figure 2: The constructions in Lemma 8. Every edge here with no labelled multiplicity has multiplicity 1.

Therefore, regardless of whether or not $k = 1$, G is a multigraph with degree sequence $(d_1, \dots, d_n)$. Moreover, we have that an edge in G has positive multiplicity if and only if it is of the form $v_1 v_i$ for $2 \leqslant i \leqslant n$, or of the form $v_{2i} v_{2i+1}$ for some $1 \leqslant i \leqslant \frac{n-2}{2}$, or it is $v_{n-1} v_n$. Hence, G is triangular, proving the even case. □

Lemma 8. *Let $n \geqslant 5$ and let $(d_1, \dots, d_n)$ be a sequence of non-negative integers satisfying*

- $d_2 \geqslant \cdots \geqslant d_n \geqslant 4$,
- $D = \sum_{i=1}^{n} (-1)^{i-1} d_i$ *is equal to* 4 *if* n *is odd, and one of* $0, 2, 4$ *if* n *is even,*

then there exists a triangular multigraph G with degree sequence $(d_1, \dots, d_n)$.

Proof. For each $i \in [n]$, let $d'_i = d_i - 4$. For each $2 \leqslant i \leqslant n$, let $D_i = \sum_{j=i}^{n} (-1)^{j-i} d'_j$, and note that $D_i \geqslant 0$.

If n is odd, then by assumption $D = 4$. Consider the multigraph G on n vertices $v_1, \dots, v_n$ with edge multiplicities $m(v_i, v_j)$ given as follows (see also Figure 2a), where we use addition mod n in the indices. For each $i \in [n-1]$, let $m(v_i, v_{i+1}) = 1 + D_{i+1}$, and let $m(v_n, v_1) = 1$. For each $i \in [n]$, let $m(v_i, v_{i+2}) = 1$. Let all other potential edges in G have multiplicity 0. Note that for each $2 \leqslant i \leqslant n-1$,

$$\begin{aligned} \deg(v_i) &= m(v_i, v_{i-2}) + m(v_i, v_{i-1}) + m(v_i, v_{i+1}) + m(v_i, v_{i+2}) \\ &= 1 + (1 + D_i) + (1 + D_{i+1}) + 1 \end{aligned}$$

$$= 4 + d_i' = d_i,$$

and similarly

$$\deg(v_1) = 4 + D_2 = 4 + \sum_{j=2}^{n} (-1)^j d_j' = 4 + \sum_{j=2}^{n} (-1)^j d_j = 4 + d_1 - D = d_1,$$
$$\deg(v_n) = 4 + D_n = 4 + d_n' = d_n.$$

Hence, G is a multigraph with degree sequence $(d_1, \ldots, d_n)$. Furthermore, since an edge in G has positive multiplicity if and only if it connects vertices whose indices have difference 1 or 2 mod n, we see that G is triangular, completing the proof when n is odd.

If n is even, then by assumption D could be 0, 2 or 4. Let G be the multigraph with the same definition as in the case when n is odd (see also Figure 2a). The same calculations show $\deg(v_i) = d_i$ for all $2 \leqslant i \leqslant n$, while

$$\begin{aligned}\deg(v_1) &= 4 + D_2 = 4 + (d_2' - d_3') + \cdots + (d_{n-2}' - d_{n-1}') + d_n' \\ &= 4 + (d_2 - d_3) + \cdots + (d_{n-2} - d_{n-1}) + (d_n - 4) = 4 + d_1 - D - 4 = d_1 - D.\end{aligned}$$

Hence, if $D = 0$, then G is a triangular multigraph with degree sequence $(d_1, \ldots, d_n)$, as required.

If $D = 2$, let G' be the multigraph obtained from G by increasing $m(v_1, v_2)$ and $m(v_1, v_n)$ by 1, and decreasing $m(v_2, v_n)$ by 1 to 0 (see also Figure 2b). Note that G' is a multigraph with degree sequence $(d_1, \ldots, d_n)$. As edge v_1v_2 is in triangle $v_1v_2v_3$ and edge v_1v_n is in triangle $v_1v_nv_{n-1}$, G' is still triangular.

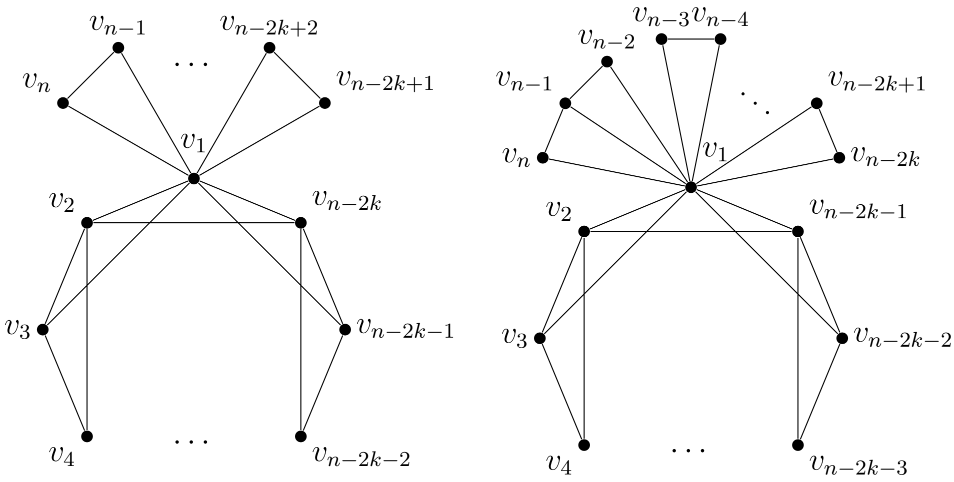
If $D = 4$, let G'' be the multigraph obtained from G by increasing $m(v_1, v_2)$ by 2 and $m(v_1, v_n)$ by 1, increasing $m(v_1, v_4)$ from 0 to 1, and decreasing both $m(v_2, v_n)$ and $m(v_2, v_4)$ by 1 to 0 (see also Figure 2c). Note that G'' is a multigraph with degree sequence $(d_1, \ldots, d_n)$. As edges v_1v_2 and v_2v_3 are in triangle $v_1v_2v_3$, edge v_3v_4 is in triangle $v_3v_4v_5$, edge v_1v_4 is in triangle $v_1v_3v_4$, and edge v_1v_n is in triangle $v_1v_nv_{n-1}$, G'' is still triangular. This completes the proof when n is even. □

As a final preparation, we deal with the $n = 3$ and $n = 4$ cases of Theorem 5 in the following lemma.

Lemma 9. *If $n = 3$ or $n = 4$ and $(d_1, \ldots, d_n)$ is a sequence of positive integers satisfying conditions (i)-(iii) of Theorem 5, then there exists a triangular multigraph G with degree sequence $(d_1, \ldots, d_n)$.*

Proof. If $n = 3$, then $D = d_1 - d_2 + d_3 \geqslant d_3 \geqslant 4 \geqslant 3 - 2$. Hence, we may apply Lemma 7 to find such a triangular multigraph G.

If $n = 4$ and $D = d_1 - d_2 + d_3 - d_4 \geqslant 2 = 4 - 2$, then we may again apply Lemma 7 to find such a triangular multigraph G. Since D is even, the only remaining case is $D = 0$, which can only happen if $d_1 = d_2$ and $d_3 = d_4$. Consider the multigraph G on v_1, v_2, v_3, v_4 given by $m(v_1, v_2) = d_1 - 2$, $m(v_3, v_4) = d_3 - 2$, and $m(v_1, v_3) = m(v_1, v_4) = m(v_2, v_3) = m(v_2, v_4) = 1$. Then G has degree sequence (d_1, d_2, d_3, d_4) and is triangular, completing the proof. □



(a) n odd (b) n even

Figure 3: The constructions in Theorem 5. Edge multiplicities are omitted here for simplicity.

We are now ready to prove Theorem 5.

Proof of Theorem 5. Recall that (i) and (ii) implies that $D = \sum_{i=1}^{n}(-1)^{i-1}d_i$ is a non-negative even integer. If $n = 3$ or $n = 4$, we are done by Lemma 9. Now assume $n \geqslant 5$. If $D \geqslant n-2$, we are done by Lemma 7. If $D \leqslant 4$ and n is even, then $D = 0, 2, 4$ and we are done by Lemma 8. If $D \leqslant 4$ and n is odd, then since $D = (d_1-d_2)+\cdots+(d_{n-2}-d_{n-1})+d_n \geqslant 4$, we must have $D = 4$ and so we are done by Lemma 8 as well. Hence, it suffices to consider the case when $n \geqslant 5$ and $6 \leqslant D \leqslant n-3$, which can only happen if $n \geqslant 9$. Let $k = \frac{1}{2}(D-4)$ and $d'_i = d_i - 4$ for all $i \in [n]$. Note that $k \geqslant 1$ and $n - 2k \geqslant 7$. Again, the triangular multigraph G we construct differs slightly depending on the parity of n.

If n is odd, let $(a_1, a_{n-2k+1}, a_{n-2k+2}, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_{n-2k})$ be degree sequences defined as follows. Let $a_1 = 2k + \sum_{i=n-2k+1}^{n}(-1)^i d_i$ and $a_i = d_i$ for all $n-2k+1 \leqslant i \leqslant n$. Let $b_1 = 4+\sum_{i=2}^{n-2k}(-1)^i d_i$ and $b_i = d_i$ for all $2 \leqslant i \leqslant n-2k$. Then $a_{n-2k+1} \geqslant \cdots \geqslant a_n \geqslant 4$, $a_1 + \sum_{i=n-2k+1}^{n} a_i$ is even, $a_1 \leqslant \sum_{i=n-2k+1}^{n}(a_i - 1)$ and $a_1 + \sum_{i=n-2k+1}^{n}(-1)^{i-1}a_i = 2k \geqslant (2k+1)-2$. Thus, by Lemma 7, there exists a triangular multigraph G_1 on vertices $v_1, v_{n-2k+1}, \ldots, v_n$ with degree sequence $(a_1, a_{n-2k+1}, \ldots, a_n)$. We also have $b_2 \geqslant \cdots \geqslant b_{n-2k} \geqslant 4$, and $\sum_{i=1}^{n-2k}(-1)^{i-1}b_i = 4$. Thus, by Lemma 8, there exists a triangular multigraph G_2 on vertices $v_1, \ldots, v_{n-2k}$ with degree sequence $(b_1, \ldots, b_{n-2k})$. Let $G = G_1 \cup G_2$. Since $a_1+b_1 = 2k+4+\sum_{i=2}^{n}(-1)^i d_i = 2k+4+d_1-D = d_1$, G is a multigraph with vertices $v_1, \ldots, v_n$ and degree sequence $(a_1+b_1, b_2, \ldots, b_{n-2k}, a_{n-2k+1}, \ldots, a_n) = (d_1, \ldots, d_n)$. Moreover, G is triangular as both G_1, G_2 are and they only share a single vertex v_1. This proves the odd case.

If n is even, let $(a_1, a_{n-2k}, a_{n-2k+1}, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_{n-2k-1})$ be degree sequences defined as follows. $a_1 = 2k + \sum_{i=n-2k}^{n}(-1)^i d_i$ and $a_i = d_i$ for all $n-2k \leqslant i \leqslant n$. $b_1 = 4+\sum_{i=2}^{n-2k-1}(-1)^i d_i$ and $b_i = d_i$ for all $2 \leqslant i \leqslant n-2k-1$. Then $a_{n-2k} \geqslant \cdots \geqslant a_n \geqslant 4$,

$a_1 + \sum_{i=n-2k}^{n} a_i$ is even, $a_1 \leqslant \sum_{i=n-2k}^{n}(a_i - 1)$ and $a_1 + \sum_{i=n-2k}^{n}(-1)^{i-1}a_i = 2k \geqslant (2k+2) - 2$. Thus, by Lemma 7, there exists a triangular multigraph G_1 on vertices $v_1, v_{n-2k}, \ldots, v_n$ with degree sequence $(a_1, a_{n-2k}, \ldots, a_n)$. We also have $b_2 \geqslant \cdots \geqslant b_{n-2k-1}$ and $\sum_{i=1}^{n-2k-1}(-1)^{i-1}b_i = 4$. Thus, by Lemma 8, there exists a triangular multigraph G_2 on vertices $v_1, \ldots, v_{n-2k-1}$ with degree sequence $(b_1, \ldots, b_{n-2k-1})$. Let $G = G_1 \cup G_2$. Since $a_1 + b_1 = 2k + 4 + \sum_{i=2}^{n}(-1)^i d_i = 2k + 4 + d_1 - D = d_1$, G is a multigraph with vertices $v_1, \ldots, v_n$ and degree sequence $(a_1+b_1, b_2, \ldots, b_{n-2k-1}, a_{n-2k}, \ldots, a_n) = (d_1, \ldots, d_n)$. Moreover, G is triangular as both G_1, G_2 are and they only share a single vertex v_1. This proves the even case. □

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